# Obtaining a Strongly Regular Design by Deleting a Vertex from a Rook Graph

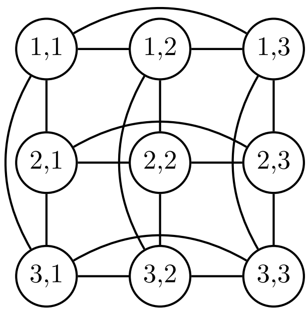


Figure 1: Rook graph $R_{3,3}$, with vertices labeled as $(i, j)$ representing the cells of a $3 \times 3$ chessboard. Edges connect vertices in the same row or column.


**Abstract**

This paper explores the relationship between rook graphs and strongly regular designs. Specifically, we analyze the structural and algebraic consequences of deleting a single vertex from a rook graph. Using adjacency matrix partitioning and coherent rank, we demonstrate how this operation results in a strongly regular design, resulting in a graph with coherent rank 10.

# 1 Introduction

Rook graphs, arising naturally from the geometry of a chessboard, provide a fascinating intersection of combinatorics and algebra. These graphs, defined by vertices corresponding to cells and edges connecting vertices in the same row or column, inherit their structure from orthogonal arrays. The inherent symmetry of these graphs leads to rich combinatorial properties.

In this paper, we investigate the effect of deleting a single vertex from a rook graph, a process that leads to the emergence of a strongly regular design. This is demonstrated using adjacency matrix partitioning and the coherent rank framework, which reflects the structural regularity of the graph.

# 2 Definition of a Rook Graph

A rook graph is defined as a graph, $R_{n,n} = (V_R, E_R)$, where:

- Vertices represent the cells of an $n \times n$ chessboard, $|V_R| = n^2$.
- Two vertices are adjacent if they lie in the same row or column on the chessboard.

An example of $R_{3,3}$ can be seen in Figure 1.

# 3 Construction of $R_{n,n}$

Alternatively, $R_{n,n}$ can be constructed using an orthogonal array of size $\mathrm{OA}(2, n)$, which has the following properties:

- $\mathrm{OA}(2, n)$ is a $2 \times n^2$ matrix, where each column corresponds to a unique pair $(x, y) \in \{1, \ldots, n\} \times \{1, \ldots, n\}$.
- For any two rows, every pair of entries appears exactly once in the same column.

We can visualise it as such:

$$\mathrm{OA}(2, n) = \begin{bmatrix} 1 & 1 & \cdots & 1 & 2 & 2 & \cdots & 2 & \cdots & n & n & \cdots & n \\ 1 & 2 & \cdots & n & 1 & 2 & \cdots & n & \cdots & 1 & 2 & \cdots & n \end{bmatrix}$$

Given $\mathrm{OA}(2, n)$, the associated block graph $G = (V, E)$ is constructed as follows:

- Each vertex $v_i$ corresponds to the $i$-th column of the orthogonal array, so $|V| = n^2$.
- Two vertices $v_i$ and $v_j$ are adjacent if they share the same value in any row of the array.

For example, for $n = 3$, the orthogonal array $\mathrm{OA}(2, 3)$ is:

$$\begin{bmatrix} 1 & 1 & 1 & 2 & 2 & 2 & 3 & 3 & 3 \\ 1 & 2 & 3 & 1 & 2 & 3 & 1 & 2 & 3 \end{bmatrix}.$$

The corresponding block graph constructed from $\mathrm{OA}(2, 3)$ would have 9 vertices in total, with each vertex being adjacent to 4 other vertices. (i.e. $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is adjacent to $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$, $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$)

It can be observed that this construction of $G$ is isomorphic to Rook graph $R_{3,3}$, and we can generalise it to $R_{n,n}$ as well.

**Proposition 1.** *A block graph construction of $OA(2, n)$ is isomorphic to $R_{n,n}$.*

*Proof.* We want to show that the block graph construction of $\mathrm{OA}(2, n)$, $G = (V, E)$, is isomorphic to the Rook graph, $R_{n,n} = (V_R, E_R)$.

Given $\mathrm{OA}(2, n) = \begin{bmatrix} 1 & 1 & \cdots & 1 & 2 & 2 & \cdots & 2 & \cdots & n & n & \cdots & n \\ 1 & 2 & \cdots & n & 1 & 2 & \cdots & n & \cdots & 1 & 2 & \cdots & n \end{bmatrix}$,
let $G = (V, E)$ be the block graph constructed using the steps defined above.

We proceed as follows:

1. **Vertex Sets**: Since the columns of $\mathrm{OA}(2,n)$ spans $\{1,2,\cdots,n\}^2$, $|V|=n^2$. Similarly, the vertices of $R_{n,n}$ correspond to the cells of an $n\times n$ chessboard, so $|V_R|=n^2$. Therefore, $|V|=|V_R|$.

2. **Edge Sets**: In $G$, two vertices are adjacent if their corresponding columns in $\mathrm{OA}(2,n)$ share the same value in at least one row. This means that:

   - If two vertices share the same value in the top row, they are adjacent.
   - If two vertices share the same value in the bottom row, they are adjacent.

   This adjacency condition matches exactly how edges are defined in $R_{n,n}$, where two cells of the chessboard are connected if they lie in the same row or column. Thus, the adjacency relationships in $G$ and $R_{n,n}$ are equivalent.

3. **Bijection**: Define a mapping $f: V \to V_R$ as follows:

   $$f(c)=(x,y),\ \text{where } c=(x,y)^T\in V \text{ and } (x,y)\in V_R.$$

   This shows a mapping $f$ where the column of $\mathrm{OA}(2,n)$, $c=(x,y)^T$, which represents the vertex in $G$, is mapped to $(x,y)$, the unique position of the vertex in $R_{n,n}$.
   To show that the mapping is bijective, we show the following:

   - **Injectivitiy**:
     Assume $f(c_1)=f(c_2)$, where $c_1=\begin{bmatrix}x_1\\y_1\end{bmatrix}$ and $c_2=\begin{bmatrix}x_2\\y_2\end{bmatrix}$. Then:

     $$f(c_1)=(x_1,y_1) \quad \text{and} \quad f(c_2)=(x_2,y_2).$$

     Since $f(c_1)=f(c_2)$, it follows that:

     $$(x_1,y_1)=(x_2,y_2).$$

     Thus, $x_1=x_2$ and $y_1=y_2$, which implies $c_1=c_2$. Therefore, $f$ is injective.
   - **Surjectivity**:
     Since we have shown injectivity, for each $c\in V$, there is a unique $f(c)\in V_R$. Thus, the set of images $\{f(c)|c\in V\}\subseteq V_R$ contains exactly $n^2$ elements. We have also shown Let $(x,y)\in V_R$ be an arbitrary vertex in $R_{n,n}$. By the construction of $\mathrm{OA}(2,n)$, there exists a column $c=\begin{bmatrix}x\\y\end{bmatrix}\in V$ such that the top row contains $x$ and the bottom row contains $y$. Thus, $f(c)=(x,y)$, meaning every vertex in $V_R$ has a preimage in $V$. Therefore, $f$ is surjective.

   Thus, we have shown that the mapping $f$ is a bijection.

4. **Adjacency Preservation**: If two vertices in $G$ are adjacent, they share the same value in the top or bottom row of $\mathrm{OA}(2,n)$. Under the mapping $f$, this means the corresponding vertices in $R_{n,n}$ share the same row or column. Similarly, if two vertices in $R_{n,n}$ are adjacent, their positions share the same row or column, which corresponds to adjacency in $G$. Thus, $f$ preserves adjacency.

Therefore, since $f$ is a bijection from $V\to V_R$ such that the edges are preserved, $G\cong R_{n,n}$.

□

# 4 Deletion of One Vertex

Rook graphs are vertex-transitive, so removing any vertex results in an equivalent graph. For simplicity, let $v_1$, corresponding to the coordinate $\begin{pmatrix}1\\1\end{pmatrix}$, be removed.

After removal:

- The degree of the $2(n-1)$ vertices originally adjacent to $v_1$ decreases by 1.
- All other vertices retain their original degrees.

This operation partitions the vertices into two subsets:

1. The $2(n-1)$ vertices adjacent to $v_1$, corresponding to the set:

$$\left\{\binom{1}{2}, \binom{1}{3}, ..., \binom{1}{n}, \binom{2}{1}, \binom{3}{1}, ..., \binom{n}{1}\right\}$$

   We shall call this set the point set, $P$.

2. The remaining $(n-1)^2$ vertices, corresponding to the set:

$$\left\{\binom{2}{2}, \binom{2}{3}, ..., \binom{2}{n}, \binom{3}{2}, \binom{3}{3}, ..., \binom{3}{n}, ..., \binom{n}{2}, \binom{n}{3}, ..., \binom{n}{n}\right\}$$

   We shall call this set the block set, $B$.

# 5 Adjacency Matrix Partitioning

By grouping the vertices corresponding to $P$ and $B$ together, we aim to show the following matrix decomposition:

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_1 & \mathbf{C} \\ \mathbf{C}^\top & \mathbf{A}_2 \end{bmatrix},$$

where:

- $\mathbf{A}_1$: Adjacency matrix of the Point graph $\Gamma_P$.
- $\mathbf{A}_2$: Adjacency matrix of the Block graph $\Gamma_B$.

In order to observe certain properties, we aim to show that the sub-graphs $\Gamma_P$ and $\Gamma_B$ have specific structures:

**Proposition 2.** *$\Gamma_P$ is the graph of two disjoint Complete graphs of size $n-1$.*

*Proof.* The point graph $\Gamma_P$ contains vertices which correspond to elements in the set:

$$P = \left\{\binom{1}{2}, \binom{1}{3}, ..., \binom{1}{n}, \binom{2}{1}, \binom{3}{1}, ..., \binom{n}{1}\right\}$$

Let us split the set into subsets $L$ and $R$:

$$L = \left\{\binom{1}{2}, \binom{1}{3}, ..., \binom{1}{n}\right\} = \left\{\binom{1}{i} \,\middle|\, i = 2, 3, \cdots, n\right\},$$

$$R = \left\{\binom{2}{1}, \binom{3}{1}, ..., \binom{n}{1}\right\} = \left\{\binom{i}{1} \,\middle|\, i = 2, 3, \cdots, n\right\}$$

- **Show disjointness of graphs**:
  Let $v_L = \binom{1}{i} \in L$ and $v_R = \binom{j}{1} \in R$, such that $i, j \in \{2, 3, \cdots, n\}$. For any $v_L$ and $v_R$, $i \neq 1$ and $j \neq 1$, and thus $v_L$ will not be adjacent to $v_R$, showing that there are no edges between the vertex sets $L$ and $R$.
  Since we have shown that the vertex sets $L$ and $R$ do not have any edges, we can conclude that the graphs from $L$ and $R$ are disjoint.

- **Show that the graphs $\Gamma_L$ and $\Gamma_R$ are both $K_{n-1}$:**
  - $\Gamma_L$: Notice that the vertices in $L$ are adjacent to each other as the top row are all equal to 1, $v_L = \binom{1}{i} \in L$. Since $|L| = n-1$, we can conclude that a Complete graph of size $n-1$ is formed, i.e. $K_{n-1}$

– $\Gamma_R$: Similar to the case of $L$, we note that the vertices in $R$ are adjacent to each other as the bottom row are all equal to 1, $v_R = \begin{pmatrix} j \\ 1 \end{pmatrix} \in R$. Since $|R| = n-1$, we can conclude that another Complete graph of size $n-1$ is formed, i.e. $K_{n-1}$

We have shown that the sub-graphs formed by vertex sets $L$ and $R$ are disjoint, and that $\Gamma_L$ and $\Gamma_R$ are both $K_{n-1}$ and thus have proved the proposition above. □

**Proposition 3.** $\Gamma_B$ *is the block graph construction of* $OA(2, n-1)$.

*Proof.* The Block graph $\Gamma_B$ contains vertices which correspond to elements in the set:

$$B = \left\{ \begin{pmatrix} 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \end{pmatrix}, ..., \begin{pmatrix} 2 \\ n \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 3 \end{pmatrix}, ..., \begin{pmatrix} 3 \\ n \end{pmatrix}, ..., \begin{pmatrix} n \\ 2 \end{pmatrix}, \begin{pmatrix} n \\ 3 \end{pmatrix}, ..., \begin{pmatrix} n \\ n \end{pmatrix} \right\}$$

We can generalise this set into:

$$\begin{aligned} B &= \left\{ \begin{pmatrix} i \\ j \end{pmatrix} \middle| \; i = \{2, 3, \cdots, n\}, j = \{2, 3, \cdots, n\} \right\} \\ &= \left\{ \begin{pmatrix} i-1+1 \\ j-1+1 \end{pmatrix} \middle| \; i-1 = \{1, 2, \cdots, n-1\}, j-1 = \{1, 2, \cdots, n-1\} \right\} \\ &= \left\{ \begin{pmatrix} i'+1 \\ j'+1 \end{pmatrix} \middle| \; i' = \{1, 2, \cdots, n-1\}, j' = \{1, 2, \cdots, n-1\} \right\} \end{aligned}$$

By mapping $\begin{pmatrix} i'+1 \\ j'+1 \end{pmatrix}$ to $\begin{pmatrix} i' \\ j' \end{pmatrix}$ by subtracting 1 from each row, we can show that $\Gamma_B$ is isomorphic to a block graph constructed by OA$(2, n-1)$. □

From this decomposition, we can tell that $\Gamma_P$ and $\Gamma_B$ both are type-3 graphs, corresponding to a $\begin{bmatrix} 3 & \\ & 3 \end{bmatrix}$ type structure of the original $\mathbf{A} = A(\Gamma)$. This design structure is akin to the Strongly regular design described by Sankey[1].

# 6 Exploring the Coherent Structure and SRD Properties of Sub-matrices

To understand why the adjacency matrix partitions as shown, we analyze each sub-matrix in relation to its structural and combinatorial properties.

## 6.1 C: Interaction Between Sub-graphs

The sub-matrix $\mathbf{C}$, of dimensions $2(n-1) \times (n-1)^2$, encodes the connections between $\mathbf{A_1}$ and $\mathbf{A_2}$. Splitting $\mathbf{C}$ into two $(n-1) \times (n-1)^2$ matrices, $\mathbf{C_1}$ and $\mathbf{C_2}$, we observe the following patterns:

**$\mathbf{C_1}$:** For $\mathbf{C_1}$, the rows correspond to:

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \ldots, \begin{pmatrix} 1 \\ n \end{pmatrix},$$

while the columns correspond to:

$$\begin{pmatrix} 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \ldots, \begin{pmatrix} 2 \\ n \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \end{pmatrix}, \ldots, \begin{pmatrix} n \\ n \end{pmatrix}.$$

Each row has $n-1$ adjacent vertices based on the bottom coordinate. For instance:

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix} \text{ is adjacent to } \begin{pmatrix} 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \end{pmatrix}, \ldots, \begin{pmatrix} n \\ 2 \end{pmatrix}.$$

This adjacency results in rows of the form:

$$[\underbrace{1 \quad 0 \quad \cdots \quad 0}_{n-1 \text{ elements}} \quad 1 \quad 0 \quad \cdots \quad 0 \quad \cdots].$$

When repeated for $\binom{1}{3}$ onwards, $\mathbf{C_1}$ is composed of $n-1$ blocks of $I_{n-1}$:

$$\mathbf{C_1} = \begin{bmatrix} I_{n-1} & I_{n-1} & \cdots \end{bmatrix}.$$

Explicitly, $\mathbf{C_1}$ looks like:

$$\mathbf{C_1} = \begin{bmatrix} 1 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & \cdots & 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & 1 & \cdots & 0 & \cdots & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \cdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 1 & \cdots & 0 & 0 & \cdots & 1 \end{bmatrix}.$$

**$\mathbf{C_2}$:** For $\mathbf{C_2}$, the rows correspond to:

$$\binom{2}{1}, \binom{3}{1}, \ldots, \binom{n}{1},$$

while the columns remain the same as for $\mathbf{C_1}$. Each row is adjacent to $n-1$ vertices, with 1's being contiguous. For instance:

$$\binom{2}{1} \text{ is adjacent to } \binom{2}{2}, \binom{2}{3}, \ldots, \binom{2}{n}.$$

This results in rows of the form:

$$[\underbrace{1 \quad 1 \quad \cdots \quad 1}_{n-1 \text{ elements}} \quad 0 \quad 0 \quad \cdots].$$

Explicitly, $\mathbf{C_2}$ looks like:

$$\mathbf{C_2} = \begin{bmatrix} 1 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 1 & 1 & \cdots & 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \cdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 1 & 1 & \cdots & 1 \end{bmatrix}.$$

**Combined Matrix C:** Putting $\mathbf{C_1}$ and $\mathbf{C_2}$ together, the complete matrix $\mathbf{C}$ is:

$$\mathbf{C} = \begin{bmatrix} \mathbf{C_1} \\ \mathbf{C_2} \end{bmatrix}.$$

Explicitly:

$$\mathbf{C} = \begin{bmatrix} 1 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & \cdots & 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & 1 & \cdots & 0 & \cdots & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \cdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 1 & \cdots & 0 & 0 & \cdots & 1 \\ \\ 1 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 1 & 1 & \cdots & 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \cdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 1 & 1 & \cdots & 1 \end{bmatrix}.$$

## 6.2 Exploring properties of C

Specifically, we note two patterns of $\mathbf{CC^T}$ and $\mathbf{C^TC}$

$\mathbf{CC^T}$ Upon calculating $\mathbf{CC^T}$, we observe this pattern

$$\begin{aligned}\mathbf{CC^T} &= \begin{bmatrix}(n-1)I_{n-1} & J_{n-1} \\ J_{n-1} & (n-1)I_{n-1}\end{bmatrix} \\ &= (n-2)I_{2(n-1)} + \mathbf{A_1} + J_{2(n-1)}\end{aligned}$$

$\mathbf{C^TC}$ Upon calculating $\mathbf{C^TC}$, we observe this pattern

$$\begin{aligned}\mathbf{C^TC} &= \begin{bmatrix}I_{n-1}+J_{n-1} & I_{n-1} & \cdots & I_{n-1} \\ I_{n-1} & I_{n-1}+J_{n-1} & \cdots & I_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ I_{n-1} & I_{n-1} & \cdots & I_{n-1}+J_{n-1}\end{bmatrix} \\ &= 2I_{(n-1)^2} + \mathbf{A_2}\end{aligned}$$

(In progress of relating it to the equations in the cited paper [1])

## Conclusion

The deletion of one vertex from a rook graph results in a strongly regular design with coherent rank 10. This partitioning of the adjacency matrix into structured submatrices highlights the combinatorial and algebraic regularity inherent in the graph's construction.

## References


[1] A. D. Sankey. On strongly regular designs admitting fusion to strongly regular decomposition. *Journal of Combinatorial Designs*, 29(10), July 2021.