Analysis of Common Neighbors in Block Graphs of Orthogonal Arrays

Definition of Orthogonal Array

An orthogonal array $\mathrm{OA}(m, n)$ is a $m \times n^2$ matrix with the following properties:

- Each row in the orthogonal array is filled with the elements $1, 2, \ldots, n$, with each element appearing exactly n times.
- For any $2 \times n^2$ submatrix, each ordered pair of elements (from different rows) appears exactly once as a pair of entries in the same column.

Let the (r, i)-th entry of the orthogonal array $\mathrm{OA}(m, n)$ be denoted by $a_{r,i}$.

Construction of the Block Graph

Given an orthogonal array $\mathrm{OA}(m, n)$, the associated block graph G is defined as follows:

- Each vertex v_i of G corresponds to the i-th column of the orthogonal array.
- Two distinct vertices v_i and v_j are adjacent in G if and only if there exists at least one row r such that the entries $a_{r,i}$ and $a_{r,j}$ in row r of columns i and j are equal.

Analysis of Common Neighbors for Non-Adjacent Vertices

Let v_i and v_j be two non-adjacent vertices in G. We aim to determine the number of common neighbors they share in G.

- Consider a vertex v_k that is a common neighbor of both v_i and v_j.
- Since v_k is adjacent to v_i, there exists a unique row r_1 such that the entries $a_{r_1,i}$ and $a_{r_1,k}$ in row r_1 of columns i and k are equal.
- Similarly, since v_k is adjacent to v_j, there must exist a unique row r_2 such that the entries $a_{r_2,j}$ and $a_{r_2,k}$ in row r_2 of columns j and k are equal.

- By the definition of non-adjacency between v_i and v_j, r_1 cannot be equal to r_2. Therefore, r_2 must be chosen from the remaining $m-1$ rows.
- The remaining $m-2$ rows do not impose further restrictions or affect the adjacency established by the chosen rows r_1 and r_2.

Since there are m possible choices for r_1 and $m-1$ choices for r_2 (for each r_1), the total number of common neighbors between v_i and v_j is:

$$\text{Number of common neighbors} = m(m-1)$$

Conclusion

The reasoning above demonstrates that for any two non-adjacent vertices v_i and v_j in the block graph G associated with the orthogonal array $\mathrm{OA}(m,n)$, the number of common neighbors is $m(m-1)$.