5

Strongly regular graphs

A graph on v vertices is a *strongly regular* graph with parameters $(v, k; a, c)$ if:

(a) it is k-regular;
(b) each pair of adjacent vertices in the graph have exactly a common neighbors;
(c) each pair of distinct nonadjacent vertices in the graph have exactly c common neighbors.

If X is a strongly regular graph, then its complement is also a strongly regular graph. A strongly regular graph X is *primitive* if both X and its complement are connected. If X is not primitive, we call it *imprimitive*. The imprimitive strongly regular graphs are exactly the disjoint unions of complete graphs and their complements, the complete multipartite graph. It is not difficult to show that a strongly regular graph is primitive if and only if $0 < c < k$. It is customary to declare that the complete and empty graphs are not strongly regular, and we follow this custom.

Two examples of strongly regular graphs are the line graphs of complete graphs and the line graphs of complete bipartite graphs. We will meet other large classes of examples as we go through this chapter – most of which arise from well-known combinatorial objects. In this chapter we provide detailed information about the cliques and cocliques for many of these graphs. We see that many interesting objects from design theory and finite geometry occur encoded as cliques and cocliques in these graphs, and this leads to interesting variants of the EKR Theorem.

5.1 An association scheme

As stated in the previous chapter, association schemes with two classes are equivalent to strongly regular graphs. Throughout this section we denote the

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adjacency matrix of a graph X by A, rather than $A(X)$, and the adjacency matrix of the complement of X by $\overline{A}$, rather than $A(\overline{X})$.

5.1.1 Lemma. *Let X be a graph. Then X is strongly regular if and only if $\mathcal{A} = \{I, A, \overline{A}\}$ is an association scheme.*

Proof. Assume that X is a strongly regular graph with parameters $(v, k; a, c)$. It is clear that the set of matrices $\mathcal{A} = \{I, A, \overline{A}\}$ satisfies properties (a) through (c) in the definition of an association scheme given in Section 3.1. We only need to confirm that the product of any two matrices in $\mathcal{A}$ is a linear combination of other matrices in $\mathcal{A}$ (in doing this, we also show that the matrices commute).

Since the (x, y)-entry in A^2 gives the number of length-2 paths between the vertices x and y, we have that

$$A^2 = kI + aA + c\overline{A}.$$

Thus A^2 is in the span of the other matrices, and we can conclude the same for $(\overline{A})^2$, since it is also an adjacency matrix for a strongly regular graph. Finally, we note that both $\overline{A}A$ and $A\overline{A}$ are equal to

$$(k-a)A + (k-c)\overline{A};$$

this also proves that the matrices A and $\overline{A}$ commute.

Conversely, assume that the matrices $\{I, A, \overline{A}\}$ are an association scheme. Then,

$$A^2 = \alpha I + \beta A + \gamma \overline{A}.$$

Let X be the graph that corresponds to A. From the equation, it is clear that any vertex in X is adjacent to exactly α vertices. Further, if two vertices of X are adjacent, then they share β common neighbors and nonadjacent vertices share γ common neighbors. Thus X is strongly regular with parameters $(v, \alpha; \beta, \gamma)$. □

5.1.2 Corollary. *In any symmetric association scheme with two classes, the classes correspond to strongly regular graphs.* □

5.2 Eigenvalues

A primitive strongly regular graph has exactly three distinct eigenvalues, and the converse of this is also true – a connected regular graph with exactly three eigenvalues must be strongly regular. We denote the eigenvalues of a strongly regular graph by k (its valency), θ and τ. We will always choose θ and τ so that

$\theta > \tau$ and denote their multiplicities by m_θ and m_τ. Since a strongly regular graph is connected, the multiplicity of k will always be 1 and the all-ones vector, **1**, is an eigenvector for this eigenvalue. We call the following matrix the *modified matrix of eigenvalues* of the strongly regular graph. The first column gives the dimensions of the eigenspaces (the multiplicities of the eigenvalues), the second column contains the eigenvalues of the graph, and the third the eigenvalues of its complement.

$$\left(\begin{array}{c|cc} 1 & k & v-1-k \\ m_\theta & \theta & -1-\theta \\ m_\tau & \tau & -1-\tau \end{array}\right)$$

It is possible to express the eigenvalues and their multiplicities in terms of the parameters of the graph. We do not include a proof of this result; rather we refer the reader to [87, Section 10.2].

5.2.1 Theorem. *If X is a primitive strongly regular graph with parameters $(v, k; a, c)$ and define*

$$\Delta = \sqrt{(a-c)^2 + 4(k-c)},$$

then the three eigenvalues of X are

$$k, \quad \theta = \frac{1}{2}(a - c + \Delta), \quad \tau = \frac{1}{2}(a - c - \Delta),$$

with respective multiplicities

$$m_k = 1, \quad m_\theta = -\frac{(v-1)\tau + k}{\theta - \tau}, \quad m_\tau = \frac{(v-1)\theta + k}{\theta - \tau}. \qquad \square$$

As $k > c$, it follows that $\theta > 0$ and $\tau < 0$; thus the maximum and minimum eigenvalues are

$$k, \quad \tau = \frac{(a-c) - \sqrt{(a-c)^2 + 4(k-c)}}{2}. \tag{5.2.1}$$

These values can be used in the ratio bound for cliques, Corollary 3.7.2, and the ratio bound for cocliques, Theorem 2.4.1. We can also apply the inertia bound, Theorem 2.9.1, to any strongly regular graph X to get that $\alpha(X) \leq m_\tau$.

Either of the ratio bound on cocliques or the inertia bound can be stronger. For example, by work of Brouwer and Haemers [33], we know that there is a unique strongly regular graph with parameters $(81, 20; 1, 6)$. Its modified

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matrix of eigenvalues is

$$\left(\begin{array}{c|cc} 1 & 20 & 60 \\ 60 & 2 & -3 \\ 20 & -7 & 6 \end{array}\right).$$

The value given by the ratio bound for cocliques in this graph is 21, while the inertia bound shows that the size of any coclique is no more than 20. Further, the ratio bound for cliques gives that a clique in this graph is no larger than $27/7$. Clearly, neither of the ratio bounds can be tight for this graph.

The remainder of this chapter is devoted to examples of strongly regular graphs in which the ratio bound for cliques holds with equality and can be interpreted as an EKR-type result.

5.3 Designs

A 2-$(n, m, 1)$ *design* is a collection of m-sets of an n-set with the property that every pair from the n-set is in exactly one set. A specific 2-$(n, m, 1)$ design is denoted by $(V, \mathcal{B})$, where V is the n-set (which we call the *base set*) and $\mathcal{B}$ is the collection of m-sets – these are called the *blocks* of the design. A 2-$(n, m, 1)$ design may also be called a 2-design. A simple counting argument shows that the number of blocks in a 2-$(n, m, 1)$ design is $\frac{n(n-1)}{m(m-1)}$ and each element of V occurs in exactly $\frac{n-1}{m-1}$ blocks (this is usually called the *replication number*). There are many references for more information on 2-designs; we simply recommend the *Handbook of Combinatorial Designs* [46] and the references within.

The blocks of a 2-design are a set system, and every pair from the base set occurs in exactly one block. Thus two distinct blocks of a 2-design must have intersection size 0 or 1. An intersecting set system from a 2-design is a set of blocks from the design in which any two have intersection of size exactly 1. The question we now ask is, what is the largest possible such set? Clearly if we take the collection of all blocks that contain a fixed element, we will have a system of size $\frac{n-1}{m-1}$. An EKR-type theorem for 2-designs would state that this is the largest possible set of intersecting blocks and determine the conditions when the only intersecting sets of blocks that has this size is the set of all blocks that contain a fixed element. (The first result would be the bound in the EKR Theorem, and the second would be the characterization.) In this section we show that the bound always holds and that the uniqueness holds in some cases. To do this, we define a graph so that the cliques in the graph are exactly the

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intersecting set systems from the design and then we determine the size and structure of the cliques.

The *block graph of a* 2-$(n, m, 1)$ *design* $(V, \mathcal{B})$ is the graph with the blocks of the design as the vertices in which two blocks are adjacent if and only if they intersect. In a 2-design, any two blocks that intersect meet in exactly one point. The block graph of a design $(V, \mathcal{B})$ is denoted by $X_{(V,\mathcal{B})}$. Alternatively, we could define a graph on the same vertex set in which two vertices are adjacent if and only if the blocks do not intersect – this graph is simply the complement of the block graph. A clique in the block graph $X_{(V,\mathcal{B})}$ (or a coclique in its complement) is an intersecting set system from $(V, \mathcal{B})$.

Fisher's inequality implies that the number of blocks in a 2-design is at least n; if equality holds, the design is said to be *symmetric* and the block graph of a symmetric 2-design is the complete graph K_n. To avoid this trivial case, we assume that our designs are not symmetric.

The block graph of a 2-$(n, m, 1)$ design is strongly regular – this can be seen by simply calculating the parameters. From these parameters the eigenvalues of the association scheme can also be calculated (this is not a pleasant task, so we leave it as an exercise!).

5.3.1 Theorem. *The block graph of a* 2-$(n, m, 1)$ *design (that is not symmetric) is strongly regular with parameters*

$$\left(\frac{n(n-1)}{m(m-1)}, \quad \frac{m(n-m)}{(m-1)}; \quad (m-1)^2 + \frac{n-1}{m-1} - 2, \quad m^2\right).$$

The modified matrix of eigenvalues is

$$\left(\begin{array}{c|cc} 1 & \frac{m(n-m)}{m-1} & \frac{(n-1)(n-m^2)}{m(m-1)} + m - 1 \\ n-1 & \frac{n-m^2}{m-1} & -1 - \frac{n-m^2}{m-1} \\ \frac{n(n-1)}{m(m-1)} - n & -m & m-1 \end{array}\right). \qquad \square$$

This association scheme is similar to the Johnson scheme in that relations are defined on sets by the size of their intersection, but since the sets are from a 2-design, there are only two possible sizes of intersections and hence only two classes.

For any nonsymmetric 2-design $(V, \mathcal{B})$ with block graph $X_{(V,\mathcal{B})}$, by the ratio bound for cliques, Corollary 3.7.2,

$$\omega(X_{(V,\mathcal{B})}) \leq 1 - \frac{k}{\tau} = 1 - \frac{\frac{m(n-m)}{(m-1)}}{-m} = \frac{n-1}{m-1}.$$

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It is not difficult to construct a clique of this size: for any $i \in \{1, \dots, n\}$ let S_i be the collection of all blocks in the design that contain i. We call the cliques S_i the *canonical cliques* of the block graph.

5.3.2 Theorem. *If $X_{(V,\mathcal{B})}$ is the block graph of 2-$(n, m, 1)$ design, then*

$$\omega(X) = \frac{n-1}{m-1}. \qquad \square$$

From this theorem, we know that a set of intersecting blocks in a 2-design is no larger than the set of all blocks that contain a common point – this is the bound for an EKR-type theorem for the blocks in a design.

5.3.3 Theorem. *The largest set of intersecting blocks from a 2-$(n, m, 1)$ design has size $\frac{n-1}{m-1}$.* $\square$

It is not known for which designs these are the only maximal intersecting sets. We can offer a partial result.

5.3.4 Theorem. *If a clique in the block graph of a 2-$(n, m, 1)$ design does not consist of all the blocks that contain a given point, then its size is at most $m^2 - m + 1$.*

Proof. Assume that C is a non-canonical clique and that the set $\{1, \dots, m\}$ is in C. Divide the other vertices in the clique into m groups, labeled G_i such that each vertex in group G_i contains the element i.

Assume that G_1 is the largest group. Since the clique is non-canonical, there is a vertex in G_i for some $i > 1$. All the vertices in G_1 must intersect this vertex so each vertex of G_1 must contain one of the $m-1$ elements in this vertex (but not the element i, as 1 and i are both in the set $\{1, \dots, m\}$). Since no two vertices of G_1 can contain the same element from the vertex in G_i, the size of G_1 can be no more than $(m-1)$. Since G_1 is the largest group the size of the clique is no more than $m(m-1)+1$. $\square$

A corollary of this is an analog of the EKR Theorem, with the characterization of maximal families, for intersecting sets of blocks in a 2-$(n, m, 1)$ design.

5.3.5 Corollary. *The only cliques of size $\frac{n-1}{m-1}$ in the block graph $X_{(V,\mathcal{B})}$ of a 2-$(n, m, 1)$ design with $n > m^3 - 2m^2 + 2m$ are the sets of blocks that contain a given point i in $\{1, \dots, n\}$.* $\square$

The characterization in this corollary may fail if $n \leq m^3 - 2m^2 + 2m$. For example, consider the projective geometry $PG(3, 2)$. The points of this geometry can be identified with the 15 nonzero vectors in a 4-dimensional vector space V over $GF(2)$, and the lines with the 35 subspaces of dimension 2. This gives us a design with parameters 2-(15, 3, 1), where each block consists of the three nonzero vectors in a 2-dimensional subspace. There are exactly 15 subspaces of V with dimension 3, and each such subspace contains exactly seven points and exactly seven lines and so provides a copy of the projective plane of order two. In the block graph, the seven lines in any one of these projective planes forms a clique of size 7. In addition, each point of the design lies on exactly seven lines, and this provides a second family of 15 cliques of size 7.

We will offer a few comments on how it might be possible to characterize all the cliques in the block graph of a 2-design. Since equality holds in the ratio bound for cliques in the block graph for any 2-$(n, m, 1)$ design, by Corollary 3.7.2, the characteristic vectors of the sets S_i are orthogonal to the τ-eigenspace. Thus these vectors lie in the sum of the eigenspace with dimension $n-1$ and the eigenspace of dimension 1. The following theorem shows that characteristic vectors of the sets S_i actually span the sum of these eigenspaces. With this fact, the characteristic vector of any clique of maximum size must be a linear combination of characteristic vectors of the sets S_i. A possible way to characterize the maximum cliques is to characterize all linear combinations of the characteristic vectors of S_i that gives a 01-vector with weight $\frac{n-1}{m-1}$. It is left as an exercise to show that this method works for the block graph of the 2-(15, 3, 1) design described earlier.

Let H be the matrix whose columns are the characteristic vectors of S_i – thus the rows of H are the characteristic vectors of the blocks in the design. The next result implies that H has rank n by proving a more general result that we will use later. A t-(n, k, λ) *design* is a collection of k-subsets (called the *blocks*) from an n-set with the property that any t-subset from the n-set is contained in exactly λ blocks.

5.3.6 Lemma. *For any t-(n, k, λ) design with $t \geq 2$ let H be the matrix whose rows are the characteristic vectors of the blocks of the design. If $n \neq 2k-1$ then H has full rank.*

Proof. By counting the number of blocks that contain a single element from the point set, and counting the number of blocks that contain a pair of elements from the point set, we get that

$$H^T H = \lambda \frac{\binom{n-1}{t-1}}{\binom{k-1}{t-1}} I_n + \lambda \frac{\binom{n-2}{t-2}}{\binom{k-2}{t-2}} (J_n - I_n).$$

This matrix has zero as an eigenvalue if and only if $n = 2k-1$. Thus $\mathrm{rk}(H) = \mathrm{rk}(H^T H) = n$, provided that $n \neq 2k-1$. □

Finally, we consider the cocliques in the block graph. Since $X_{(V,\mathcal{B})}$ is regular, the ratio bound for cocliques can be applied. Using the eigenvalues given in Theorem 5.3.1, this bound gives that $\alpha(X) \leq n/m$. Since this number is not always an integer, it is not surprising that it is not always possible to find a coclique of this size. A coclique in the block graph with size n/m is a set of blocks that partition the point set. Such a set in a 2-$(n, m, 1)$ design is called a *parallel class*. The ratio bound for cocliques holds with equality if and only if the design has a parallel class. If a design is *resolvable* then its block set can be partitioned into parallel classes, and clearly equality holds in this bound. The standard examples of resolvable designs are the affine planes, which can be defined as designs with parameters 2-$(m^2, m, 1)$. There are many other examples. In fact, Ray-Chaudhuri amd Wilson [145] proved that whenever n is large enough and divisible by m, there is a resolvable 2-$(n, m, 1)$ design.

Finally, since we have a bound on the size of the largest coclique, we also have the following bound on the chromatic number.

5.3.7 Lemma. *If $X_{(V,\mathcal{B})}$ is the block graph of a* 2-$(n, m, 1)$ *design then*

$$\chi(X) \geq \frac{n-1}{m-1}$$

and equality holds if and only if the design is resolvable. □

5.4 The Witt graph

We now consider a specific design where the method described in the previous section can be used to completely determine the cliques in the block graph of a design. The design that we use is a 3-(22, 6, 1) design – such a design can be constructed by taking the derived design of the Witt design with parameters 4-(23, 7, 1). This design has 77 blocks and each element is in exactly 21 blocks. Throughout this section we denote the block graph of this design by X. The graph X is strongly regular with parameters $(77, 60; 47, 45)$ and its eigenvalues are $\{60, -3, 5\}$ with multiplicities $\{1, 55, 21\}$, respectively. The complement of X is known as the *Witt graph*.

The ratio bound on the cliques of X is

$$1 - \frac{60}{-3} = 21.$$

The set of all blocks that contain a fixed point forms a clique of this size. We call such cliques the *canonical cliques* of X; we will show that the canonical cliques are the only maximum cliques in X.

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Define a matrix H to have the characteristic vectors of the canonical cliques as its columns. By Lemma 5.3.6, H has full rank and we conclude that the columns of H span the orthogonal complement of the (-3)-eigenspace. Since the ratio bound holds with equality, the characteristic vector of any maximum clique is in this orthogonal complement and hence is a linear combination of the columns of H. Assume that C is a maximum clique in this graph and denote its characteristic vector by v_C. Then there exists a vector y such that $Hy = v_C$. We will show that y must be the vector with all entries equal to 0 except one entry that is equal to 1.

Let B be a block in the design that is also in the clique. Order the rows of H as follows: first is the block B; then all blocks that are disjoint from B (in any order); finally all remaining blocks (in any order). Further, order the columns of H so that the characteristic vectors of the canonical clique corresponding to elements $i \in B$ occur before those that correspond to elements $i \notin B$. With this ordering we can write the matrix H and the vector y in blocks so that

$$Hy = \left(\begin{array}{c|c} \mathbf{1}_{1\times 6} & \mathbf{0}_{1\times 16} \\ \hline \mathbf{0}_{6\times 16} & M \\ \hline H_1 & H_2 \end{array}\right) \left(\begin{array}{c} y_1 \\ \hline y_2 \end{array}\right) = v_C.$$

Since none of the blocks disjoint from B can be in the clique, it must be that

$$\left(\mathbf{0}_{6\times 16} \,\middle|\, M \right) \left(\begin{array}{c} y_1 \\ \hline y_2 \end{array}\right) = \mathbf{0}_{6\times 1}.$$

At this point we need to consider the design that we used to construct the graph. This particular design has the property that the set of all blocks that do not contain any of the points from B forms a 2-(16, 6, 2) design (this is not hard to see and a proof is given in [84]). Lemma 5.3.6 shows that M is full rank, so we conclude that y_2 must be the zero vector.

Considering the multiplication of the first row of H with y, we see that the sum of the first six entries of y must be 1. Finally, the multiplication of H_1 and y_1 must produce a 01-vector. For any two columns in H_1 there is a row that has a one in each of these columns and zeros everywhere else. Considering this, some short calculations show that exactly one entry in y_1 is one and all other entries are zero. So we conclude that y must be a 01-vector with exactly one entry equal to 1 and that v_C is one of the columns of H.

The Witt graph is the graph induced by the vertices in the Higman-Sims graph that are not adjacent to a given vertex. So each clique in our graph

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corresponds to a coclique of size 22 in the Higman-Sims graph. Thus we have characterized all cocliques of size 22 in the Higman-Sims graph.

5.5 Orthogonal arrays

An *orthogonal array* $OA(m,n)$ is an $m \times n^2$ array with entries from $\{1,\dots,n\}$ with the property that the columns of any $2 \times n^2$ subarray consist of all n^2 possible pairs. In particular, between any two rows each pair from $\{1,\dots,n\}$ occurs in exactly one column. This implies that in each row, each element occurs in exactly n columns. There are many applications of orthogonal arrays, particularly to test design, and orthogonal arrays are related to many other interesting combinatorial designs (for example, they are equivalent to a set of mutually orthogonal Latin squares). See [46, Part III] for more details.

The *block graph of an orthogonal array* $OA(m,n)$ is defined to be the graph whose vertices are columns of the orthogonal array, where two columns are adjacent if there exists a row where they have the same entry. We denote the block graph for an orthogonal array $OA(m,n)$ by $X_{OA(m,n)}$.

It is well known for any $OA(m,n)$ that $m \le n+1$ and equality holds if and only if there exists a projective plane of order n (see [46, Part III, Section 3]). It is left as an exercise to show that for any n, the block graph of $OA(n+1,n)$ is isomorphic to the complete graph on n^2 vertices. So for the remainder of this section we assume that $m < n+1$.

Any two columns in an orthogonal array can have at most one row in which they have the same entry. Thus the complement of $X_{OA(m,n)}$ is the graph in which columns are adjacent if and only if there are no rows in which they have the same entry. This, as in the case of the block graph for a 2-design, produces a two-class association scheme. To prove this, we show that the block graph is a strongly regular graph. The parameters can be calculated directly (in fact, the calculations are much easier than for block graphs of 2-designs, so they are not left as an exercise).

5.5.1 Theorem. *If $OA(m,n)$ is an orthogonal array where $m < n+1$, then its block graph $X_{OA(m,n)}$ is strongly regular, with parameters*

$$\big(n^2, \quad m(n-1); \quad (m-1)(m-2)+n-2, \quad m(m-1)\big),$$

and modified matrix of eigenvalues

$$\left(\begin{array}{c|cc} 1 & m(n-1) & (n-1)(n+1-m) \\ m(n-1) & n-m & m-n-1 \\ (n-1)(n+1-m) & -m & m-1 \end{array}\right). \qquad \square$$

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We can apply the ratio bound for cliques to the block graph of an orthogonal array to see that

$$\omega(X_{OA(m,n)}) \leq 1 - \frac{m(n-1)}{-m} = n.$$

It is straightforward to construct cliques that meet this bound. If $i \in \{1, \ldots, n\}$ let $S_{r,i}$ be the set of columns of $OA(m, n)$ that have the entry i in row r. Clearly these sets are cliques, and since each element of $\{1, \ldots, n\}$ occurs exactly n times in each row, the size of $S_{r,i}$ is n for all i and r. These cliques are called the *canonical cliques* in the block graph of the orthogonal array.

If we view columns of an orthogonal array that have the same entry in the same row as *intersecting columns*, then we can view this bound as the bound in the EKR Theorem for intersecting columns of an orthogonal array. The question we ask is, under what conditions will all cliques of size n in the graph $X_{OA(m,n)}$ be canonical? The following answer can be viewed as the uniqueness part of the EKR Theorem – it tells us that if $n > (m-1)^2$, then any clique of maximal size is a canonical clique. This is equivalent to saying that the largest set of intersecting columns in an orthogonal array is the set of all columns that have the same entry in the some row, and these sets are the only maximum intersecting sets.

5.5.2 Theorem. *Let $OA(m, n)$ be an orthogonal array. If S is a* non-canonical *clique in the block graph $X_{OA(m,n)}$ then $|S| \leq (m-1)^2$.*

Proof. Assume the column of all zeros is in S (this can be done without loss of generality by swapping numbers in the array). Then every other vertex in the clique is a column with exactly one zero. Split these vertices into m classes according to which row this unique zero is in. Denote the class that has the unique zero in row i by C_i. Assume that C_1 is the largest such class.

Since S is not a canonical clique, there is an $i > 1$ such that the class C_i is non-empty; thus there is a column c that is in the clique, with a zero in the ith position. Every vertex in the first class, C_1, must be intersecting with c. There are only $m-2$ rows where this intersection can occur (since they cannot intersect where the columns have zeros) and no two distinct columns in C_1 can intersect c in the same place (since no pair can be repeated in a $2 \times n^2$ subarray).

This means that C_1 has size no bigger than $m-2$. Since there are m groups, each of size no more than $m-2$, plus the column of all zeros, the total size of clique is no more than $m(m-2)+1 = (m-1)^2$. □

5.5.3 Corollary. *If $OA(m, n)$ is an orthogonal array with $n > (m-1)^2$, then the only cliques of size n in $X_{OA(m,n)}$ are canonical cliques.* □

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Without the bound on n, it is possible to construct orthogonal arrays for which there are maximum cliques in the block graph that are not canonical cliques. Let $m-1$ be a prime power; then there exists an $OA(m, m-1)$ and, using MacNeish's construction [158, Section 6.4.2], it is possible to construct an $OA(m, (m-1)^2)$ from this array. This larger orthogonal array has $OA(m, m-1)$ as a subarray, and thus the graph $X_{OA(m,(m-1)^2)}$ has the graph $X_{OA(m,m-1)}$ as an induced subgraph. Since this subgraph is isomorphic to $K_{(m-1)^2}$, it is a clique of size $(m-1)^2$ in $X_{OA(m,(m-1)^2)}$ that is not canonical.

For example, the following $OA(3, 2)$

$$\begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

can be used to construct this $OA(3, 4)$:

$$\begin{bmatrix} 0011 & 0011 & 2233 & 2233 \\ 0101 & 2323 & 0101 & 2323 \\ 0110 & 2332 & 2332 & 0110 \end{bmatrix}.$$

The first four columns form a maximal clique that is not a canonical clique (there are several other non-canonical maximum cliques).

In general, if the array $OA(m, n)$ has a subarray with m rows that is an $OA(m, m-1)$, then the columns of this subarray form a clique in $X_{OA(m,n)}$ of size $(m-1)^2$. By the ratio bound, it must be that $(m-1)^2 \leq n$ (and if $(m-1)^2 = n$, then this would be a example of an orthogonal array whose block graph has maximum cliques that are not canonical cliques). This can be interpreted as the following result for orthogonal arrays.

5.5.4 Lemma. *Let m and n be integers with $m-1 < n < (m-1)^2$. An orthogonal array $OA(m, n)$ does not have a subarray that is an $OA(m, m-1)$.* □

For orthogonal arrays $O(m, n)$ with $n \leq (m-1)^2$ it is not known when the only maximal cliques are canonical cliques, but we now outline a method that may be useful in determining when this is the case.

Since equality holds in the clique bound, the characteristic vector of any clique of maximal size is orthogonal to the $(-m)$-eigenspace; this means that it is in the sum of the $m(n-1)$-eigenspace and the $(n-m)$-eigenspace. We will show that the characteristic vectors of the sets $S_{r,i}$ span this vector space. Thus the characteristic vector of any maximal clique is a linear combination of the characteristic vectors of $S_{r,i}$. If we can prove, for the particular orthogonal array, that the only linear combination of these vectors that gives a 01-vector with weight n is a trivial combination, then the only maximal cliques are canonical cliques. This result would be the uniqueness part of the EKR Theorem.

To show that the characteristic vectors of the sets $S_{r,i}$ span this vector space, we show that a subset of them are a basis for the $(n-m)$-eigenspace and simply note that the sum of all the characteristic vectors is a multiple of $\mathbf{1}$.

5.5.5 Theorem. *Let $OA(m,n)$ be an orthogonal array. Let N be the incidence matrix for the canonical cliques of the orthogonal array. The columns of N^T indexed by the points*

$$(r,i) \in \{1,2,\dots,m\} \times \{1,2,\dots,n-1\}$$

form a basis for the $(n-m)$-eigenspace.

Proof. The columns of N are the characteristic vectors of the sets $S_{r,i}$, so we can index the columns by (r,i). Define H be to the submatrix of N formed by the columns (r,i) with $i \leq n-1$. Order the columns so that the columns with the same value of i are together in a block, and within the blocks the columns are arranged so the value of r is increasing. Then

$$H^T H = (J_{n-1} \otimes (J_m - I_m)) + nI_{m(n-1)}.$$

The eigenvalues of this matrix are

$$\{(n-1)(m-1)+n, \quad n, \quad 1\}.$$

All the eigenvalues are nonzero and the rank of $H^T H$ is $m(n-1)$. Since the rank of H equals the rank of $H^T H$, the rank of H is $m(n-1)$, and the columns of H are a basis of the $(n-m)$-eigenspace. □

The ratio bound for cocliques also gives the bound that

$$\alpha(X_{OA(m,n)}) \leq n.$$

Unlike the ratio bound for cliques, this bound cannot be met for all orthogonal arrays. A coclique of size c in the block graph of an orthogonal array corresponds to a set of c columns that are pairwise nonintersecting (such a set of columns is called *disjoint*). Not all covering arrays have a set of n disjoint columns. It is an active area of research to determine orthogonal arrays that do, since orthogonal arrays that have a large number of disjoint columns can be used to construct covering arrays with few columns – see [157] for details.

Finally we consider the case when $m=3$. An $OA(3,n)$ is equivalent to an $n \times n$ Latin square (the first two rows of the array describe a position in the

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square, and the final row gives the entry in that position). For this reason the block graph on an orthogonal array with three rows is also known as a *Latin square graph*. If the Latin square has a *transversal* (a set of n positions, each in different rows and different columns, and each containing a different entry) then the block graph of the array has a coclique set of size n. Although it is conjectured by Ryser that every Latin square of odd order has a transversal, there are large families of Latin squares that do not have a transversal [167] (the multiplication tables for abelian groups with cyclic Sylow 2-subgroups are examples). Thus there are many orthogonal arrays whose block graph does not have a ratio tight coclique.

5.6 Partial geometries

The previous two examples can be generalized in one design. A *partial geometry*, denoted $(\mathcal{P}, \mathcal{L})$, with parameters (s, t, ν) is an incidence structure of points (denoted $\mathcal{P}$) and lines (denoted $\mathcal{L}$) such that:

(a) each pair of distinct points lies on at most one line (and so any two distinct lines have at most one point in common);
(b) each line contains exactly $s + 1$ points, and each point is on $t + 1$ lines;
(c) if x is a point and ℓ is a line not on x, then exactly ν of the lines on x contain a point on ℓ.

The first of these conditions says that our incidence structure is a *partial linear space*. The second condition says that the structure is both point-regular and line-regular. The last is equivalent to the condition that exactly ν of the points on a line ℓ are collinear with a point x not on ℓ. It follows that if $(\mathcal{P}, \mathcal{L})$ are the points and lines of a partial geometry with parameters (s, t, ν), then $(\mathcal{L}, \mathcal{P})$ is also a partial geometry with parameters (t, s, ν). The geometry $(\mathcal{L}, \mathcal{P})$ is called the *dual geometry*.

We can determine the number of points in a partial geometry with parameters (s, t, ν) by a simple counting argument. Fix a line ℓ and consider the ordered pairs (x, y) where $x \notin \ell$ and y is a point on ℓ collinear with x. Counting these pairs, we find that

$$(s + 1)ts = (|\mathcal{P}| - s - 1)\nu.$$

This gives that the number of points is

$$|\mathcal{P}| = \frac{(ts + \nu)(s + 1)}{\nu}.$$

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A similar argument counting pairs of intersecting lines (one of which contains a fixed point x) yields that the number of lines is

$$|\mathcal{L}| = \frac{(ts+\nu)(t+1)}{\nu}.$$

We have already met two important families of partial geometries. A design with parameters 2-$(n, m, 1)$ is a partial geometry with parameters

$$s = m-1, \quad t = \frac{n-m}{m-1}, \quad \nu = m.$$

Further, it can be shown that any partial geometry with $\nu = s+1$ must come from a 2-$(n, m, 1)$ design (see [28]).

Our second class of partial geometries comes from orthogonal arrays. Given an $OA(m, n)$ we construct a geometry with point set

$$\{1, \ldots, m\} \times \{1, \ldots, n\}.$$

The lines are the n^2 columns of the array, and a point (i, j) is incident with a given column if the ith entry of the column is equal to j. The parameters of this geometry are

$$s = m-1, \quad t = n-1, \quad \nu = m-1.$$

Further, any partial geometry with $\nu = s$ must come from this construction. The corresponding incidence structures are known as *transversal designs*.

There are two graphs that can be defined on a partial geometry. The *point graph* of a partial geometry is the graph on the points of the geometry, where two distinct points are adjacent if they are collinear. The *line graph* has the lines of the partial geometry as its vertices, and two lines are adjacent if they intersect; the line graph is the point graph of the dual geometry. Because of this we focus our attention mainly on the point graphs of partial geometries.

The point graph of the geometry arising from a 2-design is the complete graph, and the line graph is its block graph. The point graph of the partial geometry associated to an $OA(m, n)$ is $\overline{mK_n}$ and the line graph is the block graph of the array. We have seen that the block graphs for 2-designs and orthogonal arrays are strongly regular graphs. In this section, we see that the point and line graphs of a partial geometry are strongly regular and we determine the spectrums of these graphs. Our approach in this section is to work directly with the geometries. In the next section we re-derive this information using linear algebra.

5.6.1 Theorem. *The point graph of a partial geometry with parameters* (s, t, ν) *is a strongly regular graph with parameters*

$$((s+1)(st+\nu)/\nu, \quad (t+1)s; \quad -1+t(\nu-1), \quad (t+1)\nu)\,.$$

The modified matrix of eigenvalues of this graph is

$$\left(\begin{array}{c|cc} 1 & (t+1)s & \dfrac{st}{\nu}(s-\nu+1) \\ m_\theta & s-\nu & -1-s+\nu \\ m_\tau & -1-t & t \end{array}\right)$$

where

$$\begin{aligned} m_\theta &= \frac{(t+1)((s+1)(st+\nu)-\nu-s\nu)}{\nu(s+t+1-\nu)}, \\ m_\tau &= \frac{((s+1)(st+\nu)-\nu)(s-\nu)+\nu s(t+1)}{\nu(s+t+1-\nu)}. \end{aligned}$$

Proof. We have determined that the number of vertices in the point graph is $(st+\nu)(s+1)/\nu$. To determine the degree of the graph, let x be a point in the geometry. Then x lies on $t+1$ lines, any two of which have only x in common. Hence x is collinear with exactly $(t+1)s$ points.

To compute the parameter c, consider two noncollinear points x and y. There are $t+1$ lines on x and each of these lines contains exactly ν points collinear with y. Therefore $c = (t+1)\nu$.

To complete the list of parameters, take two distinct collinear points x and y, and let ℓ be the unique line that contains both of them. Each of the $s-1$ points on ℓ distinct from x and y is collinear with both. Also, on each of the t lines on y distinct from ℓ there are $\nu-1$ points distinct from y and collinear with x. This gives our stated value for a.

The eigenvalues and their multiplicities can now be determined from the parameters. We leave this as a straightforward but somewhat unpleasant exercise. □

Applying the ratio bound for cliques, Corollary 3.7.2, to the point graph X of a partial geometry with parameters (s, t, ν) shows that

$$\omega(X) \le 1+s.$$

For any line of the geometry, the vertices on the line form a clique of size $s+1$ in the point graph. Moreover, the set of all lines define a set of edge-disjoint cliques of size $s+1$. We define these cliques to be the *canonical cliques*, and the ratio bound proves that no clique in the graph is larger than a canonical

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clique. Since the line graph is the point graph for the dual geometry, these bounds also hold for the line graph. Specifically, the size of the maximal clique in a line graph of a partial geometry with parameters (t, s, ν) is $s + 1$. These bounds can be considered to be EKR-type theorems; we only state this as a result on line graphs, where the concept of intersection is more natural.

5.6.2 Theorem. *In a partial geometry, $(\mathcal{P}, \mathcal{L})$, the set of all lines through a fixed point of $\mathcal{P}$ is a maximum subset of $\mathcal{L}$ such that any two lines in the subset intersect.* □

Again, the next question to ask is, when are the canonical cliques the only maximal cliques in the point (or line) graph of a partial geometry? We consider this question in the next section. Our approach is similar to, but more generalized than, the approach to taken for the block graphs of 2-designs in Section 5.3, and for the block graphs of orthogonal arrays in Section 5.5.

5.7 Eigenspaces of point and line graphs

Let $(\mathcal{P}, \mathcal{L})$ be a partial geometry with parameters (s, t, ν) and let N denote its point-block incidence matrix where the blocks are the lines of the geometry. Then it is easy to see that

$$NJ = (t+1)J, \qquad N^T J = (s+1)J. \tag{5.7.1}$$

Let A be the adjacency matrix of the point graph of the geometry and let B be the adjacency matrix of the line graph. Then

$$NN^T = (t+1)I + A, \qquad N^T N = (s+1)I + B. \tag{5.7.2}$$

Now let x be a point and ℓ a line of the geometry. If $x \in \ell$, there are exactly s points on ℓ distinct from and collinear with x. If $x \notin \ell$, there are exactly ν points on ℓ distinct from and collinear with x. Thus

$$AN = sN + \nu(J - N) = (s-\nu)N + \nu J. \tag{5.7.3}$$

If we multiply the last equation on the right by N^T, we find that

$$ANN^T = (s-\nu)NN^T + \nu(t+1)J;$$

using (5.7.2) to replace A and rearranging yields

$$(NN^T)^2 - (s - \nu + t + 1)NN^T = \nu(t+1)J.$$

This, with (5.7.1), implies that the eigenvalues of NN^T are

$$0, \quad s - \nu + t + 1, \quad (s+1)(t+1).$$

Accordingly the eigenvalues of $A = NN^T - (t+1)I$ are

$$-t-1, \quad s-\nu, \quad s(t+1).$$

Now we have the eigenvalues of the adjacency matrix. We can also determine the dimensions of the eigenspaces. The only eigenvectors belonging to the eigenvalue $(t+1)s$ are the constant vectors. Thus the eigenspace is spanned by the all-ones vector and has dimension 1. If $z \in \ker(N^T)$, then

$$(A + (t+1)I)z = NN^T z = 0.$$

So if $z \neq 0$, it is an eigenvector of A with eigenvalue $-t-1$. We see that $-t-1$ has multiplicity

$$\dim(\ker(N^T)) = |\mathcal{P}| - \mathrm{rk}(N).$$

Another consequence of (5.7.3) is that the column space of N is A-invariant, and so it is spanned by eigenvectors of A. We can determine which eigenvectors are in this span. From (5.7.3), we see that if z is a balanced vector (i.e., $z^T\mathbf{1} = 0$) then Nz is an eigenvector with eigenvalue $s-\nu$. If z is balanced, then Nz is also balanced because $\mathbf{1}^T N = (s+1)\mathbf{1}^T$. Therefore the set of balanced vectors Nz is equal to the subspace of balanced vectors in $\mathrm{col}(N)$, which has codimension 1 in $\mathrm{col}(N)$. Hence we deduce that $s-\nu$ is an eigenvalue of A with multiplicity at least $\mathrm{rk}(N)-1$.

The theory just presented could be used to confirm Theorem 5.6.1 (and Theorems 5.3.1 and 5.5.1), but we leave this for the reader. However, it is worth explicitly considering the incidence matrices.

The columns of N are the characteristic vectors of the canonical cliques in the point graph. Since equality holds in the clique bound, Corollary 3.7.2, each column of N is orthogonal to the $(-1-t)$-eigenspace. Thus each column is in the sum of the $(s-\nu)$-eigenspace and the $s(t+1)$-eigenspace. The dimension of this vector space is exactly $\mathrm{rk}(N)$, and we conclude that the columns of N span this space. Since equality holds in the clique bound, any clique in the line graph is a linear combination of the columns of N. To show that the canonical cliques are the only maximum cliques, we would need to show that the only 01-vectors of weight $s+1$ in $\mathrm{col}(N)$ are the columns of N.

We now shift our focus to the cocliques of the point graph of a partial geometry. Each point in a coclique S lies on $t+1$ lines, and these sets of lines are pairwise disjoint. Thus the number of lines that intersect a point in S is bounded by the total number of lines, so

$$|S|(t+1) \leq \frac{1}{\nu}(t+1)(st+\nu),$$

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and therefore

$$\alpha(X) \le \frac{st + \nu}{\nu}.$$

A quick calculation shows that this is the same bound given by the ratio bound for cocliques. When $\nu = 1$ a coclique of this size is called an *ovoid*. We use the term ovoid for any coclique of size $(st + \nu)/\nu$ in the point graph of any partial geometry.

If an ovoid exists in the point graph of a partial geometry, then the clique-coclique bound holds with equality. This implies that the balanced characteristic vectors of an ovoid and a clique are orthogonal (see the comments following Corollary 2.1.3). We have seen that the balanced characteristic vectors of the canonical cliques always span the the θ-eigenspace. It is interesting to consider, for a partial geometry that has ovoids of size $(st + \nu)/\nu$, whether the balanced characteristic vectors of the ovoids span the τ-eigenspace.

Finally, the chromatic number of the point graph is at least $s + 1$. If equality holds then the point set of the geometry can be partitioned into $s + 1$ ovoids.

5.8 Paley graphs

The final family of strongly regular graphs that we consider is the *Paley graphs*. Let $\mathbb{F}$ be a finite field of order q. The vertices of the Paley graph, $P(q)$, are the elements of $\mathbb{F}$, and vertices are adjacent if and only if their difference is a square in $\mathbb{F}$. We will assume $q \equiv 1 \pmod 4$, since this is the only case where $P(q)$ is an undirected graph. The following theorem lists the basic properties of the Paley graphs.

5.8.1 Theorem. *Let $P(q)$ be a Paley graph with $q \equiv 1 \pmod 4$. Then*

(a) $P(q)$ is self-complementary and arc transitive;
(b) $P(q)$ is a strongly regular graph with parameters

$$(q, \quad (q-1)/2; \quad (q-5)/4, \quad (q-1)/4);$$

(c) the modified matrix of eigenvalues for $P(q)$ is

$$\left(\begin{array}{c|cc} 1 & (q-1)/2 & (q-1)/2 \\ (q-1)/2 & (-1+\sqrt{q})/2 & (-1-\sqrt{q})/2 \\ (q-1)/2 & (-1-\sqrt{q})/2 & (-1+\sqrt{q})/2 \end{array}\right).$$

Proof. Let $\mathcal{S}$ denote the set of nonzero squares in $\mathbb{F}$. Then $|\mathcal{S}| = (q-1)/2$ and $P(q)$ is the Cayley graph for the additive group of $\mathbb{F}$ relative to the connection

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set $\mathcal{S}$; hence it is regular with valency $(q-1)/2$. Since -1 is a square, $\mathcal{S}$ is closed under multiplication by -1 and the graph is not directed.

If x is not a square in $\mathbb{F}$, then $x\mathcal{S}$ is the set of all non-squares in $\mathbb{F}$ and is the complement of $\mathcal{S}$ in $\mathbb{F} \setminus 0$. Therefore the Cayley graph for $\mathbb{F}$ with connection set $x\mathcal{S}$ is the complement of $P(q)$. Thus the map that sends a vertex v of $P(q)$ to xv is an isomorphism from $P(q)$ to its complement, and $P(q)$ is self-complementary.

If z is a nonzero square in $\mathbb{F}$, then $z\mathcal{S} = \mathcal{S}$. It follows that multiplication by z is an automorphism of $P(q)$ that fixes 0 and maps 1 to z. Therefore there is an automorphism that maps any arc adjacent to 0 to any other arc adjacent to 0. From this we can conclude that $P(q)$ is arc transitive.

Similarly, if x and y are two non-squares and $z = y/x$, then z is a square. So multiplication by z is an automorphism of $P(q)$ that fixes 0 and maps x to y. It follows that the stabilizer of 0 in $\mathrm{Aut}(P(q))$ acts transitively on the vertices adjacent to 0 and on the vertices not adjacent to 0. Therefore the parameters a and c are well defined, and so $P(q)$ is strongly regular.

It remains to determine the parameters of $P(q)$. We know that $v = q$ and $k = (q-1)/2$; it remains to calculate the parameters a and c. First, we count in two different ways the number of edges that join a neighbor of 0 to a vertex at distance 2 from 0. Thus

$$k(k-a-1) = kc,$$

whence $a + c = k - 1$. If X is strongly regular with parameters $(v, k; a, c)$, then the number of triangles on an edge is a. So the number of triangles on an edge in the complement $\overline{X}$ is $v - 2 - 2k + c$. As $P(q)$ is self-complementary, we deduce that

$$a = v - 2 - 2k + c.$$

Since $v = 2k + 1$ we find that $a - c = -1$ and consequently $c = k/2$ and $a = (k-2)/2$. This yields our expressions for the parameters of $P(q)$. The formulas for the eigenvalues and their multiplicities follow from Theorem 5.2.1. □

Any strongly regular graph with the parameters

$$(v, (q-1)/2; (q-5)/4, (q-1)/4)$$

is called a *conference graph*. These graphs are characterized by the condition that the multiplicities of θ and τ are equal. There are conference graphs whose order is not a prime power and hence are not Paley graphs. For further background on conference graphs see [31, Section 1.3].

5.9 Paley graphs of square order

Since the Paley graphs are self-complementary graphs, a bound on the size of a coclique is also a bound on the size of a clique. The ratio bound on cocliques gives

$$\alpha(P(q)) \leq \frac{q}{1 - \frac{q-1}{-(1+\sqrt{q})}} = \sqrt{q}$$

and this also means that $\omega(P(q)) \leq \sqrt{q}$. Clearly this cannot be tight if q is not a square. In fact if q is a prime, the experimental evidence is that $\alpha(P(q))$ is of order $c\log(q)$ (see [45] for more details). In this section, we show that the ratio bound $\alpha(P(q^2)) \leq q$ is realized for all odd prime powers q.

5.9.1 Lemma. *If q is a prime power and 2 does not divide q, then*

$$\omega(P(q^2)) = q.$$

Proof. Let $\mathbb{F}$ be a finite field of order q^2 and $\mathbb{E}$ be the subfield of order q. The nonzero squares in $\mathbb{F}$ form a multiplicative subgroup of $\mathbb{F}^*$ with order

$$\frac{q^2-1}{2} = (q-1)\frac{q+1}{2}.$$

The group $\mathbb{F}^*$ is cyclic with order q^2-1, so for every divisor of q^2-1 there is a unique subgroup of that order.

Let $\mathcal{S}$ be the set of nonzero squares in $\mathbb{F}^*$ and define a homomorphism

$$\phi : \mathbb{F}^* \to \mathcal{S}$$

by $\phi(x) = x^2$. Then the image of ϕ is $\mathcal{S}$, which is the unique subgroup of $\mathbb{F}^*$ with order $(q-1)(q+1)/2$. Further, $\mathcal{S}$ is also cyclic, so $\mathcal{S}$ must contain a unique subgroup of order $q-1$. This subgroup is the unique subgroup of order $q-1$ in $\mathbb{F}^*$. Thus this subgroup is exactly nonzero elements in $\mathbb{E}$.

Since each element of $\mathbb{E}$ is in $\mathcal{S}$, every element in $\mathbb{E}$ is a square in $\mathbb{F}$. Thus the elements of $\mathbb{E}$ induce a clique in $P(q^2)$, and the ratio bound for cliques is tight. □

Since the Paley graphs are self-complementary, this lemma implies that $\alpha(P(q^2)) = q$. Therefore the ratio bound for cocliques and the clique-coclique bound are tight for these graphs.

We can determine the chromatic number of these graphs (in fact, we will do more than just find the chromatic number). Let $\mathcal{S}$ denote the set of nonzero squares in $\mathbb{F}$. Since $\mathbb{E}^*$ is a subgroup of index $(q+1)/2$ in $\mathcal{S}$, its multiplicative cosets partition $\mathcal{S}$ into $(q+1)/2$ subsets each of size $q-1$. Since the elements of $\mathbb{E}^*$ form a clique in $P(q^2)$, each of these subsets is also a clique in $P(q^2)$.

Adding 0 to each of these cliques produces $(q+1)/2$ cliques of size q. Further, each of these cliques forms an additive subgroup of $\mathbb{F}$.

The q additive cosets of any one of these subgroups partition $\mathbb{F}$ into q pairwise disjoint cliques. Hence, from any one of these cosets we obtain a proper q-coloring of the vertices of the complement of $P(q^2)$. Since the Paley graph is self-complementary, we conclude that

$$\chi(P(q^2)) = q.$$

We could reach the same conclusion by simply taking the clique $\mathbb{E}$ and the cosets of it in $\mathbb{F}$. But using this method we see that the cliques on 0 together with their additive cosets provide a set of $q(q+1)/2$ cliques.

We know that the size of the maximum cliques in $P(q^2)$ is q and we know that the elements of the field of order q form a clique of maximum size. But what are the other maximum cocliques in the graph? We have seen that if $\mathcal{S}$ denotes the set of nonzero squares in $\mathbb{F}$, then for any a in $\mathcal{S}$ and any b in $\mathbb{F}$, the set

$$\mathcal{S}(a,b) = \{ax+b \ : \ x \in \mathbb{E}\}$$

is a clique in $P(q^2)$. We say that these cliques are the *square translates* of $\mathbb{E}$ (the square refers to the fact that a is a square, since otherwise $\mathcal{S}(a,b)$ will be a coclique). The square translates of $\mathbb{E}$ can be considered to the *canonical* cliques in the Paley graphs. An analog of the EKR Theorem for Paley graphs would be that the only cliques of size q are the canonical cliques. Blokhuis [20] gave a very interesting proof of this using polynomials. We state this result now.

5.9.2 Theorem. *A clique in $P(q^2)$ of size q is a translate of the set of squares in $GF(q^2)$.* □

There is another possible approach to this theorem. Any clique of size q in $P(q^2)$ meets the ratio bound, and so its balanced characteristic vector lies in the θ-eigenspace of the Paley graph. If we could prove the following two results, we would have a second proof of Blokhuis's result.

(a) The balanced characteristic vectors of the canonical cliques span the θ-eigenspace of $P(q^2)$.
(b) The only balanced characteristic vectors of sets of size q in the eigenspace belonging to θ of $P(q^2)$ are the balanced characteristic vectors of the canonical cliques.

5.10 Exercises

5.1 Let X be a strongly regular graph with parameters $(v, k; a, c)$. What are the parameters of $\overline{X}$?

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5.2 Prove that a strongly regular graph is primitive if and only if $0 < c < k$.

5.3 Prove that the line graphs of the complete graph K_n and the complete bipartite graph $K_{n,n}$ are strongly regular and find the parameters of these strongly regular graphs.

5.4 Prove that a strongly regular graph is imprimitive if and only if 0 or -1 is an eigenvalue.

5.5 A strongly regular graph is also a distance-regular graph. Find the parameters a_r, b_r and c_r for the association scheme generated by a strongly regular graph with parameters $(v, k; a, c)$.

5.6 Confirm that the eigenvalues given in Theorem 5.3.1 for the block graph of a 2-$(n, m, 1)$ design are correct. The first step of this is to confirm that for this strongly regular graph

$$\Delta = \frac{n - 2m + 1}{m - 1} + 1.$$

5.7 Let $(V, \mathcal{B})$ be a 2-$(n, m, 1)$ design with $n = m^3 - 2m^2 + 2m$. Show that any non-canonical clique of size $(n-1)/(m-1)$ in the block graph of $(V, \mathcal{B})$ forms a $(m^2 - m, m, 1)$ subdesign (which is a projective plane of order $m - 1$).

5.8 Let X be the block graph of the design with parameters 2-(15, 3, 1) described after Corollary 5.3.5. Assume that S is a maximum clique in X and that b is a block in S. Let H be the matrix whose rows are the blocks in the design that do not intersect with b and whose columns are the characteristic vectors of S_i (where $i \notin b$). Find a basis for the kernel of H. Using this basis, show that any maximum clique in X is either S_i or the set of all 2-dimensional subspaces of a 3-dimensional space.

5.9 Prove that the block graph for an $OA(n + 1, n)$ is isomorphic to the complete graph on n^2 vertices.

5.10 Prove that the parameters for the block graph $X_{OA(m,n)}$ are

$$(n^2, \quad m(n-1); \quad (m-1)(m-2) + n - 2, \quad m^2).$$

5.11 Show that the vertices of block graph $X_{OA(m,n)}$ can be partitioned into n disjoint cliques of size n.

5.12 Suppose X is the block graph of an orthogonal array $OA(m, n)$ and let f be a function from $V(X)$ to the integers $1, \dots, n$. Extend the orthogonal array by adding an extra row, such that the ith entry of the row is the value of f on the ith vertex (or, equivalently, the ith column). Show that the new array is orthogonal if and only if f is a proper n-coloring.

5.13 Let $M = OA(m, n)$ be an orthogonal array. Form another orthogonal array M' by removing a single row from M. Show that the chromatic number of the block graph of M' is equal to n.

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5.14 Calculate the eigenvalues of the point graph of a partial geometry using the parameters of the graph.

5.15 A *maximal arc* with parameters (n, m) in a projective plane of order q is a set $\mathcal{M}$ of n points such that any line in the plane is either disjoint from $\mathcal{M}$ or meets it in exactly m points. Given a maximal arc $\mathcal{M}$, we define an incidence structure whose points are the points of the plane not in the arc and whose lines are the lines of the plane that contain a point of the arc. Prove that this is a partial geometry with parameters

$$\left(q-m,\ \frac{n}{m}-1,\ \frac{n}{m}-m\right) = \left(q-m,\ q+1-\frac{q}{m},\ \frac{1}{m}(q-m)(m-1)\right).$$

5.16 Prove that the only maximum cliques in the point graph of a partial geometry with parameters $(s, t, 1)$ are the canonical cliques. Using this, state a version of the EKR Theorem for generalized quadrangles.

5.17 A *linear code over* $GF(q)$ is a subspace of $GF(q)^m$, for some m. We represent a linear code with the rows of a $(d \times m)$ matrix M over $GF(q)$ (the code is the row space of M). Elements of the code are called words and the weight of a code word is the number of nonzero entries in it. A code is projective if the columns of M are distinct.

Suppose C is a projective code, with exactly two nonzero weights, w_1 and w_2. Define a graph $X(C)$ whose vertices are the words in C, where two words are adjacent if their difference has weight w_1. Prove that $X(C)$ is strongly regular by determining its parameters.

Notes

There are many additional references for strongly regular graphs. We recommend the first chapter of the classic book by Brouwer, Cohen and Neumaier [31], Chapter 9 of Brouwer and Haemers's *Spectra of Graphs* [36] and Chapter 10 of Godsil and Royle [87].

At the end of Section 5.2 we discussed a strongly regular graph with parameters $(81, 20; 1, 6)$. This graph can be constructed from a projective 2-weight code of length 10 and dimension 4 over $GF(3)$, with weights 6 and 9. For more details see [33].

A version of Corollary 5.3.5 (this is the EKR Theorem for t-designs) was proved by Rands [144]. Rands considers the more general case of s-intersecting sets of blocks from t-(n, m, λ)-designs. Provided that n is sufficiently large relative to m, t, s then the largest set of intersecting blocks is the collection of all blocks that contain a fixed s-set (Rands's proof uses a counting argument). The bound given by Rands for the case when $t = 2$ and $\lambda = 1$ is slightly weaker than the bound in Corollary 5.3.5.

Partial geometries with $\nu = 1$ are known as *generalized quadrangles*. Examples can be constructed using a nondegenerate symplectic form on $PG(3, q)$, details are given in [87, Section 5.5]. Many more examples of generalized quadrangles will be found in the book by Payne and Thas [139].

A graph is called *core complete* if the graph is itself a core or its core is a complete graph. Cameron and Kazanidis [39] asked whether all strongly regular graphs are core complete. Godsil and Royle [88] determined that many of the strongly regular graphs that we consider in this chapter are indeed core complete.