Calculating Coherent Configurations on Non-Distance Regular Graphs

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Abstract

Many regularly structured graphs, such as strongly regular graphs, have been extensively studied, and their spectral and algebraic properties are well documented in the literature. In contrast, the study of non-structured graphs remains limited, largely due to the difficulty of systematically constructing and analyzing them. In this paper, we explore whether simple graph operations—such as vertex deletion and switching—performed on strongly regular graphs can produce non-structured graphs, and examine how these operations affect their associated coherent configurations.

We focus on two well-known families of strongly regular graphs: the Rook Graph $R(n, n)$ and the Triangular Graph $T(n)$. By applying specific graph modifications, we analyze the resulting adjacency algebras and track changes in their coherent configuration structure. Our experiments reveal that certain operations consistently result in configurations of fixed rank and algebraic patterns, suggesting underlying structure even within seemingly irregular graphs.

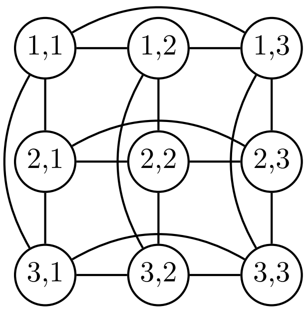

Figure 1: The Rook Graph $R(3, 3)$

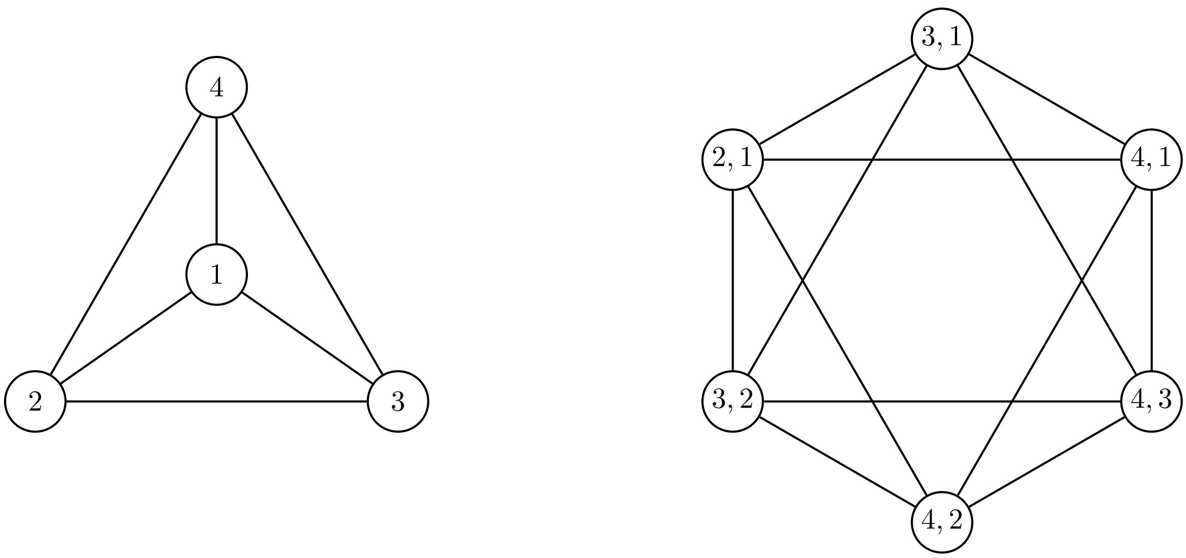

Figure 2: The complete graph K_4 and its corresponding triangular graph $T(4)$.

Acknowledgements

Contents

1 Introduction

Strongly regular graphs (SRGs) form a central class of combinatorial objects that exhibit high levels of symmetry and regularity. These graphs are characterized by fixed parameters: each vertex has the same number of neighbors, and the number of common neighbors between any two vertices depends only on whether they are adjacent or not [1]. This regularity leads to adjacency matrices that generate a commutative algebra of dimension three—known as the Bose-Mesner algebra—which naturally induces a coherent configuration [2, 3].

Among SRGs, rook graphs and triangular graphs are particularly notable. Rook graphs arise from the geometry of a chessboard, where vertices represent cells and edges connect those in the same row or column [4]. In contrast, triangular graphs can be defined as the line graphs of complete graphs, where each vertex represents an edge of the complete graph, and two vertices are adjacent if the corresponding edges in share a vertex. Equivalently, triangular graphs are formed from the 2-subsets of an n-element set, with edges between pairs that intersect [1]. These dual interpretations highlight the deep combinatorial structure of triangular graphs and their place within the family of SRGs.

Both rook and triangular graphs are highly symmetric and well-understood algebraically, making them ideal candidates for exploring how structural modifications affect underlying algebraic properties.

This paper investigates how simple graph operations, such as vertex deletion and switching, impact the coherent configurations of rook and triangular graphs. Coherent configurations—combinatorial structures that partition vertex-pairs according to regular interaction patterns—offer a rich framework for understanding algebraic changes arising from perturbations to SRGs [3].

Our core question is: Do such operations, while potentially destroying symmetry, still preserve some algebraic regularity? By analyzing the resulting coherent algebras—via adjacency matrix partitions and coherent rank computations—we aim to reveal whether predictable structures emerge despite the loss of strong regularity.

2 Notations and Definitions

2.1 Matrix Notations

- I_n represents the identity matrix of size $n \times n$.
- J_n represents the all-ones matrix of size $n \times n$.
- $M_{i,j}$ represents a matrix of size $i \times j$.
- $\otimes$ represents the Kronecker product, and is defined as:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \otimes B_{i,j} = \begin{bmatrix} aB_{i,j} & bB_{i,j} \\ cB_{i,j} & dB_{i,j} \end{bmatrix}$$

2.2 Coherent Configuration and Algebras

A *coherent configuration* [3] is a combinatorial and algebraic structure defined on a finite set V. It provides a framework for studying symmetry and regularity in graphs and other relational structures. Formally, a coherent configuration is a pair $(V, \mathcal{R})$, where $\mathcal{R} = \{R_1, \ldots, R_r\}$ is a partition of $V \times V$ into binary relations, each represented by its adjacency matrix A_i. These matrices satisfy the following axioms:

Axioms of a Coherent Configuration

Definition 2.1. *Let $\{A_1, \ldots, A_r\} \subseteq Mat_{|V|}(\{0,1\})$ be the set of adjacency matrices corresponding to the relations in $\mathcal{R}$. Then $(V, \mathcal{R})$ is called a coherent configuration of rank $r = |\mathcal{R}|$ if the following conditions hold:*

(CC1) $\sum_{i=1}^{r} A_i = J$, *where J is the all-ones matrix.*

(CC2) *For each $i \in \{1, \ldots, r\}$, there exists $j \in \{1, \ldots, r\}$ such that $A_i^T = A_j$.*

(CC3) *There exists a subset $\Delta \subseteq \{1, \ldots, r\}$ such that $\sum_{i \in \Delta} A_i = I$, where I is the identity matrix.*

(CC4) $A_i A_j = \sum_{k=1}^{r} p_{i,j}^k A_k$ *for some constants $p_{i,j}^k \in \mathbb{Z}_{\geq 0}$, for all $i, j \in \{1, \ldots, r\}$.*

Each matrix A_i corresponds to a relation class, and the set $\{A_1, \ldots, A_r\}$ forms a basis for a matrix algebra called the *coherent algebra* of the configuration.

Axioms of a Coherent Algebra

The coherent algebra $\mathcal{W}(\Gamma) \subseteq \mathrm{Mat}_V(\mathbb{C})$ associated with a coherent configuration is a matrix algebra that satisfies the following axioms:

(A1) $I, J \in \mathcal{W}(\Gamma)$, where I is the identity matrix and J is the all-ones matrix.

(A2) $M^T \in \mathcal{W}(\Gamma)$ for all $M \in \mathcal{W}(\Gamma)$.

(A3) The algebra is closed under both matrix multiplication and the Hadamard (entrywise) product: $MN \in \mathcal{W}(\Gamma)$ and $M \circ N \in \mathcal{W}(\Gamma)$.

2.3 Rook Graph

A rook graph is defined as a simple graph, $R(m,n) = (V,E), \quad m \leq n$, where:

- Vertices represent the cells of an $m \times n$ chessboard, $|V| = mn$.
- Two vertices are adjacent if they lie in the same row or column on the chessboard.
- The resulting adjacency matrix is as follows:

$$\mathbf{A}[R(m,n)] = \begin{bmatrix} (J-I)_n & I_n & \cdots & I_n \\ I_n & (J-I)_n & \cdots & I_n \\ \vdots & \vdots & \ddots & \vdots \\ I_n & I_n & \cdots & (J-I)_n \end{bmatrix}$$
$$= I_m \otimes (J-I)_n + (J-I)_m \otimes I_n$$

Square Rook Graph A square rook graph is a specific rook graph where $m = n$, which we define as $R(n) = (V,E)$, where:

- Vertices represent the cells of an $n \times n$ chessboard, $|V| = n^2$.
- Two vertices are adjacent if they lie in the same row or column on the chessboard.

An example of $R(3)$ can be seen in Figure 1.

Construction of $R(3)$ Alternatively, $R_{n,n}$ can be constructed using an orthogonal array of size $\mathrm{OA}(2,n)$, which has the following properties:

- $\mathrm{OA}(2,n)$ is a $2 \times n^2$ matrix, where each column corresponds to a unique pair $(x,y) \in \{1,\ldots,n\} \times \{1,\ldots,n\}$.
- For any two rows, every pair of entries appears exactly once in the same column.

We can visualise it as such:

$$\mathrm{OA}(2,n) = \begin{bmatrix} 1 & 1 & \cdots & 1 & 2 & 2 & \cdots & 2 & \cdots & n & n & \cdots & n \\ 1 & 2 & \cdots & n & 1 & 2 & \cdots & n & \cdots & 1 & 2 & \cdots & n \end{bmatrix}$$

Given $\mathrm{OA}(2,n)$, we define the **Orthogonal Array graph** as follows:

Definition 2.2. *The Orthogonal Array graph, $G_{OA} = (V,E)$, is constructed by $OA(2,n)$ by the following:*

- *Each vertex v_i corresponds to the i-th column of the orthogonal array, so $|V| = n^2$.*
- *Two vertices v_i and v_j are adjacent if they share the same value in any row of the array.*

For $n = 3$, the orthogonal array $\mathrm{OA}(2,3)$ is:

$$\begin{bmatrix} 1 & 1 & 1 & 2 & 2 & 2 & 3 & 3 & 3 \\ 1 & 2 & 3 & 1 & 2 & 3 & 1 & 2 & 3 \end{bmatrix}.$$

The Orthogonal Array graph of $\mathrm{OA}(2,3)$ would have 9 vertices in total, with each vertex being adjacent to 4 other vertices. (i.e. $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is adjacent to $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$, $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$)

It can be observed that this construction of G is isomorphic to Rook graph $R(3)$, and we can generalise it to $R(n)$ as well.

Theorem 2.3. *A block graph construction of $OA(2,n)$ is isomorphic to $R(n)$.*

Proof. We want to show that the block graph construction of $\mathrm{OA}(2,n)$, $G = (V_G, E_G)$, is isomorphic to the Rook graph, $R(n) = (V_R, E_R)$.

Given $\mathrm{OA}(2,n) = \begin{bmatrix} 1 & 1 & \cdots & 1 & 2 & 2 & \cdots & 2 & \cdots & n & n & \cdots & n \\ 1 & 2 & \cdots & n & 1 & 2 & \cdots & n & \cdots & 1 & 2 & \cdots & n \end{bmatrix}$,
let $G = (V_G, E_G)$ be the block graph constructed using the steps defined above.

We proceed as follows:

1. **Vertex Sets**: Since the columns of $\mathrm{OA}(2,n)$ spans $\{1, 2, \cdots, n\}^2$, $|V| = n^2$. Similarly, the vertices of $R_{n,n}$ correspond to the cells of an $n \times n$ chessboard, so $|V_R| = n^2$. Therefore, $|V_G| = |V_R|$.

2. **Edge Sets**: In G, two vertices are adjacent if their corresponding columns in $\mathrm{OA}(2,n)$ share the same value in at least one row. This means that:

   - If two vertices share the same value in the top row, they are adjacent.
   - If two vertices share the same value in the bottom row, they are adjacent.

   This adjacency condition matches exactly how edges are defined in $R(n)$, where two cells of the chessboard are connected if they lie in the same row or column. Thus, the adjacency relationships in G and $R(n)$ are equivalent.

3. **Bijection**:

   - Let $(i,j) \in V_G$ represent an arbitrary column in $\mathrm{OA}(2,n)$.
   - Let $(r_i, c_j) \in V_R$ represent a arbitrary position on a $n \times n$ chessboard which corresponds to the i-th row and j-th column.

   We now define a mapping $f : V_G \to V_R$ as follows:

   $$f((i,j)) = (r_i, c_j), \text{ where } (i,j) \in V_G \text{ and } (r_i, c_j) \in V_R.$$

   To show that the mapping is bijective, we show the following:

- **Injectivitiy**:
  Assume $f((i,j)) = f((i',j'))$. Then:
  $$f((i,j)) = (r_i, c_j) \quad \text{and} \quad f((i',j')) = (r_{i'}, c_{j'}).$$
  Since $f((i,j)) = f((i',j'))$, it follows that:
  $$(r_i, c_j) = (r_{i'}, c_{j'}).$$
  Thus, $r_i = r_{i'}$ and $c_j = c_{j'}$, which implies $(i,j) = (i',j')$. In other words, if two positions on the $n \times n$ chessboard are the same, their corresponding columns in the $\mathrm{OA}(2,n)$ must be the same.
  Therefore, f is injective.
- **Surjectivity**:
  Taking an arbitrary $(r_i, c_j) \in V_R$, we need to show that there exists a $(i,j) \in V_G$ such that $f((i,j)) = (r_i, c_j)$.
  Note that (r_i, c_j) corresponds to the position on the chessboard with row i and column j. Since f maps a column to a chessboard position uniquely, the Since we have shown injectivity, for each $(i,j) \in V_G$, there is a one-to-one $f((i,j)) = (r_i, c_j) \in V_R$. Thus, the set of images $\{f((i,j)) \mid (i,j) \in V_G\} \subseteq V_R$ contains exactly n^2 elements. We have also shown Let $(x,y) \in V_R$ be an arbitrary vertex in $R(n)$. By the construction of $\mathrm{OA}(2,n)$, there exists a column $c = \begin{bmatrix} x \\ y \end{bmatrix} \in V$ such that the top row contains x and the bottom row contains y. Thus, $f(c) = (x,y)$, meaning every vertex in V_R has a preimage in V_G.
  Therefore, f is surjective.

Thus, we have shown that the mapping f is a bijection.

4. **Adjacency Preservation**: If two vertices in G are adjacent, they share the same value in the top or bottom row of $\mathrm{OA}(2,n)$. Under the mapping f, this means the corresponding vertices in $R(n)$ share the same row or column. Similarly, if two vertices in $R(n)$ are adjacent, their positions share the same row or column, which corresponds to adjacency in G. Thus, f preserves adjacency.

Therefore, since f is a bijection from $V_G \to V_R$ such that the edges are preserved, $G \cong R_{n,n}$.

□

2.4 Triangular Graph

A triangular graph of size n, denoted $T(n)$, is defined as the line graph of the complete graph K_n. That is, each vertex in $T(n)$ corresponds to an edge of K_n, and two vertices are adjacent in $T(n)$ if the corresponding edges in K_n share a common vertex. An example can be seen in Figure 2

Equivalently, $T(n)$ is the Johnson graph $J(n,2)$, whose vertices are all 2-element subsets of the set $\{1,2,\ldots,n\}$, with adjacency defined as follows: two vertices are adjacent if and only if the corresponding subsets intersect in exactly one element.

Properties of Triangular Graphs

- $T(n)$ is vertex-transitive.
- $T(n)$ is a strongly regular graph with parameters:

$$\mathrm{srg}\left(\frac{n(n-1)}{2}, 2(n-2), n-2, 4\right).$$

- Each vertex can be labeled as $\{i, j\}$, where $i < j$ and $i, j \in \{1, \ldots, n\}$.
- Two vertices $\{i_1, j_1\}$ and $\{i_2, j_2\}$ are adjacent if and only if

$$\{i_1, j_1\} \cap \{i_2, j_2\} \neq \emptyset.$$

3 Vertex Deletion and Coherent Configurations

3.1 Deleting 1 Vertex in $R(n)$

3.1.1 Graph Construction and Deleted Vertex Selection

Rook graphs are vertex-transitive, so removing any vertex results in an equivalent graph. For simplicity, let v_1, corresponding to the coordinate $\begin{pmatrix}1\\1\end{pmatrix}$, be removed.

After removal:

- The degree of the $2(n-1)$ vertices originally adjacent to v_1 decreases by 1.
- All other vertices retain their original degrees.

This operation partitions the vertices into two subsets:

1. The $2(n-1)$ vertices adjacent to v_1, corresponding to the set:

$$\left\{\begin{pmatrix}1\\2\end{pmatrix},\begin{pmatrix}1\\3\end{pmatrix},\ldots,\begin{pmatrix}1\\n\end{pmatrix},\begin{pmatrix}2\\1\end{pmatrix},\begin{pmatrix}3\\1\end{pmatrix},\ldots,\begin{pmatrix}n\\1\end{pmatrix}\right\}$$

   We shall call this set the point set, P.

2. The remaining $(n-1)^2$ vertices, corresponding to the set:

$$\left\{\begin{pmatrix}2\\2\end{pmatrix},\begin{pmatrix}2\\3\end{pmatrix},\ldots,\begin{pmatrix}2\\n\end{pmatrix},\begin{pmatrix}3\\2\end{pmatrix},\begin{pmatrix}3\\3\end{pmatrix},\ldots,\begin{pmatrix}3\\n\end{pmatrix},\ldots,\begin{pmatrix}n\\2\end{pmatrix},\begin{pmatrix}n\\3\end{pmatrix},\ldots,\begin{pmatrix}n\\n\end{pmatrix}\right\}$$

   We shall call this set the block set, B.

By grouping the vertices corresponding to P and B together, we end up with a matrix decomposition:

$$\mathbf{A}=\begin{bmatrix}\mathbf{A}_1 & \mathbf{C}\\ \mathbf{C}^\top & \mathbf{A}_2\end{bmatrix},$$

where:

- $\mathbf{A}_1$: Adjacency matrix of the Point graph Γ_P.
- $\mathbf{A}_2$: Adjacency matrix of the Block graph Γ_B.

In order to observe certain properties, we aim to show that the sub-graphs Γ_P and Γ_B have specific structures:

Proposition 3.1. Γ_P *is the graph of two disjoint Complete graphs of size* $n-1$.

Proof. The point graph Γ_P contains vertices which correspond to elements in the set:

$$P=\left\{\begin{pmatrix}1\\2\end{pmatrix},\begin{pmatrix}1\\3\end{pmatrix},\ldots,\begin{pmatrix}1\\n\end{pmatrix},\begin{pmatrix}2\\1\end{pmatrix},\begin{pmatrix}3\\1\end{pmatrix},\ldots,\begin{pmatrix}n\\1\end{pmatrix}\right\}$$

Let us split the set into subsets L and R:

$$L = \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \end{pmatrix}, ..., \begin{pmatrix} 1 \\ n \end{pmatrix} \right\} = \left\{ \begin{pmatrix} 1 \\ i \end{pmatrix} \middle| \ i = 2, 3, \cdots, n \right\},$$

$$R = \left\{ \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \end{pmatrix}, ..., \begin{pmatrix} n \\ 1 \end{pmatrix} \right\} = \left\{ \begin{pmatrix} i \\ 1 \end{pmatrix} \middle| \ i = 2, 3, \cdots, n \right\}$$

- **Show disjointness of graphs**:
  Let $v_L = \begin{pmatrix} 1 \\ i \end{pmatrix} \in L$ and $v_R = \begin{pmatrix} j \\ 1 \end{pmatrix} \in R$, such that $i, j \in \{2, 3, \cdots, n\}$. For any v_L and v_R, $i \neq 1$ and $j \neq 1$, and thus v_L will not be adjacent to v_R, showing that there are no edges between the vertex sets L and R.
  Since we have shown that the vertex sets L and R do not have any edges, we can conclude that the graphs from L and R are disjoint.

- **Show that the graphs Γ_L and Γ_R are both K_{n-1}**:
  - Γ_L: Notice that the vertices in L are adjacent to each other as the top row are all equal to 1, $v_L = \begin{pmatrix} 1 \\ i \end{pmatrix} \in L$. Since $|L| = n - 1$, we can conclude that a Complete graph of size $n - 1$ is formed, i.e. K_{n-1}
  - Γ_R: Similar to the case of L, we note that the vertices in R are adjacent to each other as the bottom row are all equal to 1, $v_R = \begin{pmatrix} j \\ 1 \end{pmatrix} \in R$. Since $|R| = n-1$, we can conclude that another Complete graph of size $n - 1$ is formed, i.e. K_{n-1}

We have shown that the sub-graphs formed by vertex sets L and R are disjoint, and that Γ_L and Γ_R are both K_{n-1} and thus have proved the proposition above. □

Proposition 3.2. *Γ_B is the block graph construction of $OA(2, n-1)$.*

Proof. The Block graph Γ_B contains vertices which correspond to elements in the set:

$$B = \left\{ \begin{pmatrix} 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \end{pmatrix}, ..., \begin{pmatrix} 2 \\ n \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 3 \end{pmatrix}, ..., \begin{pmatrix} 3 \\ n \end{pmatrix}, ..., \begin{pmatrix} n \\ 2 \end{pmatrix}, \begin{pmatrix} n \\ 3 \end{pmatrix}, ..., \begin{pmatrix} n \\ n \end{pmatrix} \right\}$$

We can generalise this set into:

$$\begin{aligned} B &= \left\{ \begin{pmatrix} i \\ j \end{pmatrix} \middle| \ i = \{2, 3, \cdots, n\}, j = \{2, 3, \cdots, n\} \right\} \\ &= \left\{ \begin{pmatrix} i - 1 + 1 \\ j - 1 + 1 \end{pmatrix} \middle| \ i - 1 = \{1, 2, \cdots, n-1\}, j - 1 = \{1, 2, \cdots, n-1\} \right\} \\ &= \left\{ \begin{pmatrix} i' + 1 \\ j' + 1 \end{pmatrix} \middle| \ i' = \{1, 2, \cdots, n-1\}, j' = \{1, 2, \cdots, n-1\} \right\} \end{aligned}$$

By mapping $\begin{pmatrix} i'+1 \\ j'+1 \end{pmatrix}$ to $\begin{pmatrix} i' \\ j' \end{pmatrix}$ by subtracting 1 from each row, we can show that Γ_B is isomorphic to a block graph constructed by $\mathrm{OA}(2, n-1)$. $\square$

3.1.2 Coherent Configuration of the Modified Graph

From this decomposition, we can tell that Γ_P and Γ_B both are type-3 graphs, corresponding to a $\begin{bmatrix} 3 & \\ & 3 \end{bmatrix}$ type structure of the original $\mathbf{A} = A(\Gamma)$. This design structure is akin to the Strongly regular design described by Sankey[5], which has a type matrix:

$$\begin{bmatrix} 3 & 2 \\ & 3 \end{bmatrix}$$

To show that $\mathbf{A}$ decomposes into the Strongly regular design of type $\begin{bmatrix} 3 & 2 \\ & 3 \end{bmatrix}$, we need to verify that the equations from Sankey holds. Mainly, the equations are as follows:

Definition 3.3. *A strongly regular design is a finite incidence structure consisting of a set X_1 of points, a set X_2 of blocks, and an incidence relation $F \subseteq X_1 \times X_2$, such that the following are nonnegative integer constants:*

- $S_1 :=$ *number of points incident with (in) each block;*
- $S_2 :=$ *number of blocks incident with (containing) each point;*
- a_1, b_1 *:= the two distinct block intersection sizes;*
- a_2, b_2 *:= the two distinct point join sizes, that is the number of blocks containing two given points;*
- N_1 (P_1) *:= number of points adjacent to a point x and incident with a block y, provided x is (is not) incident with y;*
- N_2 (P_2) *:= number of points containing a point x and adjacent with a block y, given x is (is not) incident with y;*

Sankey also goes on to describe that the point graph, consisting of vertices from X_1, and the block graph, consisting of vertices from X_2, are strongly regular. This corresponds to the earlier propositions of Γ_P and Γ_B being type-3 graphs.
The only thing left is to verify the remaining piece of the puzzle, the incidence matrix $\mathbf{C}$ as Sankey describes it.

Definition 3.4. *Let $\mathbf{C}$ be the 0/1 incidence matrix with rows indexed by the $n_1 := |X_1|$ points and columns indexed by the $n_2 := |X_2|$ blocks. Then, letting I be the identity matrix and J be the all ones matrix of the appropriate dimensions, we have the following equations:*

1. $\mathbf{C}$ *has row sum S_2 and column sum S_1;*

2. $\mathbf{CC^T} = (S_2 - b_2)I + (a_2 - b_2)\mathbf{A_1} + b_2 J$;

3. $\mathbf{C^T C} = (S_1 - b_1)I + (a_1 - b_1)\mathbf{A_2} + b_1 J$;

4. $\mathbf{CA_2} = (N_2 - P_2)\mathbf{C} + P_2 J$;

5. $\mathbf{A_1 C} = (N_1 - P_1)\mathbf{C} + P_1 J$.

In order to check these statements, we first have to construct the matrix $\mathbf{C}$.

Construction of C We know the rows of $\mathbf{C}$ are indexed by the set P and columns are indexed by the set B. To make things simple, we first consider the top half of $\mathbf{C}$, denoted by $\mathbf{C_1}$ with rows indexed by the set L and columns indexed by B.

$\mathbf{C_1}$: For $\mathbf{C_1}$, the rows are indexed by:

$$L = \left\{ \binom{1}{2}, \binom{1}{3}, \ldots, \binom{1}{n} \right\}$$
$$= \left\{ \binom{1}{i} \middle| \; i \in \{2, 3, \cdots, n\} \right\},$$

while the columns are indexed by:

$$B = \left\{ \binom{2}{2}, \binom{2}{3}, \ldots, \binom{2}{n}, \binom{3}{2}, \ldots, \binom{n}{n} \right\}$$
$$= \left\{ \binom{j}{k} \middle| \; j, k \in \{2, 3 \cdots, n\} \right\}.$$

Proposition 3.5. *Each vertex corresponding to an element in L has exactly $n-1$ adjacent vertices corresponding to $n-1$ elements in B.*

Proof. We aim to show any $v_L = \binom{1}{i} \in L$ is adjacent to exactly $n-1$ $v_B = \binom{j}{k} \in B$.
Given $v_L = \binom{1}{i}$ and $v_B = \binom{j}{k}$, when $i = k$, v_L is adjacent to v_B. This is the only case where adjacency occurs as $j \neq 1, j \in \{2, 3, \cdots, n\}$.
Furthermore, there are $n-1$ edges for any v_L. When we set $i = k$, there are $|\{2, 3, \cdots, n\}| = n-1$ possible values of j.
Thus, for any v_L there are exactly $n-1$ adjacent vertices v_B. □

For instance, when we fix $i = k = 2$:

$$\binom{1}{2} \text{ is adjacent to } \binom{2}{2}, \binom{3}{2}, \ldots, \binom{n}{2}.$$

This adjacency results in rows of the form:

$$[\underbrace{1 \quad 0 \quad \cdots \quad 0}_{n-1 \text{ elements}} \quad 1 \quad 0 \quad \cdots \quad 0 \quad \cdots].$$

When repeated for $\begin{pmatrix}1\\3\end{pmatrix}$ onwards, $\mathbf{C_1}$ is composed of $n-1$ blocks of I_{n-1}:

$$\mathbf{C_1} = \begin{bmatrix} I_{n-1} & I_{n-1} & \cdots \end{bmatrix}.$$

Explicitly, $\mathbf{C_1}$ looks like:

$$\mathbf{C_1} = \begin{bmatrix} 1 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & \cdots & 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & 1 & \cdots & 0 & \cdots & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \cdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 1 & \cdots & 0 & 0 & \cdots & 1 \end{bmatrix}.$$

$\mathbf{C_2}$: For $\mathbf{C_2}$, the rows are indexed by:

$$\begin{aligned} R &= \left\{ \begin{pmatrix}2\\1\end{pmatrix}, \begin{pmatrix}3\\1\end{pmatrix}, \dots, \begin{pmatrix}n\\1\end{pmatrix} \right\} \\ &= \left\{ \begin{pmatrix}i\\1\end{pmatrix} \middle| \, i \in \{2, 3, \cdots, n\} \right\}, \end{aligned}$$

while the columns are still indexed by B. Following the same logic as in $\mathbf{C_1}$, we simply switch the logic from the bottom row to the top row to show adjacency.

This results in each row is adjacent to $n-1$ vertices, with 1's being contiguous. For instance:

$$\begin{pmatrix}2\\1\end{pmatrix} \text{ is adjacent to } \begin{pmatrix}2\\2\end{pmatrix}, \begin{pmatrix}2\\3\end{pmatrix}, \dots, \begin{pmatrix}2\\n\end{pmatrix}.$$

This results in rows of the form:

$$[\underbrace{1 \quad 1 \quad \cdots \quad 1}_{n-1 \text{ elements}} \quad 0 \quad 0 \quad \cdots].$$

Explicitly, $\mathbf{C_2}$ looks like:

$$\mathbf{C_2} = \begin{bmatrix} 1 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 1 & 1 & \cdots & 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \cdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 1 & 1 & \cdots & 1 \end{bmatrix}.$$

Combined Matrix $\mathbf{C}$: Putting $\mathbf{C_1}$ and $\mathbf{C_2}$ together, the complete matrix $\mathbf{C}$ is:

$$\mathbf{C} = \begin{bmatrix} \mathbf{C_1} \\ \mathbf{C_2} \end{bmatrix}.$$

Explicitly:

$$\mathbf{C} = \begin{bmatrix} 1 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & \cdots & 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & 1 & \cdots & 0 & \cdots & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \cdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 1 & \cdots & 0 & 0 & \cdots & 1 \\ \\ 1 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 1 & 1 & \cdots & 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \cdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 1 & 1 & \cdots & 1 \end{bmatrix}.$$

To simplify, we introduce a simpler notation for the matrix $\mathbf{C}$:

$$\mathbf{C} = \begin{bmatrix} I & I & I & \cdots & I \\ R_1 & R_2 & R_3 & \cdots & R_{n-1} \end{bmatrix}$$

where

$$(R_k)_{i,j} = \begin{cases} 1, & \text{if } i = k \\ 0, & \text{otherwise} \end{cases}$$

In words, R_k represents a $(n-1) \times (n-1)$ matrix where the k-th row consists of 1s and all other rows are filled with 0s.

With this constructed matrix $\mathbf{C}$, we will show that the parameters of this matrix decomposition admit a strongly regular design from definition 3.

1. $\mathbf{C}$ has row sum S_2 and column sum S_1.

   It is clear that the row sum is $n-1$ and the column sum is 2 $\Rightarrow$ $\boxed{S_1 = 2, S_2 = n-1}$

2. $\mathbf{C}\mathbf{C}^\mathbf{T} = (S_2 - b_2)I + (a_2 - b_2)\mathbf{A_1} + b_2 J$.

$$\mathbf{C}\mathbf{C}^\mathbf{T} = \begin{bmatrix} I_{n-1} & I_{n-1} & I_{n-1} & \cdots & I_{n-1} \\ R_1 & R_2 & R_3 & \cdots & R_{n-1} \end{bmatrix} \begin{bmatrix} I_{n-1} & R_1^T \\ I_{n-1} & R_2^T \\ \vdots & \vdots \\ I_{n-1} & R_{n-1}^T \end{bmatrix}$$

$$= \begin{bmatrix} \sum_{k=1}^{n-1} I_{n-1} & \sum_{k=1}^{n-1} R_k^T \\ \\ \sum_{k=1}^{n-1} R_k & \sum_{k=1}^{n-1} R_k R_k^T \end{bmatrix}$$

$$= \begin{bmatrix} (n-1)I_{n-1} & J_{n-1} \\ J_{n-1} & (n-1)I_{n-1} \end{bmatrix}$$

$$= (n-2)I_{2(n-1)} + \mathbf{A_1} + J_{2(n-1)}, \text{ (which corresponds to)}$$

$$= (S_2 - b_2)I + (a_2 - b_2)\mathbf{A_1} + b_2 J$$

3. $\mathbf{C^TC} = (S_1 - b_1)I + (a_1 - b_1)\mathbf{A_2} + b_1 J$;

4. $\mathbf{CA_2} = (N_2 - P_2)\mathbf{C} + P_2 J$;

5. $\mathbf{A_1C} = (N_1 - P_1)\mathbf{C} + P_1 J$.

We now calculate the following matrices in order to obtain the parameters and check if the equations hold:

- $\mathbf{CC^T}$

$$\begin{aligned}
\mathbf{CC^T} &= \begin{bmatrix} (n-1)I_{n-1} & J_{n-1} \\ J_{n-1} & (n-1)I_{n-1} \end{bmatrix} \\
&= (n-2)I_{2(n-1)} + \mathbf{A_1} + J_{2(n-1)}, \text{ (which corresponds to)} \\
&= (S_2 - b_2)I + (a_2 - b_2)\mathbf{A_1} + b_2 J
\end{aligned}$$

We can form equations to identify:

$$\begin{aligned}
n - 2 &= S_2 - b_2 \\
a_2 - b_2 &= 1 \\
b_2 &= 1 \\
\Rightarrow a_2 &= 2 \\
\Rightarrow n - 2 &= S_2 - 1 \\
\Rightarrow S_2 &= n - 1
\end{aligned}$$

So we obtain $\boxed{a_2 = 2, b_2 = 1, S_2 = n - 1}$.

- $\mathbf{C^TC}$

$$\begin{aligned}
\mathbf{C^TC} &= \begin{bmatrix} I_{n-1} + J_{n-1} & I_{n-1} & \cdots & I_{n-1} \\ I_{n-1} & I_{n-1} + J_{n-1} & \cdots & I_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ I_{n-1} & I_{n-1} & \cdots & I_{n-1} + J_{n-1} \end{bmatrix} \\
&= 2I_{(n-1)^2} + \mathbf{A_2}, \text{ (which corresponds to)} \\
&= (S_1 - b_1)I + (a_1 - b_1)\mathbf{A_2} + b_1 J
\end{aligned}$$

We can form equations to identify:

$$\begin{aligned}
2 &= S_1 - b_1 \\
1 &= a_1 - b_1 \\
0 &= b_1 \\
\Rightarrow a_1 &= 1 \\
\Rightarrow 2 &= S_1
\end{aligned}$$

So we obtain $\boxed{a_1 = 1, b_1 = 0, S_1 = 2}$. Note that by this point we have confirmed the values of S_1 and S_2 to be the column and row sum of $\mathbf{C}$ respectively.

- $\mathbf{CA_2}$

$$\begin{aligned}\mathbf{CA_2} &= (n-3)\mathbf{C} + J, \text{ (which corresponds to)}\\ &= (N_2 - P_2)\mathbf{C} + P_2 J\end{aligned}$$

We can form equations to identify:

$$\begin{aligned} n-3 &= N_2 - P_2 \\ 1 &= P_2 \\ \Rightarrow N_2 &= n-2 \end{aligned}$$

So we obtain $\boxed{N_2 = n-2, P_2 = 1}$.

- $\mathbf{A_1C}$

$$\begin{aligned}\mathbf{A_1C} &= -\mathbf{C} + J, \text{ (which corresponds to)}\\ &= (N_1 - P_1)\mathbf{C} + P_1 J\end{aligned}$$

We can form equations to identify:

$$\begin{aligned} -1 &= N_1 - P_1 \\ 1 &= P_1 \\ \Rightarrow N_1 &= 0 \end{aligned}$$

So we obtain $\boxed{N_1 = 0, P_1 = 1}$.

(In progress of relating it to the equations in the cited paper [5])

3.1.3 Type Matrix and Fibre Changes

3.1.4 Observed Rank Increase and Interpretation

3.2 Deleting 1 Vertex in $T(n)$

3.2.1 Graph Construction via Johnson Graph Interpretation

Since triangular graphs are vertex-transitive, we delete 1 vertex, v, from $T(n)$ and observe the resulting graph, $T'(n)$, to have the form:

$$\mathbf{A}(T'(n)) = \begin{bmatrix} \mathbf{A_1} & C \\ C^T & \mathbf{A}(R_{2,n-2}) \end{bmatrix}$$

3.2.2 Coherent Configuration and Observed Regularities

We observe this form by grouping vertices adjacent to v, denoted by the set V_a, and vertices nonadjacent to v, denoted by V_{na}.

From the properties of johnson graphs, we know that the neighborhood of any vertex in $T(n)$ is a Rook graph of size $R_{2,n-2}$, so we focus on the subgraph induced by the set V_{na}.

Subgraph of $\mathbf{A_1}$

$$\mathbf{A_1} = \begin{bmatrix} (J-I)_{n-3} & B_{n-3,n-4} & B_{n-3,n-5} & \cdots & B_{n-3,1} \\ B_{n-3,n-4}^T & (J-I)_{n-4} & B_{n-4,n-5} & \cdots & B_{n-4,1} \\ B_{n-3,n-5}^T & B_{n-4,n-5}^T & (J-I)_{n-5} & \cdots & B_{n-5,1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ B_{n-3,1}^T & B_{n-4,1}^T & B_{n-5,1}^T & \cdots & J_1 \end{bmatrix}$$

where we define $B_{i,j}, i > j$ as:

$$B_{i,j} = \begin{bmatrix} 0_{i-(j+1),j} \\ 1_{1,j} \\ I_j \end{bmatrix}$$

In words, the matrix $B_{i,j}$ of size $i \times j$ consists of $i-(j+1)$ rows of 0s, followed by 1 row of 1s, followed by the identity matrix of size j.

Findings

- For $A_1(n)$, the matrix contains $A_1(n-1)$, making the construction of this matrix recursive.
- We theorize this subgraph to be strongly regular, only left to show that the square of the graph follows the form $A^2 = kI + aA + c(J - I - A)$.
- Size of A_1 is $\dfrac{(n-2)(n-3)}{2}$ or $\binom{n-2}{2}$

Off-diagonal The matrix C would have size $\binom{n-2}{2} \times 2(n-2)$

3.2.3 Comparison with $R_{n,n}$ Deletion

4 Switching a Single Vertex and Coherent Rank Increase

4.1 Seidel Switching on 1 Vertex in $R(n)$

4.1.1 Switching Construction and Motivation

4.1.2 Coherent Configuration of the Switched Graph

4.1.3 Formation of Singleton Fibre and Type Matrix Changes

4.2 Seidel Switching on 1 Vertex in $T(n)$

4.2.1 Switching Construction via 2-subsets

4.2.2 Coherent Configuration Changes and Rank Analysis

4.2.3 Structural Role of the Switched Vertex

4.3 Explaining the Coherent Rank Increase

4.3.1 Singleton Fibre Induction and Type 1 Interactions

4.3.2 Justification of Rank Increase Based on Fibre Interaction

4.3.3 General Remarks and Observed Patterns

5 Structured Switching in $R(n)$

5.1 Switching Exactly Half the Vertices

5.1.1 Graph Construction and Symmetric Partitioning

5.1.2 Resulting Coherent Configuration of Rank 6

5.1.3 Partitioning Behavior and Interpretation of Fibres

5.2 Switching k Blocks of n Vertices

Similarly to how we switch 1 vertex from $R_{n,n}$, we switch k-blocks of n vertices ($k < \lfloor n/2 \rfloor$ by symmetry) from $R_{n,n}$ and have decomposed the matrix into the form:

$$\mathbf{A}(\Gamma) = \begin{bmatrix} \mathbf{A}(R_{k,n}) & C \\ C^T & \mathbf{A}(R_{n-k,n}) \end{bmatrix}$$

where $C = J_{k,n-k} \otimes (J - I)_n$.

5.2.1 Block-Based Switching Description and Motivation

5.2.2 Coherent Configuration of Rank 12

We want to show that this resulting graph has rank 12. We do this in 2 steps.

1. Show the upper bound by constructing a coherent algebra.
2. Show the lower bound using the Wielandt principle.

5.2.3 Showing Upper Bound

We wish to show that $\mathcal{W}(\Gamma)$ (defined below) fulfils these axioms and is therefore a coherent algebra.

$$\begin{aligned}\mathcal{W}(A_1) =& \left\langle \begin{bmatrix} I_{kn} & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} I_k \otimes (J-I)_n & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} (J-I)_k \otimes I_n & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} (J-I)_k \otimes (J-I)_n & 0 \\ 0 & 0 \end{bmatrix} \right\rangle \\ =& \langle M_{11}, M_{12}, M_{13}, M_{14} \rangle\end{aligned}$$

$$\begin{aligned}\mathcal{W}(A_2) =& \left\langle \begin{bmatrix} 0 & 0 \\ 0 & I_{(n-k)n} \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & I_{n-k} \otimes (J-I)_n \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & (J-I)_{n-k} \otimes I_n \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & (J-I)_{n-k} \otimes (J-I)_n \end{bmatrix} \right\rangle \\ =& \langle M_{21}, M_{22}, M_{23}, M_{24} \rangle\end{aligned}$$

$$\begin{aligned}\mathcal{W}(C) =& \left\langle \begin{bmatrix} 0 & J_{k,n-k} \otimes I_n \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & J_{k,n-k} \otimes (J-I)_n \\ 0 & 0 \end{bmatrix} \right\rangle \\ =& \langle M_{31}, M_{32} \rangle\end{aligned}$$

$$\begin{aligned}\mathcal{W}(C^T) =& \left\langle \begin{bmatrix} 0 & 0 \\ J_{n-k,k} \otimes I_n & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ J_{n-k,k} \otimes (J-I)_n & 0 \end{bmatrix} \right\rangle \\ =& \langle M_{41}, M_{42} \rangle\end{aligned}$$

$$\begin{aligned}\mathcal{W}(\Gamma) &= \mathcal{W}(A_1) \cup \mathcal{W}(A_2) \cup \mathcal{W}(C) \cup \mathcal{W}(C^T) \\ &= \langle M_{11}, M_{12}, M_{13}, M_{14}, M_{21}, M_{22}, M_{23}, M_{24}, M_{31}, M_{32}, M_{41}, M_{42} \rangle\end{aligned}$$

- We can observe that the elements in $\mathcal{W}(\Gamma)$ are partitions of the matrix J_{n^2}, and as such (CC1) is fulfilled.
- Each element in $\mathcal{W}(A_1)$ has its transpose in $\mathcal{W}(A_2)$, and each element in $\mathcal{W}(C)$ has its transpose in $\mathcal{W}(C^T)$, so (CC2) is also fulfilled, which also fulfils (A2).
- $M_{11} + M_{21} = I_{n^2}$, which shows the existence of a subset of $\mathcal{W}(\Gamma)$ such that the sum of the elements in the subset $\Delta = \{M_{11}, M_{21}\}$ is the identity matrix, I_{n^2}. This fulfils (CC3) and since (CC1) is also fulfilled, we have shown $I_{n^2}, J_{n^2} \in \mathcal{W}(\Gamma)$, so (A1) is fulfilled.
- We just need to show (CC4) and (A3), but showing (A3) also shows (CC4), so we will choose to show that instead of both.

Entrywise product We start with the trivial part, the entrywise product. We observe that each element in $\mathcal{W}(\Gamma)$ are partitions of J_{n^2}, with no overlapping entries that are both non-zero. As such, for any matrix $M, N \in \mathcal{W}(\Gamma)$

$$M \circ N = 0_{n^2}$$

Thus, we have shown for any matrix $M, N \in \mathcal{W}(\Gamma)$, $M \circ N \in \mathcal{W}(\Gamma)$

Matrix Multiplication We will be using the following proposition to *ignore* the commutative multiplications:

Proposition 5.1. *Given matrices A, B, C, D of compatible dimensions such that $AC = CA$ and $BD = DB$, it follows that $(A \otimes B)(C \otimes D) = (C \otimes D)(A \otimes B)$.*

Proof. Starting from that assumption that $AC = CA$ and $BD = DB$,

$$\begin{aligned}(A \otimes B)(C \otimes D) &= AC \otimes BD \\ &= CA \otimes DB \\ &= (C \otimes D)(A \otimes B)\end{aligned}$$

□

We will first consider the multiplications of the elements within their subsets, $\mathcal{W}(A_1), \mathcal{W}(A_2), \mathcal{W}(C), \mathcal{W}(C^T)$. To simplify the calculations, we will be using the respective block matrices to represent the actual matrices, as given by:

$$\begin{aligned}\mathcal{W}'(A_1) &= \langle I_{kn \times kn}, I_{k \times k} \otimes (J - I), (J - I)_{k \times k} \otimes I, (J - I)_{k \times k} \otimes (J - I)\rangle \\ &= \langle M'_{11}, M'_{12}, M'_{13}, M'_{14}\rangle\end{aligned}$$

$$\begin{aligned}\mathcal{W}'(A_2) &= \langle I_{n(n-k) \times n(n-k)}, I_{n-k \times n-k} \otimes (J - I), (J - I)_{n-k \times n-k} \otimes I, (J - I)_{n-k \times n-k} \otimes (J - I)\rangle, \\ &= \langle M'_{21}, M'_{22}, M'_{23}, M'_{24}\rangle\end{aligned}$$

$$\begin{aligned}\mathcal{W}'(C) &= \langle J_{k \times n-k} \otimes I, J_{k \times n-k} \otimes (J - I)\rangle \\ &= \langle M'_{31}, M'_{32}\rangle\end{aligned}$$

$$\begin{aligned}\mathcal{W}'(C^T) &= \langle J_{n-k \times k} \otimes I, J_{n-k \times k} \otimes (J - I)\rangle \\ &= \langle M'_{41}, M'_{42}\rangle\end{aligned}$$

- $\mathcal{W}(A_1)$
  Since the block matrices in $\mathcal{W}(A_1)$ are all in the same position, we can isolate the non-zero block of the matrices, i.e. $\begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} B & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} AB & 0 \\ 0 & 0 \end{bmatrix} \neq \mathbf{0}$, so we use the set $\mathcal{W}'(A_1)$.
  We know that $M'_{11}M'_{1i} = M'_{1i}M'_{11} = M'_{1i}$ since $M'_{11} = I$, so $M_{11}M_{1i} = M_{1i}M_{11} = M_{1i} \in \mathcal{W}(\Gamma)$.

We now consider the multiplications between M'_{12}, M'_{13} and M'_{14}.

– $M_{12}M_{13}$

$$\begin{aligned} M'_{12}M'_{13} &= (I \otimes (J-I))((J-I) \otimes I) \\ &= (I(J-I)) \otimes ((J-I)(I)) \\ &= (J-I) \otimes (J-I) \\ &= M'_{14} \end{aligned}$$

$$\Rightarrow M_{12}M_{13} = M_{14} \in \mathcal{W}(\Gamma)$$

– $M_{13}M_{12}$

Let $A = I, C = J-I, B = J-I, D = I$, we can see that $AC = CA$ and $BD = DB$. Using Proposition 5.1,

$$\begin{aligned} M'_{13}M'_{12} &= M'_{12}M'_{13} \\ &= M'_{14} \end{aligned}$$

$$\Rightarrow M_{13}M_{12} = M_{14} \in \mathcal{W}(\Gamma)$$

– $M_{12}M_{14}$

$$\begin{aligned} M'_{12}M'_{14} &= (I \otimes (J-I))((J-I) \otimes (J-I)) \\ &= (I(J-I)) \otimes ((J-I)(J-I)) \\ &= (J-I) \otimes ((n-2)J + I) \\ &= (n-2)((J-I) \otimes J) + (J-I) \otimes I \\ &= (n-2)((J-I) \otimes (J-I)) + (n-2)((J-I) \otimes I) + (J-I) \otimes I \\ &= (n-2)M'_{14} + (n-1)M'_{13} \end{aligned}$$

$$\Rightarrow M_{12}M_{14} = (n-2)M_{14} + (n-1)M_{13} \in \mathcal{W}(\Gamma)$$

– $M_{14}M_{12}$

Let $A = I, C = J-I, B = J-I, D = J-I$, we can see that $AC = CA$ and $BD = DB$.

Using Proposition 5.1,

$$\begin{aligned} M'_{14}M'_{12} &= M'_{12}M'_{14} \\ &= (n-2)M'_{14} + (n-1)M'_{13} \end{aligned}$$

$$\Rightarrow M_{14}M_{12} = (n-2)M_{14} + (n-1)M_{13} \in \mathcal{W}(\Gamma)$$

– $M_{13}M_{14}$

$$\begin{aligned} M'_{13}M'_{14} &= ((J-I)\otimes I)((J-I)\otimes(J-I)) \\ &= ((J-I)(J-I))\otimes(I(J-I)) \\ &= ((n-2)J+I)\otimes(J-I) \\ &= (n-2)(J\otimes(J-I)) + I\otimes(J-I) \\ &= (n-2)((J-I)\otimes(J-I)) + (n-2)(I\otimes(J-I)) + I\otimes(J-I) \\ &= (n-2)M'_{14} + (n-1)M'_{12} \end{aligned}$$

$$\Rightarrow M_{13}M_{14} = (n-2)M_{14} + (n-1)M_{12} \in \mathcal{W}(\Gamma)$$

Whence,

$$M_{13}M_{14} = (n-2)M_{14} + (n-1)M_{12} \in \mathcal{W}(\Gamma) \tag{1}$$

– $M_{14}M_{13}$

Let $A = J-I, C = J-I, B = I, D = J-I$, we can see that $AC = CA$ and $BD = DB$. Using Proposition 5.1,

$$\begin{aligned} M'_{14}M'_{13} &= M'_{13}M'_{14} \\ &= (n-2)M'_{14} + (n-1)M'_{12} \end{aligned}$$

$$\Rightarrow M_{14}M_{13} = (n-2)M_{14} + (n-1)M_{12} \in \mathcal{W}(\Gamma)$$

So we have shown that the matrices $M, N \in \mathcal{W}(A_1)$ satisfy the property $MN \in \mathcal{W}(A_1) \subset \mathcal{W}(\Gamma)$.

- $\mathcal{W}(A_2)$

  Since the block matrices in $\mathcal{W}(A_2)$ are all in the same position, we can isolate the non-zero block of the matrices, i.e. $\begin{bmatrix} 0 & 0 \\ 0 & A \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & B \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & AB \end{bmatrix} \neq \mathbf{0}$, so we use the set $\mathcal{W}'(A_2)$.

Note that $\mathcal{W}'(A_2) = \mathcal{W}'(A_1)$ and by following the working above, we can derive that the matrices $M, N \in \mathcal{W}(A_2)$ satisfy the property that $MN \in \mathcal{W}(A_2) \subset \mathcal{W}(\Gamma)$

- $\mathcal{W}(C)$
  Since the block matrices in $\mathcal{W}(C)$ are all in the same position, we can isolate the non-zero block of the matrices, i.e. $\begin{bmatrix} 0 & A \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & B \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. This shows that no matter which matrices $M, N \in \mathcal{W}(C)$ we choose, $MN = \mathbf{0} \in \mathcal{W}(\Gamma)$.

- $\mathcal{W}(C^T)$
  Similar to $\mathcal{W}(C)$, we show that $\begin{bmatrix} 0 & 0 \\ A & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ B & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, showing that no matter which matrices $M, N \in \mathcal{W}(C^T)$ we choose, $MN = \mathbf{0} \in \mathcal{W}(\Gamma)$.

We now consider multiplications between different subset partitions of $\mathcal{W}(\Gamma)$.

- $\mathcal{W}(A_1)$ and $\mathcal{W}(A_2)$

  – For $M \in \mathcal{W}(A_1), N \in \mathcal{W}(A_2)$, matrix multiplications would be of the form

  $$MN = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & B \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

  This shows that for any matrices $M \in \mathcal{W}(A_1), N \in \mathcal{W}(A_2)$, the product would be $\mathbf{0} \in \mathcal{W}(\Gamma)$.

  – For $M \in \mathcal{W}(A_2), N \in \mathcal{W}(A_1)$, matrix multiplications would be of the form

  $$MN = \begin{bmatrix} 0 & 0 \\ 0 & A \end{bmatrix} \begin{bmatrix} B & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

  This shows that for any matrices $M \in \mathcal{W}(A_2), N \in \mathcal{W}(A_1)$, the product would also be $\mathbf{0} \in \mathcal{W}(\Gamma)$.

  Thus, for any 2 matrices M, N from subsets $\mathcal{W}(A_1)$ and $\mathcal{W}(A_2)$, the product $MN \in \mathcal{W}(\Gamma)$.

- $\mathcal{W}(A_1)$ and $\mathcal{W}(C)$

  – For $M \in \mathcal{W}(A_1), N \in \mathcal{W}(C)$, matrix multiplications would be of the form

  $$MN = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & B \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & AB \\ 0 & 0 \end{bmatrix} \neq \mathbf{0}$$

  so we use the sets $\mathcal{W}'(A_1)$ and $\mathcal{W}'(C)$.

We know that $M'_{11}M'_{3i} = M'_{3i}$ since $M'_{11} = I$, so $M_{11}M_{31} = M_{31}$ and $M_{11}M_{32} = M_{32}$. So both $M_{11}M_{31}, M_{11}M_{32} \in \mathcal{W}(C) \subset \mathcal{W}(\Gamma)$.

* $M'_{12}M'_{31}$

$$\begin{aligned}
M'_{12}M'_{31} &= (I \otimes (J - I)(J \otimes I) \\
&= IJ \otimes (J - I)I \\
&= J \otimes (J - I) = M'_{32}
\end{aligned}$$

$$\Rightarrow M_{12}M_{31} = M_{32} \in \mathcal{W}(C) \subset \mathcal{W}(\Gamma)$$

* $M'_{12}M'_{32}$

$$\begin{aligned}
M'_{12}M'_{32} &= (I \otimes (J - I)(J \otimes (J - I)) \\
&= IJ \otimes (J - I)(J - I) \\
&= J \otimes ((n - 2)J + I) \\
&= (n - 2)J \otimes J + J \otimes I \\
&= (n - 2)J \otimes (J - I) + (n - 2)J \otimes I + J \otimes I \\
&= (n - 2)J \otimes (J - I) + (n - 1)J \otimes I \\
&= (n - 2)M'_{32} + (n - 1)M'_{31}
\end{aligned}$$

$$\Rightarrow M_{12}M_{32} = (n - 2)M_{32} + (n - 1)M_{31} \in \mathcal{W}(C) \subset \mathcal{W}(\Gamma)$$

* $M_{13}M_{31}$

$$\begin{aligned}
M'_{13}M'_{31} &= ((J - I) \otimes I)(J \otimes I) \\
&= (J - I)_{k\times k}J_{k\times n-k} \otimes II \\
&= (k - 1)J \otimes I \\
&= (k - 1)M'_{31}
\end{aligned}$$

$$\Rightarrow M_{13}M_{31} = (k - 1)M_{31} \in \mathcal{W}(C) \subset \mathcal{W}(\Gamma)$$

* $M_{13}M_{32}$

$$\begin{aligned}
M'_{13}M'_{32} &= ((J-I)\otimes I)(J\otimes (J-I)) \\
&= (J-I)_{k\times k}J_{k\times n-k}\otimes I(J-I) \\
&= (k-1)J\otimes (J-1) \\
&= (k-1)M'_{32}
\end{aligned}$$

$$\Rightarrow M_{13}M_{32} = (k-1)M_{32} \in \mathcal{W}(C) \subset \mathcal{W}(\Gamma)$$

* $M_{14}M_{31}$

$$\begin{aligned}
M'_{14}M'_{31} &= ((J-I)\otimes (J-I))(J\otimes I) \\
&= (J-I)_{k\times k}J_{k\times n-k}\otimes (J-I)I \\
&= (k-1)J\otimes (J-I) \\
&= (k-1)M'_{32}
\end{aligned}$$

$$\Rightarrow M_{14}M_{31} = (k-1)M_{32} \in \mathcal{W}(C) \subset \mathcal{W}(\Gamma)$$

* $M_{14}M_{32}$

$$\begin{aligned}
M'_{14}M'_{32} &= ((J-I)\otimes (J-I))(J\otimes (J-I)) \\
&= (J-I)_{k\times k}J_{k\times n-k}\otimes (J-I)(J-I) \\
&= (k-1)J\otimes ((n-2)J+I) \\
&= (k-1)(n-2)J\otimes J + (k-1)J\otimes I \\
&= (k-1)(n-2)J\otimes (J-I) + (k-1)(n-2)J\otimes I + (k-1)J\otimes I \\
&= (k-1)(n-2)J\otimes (J-I) + (k-1)(n-1)J\otimes I \\
&= (k-1)(n-2)M'_{32} + (k-1)(n-1)M'_{31}
\end{aligned}$$

$$\Rightarrow M_{14}M_{32} = (k-1)(n-2)M_{32} + (k-1)(n-1)M_{31} \in \mathcal{W}(C) \subset \mathcal{W}(\Gamma)$$

- For $M \in \mathcal{W}(C), N \in \mathcal{W}(A_1)$, matrix multiplications would be of the form

$$MN = \begin{bmatrix} 0 & A \\ 0 & 0 \end{bmatrix} \begin{bmatrix} B & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

  This shows that for any matrices $M \in \mathcal{W}(C), N \in \mathcal{W}(A_1)$, the product would also be $\mathbf{0} \in \mathcal{W}(\Gamma)$.

Thus, for any 2 matrices M, N from subsets $\mathcal{W}(A_1)$ and $\mathcal{W}(C)$, the product $MN \in \mathcal{W}(\Gamma)$.

- $\mathcal{W}(A_1)$ and $\mathcal{W}(C^T)$

  - For $M \in \mathcal{W}(A_1), N \in \mathcal{W}(C^T)$, matrix multiplications would be of the form

$$MN = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ B & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

    This shows that for any matrices $M \in \mathcal{W}(A_1), N \in \mathcal{W}(C^T)$, the product would also be $\mathbf{0} \in \mathcal{W}(\Gamma)$.

  - For $M \in \mathcal{W}(C^T), N \in \mathcal{W}(A_1)$, matrix multiplications would be of the form

$$MN = \begin{bmatrix} 0 & 0 \\ A & 0 \end{bmatrix} \begin{bmatrix} B & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ AB & 0 \end{bmatrix} \neq \mathbf{0}$$

    so we use the sets $\mathcal{W}'(C^T)$ and $\mathcal{W}'(A_1)$.

    We know that $M'_{4i}M'_{11} = M'_{4i}$ since $M'_{11} = I$, so $M_{41}M_{11} = M_{41}$ and $M_{42}M_{11} = M_{42}$. So both $M_{41}M_{11}, M_{42}M_{11} \in \mathcal{W}(C^T) \subset \mathcal{W}(\Gamma)$.

    * $M_{41}M_{12}$

$$\begin{aligned} M'_{41}M'_{12} &= (J_{n-k\times k} \otimes I)(I_{k\times k} \otimes (J-I)) \\ &= (J_{n-k\times k}I_{k\times k}) \otimes (I(J-I)) \\ &= J \otimes (J-I) \\ &= M'_{42} \end{aligned}$$

$$\Rightarrow M_{41}M_{12} = M_{42} \in \mathcal{W}(C^T) \subset \mathcal{W}(\Gamma)$$

    * $M_{42}M_{12}$

$$\begin{aligned} M'_{42}M'_{12} &= (J_{n-k\times k} \otimes (J-I))(I_{k\times k} \otimes (J-I)) \\ &= (J_{n-k\times k}I_{k\times k}) \otimes ((J-I)(J-I)) \\ &= J \otimes ((n-2)J + I) \\ &= (n-2)J \otimes J + J \otimes I \\ &= (n-2)J \otimes (J-I) + (n-2)J \otimes I + J \otimes I \\ &= (n-2)M'_{42} + (n-1)M'_{41} \end{aligned}$$

$$\Rightarrow M_{42}M_{12} = (n-2)M_{42} + (n-1)M_{41} \in \mathcal{W}(C^T) \subset \mathcal{W}(\Gamma)$$

* $M_{41}M_{13}$

$$\begin{aligned}
M'_{41}M'_{13} &= (J_{n-k\times k} \otimes I)((J-I)_{n\times n} \otimes I) \\
&= (J_{n-k\times k}(J-1)_{k\times k}) \otimes I(I) \\
&= (k-1)J \otimes I \\
&= (k-1)M'_{41}
\end{aligned}$$

$$\Rightarrow M_{41}M_{13} = (k-1)M_{41} \in \mathcal{W}(C^T) \subset \mathcal{W}(\Gamma)$$

* $M_{42}M_{13}$

$$\begin{aligned}
M'_{42}M'_{13} &= (J_{n-k\times k} \otimes (J-I))((J-I)_{n\times n} \otimes I) \\
&= (J_{n-k\times k}(J-1)_{k\times k}) \otimes (J-I)(I) \\
&= (k-1)J \otimes (J-I) \\
&= (k-1)M'_{42}
\end{aligned}$$

$$\Rightarrow M_{42}M_{13} = (k-1)M_{42} \in \mathcal{W}(C^T) \subset \mathcal{W}(\Gamma)$$

* $M_{41}M_{14}$

$$\begin{aligned}
M'_{41}M'_{14} &= (J_{n-k\times k} \otimes I)((J-I)_{k\times k} \otimes (J-I)) \\
&= (J_{n-k\times k}(J-1)_{k\times k}) \otimes I(J-I) \\
&= (k-1)J \otimes (J-I) \\
&= (k-1)M'_{42}
\end{aligned}$$

$$\Rightarrow M_{41}M_{14} = (k-1)M_{42} \in \mathcal{W}(C^T) \subset \mathcal{W}(\Gamma)$$

* $M_{42}M_{14}$

$$\begin{aligned}
M'_{42}M'_{14} &= (J_{n-k\times k} \otimes (J-I))((J-I)_{k\times k} \otimes (J-I)) \\
&= (J_{n-k\times k}(J-I)_{k\times k}) \otimes ((J-I)(J-I)) \\
&= (k-1)J \otimes ((n-2)J + I) \\
&= (k-1)(n-2)J \otimes J + (k-1)J \otimes I \\
&= (k-1)(n-2)J \otimes (J-I) + (k-1)(n-2)J \otimes I + (k-1)J \otimes I \\
&= (k-1)(n-2)J \otimes (J-I) + (k-1)(n-1)J \otimes I \\
&= (k-1)(n-2)M'_{42} + (k-1)(n-1)M'_{41}
\end{aligned}$$

$$\Rightarrow M_{42}M_{14} = (k-1)(n-2)M_{42} + (k-1)(n-1)M_{41} \in \mathcal{W}(C^T) \subset \mathcal{W}(\Gamma)$$

This shows that for any matrices $M \in \mathcal{W}(C^T), N \in \mathcal{W}(A_1)$, the product $MN \in \mathcal{W}(\Gamma)$.

Thus, for any 2 matrices M, N from subsets $\mathcal{W}(A_1)$ and $\mathcal{W}(C^T)$, the product $MN \in \mathcal{W}(\Gamma)$.

- $\mathcal{W}(A_2)$ and $\mathcal{W}(C)$

  - For $M \in \mathcal{W}(A_2), N \in \mathcal{W}(C)$, matrix multiplications would be of the form

    $$MN = \begin{bmatrix} 0 & 0 \\ 0 & A \end{bmatrix} \begin{bmatrix} 0 & B \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

    This shows that for any matrices $M \in \mathcal{W}(A_2), N \in \mathcal{W}(C)$, the product $MN = \mathbf{0} \in \mathcal{W}(\Gamma)$.
  - For $M \in \mathcal{W}(C), N \in \mathcal{W}(A_2)$, matrix multiplications would be of the form

    $$MN = \begin{bmatrix} 0 & A \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & B \end{bmatrix} = \begin{bmatrix} 0 & AB \\ 0 & 0 \end{bmatrix} \neq \mathbf{0}$$

    so we use the sets $\mathcal{W}'(C)$ and $\mathcal{W}'(A_2)$.
    We know that $M'_{3i}M'_{21} = M'_{3i}$ since $M'_{21} = I$, so $M_{31}M_{21} = M_{31}$ and $M_{32}M_{21} = M_{32}$. So both $M_{31}M_{21}, M_{32}M_{21} \in \mathcal{W}(C) \subset \mathcal{W}(\Gamma)$.

    * $M_{31}M_{22}$

      $$\begin{aligned}
      (J_{k\times n-k} \otimes I)(I_{n-k\times n-k} \otimes (J-I)) &= J_{k\times n-k}I_{n-k\times n-k} \otimes I(J-I) \\
      &= J_{k\times n-k} \otimes (J-I) \\
      &= M'_{32}
      \end{aligned}$$

      $$\Rightarrow M_{31}M_{12} = M_{32} \in \mathcal{W}(C) \subset \mathcal{W}(\Gamma)$$

$*$ $M_{32}M_{22}$

$$\begin{aligned}
M'_{32}M'_{22} &= (J_{k\times n-k} \otimes (J-I))(I_{n-k\times n-k} \otimes (J-I)) \\
&= J_{k\times n-k}I_{k\times n-k} \otimes (J-I)(J-I) \\
&= J_{k\times n-k} \otimes ((n-2)J+I) \\
&= (n-2)J_{k\times n-k} \otimes J + J_{k\times n-k} \otimes I \\
&= (n-2)J_{k\times n-k} \otimes (J-I) + (n-2)J_{k\times n-k} \otimes I + J_{k\times n-k} \otimes I \\
&= (n-2)M'_{32} + (n-1)M'_{31}
\end{aligned}$$

$$\Rightarrow M_{32}M_{22} = (n-2)M_{32} + (n-1)M_{31} \in \mathcal{W}(C) \subset \mathcal{W}(\Gamma)$$

$*$ $M_{31}M_{23}$

$$\begin{aligned}
M'_{31}M'_{23} &= (J_{k\times n-k} \otimes I)((J-I)_{n-k\times n-k} \otimes I) \\
&= J_{k\times n-k}(J-I)_{n-k\times n-k} \otimes I(I) \\
&= (n-k-1)J_{k\times n-k} \otimes I \\
&= (n-k-1)M'_{31}
\end{aligned}$$

$$\Rightarrow M_{31}M_{23} = (n-k-1)M_{31} \in \mathcal{W}(C) \subset \mathcal{W}(\Gamma)$$

$*$ $M_{32}M_{23}$

$$\begin{aligned}
M'_{32}M'_{23} &= (J_{k\times n-k} \otimes (J-I))((J-I)_{n-k\times n-k} \otimes I) \\
&= J_{k\times n-k}(J-I)_{n-k\times n-k} \otimes (J-I)I \\
&= (n-k-1)J_{k\times n-k} \otimes (J-I) \\
&= (n-k-1)M'_{32}
\end{aligned}$$

$$\Rightarrow M_{32}M_{23} = (n-k-1)M_{32} \in \mathcal{W}(C) \subset \mathcal{W}(\Gamma)$$

$*$ $M_{31}M_{24}$

$$\begin{aligned}
M'_{31}M'_{24} &= (J_{k\times n-k} \otimes I)((J-I)_{n-k\times n-k} \otimes (J-I)) \\
&= J_{k\times n-k}(J-I)_{n-k\times n-k} \otimes I(J-I) \\
&= (n-k-1)J_{k\times n-k} \otimes (J-I) \\
&= (n-k-1)M'_{32}
\end{aligned}$$

$$\Rightarrow M_{31}M_{24} = (n-k-1)M_{32} \in \mathcal{W}(C) \subset \mathcal{W}$$

- * $M_{32}M_{24}$

$$\begin{aligned}
M'_{32}M'_{24} &= (J_{k\times n-k}\otimes(J-I))((J-I)_{n-k\times n-k}\otimes(J-I))\\
&= J_{k\times n-k}(J-I)_{n-k\times n-k}\otimes(J-I)(J-I)\\
&= (n-k-1)J_{k\times n-k}\otimes((n-2)J+I)\\
&= (n-k-1)(n-2)J_{k\times n-k}\otimes J+(n-k-1)J_{k\times n-k}\otimes I\\
&= (n-k-1)(n-2)J_{k\times n-k}\otimes(J-I)+(n-k-1)(n-2)J_{k\times n-k}\otimes I\\
&\quad +(n-k-1)J_{k\times n-k}\otimes I\\
&= (n-k-1)(n-2)M'_{32}+(n-k-1)(n-1)M'_{31}
\end{aligned}$$

$$\Rightarrow M_{32}M_{24} = (n-k-1)(n-2)M'_{32}+(n-k-1)(n-1)M'_{31}\in\mathcal{W}(C)\subset\mathcal{W}$$

This shows that for any matrices $M\in\mathcal{W}(C), N\in\mathcal{W}(A_2)$, the product $MN\in\mathcal{W}(\Gamma)$.

Thus, for any 2 matrices M,N from subsets $\mathcal{W}(A_2)$ and $\mathcal{W}(C)$, the product $MN\in\mathcal{W}(\Gamma)$.

- $\mathcal{W}(A_2)$ and $\mathcal{W}(C^T)$

  - For $M\in\mathcal{W}(A_2), N\in\mathcal{W}(C^T)$, matrix multiplications would be of the form

$$MN=\begin{bmatrix}0&0\\0&A\end{bmatrix}\begin{bmatrix}0&0\\B&0\end{bmatrix}=\begin{bmatrix}0&0\\AB&0\end{bmatrix}\neq\mathbf{0}$$

so we use the sets $\mathcal{W}'(A_2)$ and $\mathcal{W}'(C^T)$.

- * $M_{22}M_{41}$

$$\begin{aligned}
M'_{22}M'_{41} &= (I_{n-k\times n-k}\otimes(J-I))(J_{n-k\times k}\otimes I)\\
&= I_{n-k\times n-k}J_{n-k\times k}\otimes(J-I)I\\
&= J_{n-k\times k}\otimes(J-I)\\
&= M'_{42}
\end{aligned}$$

$$\Rightarrow M_{22}M_{41}=M_{42}\in\mathcal{W}(C^T)\subset\mathcal{W}(\Gamma)$$

* $M_{22}M_{42}$

$$\begin{aligned}
M'_{22}M'_{42} &= (I_{n-k\times n-k}\otimes(J-I))(J_{n-k\times k}\otimes(J-I))\\
&= I_{n-k\times n-k}J_{n-k\times k}\otimes(J-I)(J-I)\\
&= J_{n-k\times k}\otimes((n-2)J+I)\\
&= (n-2)J_{n-k\times k}\otimes J + J\otimes I\\
&= (n-2)J_{n-k\times k}\otimes(J-I)+(n-2)J\otimes I+J\otimes I\\
&= (n-2)J_{n-k\times k}\otimes(J-I)+(n-1)J\otimes I\\
&= (n-2)M'_{42}+(n-1)M'_{41}
\end{aligned}$$

$$\Rightarrow M_{22}M_{42} = (n-2)M_{42}+(n-1)M_{41}\in\mathcal{W}(C^T)\subset\mathcal{W}(\Gamma)$$

* $M_{23}M_{41}$

$$\begin{aligned}
M'_{23}M'_{41} &= ((J-I)_{n-k\times n-k}\otimes I)(J_{n-k\times k}\otimes I)\\
&= (J-I)_{n-k\times n-k}J_{n-k\times k}\otimes I(I)\\
&= (n-k-1)J_{n-k\times k}\otimes I\\
&= (n-k-1)M'_{41}
\end{aligned}$$

$$\Rightarrow M_{23}M_{41} = (n-k-1)M_{41}\in\mathcal{W}(C^T)\subset\mathcal{W}(\Gamma)$$

* $M_{23}M_{42}$

$$\begin{aligned}
M'_{23}M'_{42} &= ((J-I)_{n-k\times n-k}\otimes I)(J_{n-k\times k}\otimes(J-I))\\
&= (J-I)_{n-k\times n-k}J_{n-k\times k}\otimes I(J-I)\\
&= (n-k-1)J_{n-k\times k}\otimes(J-I)\\
&= (n-k-1)M'_{42}
\end{aligned}$$

$$\Rightarrow M_{23}M_{42} = (n-k-1)M_{42}\in\mathcal{W}(C^T)\subset\mathcal{W}(\Gamma)$$

* $M_{23}M_{41}$

$$\begin{aligned}
M'_{24}M'_{41} &= ((J-I)_{n-k\times n-k}\otimes(J-I))(J_{n-k\times k}\otimes I)\\
&= (J-I)_{n-k\times n-k}J_{n-k\times k}\otimes(J-I)I\\
&= (n-k-1)J_{n-k\times k}\otimes(J-I)\\
&= (n-k-1)M'_{42}
\end{aligned}$$

$$\Rightarrow M_{24}M_{41} = (n-k-1)M_{42}\in\mathcal{W}(C^T)\subset\mathcal{W}(\Gamma)$$

* $M_{23}M_{42}$

$$\begin{aligned}
M'_{24}M'_{42} &= ((J-I)_{n-k\times n-k}\otimes(J-I))(J_{n-k\times k}\otimes(J-I)) \\
&= (J-I)_{n-k\times n-k}J_{n-k\times k}\otimes(J-I)(J-I) \\
&= (n-k-1)J_{n-k\times k}\otimes((n-2)J+I) \\
&= (n-k-1)(n-2)J_{n-k\times k}\otimes J+(n-k-1)J_{n-k\times k}\otimes I \\
&= (n-k-1)(n-2)J_{n-k\times k}\otimes(J-I)+(n-k-1)(n-2)J\otimes I \\
&\quad +(n-k-1)J\otimes I \\
&= (n-k-1)(n-2)M'_{42}+(n-k-1)(n-1)M'_{41}
\end{aligned}$$

$$\Rightarrow M_{24}M_{42}=(n-k-1)(n-2)M_{42}+(n-k-1)(n-1)M_{41}\in\mathcal{W}(C^T)\subset\mathcal{W}(\Gamma)$$

This shows that for any matrices $M\in\mathcal{W}(A_2), N\in\mathcal{W}(C^T)$, the product $MN\in\mathcal{W}(\Gamma)$.

- For $M\in\mathcal{W}(C^T), N\in\mathcal{W}(A_2)$, matrix multiplications would be of the form

$$MN=\begin{bmatrix}0&0\\A&0\end{bmatrix}\begin{bmatrix}0&0\\0&B\end{bmatrix}=\begin{bmatrix}0&0\\0&0\end{bmatrix}$$

This shows that for any matrices $M\in\mathcal{W}(C^T), N\in\mathcal{W}(A_2)$, the product $MN=\mathbf{0}\in\mathcal{W}(\Gamma)$.

Thus, for any 2 matrices M,N from subsets $\mathcal{W}(A_2)$ and $\mathcal{W}(C^T)$, the product $MN\in\mathcal{W}(\Gamma)$.

- $\mathcal{W}(C)$ and $\mathcal{W}(C^T)$

  - For $M\in\mathcal{W}(C), N\in\mathcal{W}(C^T)$, matrix multiplications would be of the form

$$MN=\begin{bmatrix}0&A\\0&0\end{bmatrix}\begin{bmatrix}0&0\\B&0\end{bmatrix}=\begin{bmatrix}AB&0\\0&0\end{bmatrix}\neq\mathbf{0}$$

so we use the sets $\mathcal{W}'(C)$ and $\mathcal{W}'(C^T)$.

* $M_{31}M_{41}$

$$\begin{aligned}
M'_{31}M'_{41} &= (J_{k\times(n-k)} \otimes I)(J_{(n-k)\times k} \otimes I) \\
&= J_{k\times(n-k)}J_{(n-k)\times k} \otimes I(I) \\
&= (n-k)J_{k\times k} \otimes I \\
&= (n-k)(J-I)_{k\times k} \otimes I + (n-k)I_{k\times k} \otimes I \\
&= (n-k)(J-I)_{k\times k} \otimes I + (n-k)I_{kn\times kn} \\
&= (n-k)M'_{13} + (n-k)M'_{11}
\end{aligned}$$

$$\Rightarrow M_{31}M_{41} = (n-k)M_{13} + (n-k)M_{11} \in \mathcal{W}(A_1) \subset \mathcal{W}(\Gamma)$$

* $M_{31}M_{42}$

$$\begin{aligned}
M'_{31}M'_{42} &= (J_{k\times(n-k)} \otimes I)(J_{(n-k)\times k} \otimes (J-I)) \\
&= J_{k\times(n-k)}J_{(n-k)\times k} \otimes I(J-I) \\
&= (n-k)J_{k\times k} \otimes (J-I) \\
&= (n-k)(J-I)_{k\times k} \otimes (J-I) + (n-k)I_{k\times k} \otimes (J-I) \\
&= (n-k)M'_{14} + (n-k)M'_{12}
\end{aligned}$$

$$\Rightarrow M_{31}M_{42} = (n-k)M_{14} + (n-k)M_{12} \in \mathcal{W}(A_1) \subset \mathcal{W}(\Gamma)$$

* $M_{32}M_{41}$

$$\begin{aligned}
M'_{32}M'_{41} &= (J_{k\times(n-k)} \otimes (J-I))(J_{(n-k)\times k} \otimes I) \\
&= J_{k\times(n-k)}J_{(n-k)\times k} \otimes (J-I)I \\
&= (n-k)J_{k\times k} \otimes (J-I) \\
&= (n-k)(J-I)_{k\times k} \otimes (J-I) + (n-k)I_{k\times k} \otimes (J-I) \\
&= (n-k)M'_{14} + (n-k)M'_{12}
\end{aligned}$$

$$\Rightarrow M_{32}M_{41} = (n-k)M_{14} + (n-k)M_{12} \in \mathcal{W}(A_1) \subset \mathcal{W}(\Gamma)$$

* $M_{32}M_{42}$

$$\begin{aligned}
M'_{32}M'_{42} &= (J_{k\times(n-k)} \otimes (J-I))(J_{(n-k)\times k} \otimes (J-I)) \\
&= J_{k\times(n-k)}J_{(n-k)\times k} \otimes (J-I)(J-I) \\
&= (n-k)J_{k\times k} \otimes ((n-2)J+I) \\
&= (n-k)(n-2)J_{k\times k} \otimes J + (n-k)J_{k\times k} \otimes I
\end{aligned}$$

Note that $J_{k\times k}\otimes I = M'_{13}+M'_{11}$ from $M_{31}M_{41}$, so we break down $J_{k\times k}\otimes J$:

$$\begin{aligned}J_{k\times k}\otimes J &= J_{k\times k}\otimes (J-I)+J_{k\times k}\otimes I\\ &= (J-I)_{k\times k}\otimes (J-I)+I_{k\times k}\otimes (J-I)+M'_{13}+M'_{11}\\ &= M'_{14}+M'_{12}+M'_{13}+M'_{11}\end{aligned}$$

Combining,

$$\begin{aligned}M'_{32}M'_{42} &= (n-k)(n-2)J_{k\times k}\otimes J+(n-k)J_{k\times k}\otimes I\\ &= (n-k)(n-2)(M'_{11}+M'_{12}+M'_{13}+M'_{14})+(n-k)(M'_{11}+M'_{13})\end{aligned}$$

$$\begin{aligned}&\Rightarrow M_{32}M_{42} = (n-k)(n-2)(M_{11}+M_{12}+M_{13}+M_{14})+(n-k)(M_{11}+M_{13})\\ &\Rightarrow M_{32}M_{42}\in \mathcal{W}(A_1)\subset \mathcal{W}(\Gamma)\end{aligned}$$

This shows that for any matrices $M\in\mathcal{W}(C), N\in\mathcal{W}(C^T)$, the product $MN\in\mathcal{W}(\Gamma)$.

– For $M\in\mathcal{W}(C^T), N\in\mathcal{W}(C)$, matrix multiplications would be of the form

$$MN=\begin{bmatrix}0 & 0\\ A & 0\end{bmatrix}\begin{bmatrix}0 & B\\ 0 & 0\end{bmatrix}=\begin{bmatrix}0 & 0\\ 0 & AB\end{bmatrix}\neq \mathbf{0}$$

so we use the sets $\mathcal{W}'(C^T)$ and $\mathcal{W}'(C)$.

* $M_{41}M_{31}$

$$\begin{aligned}M'_{41}M'_{31} &= (J_{(n-k)\times k}\otimes I)(J_{k\times(n-k)}\otimes I)\\ &= J_{(n-k)\times k}J_{k\times(n-k)}\otimes I(I)\\ &= kJ_{(n-k)\times(n-k)}\otimes I\\ &= k((J-I)_{(n-k)\times(n-k)}\otimes I+I_{(n-k)\times(n-k)}\otimes I)\\ &= k((J-I)_{(n-k)\times(n-k)}\otimes I+I_{(n-k)n\times(n-k)n})\\ &= k(M'_{23}+M'_{21})\end{aligned}$$

$$\Rightarrow M_{41}M_{31}=k(M_{23}+M_{21})\in\mathcal{W}(A_2)\subset\mathcal{W}(\Gamma)$$

* $M_{41}M_{32}$

$$\begin{aligned} M'_{41}M'_{32} &= (J_{(n-k)\times k} \otimes I)(J_{k\times(n-k)} \otimes (J-I)) \\ &= J_{(n-k)\times k}J_{k\times(n-k)} \otimes I(J-I) \\ &= kJ_{(n-k)\times(n-k)} \otimes (J-I) \\ &= k((J-I)_{(n-k)\times(n-k)} \otimes (J-I) + I_{(n-k)\times(n-k)} \otimes (J-I)) \\ &= k(M'_{24} + M'_{22}) \end{aligned}$$

$$\Rightarrow M_{41}M_{32} = k(M_{24} + M_{22}) \in \mathcal{W}(A_2) \subset \mathcal{W}(\Gamma)$$

* $M_{42}M_{31}$

$$\begin{aligned} M'_{42}M'_{31} &= (J_{(n-k)\times k} \otimes (J-I))(J_{k\times(n-k)} \otimes I) \\ &= J_{(n-k)\times k}J_{k\times(n-k)} \otimes (J-I)I \\ &= kJ_{(n-k)\times(n-k)} \otimes (J-I) \\ &= k((J-I)_{(n-k)\times(n-k)} \otimes (J-I) + I_{(n-k)\times(n-k)} \otimes (J-I)) \\ &= k(M'_{24} + M'_{22}) \end{aligned}$$

$$\Rightarrow M_{42}M_{31} = k(M_{24} + M_{22}) \in \mathcal{W}(A_2) \subset \mathcal{W}(\Gamma)$$

* $M_{42}M_{32}$

$$\begin{aligned} M'_{42}M'_{32} &= (J_{(n-k)\times k} \otimes (J-I))(J_{k\times(n-k)} \otimes (J-I)) \\ &= J_{(n-k)\times k}J_{k\times(n-k)} \otimes (J-I)(J-I) \\ &= kJ_{(n-k)\times(n-k)} \otimes ((n-2)J + I) \\ &= k(n-2)J_{(n-k)\times(n-k)} \otimes J + kJ_{(n-k)\times(n-k)} \otimes I \end{aligned}$$

Note that $kJ_{(n-k)\times(n-k)} \otimes I = k(M'_{23} + M'_{21})$ from $M_{41}M_{31}$, so we break down $J_{(n-k)\times(n-k)} \otimes J$:

$$\begin{aligned} J_{(n-k)\times(n-k)} \otimes J &= J_{(n-k)\times(n-k)} \otimes (J-I) + J_{(n-k)\times(n-k)} \otimes I \\ &= (J-I)_{(n-k)\times(n-k)} \otimes (J-I) + I_{(n-k)\times(n-k)} \otimes (J-I) + M'_{23} + M'_{21} \\ &= M'_{24} + M'_{22} + M'_{23} + M'_{21} \end{aligned}$$

Combining,

$$\begin{aligned} M'_{42}M'_{32} &= k(n-2)J_{(n-k)\times(n-k)} \otimes J + kJ_{(n-k)\times(n-k)} \otimes I \\ &= k(n-2)(M'_{21} + M'_{22} + M'_{23} + M'_{24}) + k(M'_{21} + M'_{23}) \end{aligned}$$

$$\begin{aligned} &\Rightarrow M_{42}M_{32} = k(n-2)(M_{21} + M_{22} + M_{23} + M_{24}) + k(M_{21} + M_{23}) \\ &\Rightarrow M_{42}M_{32} \in \mathcal{W}(A_2) \subset \mathcal{W}(\Gamma) \end{aligned}$$

This shows that for any matrices $M \in \mathcal{W}(C^T), N \in \mathcal{W}(C)$, the product $MN \in \mathcal{W}(\Gamma)$.

Thus, for any 2 matrices M, N from subsets $\mathcal{W}(C)$ and $\mathcal{W}(C^T)$, the product $MN \in \mathcal{W}(\Gamma)$.

Finally, we have shown that (A3) is fulfilled, and have shown that $\mathcal{W}(\Gamma)$ is a coherent configuration and a coherent algebra.

As a result, the coherent rank of this graph r has an upper bound of 12.

5.2.4 Showing Lower Bound

We will use the Wielandt Principle to show a lower bound for the coherent rank of the graph Γ.

Theorem 5.2. *Wielandt Principle If $M \in \mathcal{W}(\Gamma)$, then $M = \Sigma_{i=1}^{r} c_i A_i$.*

Since we know that $\mathbf{A}(\Gamma) \in \mathcal{W}(\Gamma)$, we also know that $\mathbf{A}(\Gamma)^2 \in \mathcal{W}(\Gamma)$. First, we rewrite the adjacency matrix using the coherent configuration classes:

$$\begin{aligned} \mathbf{A}(\Gamma) &= \begin{bmatrix} I_k \otimes (J-I)_n + (J-I)_k \otimes I_n & J_{k,n-k} \otimes (J-I)_n \\ J_{n-k,k} \otimes (J-I)_n & I_{n-k} \otimes (J-I)_{n-k} + (J-I)_k \end{bmatrix} \\ &= \begin{bmatrix} M'_{12} + M'_{13} & M'_{32} \\ M'_{42} & M'_{22} + M'_{23} \end{bmatrix} \end{aligned}$$

As such we take the square:

$$\mathbf{A}(\Gamma)^2 = \begin{bmatrix} (M'_{12} + M'_{13})^2 + M'_{32}M'_{42} & (M'_{12} + M'_{13})M'_{32} + M'_{32}(M'_{22} + M'_{23}) \\ M'_{42}(M'_{12} + M'_{13}) + (M'_{22} + M'_{23})M'_{42} & (M'_{22} + M'_{23})^2 + M'_{42}M'_{32} \end{bmatrix}$$

By block graph properties, we know the block graph structure of $\mathbf{A}(\Gamma)$ will be preserved, so we check each quadrant of this new matrix.

1. *Evaluating $(M'_{12} + M'_{13})^2 + M'_{32}M'_{42}$*

$$(M'_{12} + M'_{13})^2 + M'_{32}M'_{42} = (M'_{12})^2 + (M'_{13})^2 + M'_{12}M'_{13} + M'_{13}M'_{12} + M'_{32}M'_{42}$$

- $(M'_{12})^2$

$$\begin{aligned}(M'_{12})^2 &= (I_k \otimes (J-I)_n)^2 \\ &= I_k \otimes ((J-I)_n)^2 \\ &= I_k \otimes ((n-2)(J-I)_n + (n-1)I_n) \\ &= (n-2)(I_k \otimes (J-I)_n) + (n-1)(I_k \otimes I_n) \\ &= (n-2)M'_{12} + (n-1)M'_{11}\end{aligned}$$

- $(M'_{13})^2$

$$\begin{aligned}(M'_{13})^2 &= ((J-I)_k \otimes I_n)^2 \\ &= ((k-2)(J-I)_k + (k-1)I_k) \otimes I_n \\ &= (k-2)((J-I)_k \otimes I_n) + (k-1)(I_k \otimes I_n) \\ &= (k-2)M'_{13} + (k-1)M'_{11}\end{aligned}$$

- $M'_{12}M'_{13}$

$$\begin{aligned}M'_{12}M'_{13} &= M'_{13}M'_{12} \\ &= M'_{14}\end{aligned}$$

- $M'_{32}M'_{42}$

$$M'_{32}M'_{42} = (n-k)(n-2)(M'_{11} + M'_{12} + M'_{13} + M'_{14}) + (n-k)(M'_{11} + M'_{13})$$

Putting it together, we have:

$$\begin{aligned}(M'_{12} + M'_{13})^2 + M'_{32}M'_{42} &= (n-2)M'_{12} + (n-1)M'_{11} + (k-2)M'_{13} + (k-1)M'_{11} \\ &\quad + 2M'_{14} + (n-k)(n-2)(M'_{11} + M'_{12} + M'_{13} + M'_{14}) + (n-k)(M'_{11} + M'_{13}) \\ &= M'_{11}(n^2 - kn + 2k - 2) \\ &\quad + M'_{12}((n-2)(n-k+1)) \\ &\quad + M'_{13}((n-2)(n-k+1)) \\ &\quad + M'_{14}(n^2 - kn - 2n + 2k + 2)\end{aligned}$$

We can observe that by Wielandt Principle, M'_{11} and M'_{14} are definitely a class of the coherent configuration. However, since M'_{12} and M'_{13} have the same coefficient, we need to use this procedure again to show it can be "split up". We can treat this as a new matrix $\begin{bmatrix} M'_{12} + M'_{13} & 0 \\ 0 & 0 \end{bmatrix} \in \mathcal{W}$ and square it again:

$$\begin{bmatrix} M'_{12}+M'_{13} & 0 \\ 0 & 0 \end{bmatrix}^2 = \begin{bmatrix} (M'_{12})^2+(M'_{13})^2+M'_{12}M'_{13}+M'_{13}M'_{12} & 0 \\ 0 & 0 \end{bmatrix}$$
$$= (n-2)M'_{12}+(n-1)M'_{11}+(k-2)M'_{13}+(k-1)M'_{11}+2M'_{14}$$
$$= M'_{11}(n+k-2)+M'_{12}(n-2)+M'_{13}(k-2)+M'_{14}(2)$$

Now that each coefficient is distinct, by Wielandt Principle, there are 4 classes that are in the coherent configuration, $M'_{11}, M'_{12}, M'_{13}, M'_{14}$.

2. *Evaluating* $(M'_{12}+M'_{13})M'_{32}+M'_{32}(M'_{22}+M'_{23})$

$$\begin{aligned}(M'_{12}+M'_{13})M'_{32}+M'_{32}(M'_{22}+M'_{23}) &= M'_{12}M'_{32}+M'_{13}M'_{32}+M'_{32}M'_{22}+M'_{32}M'_{23} \\ &= (n-2)M'_{32}+(n-1)M'_{31}+(k-1)M'_{32} \\ &\quad +(n-2)M'_{32}+(n-1)M'_{31}+(n-k-1)M'_{32} \\ &= M'_{31}(2n-2)+M'_{32}(3n-6)\end{aligned}$$

By Wielandt Principle, we observe that there are 2 classes in the coherent configuration, M'_{31}, M'_{32}

3. *Evaluating* $M'_{42}(M'_{12}+M'_{13})+(M'_{22}+M'_{23})M'_{42}$

$$\begin{aligned}M'_{42}(M'_{12}+M'_{13})+(M'_{22}+M'_{23})M'_{42} &= M'_{42}M'_{12}+M'_{42}M'_{13}+M'_{22}M'_{42}+M'_{23}M'_{42} \\ &= (n-2)M'_{42}+(n-1)M'_{41}+(k-1)M'_{42} \\ &\quad +(n-2)M'_{42}+(n-1)M'_{41}+(n-k-1)M'_{42} \\ &= M'_{41}(2n-2)+M'_{42}(3n-6)\end{aligned}$$

By Wielandt Principle, we observe that there are 2 classes in the coherent configuration, M'_{41}, M'_{42}

4. *Evaluating* $(M'_{22}+M'_{23})^2+M'_{42}M'_{32}$

$$(M'_{22}+M'_{23})^2+M'_{42}M'_{32} = (M'_{22})^2+(M'_{23})^2+M'_{22}M'_{23}+M'_{23}M'_{22}+M'_{42}M'_{32}$$

- $(M'_{22})^2$

$$\begin{aligned}
(M'_{22})^2 &= (I_{n-k} \otimes (J-I)_n)^2 \\
&= I_{n-k} \otimes ((J-I)_n)^2 \\
&= I_{n-k} \otimes ((n-2)(J-I)_n + (n-1)I_n) \\
&= (n-2)(I_{n-k} \otimes (J-I)_n) + (n-1)(I_{n-k} \otimes I_n) \\
&= (n-2)M'_{22} + (n-1)M'_{21}
\end{aligned}$$

- $(M'_{23})^2$

$$\begin{aligned}
(M'_{23})^2 &= ((J-I)_{n-k} \otimes I_n)^2 \\
&= ((n-k-2)(J-I)_{n-k} + (n-k-1)I_{n-k}) \otimes I_n \\
&= (n-k-2)((J-I)_{n-k} \otimes I_n) + (n-k-1)(I_{n-k} \otimes I_n) \\
&= (n-k-2)M'_{23} + (n-k-1)M'_{21}
\end{aligned}$$

- $M'_{22}M'_{23}$

$$\begin{aligned}
M'_{22}M'_{23} &= M'_{23}M'_{22} \\
&= M'_{24}
\end{aligned}$$

- $M'_{42}M'_{32}$

$$M'_{42}M'_{32} = k(n-2)(M'_{21} + M'_{22} + M'_{23} + M'_{24}) + k(M'_{21} + M'_{23})$$

Putting it together, we have:

$$\begin{aligned}
(M'_{22} + M'_{23})^2 + M'_{42}M'_{32} &= (n-2)M'_{22} + (n-1)M'_{21} + (n-k-2)M'_{23} + (n-k-1)M'_{21} \\
&\quad + 2M'_{24} + k(n-2)(M'_{21} + M'_{22} + M'_{23} + M'_{24}) + k(M'_{21} + M'_{23}) \\
&= M'_{21}(kn + 2n - 2k - 2) \\
&\quad + M'_{22}(kn + n - 2k - 2) \\
&\quad + M'_{23}(kn + n - 2k - 2) \\
&\quad + M'_{24}(kn - 2k + 2)
\end{aligned}$$

We can observe that by Wielandt Principle, M'_{21} and M'_{24} are definitely a class of the coherent configuration. However, since M'_{22} and M'_{23} have the same coefficient, we need to use this procedure again to show it can be "split up". We can treat this as a new matrix $\begin{bmatrix} 0 & 0 \\ 0 & M'_{22} + M'_{23} \end{bmatrix} \in \mathcal{W}$ and square it again:

$$\begin{bmatrix} 0 & 0 \\ 0 & M'_{22} + M'_{23} \end{bmatrix}^2 = \begin{bmatrix} 0 & 0 \\ 0 & (M'_{22})^2 + (M'_{23})^2 + M'_{22}M'_{23} + M'_{23}M'_{22} \end{bmatrix}$$
$$= (n-2)M'_{22} + (n-1)M'_{21} + (n-k-2)M'_{23} + (n-k-1)M'_{21} + 2M'_{24}$$
$$= M'_{21}(2n-k-2) + M'_{22}(n-2) + M'_{23}(n-k-2) + M'_{24}(2)$$

Now that each coefficient is distinct, by Wielandt Principle, there are 4 classes that are in the coherent configuration, $M'_{21}, M'_{22}, M'_{23}, M'_{24}$.

Since the 12 classes shown have no overlap, we have shown that there is a minimum of 12 classes in the coherent configuration.

5.3 Comparison Between the Two Switching Modes

5.3.1 Rank Difference and Structural Decomposition

5.3.2 Algebraic and Combinatorial Interpretation

6 Conclusion

6.1 Summary of Observations Across Operations

6.2 Key Differences Between Rook and Triangular Graphs

6.3 Directions for Further Exploration

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