Coherent Closure on Non-Distance Regular Graphs

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Euler's 36 Officers Problem (1782):

- Arrange 36 officers in a 6×6 square.
- 6 ranks $\times$ 6 regiments.
- Each row and column must contain *exactly* one of each rank and regiment.
- Euler conjectured this is impossible (now proven true).

This inspired the study of MOLS, Mutually Orthogonal Latin Squares.

What Are MOLS?

- A **Latin square**, L of order n is a $n \times n$ grid filled with n symbols, with no repeats per row or column.
- Two Latin squares, $L^{(1)}$ and $L^{(2)}$ are **orthogonal** if pairs $(L^{(1)}_{i,j}, L^{(2)}_{i,j})$ are all distinct.
- A set of Latin squares is **mutually orthogonal** if every pair of Latin squares in the set are orthogonal to each other.

MOLS(4)

$$L^{(1)} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 4 & 1 & 2 \\ 4 & 3 & 2 & 1 \end{bmatrix}, \quad L^{(2)} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \\ 2 & 1 & 4 & 3 \\ 3 & 4 & 1 & 2 \end{bmatrix}, \quad L^{(3)} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \\ 4 & 3 & 2 & 1 \\ 2 & 1 & 4 & 3 \end{bmatrix};$$

$$L^{\{1,2\}} = \begin{bmatrix} (1,1) & (2,2) & (3,3) & (4,4) \\ (2,4) & (1,3) & (4,2) & (3,1) \\ (3,2) & (4,1) & (1,4) & (2,3) \\ (4,3) & (3,4) & (2,1) & (1,2) \end{bmatrix}.$$

MOLS(4)

$$L^{(1)} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 4 & 1 & 2 \\ 4 & 3 & 2 & 1 \end{bmatrix}, \quad L^{(2)} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \\ 2 & 1 & 4 & 3 \\ 3 & 4 & 1 & 2 \end{bmatrix}, \quad L^{(3)} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \\ 4 & 3 & 2 & 1 \\ 2 & 1 & 4 & 3 \end{bmatrix};$$

$$L^{\{1,3\}} = \begin{bmatrix} (1,1) & (2,2) & (3,3) & (4,4) \\ (2,3) & (1,4) & (4,1) & (3,2) \\ (3,4) & (4,3) & (1,2) & (2,1) \\ (4,2) & (3,1) & (2,4) & (1,3) \end{bmatrix}.$$

MOLS(4)

$$L^{(1)} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 4 & 1 & 2 \\ 4 & 3 & 2 & 1 \end{bmatrix}, \quad L^{(2)} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \\ 2 & 1 & 4 & 3 \\ 3 & 4 & 1 & 2 \end{bmatrix}, \quad L^{(3)} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \\ 4 & 3 & 2 & 1 \\ 2 & 1 & 4 & 3 \end{bmatrix};$$

$$L^{\{2,3\}} = \begin{bmatrix} (1,1) & (2,2) & (3,3) & (4,4) \\ (4,3) & (3,4) & (2,1) & (1,2) \\ (2,4) & (1,3) & (4,2) & (3,1) \\ (3,2) & (4,1) & (1,4) & (2,3) \end{bmatrix}.$$

What is an Orthogonal Array?

Definition: An **orthogonal array** $\mathrm{OA}(m,n)$ is an $m \times n^2$ array over $[n]$, such that:

- Every pair of rows contains each tuple of $[n] \times [n]$ exactly once.

We can build $\mathrm{OA}(m,n)$ by using a set of $m-2$ $\mathrm{MOLS}(n)$, $\left\{L^{(1)}, \dots, L^{(m-2)}\right\}$, and write their symbols in such a manner:

$$\mathrm{OA}(m,n) = \begin{bmatrix} 1 & 1 & \dots & r & \dots & n & n \\ 1 & 2 & \dots & c & \dots & n-1 & n \\ L^{(1)}_{1,1} & L^{(1)}_{1,2} & \dots & L^{(1)}_{r,c} & \dots & L^{(1)}_{n,n-1} & L^{(1)}_{n,n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ L^{(m-2)}_{1,1} & L^{(m-2)}_{1,2} & \dots & L^{(m-2)}_{r,c} & \dots & L^{(m-2)}_{n,n-1} & L^{(m-2)}_{n,n} \end{bmatrix}.$$

Block Graph

Definition: A **block graph** induced by an orthogonal array $\mathrm{OA}(m, n)$ is a simple graph $G = (V, E)$ with the following properties:

- $|V| = n^2$;
- Vertices are labeled by tuples (r, c) where $r, c \in [n]$;
- Two vertices (r_1, c_1) and (r_2, c_2) are adjacent, i.e., $(r_1, c_1) \sim (r_2, c_2)$, if and only if:

$$\text{the tuples } \left(r_1, c_1, L^{(1)}_{r_1,c_1}, \ldots, L^{(m-2)}_{r_1,c_1}\right) \text{ and } \left(r_2, c_2, L^{(1)}_{r_2,c_2}, \ldots, L^{(m-2)}_{r_2,c_2}\right)$$
$$\text{agree in exactly one coordinate.}$$

Example of OA(3,3)

- Consider this Latin Square of order 3:

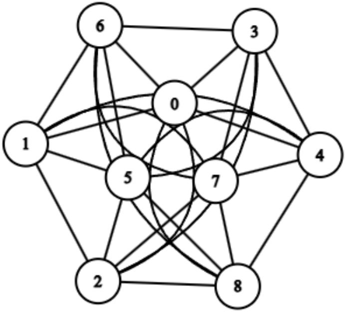
$$L = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{bmatrix}.$$

- Construct OA(3,3) with each column: $\begin{bmatrix} \text{row} \\ \text{column} \\ L_{\text{row,column}} \end{bmatrix}$

$$\text{OA}(3,3) = \begin{bmatrix} 1 & 1 & 1 & 2 & 2 & 2 & 3 & 3 & 3 \\ 1 & 2 & 3 & 1 & 2 & 3 & 1 & 2 & 3 \\ 1 & 2 & 3 & 3 & 1 & 2 & 2 & 3 & 1 \end{bmatrix}.$$

Example of OA(3,3)

OA$(3, 3)$ and its block graph

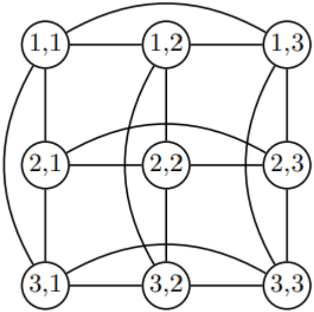


$$\mathrm{OA}(3,3) = \begin{bmatrix} 1 & 1 & 1 & 2 & 2 & 2 & 3 & 3 & 3 \\ 1 & 2 & 3 & 1 & 2 & 3 & 1 & 2 & 3 \\ 1 & 2 & 3 & 3 & 1 & 2 & 2 & 3 & 1 \end{bmatrix}.$$

Base Case: Block Graph of OA(2,3)

This smaller case of a block graph guides our construction of more complex graphs.

$$\mathrm{OA}(2,3) = \begin{bmatrix} 1 & 1 & 1 & 2 & 2 & 2 & 3 & 3 & 3 \\ 1 & 2 & 3 & 1 & 2 & 3 & 1 & 2 & 3 \end{bmatrix}$$



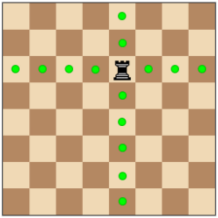
Rook Graphs: Definition

Definition: The **rook graph** $R(n)$ is the block graph induced by $\mathrm{OA}(2, n)$, $n \geq 2$.

- Vertices represent an $n \times n$ chessboard;
- Edges can be interpreted as possible moves for a rook to move on the chessboard.

$$A(R(n)) = \underbrace{\begin{bmatrix} J_n - I & I_n & \cdots & I_n \\ I_n & J_n - I & \cdots & I_n \\ \vdots & \vdots & \ddots & \vdots \\ I_n & I_n & \cdots & J_n - I \end{bmatrix}}_{n \text{ blocks}}.$$

$R(8)$



Coherent Configurations

Let V be a finite set and $\mathcal{R} = \{R_1, \ldots, R_r\}$ be a set of binary relations. For each R_i, let $W_i \in \mathrm{Mat}_V(\{0,1\})$ be defined such that its (x, y) entry is 1 if $(x, y) \in R_i$ and 0 otherwise. Suppose the following 4 conditions

- $\sum_{i=1}^{r} W_i = J$;
- For each $i \in [r]$, there exists $j \in [r]$ such that $W_i^T = W_j$;
- There exists a subset $\Delta \subseteq [r]$ such that $\sum_{i \in \Delta} W_i = I$;
- $W_i W_j = \sum_{k=1}^{r} p_{i,j}^{k} W_k$ for some constants $p_{i,j}^{k} \in \mathbb{Z}_{\geq 0}$, for all $i, j \in [r]$.

Then $(V, \mathcal{R})$ is called a **coherent configuration** of **rank** $|\mathcal{R}| = r$.

Coherent Algebras

Definition: A matrix algebra $\mathcal{A} \subset \mathrm{Mat}(\mathbb{C})$ satisfies the following:

- Spanned by unique basis $\{0,1\}$-matrices: $\{A_1, \ldots, A_r\}$;
- Closed under matrix multiplication, transpose, and Hadamard product;
- $I, J \in \mathcal{A}$.

We say $\mathcal{A}(G)$ is a **coherent algebra** containing the adjacency matrix of G when we talk about coherent algebras in this presentation.

Coherent Algebras

Key Property: If $\mathcal{A}_1$, $\mathcal{A}_2$ are coherent algebras containing the adjacency matrix of a graph G, then $\mathcal{A}' = \mathcal{A}_1 \cap \mathcal{A}_2$ is also a coherent algebra containing the adjacency matrix of a graph G.

Thus, we are motivated to find the minimal coherent algebra, which we call the coherent closure.

Coherent Closure, $\mathcal{W}(G)$

Definition: The minimal coherent algebra containing the adjacency matrix of a graph $G = (V, E)$ is called the **coherent closure**, and denoted as $\mathcal{W}(G)$.

Properties:

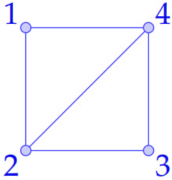
- $A(G) \in \mathcal{W}(G)$;
- Basis matrices, $\{W_1, W_2 \ldots, W_r\}$, represent structural relations in the graph;
- The number of basis matrices in the coherent closure is called the **coherent rank** of G.

$$\mathrm{cr}(G) = |\mathcal{W}(G)| = r.$$

We investigate how this algebra changes under graph operations that disrupt the regularity of a graph.

2-WL Refinement Algorithm

$$A_1 = \begin{bmatrix} a & b & c & b \\ b & a & b & b \\ c & b & a & b \\ b & b & b & a \end{bmatrix}$$



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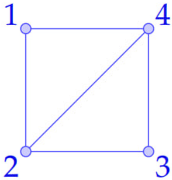

$$A_1^2 = \begin{bmatrix} a^2+2b^2+c^2 & ab+ba+b^2+cb & ac+2b^2+ca & ab+ba+b^2+cb \\ ab+ba+b^2+bc & a^2+3b^2 & ab+ba+b^2+bc & ab+ba+2b^2 \\ ac+2b^2+ca & ab+ba+b^2+cb & a^2+2b^2+c^2 & ab+ba+b^2+cb \\ ab+ba+b^2+bc & ab+ba+2b^2 & ab+ba+b^2+bc & a^2+3b^2 \end{bmatrix}$$

2-WL Refinement Algorithm

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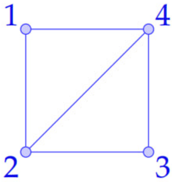

$$A_1^2 = \begin{bmatrix} \color{red}{a^2 + 2b^2 + c^2} & ab + ba + b^2 + cb & ac + 2b^2 + ca & ab + ba + b^2 + cb \\ ab + ba + b^2 + bc & a^2 + 3b^2 & ab + ba + b^2 + bc & ab + ba + 2b^2 \\ ac + 2b^2 + ca & ab + ba + b^2 + cb & \color{red}{a^2 + 2b^2 + c^2} & ab + ba + b^2 + cb \\ ab + ba + b^2 + bc & ab + ba + 2b^2 & ab + ba + b^2 + bc & a^2 + 3b^2 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} \color{red}{a} & & & \\ & & & \\ & & \color{red}{a} & \\ & & & \end{bmatrix}$$

2-WL Refinement Algorithm

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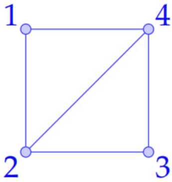

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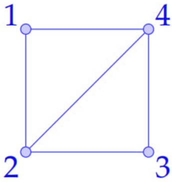

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$$A_2 = \begin{bmatrix} a & b & c & b \\ d & e & d & f \\ c & b & a & b \\ d & f & d & e \end{bmatrix}$$

2-WL Refinement Algorithm

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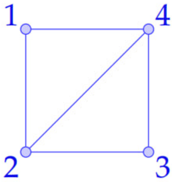

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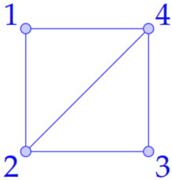

$$A_2^2 = \begin{bmatrix} a^2 + 2bd + c^2 & ab + be + bf + cb & ac + 2bd + ca & ab + be + bf + cb \\ da + dc + ed + fd & 2db + e^2 + f^2 & da + dc + ed + fd & 2db + ef + fe \\ ac + 2bc + ca & ab + be + bf + cb & a^2 + 2bd + c^2 & ab + be + bf + cb \\ da + dc + ed + fd & 2db + ef + fe & da + dc + ed + fd & 2db + e^2 + f^2 \end{bmatrix}$$

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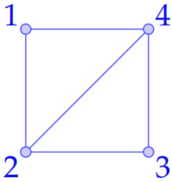

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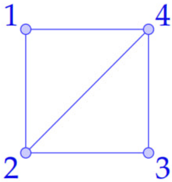
$$A_3 = \begin{bmatrix} a & b & c & b \\ d & e & d & f \\ c & b & a & b \\ d & f & d & e \end{bmatrix}$$

$$A_2 = \begin{bmatrix} a & b & c & b \\ d & e & d & f \\ c & b & a & b \\ d & f & d & e \end{bmatrix}$$



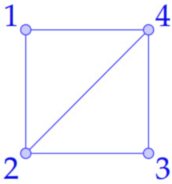
$$A_2^2 = \begin{bmatrix} a^2 + 2bd + c^2 & ab + be + bf + cb & ac + 2bd + ca & ab + be + bf + cb \\ da + dc + ed + fd & 2db + e^2 + f^2 & da + dc + ed + fd & 2db + ef + fe \\ ac + 2bc + ca & ab + be + bf + cb & a^2 + 2bd + c^2 & ab + be + bf + cb \\ da + dc + ed + fd & 2db + ef + fe & da + dc + ed + fd & 2db + e^2 + f^2 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} a & b & c & b \\ d & e & d & f \\ c & b & a & b \\ d & f & d & e \end{bmatrix} = A_2$$

2-WL Refinement Algorithm

$$\mathcal{W}(\Gamma) = \begin{bmatrix} a & b & c & b \\ d & e & d & f \\ c & b & a & b \\ d & f & d & e \end{bmatrix}$$

Γ



$$\overset{a}{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}} \quad \overset{b}{\begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}} \quad \overset{c}{\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}} \quad \overset{d}{\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}} \quad \overset{e}{\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}} \quad \overset{f}{\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}}$$

We say "Γ has **coherent rank** 6".

Choice of Base Graph

We want graphs with a known coherent rank, such as **strongly regular graphs**.

Strongly regular graphs are a class of simple graphs $G = (V, E)$, with the following properties.

- Total vertices: $|V| = v$;
- Each vertex has degree k;
- For every pair of adjacent vertices, they share λ common adjacent vertices;
- For every pair of non-adjacent vertices, they share μ common adjacent vertices;
- Known coherent rank of 3, with the coherent closure having basis matrices $\langle I, A, J - I - A \rangle$.

From here, we refer to such strongly regular graphs with the parameters above as $\mathrm{SRG}(v, k, \lambda, \mu)$.

Choice of Base Graph

Base Graph: The Rook graph $R(n)$, induced from OA(2,n).

Properties

- $|V| = n^2, |E| = 2(n-1)$;
- Strongly regular with parameters SRG$(n^2, 2(n-1), n-2, 2)$;
- Interpreted as the possible moves of a Rook on a $n \times n$ chessboard.

Graph Operations Studied

We investigate how the coherent closure $\mathcal{W}$ containing the adjacency matrix of a graph $G = (V, E)$ evolves under two structural graph modifications:

1. **Vertex Deletion**,
2. **Seidel Switching**.

Graph Operation: Vertex Deletion

Vertex Deletion:

- Choose a vertex $v \in V$,
- Remove v and all edges incident to it,
- The resulting graph is $G' = (V \setminus \{v\}, E')$.

Adjacency Matrix Perspective:

$$A(G) = \begin{bmatrix} 0 & \vec{a}^T \\ \vec{a} & A_{11} \end{bmatrix} \quad \Rightarrow \quad A(G') = A_{11}$$

where A_{11} is the adjacency matrix of the resulting graph, and $\vec{a}$ represents the adjacency between v and $V \setminus \{v\}$.

Graph Operation: Seidel Switching

Seidel Switching:

- Choose a subset $S \subseteq V$,
- Flip adjacency between S and $V \setminus S$,
- Edges within S and within $V \setminus S$ are unchanged.

Adjacency Matrix Perspective:

$$A(G) = \begin{bmatrix} A(G^S) & C \\ C^T & A(G^{V\setminus S}) \end{bmatrix} \quad \Rightarrow \quad A(G') = \begin{bmatrix} A(G^S) & J - C \\ J - C^T & A(G^{V\setminus S}) \end{bmatrix}$$

where $A(G^S)$ represents edges in the set S, $A(G^{V\setminus S})$ represents edges in the set $V \setminus S$, and C represents the adjacency between vertices in sets S and $V \setminus S$.

Theorem (Wielandt's Principle)

Let $\mathcal{A}$ be a coherent algebra and let $A \in \mathcal{A}$. For $b \in \mathbb{C}$, define the matrix B such that

$$[B]_{xy} = \begin{cases} 1, & \text{if } [A]_{xy} = b; \\ 0, & \text{otherwise.} \end{cases}$$

then, $B \in \mathcal{A}$.

Process of finding coherent closure

Showing upper bound:

1. Use 2-WL refinement algorithm for small n,
2. Notice pattern and generalise into a set of basis matrices and show they form a coherent algebra, $\mathcal{A}$.

Showing Lower bound:

1. Use the Wielandt's Principle to show that a certain number of basis matrices need to exist in the coherent closure,
2. Show that the minimum number of basis matrices is equal to $|\mathcal{A}|$, and thus $\mathcal{W} = \mathcal{A}$.

Summary of Coherent Rank and Types

Graph Operation	Coherent Rank	Type
Vertex Deletion	10	$\begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$
Switching on 1 Vertex	15	$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & 2 \\ 1 & 2 & 3 \end{bmatrix}$
Switching on 1 n-clique	10	$\begin{bmatrix} 2 & 2 \\ 2 & 4 \end{bmatrix}$
Switching on $1 < k < \frac{n}{2}$ n-cliques	12	$\begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$
Switching on $\frac{n}{2}$ n-cliques	6	$[6]$

$$A(R(n)) = \begin{bmatrix} J_n - I & I_n & \cdots & I_n \\ I_n & J_n - I & \cdots & I_n \\ \vdots & \vdots & \ddots & \vdots \\ I_n & I_n & \cdots & J_n - I \end{bmatrix}$$

Interpreting Coherent Rank Patterns

Other graphs considered:

Block Graph of OA$(3, n)$

```
n:3, rank 7
n:4, rank 24
n:5, rank 28
n:6, rank 116
n:7, rank 74
n:8, rank 200
n:9, rank 194
n:10, rank 430
n:11, rank 250
n:12, rank 974
n:13, rank 412
n:14, rank 1090
n:15, rank 1112
n:16, rank 1516
n:17, rank 880
n:18, rank 3024
n:19, rank 1230
```

Triangular Graph, $T(n)$

```
n:3, rank 2
n:4, rank 6
n:5, rank 10
n:6, rank 11
n:7, rank 11
n:8, rank 11
n:9, rank 11
n:10, rank 11
n:11, rank 11
n:12, rank 11
n:13, rank 11
n:14, rank 11
n:15, rank 11
n:16, rank 11
n:17, rank 11
n:18, rank 11
n:19, rank 11
```

Paley Graph, $P(n)$

```
n:5, rank 8
n:13, rank 24
n:17, rank 32
n:29, rank 56
n:37, rank 72
n:41, rank 80
n:53, rank 104
n:61, rank 120
n:73, rank 144
n:89, rank 176
n:97, rank 192
n:101, rank 200
n:109, rank 216
n:113, rank 224
```

Conclusion and Future Directions

Summary:

- Studied coherent closures of modified rook graphs;
- Applied vertex deletion and Seidel switching;
- Observed fixed coherent rank and algebraic structure under graph operations.

Future Work:

- Extend analysis to larger k, higher-order $\mathrm{OA}(k, n)$;
- Extend analysis to other strongly regular graphs, such as Triangular graphs or Paley graphs.

Thank you! I welcome your questions.