Chapter 6. Mutually Orthogonal Latin Squares

Note. In this chapter, we define orthogonal latin squares and explore their existence. We ultimately show that a pair of orthogonal latin squares of order n exists for all $n \geq 3$ except $n = 6$. We construct a "complete" collection of mutually orthogonal latin squares of orders a power of a prime (in Section 6.2). Along the way, we state and disprove two conjectures concerning the number of mutually orthogonal latin squares of various orders.

6.1. Introduction

Note. In this section, we define what it means for two latin squares of the same order to be orthogonal, and what it means for a collection of latin squares of the same order to be mutually orthogonal. We give examples and history, and put a bound on the number of mutually orthogonal latin squares of order n.

Note. Recall from Section 1.2. $v \equiv 3 \pmod 6$: The Bose Construction that a *latin square* of order n is an $n \times n$ array, each cell of which contains exactly one of the symbols in $\{1, 2, ..., n\}$, such that each row and each column of the array contains each of the symbols in $\{1, 2, ..., n\}$ exactly once. We refer to the entry in row i and column j as being in cell (i, j).

Definition. Two latin squares L_1 and L_2 of the same order n are *orthogonal* if for each $(x, y) \in \{1, 2, \ldots, n\} \times \{1, 2, \ldots, n\}$ there is exactly one ordered pair (i, j) such that cell (i, j) of L_1 contains the symbol x and cell (i, j) of L_2 contains the symbol y. Latin squares $L_1, L_2, \ldots, L_t$ are *mutually orthogonal* if for $1 \leq a \leq t$, $1 \leq b \leq t$, $a \neq b$, we have that L_a and L_b are orthogonal.

Note 6.1.A. Equivalently, we have that latin squares L_1 and L_2 are orthogonal if placing L_1 on top of L_2 and considering the resulting n^2 ordered pairs of elements of $\{1, 2, \ldots, n\}$ results in all possible n^2 ordered pairs. Lindner and Rodger describe "The Two-Finger Rule" to check if latin squares L and M are orthogonal (see page 119):

> Let L and M be latin squares of order n. Then L and M are orthogonal if and only if whenever a pair of cells are occupied by the same symbol in L, they are occupied by different symbols in M.

See Figure 6.1

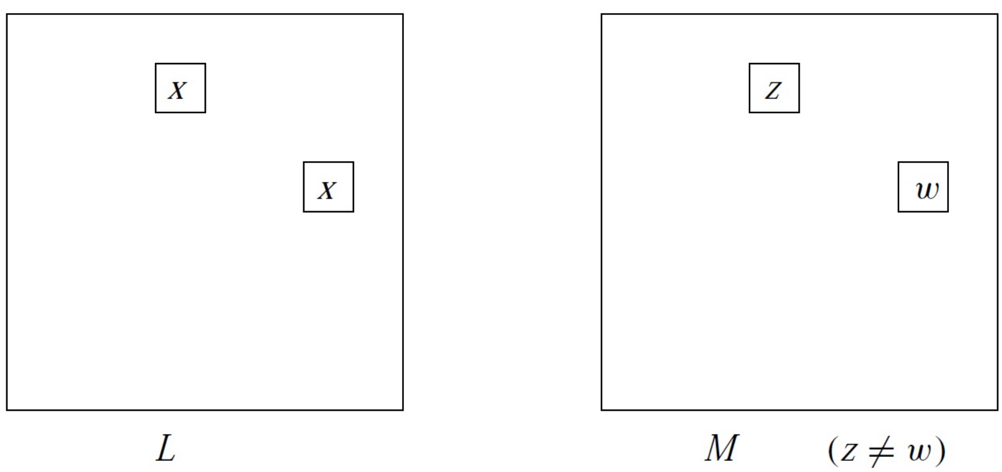

Figure 6.1: The Two-Finger Rule.

Example 6.1.1. **(a)** The latin squares L_1 and L_2 are orthogonal (and of order 3) where:

$$L_1 = \begin{array}{|c|c|c|} \hline 1 & 3 & 2 \\ \hline 3 & 2 & 1 \\ \hline 2 & 1 & 3 \\ \hline \end{array} \qquad L_2 = \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 3 & 1 & 2 \\ \hline 2 & 3 & 1 \\ \hline \end{array}.$$

This example shows that orthogonal latin squares exist.

(b) Latin squares L_3, L_4, and L_5 are three mutually orthogonal latin squares of order 4:

$$L_3 = \begin{array}{|c|c|c|c|} \hline 1 & 3 & 4 & 2 \\ \hline 4 & 2 & 1 & 3 \\ \hline 2 & 4 & 3 & 1 \\ \hline 3 & 1 & 2 & 4 \\ \hline \end{array} \qquad L_4 = \begin{array}{|c|c|c|c|} \hline 1 & 4 & 2 & 3 \\ \hline 3 & 2 & 4 & 1 \\ \hline 4 & 1 & 3 & 2 \\ \hline 2 & 3 & 1 & 4 \\ \hline \end{array} \qquad L_5 = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 4 \\ \hline 2 & 1 & 4 & 3 \\ \hline 3 & 4 & 1 & 2 \\ \hline 4 & 3 & 2 & 1 \\ \hline \end{array}.$$

Note. One of the early problems related to orthogonal latin squares was addressed by Leonhard Euler:

> **The Euler Officer Problem.** Six officers from each of six different regiments are selected so that the six officers from each regiment are of six different ranks, the same six ranks being represented by each regiment. Is it possible to arrange these 36 officers in a 6×6 array so that each regiment and each rank is represented exactly once in each row and column of this array?

This appears in: Leonhard Euler, Recherches sur une nouvelie espèce de quarres magiques, *Vehandlingen Zeeuwach Genootschap Wetenshapper Vlissengen*, **9**, 85–239 (1782). A copy (in French) is online on The Euler Archive webpage. The symbols used are not relevant. We take the ranks to be 1, 2, 3, 4, 5, 6 and the regiments to be 1, 2, 3, 4, 5, 6. Then each officer is represented by a unique ordered pair $(x, y) \in \{1, 2, 3, 4, 5, 6\} \times \{1, 2, 3, 4, 5, 6\}$ (the fact that the rank appears in the first entry and the regiment appears in the second means that the use of the same symbols is not a problem). If a solution to the Euler Officer Problem exists, then collection of 36 ordered pairs (x, y) forms a latin square of order 6. Also, the first coordinates form a latin square of order 6, and the second coordinates form a latin square of order 6. So if a solution exists, then two orthogonal latin squares of order 6 exist. But, no such structures exist as was first shown by G. Tarry "by brute force" in 1900. An efficient proof based on transversal designs and pariwise balanced designs was given by Doug Stinson in "A short proof of the nonexistence of a pair of orthogonal latin squares of order six," *Journal of Combinatorial Theory, Series A*, **36**(3), 373–376 (1984); a copy is available online at ScienceDirect.com. These webpages were accessed 5/27/2022.

Definition. A latin square of order n is in *standard form* if for $1 \le i \le n$, cell$(1, i)$ contains the symbol i.

Note 6.1.B. In Exercise 6.1.2, it is shown that if latin squares L_1 and L_2 are orthogonal, and if L_1' and L_2' are produced from L_1 and L_2 by renaming the symbols, then L_1' and L_2' are orthogonal. So, by induction, in any collection of mutually orthogonal latin squares we can rename the symbols in such a way that each of the latin squares is in standard form. This observation allows us to count the possible number of mutually orthogonal latin squares of order n; we denote such a collection as MOLS(n).

Lemma 6.1.A. If $\{L_1, L_2, \ldots, L_t\}$ is a collection of mutually orthogonal latin squares of order $n \geq 2$, MOLS(n), then the number of MOLS(n) satisfies $t \leq n-1$.

Note. We saw $t = 2$ MOLS(3) and $t = 3$ MOLS(4) in Example 6.1.1, so the bound given in Lemma 6.1.A can be attained (that is, the result is *sharp*).

Definition. A *complete set* of MOLS(n) is a set of $n-1$ MOLS(n).

Note. In the next section, we use finite fields to construct a complete set of MOLS(n) when n is a prime power; see Theorem 6.2.2.

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