Obtaining Coherent Configurations on Non-Distance Regular Graphs

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Abstract

Many regularly structured graphs, such as strongly regular graphs, have been extensively studied, and their spectral and algebraic properties are well documented in the literature. In contrast, the study of non-structured graphs remains limited, largely due to the difficulty of systematically constructing and analyzing them. In this paper, we explore whether simple graph operations—such as vertex deletion and switching—performed on strongly regular graphs can produce non-structured graphs, and examine how these operations affect their associated coherent configurations.

We focus on two well-known families of strongly regular graphs: the Rook Graph $R(n, n)$ and the Triangular Graph $T(n)$. By applying specific graph modifications, we analyze the resulting adjacency algebras and track changes in their coherent configuration structure. Our experiments reveal that certain operations consistently result in configurations of fixed rank and algebraic patterns, suggesting underlying structure even within seemingly irregular graphs.

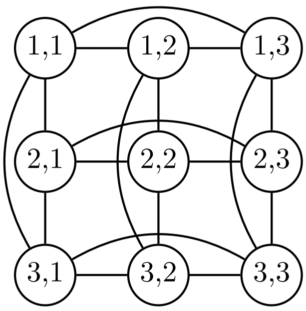

Figure 1: The Rook Graph $R(3)$

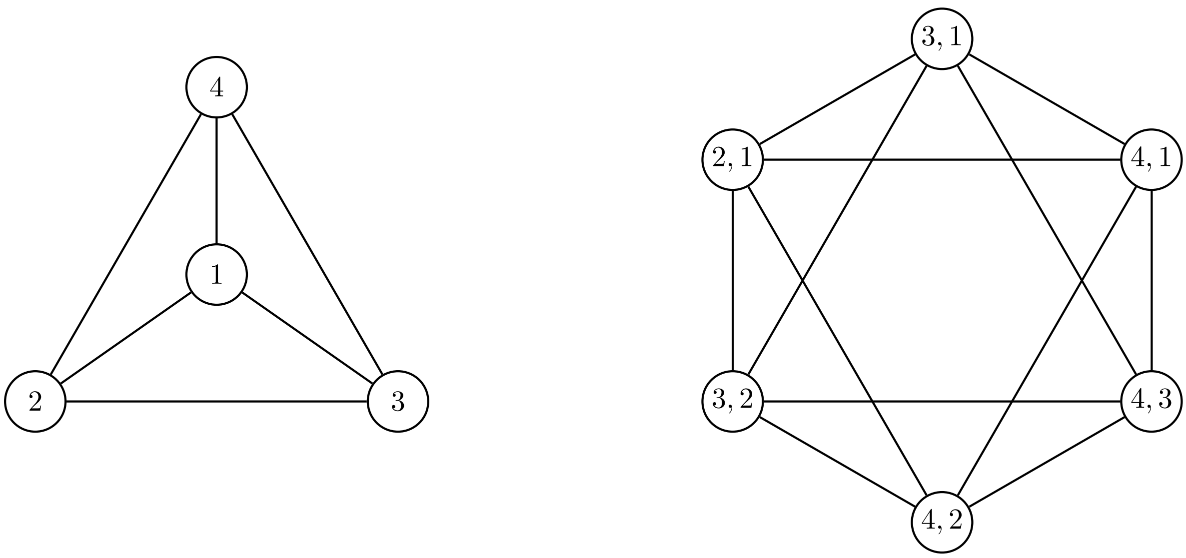

Figure 2: The complete graph K_4 and its corresponding triangular graph $T(4)$.

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1 Introduction

In this paper, we investigate the coherent closure of graphs derived by performing graph operations on strongly regular graphs. We first discuss the motivation and why we choose to investigate such properties.

1.1 Historical Motivation

In 1782, the mathematician Leonhard Euler posed a question: There are 6 army regiments, each with 6 officers of varying ranks. Is there a way to arrange the 36 officers in a 6-by-6 square such that no row nor column have any repeated army regiments or ranks? [1]. This question, now known as Euler's 36 Officer problem, was deemed impossible at the time, but inspired the study of what we now know as Mutually Orthogonal Latin Squares. By definition, a Latin Square is a n-by-n array filled with n different symbols such that no rows or columns have a duplicate symbol. A common example is the famous Sudoku games, though it has a stronger restriction that each block can have no repeating symbols as well. Relating it back to the 36 officer problem, we introduce the concept of Mutually Orthogonal Latin Squares, where there are a collection of Latin Squares of order n that when superimposed, do not have any repetition of symbols in 2 cells. We explain with an example.

$$L_1 = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 4 & 1 & 2 \\ 4 & 3 & 2 & 1 \end{bmatrix} \qquad L_2 = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \\ 2 & 1 & 4 & 3 \\ 3 & 4 & 1 & 2 \end{bmatrix} \qquad L_3 = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \\ 4 & 3 & 2 & 1 \\ 2 & 1 & 4 & 3 \end{bmatrix}$$

Here we have a set of Mutually Orthogonal Latin Squares of order 4, or MOLS(4). We can indeed verify that all 3 matrices are Latin Squares as no row or column has a repeating element. We now show orthogonality for the first 2 matrices by this rule:

$$\begin{aligned} L_{1,2} &= [(a_{ij}, b_{ij})] \\ &= \begin{bmatrix} (1,1) & (2,2) & (3,3) & (4,4) \\ (2,4) & (1,3) & (4,2) & (3,1) \\ (3,2) & (4,1) & (1,4) & (2,3) \\ (4,3) & (3,4) & (2,1) & (1,2) \end{bmatrix} \end{aligned}$$

where $L_1 = [a_{ij}]$ and $L_2 = [b_{ij}]$ are the first pair of orthogonal latin squares shown above. The same procedure can be repeated to show orthogonality between L_1, L_3 and L_2, L_3. As we can see, this is clearly the problem that Euler deemed impossible, just of an order of 6.

Interestingly, it has been proven that for any prime power $q = p^k$, there exists $q - 1$ MOLS(q). The proof is done using finite fields of size q.

For any Latin Square $L = [a_{ij}]$, we write it as an array of the following form:

$$\begin{bmatrix} 1 & 1 & \dots & 1 & 2 & 2 & \dots & 3 & \dots & n & n & \dots & n \\ 1 & 2 & \dots & n & 1 & 2 & \dots & 1 & \dots & 1 & 2 & \dots & n \\ a_{11} & a_{12} & \dots & a_{1n} & a_{21} & a_{22} & \dots & a_{31} & \dots & a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \in \mathbb{N}^{3 \times n^2}$$

where the first and second row correspond to the (i, j)-th position of the element a_{ij}, which is positioned on the third row. This is an example of a orthogonal array of size $(3, n)$. Orthogonal arrays can actually be generalised to sizes (m, n), and we study the properties by using incidence structures.

1.2 From Orthogonal Arrays to Graphs

For any orthogonal array $\mathrm{OA}(m, n)$, we can construct block graphs using the columns of $\mathrm{OA}(m.n)$, where 2 columns are adjacent if the columns have overlapping entries in any row. Interestingly enough, this construction of the orthogonal array block graph results in a Strongly Regular Graph [2]. In this paper we focus on the base case $\mathrm{OA}(2, n)$.

We first display $\mathrm{OA}(2, n)$:

$$\mathrm{OA}(2,n)\begin{bmatrix} 1 & 1 & \dots & 1 & 2 & 2 & \dots & 2 & 3 & \dots & n & n & \dots & n \\ 1 & 2 & \dots & n & 1 & 2 & \dots & n & 1 & \dots & 1 & 2 & \dots & n \end{bmatrix} \in \mathbb{N}^{2\times n^2}$$

We can interpret this combinatorially as sets of ordered pairs $\{1, 2, \dots, n\} \times \{1, 2, \dots, n\}$, each appearing once in each column of $\mathrm{OA}(2, n)$. Following the block graph construction of this graph, any 2 columns are adjacent if the top row of the 2 columns are the same or the bottom row of the 2 columns are the same. This block graph construction of $\mathrm{OA}(2, n)$ is precisely the Rook's Graph $R(n)$, which is the graph of how a rook moves on a $n \times n$ chessboard. We can see that each column can represent the (i, j)-th position of the chessboard, each columns joined by where the rook can move next.

1.3 Strongly Regular Graphs

Along with Rook Graphs, we also consider the Triangular Graph $T(n)$, which is also a Strongly Regular Graph as our family of graphs in this paper. A strongly regular graph with parameters (v, k, λ, μ) is a simple, undirected graph on v vertices such that each vertex has exactly k neighbors, every pair of adjacent vertices shares λ common neighbors, and every pair of non-adjacent vertices shares μ common neighbors. [3]. Strongly regular graphs have many interesting structures and properties, one important relating its adjacency matrix A to its SRG parameters, namely $A^2 = kI + \lambda A + \mu(J - I - A)$. This regularity leads to adjacency matrices that generate a commutative algebra of dimension three, which induces a coherent configuration [4, 5].

1.4 Primary Goal

In their paper, Greaves and Yip [5] studied graphs with 3 eigenvalues and the change in coherent rank upon switching blocks of the respective graphs. There they discovered that switching graphs of 3 eigenvalues resulted in large coherent rank, with some ranks being unbounded. As such, we choose to investigate if it is possible to obtain a general coherent closure when switching strongly regular graphs, as strongly regular graphs are a subset of graphs with 3 eigenvalues. We choose the rook graph and triangular graph, well-known examples of strongly regular graphs, and destroy their symmetry to obtain a more general coherent configuration. In particular, we use Seidel switching and vertex deletion to investigate the coherent rank of graphs modified from strongly regular graphs.

2 Notations and Definitions

2.1 Notations

- We denote the set comprising of integers from 1 to n as $[n]$.
- We denote by I_n, J_n, O_n, and $\mathbf{1}_n$ the identity matrix, all-ones matrix, zero matrix, and all-ones (column) vector of order n, respectively. We simply write I, J, O, and $\mathbf{1}$ when the order is clear from context, or in the case of J and O, when the matrix is not square.
- We denote by $e_{i,n}$ the elementary vector of size n with a 1 in the i-th position and 0 elsewhere.
- K_n denotes a Complete Graph of size n.
- $A \otimes B$ denotes the Kronecker product of matrices A and B. If $A \in \mathbb{Q}^{m\times n}$ and $B \in \mathbb{Q}^{p\times q}$, then:
$$A \otimes B = \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{bmatrix} \in \mathbb{Q}^{mp\times nq}.$$
- For any graph Γ, we denote the adjacency matrix of the graph as $A(\Gamma)$.
- Let $\langle A_1, \ldots, A_n\rangle$ denote $\text{span}\{A_1, \ldots, A_n\}$, for some matrices A_i.
- We denote the Complete Graph on n vertices as K_n.
- We will use the symbol $v_1 \sim v_2$ to show adjacency between vertices v_1 and v_2.

2.2 Coherent Configuration and Algebras

A *coherent configuration* [5] is a combinatorial and algebraic structure defined on a finite set V. It provides a framework for studying symmetry and regularity in graphs and other relational structures. Formally, a coherent configuration is a pair $(V, \mathcal{R})$, where $\mathcal{R} = \{R_1, \ldots, R_r\}$ is a partition of $V \times V$ into binary relations, each represented by its adjacency matrix A_i. These matrices satisfy the following axioms:

Axioms of a Coherent Configuration

Definition 2.1. *Let V be a finite set and $\mathcal{R} = \{R_1, \ldots, R_r\}$ be a set of binary relations. For each R_i, let $W_i \in \mathrm{Mat}_V(\{0,1\})$ be defined such that its (x, y)-th entry is 1 if $(x, y) \in R_i$ and 0 otherwise. Suppose the following 4 conditions*

(CC1) $\sum_{i=1}^{r} W_i = J$;

(CC2) *For each $i \in [r]$, there exists $j \in [r]$ such that $W_i^T = W_j$;*

(CC3) *There exists a subset $\Delta \subseteq \{1, \ldots, r\}$ such that $\sum_{i\in\Delta} W_i = I$;*

(CC4) $W_iW_j = \sum_{k=1}^{r} p_{i,j}^k W_k$ *for some constants $p_{i,j}^k \in \mathbb{Z}_{\geq 0}$, for all $i, j \in [r]$.*

Then $(V, \mathcal{R})$ is called a ***coherent configuration*** *of* ***rank*** $|\mathcal{R}| = r$*. The set V is called the* ***point-set*** *of the coherent configuration*

For each $i \in \Delta$, we call the subset $V_i := \{x \in V : (x,x) \in R_i\}$ a **fibre** of the coherent configuration. It can be observed that the fibres form a partition of the point-set V. When $|\Delta| = 1$, the coherent configuration (V, R) is called an **association scheme**. It follows from (CC4) that, for each $k \in [r]$, there exists i and j such that $R_k \subset V_i \times V_j$. Thus, each subset $\Delta' \subset \Delta$ induces a coherent configuration with point-set $\bigcup_{i \in \Delta'} V_i$. The **type** of $(V, \mathcal{R})$ is defined to be the matrix in $\mathrm{Mat}_\Delta(\mathbb{N})$ whose (i,j)-entry t_{ij} is equal to the cardinality $|\{k : R_k \subset V_i \times V_j\}|$. Note that the sum of the entries of the type matrix is equal to r. Furthermore, since the type matrix must be symmetric, we omit the entries below the diagonal. Higman [6] established the following restriction on the type matrix.

Lemma 2.2. *For each $i, j \in \Delta$, if $t_{ii} \leq 5$ and $t_{jj} \leq 5$ then $t_{ij} \leq \min(t_{ii}, t_{jj})$.*

Definition 2.3. *A* ***coherent algebra*** *is a matrix algebra $\mathcal{A} \subset \mathrm{Mat}_V(\mathbb{C})$ that satisfies the following axioms.*

- *$I, J \in \mathcal{A}$;*
- *$M^\top \in \mathcal{A}$ for each $M \in \mathcal{A}$;*
- *$MN \in \mathcal{A}$ and $M \circ N \in \mathcal{A}$ for each $M, N \in \mathcal{A}$, where $\circ$ denotes the entrywise product.*

Each coherent algebra $\mathcal{A}$ has a unique basis of $\{0,1\}$-matrices $\{W_1, \dots, W_r\}$ that corresponds to a coherent configuration $(V_\mathcal{A}, \mathcal{R}_\mathcal{A})$. We denote by $\mathcal{F}_\mathcal{A}$ the set of fibres of the coherent configuration $(V_\mathcal{A}, \mathcal{R}_\mathcal{A})$, and we define the **type** of $\mathcal{A}$ to be that of $(V_\mathcal{A}, \mathcal{R}_\mathcal{A})$. We note that the intersection of any two coherent algebras is itself a coherent algebra. Thus we define the **coherent closure** $\mathcal{W}(\Gamma)$ of Γ to be the minimal coherent algebra that contains the adjacency matrix $A(\Gamma)$ of Γ. We write $\mathcal{W}(\Gamma) = \langle W_1, \dots, W_r \rangle$, where $\{W_1, \dots, W_r\}$ is the unique basis of $\{0,1\}$-matrices for $\mathcal{W}(\Gamma)$.

To show that a coherent algebra is minimal, we use the Wielandt Principle [7] to derive a lower bound of the coherent rank.

Theorem 2.4. *Let $\mathcal{W}(\Gamma)$ be the coherent closure and of the graph Γ. Let $\mathcal{A}(\Gamma)$ be a coherent algebra of Γ with basis $\{W_i : i \in [r]\}$. Let there be a matrix $M \in \mathcal{A}(\Gamma)$, such that $M = \sum_{i=1}^r c_i W_i$. Then*

$$\sum_{\{i | c_i = c\}} W_i \in \mathcal{W}(\Gamma), \quad \forall c \in \mathbb{C}.$$

2.3 Weisfeiler-Lehman Algorithm

The Weisfeiler-Leman (WL) refinement algorithm is a combinatorial method originally developed for graph isomorphism testing, which iteratively refines colorings on tuples of vertices based on their neighborhoods. For a given dimension k, the k-WL algorithm operates on k-tuples in V^k and produces increasingly fine partitions of the tuple space as the algorithm stabilizes. In the context of coherent closure, the case $k = 2$ is of particular interest.

When applied to a graph $\Gamma = (V, E)$, the 2-WL algorithm refines the coloring on $V \times V$, beginning from an initial coloring that distinguishes edges, non-edges, and diagonal elements. At each iteration, the coloring of a pair (x, y) is updated based on the multiset of colors of vertices z such that (x, z) and (z, y) are considered. This refinement continues until a stable partition is reached.

The key significance of the 2-WL algorithm lies in its equivalence to generating the *coherent closure* of a graph. That is, the final coloring produced by 2-WL corresponds to a coherent configuration whose basis relations partition $V \times V$ in a way that is closed under transpose and composition — the defining properties of a coherent configuration.

This connection is formalized by the following result, adapted from Theorem 4.6.19 in [8], where it implies that the 2-WL refinement captures the same structure as the 2-closure of a graph, and thus the coherent closure of Γ may be computed via the 2-dimensional WL algorithm.

To support our theoretical investigation, we leveraged computational tools to compute the coherent closure of graphs under various operations. Specifically, we utilised SageMath alongside the C++ implementation of the k-WL refinement algorithm available at `https://github.com/sven-reichard/stabilization/blob/master/weisfeiler.org` [9]. This allowed us to efficiently compute the 2-WL stabilization of graphs and directly obtain their coherent closures for small values of n. Through these computations, we observed consistent patterns in the resulting coherent ranks across different graph modifications, which in turn guided the formal proofs presented in the following sections.

2.4 Rook Graph

General Rook Graph The **rook graph** $R(m, n) = (V, E)$, where $m \leq n$, is defined as the simple undirected graph formed possible moves of a rook on each cell of an $m \times n$ chessboard. Formally, let

$$V = \left\{ \begin{pmatrix} i \\ j \end{pmatrix} : \quad i \in [m], j \in [n] \right\};$$

be the vertex set representing all cells on the $m \times n$ chessboard. The edge set is given by

$$E = \left\{ \left\{ \begin{pmatrix} i \\ j \end{pmatrix}, \begin{pmatrix} k \\ l \end{pmatrix} \right\} : \quad i = k \text{ or } j = l \right\}.$$

Then $R(m, n)$ is the rook graph.

- Each vertex corresponds to a cell on the chessboard, so the total number of vertices is $|V| = mn$.
- Two vertices are adjacent if and only if they lie in the same row or the same column of the chessboard.

The adjacency matrix of $R(m,n)$ can be written in block form as:

$$\mathbf{A}[R(m,n)] = \underbrace{\begin{bmatrix} J_n - I & I_n & \cdots & I_n \\ I_n & J_n - I & \cdots & I_n \\ \vdots & \vdots & \ddots & \vdots \\ I_n & I_n & \cdots & J_n - I \end{bmatrix}}_{m \text{ blocks}}.$$

Square Rook Graph A **square rook graph** is the special case where $m = n$. We denote this as $R(n)$. An illustration of $R(3)$ is provided in Figure 1.

Properties of the square rook graph

- $R(n)$ is a strongly regular graph with parameters:

$$\mathrm{SRG}(n^2, 2(n-1), n-2, 2) \quad \text{for } n \geq 3;$$

- Let A be the adjacency matrix $R(n)$. Since it is strongly regular, $A^2 = 2(n-1)I + (n-2)A + 2(J - I - A)$.

2.5 Triangular Graph

The **triangular graph** $T(n)$ is defined as the graph whose vertices correspond to the 2-element subsets of an n-element set. Two vertices are adjacent if and only if the corresponding subsets intersect in exactly one element.

Formally, let

$$V = \{\{i,j\} : \quad 1 \leq i < j \leq n\}$$

be the vertex set of all unordered pairs from the set $[n] = \{1, 2, \ldots, n\}$. The edge set is given by

$$E = \{\{\{i,j\}, \{k,\ell\}\} : \quad |\{i,j\} \cap \{k,\ell\}| = 1\}.$$

Then $T(n) = (V, E)$ is the triangular graph.

Properties

- $T(n)$ has $\binom{n}{2}$ vertices.

- $T(n)$ is a *strongly regular graph* with parameters

$$\mathrm{SRG}\left(\frac{n(n-1)}{2}, 2(n-2), n-2, 4\right), \quad \text{for } n \geq 4.$$

3 Vertex Deletion and Coherent Configurations

In this section, we perform vertex deletion on $R(n)$ and $T(n)$ and investigate its coherent closure.

3.1 Deleting 1 Vertex in $R(n)$

Here, we apply vertex deletion on $R(n)$ witth respect to 1 vertex. We then investigate its resulting adjacency matrix structure and make a claim about its coherent closure.

3.1.1 Graph Construction

Rook graphs are vertex-transitive [3], so we choose any vertex to be deleted. For simplicity, let v_1, corresponding to the cell $\begin{pmatrix}1\\1\end{pmatrix}$, be deleted. This resulting graph will be denoted as Γ_1.

We choose to partition the remaining vertices according to their adjacency with the chosen v_1:

1. The $2(n-1)$ vertices adjacent to v_1, corresponding to the set:

$$\left\{\begin{pmatrix}1\\2\end{pmatrix},\begin{pmatrix}1\\3\end{pmatrix},\ldots,\begin{pmatrix}1\\n\end{pmatrix},\begin{pmatrix}2\\1\end{pmatrix},\begin{pmatrix}3\\1\end{pmatrix},\ldots,\begin{pmatrix}n\\1\end{pmatrix}\right\}.$$

   We shall call this set V_1.

2. The remaining $(n-1)^2$ vertices, corresponding to the set:

$$\left\{\begin{pmatrix}2\\2\end{pmatrix},\begin{pmatrix}2\\3\end{pmatrix},\ldots,\begin{pmatrix}2\\n\end{pmatrix},\begin{pmatrix}3\\2\end{pmatrix},\begin{pmatrix}3\\3\end{pmatrix},\ldots,\begin{pmatrix}3\\n\end{pmatrix},\ldots,\begin{pmatrix}n\\2\end{pmatrix},\begin{pmatrix}n\\3\end{pmatrix},\ldots,\begin{pmatrix}n\\n\end{pmatrix}\right\}.$$

   We shall call this set V_2.

By grouping the vertices corresponding to V_1 and V_2 together, we end up with a matrix decomposition:

$$A(\Gamma_1)=\begin{bmatrix}A_1 & C\\ C^T & A_2\end{bmatrix},$$

We will now aim to obtain the adjacency matrix $A(\Gamma_1)$ by determining the structure of matrices A_1, A_2 and C.

Proposition 3.1. *A_1 is the adjacency matrix of two disjoint K_{n-1}.*

Proof. The set V_1 contains vertices:

$$V_1=\left\{\begin{pmatrix}1\\2\end{pmatrix},\begin{pmatrix}1\\3\end{pmatrix},\ldots,\begin{pmatrix}1\\n\end{pmatrix},\begin{pmatrix}2\\1\end{pmatrix},\begin{pmatrix}3\\1\end{pmatrix},\ldots,\begin{pmatrix}n\\1\end{pmatrix}\right\}$$

Let us split the set into disjoint subsets L and R:

$$L = \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \end{pmatrix}, ..., \begin{pmatrix} 1 \\ n \end{pmatrix} \right\} = \left\{ \begin{pmatrix} 1 \\ i \end{pmatrix} \middle| \; i \in [n] \setminus \{1\} \right\},$$

$$R = \left\{ \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \end{pmatrix}, ..., \begin{pmatrix} n \\ 1 \end{pmatrix} \right\} = \left\{ \begin{pmatrix} j \\ 1 \end{pmatrix} \middle| \; j \in [n] \setminus \{1\} \right\}$$

- **Show disjointness of graphs**:
  Let $v_L = \begin{pmatrix} 1 \\ i \end{pmatrix} \in L$ and $v_R = \begin{pmatrix} j \\ 1 \end{pmatrix} \in R$, such that $i, j \in [n] \setminus \{1\}$. For any v_L and v_R, $i \neq 1$ and $j \neq 1$, and thus v_L will not be adjacent to v_R, showing that there are no edges between the vertex sets L and $R \implies$ Graphs induced by vertices in L and R are disjoint.

- **Show that the disjoint graphs are both K_{n-1}**:

  - For the vertex set L
    Notice that the vertices in L are adjacent to each other as the top row are all equal to 1, $v_L = \begin{pmatrix} 1 \\ i \end{pmatrix} \in L$. Since $|L| = n - 1$, we conclude that the $n - 1$ vertices in L are adjacent to each other, which is the definition of K_{n-1}.
  - For the vertex set R
    Similar to the case of L, we note that the vertices in R are adjacent to each other as the bottom row are all equal to 1, $v_R = \begin{pmatrix} j \\ 1 \end{pmatrix} \in R$. Since $|R| = n - 1$, we conclude that the $n - 1$ vertices in R are adjacent to each other, which is the definition of K_{n-1}.

We have shown that the graphs formed by vertex sets L and R are disjoint, and that each graph formed is K_{n-1}. Thus we have:

$$A_1 = \begin{bmatrix} J_{n-1} - I & O_{n-1} \\ O_{n-1} & J_{n-1} - I \end{bmatrix}$$

□

Proposition 3.2. *A_2 is the adjacency matrix of the square rook graph $R(n-1)$.*

Proof. The set V_2 contains vertices:

$$V_2 = \left\{ \begin{pmatrix} 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \end{pmatrix}, ..., \begin{pmatrix} 2 \\ n \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 3 \end{pmatrix}, ..., \begin{pmatrix} 3 \\ n \end{pmatrix}, ..., \begin{pmatrix} n \\ 2 \end{pmatrix}, \begin{pmatrix} n \\ 3 \end{pmatrix}, ..., \begin{pmatrix} n \\ n \end{pmatrix} \right\}$$

We can generalise this set into:

$$
\begin{aligned}
V_2 &= \left\{ \begin{pmatrix} i \\ j \end{pmatrix} : \quad i \in [n] \setminus \{1\}, \quad j \in [n] \setminus \{1\} \right\} \\
&= \left\{ \begin{pmatrix} i \\ j \end{pmatrix} : \quad i - 1 \in [n-1], \quad j - 1 \in [n-1] \right\}
\end{aligned}
$$

We will show that it is isomorphic to $R(n-1) = (V, E)$ by forming a bijection between V_2 and V. First we state the definition of V:

$$
V = \left\{ \begin{pmatrix} i' \\ j' \end{pmatrix} \middle| \; i' \in [n-1], j' \in [n-1] \right\}
$$

The bijection used here is

$$
f : V_2 \to V, f\left(\begin{pmatrix} i \\ j \end{pmatrix} \right) = \begin{pmatrix} i-1 \\ j-1 \end{pmatrix}
$$

In words, each cell $\begin{pmatrix} i \\ j \end{pmatrix} \in V_2$ is mapped to the cell $\begin{pmatrix} i' \\ j' \end{pmatrix} \in V$ where $\begin{pmatrix} i' \\ j' \end{pmatrix} = \begin{pmatrix} i-1 \\ j-1 \end{pmatrix}$.

- Show f is injective

  Let $\begin{pmatrix} i'_1 \\ j'_1 \end{pmatrix}, \begin{pmatrix} i'_2 \\ j'_2 \end{pmatrix} \in V$. We want to show if $\begin{pmatrix} i'_1 \\ j'_1 \end{pmatrix} = \begin{pmatrix} i'_2 \\ j'_2 \end{pmatrix}$, then $\begin{pmatrix} i_1 \\ j_1 \end{pmatrix} = \begin{pmatrix} i_2 \\ j_2 \end{pmatrix}$:

  $$
  \begin{pmatrix} i'_1 \\ j'_1 \end{pmatrix} = \begin{pmatrix} i'_2 \\ j'_2 \end{pmatrix} \longrightarrow \begin{pmatrix} i_1 - 1 \\ j_1 - 1 \end{pmatrix} = \begin{pmatrix} i_2 - 1 \\ j_2 - 1 \end{pmatrix} \longrightarrow \begin{pmatrix} i_1 \\ j_1 \end{pmatrix} = \begin{pmatrix} i_2 \\ j_2 \end{pmatrix}
  $$

  Thus, f is injective.

- Show f surjective

  We want to show

  $$
  \forall \begin{pmatrix} i' \\ j' \end{pmatrix} \in V, \quad \exists \begin{pmatrix} i \\ j \end{pmatrix} \in V_2 \quad \text{such that} \quad f\left(\begin{pmatrix} i \\ j \end{pmatrix} \right) = \begin{pmatrix} i' \\ j' \end{pmatrix}
  $$

  We have shown that f is injective. Since the domain V_2 and codomain V both have cardinality $(n-1)^2$, it follows that the image $f(V_2) \subseteq V$ must also have size $(n-1)^2$.

  Thus, $f(V_2) = V$, and f is surjective.

- Show Adjacency preservation

  Let E_2 be the edge set of the graph with adjacency matrix A_2. Let $\left\{ \begin{pmatrix} i_1 \\ j_1 \end{pmatrix}, \begin{pmatrix} i_2 \\ j_2 \end{pmatrix} \right\} \in E_2$.

This implies $i_1 = i_2$ or $j_1 = j_2$. Under f,

$$\left\{ f\left(\begin{pmatrix} i_1 \\ j_1 \end{pmatrix}\right), f\left(\begin{pmatrix} i_2 \\ j_2 \end{pmatrix}\right) \right\} = \left\{ \begin{pmatrix} i_1 - 1 \\ j_1 - 1 \end{pmatrix}, \begin{pmatrix} i_2 - 1 \\ j_2 - 1 \end{pmatrix} \right\}$$

Since $i_1 = i_2$ or $j_1 = j_2$, $i_1 - 1 = i_2 - 1$ or $j_1 - 1 = j_2 - 1$ and so

$$\left\{ \begin{pmatrix} i_1 - 1 \\ j_1 - 1 \end{pmatrix}, \begin{pmatrix} i_2 - 1 \\ j_2 - 1 \end{pmatrix} \right\} \in E$$

Thus adjacency is preserved under f as well.

Since V_2 has a bijective mapping to V and adjacency is preserved under said mapping, we have shown that the graph with adjacency matrix A_2 is isomorphic to $R(n-1)$. So we conclude that:

$$A_2 = \underbrace{\begin{bmatrix} J_{n-1} - I & I_{n-1} & \cdots & I_{n-1} \\ I_{n-1} & J_{n-1} - I & \cdots & I_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ I_{n-1} & I_{n-1} & \cdots & J_{n-1} - I \end{bmatrix}}_{n-1 \text{ blocks}}$$

$\square$

We now aim to construct the matrix C.

Construction of C We know the rows of C are indexed by the set V_1 and columns are indexed by the set V_2. To make things simple, we consider this decomposition of C:

$$C = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}$$

where the top half of C is denoted by C_1 with rows indexed by the set L and columns indexed by V_2, while the bottom half is denoted by C_2 with rows indexed by the set R and columns indexed by V_2

C_1 For C_1, the rows are indexed by:

$$\begin{aligned} L &= \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \ldots, \begin{pmatrix} 1 \\ n \end{pmatrix} \right\} \\ &= \left\{ \begin{pmatrix} 1 \\ i \end{pmatrix} : \quad i \in [n] \setminus \{1\} \right\}, \end{aligned}$$

while the columns are indexed by:

$$V_2 = \left\{ \begin{pmatrix} 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \dots, \begin{pmatrix} 2 \\ n \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \end{pmatrix}, \dots, \begin{pmatrix} n \\ n \end{pmatrix} \right\}$$
$$= \left\{ \begin{pmatrix} j \\ k \end{pmatrix} : \quad j, k \in [n] \setminus \{1\} \right\}.$$

Proposition 3.3. *Each vertex corresponding to an element in* L *has exactly* $n-1$ *adjacent vertices corresponding to* $n-1$ *elements in* V_2*.*

Proof. We aim to show any $v_L = \begin{pmatrix} 1 \\ i \end{pmatrix} \in L$ is adjacent to exactly $n-1$ $v_2 = \begin{pmatrix} j \\ k \end{pmatrix} \in V_2$.
Given $v_L = \begin{pmatrix} 1 \\ i \end{pmatrix}$ and $v_2 = \begin{pmatrix} j \\ k \end{pmatrix}$, when $i = k$, v_L is adjacent to v_2. This is the only case where adjacency occurs as $j \neq 1, j \in [n] \setminus \{1\}$.
Furthermore, there are $n-1$ edges for any v_L. When we set $i = k$, there are $|[n] \setminus \{1\}| = n-1$ possible values of j.
Thus, for any v_L there are exactly $n-1$ adjacent vertices v_2. □

For instance, when we fix $i = k = 2$:

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix} \text{ is adjacent to } \begin{pmatrix} 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \end{pmatrix}, \dots, \begin{pmatrix} n \\ 2 \end{pmatrix}.$$

This adjacency results in rows of the form:

$$[\underbrace{1 \quad 0 \quad \cdots \quad 0}_{n-1 \text{ elements}} \quad 1 \quad 0 \quad \cdots \quad 0 \quad \cdots].$$

When repeated for $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$ onwards, C_1 is composed of $n-1$ blocks of $I^{(n-1)}$:

$$C_1 = \begin{bmatrix} I_{n-1} & I_{n-1} & \cdots & I_{n-1} \end{bmatrix}.$$

Explicitly, $\mathbf{C_1}$ looks like:

$$C_1 = \begin{bmatrix} 1 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & \cdots & 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & 1 & \cdots & 0 & \cdots & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \cdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 1 & \cdots & 0 & 0 & \cdots & 1 \end{bmatrix}.$$

C_2 For C_2, the rows are indexed by:

$$R = \left\{ \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \dots, \begin{pmatrix} n \\ 1 \end{pmatrix} \right\}$$
$$= \left\{ \begin{pmatrix} i \\ 1 \end{pmatrix} : \quad i \in [n] \setminus \{1\} \right\},$$

while the columns are still indexed by V_2. Following the same logic as in C_1, we simply switch the logic from the bottom row to the top row to show adjacency.

This results in each row is adjacent to $n-1$ vertices, with 1's being contiguous. For instance:

$$\begin{pmatrix} 2 \\ 1 \end{pmatrix} \text{ is adjacent to } \begin{pmatrix} 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \dots, \begin{pmatrix} 2 \\ n \end{pmatrix}.$$

This results in rows of the form:

$$[\underbrace{1 \quad 1 \quad \cdots \quad 1}_{n-1 \text{ elements}} \quad 0 \quad 0 \quad \cdots].$$

Explicitly, C_2 looks like:

$$C_2 = \begin{bmatrix} 1 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 1 & 1 & \cdots & 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \cdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 1 & 1 & \cdots & 1 \end{bmatrix}.$$

We can also condense this matrix C_2 into block form, denoted by $M_i \in \mathbb{R}^{(n-1)\times(n-1)}$, where the i-th row consists of 1s and 0s elsewhere.

$$C_2 = \begin{bmatrix} M_1 & M_2 & M_3 & \dots & M_{n-1} \end{bmatrix}$$

Since $M_i \in \mathbb{R}^{(n-1)\times(n-1)}$, we can represent it as:

$$\begin{aligned} M_i = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ 1 & 1 & 1 & 1 & 1 & \dots & 1 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 \end{bmatrix} &= \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & \cdots & 1 \end{bmatrix} \\ &= e_{i,n-1} \otimes \mathbf{1}_{n-1}^T \end{aligned}$$

So finally we have

$$C_2 = \begin{bmatrix} e_{1,n-1} \otimes \mathbf{1}_{n-1}^T & e_{2,n-1} \otimes \mathbf{1}_{n-1}^T & e_{3,n-1} \otimes \mathbf{1}_{n-1}^T & \dots & e_{n-1,n-1} \otimes \mathbf{1}_{n-1}^T \end{bmatrix}$$

C, combined Putting C_1 and C_2 together, the complete matrix C is:

$$C = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}.$$

Explicitly:

$$C = \begin{bmatrix} 1 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & \cdots & 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & 1 & \cdots & 0 & \cdots & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \cdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 1 & \cdots & 0 & 0 & \cdots & 1 \\ \\ 1 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 1 & 1 & \cdots & 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \cdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 1 & 1 & \cdots & 1 \end{bmatrix}.$$

Or as its block representation,

$$C = \begin{bmatrix} I_{n-1} & I_{n-1} & I_{n-1} & \cdots & I_{n-1} \\ e_{1,n-1} \otimes \mathbf{1}_{n-1}^T & e_{2,n-1} \otimes \mathbf{1}_{n-1}^T & e_{3,n-1} \otimes \mathbf{1}_{n-1}^T & \dots & e_{n-1,n-1} \otimes \mathbf{1}_{n-1}^T \end{bmatrix}$$

3.1.2 Coherent Algebra

We claim that the following 10 matrices form a coherent algebra of Γ_1.

$$\mathcal{A}(\Gamma_1) = \langle W_i : i \in [10] \rangle, \quad \text{where}$$

$$W_1 = \begin{bmatrix} I_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix}, \quad W_2 = \begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & I_{(n-1)^2} \end{bmatrix}$$

$$W_3 = \begin{bmatrix} A_1 & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix}, \quad W_4 = \begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & A_2 \end{bmatrix}$$

$$W_5 = \begin{bmatrix} J - I - A_1 & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix}, \quad W_6 = \begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & J - I - A_2 \end{bmatrix}$$

$$W_7 = \begin{bmatrix} O_{2(n-1)} & C \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix}, \quad W_8 = \begin{bmatrix} O_{2(n-1)} & J - C \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix}$$

$$W_9 = \begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ C^T & O_{(n-1)^2} \end{bmatrix}, \quad W_{10} = \begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ J - C^T & O_{(n-1)^2} \end{bmatrix}$$

Closure under Identity Since $W_1 + W_2 = I_{n^2-1}$, $I \in \mathcal{A}(\Gamma_1)$

Closure under Transpose It can be observed that $W_i, i \in [6]$ are self-tranpose, so we show for $i \in \{7, 8, 9, 10\}$:

$$W_7^T = W_9, W_8^T = W_{10}$$

So $\mathcal{A}(\Gamma_1)$ is closed under transposition.

Closure under all-ones matrix If we sum all the matrices $\sum_{i=1}^{10} W_i$, we actually get J_{n^2-1}, so the set $\mathcal{A}(\Gamma_1)$ does contain the all-ones matrix.

Closed under matrix multiplication Here we have to show for each pair-wise multiplication, its product is still contained in $\mathcal{A}(\Gamma_1)$.

In the Appendix (6.2), we rigourously show that $\mathcal{A}(\Gamma_1)$ is closed under matrix multiplication.

Thus, $\mathcal{A}(\Gamma_1)$ is a coherent algebra. So we know the coherent closure $\mathcal{W}(\Gamma_1) \subseteq \mathcal{A}(\Gamma_1)$.

3.1.3 Showing Minimal Coherent Algebra

To use the Wielandt Principle 2.4, we let $A = A(\Gamma_1) \in \mathcal{A}$, so we know that $A^2 \in \mathcal{A}$ as well.

The detailed work can be found in the Appendix ().

By Wielandt Principle, $\mathcal{W}(\Gamma_1) = \langle W_i : \quad i \in [10] \rangle = \mathcal{A}(\Gamma_1)$ so we have proven that the coherent rank of Γ_1 is 10.

3.2 Deleting 1 Vertex in $T(n)$

Similarly, we apply vertex deletion on $T(n)$ with respect to 1 vertex. We then investigate its resulting adjacency matrix structure and make a claim about its coherent closure.

3.2.1 Graph Construction

Since triangular graphs are vertex-transitive, we delete 1 vertex, v, from $T(n)$ and observe the resulting graph, Γ_2, to have the form:

$$A(\Gamma_2) = \begin{bmatrix} A(T(n-2)) & C \\ C^T & A(R(2, n-2)) \end{bmatrix}$$

We will show why the subgraphs are isomorphic to $R(2, n-2)$ and $T(n-2)$.

Proposition 3.4. *The neighbourhood of any vertex in $T(n)$ is $R(2, n-2)$.*

Proof. Let $v = \{a, b\}$ be an arbitrary vertex in $T(n)$. Two vertices in $T(n)$ are adjacent if and only if the corresponding sets intersect in exactly one element.The neighbors of $v = \{a, b\}$ are all 2-element subsets of $[n]$ that share exactly one element with $\{a, b\}$. These are:

$$\mathcal{N}(v) = \{\{a, x\} : \quad x \in [n] \setminus \{a, b\}\} \ \cup \ \{\{x, b\} : \quad x \in [n] \setminus \{a, b\}\}$$

There are exactly $2(n-2)$ such vertices.

We can also rewrite the set $\mathcal{N}(v)$ as:

$$\mathcal{N}(v) = \{\{a, x_i\}|i \in [n-2]\} \cup \{\{x_i, b\}|i \in [n-2]\}$$

where $\{x_1, x_2, \dots, x_{n-2}\} = [n] \setminus \{a, b\}$.

Let $\phi : \mathcal{N}(v) \to [2] \times [n-2]$ be a mapping with the following rule:

$$\phi(\{a, x_i\}) = (1, i) \quad \text{and} \quad \phi(\{x_i, b\}) = (2, i)$$

We will show why ϕ is a bijection and preserves adjacency.

- Showing Injectivity
  Suppose $\phi(u_1) = \phi(u_2)$. Then both u_1 and u_2 must be mapped to the same (r, i) for some $r \in [2]$ and $i \in [n-2]$.

  1. $r = 1$ For any i, $\phi(u_1) = \phi(u_2) \iff (1, i) = (1, i) \iff \{a, x_i\} = \{a, x_i\} \iff u_1 = u_2$.
  2. $r = 2$ For any i, $\phi(u_1) = \phi(u_2) \iff (2, i) = (2, i) \iff \{x_i, b\} = \{x_i, b\} \iff u_1 = u_2$.

  In both cases, the injectivity condition is satisfied, thus ϕ is injective.

- Showing Surjectivity
  Let $(r, i) \in [2] \times [n-2]$. We can also split r into 2 cases:

  1. $r = 1$ For any $(1, i)$, choose $u = a, x_i \in \mathcal{N}(v)$, then $\phi(u) = (1, i)$.

2. $r = 2$ For any $(2, i)$, choose $u = x_i, b \in \mathcal{N}(v)$, then $\phi(u) = (2, i)$.

In both cases, the surjectivity condition is satisfied, thus ϕ is surjective.

- Showing Adjacency Preservation
  We want to show that if $u_1 \sim u_2$ in $T(n)$, then $\phi(u_1) \sim \phi(u_2)$ in $R(2, n-2)$ and if $u_1 \nsim u_2$ in $T(n)$, then $\phi(u_1) \nsim \phi(u_2)$ in $R(2, n-2)$. We do this by splitting into cases:

  1. $u_1 = \{a, x_i\}, u_2 = \{a, x_j\}, i \neq j$
     In $T(n)$, $u_1 \sim u_2$ as they share an element a. Under ϕ, $\phi(u_1) = (1, i)$, $\phi(u_2) = (1, j)$. These 2 vertices in $R(2, n-2)$ also share the same row position, leading to $\phi(u_1) \sim \phi(u_2)$. Thus adjacency is preserved.
  2. $u_1 = \{a, x_i\}, u_2 = \{x_j, b\}, i \neq j$
     In $T(n)$, $u_1 \nsim u_2$ as they do not share any element. Under ϕ, $\phi(u_1) = (1, i)$, $\phi(u_2) = (2, j)$. These 2 vertices in $R(2, n-2)$ also do not have any common elements in the respective row and column positions, leading to $\phi(u_1) \nsim \phi(u_2)$.
  3. $u_1 = \{a, x_i\}, u_2 = \{x_i, b\}$
     In $T(n)$, $u_1 \sim u_2$ as they share an element x_i. Under ϕ, $\phi(u_1) = (1, i)$, $\phi(u_2) = (2, i)$. These 2 vertices in $R(2, n-2)$ also share the same column position, leading to $\phi(u_1) \sim \phi(u_2)$. Thus adjacency is preserved.
  4. $u_1 = \{x_i, b\}, u_2 = \{x_j, b\}, i \neq j$
     In $T(n)$, $u_1 \sim u_2$ as they share an element b. Under ϕ, $\phi(u_1) = (2, i)$, $\phi(u_2) = (2, j)$. These 2 vertices in $R(2, n-2)$ also share the same row position, leading to $\phi(u_1) \sim \phi(u_2)$. Thus adjacency is preserved.

Since ϕ is a bijection from $\mathcal{N}(v) \to [2] \times [n-2]$ and preserves adjacency, We conclude that the graph induced by $\mathcal{N}(v) \cong R(2, n-2)$ □

Proposition 3.5. *The non-adjacent neighbourhood of any vertex in $T(n)$ is $T(n-2)$.*

Proof. Let $v = \{a, b\}$ be an arbitrary vertex in $T(n)$. Two vertices in $T(n)$ are adjacent if and only if the corresponding sets intersect in exactly one element. The non-neighbors of v are all 2-element subsets of $[n] \setminus \{a, b\}$, since any vertex disjoint from v cannot be adjacent to it. That is,

$$\mathcal{N}^c(v) = \{\{i, j\} : \quad i, j \in [n] \setminus \{a, b\}\}$$

The set of non-neighbours is exactly $(n-2)(n-3)/2$ or $\binom{n-2}{2}$.

This set is precisely the vertex set of $T(n-2)$, since it contains all 2-element subsets of $[n-2]$.

Furthermore, the adjacency condition on $T(n)$ extends to this subgraph, which is the same adjacency condition in $T(n-2)$. It can therefore be seen that the subgraph formed by non-neighbouring vertices of v form $T(n-2)$. □

3.2.2 Coherent Algebra

We hypothesise that the type matrix of Γ_2 has the following form:

$$\begin{bmatrix} 3 & t_{12} \\ & 4 \end{bmatrix}.$$

We observe this as t_{11} corresponds to the subset of vertices in the non-neighbourhood of v, and we have shown the induced subgraph is $T(n-2)$, which is strongly regular. Since it is strongly regular, it has a coherent closure of $\langle I, A(T(n-2)), J\rangle$, which has a rank of 3 [5].

Similarly for t_{22}, it corresponds to the subset of vertices in the neighbourhood of v, which is $R(2, n-2)$. The non-square rook graph is known to have 4 eigenvalues, which motivates the hypothesis that its coherent closure is of the following form: $\langle I, I\otimes(J-I), (J-I)\otimes I, (J-I)\otimes(J-I)\rangle$, which has a rank of 4.

Using our implementation in SageMath, the 2-WL algorithm returned a set of 11 matrices:

$$\begin{aligned}
W_1 &= \begin{bmatrix} I & O \\ O & O \end{bmatrix}, \quad W_2 = \begin{bmatrix} A(T(n-2)) & O \\ O & O \end{bmatrix} \\
W_3 &= \begin{bmatrix} J-I-A(T(n-2)) & O \\ O & O \end{bmatrix}, \quad W_4 = \begin{bmatrix} O & O \\ O & I \end{bmatrix} \\
W_5 &= \begin{bmatrix} O & O \\ O & I\otimes(J-I) \end{bmatrix}, \quad W_6 = \begin{bmatrix} O & O \\ O & (J-I)\otimes I \end{bmatrix} \\
W_7 &= \begin{bmatrix} O & O \\ O & (J-I)\otimes(J-I) \end{bmatrix}, \qquad W_8 = \begin{bmatrix} O & C \\ O & O \end{bmatrix} \\
W_9 &= \begin{bmatrix} O & J-C \\ O & O \end{bmatrix}, \qquad W_{10} = \begin{bmatrix} O & O \\ C^T & O \end{bmatrix} \\
W_{11} &= \begin{bmatrix} O & O \\ J-C^T & O \end{bmatrix}
\end{aligned}$$

However, we were unable to generate the structure of the matrix C for a general case, and recursive techniques may be needed to fully compute the matrix multiplications. We leave this as a consideration for future work.

4 Switching a Single Vertex and Coherent Rank Increase

Definition 4.1 (Seidel Switching)**.** *Let $G = (V, E)$ be a simple undirected graph on n vertices with adjacency matrix $A \in \{0,1\}^{n \times n}$, and let $S \subseteq V$ be a subset of the vertices. Then, the Seidel switching of G with respect to S is a new graph $G_S = (V, E^S)$, obtained by modifying G as follows:*

- *For each pair of vertices $u, v \in V$:*
  - *If **both** $u, v \in S$, or both $u, v \in V - S$, then $\{u, v\} \in E \Rightarrow \{u, v\} \in E^S$ (adjacency within the sets S and $V - S$).*
  - *If **exactly** one of u or v belongs to S, then:*
    - *If $\{u, v\} \in E$, then $\{u, v\} \notin E^S$ (remove the edge).*
    - *If $\{u, v\} \notin E$, then $\{u, v\} \in E^S$ (add the edge).*

This operation toggles the adjacency between S and $V - S$, whilst preserving adjacency within the respective sets S and $V - S$.

4.1 Switching 1 Vertex in $R(n)$

In this section, we consider switching on $R(n)$ a single vertex v. We then observe the change in coherent configurations.

4.1.1 Switching Construction

As mentioned before, $R(n)$ is vertex transitive so without loss of generality, we choose an arbitrary vertex $v_{1,1}$ corresponding to the cell $\binom{1}{1}$ to be switched. As a result, we can partition the graph based on the degree sequence.

Before switching, $R(n)$ is strongly regular, with a common degree of $2(n-1)$ across all vertices. After choosing to switch the vertex v, we end up with a degree sequence as follows:

Vertex	**Degree after switching**
$v_{1,1}$	$(n-1)^2$
$v_{1,2}, v_{1,3}, \ldots, v_{1,n}$	$2n-3$
$v_{2,2}, v_{2,3}, \ldots, v_{n-1,n-1}$	$2n-1$

Table 1: Degree sequence of vertices in $R(n)$ after switching vertex $v = v_{1,1}$

The change in degree of each vertex can be explained by the adjacency with $v_{1,1}$.

- For $v_{1,1}$

  By switching on this vertex, its degree would be changed to

$$
\begin{aligned}
n^2 - 1 - 2(n-1) &= n^2 - 2n + 1 \\
&= (n-1)^2
\end{aligned}
$$

- For vertices adjacent to $v_{1,1}$ in $R(n)$

  Since these vertices are adjacent to $v_{1,1}$ in $R(n)$, after switching in Γ_3, the adjacency will be removed, so their degree would be changed to

$$2(n-1)-1=2n-3$$

- For vertices not adjacent to $v_{1,1}$ in $R(n)$
  Since these vertices are not adjacent to $v_{1,1}$ in $R(n)$, after switching in Γ_3, the adjacency will be added, so their degree would be changed to

$$2(n-1)+1=2n-1$$

We note that $(n-1)^2 = 2n-3$ in the case of $n = 2$, which would lead to $R(2)$, which is isomorphic to the Cycle graph on 4 vertices. For the purposes of our discussion we restrict $n \geq 3$.

By grouping the vertices together based on their degree sequence, we will have this matrix decomposition of the switched graph, Γ_3.

$$A(\Gamma_3) = \begin{bmatrix} O_1 & O_{1,2(n-1)} & \mathbf{1}_{n-1}^T \\ O_{2(n-1),1} & A_1 & C \\ \mathbf{1}_{n-1} & C^T & A_2 \end{bmatrix}$$

Remark. The matrices A_1, A_2 and C are the same as the ones in graph Γ_1. This is due to the structure of $A(\Gamma_1)$ being enclosed within $A(\Gamma_3)$.

We can take the results from the earlier section on vertex deletion to simplify our computation of the coherent configuration of the graph Γ_3. We first propose the following lemma.

Lemma 4.2. *In a coherent configuration, any fibre consisting of a single vertex supports only the identity relation. That is, the only type within such a fibre is type 1.*

Proof. Let X be a coherent configuration with a fibre $F = \{v\}$. By definition, the set of relations R_i on X partition $X \times X$, and the identity relation $\{(v, v)\}$ is always one of them.

Since F contains only one vertex, there can be no other ordered pairs in $F \times F$ besides (v, v). Therefore, the only basis relation supported on $F \times F$ is the identity relation, which must be of rank 1.

Thus, any singleton fibre supports only the identity type, [1]. □

Using this lemma, we can show that the fibre formed by vertex $v_{1,1}$ only has the identity, which implies that it is of rank 1.

We also employ Higman's lemma 2.2 to help determine the off-diagonal type as well.

From what we already know about the type-matrix of Γ_3, we have this form:

$$\begin{bmatrix} 1 & t_{12} & t_{13} \\ & 3 & 2 \\ & & 3 \end{bmatrix}$$

By , $t_{12} \leq \min(t_{11}, t_{22}) = 1$ and $t_{13} \leq \min(t_{11}, t_{33}) = 1$. So we conclude that the type-matrix of Γ_3 has the form

$$\begin{bmatrix} 1 & 1 & 1 \\ & 3 & 2 \\ & & 3 \end{bmatrix}$$

By symmetry of the type matrix, we can compute that the coherent rank of Γ_3 is 15.

4.1.2 Resulting Coherent Closure

Since we know that the coherent rank of the graph Γ_3 is 15, we use the results found in the earlier section and derive the coherent closure $\mathcal{W}(\Gamma_3)$.

We claim that the following 15 matrices form a coherent closure of our modified rook graph.

$$W_1 = \begin{bmatrix} O_1 & O_{1,2(n-1)} & O_{1,(n-1)^2} \\ O_{2(n-1),1} & I_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,1} & O_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix}, \quad W_2 = \begin{bmatrix} O_1 & O_{1,2(n-1)} & O_{1,(n-1)^2} \\ O_{2(n-1),1} & O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,1} & O_{(n-1)^2,2(n-1)} & I_{(n-1)^2} \end{bmatrix}$$

$$W_3 = \begin{bmatrix} O_1 & O_{1,2(n-1)} & O_{1,(n-1)^2} \\ O_{2(n-1),1} & A_1 & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,1} & O_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix}, \quad W_4 = \begin{bmatrix} O_1 & O_{1,2(n-1)} & O_{1,(n-1)^2} \\ O_{2(n-1),1} & O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,1} & O_{(n-1)^2,2(n-1)} & A_2 \end{bmatrix}$$

$$W_5 = \begin{bmatrix} O_1 & O_{1,2(n-1)} & O_{1,(n-1)^2} \\ O_{2(n-1),1} & J-I-A_1 & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,1} & O_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix}, \quad W_6 = \begin{bmatrix} O_1 & O_{1,2(n-1)} & O_{1,(n-1)^2} \\ O_{2(n-1),1} & O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,1} & O_{(n-1)^2,2(n-1)} & J-I-A_2 \end{bmatrix}$$

$$W_7 = \begin{bmatrix} O_1 & O_{1,2(n-1)} & O_{1,(n-1)^2} \\ O_{2(n-1),1} & O_{2(n-1)} & C \\ O_{(n-1)^2,1} & O_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix}, \quad W_8 = \begin{bmatrix} O_1 & O_{1,2(n-1)} & O_{1,(n-1)^2} \\ O_{2(n-1),1} & O_{2(n-1)} & J-C \\ O_{(n-1)^2,1} & O_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix}$$

$$W_9 = \begin{bmatrix} O_1 & O_{1,2(n-1)} & O_{1,(n-1)^2} \\ O_{2(n-1),1} & O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,1} & C^T & O_{(n-1)^2} \end{bmatrix}, \quad W_{10} = \begin{bmatrix} O_1 & O_{1,2(n-1)} & O_{1,(n-1)^2} \\ O_{2(n-1),1} & O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,1} & J-C^T & O_{(n-1)^2} \end{bmatrix}$$

$$W_{11} = \begin{bmatrix} 1 & O_{1,2(n-1)} & O_{1,(n-1)^2} \\ O_{2(n-1),1} & O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,1} & O_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix}, \quad W_{12} = \begin{bmatrix} O_1 & \mathbf{1}^T_{2(n-1)} & O_{1,(n-1)^2} \\ O_{2(n-1),1} & O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,1} & O_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix}$$

$$W_{13} = \begin{bmatrix} O_1 & O_{1,2(n-1)} & \mathbf{1}^T_{(n-1)^2} \\ O_{2(n-1),1} & O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,1} & O_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix}, \quad W_{14} = \begin{bmatrix} O_1 & O_{1,2(n-1)} & O_{1,(n-1)^2} \\ \mathbf{1}_{2(n-1)} & O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,1} & O_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix}$$

$$W_{15} = \begin{bmatrix} O_1 & O_{1,2(n-1)} & O_{1,(n-1)^2} \\ O_{2(n-1),1} & O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ \mathbf{1}_{(n-1)^2} & O_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix}$$

This approach demonstrates that key features of the coherent algebra can be recovered without direct matrix computations, using only the combinatorial structure encoded in the type matrix and fibre decomposition.

5 Block Switching in $R(n)$

Similar to the previous section, we consider Seidel switching on $R(n)$, but instead of a single vertex we focus on switching on cliques of size n instead.

The resulting adjacency matrix is as follows:

$$A(R'(n)) = \begin{bmatrix} A(R_{k,n}) & C \\ C^T & A(R_{n-k,n}) \end{bmatrix}$$

where $C = J_{k,n-k} \otimes (J - I)_n$.

5.1 Switching Exactly Half the Vertices

We first consider a specific case where n is even, and we switch k n-cliques, $k = n/2$.

5.1.1 Graph Construction and Symmetric Partitioning

Since $k = n/2$, we rewrite our adjacency matrix as:

$$A(\Gamma_4) = \begin{bmatrix} A(R_{k,2k}) & C \\ C^T & A(R_{k,2k}) \end{bmatrix}$$

where $C = J_k \otimes (J - I)_{2k}$.

5.1.2 Coherent Algebra

We claim that the following 6 matrices form a basis for a coherent algebra $\mathcal{A}(\Gamma_4)$ for Γ_4:

$$W_1 = \begin{bmatrix} I_{2k^2} & O_{2k^2} \\ O_{2k^2} & I_{2k^2} \end{bmatrix}, \qquad W_2 = \begin{bmatrix} (J_k - I) \otimes (J_{2k} - I) & O_{2k^2} \\ O_{2k^2} & (J_k - I) \otimes (J_{2k} - I) \end{bmatrix},$$

$$W_3 = \begin{bmatrix} O_{2k^2} & J_k \otimes I_{2k} \\ J_k \otimes I_{2k} & O_{2k^2} \end{bmatrix}, \qquad W_4 = \begin{bmatrix} I_k \otimes (J_{2k} - I) & O_{2k^2} \\ O_{2k^2} & I_k \otimes (J_{2k} - I) \end{bmatrix},$$

$$W_5 = \begin{bmatrix} (J_k - I) \otimes I_{2k} & O_{2k^2} \\ O_{2k^2} & (J_k - I) \otimes I_{2k} \end{bmatrix}, \qquad W_6 = \begin{bmatrix} O_{2k^2} & J_k \otimes (J_{2k} - I) \\ J_k \otimes (J_{2k} - I) & O_{2k^2} \end{bmatrix}.$$

Closure under Identity Since $\mathcal{W}_1 = I_{(2k)^2}$, the identity matrix exists in $\mathcal{A}(\Gamma_4)$.

Closure under Transpose It can be observed that $W_i, \quad i \in [6]$ are self-tranpose, so $\mathcal{A}(\Gamma_4)$ is closed under transposition.

Closure under all-ones matrix If we sum all the matrices $\sum_{i=1}^{6} W_i$, we actually get $J_{(2k)^2}$, so the set $\mathcal{A}(\Gamma_4)$ contains the all-ones matrix.

Closure under matrix multiplication Here we have to show for each pair-wise multiplication, its product is still contained in $\mathcal{A}(\Gamma_4)$.

In the appendix 6.2, we rigourously show that $\mathcal{A}(\Gamma_4)$ is closed under matrix multiplication.

Thus, $\mathcal{A}(\Gamma_4)$ is a coherent algebra. So we know the coherent closure $\mathcal{W}(\Gamma_4) \subseteq \mathcal{A}(\Gamma_4)$.

5.1.3 Showing Minimal Coherent Algebra

To use the Wielandt Principle 2.4, we let $A = A(\Gamma_4) \in \mathcal{A}$, so we know that $A^2 \in \mathcal{A}$ as well.

Detailed workings can be found in the Appendix 6.2. For simplicity we will just state the relevant conclusions.

$$A^2 = \begin{bmatrix} I_k \otimes (J_{2k} - I) + (J_k - I) \otimes I_{2k} & J_k \otimes (J_{2k} - I) \\ J_k \otimes (J_{2k} - I) & I_k \otimes (J_{2k} - I) + (J_k - I) \otimes I_{2k} \end{bmatrix}^2$$

$$\begin{aligned} &= (2k^2 + 2k - 2) \begin{bmatrix} I_k \otimes I_{2k} & O \\ O & I_k \otimes I_{2k} \end{bmatrix} \\ &\quad + (2k^2 - 2k + 2) \begin{bmatrix} (J_k - I) \otimes (J_{2k} - I) & O \\ O & (J_k - I) \otimes (J_{2k} - I) \end{bmatrix} \\ &\quad (2k^2 - 2) \begin{bmatrix} (I_k \otimes (J_{2k} - I) + (J_k - I) \otimes I_{2k}) & O \\ O & (I_k \otimes (J_{2k} - I) + (J_k - I) \otimes I_{2k}) \end{bmatrix} \\ &\quad + (6k - 6) \begin{bmatrix} O & J_k \otimes (J_{2k} - I) \\ J_k \otimes (J_{2k} - I) & O \end{bmatrix} + (4k - 2) \begin{bmatrix} O & J_k \otimes I_{2k} \\ J_k \otimes I_{2k} & O \end{bmatrix} \end{aligned}$$

By Wielandt's Principle,

$$\begin{aligned} &\begin{bmatrix} I_k \otimes I_{2k} & O \\ O & I_k \otimes I_{2k} \end{bmatrix}, \begin{bmatrix} (J_k - I) \otimes (J_{2k} - I) & O \\ O & (J_k - I) \otimes (J_{2k} - I) \end{bmatrix}, \\ &\begin{bmatrix} O & J_k \otimes (J_{2k} - I) \\ J_k \otimes (J_{2k} - I) & O \end{bmatrix}, \begin{bmatrix} O & J_k \otimes I_{2k} \\ J_k \otimes I_{2k} & O \end{bmatrix}, \\ &\begin{bmatrix} (I_k \otimes (J_{2k} - I) + (J_k - I) \otimes I_{2k}) & O \\ O & (I_k \otimes (J_{2k} - I) + (J_k - I) \otimes I_{2k}) \end{bmatrix} \in \mathcal{W}(\Gamma_4) \end{aligned}$$

Applying it one more time using the matrix

$$\begin{bmatrix} (I_k \otimes (J_{2k} - I) + (J_k - I) \otimes I_{2k}) & O \\ O & (I_k \otimes (J_{2k} - I) + (J_k - I) \otimes I_{2k}) \end{bmatrix} \in \mathcal{W}(\Gamma_4) \subseteq \mathcal{A},$$

$$\begin{bmatrix} (I_k \otimes (J_{2k}-I)+(J_k-I)\otimes I_{2k}) & O \\ O & (I_k \otimes (J_{2k}-I)+(J_k-I)\otimes I_{2k}) \end{bmatrix}^2 \in \mathcal{A}(\Gamma_4)$$

$$\begin{aligned} =&(3k-2)\begin{bmatrix} I_k\otimes I_{2k} & O \\ O & I_k \otimes I_{2k}\end{bmatrix} + (2k-2)\begin{bmatrix} I_k\otimes (J_{2k}-I) & O \\ O & I_k \otimes (J_{2k}-I)\end{bmatrix} \\ +&(k-2)\begin{bmatrix} (J_k-I)\otimes I_{2k} & O \\ O & (J_k-I)\otimes I_{2k}\end{bmatrix} + 2\begin{bmatrix} (J_k-I)\otimes (J_{2k}-I) & O \\ O & (J_k-I)\otimes (J_{2k}-I)\end{bmatrix} \end{aligned}$$

By Wielandt's Principle,

$$\begin{aligned} &\begin{bmatrix} I_k\otimes I_{2k} & O \\ O & I_k \otimes I_{2k}\end{bmatrix}, \begin{bmatrix} I_k\otimes (J_{2k}-I) & O \\ O & I_k \otimes (J_{2k}-I)\end{bmatrix} \\ &\begin{bmatrix} (J_k-I)\otimes I_{2k} & O \\ O & (J_k-I)\otimes I_{2k}\end{bmatrix}, \begin{bmatrix} (J_k-I)\otimes (J_{2k}-I) & O \\ O & (J_k-I)\otimes (J_{2k}-I)\end{bmatrix} \\ &\begin{bmatrix} O & J_k\otimes (J_{2k}-I) \\ J_k\otimes (J_{2k}-I) & O\end{bmatrix}, \begin{bmatrix} O & J_k\otimes I_{2k} \\ J_k\otimes I_{2k} & O\end{bmatrix} \in \mathcal{W}(\Gamma_4) \end{aligned}$$

We have shown that there are at least 6 different classes in the coherent closure, so the coherent rank of this switched graph is $|\mathcal{W}(\Gamma_4)| = 6$. We can also conclude that

$$\mathcal{W}(\Gamma_4) = \langle W_1, W_2, W_3, W_4, W_5, W_6 \rangle.$$

5.2 Switching k Blocks of n Vertices

Now for a more general case, we choose to switch k n-cliques from $R(n)$, $1 < k < \lfloor n/2 \rfloor$.

Remark. We restrict $k < \lfloor n/2 \rfloor$ as switching k n-cliques is equivalent to switching $n-k$ n-cliques by symmetry of $R(n)$.

5.2.1 Graph Construction

The resulting adjacency matrix is as follows:

$$A(\Gamma_5) = \begin{bmatrix} A(R_{k,n}) & C \\ C^T & A(R_{n-k,n}) \end{bmatrix}$$

where $C = J_{k,n-k} \otimes (J-I)_n$.

5.2.2 Resulting Coherent Configuration

We claim that the following 12 matrices form a basis for a coherent algebra $\mathcal{A}(\Gamma_5)$ for Γ_5:

$$W_1 = \begin{bmatrix} I_{kn} & O \\ O & O \end{bmatrix}, \qquad W_2 = \begin{bmatrix} I_k \otimes (J_n - I) & O \\ O & O \end{bmatrix},$$

$$W_3 = \begin{bmatrix} (J_k - I) \otimes I_n & O \\ O & O \end{bmatrix}, \qquad W_4 = \begin{bmatrix} (J_k - I) \otimes (J_n - I) & O \\ O & O \end{bmatrix},$$

$$W_5 = \begin{bmatrix} O & O \\ O & I_{(n-k)n} \end{bmatrix}, \qquad W_6 = \begin{bmatrix} O & O \\ O & I_{n-k} \otimes (J_n - I) \end{bmatrix},$$

$$W_7 = \begin{bmatrix} O & O \\ O & (J_{n-k} - I) \otimes I_n \end{bmatrix}, \qquad W_8 = \begin{bmatrix} O & O \\ O & (J_{n-k} - I) \otimes (J_n - I) \end{bmatrix},$$

$$W_9 = \begin{bmatrix} O & J_{k,n-k} \otimes I_n \\ O & O \end{bmatrix}, \qquad W_{10} = \begin{bmatrix} O & J_{k,n-k} \otimes (J_n - I) \\ O & O \end{bmatrix},$$

$$W_{11} = \begin{bmatrix} O & O \\ J_{n-k,k} \otimes I_n & O \end{bmatrix}, \qquad W_{12} = \begin{bmatrix} O & O \\ J_{n-k,k} \otimes (J_n - I) & O \end{bmatrix}$$

Closure under Identity Since $\mathcal{W}_1 + \mathcal{W}_5 = I_{n^2}$, the identity matrix exists in $\mathcal{A}(\Gamma_5)$.

Closure under Transpose It can be observed that $W_i, \quad i \in [8]$ are self-tranpose, and we verify that

$$W_9^T = W_{11} \quad \text{and} \quad W_{10}^T = W_{12}$$

so $\mathcal{A}(\Gamma_5)$ is closed under transposition.

Closure under all-ones matrix If we sum all the matrices $\sum_{i=1}^{12} W_i$, we actually get J_{n^2}, so the set $\mathcal{A}(\Gamma_5)$ contains the all-ones matrix.

Closure under matrix multiplication Here we have to show for each pair-wise multiplication, its product is still contained in $\mathcal{A}(\Gamma_5)$.

In the appendix 6.2, we rigourously show that $\mathcal{A}(\Gamma_5)$ is closed under matrix multiplication.

Thus, $\mathcal{A}(\Gamma_5)$ is a coherent algebra. So we know the coherent closure $\mathcal{W}(\Gamma_5) \subseteq \mathcal{A}(\Gamma_5)$.

5.2.3 Showing Minimal Coherent Algebra

To use the Wielandt Principle 2.4, we let $A = A(\Gamma_5) \in \mathcal{A}$, so we know that $A^2 \in \mathcal{A}$ we well.

Detailed workings can be found in the appendix 6.2. For simplicity we will just state the relevant conclusions.

We have shown that there are at least 12 different classes in the coherent closure, so the coherent rank of this switched graph is $|\mathcal{W}(\Gamma_5)| = 12$. We can also conclude that

$$\mathcal{W}(\Gamma_5) = \langle W_1, W_2, W_3, W_4, W_5, W_6, W_7, W_8, W_9, W_{10}, W_{11}, W_{12} \rangle.$$

6 Conclusion

6.1 Summary of Observations Across Operations

This paper explored the consequences of applying specific graph operations, mainly seidel switching and vertex deletion, to the Rook Graph $R(n)$, with a focus on how these operations modify the structure of the resulting coherent closure of the respective graphs. The overarching goal - to obtain a general coherent closure when switching strongly regular graphs - was achieved through a deliberate sequence of steps. Rather than rely on pure computation, we applied well-defined graph operations, computed the resulting adjacency matrix and coherent algebras, and finally shown why the coherent algebra was minimal, allowing us to directly infer the coherent closure and rank of the graph. Furthermore, we did this both by doing it the tedious way of matrix multiplications, as well as using known structural properties of type matrices and fibre decompositions. This pipeline ensures that each result is not just observed, but mathematically justified.

The main results can be summarised as follows:

1. **Vertex deletion**: Removing a single vertex from $R(n)$ yields a coherent closure of rank 10.
2. **Seidel switching** (Single vertex): Applying Seidel switching to a single vertex results in a coherent closure of rank 15.
3. **Clique switching** (even case): Switching $n/2$ n-cliques in $R(n)$, where n is even, yields a coherent closure of rank 6.
4. **Clique switching** (general case): For $1 < k < n/2$, switching k n-cliques in $R(n)$, regardless of whether n is odd or even, yields a coherent closure of rank 12.

These findings emphasise that although symmetry and regularity of the strongly regular graph $R(n)$ is deliberately broken, the resulting structures still admit a well-defined coherent closure of finite rank.

6.2 Directions for Further Exploration

The result of obtaining a general coherent closure despite disturbing the symmetry and regularity of strongly regular graph suggests a number of interesting paths for future work. One such path is to extend this analysis to other families of strongly regular or distance regular graphs. For example, in Section 3.2, the computation of the coherent closure of $T(n)$ proved to be a more tedious and painful way than that of $R(n)$. Although a hypothesis was formed using computational power of type matrix $\begin{bmatrix} 3 & 2 \\ & 4 \end{bmatrix}$, we were not able to prove it. Extensions to Paley Graphs and Latin Square Graphs are also worth further investigation due to their strongly regular structures as well.

In summary, this work opens the floor to understanding the method to obtain a general coherent closure for graphs that have their symmetry and regularity disturbed.

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Appendix

(For draft purposes, the final appendix is not finalised. The final display of workings will be added in by the final submission. Do note that the statements in the sections above are indeed verified thoroughly.)

Matrix Multiplication for Section 3.1

- Evaluating $\mathbf{CC}^T$

$$
\begin{aligned}
\mathbf{CC^T} &= \begin{bmatrix} I_{n-1} & I_{n-1} & I_{n-1} & \cdots & I_{n-1} \\ M_1 & M_2 & M_3 & \cdots & M_{n-1} \end{bmatrix} \begin{bmatrix} I_{n-1} & M_1^T \\ I_{n-1} & M_2^T \\ \vdots & \vdots \\ I_{n-1} & M_{n-1}^T \end{bmatrix} \\
&= \begin{bmatrix} (n-1)I_{n-1} & \sum_{k=1}^{n-1} M_k^T \\ & \\ \sum_{k=1}^{n-1} M_k & \sum_{k=1}^{n-1} M_k M_k^T \end{bmatrix}
\end{aligned}
$$

We evaluate the terms involving M_k:

$$\sum_{k=1}^{n-1} M_k = \sum_{k=1}^{n-1} M_k^T = J_{n-1}$$

and

$$
\begin{aligned}
M_k M_k^T &= (e_{k,n-1} \otimes \mathbf{1}_{n-1}^T)(e_{k,n-1}^T \otimes \mathbf{1}_{n-1}) \\
&= (e_{k,n-1} e_{k,n-1}^T) \otimes (\mathbf{1}_{n-1}^T \mathbf{1}_{n-1}) \\
&= E_{k,k} \otimes (n-1) \\
&= (n-1)E_{k,k}
\end{aligned}
$$

where $E_{k,k} \in \mathbb{R}^{(n-1)\times(n-1)}$ has 1 at its (k,k) position and 0 elsewhere. As such, $\sum_{k=1}^{n-1} M_k M_k^T = \sum_{k=1}^{n-1}(n-1)E_{k,k} = (n-1)I_{n-1}$.

$$
\begin{aligned}
&\Rightarrow \begin{bmatrix} (n-1)I_{n-1} & \sum_{k=1}^{n-1} M_k^T \\ & \\ \sum_{k=1}^{n-1} M_k & \sum_{k=1}^{n-1} M_k M_k^T \end{bmatrix} \\
&= \begin{bmatrix} (n-1)I_{n-1} & J_{n-1} \\ J_{n-1} & (n-1)I_{n-1} \end{bmatrix} \\
&= (n-2)I_{2(n-1)} - \mathbf{A_1} + J_{2(n-1)}
\end{aligned}
$$

- Evaluating $\mathbf{C^T C}$

$$\mathbf{C^T C} = \begin{bmatrix} I_{n-1} & M_1^T \\ I_{n-1} & M_2^T \\ \vdots & \vdots \\ I_{n-1} & M_{n-1}^T \end{bmatrix} \begin{bmatrix} I_{n-1} & I_{n-1} & \cdots & I_{n-1} \\ M_1 & M_2 & \cdots & M_{n-1} \end{bmatrix}$$

$$= \begin{bmatrix} I_{n-1} + M_1^T M_1 & I_{n-1} + M_1^T M_2 & I_{n-1} + M_1^T M_3 & \cdots & I_{n-1} + M_1^T M_{n-1} \\ I_{n-1} + M_2^T M_1 & I_{n-1} + M_2^T M_2 & I_{n-1} + M_2^T M_3 & \cdots & I_{n-1} + M_2^T M_{n-1} \\ I_{n-1} + M_3^T M_1 & I_{n-1} + M_3^T M_2 & I_{n-1} + M_3^T M_3 & \cdots & I_{n-1} + M_3^T M_{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ I_{n-1} + M_{n-1}^T M_1 & I_{n-1} + M_{n-1}^T M_2 & I_{n-1} + M_{n-1}^T M_3 & \cdots & I_{n-1} + M_{n-1}^T M_{n-1} \end{bmatrix}$$

Note that $M_i^T M_j$ has the following expression:

$$\begin{aligned} M_i^T M_j &= (e_{i,n-1}^T \otimes \mathbf{1}_{n-1})(e_{j,n-1} \otimes \mathbf{1}_{n-1}^T) \\ &= (e_{i,n-1}^T e_{j,n-1}) \otimes (\mathbf{1}_{n-1} \mathbf{1}_{n-1}^T) \\ &= \begin{cases} O_1 \otimes J_{n-1} & \text{if } i \neq j \\ \mathbf{1}_1 \otimes J_{n-1}, & \text{if } i = j \end{cases} \\ &= \begin{cases} O_{n-1}, & \text{if } i \neq j \\ J_{n-1}, & \text{if } i = j \end{cases} \end{aligned}$$

Thus,

$$\begin{aligned} \mathbf{C^T C} &= \begin{bmatrix} I_{n-1} + J & I_{n-1} & \cdots & I_{n-1} \\ I_{n-1} & I_{n-1} + J & \cdots & I_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ I_{n-1} & I_{n-1} & \cdots & I_{n-1} + J \end{bmatrix} \\ &= 2I_{(n-1)^2} + \mathbf{A_2} \end{aligned}$$

- Evaluating $\mathbf{CA_2}$

$\mathbf{CA_2}$

$$= \begin{bmatrix} I_{n-1} & I_{n-1} & I_{n-1} & \cdots & I_{n-1} \\ M_1 & M_2 & M_3 & \cdots & M_{n-1} \end{bmatrix} \begin{bmatrix} J_{n-1}-I & I_{n-1} & \cdots & I_{n-1} \\ I_{n-1} & J_{n-1}-I & \cdots & I_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ I_{n-1} & I_{n-1} & \cdots & J_{n-1}-I \end{bmatrix}$$

$$= \begin{bmatrix} J_{n-1}-I+(n-2)I & J_{n-1}-I+(n-2)I & \cdots & J_{n-1}-I+(n-2)I \\ \\ M_1(J_{n-1}-I) & M_2(J_{n-1}-I) & & M_{n-1}(J_{n-1}-I) \\ +\left(\sum_{k=1}^{n-1} M_k\right)-M_1 & +\left(\sum_{k=1}^{n-1} M_k\right)-M_2 & \cdots & +\left(\sum_{k=1}^{n-1} M_k\right)-M_{n-1} \end{bmatrix}$$

$$= \begin{bmatrix} J_{n-1} & J_{n-1} & \cdots & J_{n-1} \\ J_{n-1} & J_{n-1} & \cdots & J_{n-1} \end{bmatrix} + \begin{bmatrix} (n-3)I_{n-1} & (n-3)I_{n-1} & \cdots & (n-3)I_{n-1} \\ M_1J_{n-1}-2M_1 & M_2J_{n-1}-2M_2 & \cdots & M_{n-1}J_{n-1}-2M_{n-1} \end{bmatrix}$$

$$= J_{2(n-1),(n-1)^2} + \begin{bmatrix} (n-3)I_{n-1} & (n-3)I_{n-1} & \cdots & (n-3)I_{n-1} \\ (n-1)M_1-2M_1 & (n-1)M_2-2M_2 & \cdots & (n-1)M_{n-1}-2M_{n-1} \end{bmatrix}$$

$$= J_{2(n-1),(n-1)^2} + (n-3)\begin{bmatrix} I_{n-1} & I_{n-1} & \cdots & I_{n-1} \\ M_1 & M_2 & \cdots & M_{n-1} \end{bmatrix}$$

$$= J_{2(n-1),(n-1)^2} + (n-3)\mathbf{C}$$

- Evaluating $\mathbf{A_1C}$

$$\begin{aligned} \mathbf{A_1C} &= \begin{bmatrix} J_{n-1}-I & O_{n-1} \\ O_{n-1} & J_{n-1}-I \end{bmatrix} \begin{bmatrix} I_{n-1} & I_{n-1} & \dots & I_{n-1} \\ M_1 & M_2 & \dots & M_{n-1} \end{bmatrix} \\ &= \begin{bmatrix} J_{n-1}-I & J_{n-1}-I & \dots & J_{n-1}-I \\ J_{n-1}M_1-M_1 & J_{n-1}M_2-M_2 & \dots & J_{n-1}M_{n-1}-M_{n-1} \end{bmatrix} \\ &= -\begin{bmatrix} I_{n-1} & I_{n-1} & \dots & I_{n-1} \\ M_1 & M_2 & \dots & M_{n-1} \end{bmatrix} + \begin{bmatrix} J_{n-1} & J_{n-1} & \dots & J_{n-1} \\ J_{n-1}M_1 & J_{n-1}M_2 & \dots & J_{n-1}M_{n-1} \end{bmatrix} \end{aligned}$$

Note that

$$\begin{aligned} J_{n-1}M_k &= (J_{n-1}\otimes \mathbf{1}_1)(e_{k,n-1}\otimes \mathbf{1}_{n-1}^T) \\ &= (J_{n-1}e_{k,n-1})\otimes(\mathbf{1}_1\mathbf{1}_{n-1}^T) \\ &= \mathbf{1}_{n-1}\otimes \mathbf{1}_{n-1}^T \\ &= J_{n-1} \end{aligned}$$

So

$$\begin{aligned} \mathbf{A_1C} &= -\begin{bmatrix} I_{n-1} & I_{n-1} & \dots & I_{n-1} \\ M_1 & M_2 & \dots & M_{n-1} \end{bmatrix} + \begin{bmatrix} J_{n-1} & J_{n-1} & \dots & J_{n-1} \\ J_{n-1} & J_{n-1} & \dots & J_{n-1} \end{bmatrix} \\ &= -\mathbf{C} + J_{2(n-1),(n-1)^2} \end{aligned}$$

- Evaluating $\mathcal{W}_1\mathcal{W}_1$

$$\begin{aligned}\mathcal{W}_1\mathcal{W}_1 &= \begin{bmatrix} I_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix} \begin{bmatrix} I_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix} \\ &= \begin{bmatrix} I_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix} \\ &= \mathcal{W}_1 \in \mathcal{W}(\Gamma_1)\end{aligned}$$

- Evaluating $\mathcal{W}_1\mathcal{W}_2$

$$\begin{aligned}\mathcal{W}_1\mathcal{W}_2 &= \begin{bmatrix} I_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix} \begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & I_{(n-1)^2} \end{bmatrix} \\ &= \begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix} \\ &= O_{n^2-1} \in \mathcal{W}(\Gamma_1)\end{aligned}$$

- Evaluating $\mathcal{W}_1\mathcal{W}_3$

$$\begin{aligned}\mathcal{W}_1\mathcal{W}_3 &= \begin{bmatrix} I_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix} \begin{bmatrix} \mathbf{A_1} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{A_1} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix} \\ &= \mathcal{W}_3 \in \mathcal{W}(\Gamma_1)\end{aligned}$$

- Evaluating $\mathcal{W}_1\mathcal{W}_4$

$$\begin{aligned}\mathcal{W}_1\mathcal{W}_4 &= \begin{bmatrix} I_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix} \begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & \mathbf{A_2} \end{bmatrix} \\ &= \begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix} \\ &= O_{n^2-1} \in \mathcal{W}(\Gamma_1)\end{aligned}$$

- Evaluating $\mathcal{W}_1\mathcal{W}_5$

$$\begin{aligned}\mathcal{W}_1\mathcal{W}_5 &= \begin{bmatrix} I_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix} \begin{bmatrix} J - I - \mathbf{A_1} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix} \\ &= \begin{bmatrix} J - I - \mathbf{A_1} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix} \\ &= \mathcal{W}_5 \in \mathcal{W}(\Gamma_1)\end{aligned}$$

- Evaluating $\mathcal{W}_1\mathcal{W}_6$

$$
\begin{aligned}
\mathcal{W}_1\mathcal{W}_6 &= \begin{bmatrix} I_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix} \begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & J - I - \mathbf{A_2} \end{bmatrix} \\
&= \begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix} \\
&= O_{n^2-1} \in \mathcal{W}(\Gamma_1)
\end{aligned}
$$

- Evaluating $\mathcal{W}_1\mathcal{W}_7$

$$
\begin{aligned}
\mathcal{W}_1\mathcal{W}_7 &= \begin{bmatrix} I_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix} \begin{bmatrix} O_{2(n-1)} & \mathbf{C} \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix} \\
&= \begin{bmatrix} O_{2(n-1)} & \mathbf{C} \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix} \\
&= \mathcal{W}_7 \in \mathcal{W}(\Gamma_1)
\end{aligned}
$$

- Evaluating $\mathcal{W}_1\mathcal{W}_8$

$$
\begin{aligned}
\mathcal{W}_1\mathcal{W}_8 &= \begin{bmatrix} I_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix} \begin{bmatrix} O_{2(n-1)} & J - \mathbf{C} \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix} \\
&= \begin{bmatrix} O_{2(n-1)} & J - \mathbf{C} \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix} \\
&= \mathcal{W}_8 \in \mathcal{W}(\Gamma_1)
\end{aligned}
$$

- Evaluating $\mathcal{W}_1\mathcal{W}_9$

$$
\begin{aligned}
\mathcal{W}_1\mathcal{W}_9 &= \begin{bmatrix} I_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix} \begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ \mathbf{C^T} & O_{(n-1)^2} \end{bmatrix} \\
&= \begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix} \\
&= O_{n^2-1} \in \mathcal{W}(\Gamma_1)
\end{aligned}
$$

- Evaluating $\mathcal{W}_1\mathcal{W}_{10}$

$$
\begin{aligned}
\mathcal{W}_1\mathcal{W}_{10} &= \begin{bmatrix} I_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix} \begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ J - \mathbf{C^T} & O_{(n-1)^2} \end{bmatrix} \\
&= \begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix} \\
&= O_{n^2-1} \in \mathcal{W}(\Gamma_1)
\end{aligned}
$$

- Evaluating $\mathcal{W}_2\mathcal{W}_1$

$$\begin{aligned}\mathcal{W}_2\mathcal{W}_1 &= \begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & I_{(n-1)^2} \end{bmatrix} \begin{bmatrix} I_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix} \\ &= \begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix} \\ &= O_{n^2-1} \in \mathcal{W}(\Gamma_1)\end{aligned}$$

- Evaluating $\mathcal{W}_2\mathcal{W}_2$

$$\begin{aligned}\mathcal{W}_2\mathcal{W}_2 &= \begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & I_{(n-1)^2} \end{bmatrix} \begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & I_{(n-1)^2} \end{bmatrix} \\ &= \begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & I_{(n-1)^2} \end{bmatrix} \\ &= \mathcal{W}_2 \in \mathcal{W}(\Gamma_1)\end{aligned}$$

- Evaluating $\mathcal{W}_2\mathcal{W}_3$

$$\begin{aligned}\mathcal{W}_2\mathcal{W}_3 &= \begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & I_{(n-1)^2} \end{bmatrix} \begin{bmatrix} \mathbf{A_1} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix} \\ &= \begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix} \\ &= O_{n^2-1} \in \mathcal{W}(\Gamma_1)\end{aligned}$$

- Evaluating $\mathcal{W}_2\mathcal{W}_4$

$$\begin{aligned}\mathcal{W}_2\mathcal{W}_4 &= \begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & I_{(n-1)^2} \end{bmatrix} \begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & \mathbf{A_2} \end{bmatrix} \\ &= \begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & \mathbf{A_2} \end{bmatrix} \\ &= \mathcal{W}_4 \in \mathcal{W}(\Gamma_1)\end{aligned}$$

- Evaluating $\mathcal{W}_2\mathcal{W}_5$

$$\begin{aligned}\mathcal{W}_2\mathcal{W}_5 &= \begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & I_{(n-1)^2} \end{bmatrix} \begin{bmatrix} J - I - \mathbf{A_1} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix} \\ &= \begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix} \\ &= O_{n^2-1} \in \mathcal{W}(\Gamma_1)\end{aligned}$$

- Evaluating $\mathcal{W}_2\mathcal{W}_6$

$$\begin{aligned}\mathcal{W}_2\mathcal{W}_6 &= \begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & I_{(n-1)^2} \end{bmatrix} \begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & J-I-\mathbf{A_2} \end{bmatrix} \\ &= \begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & J-I-\mathbf{A_2} \end{bmatrix} \\ &= \mathcal{W}_6 \in \mathcal{W}(\Gamma_1)\end{aligned}$$

- Evaluating $\mathcal{W}_2\mathcal{W}_7$

$$\begin{aligned}\mathcal{W}_2\mathcal{W}_7 &= \begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & I_{(n-1)^2} \end{bmatrix} \begin{bmatrix} O_{2(n-1)} & \mathbf{C} \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix} \\ &= \begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix} \\ &= O_{n^2-1} \in \mathcal{W}(\Gamma_1)\end{aligned}$$

- Evaluating $\mathcal{W}_2\mathcal{W}_8$

$$\begin{aligned}\mathcal{W}_2\mathcal{W}_8 &= \begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & I_{(n-1)^2} \end{bmatrix} \begin{bmatrix} O_{2(n-1)} & J-\mathbf{C} \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix} \\ &= \begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix} \\ &= O_{n^2-1} \in \mathcal{W}(\Gamma_1)\end{aligned}$$

- Evaluating $\mathcal{W}_2\mathcal{W}_9$

$$\begin{aligned}\mathcal{W}_2\mathcal{W}_9 &= \begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & I_{(n-1)^2} \end{bmatrix} \begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ \mathbf{C^T} & O_{(n-1)^2} \end{bmatrix} \\ &= \begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ \mathbf{C^T} & O_{(n-1)^2} \end{bmatrix} \\ &= \mathcal{W}_9 \in \mathcal{W}(\Gamma_1)\end{aligned}$$

- Evaluating $\mathcal{W}_2\mathcal{W}_{10}$

$$\begin{aligned}\mathcal{W}_2\mathcal{W}_{10} &= \begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & I_{(n-1)^2} \end{bmatrix} \begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ J-\mathbf{C^T} & O_{(n-1)^2} \end{bmatrix} \\ &= \begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ J-\mathbf{C^T} & O_{(n-1)^2} \end{bmatrix} \\ &= \mathcal{W}_{10} \in \mathcal{W}(\Gamma_1)\end{aligned}$$

- Evaluating $\mathcal{W}_3\mathcal{W}_1$

$$\begin{aligned}
\mathcal{W}_3\mathcal{W}_1 &= \begin{bmatrix} \mathbf{A_1} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix} \begin{bmatrix} I_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix} \\
&= \begin{bmatrix} \mathbf{A_1} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix} \\
&= \mathcal{W}_3 \in \mathcal{W}(\Gamma_1)
\end{aligned}$$

- Evaluating $\mathcal{W}_3\mathcal{W}_2$

$$\begin{aligned}
\mathcal{W}_3\mathcal{W}_2 &= \begin{bmatrix} \mathbf{A_1} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix} \begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & I_{(n-1)^2} \end{bmatrix} \\
&= \begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix} \\
&= O_{n^2-1} \in \mathcal{W}(\Gamma_1)
\end{aligned}$$

- Evaluating $\mathcal{W}_3\mathcal{W}_3$

$$\begin{aligned}
\mathcal{W}_3\mathcal{W}_3 &= \begin{bmatrix} \mathbf{A_1} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix} \begin{bmatrix} \mathbf{A_1} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix} \\
&= \begin{bmatrix} \mathbf{A_1}^2 & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
\text{Since} \quad \mathbf{A_1} &= \begin{bmatrix} J_{n-1} - I & O_{n-1} \\ O_{n-1} & J_{n-1} - I \end{bmatrix}, \\
\mathbf{A_1^2} &= \begin{bmatrix} (J_{n-1} - I)^2 & O_{n-1} \\ O_{n-1} & (J_{n-1} - I)^2 \end{bmatrix} \\
&= \begin{bmatrix} J_{n-1}^2 - 2J + I & O_{n-1} \\ O_{n-1} & J_{n-1}^2 - 2J + I \end{bmatrix} \\
&= \begin{bmatrix} (n-1)J_{n-1} - 2J + I & O_{n-1} \\ O_{n-1} & (n-1)J_{n-1} - 2J + I \end{bmatrix} \\
&= \begin{bmatrix} (n-3)J_{n-1} + I & O_{n-1} \\ O_{n-1} & (n-3)J_{n-1} + I \end{bmatrix} \\
&= (n-3) \begin{bmatrix} J_{n-1} - I & O_{n-1} \\ O_{n-1} & J_{n-1} - I \end{bmatrix} + (n-2) \begin{bmatrix} I_{n-1} & O_{n-1} \\ O_{n-1} & I_{n-1} \end{bmatrix} \\
&= (n-3)\mathbf{A_1} + (n-2)I_{2(n-1)}
\end{aligned}$$

So we have

$$\begin{aligned}\mathcal{W}_3\mathcal{W}_3 &= \begin{bmatrix}(n-3)\mathbf{A_1}+(n-2)I_{2(n-1)} & O_{2(n-1),(n-1)^2}\\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2}\end{bmatrix}\\ &= (n-3)\mathcal{W}_3+(n-2)\mathcal{W}_1 \in \mathcal{W}(\Gamma_1)\end{aligned}$$

- Evaluating $\mathcal{W}_3\mathcal{W}_4$

$$\begin{aligned}\mathcal{W}_3\mathcal{W}_4 &= \begin{bmatrix}\mathbf{A_1} & O_{2(n-1),(n-1)^2}\\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2}\end{bmatrix}\begin{bmatrix}O_{2(n-1)} & O_{2(n-1),(n-1)^2}\\ O_{(n-1)^2,2(n-1)} & \mathbf{A_2}\end{bmatrix}\\ &= \begin{bmatrix}O_{2(n-1)} & O_{2(n-1),(n-1)^2}\\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2}\end{bmatrix}\\ &= O_{n^2-1} \in \mathcal{W}(\Gamma_1)\end{aligned}$$

- Evaluating $\mathcal{W}_3\mathcal{W}_5$

$$\begin{aligned}\mathcal{W}_3\mathcal{W}_5 &= \begin{bmatrix}\mathbf{A_1} & O_{2(n-1),(n-1)^2}\\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2}\end{bmatrix}\begin{bmatrix}J-I-\mathbf{A_1} & O_{2(n-1),(n-1)^2}\\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2}\end{bmatrix}\\ &= \begin{bmatrix}\mathbf{A_1}J_{n-1}-\mathbf{A_1}-\mathbf{A_1}^2 & O_{2(n-1),(n-1)^2}\\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2}\end{bmatrix}\end{aligned}$$

$$\begin{aligned}\mathbf{A_1}J_{n-1} &= \begin{bmatrix}J_{n-1}-I & O_{n-1}\\ O_{n-1} & J_{n-1}-I\end{bmatrix}\begin{bmatrix}J_{n-1} & J_{n-1}\\ J_{n-1} & J_{n-1}\end{bmatrix}\\ &= \begin{bmatrix}J_{n-1}^2-J & J_{n-1}^2-J\\ J_{n-1}^2-J & J_{n-1}^2-J\end{bmatrix}\\ &= \begin{bmatrix}(n-1)J_{n-1}-J & (n-1)J_{n-1}-J\\ (n-1)J_{n-1}-J & (n-1)J_{n-1}-J\end{bmatrix}\\ &= (n-2)J_{2(n-1)}\end{aligned}$$

We now have

$$\begin{aligned}\mathcal{W}_3\mathcal{W}_5 &= \begin{bmatrix}\mathbf{A_1}J_{n-1}-\mathbf{A_1}-\mathbf{A_1}^2 & O_{2(n-1),(n-1)^2}\\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2}\end{bmatrix}\\ &= \begin{bmatrix}(n-2)J-\mathbf{A_1}-(n-3)\mathbf{A_1}-(n-2)I & O_{2(n-1),(n-1)^2}\\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2}\end{bmatrix}\\ &= (n-2)\begin{bmatrix}J-I-\mathbf{A_1} & O_{2(n-1),(n-1)^2}\\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2}\end{bmatrix}\\ &= (n-2)\mathcal{W}_5 \in \mathcal{W}(\Gamma_1)\end{aligned}$$

- Evaluating $\mathcal{W}_3\mathcal{W}_6$

$$\begin{aligned}\mathcal{W}_3\mathcal{W}_6 &= \begin{bmatrix} \mathbf{A_1} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix} \begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & J - I - \mathbf{A_2} \end{bmatrix} \\ &= \begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix} \\ &= O_{n^2-1} \in \mathcal{W}(\Gamma_1)\end{aligned}$$

- Evaluating $\mathcal{W}_3\mathcal{W}_7$

$$\begin{aligned}\mathcal{W}_3\mathcal{W}_7 &= \begin{bmatrix} \mathbf{A_1} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix} \begin{bmatrix} O_{2(n-1)} & \mathbf{C} \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix} \\ &= \begin{bmatrix} O_{2(n-1)} & \mathbf{A_1 C} \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix} \\ &= \begin{bmatrix} O_{2(n-1)} & J - \mathbf{C} \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix} \\ &= \mathcal{W}_8 \in \mathcal{W}(\Gamma_1)\end{aligned}$$

- Evaluating $\mathcal{W}_3\mathcal{W}_8$

$$\begin{aligned}\mathcal{W}_3\mathcal{W}_8 &= \begin{bmatrix} \mathbf{A_1} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix} \begin{bmatrix} O_{2(n-1)} & J - \mathbf{C} \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix} \\ &= \begin{bmatrix} O_{2(n-1)} & \mathbf{A_1}J - \mathbf{A_1 C} \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix} \\ &= \begin{bmatrix} O_{2(n-1)} & (n-2)J - (J - \mathbf{C}) \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix} \\ &= (n-3) \begin{bmatrix} O_{2(n-1)} & J - \mathbf{C} \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix} + (n-2) \begin{bmatrix} O_{2(n-1)} & \mathbf{C} \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix} \\ &= (n-3)\mathcal{W}_8 + (n-2)\mathcal{W}_7 \in \mathcal{W}(\Gamma_1)\end{aligned}$$

- Evaluating $\mathcal{W}_3\mathcal{W}_9$

$$\begin{aligned}\mathcal{W}_3\mathcal{W}_9 &= \begin{bmatrix} \mathbf{A_1} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix} \begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ \mathbf{C^T} & O_{(n-1)^2} \end{bmatrix} \\ &= \begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix} \\ &= O_{n^2-1} \in \mathcal{W}(\Gamma_1)\end{aligned}$$

- Evaluating $\mathcal{W}_3\mathcal{W}_{10}$

$$\begin{aligned}\mathcal{W}_3\mathcal{W}_{10} &= \begin{bmatrix}\mathbf{A_1} & O_{2(n-1),(n-1)^2}\\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2}\end{bmatrix}\begin{bmatrix}O_{2(n-1)} & O_{2(n-1),(n-1)^2}\\ J-\mathbf{C^T} & O_{(n-1)^2}\end{bmatrix}\\ &= \begin{bmatrix}O_{2(n-1)} & O_{2(n-1),(n-1)^2}\\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2}\end{bmatrix}\\ &= O_{n^2-1} \in \mathcal{W}(\Gamma_1)\end{aligned}$$

- Evaluating $\mathcal{W}_4\mathcal{W}_1$

$$\begin{aligned}\mathcal{W}_4\mathcal{W}_1 &= \begin{bmatrix}O_{2(n-1)} & O_{2(n-1),(n-1)^2}\\ O_{(n-1)^2,2(n-1)} & \mathbf{A_2}\end{bmatrix}\begin{bmatrix}I_{2(n-1)} & O_{2(n-1),(n-1)^2}\\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2}\end{bmatrix}\\ &= O_{n^2-1} \in \mathcal{W}(\Gamma_1)\end{aligned}$$

- Evaluating $\mathcal{W}_4\mathcal{W}_2$

$$\begin{aligned}\mathcal{W}_4\mathcal{W}_2 &= \begin{bmatrix}O_{2(n-1)} & O_{2(n-1),(n-1)^2}\\ O_{(n-1)^2,2(n-1)} & \mathbf{A_2}\end{bmatrix}\begin{bmatrix}O_{2(n-1)} & O_{2(n-1),(n-1)^2}\\ O_{(n-1)^2,2(n-1)} & I_{(n-1)^2}\end{bmatrix}\\ &= \begin{bmatrix}O_{2(n-1)} & O_{2(n-1),(n-1)^2}\\ O_{(n-1)^2,2(n-1)} & \mathbf{A_2}\end{bmatrix}\\ &= \mathcal{W}_4 \in \mathcal{W}(\Gamma_1)\end{aligned}$$

- Evaluating $\mathcal{W}_4\mathcal{W}_3$

$$\begin{aligned}\mathcal{W}_4\mathcal{W}_3 &= \begin{bmatrix}O_{2(n-1)} & O_{2(n-1),(n-1)^2}\\ O_{(n-1)^2,2(n-1)} & \mathbf{A_2}\end{bmatrix}\begin{bmatrix}\mathbf{A_1} & O_{2(n-1),(n-1)^2}\\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2}\end{bmatrix}\\ &= O_{n^2-1} \in \mathcal{W}(\Gamma_1)\end{aligned}$$

- Evaluating $\mathcal{W}_4\mathcal{W}_4$

$$\begin{aligned}\mathcal{W}_4\mathcal{W}_4 &= \begin{bmatrix}O_{2(n-1)} & O_{2(n-1),(n-1)^2}\\ O_{(n-1)^2,2(n-1)} & \mathbf{A_2}\end{bmatrix}\begin{bmatrix}O_{2(n-1)} & O_{2(n-1),(n-1)^2}\\ O_{(n-1)^2,2(n-1)} & \mathbf{A_2}\end{bmatrix}\\ &= \begin{bmatrix}O_{2(n-1)} & O_{2(n-1),(n-1)^2}\\ O_{(n-1)^2,2(n-1)} & \mathbf{A_2}^2\end{bmatrix}\end{aligned}$$

Recall that $\Gamma_{\mathbf{A_2}}$ is the adjacency matrix of a square rook's graph $R(n-1)$. Thus,

$$\mathbf{A_2^2} = 2(n-2)I + (n-3)\mathbf{A_2} + 2(J - I - \mathbf{A_2})$$

We now have

$$\mathcal{W}_4\mathcal{W}_4 = \begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & 2(n-2)I + (n-3)\mathbf{A_2} + 2(J - I - \mathbf{A_2}) \end{bmatrix}$$
$$= 2(n-2)\mathcal{W}_2 + (n-3)\mathcal{W}_4 + 2\mathcal{W}_6 \in \mathcal{W}(\Gamma_1)$$

- Evaluating $\mathcal{W}_4\mathcal{W}_5$

$$\mathcal{W}_4\mathcal{W}_5 = \begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & \mathbf{A_2} \end{bmatrix} \begin{bmatrix} J - I - \mathbf{A_1} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix}$$
$$= O_{n^2-1} \in \mathcal{W}(\Gamma_1)$$

- Evaluating $\mathcal{W}_4\mathcal{W}_6$

$$\mathcal{W}_4\mathcal{W}_6 = \begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & \mathbf{A_2} \end{bmatrix} \begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & J - I - \mathbf{A_2} \end{bmatrix}$$
$$= \begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & \mathbf{A_2}J_{(n-1)^2} - \mathbf{A_2} - \mathbf{A_2^2} \end{bmatrix}$$

$$\mathbf{A_2}J_{(n-1)^2} = \begin{bmatrix} J_{n-1} - I & I_{n-1} & \cdots & I_{n-1} \\ I_{n-1} & J_{n-1} - I & \cdots & I_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ I_{n-1} & I_{n-1} & \cdots & J_{n-1} - I \end{bmatrix} \begin{bmatrix} J_{n-1} & J_{n-1} & \cdots & J_{n-1} \\ J_{n-1} & J_{n-1} & \cdots & J_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ J_{n-1} & J_{n-1} & \cdots & J_{n-1} \end{bmatrix}$$
$$= \begin{bmatrix} J_{n-1}^2 - J - (n-2)J & J_{n-1}^2 - J - (n-2)J & \cdots & J_{n-1}^2 - J - (n-2)J \\ J_{n-1}^2 - J - (n-2)J & J_{n-1}^2 - J - (n-2)J & \cdots & J_{n-1}^2 - J - (n-2)J \\ \vdots & \vdots & \ddots & \vdots \\ J_{n-1}^2 - J - (n-2)J & J_{n-1}^2 - J - (n-2)J & \cdots & J_{n-1}^2 - J - (n-2)J \end{bmatrix}$$
$$= J_{n-1} \otimes (J_{n-1}^2 - (n-3)J)$$
$$= J_{n-1} \otimes ((n-1)J_{n-1} - (n-3)J)$$
$$= J_{n-1} \otimes (2(n-2)J_{n-1})$$
$$= 2(n-2)J_{(n-1)^2}$$

We now have

$$\begin{aligned}
\mathcal{W}_4\mathcal{W}_6 &= \begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & 2(n-2)J_{(n-1)^2} - \mathbf{A_2} - 2(n-2)I + (n-3)\mathbf{A_2} + 2(J-I-\mathbf{A_2}) \end{bmatrix} \\
&= \begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & 2(n-2)(J-I-\mathbf{A_2}) - n\mathbf{A_2} + 2(J-I-\mathbf{A_2}) \end{bmatrix} \\
&= \begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & 2(n-1)(J-I-\mathbf{A_2}) - n\mathbf{A_2} \end{bmatrix} \\
&= 2(n-1)\mathcal{W}_6 - n\mathcal{W}_4 \in \mathcal{W}(\Gamma_1)
\end{aligned}$$

- Evaluating $\mathcal{W}_4\mathcal{W}_7$

$$\begin{aligned}
\mathcal{W}_4\mathcal{W}_7 &= \begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & \mathbf{A_2} \end{bmatrix} \begin{bmatrix} O_{2(n-1)} & \mathbf{C} \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix} \\
&= \begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix} \\
&= O_{n^2-1} \in \mathcal{W}(\Gamma_1)
\end{aligned}$$

- Evaluating $\mathcal{W}_4\mathcal{W}_8$

$$\begin{aligned}
\mathcal{W}_4\mathcal{W}_8 &= \begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & \mathbf{A_2} \end{bmatrix} \begin{bmatrix} O_{2(n-1)} & J-\mathbf{C} \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix} \\
&= \begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix} \\
&= O_{n^2-1} \in \mathcal{W}(\Gamma_1)
\end{aligned}$$

- Evaluating $\mathcal{W}_4\mathcal{W}_9$

$$
\begin{aligned}
\mathcal{W}_4\mathcal{W}_9 &= \begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & \mathbf{A_2} \end{bmatrix} \begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ \mathbf{C}^T & O_{(n-1)^2} \end{bmatrix} \\
&= \begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ \mathbf{A_2}\mathbf{C}^T & O_{(n-1)^2} \end{bmatrix} \\
&= \begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ (\mathbf{C}\mathbf{A_2})^T & O_{(n-1)^2} \end{bmatrix} \quad \text{(since } A_2 \text{ is self-transpose)} \\
&= \begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ (J+(n-3)\mathbf{C})^T & O_{(n-1)^2} \end{bmatrix} \\
&= \begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ J+(n-3)\mathbf{C}^T & O_{(n-1)^2} \end{bmatrix} \\
&= \begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ (J-\mathbf{C}^T)+(n-2)\mathbf{C}^T & O_{(n-1)^2} \end{bmatrix} \\
&= \mathcal{W}_{10} + (n-2)\mathcal{W}_9 \in \mathcal{W}(\Gamma_1)
\end{aligned}
$$

- Evaluating $\mathcal{W}_4\mathcal{W}_{10}$

$$
\begin{aligned}
\mathcal{W}_4\mathcal{W}_9 &= \begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & \mathbf{A_2} \end{bmatrix} \begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ J-\mathbf{C}^T & O_{(n-1)^2} \end{bmatrix} \\
&= \begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ \mathbf{A_2}J-\mathbf{A_2}\mathbf{C}^T & O_{(n-1)^2} \end{bmatrix} \\
&= \begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ 2(n-2)J-(J-\mathbf{C}^T)-(n-2)\mathbf{C}^T & O_{(n-1)^2} \end{bmatrix} \\
&= \begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ (2n-5)(J-\mathbf{C}^T)+(n-2)\mathbf{C}^T & O_{(n-1)^2} \end{bmatrix} \\
&= (2n-5)\mathcal{W}_{10} + (n-2)\mathcal{W}_9 \in \mathcal{W}(\Gamma_1)
\end{aligned}
$$

- Evaluating $\mathcal{W}_5\mathcal{W}_1$

$$
\begin{aligned}
\mathcal{W}_5\mathcal{W}_1 &= \begin{bmatrix} J-I-\mathbf{A_1} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix} \begin{bmatrix} I_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix} \\
&= \begin{bmatrix} J-I-\mathbf{A_1} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix} \\
&= \mathcal{W}_5 \in \mathcal{W}(\Gamma_1)
\end{aligned}
$$

- Evaluating $\mathcal{W}_5\mathcal{W}_2$

$$\begin{aligned}\mathcal{W}_5\mathcal{W}_2 &= \begin{bmatrix} J-I-\mathbf{A_1} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix} \begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & I_{(n-1)^2} \end{bmatrix} \\ &= O_{n^2-1} \in \mathcal{W}(\Gamma_1)\end{aligned}$$

- Evaluating $\mathcal{W}_5\mathcal{W}_3$

$$\begin{aligned}\mathcal{W}_5\mathcal{W}_3 &= \begin{bmatrix} J-I-\mathbf{A_1} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix} \begin{bmatrix} \mathbf{A_1} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix} \\ &= \begin{bmatrix} J\mathbf{A_1}-\mathbf{A_1}-\mathbf{A_1^2} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix}\end{aligned}$$

Since $J_{2(n-1)}\mathbf{A_1} = (\mathbf{A_1}^T J_{2(n-1)}^T)^T = (\mathbf{A_1}J_{2(n-1)})^T = ((n-2)J_{2(n-1)})^T = (n-2)J_{2(n-1)}$,

$$\begin{aligned}\mathcal{W}_5\mathcal{W}_3 &= \begin{bmatrix} (n-2)J-\mathbf{A_1}-(n-3)\mathbf{A_1}-(n-2)I & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix} \\ &= \begin{bmatrix} (n-2)(J-I-\mathbf{A_1}) & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix} \\ &= (n-2)\mathcal{W}_5 \in \mathcal{W}(\Gamma_1)\end{aligned}$$

- Evaluating $\mathcal{W}_5\mathcal{W}_4$

$$\begin{aligned}\mathcal{W}_5\mathcal{W}_4 &= \begin{bmatrix} J-I-\mathbf{A_1} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix} \begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & \mathbf{A_2} \end{bmatrix} \\ &= O_{n^2-1} \in \mathcal{W}(\Gamma_1)\end{aligned}$$

- Evaluating $\mathcal{W}_5\mathcal{W}_5$

$$\begin{aligned}\mathcal{W}_5\mathcal{W}_5 &= \begin{bmatrix} J_{2(n-1)}-I-\mathbf{A_1} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix} \begin{bmatrix} J_{2(n-1)}-I-\mathbf{A_1} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix} \\ &= \begin{bmatrix} J_{2(n-1)}^2-J-J\mathbf{A_1}-J+I+\mathbf{A_1}-\mathbf{A_1}J+\mathbf{A_1}+\mathbf{A_1^2} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix} \\ &= \begin{bmatrix} J_{2(n-1)}^2-2J-\mathbf{A_1}J-J\mathbf{A_1}+I+2\mathbf{A_1}+\mathbf{A_1^2} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix} \\ &= \begin{bmatrix} 2(n-1)J_{2(n-1)}-2J-2(n-2)J+I+2\mathbf{A_1}+(n-3)\mathbf{A_1}+(n-2)I & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix} \\ &= \begin{bmatrix} (n-1)I+(n-1)\mathbf{A_1} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix} \\ &= (n-1)\mathcal{W}_1+(n-1)\mathcal{W}_3 \in \mathcal{W}(\Gamma_1)\end{aligned}$$

- Evaluating $\mathcal{W}_5\mathcal{W}_6$

$$\begin{aligned}\mathcal{W}_5\mathcal{W}_6 &= \begin{bmatrix} J-I-\mathbf{A_1} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2}\end{bmatrix}\begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & J-I-\mathbf{A_2}\end{bmatrix} \\ &= O_{n^2-1} \in \mathcal{W}(\Gamma_1)\end{aligned}$$

- Evaluating $\mathcal{W}_5\mathcal{W}_7$

$$\begin{aligned}\mathcal{W}_5\mathcal{W}_7 &= \begin{bmatrix} J-I-\mathbf{A_1} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2}\end{bmatrix}\begin{bmatrix} O_{2(n-1)} & \mathbf{C} \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2}\end{bmatrix} \\ &= \begin{bmatrix} O_{2(n-1)} & J_{2(n-1)}\mathbf{C}-\mathbf{C}-\mathbf{A_1}\mathbf{C} \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2}\end{bmatrix}\end{aligned}$$

$$\begin{aligned}J_{2(n-1)}\mathbf{C} &= \begin{bmatrix} J_{n-1} & J_{n-1} \\ J_{n-1} & J_{n-1}\end{bmatrix}\begin{bmatrix} I_{n-1} & I_{n-1} & \dots & I_{n-1} \\ M_1 & M_2 & \dots & M_{n-1}\end{bmatrix} \\ &= \begin{bmatrix} J_{n-1}+JM_1 & J_{n-1}+JM_2 & \dots & J_{n-1}+JM_{n-1} \\ J_{n-1}+JM_1 & J_{n-1}+JM_2 & \dots & J_{n-1}+JM_{n-1}\end{bmatrix}\end{aligned}$$

Since $J_{n-1}M_k = J_{n-1}$,

$$\begin{aligned}J_{2(n-1)}\mathbf{C} &= \begin{bmatrix} J_{n-1}+J & J_{n-1}+J & \dots & J_{n-1}+J \\ J_{n-1}+J & J_{n-1}+J & \dots & J_{n-1}+J\end{bmatrix} \\ &= 2J_{2(n-1)}\end{aligned}$$

We now have

$$\begin{aligned}\mathcal{W}_5\mathcal{W}_7 &= \begin{bmatrix} O_{2(n-1)} & 2J-\mathbf{C}-(J-\mathbf{C}) \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2}\end{bmatrix} \\ &= \begin{bmatrix} O_{2(n-1)} & \mathbf{C}+J-\mathbf{C} \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2}\end{bmatrix} \\ &= \mathcal{W}_7+\mathcal{W}_8 \in \mathcal{W}(\Gamma_1)\end{aligned}$$

- Evaluating $\mathcal{W}_5\mathcal{W}_8$

$$\begin{aligned}
\mathcal{W}_5\mathcal{W}_8 &= \begin{bmatrix} J-I-\mathbf{A_1} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix} \begin{bmatrix} O_{2(n-1)} & J-\mathbf{C} \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix} \\
&= \begin{bmatrix} O_{2(n-1)} & J_{2(n-1)}J_{2(n-1),(n-1)^2} - J\mathbf{C} - J + \mathbf{C} - \mathbf{A_1}J + \mathbf{A_1}\mathbf{C} \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix} \\
&= \begin{bmatrix} O_{2(n-1)} & 2(n-1)J_{2(n-1),(n-1)^2} - 2J - J + \mathbf{C} - (n-2)J + (J-\mathbf{C}) \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix} \\
&= \begin{bmatrix} O_{2(n-1)} & (n-2)J_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix} \\
&= (n-2)\mathcal{W}_7 + (n-2)\mathcal{W}_8 \in \mathcal{W}(\Gamma_1)
\end{aligned}$$

- Evaluating $\mathcal{W}_5\mathcal{W}_9$

$$\begin{aligned}
\mathcal{W}_5\mathcal{W}_9 &= \begin{bmatrix} J-I-\mathbf{A_1} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix} \begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ \mathbf{C}^T & O_{(n-1)^2} \end{bmatrix} \\
&= O_{n^2-1} \in \mathcal{W}(\Gamma_1)
\end{aligned}$$

- Evaluating $\mathcal{W}_5\mathcal{W}_{10}$

$$\begin{aligned}
\mathcal{W}_5\mathcal{W}_{10} &= \begin{bmatrix} J-I-\mathbf{A_1} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix} \begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ J-\mathbf{C}^T & O_{(n-1)^2} \end{bmatrix} \\
&= O_{n^2-1} \in \mathcal{W}(\Gamma_1)
\end{aligned}$$

- Evaluating $\mathcal{W}_6\mathcal{W}_1$

$$\begin{aligned}
\mathcal{W}_6\mathcal{W}_1 &= \begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & J-I-\mathbf{A_2} \end{bmatrix} \begin{bmatrix} I_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix} \\
&= O_{n^2-1} \in \mathcal{W}(\Gamma_1)
\end{aligned}$$

- Evaluating $\mathcal{W}_6\mathcal{W}_2$

$$\begin{aligned}
\mathcal{W}_6\mathcal{W}_2 &= \begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & J-I-\mathbf{A_2} \end{bmatrix} \begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & I_{(n-1)^2} \end{bmatrix} \\
&= \begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & J-I-\mathbf{A_2} \end{bmatrix} \\
&= \mathcal{W}_6 \in \mathcal{W}(\Gamma_1)
\end{aligned}$$

- Evaluating $\mathcal{W}_6\mathcal{W}_3$

$$\begin{aligned}\mathcal{W}_6\mathcal{W}_3 &= \begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & J-I-\mathbf{A_2} \end{bmatrix}\begin{bmatrix} \mathbf{A_1} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix} \\ &= O_{n^2-1} \in \mathcal{W}(\Gamma_1)\end{aligned}$$

- Evaluating $\mathcal{W}_6\mathcal{W}_4$

$$\begin{aligned}\mathcal{W}_6\mathcal{W}_4 &= \begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & J-I-\mathbf{A_2} \end{bmatrix}\begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & \mathbf{A_2} \end{bmatrix} \\ &= \begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & J\mathbf{A_2}-\mathbf{A_2}-\mathbf{A_2}^2 \end{bmatrix}\end{aligned}$$

Since $J_{(n-1)^2}\mathbf{A_2} = (\mathbf{A_2^T} J^T_{(n-1)^2})^T = (\mathbf{A_2} J_{(n-1)^2})^T = 2(n-2)J^T_{(n-1)^2} = 2(n-2)J_{(n-1)^2}$,

$$\begin{aligned}\mathcal{W}_6\mathcal{W}_4 &= \begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2} & 2(n-2)J_{(n-1)^2}-\mathbf{A_2}-2(n-2)I-(n-3)\mathbf{A_2}-2(J-I-\mathbf{A_2}) \end{bmatrix} \\ &= \begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2} & (2n-6)J_{(n-1)^2}-(2n-6)I-(n-4)\mathbf{A_2} \end{bmatrix} \\ &= \begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2} & (2n-6)(J_{(n-1)^2}-I-\mathbf{A_2})+(n-2)\mathbf{A_2} \end{bmatrix} \\ &= 2(n-3)\mathcal{W}_6 + (n-2)\mathbf{W}_4 \in \mathcal{W}(\Gamma_1)\end{aligned}$$

- Evaluating $\mathcal{W}_6\mathcal{W}_5$

$$\begin{aligned}\mathcal{W}_6\mathcal{W}_5 &= \begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & J-I-\mathbf{A_2} \end{bmatrix}\begin{bmatrix} J-I-\mathbf{A_1} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix} \\ &= O_{n^2-1} \in \mathcal{W}(\Gamma_1)\end{aligned}$$

- Evaluating $\mathcal{W}_6\mathcal{W}_6$

$$\begin{aligned}
\mathcal{W}_6\mathcal{W}_6 &= \begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & J-I-\mathbf{A_2} \end{bmatrix} \begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & J-I-\mathbf{A_2} \end{bmatrix} \\
&= \begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & J^2_{(n-1)^2} - J - J\mathbf{A_2} - J + I + \mathbf{A_2} - \mathbf{A_2}J + \mathbf{A_2} + \mathbf{A_2^2} \end{bmatrix} \\
&= \begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & \begin{matrix}(n-1)^2 J_{(n-1)^2} - 2J - 2(2(n-2)J) + I + 2\mathbf{A_2} \\ +2(n-2)I + (n-3)\mathbf{A_2} + 2(J-I-\mathbf{A_2})\end{matrix} \end{bmatrix} \\
&= \begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & [(n-1)^2 - 4n + 8]J + (2n-5)I + (n-3)\mathbf{A_2} \end{bmatrix} \\
&= \begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & (n^2-6n+9)J + (2n-5)I + (n-3)\mathbf{A_2} \end{bmatrix} \\
&= \begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & (n^2-6n+9)(J-I-\mathbf{A_2}) + (n^2-4n+4)I + (n^2-5n+6)\mathbf{A_2} \end{bmatrix} \\
&= (n^2-6n+9)\mathcal{W}_6 + (n^2-4n+4)\mathcal{W}_2 + (n^2-5n+6)\mathcal{W}_4 \in \mathcal{W}(\Gamma_1)
\end{aligned}$$

- Evaluating $\mathcal{W}_6\mathcal{W}_7$

$$\begin{aligned}
\mathcal{W}_6\mathcal{W}_7 &= \begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & J-I-\mathbf{A_2} \end{bmatrix} \begin{bmatrix} O_{2(n-1)} & \mathbf{C} \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix} \\
&= O_{n^2-1} \in \mathcal{W}(\Gamma_1)
\end{aligned}$$

- Evaluating $\mathcal{W}_6\mathcal{W}_8$

$$\begin{aligned}
\mathcal{W}_6\mathcal{W}_8 &= \begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & J-I-\mathbf{A_2} \end{bmatrix} \begin{bmatrix} O_{2(n-1)} & J-\mathbf{C} \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix} \\
&= O_{n^2-1} \in \mathcal{W}(\Gamma_1)
\end{aligned}$$

- Evaluating $\mathcal{W}_6\mathcal{W}_9$

$$\begin{aligned}
\mathcal{W}_6\mathcal{W}_9 &= \begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & J-I-\mathbf{A_2} \end{bmatrix} \begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ \mathbf{C}^T & O_{(n-1)^2} \end{bmatrix} \\
&= \begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ J_{(n-1)^2}\mathbf{C}^T - \mathbf{C}^T - \mathbf{A_2}\mathbf{C}^T & O_{(n-1)^2} \end{bmatrix}
\end{aligned}$$

$$J_{(n-1)^2}\mathbf{C}^T = \begin{bmatrix} J_{n-1} & J_{n-1} & \dots & J_{n-1} \\ J_{n-1} & J_{n-1} & \dots & J_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ J_{n-1} & J_{n-1} & \dots & J_{n-1} \end{bmatrix} \begin{bmatrix} I_{n-1} & M_1^T \\ I_{n-1} & M_2^T \\ \vdots & \vdots \\ I_{n-1} & M_{n-1}^T \end{bmatrix}$$
$$= \begin{bmatrix} (n-1)J_{n-1} & \sum_{k=1}^{n-1} J_{n-1}M_k^T \\ (n-1)J_{n-1} & \sum_{k=1}^{n-1} J_{n-1}M_k^T \\ \vdots & \vdots \\ (n-1)J_{n-1} & \sum_{k=1}^{n-1} J_{n-1}M_k^T \end{bmatrix}$$

$$\begin{aligned} J_{n-1}M_k^T &= (J_{n-1} \otimes \mathbf{1}_1)(\mathbf{1}_{n-1} \otimes e_{k,n-1}^T) \\ &= J_{n-1}\mathbf{1}_{n-1} \otimes \mathbf{1}_1 e_{k,n-1}^T \\ &= (n-1)\mathbf{1}_{n-1} \otimes e_{k,n-1}^T \\ &= (n-1)M_k^T \\ \Rightarrow \quad & \sum_{k=1}^{n-1} J_{n-1}M_k^T = (n-1)\sum_{k=1}^{n-1} M_k^T = (n-1)J_{n-1} \\ \Rightarrow \quad & J_{(n-1)^2}\mathbf{C}^T = (n-1)J_{(n-1)^2,2(n-1)} \end{aligned}$$

We now have

$$\begin{aligned} \mathcal{W}_6\mathcal{W}_9 &= \begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ (n-1)J_{(n-1)^2,2(n-1)} - \mathbf{C}^T - J - (n-3)\mathbf{C}^T & O_{(n-1)^2} \end{bmatrix} \\ &= \begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ (n-2)(J - \mathbf{C}^T) & O_{(n-1)^2} \end{bmatrix} \\ &= (n-2)\mathcal{W}_{10} \in \mathcal{W}(\Gamma_1) \end{aligned}$$

- Evaluating $\mathcal{W}_6\mathcal{W}_{10}$

$$\begin{aligned}
\mathcal{W}_6\mathcal{W}_{10} &= \begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & J-I-\mathbf{A_2} \end{bmatrix} \begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ J-\mathbf{C}^T & O_{(n-1)^2} \end{bmatrix} \\
&= \begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & J_{(n-1)^2}J_{(n-1)^2,2(n-1)} - J - \mathbf{A_2}J_{(n-1)^2,2(n-1)} \\ & -J_{(n-1)^2}\mathbf{C}^T + \mathbf{C}^T + \mathbf{A_2}\mathbf{C}^T \end{bmatrix} \\
&= \begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & (n^2-5n+6)J + (n-2)\mathbf{C}^T \end{bmatrix} \\
&= \begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & (n^2-5n+6)(J-\mathbf{C}^T) + (n-2)\mathbf{C}^T \end{bmatrix} \\
&= \begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & (n^2-5n+6)(J-\mathbf{C}^T) + (n^2-4n+4)\mathbf{C}^T \end{bmatrix} \\
&= (n^2-5n+6)\mathcal{W}_{10} + (n^2-4n+4)\mathcal{W}_9 \in \mathcal{W}(\Gamma_1)
\end{aligned}$$

- Evaluating $\mathcal{W}_7\mathcal{W}_1$

$$\begin{aligned}
\mathcal{W}_7\mathcal{W}_1 &= \begin{bmatrix} O_{2(n-1)} & \mathbf{C} \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix} \begin{bmatrix} I_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix} \\
&= O_{n^2-1} \in \mathcal{W}(\Gamma_1)
\end{aligned}$$

- Evaluating $\mathcal{W}_7\mathcal{W}_2$

$$\begin{aligned}
\mathcal{W}_7\mathcal{W}_2 &= \begin{bmatrix} O_{2(n-1)} & \mathbf{C} \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix} \begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & I_{(n-1)^2} \end{bmatrix} \\
&= \begin{bmatrix} O_{2(n-1)} & \mathbf{C} \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix} \\
&= \mathcal{W}_7 \in \mathcal{W}(\Gamma_1)
\end{aligned}$$

- Evaluating $\mathcal{W}_7\mathcal{W}_3$

$$\begin{aligned}
\mathcal{W}_7\mathcal{W}_3 &= \begin{bmatrix} O_{2(n-1)} & \mathbf{C} \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix} \begin{bmatrix} \mathbf{A_1} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix} \\
&= O_{n^2-1} \in \mathcal{W}(\Gamma_1)
\end{aligned}$$

- Evaluating $\mathcal{W}_7\mathcal{W}_4$

$$\begin{aligned}
\mathcal{W}_7\mathcal{W}_4 &= \begin{bmatrix} O_{2(n-1)} & \mathbf{C} \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix} \begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & \mathbf{A_2} \end{bmatrix} \\
&= \begin{bmatrix} O_{2(n-1)} & \mathbf{CA_2} \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix} \\
&= \begin{bmatrix} O_{2(n-1)} & J + (n-3)\mathbf{C} \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix} \\
&= \begin{bmatrix} O_{2(n-1)} & J - \mathbf{C} + (n-2)\mathbf{C} \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix} \\
&= \mathcal{W}_8 + (n-2)\mathcal{W}_7 \in \mathcal{W}(\Gamma_1)
\end{aligned}$$

- Evaluating $\mathcal{W}_7\mathcal{W}_5$

$$\begin{aligned}
\mathcal{W}_7\mathcal{W}_5 &= \begin{bmatrix} O_{2(n-1)} & \mathbf{C} \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix} \begin{bmatrix} J - I - \mathbf{A_1} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix} \\
&= O_{n^2-1} \in \mathcal{W}(\Gamma_1)
\end{aligned}$$

- Evaluating $\mathcal{W}_7\mathcal{W}_6$

$$\begin{aligned}
\mathcal{W}_7\mathcal{W}_6 &= \begin{bmatrix} O_{2(n-1)} & \mathbf{C} \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix} \begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & J - I\mathbf{A_2} \end{bmatrix} \\
&= \begin{bmatrix} O_{2(n-1)} & \mathbf{C}J_{(n-1)^2} - \mathbf{C} - \mathbf{CA_2} \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix} \\
&= \begin{bmatrix} O_{2(n-1)} & (J_{(n-1)^2}^T\mathbf{C}^T)^T - \mathbf{C} - J - (n-3)\mathbf{C} \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix} \\
&= \begin{bmatrix} O_{2(n-1)} & (J_{(n-1)^2}\mathbf{C}^T)^T - J - (n-2)\mathbf{C} \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix} \\
&= \begin{bmatrix} O_{2(n-1)} & ((n-1)J_{(n-1)^2,2(n-1)})^T - J - (n-2)\mathbf{C} \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix} \\
&= \begin{bmatrix} O_{2(n-1)} & (n-1)J_{2(n-1),(n-1)^2} - J - (n-2)\mathbf{C} \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix} \\
&= \begin{bmatrix} O_{2(n-1)} & (n-2)(J - \mathbf{C}) \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix} \\
&= (n-2)\mathcal{W}_8 \in \mathcal{W}(\Gamma_1)
\end{aligned}$$

- Evaluating $\mathcal{W}_7\mathcal{W}_7$

$$\begin{aligned}\mathcal{W}_7\mathcal{W}_7 &= \begin{bmatrix} O_{2(n-1)} & \mathbf{C} \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2}\end{bmatrix}\begin{bmatrix} O_{2(n-1)} & \mathbf{C} \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2}\end{bmatrix} \\ &= O_{n^2-1} \in \mathcal{W}(\Gamma_1)\end{aligned}$$

- Evaluating $\mathcal{W}_7\mathcal{W}_8$

$$\begin{aligned}\mathcal{W}_7\mathcal{W}_8 &= \begin{bmatrix} O_{2(n-1)} & \mathbf{C} \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2}\end{bmatrix}\begin{bmatrix} O_{2(n-1)} & J-\mathbf{C} \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2}\end{bmatrix} \\ &= O_{n^2-1} \in \mathcal{W}(\Gamma_1)\end{aligned}$$

- Evaluating $\mathcal{W}_7\mathcal{W}_9$

$$\begin{aligned}\mathcal{W}_7\mathcal{W}_9 &= \begin{bmatrix} O_{2(n-1)} & \mathbf{C} \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2}\end{bmatrix}\begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ \mathbf{C}^T & O_{(n-1)^2}\end{bmatrix} \\ &= \begin{bmatrix} \mathbf{C}\mathbf{C}^T & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2}\end{bmatrix} \\ &= \begin{bmatrix} (n-2)I-\mathbf{A_1}+J & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2}\end{bmatrix} \\ &= \begin{bmatrix} (n-1)I+(J-I-\mathbf{A_1}) & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2}\end{bmatrix} \\ &= (n-1)\mathcal{W}_1+\mathcal{W}_5 \in \mathcal{W}(\Gamma_1)\end{aligned}$$

- Evaluating $\mathcal{W}_7\mathcal{W}_{10}$

$$\begin{aligned}\mathcal{W}_7\mathcal{W}_{10} &= \begin{bmatrix} O_{2(n-1)} & \mathbf{C} \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2}\end{bmatrix}\begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ J-\mathbf{C}^T & O_{(n-1)^2}\end{bmatrix} \\ &= \begin{bmatrix} \mathbf{C}J_{(n-1)^2,2(n-1)}-\mathbf{C}\mathbf{C}^T & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2}\end{bmatrix}\end{aligned}$$

$$\mathbf{C}J_{(n-1)^2,2(n-1)} = \begin{bmatrix} I_{n-1} & I_{n-1} & \dots & I_{n-1} \\ M_1 & M_2 & \dots & M_{n-1} \end{bmatrix} \begin{bmatrix} J_{n-1} & J_{n-1} \\ J_{n-1} & J_{n-1} \\ \vdots & \vdots \\ J_{n-1} & J_{n-1} \end{bmatrix}$$

$$= \begin{bmatrix} (n-1)J_{n-1} & (n-1)J_{n-1} \\ \sum_{k=1}^{n-1} M_k J_{n-1} & \sum_{k=1}^{n-1} M_k J_{n-1} \end{bmatrix}$$

$$= \begin{bmatrix} (n-1)J_{n-1} & (n-1)J_{n-1} \\ \sum_{k=1}^{n-1} (J_{n-1}^T M_k^T)^T & \sum_{k=1}^{n-1} (J_{n-1}^T M_k^T)^T \end{bmatrix}$$

$$= \begin{bmatrix} (n-1)J_{n-1} & (n-1)J_{n-1} \\ \sum_{k=1}^{n-1} (J_{n-1} M_k^T)^T & \sum_{k=1}^{n-1} (J_{n-1} M_k^T)^T \end{bmatrix}$$

$$= \begin{bmatrix} (n-1)J_{n-1} & (n-1)J_{n-1} \\ \sum_{k=1}^{n-1} ((n-1) M_k^T)^T & \sum_{k=1}^{n-1} ((n-1) M_k^T)^T \end{bmatrix}$$

$$= \begin{bmatrix} (n-1)J_{n-1} & (n-1)J_{n-1} \\ \sum_{k=1}^{n-1} (n-1) M_k & \sum_{k=1}^{n-1} (n-1) M_k \end{bmatrix}$$

$$= \begin{bmatrix} (n-1)J_{n-1} & (n-1)J_{n-1} \\ (n-1)J_{n-1} & (n-1)J_{n-1} \end{bmatrix}$$

$$= (n-1)J_{2(n-1)}$$

We now have

$$\mathcal{W}_7\mathcal{W}_{10} = \begin{bmatrix} \mathbf{C}J_{(n-1)^2,2(n-1)} - \mathbf{C}\mathbf{C}^T & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix}$$

$$= \begin{bmatrix} (n-1)J - (n-2)I + \mathbf{A_1} - J & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix}$$

$$= \begin{bmatrix} (n-2)(J - I - \mathbf{A_1}) + (n-1)\mathbf{A_1} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix}$$

$$= (n-2)\mathcal{W}_5 + (n-1)\mathcal{W}_3 \in \mathcal{W}(\Gamma_1)$$

- Evaluating $\mathcal{W}_8\mathcal{W}_1$

$$\mathcal{W}_8\mathcal{W}_1 = \begin{bmatrix} O_{2(n-1)} & J - \mathbf{C} \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix} \begin{bmatrix} I_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix}$$

$$= O_{n^2-1} \in \mathcal{W}(\Gamma_1)$$

- Evaluating $\mathcal{W}_8\mathcal{W}_2$

$$\begin{aligned}\mathcal{W}_8\mathcal{W}_2 &= \begin{bmatrix} O_{2(n-1)} & J-\mathbf{C} \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix} \begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & I_{(n-1)^2} \end{bmatrix} \\ &= \begin{bmatrix} O_{2(n-1)} & J-\mathbf{C} \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix} \\ &= \mathcal{W}_8 \in \mathcal{W}(\Gamma_1)\end{aligned}$$

- Evaluating $\mathcal{W}_8\mathcal{W}_3$

$$\begin{aligned}\mathcal{W}_8\mathcal{W}_3 &= \begin{bmatrix} O_{2(n-1)} & J-\mathbf{C} \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix} \begin{bmatrix} \mathbf{A_1} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix} \\ &= O_{n^2-1} \in \mathcal{W}(\Gamma_1)\end{aligned}$$

- Evaluating $\mathcal{W}_8\mathcal{W}_4$

$$\begin{aligned}\mathcal{W}_8\mathcal{W}_4 &= \begin{bmatrix} O_{2(n-1)} & J-\mathbf{C} \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix} \begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & \mathbf{A_2} \end{bmatrix} \\ &= \begin{bmatrix} O_{2(n-1)} & J_{2(n-1),(n-1)^2}\mathbf{A_2} - \mathbf{C}\mathbf{A_2} \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix} \\ &= \begin{bmatrix} O_{2(n-1)} & (\mathbf{A_2}^T J^T_{2(n-1),(n-1)^2})^T - (J-(n-3)\mathbf{C}) \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix} \\ &= \begin{bmatrix} O_{2(n-1)} & (\mathbf{A_2} J_{(n-1)^2,2(n-1)})^T - (J-(n-3)\mathbf{C}) \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix} \\ &= \begin{bmatrix} O_{2(n-1)} & (2(n-2) J_{(n-1)^2,2(n-1)})^T - (J-(n-3)\mathbf{C}) \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix} \\ &= \begin{bmatrix} O_{2(n-1)} & 2(n-2) J_{2(n-1),(n-1)^2} - (J-(n-3)\mathbf{C}) \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix} \\ &= \begin{bmatrix} O_{2(n-1)} & (2n-5)(J-\mathbf{C}) + (n-2)\mathbf{C} \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix} \\ &= (2n-5)\mathcal{W}_8 + (n-2)\mathcal{W}_7 \in \mathcal{W}(\Gamma_1)\end{aligned}$$

- Evaluating $\mathcal{W}_8\mathcal{W}_5$

$$\begin{aligned}\mathcal{W}_8\mathcal{W}_5 &= \begin{bmatrix} O_{2(n-1)} & J-\mathbf{C} \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix} \begin{bmatrix} J-I-\mathbf{A_1} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix} \\ &= O_{n^2-1} \in \mathcal{W}(\Gamma_1)\end{aligned}$$

- Evaluating $\mathcal{W}_8\mathcal{W}_6$

$$\begin{aligned}
\mathcal{W}_8\mathcal{W}_6 &= \begin{bmatrix} O_{2(n-1)} & J-\mathbf{C} \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix} \begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & J-I\mathbf{A_2} \end{bmatrix} \\
&= \begin{bmatrix} O_{2(n-1)} & J_{2(n-1),(n-1)^2}J_{(n-1)^2} - J - J\mathbf{A_2} \\ & -\mathbf{C}J + \mathbf{C} + \mathbf{C}\mathbf{A_2} \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix} \\
&= \begin{bmatrix} O_{2(n-1)} & (n-1)^2 J_{2(n-1),(n-1)^2} - J - 2(n-2)J \\ & -(n-1)J + \mathbf{C} + J + (n-3)\mathbf{C} \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix} \\
&= \begin{bmatrix} O_{2(n-1)} & (n^2-5n+6)J + (n-2)\mathbf{C} \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix} \\
&= \begin{bmatrix} O_{2(n-1)} & (n^2-5n+6)(J-\mathbf{C}) + (n^2-4n+4)\mathbf{C} \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix} \\
&= (n^2-5n+6)\mathcal{W}_8 + (n^2-4n+4)\mathcal{W}_7 \in \mathcal{W}(\Gamma_1)
\end{aligned}$$

- Evaluating $\mathcal{W}_8\mathcal{W}_7$

$$\begin{aligned}
\mathcal{W}_8\mathcal{W}_7 &= \begin{bmatrix} O_{2(n-1)} & J-\mathbf{C} \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix} \begin{bmatrix} O_{2(n-1)} & \mathbf{C} \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix} \\
&= O_{n^2-1} \in \mathcal{W}(\Gamma_1)
\end{aligned}$$

- Evaluating $\mathcal{W}_8\mathcal{W}_8$

$$\begin{aligned}
\mathcal{W}_8\mathcal{W}_7 &= \begin{bmatrix} O_{2(n-1)} & J-\mathbf{C} \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix} \begin{bmatrix} O_{2(n-1)} & J-\mathbf{C} \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix} \\
&= O_{n^2-1} \in \mathcal{W}(\Gamma_1)
\end{aligned}$$

- Evaluating $\mathcal{W}_8\mathcal{W}_9$

$$\begin{aligned}
\mathcal{W}_8\mathcal{W}_9 &= \begin{bmatrix} O_{2(n-1)} & J-\mathbf{C} \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix} \begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ \mathbf{C}^T & O_{(n-1)^2} \end{bmatrix} \\
&= \begin{bmatrix} J_{2(n-1),(n-1)^2}\mathbf{C}^T - \mathbf{C}\mathbf{C}^T & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix} \\
&= \begin{bmatrix} (n-1)J_{2(n-1)} - (n-2)I + \mathbf{A_1} - J & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix} \\
&= \begin{bmatrix} (n-2)(J_{2(n-1)} - I - \mathbf{A_1}) + (n-1)\mathbf{A_1} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix} \\
&= (n-2)\mathcal{W}_5 + (n-1)\mathcal{W}_3 \in \mathcal{W}(\Gamma_1)
\end{aligned}$$

- Evaluating $\mathcal{W}_8\mathcal{W}_{10}$

$$\begin{aligned}
\mathcal{W}_8\mathcal{W}_{10} &= \begin{bmatrix} O_{2(n-1)} & J-\mathbf{C} \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix} \begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ J-\mathbf{C}^T & O_{(n-1)^2} \end{bmatrix} \\
&= \begin{bmatrix} J_{2(n-1),(n-1)^2}J_{(n-1)^2,2(n-1)} - \mathbf{C}J_{(n-1)^2,2(n-1)} - J_{2(n-1),(n-1)^2}\mathbf{C}^T + \mathbf{C}\mathbf{C}^T & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix} \\
&= \begin{bmatrix} (n-1)^2J_{2(n-1)} - (n-1)J - (n-1)J + (n-2)I - \mathbf{A_1} + J & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix} \\
&= \begin{bmatrix} (n^2-4n+4)J_{2(n-1)} + (n-2)I - \mathbf{A_1} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix} \\
&= \begin{bmatrix} (n^2-4n+4)(J-I-\mathbf{A_1}) + (n^2-3n+2)I + (n^2-4n+3)\mathbf{A_1} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix} \\
&= (n^2-4n+4)\mathcal{W}_5 + (n^2-3n+2)\mathcal{W}_1 + (n^2-4n+3)\mathcal{W}_3
\end{aligned}$$

- Evaluating $\mathcal{W}_9\mathcal{W}_1$

$$\begin{aligned}
\mathcal{W}_9\mathcal{W}_1 &= \begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ \mathbf{C^T} & O_{(n-1)^2} \end{bmatrix} \begin{bmatrix} I_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix} \\
&= \begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ \mathbf{C^T} & O_{(n-1)^2} \end{bmatrix} \\
&= \mathcal{W}_9 \in \mathcal{W}(\Gamma_1)
\end{aligned}$$

- Evaluating $\mathcal{W}_9\mathcal{W}_2$

$$\begin{aligned}
\mathcal{W}_9\mathcal{W}_2 &= \begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ \mathbf{C^T} & O_{(n-1)^2} \end{bmatrix} \begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & I_{(n-1)^2} \end{bmatrix} \\
&= \begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix} \\
&= O_{n^2-1} \in \mathcal{W}(\Gamma_1)
\end{aligned}$$

- Evaluating $\mathcal{W}_9\mathcal{W}_3$

$$
\begin{aligned}
\mathcal{W}_9\mathcal{W}_3 &= \begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ \mathbf{C^T} & O_{(n-1)^2} \end{bmatrix} \begin{bmatrix} \mathbf{A_1} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix} \\
&= \begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ \mathbf{C}^T\mathbf{A_1} & O_{(n-1)^2} \end{bmatrix} \\
&= \begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ (\mathbf{A_1}^T\mathbf{C})^T & O_{(n-1)^2} \end{bmatrix} \\
&= \begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ (\mathbf{A_1}\mathbf{C})^T & O_{(n-1)^2} \end{bmatrix} \\
&= \begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ (-\mathbf{C} + J_{2(n-1),(n-1)^2})^T & O_{(n-1)^2} \end{bmatrix} \\
&= \begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ -\mathbf{C}^T + J_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix} \\
&= \mathcal{W}_{10} \in \mathcal{W}(\Gamma_1)
\end{aligned}
$$

- Evaluating $\mathcal{W}_9\mathcal{W}_4$

$$
\begin{aligned}
\mathcal{W}_9\mathcal{W}_4 &= \begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ \mathbf{C^T} & O_{(n-1)^2} \end{bmatrix} \begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & \mathbf{A_2} \end{bmatrix} \\
&= \begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix} \\
&= O_{n^2-1} \in \mathcal{W}(\Gamma_1)
\end{aligned}
$$

- Evaluating $\mathcal{W}_9\mathcal{W}_5$

$$
\begin{aligned}
\mathcal{W}_9\mathcal{W}_5 &= \begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ \mathbf{C^T} & O_{(n-1)^2} \end{bmatrix} \begin{bmatrix} J - I - \mathbf{A_1} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix} \\
&= \begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ \mathbf{C}^T J - \mathbf{C}^T - \mathbf{C}^T\mathbf{A_1} & O_{(n-1)^2} \end{bmatrix} \\
&= \begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ 2J - \mathbf{C}^T + \mathbf{C}^T - J_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix} \\
&= \begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ J - \mathbf{C}^T + \mathbf{C}^T & O_{(n-1)^2} \end{bmatrix} \\
&= \mathcal{W}_9 + \mathcal{W}_{10} \in \mathcal{W}(\Gamma_1)
\end{aligned}
$$

- Evaluating $\mathcal{W}_9\mathcal{W}_6$

$$\begin{aligned}
\mathcal{W}_9\mathcal{W}_2 &= \begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ \mathbf{C^T} & O_{(n-1)^2} \end{bmatrix} \begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & J - I - \mathbf{A_2} \end{bmatrix} \\
&= \begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix} \\
&= O_{n^2-1} \in \mathcal{W}(\Gamma_1)
\end{aligned}$$

- Evaluating $\mathcal{W}_9\mathcal{W}_7$

$$\begin{aligned}
\mathcal{W}_9\mathcal{W}_7 &= \begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ \mathbf{C^T} & O_{(n-1)^2} \end{bmatrix} \begin{bmatrix} O_{2(n-1)} & \mathbf{C} \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix} \\
&= \begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & \mathbf{C}^T\mathbf{C} \end{bmatrix} \\
&= \begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & 2I + \mathbf{A_2} \end{bmatrix} \\
&= 2\mathcal{W}_2 + \mathcal{W}_4 \in \mathcal{W}(\Gamma_1)
\end{aligned}$$

- Evaluating $\mathcal{W}_9\mathcal{W}_8$

$$\begin{aligned}
\mathcal{W}_9\mathcal{W}_8 &= \begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ \mathbf{C^T} & O_{(n-1)^2} \end{bmatrix} \begin{bmatrix} O_{2(n-1)} & J - \mathbf{C} \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix} \\
&= \begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & \mathbf{C}^T J - \mathbf{C}^T\mathbf{C} \end{bmatrix} \\
&= \begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & 2J - 2I - \mathbf{A_2} \end{bmatrix} \\
&= \begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & 2(J - I - \mathbf{A_2}) + \mathbf{A_2} \end{bmatrix} \\
&= 2\mathcal{W}_6 + \mathcal{W}_4 \in \mathcal{W}(\Gamma_1)
\end{aligned}$$

- Evaluating $\mathcal{W}_9\mathcal{W}_9$

$$\begin{aligned}
\mathcal{W}_9\mathcal{W}_9 &= \begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ \mathbf{C^T} & O_{(n-1)^2} \end{bmatrix} \begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ \mathbf{C^T} & O_{(n-1)^2} \end{bmatrix} \\
&= O_{n^2-1} \in \mathcal{W}(\Gamma_1)
\end{aligned}$$

- Evaluating $\mathcal{W}_9\mathcal{W}_{10}$

$$\begin{aligned}
\mathcal{W}_9\mathcal{W}_{10} &= \begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ \mathbf{C^T} & O_{(n-1)^2} \end{bmatrix} \begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ J - \mathbf{C^T} & O_{(n-1)^2} \end{bmatrix} \\
&= O_{n^2-1} \in \mathcal{W}(\Gamma_1)
\end{aligned}$$

- Evaluating $\mathcal{W}_{10}\mathcal{W}_1$

$$\begin{aligned}
\mathcal{W}_{10}\mathcal{W}_1 &= \begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ J-\mathbf{C^T} & O_{(n-1)^2} \end{bmatrix} \begin{bmatrix} I_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix} \\
&= \begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ J-\mathbf{C^T} & O_{(n-1)^2} \end{bmatrix} \\
&= \mathcal{W}_{10} \in \mathcal{W}(\Gamma_1)
\end{aligned}$$

- Evaluating $\mathcal{W}_{10}\mathcal{W}_2$

$$\begin{aligned}
\mathcal{W}_{10}\mathcal{W}_2 &= \begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ J-\mathbf{C^T} & O_{(n-1)^2} \end{bmatrix} \begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & I_{(n-1)^2} \end{bmatrix} \\
&= O_{n^2-1} \in \mathcal{W}(\Gamma_1)
\end{aligned}$$

- Evaluating $\mathcal{W}_{10}\mathcal{W}_3$

$$\begin{aligned}
\mathcal{W}_{10}\mathcal{W}_3 &= \begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ J-\mathbf{C^T} & O_{(n-1)^2} \end{bmatrix} \begin{bmatrix} \mathbf{A_1} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix} \\
&= \begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ J\mathbf{A_1}-\mathbf{C^T}\mathbf{A_1} & O_{(n-1)^2} \end{bmatrix} \\
&= \begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ (n-2)J-(J-\mathbf{C})^T & O_{(n-1)^2} \end{bmatrix} \\
&= \begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ (n-2)J-J+\mathbf{C}^T & O_{(n-1)^2} \end{bmatrix} \\
&= \begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ (n-3)(J-\mathbf{C}^T)+(n-2)\mathbf{C}^T & O_{(n-1)^2} \end{bmatrix} \\
&= (n-3)\mathcal{W}_{10}+(n-2)\mathcal{W}_9 \in \mathcal{W}(\Gamma_1)
\end{aligned}$$

- Evaluating $\mathcal{W}_{10}\mathcal{W}_4$

$$\begin{aligned}
\mathcal{W}_{10}\mathcal{W}_4 &= \begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ J-\mathbf{C^T} & O_{(n-1)^2} \end{bmatrix} \begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & \mathbf{A_2} \end{bmatrix} \\
&= O_{n^2-1} \in \mathcal{W}(\Gamma_1)
\end{aligned}$$

- Evaluating $\mathcal{W}_{10}\mathcal{W}_5$

$$\begin{aligned}
\mathcal{W}_{10}\mathcal{W}_5 &= \begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ J-\mathbf{C^T} & O_{(n-1)^2} \end{bmatrix} \begin{bmatrix} J-I-\mathbf{A_1} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix} \\
&= \begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ J_{(n-1)^2,2(n-1)}J_{2(n-1)} - J - J\mathbf{A_1} & O_{(n-1)^2} \\ -\mathbf{C}^TJ + \mathbf{C}^T + \mathbf{C}^T\mathbf{A_1} & \end{bmatrix} \\
&= \begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ 2(n-1)J_{(n-1)^2,2(n-1)} - J - (n-2)J & O_{(n-1)^2} \\ -2J + \mathbf{C}^T + J - \mathbf{C}^T & \end{bmatrix} \\
&= \begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ (n-2)(J-\mathbf{C}^T) + (n-2)\mathbf{C}^T & O_{(n-1)^2} \end{bmatrix} \\
&= (n-2)\mathcal{W}_{10} + (n-2)\mathcal{W}_9 \in \mathcal{W}(\Gamma_1)
\end{aligned}$$

- Evaluating $\mathcal{W}_{10}\mathcal{W}_6$

$$\begin{aligned}
\mathcal{W}_{10}\mathcal{W}_6 &= \begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ J-\mathbf{C^T} & O_{(n-1)^2} \end{bmatrix} \begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & J-I-\mathbf{A_2} \end{bmatrix} \\
&= O_{n^2-1} \in \mathcal{W}(\Gamma_1)
\end{aligned}$$

- Evaluating $\mathcal{W}_{10}\mathcal{W}_7$

$$\begin{aligned}
\mathcal{W}_{10}\mathcal{W}_7 &= \begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ J-\mathbf{C}^T & O_{(n-1)^2} \end{bmatrix} \begin{bmatrix} O_{2(n-1)} & \mathbf{C} \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix} \\
&= \begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & J_{(n-1)^2,2(n-1)}\mathbf{C} - \mathbf{C}^T\mathbf{C} \end{bmatrix} \\
&= \begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & 2J_{(n-1)^2} - 2I - \mathbf{A_2} \end{bmatrix} \\
&= \begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & 2(J_{(n-1)^2} - I - \mathbf{A_2}) + \mathbf{A_2} \end{bmatrix} \\
&= 2\mathcal{W}_6 + \mathcal{W}_4 \in \mathcal{W}(\Gamma_1)
\end{aligned}$$

- Evaluating $\mathcal{W}_{10}\mathcal{W}_8$

$$\begin{aligned}\mathcal{W}_{10}\mathcal{W}_8 &= \begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ J-\mathbf{C}^T & O_{(n-1)^2} \end{bmatrix}\begin{bmatrix} O_{2(n-1)} & J-\mathbf{C} \\ O_{(n-1)^2,2(n-1)} & O_{(n-1)^2} \end{bmatrix} \\ &= \begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & J_{(n-1)^2,2(n-1)}J_{2(n-1),(n-1)^2} - J_{(n-1)^2,2(n-1)}\mathbf{C} - \mathbf{C}^T J_{2(n-1),(n-1)^2} + \mathbf{C}^T\mathbf{C} \end{bmatrix} \\ &= \begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & 2(n-1)J - 4J + 2I + \mathbf{A_2} \end{bmatrix} \\ &= \begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ O_{(n-1)^2,2(n-1)} & (2n-6)(J-I-\mathbf{A_2}) + (2n-4)I + (2n-5)\mathbf{A_2} \end{bmatrix} \\ &= (2n-6)\mathcal{W}_6 + (2n-5)\mathcal{W}_4 + (2n-4)\mathcal{W}_2 \in \mathcal{W}(\Gamma_1)\end{aligned}$$

- Evaluating $\mathcal{W}_{10}\mathcal{W}_9$

$$\begin{aligned}\mathcal{W}_{10}\mathcal{W}_9 &= \begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ J-\mathbf{C^T} & O_{(n-1)^2} \end{bmatrix}\begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ \mathbf{C}^T & O_{(n-1)^2} \end{bmatrix} \\ &= O_{n^2-1} \in \mathcal{W}(\Gamma_1)\end{aligned}$$

- Evaluating $\mathcal{W}_{10}\mathcal{W}_{10}$

$$\begin{aligned}\mathcal{W}_{10}\mathcal{W}_{10} &= \begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ J-\mathbf{C^T} & O_{(n-1)^2} \end{bmatrix}\begin{bmatrix} O_{2(n-1)} & O_{2(n-1),(n-1)^2} \\ J-\mathbf{C^T} & O_{(n-1)^2} \end{bmatrix} \\ &= O_{n^2-1} \in \mathcal{W}(\Gamma_1)\end{aligned}$$

Wielandt Principle for Secion 3.1 The working will be in the final submission.

Matrix Multiplication for Section 5.1

- For any matrix multiplication with $\mathcal{W}_1$, $\mathcal{W}_i\mathcal{W}_1 = \mathcal{W}_1\mathcal{W}_i = \mathcal{W}_i$ since $\mathcal{W}_1 = I_{(2k)^2}$. So we deal with matrix multiplications from $i, j \in [6] \setminus \{1\}$
- Evaluating $\mathcal{W}_2\mathcal{W}_2$

$$\begin{aligned}\mathcal{W}_2\mathcal{W}_2 &= \begin{bmatrix} (J_k-I)\otimes(J_{2k}-I) & O_{2k^2} \\ O_{2k^2} & (J_k-I)\otimes(J_{2k}-I) \end{bmatrix}\begin{bmatrix} (J_k-I)\otimes(J_{2k}-I) & O_{2k^2} \\ O_{2k^2} & (J_k-I)\otimes(J_{2k}-I) \end{bmatrix} \\ &= I_2 \otimes [(J_k-I)\otimes(J_{2k}-I)][(J_k-I)\otimes(J_{2k}-I)] \\ &= I_2 \otimes [(J_k-I)(J_k-I)\otimes(J_{2k}-I)(J_{2k}-I)] \\ &= I_2 \otimes [(k-2)(J_k-I)+(k-1)I_k] \otimes [(2k-2)(J_{2k}-I)+(2k-1)I_{2k}] \\ &= I_2 \otimes [(k-2)(2k-2)(J_k-I)\otimes(J_{2k}-I) + (k-1)(2k-1)(J_k-I)\otimes(I_{2k}) \\ &\qquad + (k-1)(2k-2)I_k\otimes(J_{2k}-I) + (k-1)(2k-1)I_k\otimes I_{2k}] \\ &= (k-2)(2k-2)\mathcal{W}_2 + (k-1)(2k-1)\mathcal{W}_5 + (k-1)(2k-2)\mathcal{W}_4 + (k-1)(2k-1)\mathcal{W}_1 \in \mathcal{W}(\Gamma_4)\end{aligned}$$

- Evaluating $\mathcal{W}_2\mathcal{W}_3$

$$\begin{aligned}
\mathcal{W}_2\mathcal{W}_3 &= \begin{bmatrix} (J_k - I)\otimes(J_{2k}-I) & O_{2k^2} \\ O_{2k^2} & (J_k - I)\otimes(J_{2k}-I) \end{bmatrix} \begin{bmatrix} O_{2k^2} & J_k \otimes I_{2k} \\ J_k \otimes I_{2k} & O_{2k^2} \end{bmatrix} \\
&= (J_2 - I) \otimes [(J_k - I)\otimes (J_{2k}-I)][J_k \otimes I_{2k}] \\
&= (J_2 - I) \otimes [(J_k - I)J_k \otimes (J_{2k}-I)I_{2k}] \\
&= (J_2 - I) \otimes [(k-1)J_k \otimes (J_{2k}-I)] \\
&= (k-1)\mathcal{W}_6 \in \mathcal{W}(\Gamma_4
\end{aligned}$$

- Evaluating $\mathcal{W}_2\mathcal{W}_4$

$$\begin{aligned}
\mathcal{W}_2\mathcal{W}_4 &= \begin{bmatrix} (J_k - I)\otimes(J_{2k}-I) & O_{2k^2} \\ O_{2k^2} & (J_k - I)\otimes(J_{2k}-I) \end{bmatrix} \begin{bmatrix} I_k \otimes (J_{2k}-I) & O_{2k^2} \\ O_{2k^2} & I_k \otimes (J_{2k}-I) \end{bmatrix} \\
&= I_2 \otimes [(J_k - I)\otimes (J_{2k}-I)][I_k \otimes (J_{2k}-I)] \\
&= I_2 \otimes (J_k - I)\otimes (J_{2k}-I)(J_{2k}-I) \\
&= I_2 \otimes (J_k - I)\otimes [(2k-2)(J_{2k}-I) + (2k-1)I_{2k}] \\
&= (2k-2)\begin{bmatrix} (J_k - I)\otimes(J_{2k}-I) & O_{2k^2} \\ O_{2k^2} & (J_k - I)\otimes(J_{2k}-I) \end{bmatrix} \\
&\quad + (2k-1)\begin{bmatrix} (J_k - I)\otimes I_{2k} & O_{2k^2} \\ O_{2k^2} & (J_k - I)\otimes I_{2k} \end{bmatrix} \\
&= (2k-2)\mathcal{W}_2 + (2k-1)\mathcal{W}_5 \in \mathcal{W}(\Gamma_4)
\end{aligned}$$

- Evaluating $\mathcal{W}_2\mathcal{W}_5$

$$\begin{aligned}
\mathcal{W}_2\mathcal{W}_5 &= \begin{bmatrix} (J_k - I)\otimes(J_{2k}-I) & O_{2k^2} \\ O_{2k^2} & (J_k - I)\otimes(J_{2k}-I) \end{bmatrix} \begin{bmatrix} (J_k - I) \otimes I_{2k} & O_{2k^2} \\ O_{2k^2} & (J_k - I) \otimes I_{2k} \end{bmatrix} \\
&= I_2 \otimes [(J_k - I)\otimes (J_{2k}-I)][(J_k - I) \otimes I_{2k}] \\
&= I_2 \otimes (J_k - I)^2 \otimes (J_{2k}-I) \\
&= I_2 \otimes [(k-2)(J_k - I) + (k-1)I_k] \otimes (J_{2k}-I) \\
&= (k-2)\begin{bmatrix} (J_k - I)\otimes(J_{2k}-I) & O_{2k^2} \\ O_{2k^2} & (J_k - I)\otimes(J_{2k}-I) \end{bmatrix} \\
&\quad + (k-1)\begin{bmatrix} I_k \otimes (J_{2k}-I) & O_{2k^2} \\ O_{2k^2} & I_k \otimes (J_{2k}-I) \end{bmatrix} \\
&= (k-2)\mathcal{W}_2 + (k-1)\mathcal{W}_4 \in \mathcal{W}(\Gamma_4)
\end{aligned}$$

- Evaluating $\mathcal{W}_2\mathcal{W}_6$

$$\begin{aligned}
\mathcal{W}_2\mathcal{W}_6 &= \begin{bmatrix} (J_k - I)\otimes(J_{2k}-I) & O_{2k^2} \\ O_{2k^2} & (J_k - I)\otimes(J_{2k}-I) \end{bmatrix} \begin{bmatrix} O_{2k^2} & J_k\otimes(J_{2k}-I) \\ J_k\otimes(J_{2k}-I) & O_{2k^2} \end{bmatrix} \\
&= (J_2 - I)\otimes[(J_k - I)\otimes(J_{2k}-I)][J_k\otimes(J_{2k}-I)] \\
&= (J_2 - I)\otimes[(J_k - I)J_k\otimes(J_{2k}-I)^2] \\
&= (J_2 - I)\otimes[(k-1)J_k\otimes((2k-2)(J_{2k}-I)+(2k-1)I_{2k})] \\
&= (k-1)(2k-2)\mathcal{W}_6 + (k-1)(2k-1)\mathcal{W}_3 \in \mathcal{W}(\Gamma_4)
\end{aligned}$$

- Evaluating $\mathcal{W}_3\mathcal{W}_2$

$$\begin{aligned}
\mathcal{W}_3\mathcal{W}_2 &= \begin{bmatrix} O_{2k^2} & J_k\otimes I_{2k} \\ J_k\otimes I_{2k} & O_{2k^2} \end{bmatrix} \begin{bmatrix} (J_k - I)\otimes(J_{2k}-I) & O_{2k^2} \\ O_{2k^2} & (J_k - I)\otimes(J_{2k}-I) \end{bmatrix} \\
&= (J_2 - I)\otimes[J_k\otimes I_{2k}][(J_k - I)\otimes(J_{2k}-I)] \\
&= (J_2 - I)\otimes[J_k(J_k - I)\otimes(J_{2k}-I)] \\
&= (J_2 - I)\otimes[(k-1)J_k\otimes(J_{2k}-I)] \\
&= (k-1)\mathcal{W}_6(\Gamma_4)
\end{aligned}$$

- Evaluating $\mathcal{W}_3\mathcal{W}_3$

$$\begin{aligned}
\mathcal{W}_3\mathcal{W}_3 &= \begin{bmatrix} O_{2k^2} & J_k\otimes I_{2k} \\ J_k\otimes I_{2k} & O_{2k^2} \end{bmatrix} \begin{bmatrix} O_{2k^2} & J_k\otimes I_{2k} \\ J_k\otimes I_{2k} & O_{2k^2} \end{bmatrix} \\
&= I_2\otimes[J_k\otimes I_{2k}][J_k\otimes I_{2k}] \\
&= I_2\otimes[J_k^2\otimes I_{2k}] \\
&= I_2\otimes[k(J_k - I)+kI_k]\otimes I_{2k} \\
&= k\mathcal{W}_5 + k\mathcal{W}_1 \in \mathcal{W}(\Gamma_4)
\end{aligned}$$

- Evaluating $\mathcal{W}_3\mathcal{W}_4$

$$\begin{aligned}
\mathcal{W}_3\mathcal{W}_4 &= \begin{bmatrix} O_{2k^2} & J_k\otimes I_{2k} \\ J_k\otimes I_{2k} & O_{2k^2} \end{bmatrix} \begin{bmatrix} I_k\otimes(J_{2k}-I) & O_{2k^2} \\ O_{2k^2} & I_k\otimes(J_{2k}-I) \end{bmatrix} \\
&= (J_2 - I)\otimes[J_k\otimes I_{2k}][I_k\otimes(J_{2k}-I)] \\
&= (J_2 - I)\otimes[J_k\otimes(J_{2k}-I)] \\
&= \mathcal{W}_6 \in \mathcal{W}(\Gamma_4)
\end{aligned}$$

- Evaluating $\mathcal{W}_3\mathcal{W}_5$

$$
\begin{aligned}
\mathcal{W}_3\mathcal{W}_5 &= \begin{bmatrix} O_{2k^2} & J_k \otimes I_{2k} \\ J_k \otimes I_{2k} & O_{2k^2} \end{bmatrix} \begin{bmatrix} (J_k - I) \otimes I_{2k} & O_{2k^2} \\ O_{2k^2} & (J_k - I) \otimes I_{2k} \end{bmatrix} \\
&= (J_2 - I) \otimes [J_k \otimes I_{2k}][(J_k - I) \otimes I_{2k}] \\
&= (J_2 - I) \otimes [J_k(J_k - I) \otimes I_{2k}] \\
&= (J_2 - I) \otimes [(k-1)J_k \otimes I_{2k}] \\
&= (k-1)\mathcal{W}_3 \in \mathcal{W}(\Gamma_4)
\end{aligned}
$$

- Evaluating $\mathcal{W}_3\mathcal{W}_6$

$$
\begin{aligned}
\mathcal{W}_3\mathcal{W}_6 &= \begin{bmatrix} O_{2k^2} & J_k \otimes I_{2k} \\ J_k \otimes I_{2k} & O_{2k^2} \end{bmatrix} \begin{bmatrix} O_{2k^2} & J_k \otimes (J_{2k} - I) \\ J_k \otimes (J_{2k} - I) & O_{2k^2} \end{bmatrix} \\
&= I_2 \otimes [J_k \otimes I_{2k}][J_k \otimes (J_{2k} - I)] \\
&= I_2 \otimes [J_k^2 \otimes (J_{2k} - I)] \\
&= I_2 \otimes [kJ_k \otimes (J_{2k} - I)] \\
&= k\mathcal{W}_2 + k\mathcal{W}_4 \in \mathcal{W}(\Gamma_4)
\end{aligned}
$$

- Evaluating $\mathcal{W}_4\mathcal{W}_2$

$$
\begin{aligned}
\mathcal{W}_4\mathcal{W}_2 &= \begin{bmatrix} I_k \otimes (J_{2k} - I) & O_{2k^2} \\ O_{2k^2} & I_k \otimes (J_{2k} - I) \end{bmatrix} \begin{bmatrix} (J_k - I) \otimes (J_{2k} - I) & O_{2k^2} \\ O_{2k^2} & (J_k - I) \otimes (J_{2k} - I) \end{bmatrix} \\
&= I_2 \otimes [I_k \otimes (J_{2k} - I)][(J_k - I) \otimes (J_{2k} - I)] \\
&= I_2 \otimes (J_k - I) \otimes (J_{2k} - I)^2 \\
&= I_2 \otimes (J_k - I) \otimes [(2k-2)(J_{2k} - I) + (2k-1)I_{2k}] \\
&= (2k-2)\mathcal{W}_2 + (2k-1)\mathcal{W}_5 \in \mathcal{W}(\Gamma_4)
\end{aligned}
$$

- Evaluating $\mathcal{W}_4\mathcal{W}_3$

$$
\begin{aligned}
\mathcal{W}_4\mathcal{W}_3 &= \begin{bmatrix} I_k \otimes (J_{2k} - I) & O_{2k^2} \\ O_{2k^2} & I_k \otimes (J_{2k} - I) \end{bmatrix} \begin{bmatrix} O_{2k^2} & J_k \otimes I_{2k} \\ J_k \otimes I_{2k} & O_{2k^2} \end{bmatrix} \\
&= (J_2 - I) \otimes [I_k \otimes (J_{2k} - I)][J_k \otimes I_{2k}] \\
&= (J_2 - I) \otimes [J_k \otimes (J_{2k} - I)] \\
&= \mathcal{W}_6
\end{aligned}
$$

- Evaluating $\mathcal{W}_4\mathcal{W}_4$

$$
\begin{aligned}
\mathcal{W}_4\mathcal{W}_4 &= \begin{bmatrix} I_k \otimes (J_{2k}-I) & O_{2k^2} \\ O_{2k^2} & I_k \otimes (J_{2k}-I) \end{bmatrix} \begin{bmatrix} I_k \otimes (J_{2k}-I) & O_{2k^2} \\ O_{2k^2} & I_k \otimes (J_{2k}-I) \end{bmatrix} \\
&= I_2 \otimes [I_k \otimes (J_{2k}-I)][I_k \otimes (J_{2k}-I)] \\
&= I_2 \otimes [I_k \otimes (J_{2k}-I)^2] \\
&= I_2 \otimes I_k \otimes ((2k-2)(J_{2k}-I) + (2k-1)I_{2k}) \\
&= (2k-2)\mathcal{W}_4 + (2k-1)\mathcal{W}_1 \in \mathcal{W}(\Gamma_4)
\end{aligned}
$$

- Evaluating $\mathcal{W}_4\mathcal{W}_5$

$$
\begin{aligned}
\mathcal{W}_4\mathcal{W}_5 &= \begin{bmatrix} I_k \otimes (J_{2k}-I) & O_{2k^2} \\ O_{2k^2} & I_k \otimes (J_{2k}-I) \end{bmatrix} \begin{bmatrix} (J_k-I) \otimes I_{2k} & O_{2k^2} \\ O_{2k^2} & (J_k-I) \otimes I_{2k} \end{bmatrix} \\
&= I_2 \otimes [I_k \otimes (J_{2k}-I)][(J_k-I) \otimes I_{2k}] \\
&= I_2 \otimes [(J_k-I) \otimes (J_{2k}-I)] \\
&= \mathcal{W}_2 \in \mathcal{W}(\Gamma_4)
\end{aligned}
$$

- Evaluating $\mathcal{W}_4\mathcal{W}_6$

$$
\begin{aligned}
\mathcal{W}_4\mathcal{W}_6 &= \begin{bmatrix} I_k \otimes (J_{2k}-I) & O_{2k^2} \\ O_{2k^2} & I_k \otimes (J_{2k}-I) \end{bmatrix} \begin{bmatrix} O_{2k^2} & J_k \otimes (J_{2k}-I) \\ J_k \otimes (J_{2k}-I) & O_{2k^2} \end{bmatrix} \\
&= (J_2-I) \otimes [I_k \otimes (J_{2k}-I)][J_k \otimes (J_{2k}-I)] \\
&= (J_2-I) \otimes [J_k \otimes (J_{2k}-I)^2] \\
&= (J_2-I) \otimes J_k \otimes [(2k-2)(J_{2k}-I) + (2k-1)I_{2k}] \\
&= (2k-2)\mathcal{W}_6 + (2k-1)\mathcal{W}_3 \in \mathcal{W}(\Gamma_4)
\end{aligned}
$$

- Evaluating $\mathcal{W}_5\mathcal{W}_2$

$$
\begin{aligned}
\mathcal{W}_5\mathcal{W}_2 &= \begin{bmatrix} (J_k-I) \otimes I_{2k} & O_{2k^2} \\ O_{2k^2} & (J_k-I) \otimes I_{2k} \end{bmatrix} \begin{bmatrix} (J_k-I) \otimes (J_{2k}-I) & O_{2k^2} \\ O_{2k^2} & (J_k-I) \otimes (J_{2k}-I) \end{bmatrix} \\
&= I_2 \otimes [(J_k-I) \otimes I_{2k}][(J_k-I) \otimes (J_{2k}-I)] \\
&= I_2 \otimes [(k-2)(J_k-I) + (k-1)I_k] \otimes (J_{2k}-I) \\
&= (k-2)\mathcal{W}_2 + (k-1)\mathcal{W}_4 \in \mathcal{W}(\Gamma_4)
\end{aligned}
$$

- Evaluating $\mathcal{W}_5\mathcal{W}_3$

$$\begin{aligned}
\mathcal{W}_5\mathcal{W}_3 &= \begin{bmatrix} (J_k - I)\otimes I_{2k} & O_{2k^2} \\ O_{2k^2} & (J_k - I)\otimes I_{2k} \end{bmatrix} \begin{bmatrix} O_{2k^2} & J_k \otimes I_{2k} \\ J_k \otimes I_{2k} & O_{2k^2} \end{bmatrix} \\
&= (J_2 - I) \otimes [(J_k - I)\otimes I_{2k}][J_k \otimes I_{2k}] \\
&= (J_2 - I) \otimes (J_k - I)J_k \otimes I_{2k} \\
&= (J_2 - I) \otimes (k-1)J_k \otimes I_{2k} \\
&= (k-1)\mathcal{W}_3 \in \mathcal{W}(\Gamma_4)
\end{aligned}$$

- Evaluating $\mathcal{W}_5\mathcal{W}_4$

$$\begin{aligned}
\mathcal{W}_5\mathcal{W}_4 &= \begin{bmatrix} (J_k - I)\otimes I_{2k} & O_{2k^2} \\ O_{2k^2} & (J_k - I)\otimes I_{2k} \end{bmatrix} \begin{bmatrix} I_k \otimes (J_{2k} - I) & O_{2k^2} \\ O_{2k^2} & I_k \otimes (J_{2k} - I) \end{bmatrix} \\
&= I_2 \otimes [(J_k - I)\otimes I_{2k}][I_k \otimes (J_{2k} - I)] \\
&= I_2 \otimes [(J_k - I)\otimes (J_{2k} - I)] \\
&= \mathcal{W}_2 \in \mathcal{W}(\Gamma_4)
\end{aligned}$$

- Evaluating $\mathcal{W}_5\mathcal{W}_5$

$$\begin{aligned}
\mathcal{W}_5\mathcal{W}_5 &= \begin{bmatrix} (J_k - I)\otimes I_{2k} & O_{2k^2} \\ O_{2k^2} & (J_k - I)\otimes I_{2k} \end{bmatrix} \begin{bmatrix} (J_k - I)\otimes I_{2k} & O_{2k^2} \\ O_{2k^2} & (J_k - I)\otimes I_{2k} \end{bmatrix} \\
&= I_2 \otimes [(J_k - I)^2 \otimes I_{2k}] \\
&= I_2 \otimes [(k-2)(J_k - I) + (k-1)I_k] \otimes I_{2k} \\
&= (k-2)\mathcal{W}_5 + (k-1)\mathcal{W}_1 \in \mathcal{W}(\Gamma_4)
\end{aligned}$$

- Evaluating $\mathcal{W}_5\mathcal{W}_6$

$$\begin{aligned}
\mathcal{W}_5\mathcal{W}_6 &= \begin{bmatrix} (J_k - I)\otimes I_{2k} & O_{2k^2} \\ O_{2k^2} & (J_k - I)\otimes I_{2k} \end{bmatrix} \begin{bmatrix} O_{2k^2} & J_k \otimes (J_{2k} - I) \\ J_k \otimes (J_{2k} - I) & O_{2k^2} \end{bmatrix} \\
&= (J_2 - I) \otimes [(J_k - I)\otimes I_{2k}][J_k \otimes (J_{2k} - I)] \\
&= (J_2 - I) \otimes (J_k - I)J_k \otimes (J_{2k} - I) \\
&= (J_2 - I) \otimes (k-1)J_k \otimes (J_{2k} - I) \\
&= (k-1)\mathcal{W}_6 \in \mathcal{W}(\Gamma_4)
\end{aligned}$$

- Evaluating $\mathcal{W}_6\mathcal{W}_2$

$$\begin{aligned}
\mathcal{W}_6\mathcal{W}_2 &= \begin{bmatrix} O_{2k^2} & J_k \otimes (J_{2k}-I) \\ J_k \otimes (J_{2k}-I) & O_{2k^2} \end{bmatrix} \begin{bmatrix} (J_k - I) \otimes (J_{2k}-I) & O_{2k^2} \\ O_{2k^2} & (J_k - I) \otimes (J_{2k}-I) \end{bmatrix} \\
&= (J_2 - I) \otimes [J_k \otimes (J_{2k}-I)][(J_k - I) \otimes (J_{2k}-I)] \\
&= (J_2 - I) \otimes [J_k(J_k - I) \otimes (J_{2k}-I)^2] \\
&= (J_2 - I) \otimes (k-1)J_k \otimes [(2k-2)(J_{2k}-I) + (2k-1)I_{2k}] \\
&= (k-1)(2k-2)\mathcal{W}_6 + (k-1)(2k-1)\mathcal{W}_3 \in \mathcal{W}(\Gamma_4)
\end{aligned}$$

- Evaluating $\mathcal{W}_6\mathcal{W}_3$

$$\begin{aligned}
\mathcal{W}_6\mathcal{W}_3 &= \begin{bmatrix} O_{2k^2} & J_k \otimes (J_{2k}-I) \\ J_k \otimes (J_{2k}-I) & O_{2k^2} \end{bmatrix} \begin{bmatrix} O_{2k^2} & J_k \otimes I_{2k} \\ J_k \otimes I_{2k} & O_{2k^2} \end{bmatrix} \\
&= I_2 \otimes [J_k \otimes (J_{2k}-I)][J_k \otimes I_{2k}] \\
&= I_2 \otimes [J_k^2 \otimes (J_{2k}-I)] \\
&= I_2 \otimes (k(J_k - I) + kI_k) \otimes (J_{2k}-I) \\
&= k\mathcal{W}_2 + k\mathcal{W}_4 \in \mathcal{W}(\Gamma_4)
\end{aligned}$$

- Evaluating $\mathcal{W}_6\mathcal{W}_4$

$$\begin{aligned}
\mathcal{W}_6\mathcal{W}_4 &= \begin{bmatrix} O_{2k^2} & J_k \otimes (J_{2k}-I) \\ J_k \otimes (J_{2k}-I) & O_{2k^2} \end{bmatrix} \begin{bmatrix} I_k \otimes (J_{2k}-I) & O_{2k^2} \\ O_{2k^2} & I_k \otimes (J_{2k}-I) \end{bmatrix} \\
&= (J_2 - I) \otimes [J_k \otimes (J_{2k}-I)][I_k \otimes (J_{2k}-I)] \\
&= (J_2 - I) \otimes [J_k \otimes (J_{2k}-I)^2] \\
&= (J_2 - I) \otimes J_k \otimes [(2k-2)(J_{2k}-I) + (2k-1)I_{2k}] \\
&= (2k-2)\mathcal{W}_6 + (2k-1)\mathcal{W}_3 \in \mathcal{W}(\Gamma_4)
\end{aligned}$$

- Evaluating $\mathcal{W}_6\mathcal{W}_5$

$$\begin{aligned}
\mathcal{W}_6\mathcal{W}_2 &= \begin{bmatrix} O_{2k^2} & J_k \otimes (J_{2k}-I) \\ J_k \otimes (J_{2k}-I) & O_{2k^2} \end{bmatrix} \begin{bmatrix} (J_k - I) \otimes I_{2k} & O_{2k^2} \\ O_{2k^2} & (J_k - I) \otimes I_{2k} \end{bmatrix} \\
&= (J_2 - I) \otimes [J_k \otimes (J_{2k}-I)][(J_k - I) \otimes I_{2k}] \\
&= (J_2 - I) \otimes [J_k(J_k - I) \otimes (J_{2k}-I)] \\
&= (J_2 - I) \otimes (k-1)J_k \otimes (J_{2k}-I) \\
&= (k-1)\mathcal{W}_6 \in \mathcal{W}(\Gamma_4)
\end{aligned}$$

- Evaluating $\mathcal{W}_6\mathcal{W}_6$

$$
\begin{aligned}
\mathcal{W}_6\mathcal{W}_6 &= \begin{bmatrix} O_{2k^2} & J_k \otimes (J_{2k}-I) \\ J_k \otimes (J_{2k}-I) & O_{2k^2} \end{bmatrix} \begin{bmatrix} O_{2k^2} & J_k \otimes (J_{2k}-I) \\ J_k \otimes (J_{2k}-I) & O_{2k^2} \end{bmatrix} \\
&= I_2 \otimes [J_k \otimes (J_{2k}-I)][J_k \otimes (J_{2k}-I)] \\
&= I_2 \otimes [J_k^2 \otimes (J_{2k}-I)^2] \\
&= I_2 \otimes (k(J_k - I) + kI_k) \otimes [(2k-2)(J_{2k}-I) + (2k-1)I_{2k}] \\
&= k(2k-2)\mathcal{W}_2 + k(2k-1)\mathcal{W}_5 + k(2k-2)\mathcal{W}_4 + k(2k-1)\mathcal{W}_1 \in \mathcal{W}(\Gamma_4)
\end{aligned}
$$

Wielandt Principle for Secion 5.1

$$
\begin{aligned}
A[\Gamma_4]^2 &= \begin{bmatrix} I_k \otimes (J_{2k}-I) + (J_k - I) \otimes I_{2k} & J_k \otimes (J_{2k}-I) \\ J_k \otimes (J_{2k}-I) & I_k \otimes (J_{2k}-I) + (J_k - I) \otimes I_{2k} \end{bmatrix}^2 \\
&= \begin{bmatrix} M_1 & M_2 \\ M_2 & M_1 \end{bmatrix}^2 \\
&= \begin{bmatrix} M_1^2 + M_2^2 & M_1M_2 + M_2M_1 \\ M_1M_2 + M_2M_1 & M_1^2 + M_2^2 \end{bmatrix}
\end{aligned}
$$

We isolate the terms and solve it before substituting back into the matrix:

$$
\begin{aligned}
M_1^2 + M_2^2 &= (I_k \otimes (J_{2k}-I) + (J_k - I) \otimes I_{2k})(I_k \otimes (J_{2k}-I) + (J_k - I) \otimes I_{2k}) \\
&\quad + (J_k \otimes (J_{2k}-I))(J_k \otimes (J_{2k}-I)) \\
&= (I_k \otimes (J_{2k}-I))(I_k \otimes (J_{2k}-I)) + (I_k \otimes (J_{2k}-I))(J_k - I) \otimes I_{2k} \\
&\quad + ((J_k - I) \otimes I_{2k})(I_k \otimes (J_{2k}-I)) + ((J_k - I) \otimes I_{2k})((J_k - I) \otimes I_{2k}) \\
&\quad + (J_k^2 \otimes (J_{2k}-I)^2) \\
&= (I_k \otimes (J_{2k}-I)^2) + ((J_k - I) \otimes (J_{2k}-I)) \\
&\quad + ((J_k - I) \otimes (J_{2k}-I)) + ((J_k - I)^2 \otimes I_{2k}) \\
&\quad + (k(J_k - I) + k(I_k)) \otimes ((2k-2)(J_{2k}-I) + (2k-1)(I_{2k})) \\
&= (I_k \otimes ((2k-2)(J_{2k}-I) + (2k-1)I_{2k})) \\
&\quad + 2((J_k - I) \otimes (J_{2k}-I)) + ((k-2)(J_k - I) + (k-1)I_k) \otimes I_{2k} \\
&\quad + k(2k-2)(J_k - I) \otimes (J_{2k}-I) + k(2k-2)I_k \otimes (J_{2k}-I) \\
&\quad + k(2k-1)(J_k - I) \otimes I_{2k} + k(2k-1)I_k \otimes I_{2k} \\
&= (2k^2 + 2k - 2)I_k \otimes I_{2k} \\
&\quad + (2k^2 - 2)(I_k \otimes (J_{2k}-I) + (J_k - I) \otimes I_{2k}) \\
&\quad + (2k^2 - 2k + 2)(J_k - I) \otimes (J_{2k}-I)
\end{aligned}
$$

$$
\begin{aligned}
M_1M_2+M_2M_1 &= (I_k\otimes(J_{2k}-I)+(J_k-I)\otimes I_{2k})(J_k\otimes(J_{2k}-I))\\
&\qquad +(J_k\otimes(J_{2k}-I))(I_k\otimes(J_{2k}-I)+(J_k-I)\otimes I_{2k})\\
&= J_k\otimes(J_{2k}-I)^2+(J_k-I)J_k\otimes(J_{2k}-I)\\
&\qquad + J_k\otimes(J_{2k}-I)^2+J_k(J_k-I)\otimes(J_{2k}-I)\\
&= 2(J_k\otimes(J_{2k}-I)^2+(J_k-I)J_k\otimes(J_{2k}-I))\\
&= 2(J_k\otimes[(2k-2)(J_{2k}-I)+(2k-1)I_{2k}]+(k-1)J_k\otimes(J_{2k}-I))\\
&= 2((2k-2+k-1)J_k\otimes(J_{2k}-I)+(2k-1)J_k\otimes I_{2k})\\
&= (6k-6)J_k\otimes(J_{2k}-I)+(4k-2)J_k\otimes I_{2k}
\end{aligned}
$$

So we have

$$
\begin{aligned}
A[\Gamma_4]^2 &= \begin{bmatrix} M_1^2+M_2^2 & M_1M_2+M_2M_1\\ M_1M_2+M_2M_1 & M_1^2+M_2^2\end{bmatrix}\\
&= \begin{bmatrix}
\begin{array}{c}(2k^2+2k-2)I_k\otimes I_{2k}\\ +(2k^2-2)(I_k\otimes(J_{2k}-I)+(J_k-I)\otimes I_{2k})\\ +(2k^2-2k+2)(J_k-I)\otimes(J_{2k}-I)\end{array} & \begin{array}{c}(6k-6)J_k\otimes(J_{2k}-I)\\ +(4k-2)J_k\otimes I_{2k}\end{array}\\[2em]
\begin{array}{c}(6k-6)J_k\otimes(J_{2k}-I)\\ +(4k-2)J_k\otimes I_{2k}\end{array} & \begin{array}{c}(2k^2+2k-2)I_k\otimes I_{2k}\\ +(2k^2-2)(I_k\otimes(J_{2k}-I)+(J_k-I)\otimes I_{2k})\\ +(2k^2-2k+2)(J_k-I)\otimes(J_{2k}-I)\end{array}
\end{bmatrix}\\
&= (2k^2+2k-2)\begin{bmatrix} I_k\otimes I_{2k} & O\\ O & I_k\otimes I_{2k}\end{bmatrix}\\
&\quad + (2k^2-2k+2)\begin{bmatrix}(J_k-I)\otimes(J_{2k}-I) & O\\ O & (J_k-I)\otimes(J_{2k}-I)\end{bmatrix}\\
&\quad (2k^2-2)\begin{bmatrix}(I_k\otimes(J_{2k}-I)+(J_k-I)\otimes I_{2k}) & O\\ O & (I_k\otimes(J_{2k}-I)+(J_k-I)\otimes I_{2k})\end{bmatrix}\\
&\quad + (6k-6)\begin{bmatrix} O & J_k\otimes(J_{2k}-I)\\ J_k\otimes(J_{2k}-I) & O\end{bmatrix} + (4k-2)\begin{bmatrix} O & J_k\otimes I_{2k}\\ J_k\otimes I_{2k} & O\end{bmatrix}
\end{aligned}
$$

$$\begin{bmatrix} (I_k \otimes (J_{2k} - I) + (J_k - I) \otimes I_{2k}) & O \\ O & (I_k \otimes (J_{2k} - I) + (J_k - I) \otimes I_{2k}) \end{bmatrix}^2 \in \mathcal{W}(\Gamma_4)$$

$$\begin{aligned}
\Rightarrow &\begin{bmatrix} (I_k \otimes (J_{2k} - I) + (J_k - I) \otimes I_{2k})^2 & O \\ O & (I_k \otimes (J_{2k} - I) + (J_k - I) \otimes I_{2k})^2 \end{bmatrix} \\
= &\begin{bmatrix} \begin{matrix} I_k \otimes (J_{2k} - I)^2 + (J_k - I)^2 \otimes I_{2k} \\ +2(J_k - I) \otimes (J_{2k} - I) \end{matrix} & O \\ O & \begin{matrix} I_k \otimes (J_{2k} - I)^2 + (J_k - I)^2 \otimes I_{2k} \\ +2(J_k - I) \otimes (J_{2k} - I) \end{matrix} \end{bmatrix} \\
= &\begin{bmatrix} \begin{matrix} I_k \otimes ((2k-2)(J_{2k} - I) + (2k-1)I_{2k}) \\ +((k-2)(J_k - I) + (k-1)I_k) \otimes I_{2k} \\ +2(J_k - I) \otimes (J_{2k} - I) \end{matrix} & O \\ O & \begin{matrix} I_k \otimes ((2k-2)(J_{2k} - I) \\ +((k-2)(J_k - I) + (k-1)I_k) \otimes I_{2k} \\ +2(J_k - I) \otimes (J_{2k} - I) \end{matrix} \end{bmatrix} \\
= &\begin{bmatrix} \begin{matrix} (3k-2)I_k \otimes I_{2k} \\ +(2k-2)I_k \otimes (J_{2k} - I) \\ +(k-2)(J_k - I) \otimes I_{2k} \\ +2(J_k - I) \otimes (J_{2k} - I) \end{matrix} & O \\ O & \begin{matrix} (3k-2)I_k \otimes I_{2k} \\ +(2k-2)I_k \otimes (J_{2k} - I) \\ +(k-2)(J_k - I) \otimes I_{2k} \\ +2(J_k - I) \otimes (J_{2k} - I) \end{matrix} \end{bmatrix} \\
= &(3k-2) \begin{bmatrix} I_k \otimes I_{2k} & O \\ O & I_k \otimes I_{2k} \end{bmatrix} + (2k-2) \begin{bmatrix} I_k \otimes (J_{2k} - I) & O \\ O & I_k \otimes (J_{2k} - I) \end{bmatrix} \\
&+(k-2) \begin{bmatrix} (J_k - I) \otimes I_{2k} & O \\ O & (J_k - I) \otimes I_{2k} \end{bmatrix} + 2 \begin{bmatrix} (J_k - I) \otimes (J_{2k} - I) & O \\ O & (J_k - I) \otimes (J_{2k} - I) \end{bmatrix}
\end{aligned}$$

Matrix Multiplication for Section 5.2 We will be using the following proposition to simplify the commutative multiplications:

Proposition .1. *Given matrices A, B, C, D of compatible dimensions such that $AC = CA$ and $BD = DB$, it follows that $(A \otimes B)(C \otimes D) = (C \otimes D)(A \otimes B)$.*

Proof. Starting from that assumption that $AC = CA$ and $BD = DB$,

$$\begin{aligned}
(A \otimes B)(C \otimes D) &= AC \otimes BD \\
&= CA \otimes DB \\
&= (C \otimes D)(A \otimes B)
\end{aligned}$$

□

We will first consider the multiplications of the elements within their subsets, $\mathcal{W}(A_1), \mathcal{W}(A_2), \mathcal{W}(C), \mathcal{W}(C^T)$. To simplify the calculations, we will be using the respective block matrices to represent the actual matrices, as given by:

$$\begin{aligned}\mathcal{W}'(A_1) &= \langle I_{kn\times kn}, I_{k\times k} \otimes (J-I), (J-I)_{k\times k} \otimes I, (J-I)_{k\times k} \otimes (J-I)\rangle \\ &= \langle M'_{11}, M'_{12}, M'_{13}, M'_{14}\rangle\end{aligned}$$

$$\begin{aligned}\mathcal{W}'(A_2) &= \langle I_{n(n-k)\times n(n-k)}, I_{n-k\times n-k} \otimes (J-I), (J-I)_{n-k\times n-k} \otimes I, (J-I)_{n-k\times n-k} \otimes (J-I)\rangle, \\ &= \langle M'_{21}, M'_{22}, M'_{23}, M'_{24}\rangle\end{aligned}$$

$$\begin{aligned}\mathcal{W}'(C) &= \langle J_{k\times n-k} \otimes I, J_{k\times n-k} \otimes (J-I)\rangle \\ &= \langle M'_{31}, M'_{32}\rangle\end{aligned}$$

$$\begin{aligned}\mathcal{W}'(C^T) &= \langle J_{n-k\times k} \otimes I, J_{n-k\times k} \otimes (J-I)\rangle \\ &= \langle M'_{41}, M'_{42}\rangle\end{aligned}$$

- $\mathcal{W}(A_1)$

  Since the block matrices in $\mathcal{W}(A_1)$ are all in the same position, we can isolate the non-zero block of the matrices, i.e. $\begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} B & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} AB & 0 \\ 0 & 0 \end{bmatrix} \neq \mathbf{0}$, so we use the set $\mathcal{W}'(A_1)$.

  We know that $M'_{11}M'_{1i} = M'_{1i}M'_{11} = M'_{1i}$ since $M'_{11} = I$, so $M_{11}M_{1i} = M_{1i}M_{11} = M_{1i} \in \mathcal{W}(\Gamma)$.

  We now consider the multiplications between M'_{12}, M'_{13} and M'_{14}.

  - $M_{12}M_{13}$

$$\begin{aligned}M'_{12}M'_{13} &= (I \otimes (J-I))((J-I) \otimes I) \\ &= (I(J-I)) \otimes ((J-I)(I)) \\ &= (J-I) \otimes (J-I) \\ &= M'_{14}\end{aligned}$$

$$\Rightarrow M_{12}M_{13} = M_{14} \in \mathcal{W}(\Gamma)$$

  - $M_{13}M_{12}$

    Let $A = I, C = J-I, B = J-I, D = I$, we can see that $AC = CA$ and $BD = DB$.

Using Proposition .1,

$$\begin{aligned} M'_{13}M'_{12} &= M'_{12}M'_{13} \\ &= M'_{14} \end{aligned}$$

$$\Rightarrow M_{13}M_{12} = M_{14} \in \mathcal{W}(\Gamma)$$

– $M_{12}M_{14}$

$$\begin{aligned} M'_{12}M'_{14} &= (I \otimes (J-I))((J-I) \otimes (J-I)) \\ &= (I(J-I)) \otimes ((J-I)(J-I)) \\ &= (J-I) \otimes ((n-2)J + I) \\ &= (n-2)((J-I) \otimes J) + (J-I) \otimes I \\ &= (n-2)((J-I) \otimes (J-I)) + (n-2)((J-I) \otimes I) + (J-I) \otimes I \\ &= (n-2)M'_{14} + (n-1)M'_{13} \end{aligned}$$

$$\Rightarrow M_{12}M_{14} = (n-2)M_{14} + (n-1)M_{13} \in \mathcal{W}(\Gamma)$$

– $M_{14}M_{12}$

Let $A = I, C = J-I, B = J-I, D = J-I$, we can see that $AC = CA$ and $BD = DB$. Using Proposition .1,

$$\begin{aligned} M'_{14}M'_{12} &= M'_{12}M'_{14} \\ &= (n-2)M'_{14} + (n-1)M'_{13} \end{aligned}$$

$$\Rightarrow M_{14}M_{12} = (n-2)M_{14} + (n-1)M_{13} \in \mathcal{W}(\Gamma)$$

– $M_{13}M_{14}$

$$\begin{aligned}
M'_{13}M'_{14} &= ((J-I)\otimes I)((J-I)\otimes(J-I)) \\
&= ((J-I)(J-I))\otimes(I(J-I)) \\
&= ((n-2)J+I)\otimes(J-I) \\
&= (n-2)(J\otimes(J-I)) + I\otimes(J-I) \\
&= (n-2)((J-I)\otimes(J-I)) + (n-2)(I\otimes(J-I)) + I\otimes(J-I) \\
&= (n-2)M'_{14} + (n-1)M'_{12}
\end{aligned}$$

$$\Rightarrow M_{13}M_{14} = (n-2)M_{14} + (n-1)M_{12} \in \mathcal{W}(\Gamma)$$

Whence,

$$M_{13}M_{14} = (n-2)M_{14} + (n-1)M_{12} \in \mathcal{W}(\Gamma) \tag{1}$$

– $M_{14}M_{13}$

Let $A = J-I, C = J-I, B = I, D = J-I$, we can see that $AC = CA$ and $BD = DB$. Using Proposition .1,

$$\begin{aligned}
M'_{14}M'_{13} &= M'_{13}M'_{14} \\
&= (n-2)M'_{14} + (n-1)M'_{12}
\end{aligned}$$

$$\Rightarrow M_{14}M_{13} = (n-2)M_{14} + (n-1)M_{12} \in \mathcal{W}(\Gamma)$$

So we have shown that the matrices $M, N \in \mathcal{W}(A_1)$ satisfy the property $MN \in \mathcal{W}(A_1) \subset \mathcal{W}(\Gamma)$.

- $\mathcal{W}(A_2)$
  Since the block matrices in $\mathcal{W}(A_2)$ are all in the same position, we can isolate the non-zero block of the matrices, i.e. $\begin{bmatrix} 0 & 0 \\ 0 & A \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & B \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & AB \end{bmatrix} \neq \mathbf{0}$, so we use the set $\mathcal{W}'(A_2)$.

  Note that $\mathcal{W}'(A_2) = \mathcal{W}'(A_1)$ and by following the working above, we can derive that the matrices $M, N \in \mathcal{W}(A_2)$ satisfy the property that $MN \in \mathcal{W}(A_2) \subset \mathcal{W}(\Gamma)$

- $\mathcal{W}(C)$
  Since the block matrices in $\mathcal{W}(C)$ are all in the same position, we can isolate the non-zero block of the matrices, i.e. $\begin{bmatrix} 0 & A \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & B \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. This shows that no matter which matrices $M, N \in \mathcal{W}(C)$ we choose, $MN = \mathbf{0} \in \mathcal{W}(\Gamma)$.

- $\mathcal{W}(C^T)$

Similar to $\mathcal{W}(C)$, we show that $\begin{bmatrix} 0 & 0 \\ A & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ B & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, showing that no matter which matrices $M, N \in \mathcal{W}(C^T)$ we choose, $MN = \mathbf{0} \in \mathcal{W}(\Gamma)$.

We now consider multiplications between different subset partitions of $\mathcal{W}(\Gamma)$.

- $\mathcal{W}(A_1)$ and $\mathcal{W}(A_2)$

  - For $M \in \mathcal{W}(A_1), N \in \mathcal{W}(A_2)$, matrix multiplications would be of the form

    $$MN = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & B \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

    This shows that for any matrices $M \in \mathcal{W}(A_1), N \in \mathcal{W}(A_2)$, the product would be $\mathbf{0} \in \mathcal{W}(\Gamma)$.
  - For $M \in \mathcal{W}(A_2), N \in \mathcal{W}(A_1)$, matrix multiplications would be of the form

    $$MN = \begin{bmatrix} 0 & 0 \\ 0 & A \end{bmatrix}\begin{bmatrix} B & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

    This shows that for any matrices $M \in \mathcal{W}(A_2), N \in \mathcal{W}(A_1)$, the product would also be $\mathbf{0} \in \mathcal{W}(\Gamma)$.

  Thus, for any 2 matrices M, N from subsets $\mathcal{W}(A_1)$ and $\mathcal{W}(A_2)$, the product $MN \in \mathcal{W}(\Gamma)$.

- $\mathcal{W}(A_1)$ and $\mathcal{W}(C)$

  - For $M \in \mathcal{W}(A_1), N \in \mathcal{W}(C)$, matrix multiplications would be of the form

    $$MN = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & B \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & AB \\ 0 & 0 \end{bmatrix} \neq \mathbf{0}$$

    so we use the sets $\mathcal{W}'(A_1)$ and $\mathcal{W}'(C)$.
    We know that $M'_{11}M'_{3i} = M'_{3i}$ since $M'_{11} = I$, so $M_{11}M_{31} = M_{31}$ and $M_{11}M_{32} = M_{32}$. So both $M_{11}M_{31}, M_{11}M_{32} \in \mathcal{W}(C) \subset \mathcal{W}(\Gamma)$.
    - $M'_{12}M'_{31}$

$$\begin{aligned}M'_{12}M'_{31} &= (I \otimes (J-I)(J \otimes I)\\ &= IJ \otimes (J-I)I\\ &= J \otimes (J-I) = M'_{32}\end{aligned}$$

$$\Rightarrow M_{12}M_{31} = M_{32} \in \mathcal{W}(C) \subset \mathcal{W}(\Gamma)$$

$*$ $M'_{12}M'_{32}$

$$\begin{aligned}M'_{12}M'_{32} &= (I \otimes (J-I)(J \otimes (J-I))\\ &= IJ \otimes (J-I)(J-I)\\ &= J \otimes ((n-2)J+I)\\ &= (n-2)J \otimes J + J \otimes I\\ &= (n-2)J \otimes (J-I) + (n-2)J \otimes I + J \otimes I\\ &= (n-2)J \otimes (J-I) + (n-1)J \otimes I\\ &= (n-2)M'_{32} + (n-1)M'_{31}\end{aligned}$$

$$\Rightarrow M_{12}M_{32} = (n-2)M_{32} + (n-1)M_{31} \in \mathcal{W}(C) \subset \mathcal{W}(\Gamma)$$

$*$ $M_{13}M_{31}$

$$\begin{aligned}M'_{13}M'_{31} &= ((J-I) \otimes I)(J \otimes I)\\ &= (J-I)_{k\times k}J_{k\times n-k} \otimes II\\ &= (k-1)J \otimes I\\ &= (k-1)M'_{31}\end{aligned}$$

$$\Rightarrow M_{13}M_{31} = (k-1)M_{31} \in \mathcal{W}(C) \subset \mathcal{W}(\Gamma)$$

$*$ $M_{13}M_{32}$

$$
\begin{aligned}
M'_{13}M'_{32} &= ((J-I)\otimes I)(J\otimes(J-I)) \\
&= (J-I)_{k\times k}J_{k\times n-k}\otimes I(J-I) \\
&= (k-1)J\otimes(J-1) \\
&= (k-1)M'_{32}
\end{aligned}
$$

$$
\Rightarrow M_{13}M_{32} = (k-1)M_{32} \in \mathcal{W}(C) \subset \mathcal{W}(\Gamma)
$$

* $M_{14}M_{31}$

$$
\begin{aligned}
M'_{14}M'_{31} &= ((J-I)\otimes(J-I))(J\otimes I) \\
&= (J-I)_{k\times k}J_{k\times n-k}\otimes(J-I)I \\
&= (k-1)J\otimes(J-I) \\
&= (k-1)M'_{32}
\end{aligned}
$$

$$
\Rightarrow M_{14}M_{31} = (k-1)M_{32} \in \mathcal{W}(C) \subset \mathcal{W}(\Gamma)
$$

* $M_{14}M_{32}$

$$
\begin{aligned}
M'_{14}M'_{32} &= ((J-I)\otimes(J-I))(J\otimes(J-I)) \\
&= (J-I)_{k\times k}J_{k\times n-k}\otimes(J-I)(J-I) \\
&= (k-1)J\otimes((n-2)J+I) \\
&= (k-1)(n-2)J\otimes J+(k-1)J\otimes I \\
&= (k-1)(n-2)J\otimes(J-I)+(k-1)(n-2)J\otimes I+(k-1)J\otimes I \\
&= (k-1)(n-2)J\otimes(J-I)+(k-1)(n-1)J\otimes I \\
&= (k-1)(n-2)M'_{32}+(k-1)(n-1)M'_{31}
\end{aligned}
$$

$$
\Rightarrow M_{14}M_{32} = (k-1)(n-2)M_{32}+(k-1)(n-1)M_{31} \in \mathcal{W}(C) \subset \mathcal{W}(\Gamma)
$$

– For $M \in \mathcal{W}(C), N \in \mathcal{W}(A_1)$, matrix multiplications would be of the form

$$
MN = \begin{bmatrix} 0 & A \\ 0 & 0 \end{bmatrix}\begin{bmatrix} B & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}
$$

This shows that for any matrices $M \in \mathcal{W}(C), N \in \mathcal{W}(A_1)$, the product would also be $\mathbf{0} \in \mathcal{W}(\Gamma)$.

Thus, for any 2 matrices M, N from subsets $\mathcal{W}(A_1)$ and $\mathcal{W}(C)$, the product $MN \in \mathcal{W}(\Gamma)$.

- $\mathcal{W}(A_1)$ and $\mathcal{W}(C^T)$

  - For $M \in \mathcal{W}(A_1), N \in \mathcal{W}(C^T)$, matrix multiplications would be of the form

$$MN = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ B & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

    This shows that for any matrices $M \in \mathcal{W}(A_1), N \in \mathcal{W}(C^T)$, the product would also be $\mathbf{0} \in \mathcal{W}(\Gamma)$.

  - For $M \in \mathcal{W}(C^T), N \in \mathcal{W}(A_1)$, matrix multiplications would be of the form

$$MN = \begin{bmatrix} 0 & 0 \\ A & 0 \end{bmatrix} \begin{bmatrix} B & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ AB & 0 \end{bmatrix} \neq \mathbf{0}$$

    so we use the sets $\mathcal{W}'(C^T)$ and $\mathcal{W}'(A_1)$.

    We know that $M'_{4i}M'_{11} = M'_{4i}$ since $M'_{11} = I$, so $M_{41}M_{11} = M_{41}$ and $M_{42}M_{11} = M_{42}$. So both $M_{41}M_{11}, M_{42}M_{11} \in \mathcal{W}(C^T) \subset \mathcal{W}(\Gamma)$.

    * $M_{41}M_{12}$

$$\begin{aligned} M'_{41}M'_{12} &= (J_{n-k\times k} \otimes I)(I_{k\times k} \otimes (J - I)) \\ &= (J_{n-k\times k}I_{k\times k}) \otimes (I(J - I)) \\ &= J \otimes (J - I) \\ &= M'_{42} \end{aligned}$$

$$\Rightarrow M_{41}M_{12} = M_{42} \in \mathcal{W}(C^T) \subset \mathcal{W}(\Gamma)$$

    * $M_{42}M_{12}$

$$\begin{aligned} M'_{42}M'_{12} &= (J_{n-k\times k} \otimes (J - I))(I_{k\times k} \otimes (J - I)) \\ &= (J_{n-k\times k}I_{k\times k}) \otimes ((J - I)(J - I)) \\ &= J \otimes ((n-2)J + I) \\ &= (n-2)J \otimes J + J \otimes I \\ &= (n-2)J \otimes (J - I) + (n-2)J \otimes I + J \otimes I \\ &= (n-2)M'_{42} + (n-1)M'_{41} \end{aligned}$$

$$\Rightarrow M_{42}M_{12} = (n-2)M_{42} + (n-1)M_{41} \in \mathcal{W}(C^T) \subset \mathcal{W}(\Gamma)$$

* $M_{41}M_{13}$

$$\begin{aligned}
M'_{41}M'_{13} &= (J_{n-k\times k}\otimes I)((J-I)_{n\times n}\otimes I)\\
&= (J_{n-k\times k}(J-1)_{k\times k})\otimes I(I)\\
&= (k-1)J\otimes I\\
&= (k-1)M'_{41}
\end{aligned}$$

$$\Rightarrow M_{41}M_{13} = (k-1)M_{41} \in \mathcal{W}(C^T) \subset \mathcal{W}(\Gamma)$$

* $M_{42}M_{13}$

$$\begin{aligned}
M'_{42}M'_{13} &= (J_{n-k\times k}\otimes (J-I))((J-I)_{n\times n}\otimes I)\\
&= (J_{n-k\times k}(J-1)_{k\times k})\otimes (J-I)(I)\\
&= (k-1)J\otimes (J-I)\\
&= (k-1)M'_{42}
\end{aligned}$$

$$\Rightarrow M_{42}M_{13} = (k-1)M_{42} \in \mathcal{W}(C^T) \subset \mathcal{W}(\Gamma)$$

* $M_{41}M_{14}$

$$\begin{aligned}
M'_{41}M'_{14} &= (J_{n-k\times k}\otimes I)((J-I)_{k\times k}\otimes (J-I))\\
&= (J_{n-k\times k}(J-1)_{k\times k})\otimes I(J-I)\\
&= (k-1)J\otimes (J-I)\\
&= (k-1)M'_{42}
\end{aligned}$$

$$\Rightarrow M_{41}M_{14} = (k-1)M_{42} \in \mathcal{W}(C^T) \subset \mathcal{W}(\Gamma)$$

* $M_{42}M_{14}$

$$
\begin{aligned}
M'_{42}M'_{14} &= (J_{n-k\times k} \otimes (J-I))((J-I)_{k\times k} \otimes (J-I)) \\
&= (J_{n-k\times k}(J-I)_{k\times k}) \otimes ((J-I)(J-I)) \\
&= (k-1)J \otimes ((n-2)J+I) \\
&= (k-1)(n-2)J \otimes J + (k-1)J \otimes I \\
&= (k-1)(n-2)J \otimes (J-I) + (k-1)(n-2)J \otimes I + (k-1)J \otimes I \\
&= (k-1)(n-2)J \otimes (J-I) + (k-1)(n-1)J \otimes I \\
&= (k-1)(n-2)M'_{42} + (k-1)(n-1)M'_{41}
\end{aligned}
$$

$$
\Rightarrow M_{42}M_{14} = (k-1)(n-2)M_{42} + (k-1)(n-1)M_{41} \in \mathcal{W}(C^T) \subset \mathcal{W}(\Gamma)
$$

This shows that for any matrices $M \in \mathcal{W}(C^T), N \in \mathcal{W}(A_1)$, the product $MN \in \mathcal{W}(\Gamma)$.

Thus, for any 2 matrices M, N from subsets $\mathcal{W}(A_1)$ and $\mathcal{W}(C^T)$, the product $MN \in \mathcal{W}(\Gamma)$.

- $\mathcal{W}(A_2)$ and $\mathcal{W}(C)$

  - For $M \in \mathcal{W}(A_2), N \in \mathcal{W}(C)$, matrix multiplications would be of the form

    $$
    MN = \begin{bmatrix} 0 & 0 \\ 0 & A \end{bmatrix} \begin{bmatrix} 0 & B \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}
    $$

    This shows that for any matrices $M \in \mathcal{W}(A_2), N \in \mathcal{W}(C)$, the product $MN = \mathbf{0} \in \mathcal{W}(\Gamma)$.

  - For $M \in \mathcal{W}(C), N \in \mathcal{W}(A_2)$, matrix multiplications would be of the form

    $$
    MN = \begin{bmatrix} 0 & A \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & B \end{bmatrix} = \begin{bmatrix} 0 & AB \\ 0 & 0 \end{bmatrix} \neq \mathbf{0}
    $$

    so we use the sets $\mathcal{W}'(C)$ and $\mathcal{W}'(A_2)$.
    We know that $M'_{3i}M'_{21} = M'_{3i}$ since $M'_{21} = I$, so $M_{31}M_{21} = M_{31}$ and $M_{32}M_{21} = M_{32}$. So both $M_{31}M_{21}, M_{32}M_{21} \in \mathcal{W}(C) \subset \mathcal{W}(\Gamma)$.

    * $M_{31}M_{22}$

      $$
      \begin{aligned}
      (J_{k\times n-k} \otimes I)(I_{n-k\times n-k} \otimes (J-I)) &= J_{k\times n-k}I_{n-k\times n-k} \otimes I(J-I) \\
      &= J_{k\times n-k} \otimes (J-I) \\
      &= M'_{32}
      \end{aligned}
      $$

      $$
      \Rightarrow M_{31}M_{12} = M_{32} \in \mathcal{W}(C) \subset \mathcal{W}(\Gamma)
      $$

* $M_{32}M_{22}$

$$\begin{aligned}
M'_{32}M'_{22} &= (J_{k\times n-k} \otimes (J-I))(I_{n-k\times n-k} \otimes (J-I)) \\
&= J_{k\times n-k}I_{k\times n-k} \otimes (J-I)(J-I) \\
&= J_{k\times n-k} \otimes ((n-2)J+I) \\
&= (n-2)J_{k\times n-k} \otimes J + J_{k\times n-k} \otimes I \\
&= (n-2)J_{k\times n-k} \otimes (J-I) + (n-2)J_{k\times n-k} \otimes I + J_{k\times n-k} \otimes I \\
&= (n-2)M'_{32} + (n-1)M'_{31}
\end{aligned}$$

$$\Rightarrow M_{32}M_{22} = (n-2)M_{32} + (n-1)M_{31} \in \mathcal{W}(C) \subset \mathcal{W}(\Gamma)$$

* $M_{31}M_{23}$

$$\begin{aligned}
M'_{31}M'_{23} &= (J_{k\times n-k} \otimes I)((J-I)_{n-k\times n-k} \otimes I) \\
&= J_{k\times n-k}(J-I)_{n-k\times n-k} \otimes I(I) \\
&= (n-k-1)J_{k\times n-k} \otimes I \\
&= (n-k-1)M'_{31}
\end{aligned}$$

$$\Rightarrow M_{31}M_{23} = (n-k-1)M_{31} \in \mathcal{W}(C) \subset \mathcal{W}(\Gamma)$$

* $M_{32}M_{23}$

$$\begin{aligned}
M'_{32}M'_{23} &= (J_{k\times n-k} \otimes (J-I))((J-I)_{n-k\times n-k} \otimes I) \\
&= J_{k\times n-k}(J-I)_{n-k\times n-k} \otimes (J-I)I \\
&= (n-k-1)J_{k\times n-k} \otimes (J-I) \\
&= (n-k-1)M'_{32}
\end{aligned}$$

$$\Rightarrow M_{32}M_{23} = (n-k-1)M_{32} \in \mathcal{W}(C) \subset \mathcal{W}(\Gamma)$$

* $M_{31}M_{24}$

$$\begin{aligned}
M'_{31}M'_{24} &= (J_{k\times n-k} \otimes I)((J-I)_{n-k\times n-k} \otimes (J-I)) \\
&= J_{k\times n-k}(J-I)_{n-k\times n-k} \otimes I(J-I) \\
&= (n-k-1)J_{k\times n-k} \otimes (J-I) \\
&= (n-k-1)M'_{32}
\end{aligned}$$

$$\Rightarrow M_{31}M_{24} = (n-k-1)M_{32} \in \mathcal{W}(C) \subset \mathcal{W}$$

* $M_{32}M_{24}$

$$\begin{aligned}
M'_{32}M'_{24} &= (J_{k\times n-k}\otimes(J-I))((J-I)_{n-k\times n-k}\otimes(J-I))\\
&= J_{k\times n-k}(J-I)_{n-k\times n-k}\otimes(J-I)(J-I)\\
&= (n-k-1)J_{k\times n-k}\otimes((n-2)J+I)\\
&= (n-k-1)(n-2)J_{k\times n-k}\otimes J+(n-k-1)J_{k\times n-k}\otimes I\\
&= (n-k-1)(n-2)J_{k\times n-k}\otimes(J-I)+(n-k-1)(n-2)J_{k\times n-k}\otimes I\\
&\quad +(n-k-1)J_{k\times n-k}\otimes I\\
&= (n-k-1)(n-2)M'_{32}+(n-k-1)(n-1)M'_{31}
\end{aligned}$$

$$\Rightarrow M_{32}M_{24}=(n-k-1)(n-2)M'_{32}+(n-k-1)(n-1)M'_{31}\in\mathcal{W}(C)\subset\mathcal{W}$$

This shows that for any matrices $M\in\mathcal{W}(C), N\in\mathcal{W}(A_2)$, the product $MN\in\mathcal{W}(\Gamma)$.

Thus, for any 2 matrices M,N from subsets $\mathcal{W}(A_2)$ and $\mathcal{W}(C)$, the product $MN\in\mathcal{W}(\Gamma)$.

- $\mathcal{W}(A_2)$ and $\mathcal{W}(C^T)$

  - For $M\in\mathcal{W}(A_2), N\in\mathcal{W}(C^T)$, matrix multiplications would be of the form

$$MN=\begin{bmatrix}0&0\\0&A\end{bmatrix}\begin{bmatrix}0&0\\B&0\end{bmatrix}=\begin{bmatrix}0&0\\AB&0\end{bmatrix}\neq\mathbf{0}$$

so we use the sets $\mathcal{W}'(A_2)$ and $\mathcal{W}'(C^T)$.

* $M_{22}M_{41}$

$$\begin{aligned}
M'_{22}M'_{41} &= (I_{n-k\times n-k}\otimes(J-I))(J_{n-k\times k}\otimes I)\\
&= I_{n-k\times n-k}J_{n-k\times k}\otimes(J-I)I\\
&= J_{n-k\times k}\otimes(J-I)\\
&= M'_{42}
\end{aligned}$$

$$\Rightarrow M_{22}M_{41}=M_{42}\in\mathcal{W}(C^T)\subset\mathcal{W}(\Gamma)$$

* $M_{22}M_{42}$

$$\begin{aligned} M'_{22}M'_{42} &= (I_{n-k\times n-k} \otimes (J-I))(J_{n-k\times k} \otimes (J-I)) \\ &= I_{n-k\times n-k}J_{n-k\times k} \otimes (J-I)(J-I) \\ &= J_{n-k\times k} \otimes ((n-2)J+I) \\ &= (n-2)J_{n-k\times k} \otimes J + J \otimes I \\ &= (n-2)J_{n-k\times k} \otimes (J-I) + (n-2)J \otimes I + J \otimes I \\ &= (n-2)J_{n-k\times k} \otimes (J-I) + (n-1)J \otimes I \\ &= (n-2)M'_{42} + (n-1)M'_{41} \end{aligned}$$

$$\Rightarrow M_{22}M_{42} = (n-2)M_{42} + (n-1)M_{41} \in \mathcal{W}(C^T) \subset \mathcal{W}(\Gamma)$$

* $M_{23}M_{41}$

$$\begin{aligned} M'_{23}M'_{41} &= ((J-I)_{n-k\times n-k} \otimes I)(J_{n-k\times k} \otimes I) \\ &= (J-I)_{n-k\times n-k}J_{n-k\times k} \otimes I(I) \\ &= (n-k-1)J_{n-k\times k} \otimes I \\ &= (n-k-1)M'_{41} \end{aligned}$$

$$\Rightarrow M_{23}M_{41} = (n-k-1)M_{41} \in \mathcal{W}(C^T) \subset \mathcal{W}(\Gamma)$$

* $M_{23}M_{42}$

$$\begin{aligned} M'_{23}M'_{42} &= ((J-I)_{n-k\times n-k} \otimes I)(J_{n-k\times k} \otimes (J-I)) \\ &= (J-I)_{n-k\times n-k}J_{n-k\times k} \otimes I(J-I) \\ &= (n-k-1)J_{n-k\times k} \otimes (J-I) \\ &= (n-k-1)M'_{42} \end{aligned}$$

$$\Rightarrow M_{23}M_{42} = (n-k-1)M_{42} \in \mathcal{W}(C^T) \subset \mathcal{W}(\Gamma)$$

* $M_{23}M_{41}$

$$\begin{aligned} M'_{24}M'_{41} &= ((J-I)_{n-k\times n-k} \otimes (J-I))(J_{n-k\times k} \otimes I) \\ &= (J-I)_{n-k\times n-k}J_{n-k\times k} \otimes (J-I)I \\ &= (n-k-1)J_{n-k\times k} \otimes (J-I) \\ &= (n-k-1)M'_{42} \end{aligned}$$

$$\Rightarrow M_{24}M_{41} = (n-k-1)M_{42} \in \mathcal{W}(C^T) \subset \mathcal{W}(\Gamma)$$

* $M_{23}M_{42}$

$$\begin{aligned}
M'_{24}M'_{42} &= ((J-I)_{n-k\times n-k} \otimes (J-I))(J_{n-k\times k} \otimes (J-I)) \\
&= (J-I)_{n-k\times n-k}J_{n-k\times k} \otimes (J-I)(J-I) \\
&= (n-k-1)J_{n-k\times k} \otimes ((n-2)J+I) \\
&= (n-k-1)(n-2)J_{n-k\times k} \otimes J + (n-k-1)J_{n-k\times k} \otimes I \\
&= (n-k-1)(n-2)J_{n-k\times k} \otimes (J-I) + (n-k-1)(n-2)J \otimes I \\
&\quad + (n-k-1)J \otimes I \\
&= (n-k-1)(n-2)M'_{42} + (n-k-1)(n-1)M'_{41}
\end{aligned}$$

$$\Rightarrow M_{24}M_{42} = (n-k-1)(n-2)M_{42} + (n-k-1)(n-1)M_{41} \in \mathcal{W}(C^T) \subset \mathcal{W}(\Gamma)$$

This shows that for any matrices $M \in \mathcal{W}(A_2), N \in \mathcal{W}(C^T)$, the product $MN \in \mathcal{W}(\Gamma)$.

- For $M \in \mathcal{W}(C^T), N \in \mathcal{W}(A_2)$, matrix multiplications would be of the form

$$MN = \begin{bmatrix} 0 & 0 \\ A & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & B \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

This shows that for any matrices $M \in \mathcal{W}(C^T), N \in \mathcal{W}(A_2)$, the product $MN = \mathbf{0} \in \mathcal{W}(\Gamma)$.

Thus, for any 2 matrices M, N from subsets $\mathcal{W}(A_2)$ and $\mathcal{W}(C^T)$, the product $MN \in \mathcal{W}(\Gamma)$.

- $\mathcal{W}(C)$ and $\mathcal{W}(C^T)$

  - For $M \in \mathcal{W}(C), N \in \mathcal{W}(C^T)$, matrix multiplications would be of the form

$$MN = \begin{bmatrix} 0 & A \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ B & 0 \end{bmatrix} = \begin{bmatrix} AB & 0 \\ 0 & 0 \end{bmatrix} \neq \mathbf{0}$$

so we use the sets $\mathcal{W}'(C)$ and $\mathcal{W}'(C^T)$.

* $M_{31}M_{41}$

$$\begin{aligned}
M'_{31}M'_{41} &= (J_{k\times(n-k)} \otimes I)(J_{(n-k)\times k} \otimes I) \\
&= J_{k\times(n-k)}J_{(n-k)\times k} \otimes I(I) \\
&= (n-k)J_{k\times k} \otimes I \\
&= (n-k)(J-I)_{k\times k} \otimes I + (n-k)I_{k\times k} \otimes I \\
&= (n-k)(J-I)_{k\times k} \otimes I + (n-k)I_{kn\times kn} \\
&= (n-k)M'_{13} + (n-k)M'_{11}
\end{aligned}$$

$$\Rightarrow M_{31}M_{41} = (n-k)M_{13} + (n-k)M_{11} \in \mathcal{W}(A_1) \subset \mathcal{W}(\Gamma)$$

* $M_{31}M_{42}$

$$\begin{aligned}
M'_{31}M'_{42} &= (J_{k\times(n-k)} \otimes I)(J_{(n-k)\times k} \otimes (J-I)) \\
&= J_{k\times(n-k)}J_{(n-k)\times k} \otimes I(J-I) \\
&= (n-k)J_{k\times k} \otimes (J-I) \\
&= (n-k)(J-I)_{k\times k} \otimes (J-I) + (n-k)I_{k\times k} \otimes (J-I) \\
&= (n-k)M'_{14} + (n-k)M'_{12}
\end{aligned}$$

$$\Rightarrow M_{31}M_{42} = (n-k)M_{14} + (n-k)M_{12} \in \mathcal{W}(A_1) \subset \mathcal{W}(\Gamma)$$

* $M_{32}M_{41}$

$$\begin{aligned}
M'_{32}M'_{41} &= (J_{k\times(n-k)} \otimes (J-I))(J_{(n-k)\times k} \otimes I) \\
&= J_{k\times(n-k)}J_{(n-k)\times k} \otimes (J-I)I \\
&= (n-k)J_{k\times k} \otimes (J-I) \\
&= (n-k)(J-I)_{k\times k} \otimes (J-I) + (n-k)I_{k\times k} \otimes (J-I) \\
&= (n-k)M'_{14} + (n-k)M'_{12}
\end{aligned}$$

$$\Rightarrow M_{32}M_{41} = (n-k)M_{14} + (n-k)M_{12} \in \mathcal{W}(A_1) \subset \mathcal{W}(\Gamma)$$

* $M_{32}M_{42}$

$$\begin{aligned}
M'_{32}M'_{42} &= (J_{k\times(n-k)} \otimes (J-I))(J_{(n-k)\times k} \otimes (J-I)) \\
&= J_{k\times(n-k)}J_{(n-k)\times k} \otimes (J-I)(J-I) \\
&= (n-k)J_{k\times k} \otimes ((n-2)J + I) \\
&= (n-k)(n-2)J_{k\times k} \otimes J + (n-k)J_{k\times k} \otimes I
\end{aligned}$$

Note that $J_{k\times k} \otimes I = M'_{13} + M'_{11}$ from $M_{31}M_{41}$, so we break down $J_{k\times k} \otimes J$:

$$\begin{aligned}
J_{k\times k} \otimes J &= J_{k\times k} \otimes (J - I) + J_{k\times k} \otimes I \\
&= (J - I)_{k\times k} \otimes (J - I) + I_{k\times k} \otimes (J - I) + M'_{13} + M'_{11} \\
&= M'_{14} + M'_{12} + M'_{13} + M'_{11}
\end{aligned}$$

Combining,

$$\begin{aligned}
M'_{32}M'_{42} &= (n-k)(n-2)J_{k\times k} \otimes J + (n-k)J_{k\times k} \otimes I \\
&= (n-k)(n-2)(M'_{11} + M'_{12} + M'_{13} + M'_{14}) + (n-k)(M'_{11} + M'_{13})
\end{aligned}$$

$$\begin{aligned}
&\Rightarrow M_{32}M_{42} = (n-k)(n-2)(M_{11} + M_{12} + M_{13} + M_{14}) + (n-k)(M_{11} + M_{13}) \\
&\Rightarrow M_{32}M_{42} \in \mathcal{W}(A_1) \subset \mathcal{W}(\Gamma)
\end{aligned}$$

This shows that for any matrices $M \in \mathcal{W}(C), N \in \mathcal{W}(C^T)$, the product $MN \in \mathcal{W}(\Gamma)$.

– For $M \in \mathcal{W}(C^T), N \in \mathcal{W}(C)$, matrix multiplications would be of the form

$$MN = \begin{bmatrix} 0 & 0 \\ A & 0 \end{bmatrix} \begin{bmatrix} 0 & B \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & AB \end{bmatrix} \neq \mathbf{0}$$

so we use the sets $\mathcal{W}'(C^T)$ and $\mathcal{W}'(C)$.

* $M_{41}M_{31}$

$$\begin{aligned}
M'_{41}M'_{31} &= (J_{(n-k)\times k} \otimes I)(J_{k\times(n-k)} \otimes I) \\
&= J_{(n-k)\times k}J_{k\times(n-k)} \otimes I(I) \\
&= kJ_{(n-k)\times(n-k)} \otimes I \\
&= k((J - I)_{(n-k)\times(n-k)} \otimes I + I_{(n-k)\times(n-k)} \otimes I) \\
&= k((J - I)_{(n-k)\times(n-k)} \otimes I + I_{(n-k)n\times(n-k)n}) \\
&= k(M'_{23} + M'_{21})
\end{aligned}$$

$$\Rightarrow M_{41}M_{31} = k(M_{23} + M_{21}) \in \mathcal{W}(A_2) \subset \mathcal{W}(\Gamma)$$

* $M_{41}M_{32}$

$$\begin{aligned}
M'_{41}M'_{32} &= (J_{(n-k)\times k} \otimes I)(J_{k\times(n-k)} \otimes (J-I)) \\
&= J_{(n-k)\times k}J_{k\times(n-k)} \otimes I(J-I) \\
&= kJ_{(n-k)\times(n-k)} \otimes (J-I) \\
&= k((J-I)_{(n-k)\times(n-k)} \otimes (J-I) + I_{(n-k)\times(n-k)} \otimes (J-I)) \\
&= k(M'_{24} + M'_{22})
\end{aligned}$$

$$\Rightarrow M_{41}M_{32} = k(M_{24} + M_{22}) \in \mathcal{W}(A_2) \subset \mathcal{W}(\Gamma)$$

* $M_{42}M_{31}$

$$\begin{aligned}
M'_{42}M'_{31} &= (J_{(n-k)\times k} \otimes (J-I))(J_{k\times(n-k)} \otimes I) \\
&= J_{(n-k)\times k}J_{k\times(n-k)} \otimes (J-I)I \\
&= kJ_{(n-k)\times(n-k)} \otimes (J-I) \\
&= k((J-I)_{(n-k)\times(n-k)} \otimes (J-I) + I_{(n-k)\times(n-k)} \otimes (J-I)) \\
&= k(M'_{24} + M'_{22})
\end{aligned}$$

$$\Rightarrow M_{42}M_{31} = k(M_{24} + M_{22}) \in \mathcal{W}(A_2) \subset \mathcal{W}(\Gamma)$$

* $M_{42}M_{32}$

$$\begin{aligned}
M'_{42}M'_{32} &= (J_{(n-k)\times k} \otimes (J-I))(J_{k\times(n-k)} \otimes (J-I)) \\
&= J_{(n-k)\times k}J_{k\times(n-k)} \otimes (J-I)(J-I) \\
&= kJ_{(n-k)\times(n-k)} \otimes ((n-2)J + I) \\
&= k(n-2)J_{(n-k)\times(n-k)} \otimes J + kJ_{(n-k)\times(n-k)} \otimes I
\end{aligned}$$

Note that $kJ_{(n-k)\times(n-k)} \otimes I = k(M'_{23} + M'_{21})$ from $M_{41}M_{31}$, so we break down $J_{(n-k)\times(n-k)} \otimes J$:

$$\begin{aligned}
J_{(n-k)\times(n-k)} \otimes J &= J_{(n-k)\times(n-k)} \otimes (J-I) + J_{(n-k)\times(n-k)} \otimes I \\
&= (J-I)_{(n-k)\times(n-k)} \otimes (J-I) + I_{(n-k)\times(n-k)} \otimes (J-I) + M'_{23} + M'_{21} \\
&= M'_{24} + M'_{22} + M'_{23} + M'_{21}
\end{aligned}$$

Combining,

$$\begin{aligned}
M'_{42}M'_{32} &= k(n-2)J_{(n-k)\times(n-k)} \otimes J + kJ_{(n-k)\times(n-k)} \otimes I \\
&= k(n-2)(M'_{21} + M'_{22} + M'_{23} + M'_{24}) + k(M'_{21} + M'_{23})
\end{aligned}$$

$$\begin{aligned}
&\Rightarrow M_{42}M_{32} = k(n-2)(M_{21} + M_{22} + M_{23} + M_{24}) + k(M_{21} + M_{23}) \\
&\Rightarrow M_{42}M_{32} \in \mathcal{W}(A_2) \subset \mathcal{W}(\Gamma)
\end{aligned}$$

This shows that for any matrices $M \in \mathcal{W}(C^T), N \in \mathcal{W}(C)$, the product $MN \in \mathcal{W}(\Gamma)$.

Thus, for any 2 matrices M, N from subsets $\mathcal{W}(C)$ and $\mathcal{W}(C^T)$, the product $MN \in \mathcal{W}(\Gamma)$.

Wielandt Principle for Section 5.2 First, we rewrite the adjacency matrix using the coherent configuration classes:

$$\begin{aligned}
\mathbf{A}(\Gamma) &= \begin{bmatrix} I_k \otimes (J-I)_n + (J-I)_k \otimes I_n & J_{k,n-k} \otimes (J-I)_n \\ J_{n-k,k} \otimes (J-I)_n & I_{n-k} \otimes (J-I)_{n-k} + (J-I)_k \end{bmatrix} \\
&= \begin{bmatrix} M'_{12} + M'_{13} & M'_{32} \\ M'_{42} & M'_{22} + M'_{23} \end{bmatrix}
\end{aligned}$$

As such we take the square:

$$\mathbf{A}(\Gamma)^2 = \begin{bmatrix} (M'_{12} + M'_{13})^2 + M'_{32}M'_{42} & (M'_{12} + M'_{13})M'_{32} + M'_{32}(M'_{22} + M'_{23}) \\ M'_{42}(M'_{12} + M'_{13}) + (M'_{22} + M'_{23})M'_{42} & (M'_{22} + M'_{23})^2 + M'_{42}M'_{32} \end{bmatrix}$$

By block graph properties, we know the block graph structure of $\mathbf{A}(\Gamma)$ will be preserved, so we check each quadrant of this new matrix.

1. Evaluating $(M'_{12} + M'_{13})^2 + M'_{32}M'_{42}$

$$(M'_{12} + M'_{13})^2 + M'_{32}M'_{42} = (M'_{12})^2 + (M'_{13})^2 + M'_{12}M'_{13} + M'_{13}M'_{12} + M'_{32}M'_{42}$$

- $(M'_{12})^2$

$$\begin{aligned}
(M'_{12})^2 &= (I_k \otimes (J-I)_n)^2 \\
&= I_k \otimes ((J-I)_n)^2 \\
&= I_k \otimes ((n-2)(J-I)_n + (n-1)I_n) \\
&= (n-2)(I_k \otimes (J-I)_n) + (n-1)(I_k \otimes I_n) \\
&= (n-2)M'_{12} + (n-1)M'_{11}
\end{aligned}$$

- $(M'_{13})^2$

$$\begin{aligned}(M'_{13})^2 &= ((J-I)_k \otimes I_n)^2 \\ &= ((k-2)(J-I)_k + (k-1)I_k) \otimes I_n \\ &= (k-2)((J-I)_k \otimes I_n) + (k-1)(I_k \otimes I_n) \\ &= (k-2)M'_{13} + (k-1)M'_{11}\end{aligned}$$

- $M'_{12}M'_{13}$

$$\begin{aligned}M'_{12}M'_{13} &= M'_{13}M'_{12} \\ &= M'_{14}\end{aligned}$$

- $M'_{32}M'_{42}$

$$M'_{32}M'_{42} = (n-k)(n-2)(M'_{11} + M'_{12} + M'_{13} + M'_{14}) + (n-k)(M'_{11} + M'_{13})$$

Putting it together, we have:

$$\begin{aligned}(M'_{12} + M'_{13})^2 + M'_{32}M'_{42} &= (n-2)M'_{12} + (n-1)M'_{11} + (k-2)M'_{13} + (k-1)M'_{11} \\ &\quad + 2M'_{14} + (n-k)(n-2)(M'_{11} + M'_{12} + M'_{13} + M'_{14}) + (n-k)(M'_{11} + M'_{13}) \\ &= M'_{11}(n^2 - kn + 2k - 2) \\ &\quad + M'_{12}((n-2)(n-k+1)) \\ &\quad + M'_{13}((n-2)(n-k+1)) \\ &\quad + M'_{14}(n^2 - kn - 2n + 2k + 2)\end{aligned}$$

We can observe that by Wielandt Principle, M'_{11} and M'_{14} are definitely a class of the coherent configuration. However, since M'_{12} and M'_{13} have the same coefficient, we need to use this procedure again to show it can be "split up". We can treat this as a new matrix $\begin{bmatrix} M'_{12} + M'_{13} & 0 \\ 0 & 0 \end{bmatrix} \in \mathcal{W}$ and square it again:

$$\begin{aligned}\begin{bmatrix} M'_{12} + M'_{13} & 0 \\ 0 & 0 \end{bmatrix}^2 &= \begin{bmatrix} (M'_{12})^2 + (M'_{13})^2 + M'_{12}M'_{13} + M'_{13}M'_{12} & 0 \\ 0 & 0 \end{bmatrix} \\ &= (n-2)M'_{12} + (n-1)M'_{11} + (k-2)M'_{13} + (k-1)M'_{11} + 2M'_{14} \\ &= M'_{11}(n+k-2) + M'_{12}(n-2) + M'_{13}(k-2) + M'_{14}(2)\end{aligned}$$

Now that each coefficient is distinct, by Wielandt Principle, there are 4 classes that are in the coherent configuration, $M'_{11}, M'_{12}, M'_{13}, M'_{14}$.

2. Evaluating $(M'_{12} + M'_{13})M'_{32} + M'_{32}(M'_{22} + M'_{23})$

$$\begin{aligned}(M'_{12}+M'_{13})M'_{32}+M'_{32}(M'_{22}+M'_{23}) &= M'_{12}M'_{32}+M'_{13}M'_{32}+M'_{32}M'_{22}+M'_{32}M'_{23}\\ &= (n-2)M'_{32}+(n-1)M'_{31}+(k-1)M'_{32}\\ &\quad +(n-2)M'_{32}+(n-1)M'_{31}+(n-k-1)M'_{32}\\ &= M'_{31}(2n-2)+M'_{32}(3n-6)\end{aligned}$$

By Wielandt Principle, we observe that there are 2 classes in the coherent configuration, M'_{31}, M'_{32}

3. Evaluating $M'_{42}(M'_{12}+M'_{13})+(M'_{22}+M'_{23})M'_{42}$

$$\begin{aligned}M'_{42}(M'_{12}+M'_{13})+(M'_{22}+M'_{23})M'_{42} &= M'_{42}M'_{12}+M'_{42}M'_{13}+M'_{22}M'_{42}+M'_{23}M'_{42}\\ &= (n-2)M'_{42}+(n-1)M'_{41}+(k-1)M'_{42}\\ &\quad +(n-2)M'_{42}+(n-1)M'_{41}+(n-k-1)M'_{42}\\ &= M'_{41}(2n-2)+M'_{42}(3n-6)\end{aligned}$$

By Wielandt Principle, we observe that there are 2 classes in the coherent configuration, M'_{41}, M'_{42}

4. Evaluating $(M'_{22}+M'_{23})^2+M'_{42}M'_{32}$

$$(M'_{22}+M'_{23})^2+M'_{42}M'_{32} = (M'_{22})^2+(M'_{23})^2+M'_{22}M'_{23}+M'_{23}M'_{22}+M'_{42}M'_{32}$$

- $(M'_{22})^2$

$$\begin{aligned}(M'_{22})^2 &= (I_{n-k}\otimes(J-I)_n)^2\\ &= I_{n-k}\otimes((J-I)_n)^2\\ &= I_{n-k}\otimes((n-2)(J-I)_n+(n-1)I_n)\\ &= (n-2)(I_{n-k}\otimes(J-I)_n)+(n-1)(I_{n-k}\otimes I_n)\\ &= (n-2)M'_{22}+(n-1)M'_{21}\end{aligned}$$

- $(M'_{23})^2$

$$\begin{aligned}(M'_{23})^2 &= ((J-I)_{n-k} \otimes I_n)^2 \\ &= ((n-k-2)(J-I)_{n-k} + (n-k-1)I_{n-k}) \otimes I_n \\ &= (n-k-2)((J-I)_{n-k} \otimes I_n) + (n-k-1)(I_{n-k} \otimes I_n) \\ &= (n-k-2)M'_{23} + (n-k-1)M'_{21}\end{aligned}$$

- $M'_{22}M'_{23}$

$$\begin{aligned}M'_{22}M'_{23} &= M'_{23}M'_{22} \\ &= M'_{24}\end{aligned}$$

- $M'_{42}M'_{32}$

$$M'_{42}M'_{32} = k(n-2)(M'_{21} + M'_{22} + M'_{23} + M'_{24}) + k(M'_{21} + M'_{23})$$

Putting it together, we have:

$$\begin{aligned}(M'_{22} + M'_{23})^2 + M'_{42}M'_{32} &= (n-2)M'_{22} + (n-1)M'_{21} + (n-k-2)M'_{23} + (n-k-1)M'_{21} \\ &\quad + 2M'_{24} + k(n-2)(M'_{21} + M'_{22} + M'_{23} + M'_{24}) + k(M'_{21} + M'_{23}) \\ &= M'_{21}(kn + 2n - 2k - 2) \\ &\quad + M'_{22}(kn + n - 2k - 2) \\ &\quad + M'_{23}(kn + n - 2k - 2) \\ &\quad + M'_{24}(kn - 2k + 2)\end{aligned}$$

We can observe that by Wielandt Principle, M'_{21} and M'_{24} are definitely a class of the coherent configuration. However, since M'_{22} and M'_{23} have the same coefficient, we need to use this procedure again to show it can be "split up". We can treat this as a new matrix $\begin{bmatrix} 0 & 0 \\ 0 & M'_{22} + M'_{23} \end{bmatrix} \in \mathcal{W}$ and square it again:

$$\begin{aligned}\begin{bmatrix} 0 & 0 \\ 0 & M'_{22} + M'_{23} \end{bmatrix}^2 &= \begin{bmatrix} 0 & 0 \\ 0 & (M'_{22})^2 + (M'_{23})^2 + M'_{22}M'_{23} + M'_{23}M'_{22} \end{bmatrix} \\ &= (n-2)M'_{22} + (n-1)M'_{21} + (n-k-2)M'_{23} + (n-k-1)M'_{21} + 2M'_{24} \\ &= M'_{21}(2n-k-2) + M'_{22}(n-2) + M'_{23}(n-k-2) + M'_{24}(2)\end{aligned}$$

Now that each coefficient is distinct, by Wielandt Principle, there are 4 classes that are in the coherent configuration, $M'_{21}, M'_{22}, M'_{23}, M'_{24}$.