$$\underset{\substack{R1+(-2)R2 \\ R3+(-1)R2}}{\approx} \begin{bmatrix} 1 & 0 & -1 & -1 & 2 & 4 \\ 0 & 1 & 0 & 0 & 2 & -2 \\ 0 & 0 & 2 & 2 & -2 & -2 \end{bmatrix} \underset{(1/2)R3}{\approx} \begin{bmatrix} 1 & 0 & -1 & -1 & 2 & 4 \\ 0 & 1 & 0 & 0 & 2 & -2 \\ 0 & 0 & 1 & 1 & -1 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 3 \\ 0 & 1 & 0 & 0 & 2 & -2 \\ 0 & 0 & 1 & 1 & -1 & -1 \end{bmatrix},$$

so the solutions are in turn $x_1 = 0$, $x_2 = 0$, $x_3 = 1$; $x_1 = 1$, $x_2 = 2$, $x_3 = -1$; and $x_1 = 3$, $x_2 = -2$, $x_3 = -1$.

Exercise Set 1.2

1. (a) Yes. (b) Yes.

   (c) No. The second column contains a leading 1, so other elements in that column should be zero.

   (d) No. The second row does not have 1 as the first nonzero number.

   (e) Yes. (f) Yes. (g) Yes.

   (h) No. The second row does not have 1 as the first nonzero number. (i) Yes.

2. (a) No. The leading 1 in row 3 is not to the right of the leading 1 in row 2.

   (b) Yes. (c) Yes.

   (d) No. The fourth and fifth columns contain leading 1s, so the other numbers in those columns should be zeros.

   (e) No. The row containing all zeros should be at the bottom of the matrix.

   (f) Yes.

   (g) No. The leading 1 in row 3 is not to the right of the leading 1 in row 2. Also, since column 3 contains a leading 1, all other numbers in that column should be zero.

   (h) No. The leading 1 in row 3 is not to the right of the leading 1s in rows 1 and 2.

(i) Yes.

3. (a) $x_1 = 2,\ x_2 = 4,\ x_3 = -3.$ (b) $x_1 = 3r + 4,\ x_2 = -2r + 8,\ x_3 = r.$

(c) $x_1 = -3r + 6,\ x_2 = r,\ x_3 = -2.$ (d) There is no solution. The last row gives 0 = 1.

(e) $x_1 = -5r + 3,\ x_2 = -6r - 2,\ x_3 = -2r - 4,\ x_4 = r.$

(f) $x_1 = -3r + 2,\ x_2 = r,\ x_3 = 4,\ x_4 = 5.$

4. (a) $x_1 = -2r - 4s + 1,\ x_2 = 3r - 5s - 6,\ x_3 = r,\ x_4 = s.$

(b) $x_1 = 3r - 2s + 4,\ x_2 = r,\ x_3 = s,\ x_4 = -7.$

(c) $x_1 = 2r - 3s + 4,\ x_2 = r,\ x_3 = -2s + 9,\ x_4 = s,\ x_5 = 8.$

(d) $x_1 = -2r - 3s + 6,\ x_2 = -5r - 4s + 7,\ x_3 = r,\ x_4 = -9s - 3,\ x_5 = s.$

5. (a) $$\begin{bmatrix}1 & 4 & 3 & 1\\ 2 & 8 & 11 & 7\\ 1 & 6 & 7 & 3\end{bmatrix} \underset{\substack{R2+(-2)R1\\ R3+(-1)R1}}{\approx} \begin{bmatrix}1 & 4 & 3 & 1\\ 0 & 0 & 5 & 5\\ 0 & 2 & 4 & 2\end{bmatrix} \underset{R2 \Leftrightarrow R3}{\approx} \begin{bmatrix}1 & 4 & 3 & 1\\ 0 & 2 & 4 & 2\\ 0 & 0 & 5 & 5\end{bmatrix}$$

$$\underset{(1/2)R2}{\approx} \begin{bmatrix}1 & 4 & 3 & 1\\ 0 & 1 & 2 & 1\\ 0 & 0 & 5 & 5\end{bmatrix} \underset{R1+(-4)R2}{\approx} \begin{bmatrix}1 & 0 & -5 & -3\\ 0 & 1 & 2 & 1\\ 0 & 0 & 5 & 5\end{bmatrix} \underset{(1/5)R3}{\approx} \begin{bmatrix}1 & 0 & -5 & -3\\ 0 & 1 & 2 & 1\\ 0 & 0 & 1 & 1\end{bmatrix}$$

$$\underset{\substack{R1+(5)R3\\ R2+(-2)R3}}{\approx} \begin{bmatrix}1 & 0 & 0 & 2\\ 0 & 1 & 0 & -1\\ 0 & 0 & 1 & 1\end{bmatrix}$$, so the solution is $x_1 = 2,\ x_2 = -1,\ x_3 = 1.$

(b) $$\begin{bmatrix}1&2&4&15\\2&4&9&33\\1&3&5&20\end{bmatrix}\underset{\substack{R2+(-2)R1\\R3+(-1)R1}}{\approx}\begin{bmatrix}1&2&4&15\\0&0&1&3\\0&1&1&5\end{bmatrix}\underset{R2\Leftrightarrow R3}{\approx}\begin{bmatrix}1&2&4&15\\0&1&1&5\\0&0&1&3\end{bmatrix}$$

$$\underset{R1+(-2)R2}{\approx}\begin{bmatrix}1&0&2&5\\0&1&1&5\\0&0&1&3\end{bmatrix}\underset{\substack{R1+(-2)R3\\R2+(-1)R3}}{\approx}\begin{bmatrix}1&0&0&-1\\0&1&0&2\\0&0&1&3\end{bmatrix},$$

so the solution is $x_1 = -1$, $x_2 = 2$, $x_3 = 3$.

(c) $$\begin{bmatrix}1&1&1&7\\2&3&1&18\\-1&1&-3&1\end{bmatrix}\underset{\substack{R2+(-2)R1\\R3+R1}}{\approx}\begin{bmatrix}1&1&1&7\\0&1&-1&4\\0&2&-2&8\end{bmatrix}\underset{\substack{R1+(-1)R2\\R3+(-2)R3}}{\approx}\begin{bmatrix}1&0&2&3\\0&1&-1&4\\0&0&0&0\end{bmatrix},$$

so $x_1 + 2x_3 = 3$ and $x_2 - x_3 = 4$.
Thus the general solution is $x_1 = 3 - 2r$, $x_2 = 4 + r$, $x_3 = r$.

(d) $$\begin{bmatrix}1&4&1&2\\1&2&-1&0\\2&6&0&3\end{bmatrix}\underset{\substack{R2+(-1)R1\\R3+(-2)R1}}{\approx}\begin{bmatrix}1&4&1&2\\0&-2&-2&-2\\0&-2&-2&-1\end{bmatrix}\underset{(-1/2)R2}{\approx}\begin{bmatrix}1&4&1&2\\0&1&1&1\\0&-2&-2&-1\end{bmatrix}$$

$$\underset{\substack{R1+(-4)R2\\R3+(2)R2}}{\approx}\begin{bmatrix}1&0&-3&-2\\0&1&1&1\\0&0&0&1\end{bmatrix},$$ so there is no solution, since the last row of the

matrix corresponds to the equation 0 = 1.

(e) $$\begin{bmatrix}1&-1&1&3\\2&-1&4&7\\3&-5&-1&7\end{bmatrix}\underset{\substack{R2+(-2)R1\\R3+(-3)R1}}{\approx}\begin{bmatrix}1&-1&1&3\\0&1&2&1\\0&-2&-4&-2\end{bmatrix}\underset{\substack{R1+R2\\R3+(2)R2}}{\approx}\begin{bmatrix}1&0&3&4\\0&1&2&1\\0&0&0&0\end{bmatrix},$$

so $x_1 + 3x_3 = 4$ and $x_2 + 2x_3 = 1$.
Thus the general solution is $x_1 = 4 - 3r$, $x_2 = 1 - 2r$, $x_3 = r$.

(f) $\begin{bmatrix} 3 & -3 & 9 & 24 \\ 2 & -2 & 7 & 17 \\ -1 & 2 & -4 & -11 \end{bmatrix} \underset{(1/3)R1}{\approx} \begin{bmatrix} 1 & -1 & 3 & 8 \\ 2 & -2 & 7 & 17 \\ -1 & 2 & -4 & -11 \end{bmatrix} \underset{\substack{R2+(-2)R1 \\ R2+R1}}{\approx} \begin{bmatrix} 1 & -1 & 3 & 8 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & -1 & -3 \end{bmatrix}$

$\underset{R2 \Leftrightarrow R3}{\approx} \begin{bmatrix} 1 & -1 & 3 & 8 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 1 & 1 \end{bmatrix} \underset{R1+R2}{\approx} \begin{bmatrix} 1 & 0 & 2 & 5 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 1 & 1 \end{bmatrix} \underset{\substack{R1+(-2)R3 \\ R2+R3}}{\approx} \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \end{bmatrix}$,

so the solution is $x_1 = 3$, $x_2 = -2$, $x_3 = 1$.

6. (a) $\begin{bmatrix} 3 & 6 & -3 & 6 \\ -2 & -4 & -3 & -1 \\ 3 & 6 & -2 & 10 \end{bmatrix} \underset{(1/3)R1}{\approx} \begin{bmatrix} 1 & 2 & -1 & 2 \\ -2 & -4 & -3 & -1 \\ 3 & 6 & -2 & 10 \end{bmatrix} \underset{\substack{R2+(2)R1 \\ R3+(-3)R1}}{\approx} \begin{bmatrix} 1 & 2 & -1 & 2 \\ 0 & 0 & -5 & 3 \\ 0 & 0 & 1 & 4 \end{bmatrix}$

It is now clear that there is no solution. The last two rows give $-5x_3 = 3$ and $x_3 = 4$.

(b) $\begin{bmatrix} 1 & 2 & 1 & 7 \\ 1 & 2 & 2 & 11 \\ 2 & 4 & 3 & 18 \end{bmatrix} \underset{\substack{R2+(-1)R1 \\ R3+(-2)R1}}{\approx} \begin{bmatrix} 1 & 2 & 1 & 7 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 1 & 4 \end{bmatrix} \underset{\substack{R1+(-1)R2 \\ R3+(-1)R2}}{\approx} \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

so $x_1 + 2x_2 = 3$ and $x_3 = 4$. Thus the general solution is
$x_1 = 3 - 2r$, $x_2 = r$, $x_3 = 4$.

(c) $\begin{bmatrix} 1 & 2 & -1 & 3 \\ 2 & 4 & -2 & 6 \\ 3 & 6 & 2 & -1 \end{bmatrix} \underset{\substack{R2+(-2)R1 \\ R3+(-3)R1}}{\approx} \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & -10 \end{bmatrix} \underset{R2 \Leftrightarrow R3}{\approx} \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 0 & 5 & -10 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\underset{(1/5)R2}{\approx} \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \underset{R1+R2}{\approx} \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, so $x_1 + 2x_2 = 1$, $x_3 = -2$.

Thus the general solution is $x_1 = 1 - 2r$, $x_2 = r$, $x_3 = -2$.

(d) $\begin{bmatrix}1&2&3&8\\3&7&9&26\\2&0&6&11\end{bmatrix}\underset{\substack{R2+(-3)R1\\R3+(-2)R1}}{\approx}\begin{bmatrix}1&2&3&8\\0&1&0&2\\0&-4&0&-5\end{bmatrix}$, so there is no solution since the

last two rows give $x_2 = 2$ and $-4x_2 = -5$.

(e) $\begin{bmatrix}0&1&2&5\\1&2&5&13\\1&0&2&4\end{bmatrix}\underset{R1\Leftrightarrow R2}{\approx}\begin{bmatrix}1&2&5&13\\0&1&2&5\\1&0&2&4\end{bmatrix}\underset{R3+(-1)R1}{\approx}\begin{bmatrix}1&2&5&13\\0&1&2&5\\0&-2&-3&-9\end{bmatrix}$

$\underset{\substack{R1+(-2)R2\\R3+(2)R2}}{\approx}\begin{bmatrix}1&0&1&3\\0&1&2&5\\0&0&1&1\end{bmatrix}\underset{\substack{R1+(-1)R3\\R2+(-2)R3}}{\approx}\begin{bmatrix}1&0&0&2\\0&1&0&3\\0&0&1&1\end{bmatrix}$,

so the solution is $x_1 = 2$, $x_2 = 3$, $x_3 = 1$.

(f) $\begin{bmatrix}1&2&8&7\\2&4&16&14\\0&1&3&4\end{bmatrix}\underset{R2+(-2)R1}{\approx}\begin{bmatrix}1&2&8&7\\0&0&0&0\\0&1&3&4\end{bmatrix}\underset{R2\Leftrightarrow R3}{\approx}\begin{bmatrix}1&2&8&7\\0&1&3&4\\0&0&0&0\end{bmatrix}$

$\underset{R1+(-2)R2}{\approx}\begin{bmatrix}1&0&2&-1\\0&1&3&4\\0&0&0&0\end{bmatrix}$, so $x_1 + 2x_3 = -1$, $x_2 + 3x_3 = 4$.

Thus the general solution is $x_1 = -1 - 2r$, $x_2 = 4 - 3r$, $x_3 = r$.

7. (a) $\begin{bmatrix}1&1&-3&10\\-3&-2&4&-24\end{bmatrix}\underset{R2+(3)R1}{\approx}\begin{bmatrix}1&1&-3&10\\0&1&-5&6\end{bmatrix}\underset{R1+(-1)R2}{\approx}\begin{bmatrix}1&0&2&4\\0&1&-5&6\end{bmatrix}$,

so $x_1 + 2x_3 = 4$ and $x_2 - 5x_3 = 6$. Thus the general solution is
$x_1 = 4 - 2r$, $x_2 = 6 + 5r$, $x_3 = r$.

(b) $\begin{bmatrix}2&-6&-14&38\\-3&7&15&-37\end{bmatrix}\underset{(1/2)R1}{\approx}\begin{bmatrix}1&-3&-7&19\\-3&7&15&-37\end{bmatrix}\underset{R2+(3)R1}{\approx}\begin{bmatrix}1&-3&-7&19\\0&-2&-6&20\end{bmatrix}$

$$\underset{(-1/2)R2}{\approx}\begin{bmatrix}1 & -3 & -7 & 19\\ 0 & 1 & 3 & -10\end{bmatrix}\underset{R1+(3)R2}{\approx}\begin{bmatrix}1 & 0 & 2 & -11\\ 0 & 1 & 3 & -10\end{bmatrix},$$

so $x_1 + 2x_3 = -11$ and $x_2 + 3x_3 = -10$. Thus the general solution is $x_1 = -11 - 2r$, $x_2 = -10 - 3r$, $x_3 = r$.

(c) $$\begin{bmatrix}1 & 2 & -1 & -1 & 0\\ 1 & 2 & 0 & 1 & 4\\ -1 & -2 & 2 & 4 & 5\end{bmatrix}\underset{\substack{R2+(-1)R1\\ R3+R1}}{\approx}\begin{bmatrix}1 & 2 & -1 & -1 & 0\\ 0 & 0 & 1 & 2 & 4\\ 0 & 0 & 1 & 3 & 5\end{bmatrix}\underset{\substack{R1+R2\\ R3+(-1)R2}}{\approx}\begin{bmatrix}1 & 2 & 0 & 1 & 4\\ 0 & 0 & 1 & 2 & 4\\ 0 & 0 & 0 & 1 & 1\end{bmatrix}$$

$$\underset{\substack{R1+(-1)R3\\ R2+(-2)R3}}{\approx}\begin{bmatrix}1 & 2 & 0 & 0 & 3\\ 0 & 0 & 1 & 0 & 2\\ 0 & 0 & 0 & 1 & 1\end{bmatrix},$$ so $x_1 + 2x_2 = 3$ and $x_3 = 2$, and $x_4 = 1$.

Thus the general solution is $x_1 = 3 - 2r$, $x_2 = r$, $x_3 = 2$, and $x_4 = 1$.

(d) $$\begin{bmatrix}1 & 2 & 0 & 4 & 0\\ -2 & -4 & 3 & -2 & 0\end{bmatrix}\underset{R2+(2)R1}{\approx}\begin{bmatrix}1 & 2 & 0 & 4 & 0\\ 0 & 0 & 3 & 6 & 0\end{bmatrix}\underset{(1/3)R2}{\approx}\begin{bmatrix}1 & 2 & 0 & 4 & 0\\ 0 & 0 & 1 & 2 & 0\end{bmatrix},$$

so $x_1 + 2x_2 + 4x_4 = 0$ and $x_3 + 2x_4 = 0$. Thus the general solution is $x_1 = -2r - 4s$, $x_2 = r$, $x_3 = -2s$, $x_4 = s$.

(e) $$\begin{bmatrix}0 & 1 & -3 & 1 & 0\\ 1 & 1 & -1 & 4 & 0\\ -2 & -2 & 2 & -8 & 0\end{bmatrix}\underset{R1\Leftrightarrow R2}{\approx}\begin{bmatrix}1 & 1 & -1 & 4 & 0\\ 0 & 1 & -3 & 1 & 0\\ -2 & -2 & 2 & -8 & 0\end{bmatrix}$$

$$\underset{R3+(2)R1}{\approx}\begin{bmatrix}1 & 1 & -1 & 4 & 0\\ 0 & 1 & -3 & 1 & 0\\ 0 & 0 & 0 & 0 & 0\end{bmatrix}\underset{R1+(-1)R2}{\approx}\begin{bmatrix}1 & 0 & 2 & 3 & 0\\ 0 & 1 & -3 & 1 & 0\\ 0 & 0 & 0 & 0 & 0\end{bmatrix},$$

so $x_1 + 2x_3 + 3x_4 = 0$ and $x_2 - 3x_3 + x_4 = 0$. Thus the general solution is $x_1 = -2r - 3s$, $x_2 = 3r - s$, $x_3 = r$, $x_4 = s$.

8. (a) $$\begin{bmatrix}1 & 1 & 1 & -1 & -3\\ 2 & 3 & 1 & -5 & -9\\ 1 & 3 & -1 & -6 & -7\\ -1 & -1 & -1 & 0 & 1\end{bmatrix}\underset{\substack{R2+(-2)R1\\ R3+(-1)R1\\ R4+R1}}{\approx}\begin{bmatrix}1 & 1 & 1 & -1 & -3\\ 0 & 1 & -1 & -3 & -3\\ 0 & 2 & -2 & -5 & -4\\ 0 & 0 & 0 & -1 & -2\end{bmatrix}$$

$$\underset{\substack{R1+(-1)R2\\R3+(-2)R2}}{\approx}\begin{bmatrix}1&0&2&2&0\\0&1&-1&-3&-3\\0&0&0&1&2\\0&0&0&-1&-2\end{bmatrix}\underset{\substack{R1+(2)R3\\R2+(-3)R3\\R4+R3}}{\approx}\begin{bmatrix}1&0&2&0&-4\\0&1&-1&0&3\\0&0&0&1&2\\0&0&0&0&0\end{bmatrix},$$

so $x_1 + 2x_3 = -4$, $x_2 - x_3 = 3$, $x_4 = 2$.
The general solution is $x_1 = -2r - 4$, $x_2 = r + 3$, $x_3 = r$, $x_4 = 2$.

(b) $$\begin{bmatrix}0&1&2&7\\1&-2&-6&-18\\-1&-1&-2&-5\\2&-5&-15&-46\end{bmatrix}\underset{R1\Leftrightarrow R2}{\approx}\begin{bmatrix}1&-2&-6&-18\\0&1&2&7\\-1&-1&-2&-5\\2&-5&-15&-46\end{bmatrix}$$

$$\underset{\substack{R3+R1\\R4+(-2)R1}}{\approx}\begin{bmatrix}1&-2&-6&-18\\0&1&2&7\\0&-3&-8&-23\\0&-1&-3&-10\end{bmatrix}\underset{\substack{R1+(2)R2\\R3+(3)R2\\R4+R2}}{\approx}\begin{bmatrix}1&0&-2&4\\0&1&2&7\\0&0&-2&-2\\0&0&-1&-3\end{bmatrix},$$ and there are no

solutions because the last two rows of the matrix give, respectively,
$x_3 = 1$ and $x_3 = 3$.

(c) $$\begin{bmatrix}2&-4&16&-14&10\\-1&5&-17&19&-2\\1&-3&11&-11&4\\3&-4&18&-13&17\end{bmatrix}\underset{(1/2)R1}{\approx}\begin{bmatrix}1&-2&8&-7&5\\-1&5&-17&19&-2\\1&-3&11&-11&4\\3&-4&18&-13&17\end{bmatrix}$$

$$\underset{\substack{R2+R1\\R3+(-1)R1\\R4+(-3)R1}}{\approx}\begin{bmatrix}1&-2&8&-7&5\\0&3&-9&12&3\\0&-1&3&-4&-1\\0&2&-6&8&2\end{bmatrix}\underset{(1/3)R2}{\approx}\begin{bmatrix}1&-2&8&-7&5\\0&1&-3&4&1\\0&-1&3&-4&-1\\0&2&-6&8&2\end{bmatrix}$$

$$\underset{\substack{R1+(2)R2\\R3+R2\\R4+(-2)R2}}{\approx}\begin{bmatrix}1&0&2&1&7\\0&1&-3&4&1\\0&0&0&0&0\\0&0&0&0&0\end{bmatrix},$$ so $x_1+2x_3+x_4 = 7$ and $x_2-3x_3+4x_4=1$.

Thus the general solution is $x_1 = 7 - 2r - s$, $x_2 = 1 + 3r - 4s$, $x_3 = r$, $x_4 = s$.

(d) $$\begin{bmatrix}1 & -1 & 2 & 0 & 7\\ 2 & -2 & 2 & -4 & 12\\ -1 & 1 & -1 & 2 & -4\\ -3 & 1 & -8 & -10 & -29\end{bmatrix}\begin{matrix}\approx\\ R2+(-2)R1\\ R3+R1\\ R4+(3)R1\end{matrix}\begin{bmatrix}1 & -1 & 2 & 0 & 7\\ 0 & 0 & -2 & -4 & -2\\ 0 & 0 & 1 & 2 & 3\\ 0 & -2 & -2 & -10 & -8\end{bmatrix}$$

$$\begin{matrix}\approx\\ R2\Leftrightarrow R4\end{matrix}\begin{bmatrix}1 & -1 & 2 & 0 & 7\\ 0 & -2 & -2 & -10 & -8\\ 0 & 0 & 1 & 2 & 3\\ 0 & 0 & -2 & -4 & -2\end{bmatrix}\begin{matrix}\approx\\ (-1/2)R2\end{matrix}\begin{bmatrix}1 & -1 & 2 & 0 & 7\\ 0 & 1 & 1 & 5 & 4\\ 0 & 0 & 1 & 2 & 3\\ 0 & 0 & -2 & -4 & -2\end{bmatrix}$$

$$\begin{matrix}\approx\\ R1+R2\end{matrix}\begin{bmatrix}1 & 0 & 3 & 5 & 11\\ 0 & 1 & 1 & 5 & 4\\ 0 & 0 & 1 & 2 & 3\\ 0 & 0 & -2 & -4 & -2\end{bmatrix}\begin{matrix}\approx\\ R1+(-3)R3\\ R2+(-1)R3\\ R4+(2R3)\end{matrix}\begin{bmatrix}1 & 0 & 0 & -1 & 2\\ 0 & 1 & 0 & 3 & 1\\ 0 & 0 & 1 & 2 & 3\\ 0 & 0 & 0 & 0 & 4\end{bmatrix}$$

The last row gives 0 = 4, so there is no solution.

(e) $$\begin{bmatrix}1 & 6 & -1 & -4 & 0\\ -2 & -12 & 5 & 17 & 0\\ 3 & 18 & -1 & -6 & 0\end{bmatrix}\begin{matrix}\approx\\ R2+(2)R1\\ R3+(-3)R1\end{matrix}\begin{bmatrix}1 & 6 & -1 & -4 & 0\\ 0 & 0 & 3 & 9 & 0\\ 0 & 0 & 2 & 6 & 0\end{bmatrix}$$

$$(1/3)R2\begin{bmatrix}1 & 6 & -1 & -4 & 0\\ 0 & 0 & 1 & 3 & 0\\ 0 & 0 & 2 & 6 & 0\end{bmatrix}\begin{matrix}\approx\\ R1+R2\\ R3+(-2)R2\end{matrix}\begin{bmatrix}1 & 6 & 0 & -1 & 0\\ 0 & 0 & 1 & 3 & 0\\ 0 & 0 & 0 & 0 & 0\end{bmatrix},$$

so $x_1 + 6x_2 - x_4 = 0$ and $x_3 + 3x_4 = 0$.
Thus the general solution is $x_1 = -6r + s$, $x_2 = r$, $x_3 = -3s$, $x_4 = s$.

(f) $$\begin{bmatrix}4 & 8 & -12 & 28\\ -1 & -2 & 3 & -7\\ 2 & 4 & -8 & 16\\ -3 & -6 & 9 & -21\end{bmatrix}\begin{matrix}\approx\\ (1/4)R1\end{matrix}\begin{bmatrix}1 & 2 & -3 & 7\\ -1 & -2 & 3 & -7\\ 2 & 4 & -8 & 16\\ -3 & -6 & 9 & -21\end{bmatrix}$$

$$\begin{matrix}\approx\\ R2+R1\\ R3+(-2)R1\\ R4+(3)R1\end{matrix}\begin{bmatrix}1 & 2 & -3 & 7\\ 0 & 0 & 0 & 0\\ 0 & 0 & -2 & 2\\ 0 & 0 & 0 & 0\end{bmatrix}\begin{matrix}\approx\\ R2\Leftrightarrow R3\end{matrix}\begin{bmatrix}1 & 2 & -3 & 7\\ 0 & 0 & -2 & 2\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}$$

$$\underset{(-1/2)R2}{\approx}\begin{bmatrix}1 & 2 & -3 & 7\\ 0 & 0 & 1 & -1\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}\underset{R1+(3)R2}{\approx}\begin{bmatrix}1 & 2 & 0 & 4\\ 0 & 0 & 1 & -1\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}$$, so $x_1 + 2x_2 = 4$ and $x_3 = -1$.

Thus the general solution is $x_1 = 4 - 2r$, $x_2 = r$, $x_3 = -1$.

(g) $$\begin{bmatrix}1 & 1 & 2\\ 2 & 3 & 3\\ 1 & 3 & 0\\ 1 & 2 & 1\end{bmatrix}\underset{\substack{R2+(-2)R1\\ R3+(-1)R1\\ R4+(-1)R1}}{\approx}\begin{bmatrix}1 & 1 & 2\\ 0 & 1 & -1\\ 0 & 2 & -2\\ 0 & 1 & -1\end{bmatrix}\underset{\substack{R1+(-1)R2\\ R3+(-2)R2\\ R4+(-1)R2}}{\approx}\begin{bmatrix}1 & 0 & 3\\ 0 & 1 & -1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix},$$

so $x_1 = 3$, $x_2 = -1$.

9. (a) The system of equations

$$\begin{aligned}3x_1 + 2x_2 - x_3 + x_4 &= 4\\ 3x_1 + 2x_2 - x_3 + x_4 &= 1\end{aligned}$$

clearly has no solution, since the equations are inconsistent. To make a system that is less obvious, add another equation to the system and perform an elementary transformation on this new system of three equations. For example, replace the second equation by the sum of the second equation and some multiple (2 in the example below) of the third equation:

$$\begin{aligned}3x_1 + 2x_2 - x_3 + x_4 &= 4\\ 5x_1 + 4x_2 - x_3 - x_4 &= 1\\ x_1 + x_2 - x_4 &= 0\end{aligned}$$

(b) Choose a solution, e.g., $x_1 = 1$, $x_2 = 2$. Now make up equations thinking of x_1 as 1 and x_2 as 2:

$$\begin{aligned}x_1 + x_2 &= 3\\ x_1 + 2x_2 &= 5\\ x_1 - 2x_2 &= -3\end{aligned}$$

An easy way to ensure that there are no additional solutions is to include $x_1 = 1$ or $x_2 = 2$ as an equation in the system.

10. (a) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ no solution $\quad \begin{bmatrix} 1 & 0 & * \\ 0 & 1 & * \\ 0 & 0 & 0 \end{bmatrix}$ unique solution $\quad \begin{bmatrix} 1 & * & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ no solution $\quad \begin{bmatrix} 1 & * & * \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ many solutions

(b) $\begin{bmatrix} 1 & 0 & 0 & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \end{bmatrix}$ unique solution $\quad \begin{bmatrix} 1 & 0 & * & 0 \\ 0 & 1 & * & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ no solution $\quad \begin{bmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \end{bmatrix}$ many solutions $\quad \begin{bmatrix} 1 & * & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ no solution

$\begin{bmatrix} 1 & * & 0 & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 \end{bmatrix}$ many solutions $\quad \begin{bmatrix} 1 & * & * & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ no solution $\quad \begin{bmatrix} 1 & * & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ many solutions

11. (a) If $ax_0 + by_0 = 0$ then $k(ax_0 + by_0) = 0$ so that $a(kx_0) + b(ky_0) = 0$. Thus $x = kx_0$, $y = ky_0$ is a solution. Likewise for the equation $cx + dy = 0$.

(b) If $ax_0 + by_0 = 0$ and $ax_1 + by_1 = 0$ then $ax_0 + by_0 + ax_1 + by_1 = 0 + 0 = 0$. But $ax_0 + by_0 + ax_1 + by_1 = ax_0 + ax_1 + by_0 + by_1 = a(x_0 + x_1) + b(y_0 + y_1)$ so that $a(x_0 + x_1) + b(y_0 + y_1) = 0$. Thus $x = x_0 + x_1$, $y = y_0 + y_1$ is a solution. Likewise for the equation $cx + dy = 0$.

12. $a(0) + b(0) = 0$ and $c(0) + d(0) = 0$, so $x = 0$, $y = 0$ is a solution.
Multiply 1st equation by c, 2nd by a to eliminate x. Get
$cax+cby=0$ and $acx+ady=0$. Subtract, $ady-cby=0$, $(ad-bc)y=0$. Similarly $(ad-bc)x=0$.
If $ad-bc\neq0$, $x=0,y=0$. If $ad-bc=0$ the x and y can be anything; thus many solutions.
Therefore $x=0,y=0$ is the only solution if and only if $ad-bc\neq0$.

13. (a) and (b), No. If the first system of equations has a unique solution, then the reduced echelon form of the matrix $[A:B_1]$ will be $[I_3 :X]$. The reduced echelon form of $[A:B_2]$ must therefore be $[I_3 :Y]$. So the second system must also have a unique solution.

(c) If the first system of equations has many solutions, then at least one row of the reduced echelon form of $[A{:}B_1]$ will consist entirely of zeros. Therefore the corresponding row(s) of the reduced echelon form of $[A{:}B_2]$ will have zeros in the first three columns. If any such row has a nonzero number in the fourth column, the system will have no solution.

14. (a) $$\begin{bmatrix}1&1&5&2&3\\1&2&8&5&2\\2&4&16&10&4\end{bmatrix}\underset{\substack{R2+(-1)R1\\R3+(-2)R1}}{\approx}\begin{bmatrix}1&1&5&2&3\\0&1&3&3&-1\\0&2&6&6&-2\end{bmatrix}\underset{\substack{R1+(-1)R2\\R3+(-2)R2}}{\approx}\begin{bmatrix}1&0&2&-1&4\\0&1&3&3&-1\\0&0&0&0&0\end{bmatrix},$$

so the general solution to the first system is $x_1 = -1 - 2r$, $x_2 = 3 - 3r$, $x_3 = r$, and the general solution to the second system is $x_1 = 4 - 2r$, $x_2 = -1 - 3r$, $x_3 = r$.

(b) $$\begin{bmatrix}1&2&4&8&5\\1&1&2&5&3\\2&3&6&13&11\end{bmatrix}\underset{\substack{R2+(-1)R1\\R3+(-2)R1}}{\approx}\begin{bmatrix}1&2&4&8&5\\0&-1&-2&-3&-2\\0&-1&-2&-3&1\end{bmatrix}\underset{(-1)R2}{\approx}\begin{bmatrix}1&2&4&8&5\\0&1&2&3&2\\0&-1&-2&-3&1\end{bmatrix},$$

$$\underset{\substack{R1+(-2)R2\\R3+R2}}{\approx}\begin{bmatrix}1&0&0&2&1\\0&1&2&3&2\\0&0&0&0&3\end{bmatrix},$$ so the general solution to the first system is

$x_1 = 2$, $x_2 = 3 - 2r$, $x_3 = r$, and the second system has no solution.

15. A 3x3 matrix represents the equations of three lines in a plane. In order for there to be a unique solution, the three lines would have to meet in a point. For there to be many solutions, the three lines would all have to be the same. It is far more likely that the lines will meet in pairs (or that one pair will be parallel), i.e., that there will be no solution, the situation represented by the reduced echelon form I_3 .

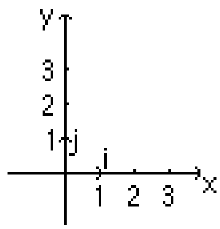
16. A 3x4 matrix represents the equations of three planes. In order for there to be many solutions, the three planes must have at least one line in common. For there to be no solutions, either at least two of the three planes must be parallel or the line of intersection of two of the planes must lie in a plane that is parallel to the third plane. It is more likely that the three planes will meet in a single point, i.e., that there will be a unique solution. The reduced echelon form therefore will be $[I_3{:}X]$.

17. The difference between no solution and at least one solution is the presence of a nonzero number in the last position of a row that otherwise consists entirely of zeros. Round-off error is more likely to produce a nonzero number when there should be a

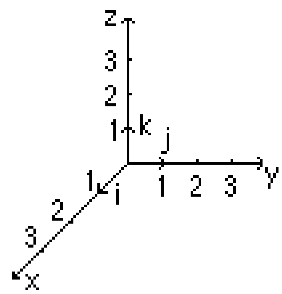
zero than the reverse. Thus the answer is (b). Thinking geometrically, a small move by one or more of the linear surfaces (round-off error) may destroy a solution if there is one, but probably won't produce a solution if there is none.

Exercise Set 1.3

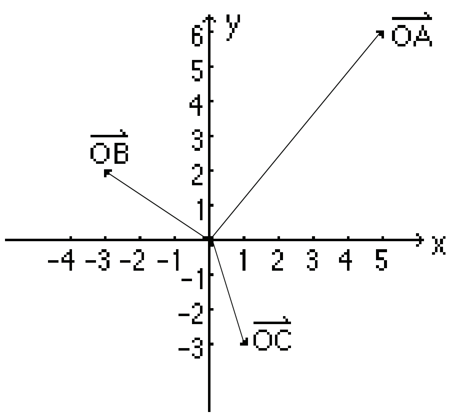
1.



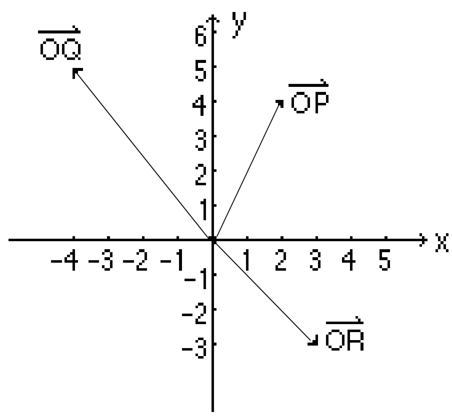
2.



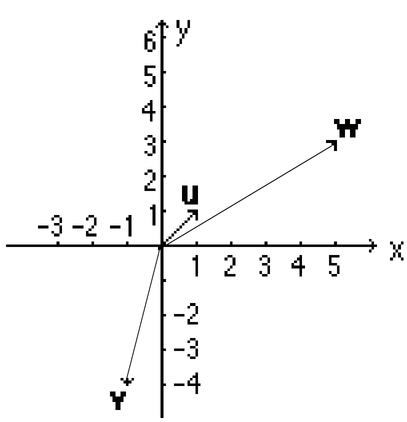
3. (a)



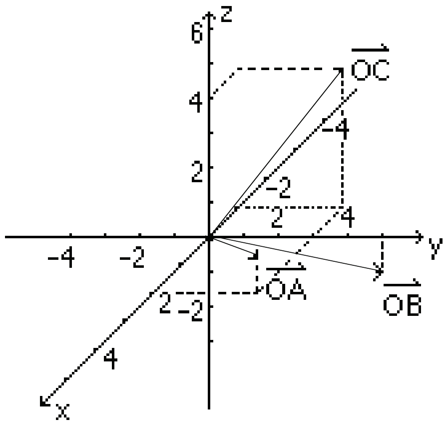
(b)



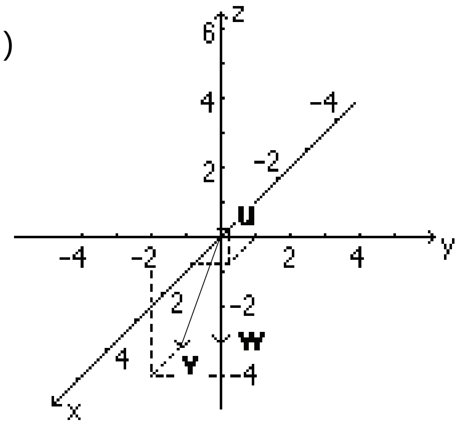
(c)



4. (a)



(b)



5. (a) $3(1,4) = (3,12)$. (b) $-2(-1,3) = (2,-6)$. (c) $(1/2)(2,6) = (1,3)$.

(d) $(-1/2)(2,4,2) = (-1,-2,-1)$. (e) $3(-1,2,3) = (-3,6,9)$.

(f) $4(-1,2,3,-2) = (-4,8,12,-8)$. (g) $-5(1,-4,3,-2,5) = (-5,20,-15,10,-25)$.

(h) $3(3,0,4,2,-1) = (9,0,12,6,-3)$.

6. (a) $\mathbf{u} + \mathbf{w} = (1,2) + (-3,5) = (-2,7)$, (b) $\mathbf{u} + 3\mathbf{v} = (1,2) + 3(4,-1) = (13,-1)$.

(c) $\mathbf{v} + \mathbf{w} = (4,-1) + (-3,5) = (1,4)$.

(d) $2\mathbf{u} + 3\mathbf{v} - \mathbf{w} = 2(1,2) + 3(4,-1) - (-3,5) = (17,-4)$.

(e) $-3\mathbf{u} + 4\mathbf{v} - 2\mathbf{w} = -3(1,2) + 4(4,-1) - 2(-3,5) = (19,-20)$.

7. (a) $\mathbf{u} + \mathbf{w} = (2,1,3) + (2,4,-2) = (4,5,1)$. (b) $2\mathbf{u} + \mathbf{v} = 2(2,1,3) + (-1,3,2) = (3,5,8)$.

(c) $\mathbf{u} + 3\mathbf{w} = (2,1,3) + 3(2,4,-2) = (8,13,-3)$.

(d) $5\mathbf{u} - 2\mathbf{v} + 6\mathbf{w} = 5(2,1,3) - 2(-1,3,2) + 6(2,4,-2) = (24,23,-1)$.

(e) $2\mathbf{u} - 3\mathbf{v} - 4\mathbf{w} = 2(2,1,3) - 3(-1,3,2) - 4(2,4,-2) = (-1,-23,8)$.

8. (a) $\mathbf{u} + \mathbf{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} + \begin{bmatrix} -1 \\ -4 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. (b) $2\mathbf{v} - 3\mathbf{w} = 2\begin{bmatrix} -1 \\ -4 \end{bmatrix} - 3\begin{bmatrix} 4 \\ -6 \end{bmatrix} = \begin{bmatrix} -14 \\ 10 \end{bmatrix}$.

(c) $2\mathbf{u} + 4\mathbf{v} - \mathbf{w} = 2\begin{bmatrix} 2 \\ 3 \end{bmatrix} + 4\begin{bmatrix} -1 \\ -4 \end{bmatrix} - \begin{bmatrix} 4 \\ -6 \end{bmatrix} = \begin{bmatrix} -4 \\ -4 \end{bmatrix}$.

(d) $-3\mathbf{u} - 2\mathbf{v} + 4\mathbf{w} = -3\begin{bmatrix} 2 \\ 3 \end{bmatrix} - 2\begin{bmatrix} -1 \\ -4 \end{bmatrix} + 4\begin{bmatrix} 4 \\ -6 \end{bmatrix} = \begin{bmatrix} 12 \\ -25 \end{bmatrix}$.

9. (a) $\mathbf{u} + 2\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + 2\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ 2 \\ 1 \end{bmatrix}$. (b) $-4\mathbf{v} + 3\mathbf{w} = -4\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} + 3\begin{bmatrix} -1 \\ 0 \\ 5 \end{bmatrix} = \begin{bmatrix} -15 \\ 0 \\ 11 \end{bmatrix}$.

(c) $3\mathbf{u} - 2\mathbf{v} + 4\mathbf{w} = 3\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} - 2\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} + 4\begin{bmatrix} -1 \\ 0 \\ 5 \end{bmatrix} = \begin{bmatrix} -7 \\ 6 \\ 15 \end{bmatrix}$.

$$2\mathbf{u} + 3\mathbf{v} - 8\mathbf{w} = 2\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + 3\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} - 8\begin{bmatrix} -1 \\ 0 \\ 5 \end{bmatrix} = \begin{bmatrix} 19 \\ 4 \\ -39 \end{bmatrix}.$$

10. (a) a(1,2) + b(-1,3) = (1,7). a-b=1, 2a+3b=7. Unique solution a=2, b=1. (1,7)=2(1,2) + (-1,3).
(1,7) is a linear combination of (1,2) and (-1,3).

(b) a(1,1) + b(3,2) = (1,2). a+3b=1, a+2b=2. Unique solution a=4,b=-1. (1,2)=4(1,1) - (3,2).
(1,2) is a linear combination of (1,1) and (3,2).

(c) a(1,-3) + b(-2,6) = (3,5). a-2b=3, -3a+6b=5. No solution.
(3,5) is not a linear combination of (1,-3) and (-2,6).

(d) a(2,4) + b(-4,-8) = (6,2). 2a-4b=6, 4a-8b=2. No solution.
(6,2) is not a linear combination of (2,4) and (-4,-8).

11. (a) a(1,1,2) + b(1,2,1) + c(2,3,4) = (7,9,15). a+b+2c = 7, a+2b+3c = 9, 2a+b+4c = 15.
Unique solution, a=2, b=-1, c=3. Is a linear combination.

(b) a(1,1,1) + b(1,2,4) + c(0,1,-3) = (6,13,9). a+b = 6, a+2b+c = 13, a+4b-3c = 9.
Unique solution, a=2, b=4, c=3. Is a linear combination.

(c) a(1,2,0) + b(-1,-1,2) + c(1,3,2) = (1,2,-1). a-b+c = 1, 2a-b+3c = 2, 2b+2c = -1.
No solution . Not a linear combination.

(d) $a(1,2,3) + b(2,5,7) + c(0,0,1) = (-1,-1,2)$. $a+2b = -1$, $2a+5b = -1$, $3a+7b+c = 2$.
Unique solution, $a=-3$, $b=1$, $c=4$. Is a linear combination.

(e) $a(1,1,1) + b(1,2,-1) + c(5,7,1) = (5,8,1)$. $a+b+5c = 5$, $a+2b+7c = 8$, $a-b+c = 1$.
No solution . Not a linear combination.

(f) $a(1,1,2) + b(2,2,4) + c(1,-1,1) = (5,-1,7)$. $a+2b+c = 5$, $a+2b-c = -1$, $2a+4b+c = 7$.
Many solutions, $a=-2r+2$, $b=r$, $c=3$. $(5,-1,7) = (-2r+2)(1,1,2) + r(2,2,4) + 3(1,-1,1)$.
There are many linear combinations.
For example, when $r=1$ we get $(5,-1,7) = 0(1,1,2) + (2,2,4) + 3(1,-1,1)$.
When $r=2$, $(5,-1,7) = -2(1,1,2) + 2(2,2,4) + 3(1,-1,1)$.

12. (a) $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (u_1, u_2, \ldots, u_n) + ((v_1, v_2, \ldots, v_n) + (w_1, w_2, \ldots, w_n))$
$= (u_1, u_2, \ldots, u_n) + (v_1+w_1, v_2+w_2, \ldots, v_n+w_n)$
$= (u_1+(v_1+w_1), u_2+(v_2+w_2), \ldots, u_n+(v_n+w_n))$
$= ((u_1+v_1)+w_1, (u_2+v_2)+w_2, \ldots, (u_n+v_n)+w_n)$
$= ((u_1+v_1), (u_2+v_2), \ldots, (u_n+v_n))+ (w_1, w_2, \ldots, w_n)$
$= ((u_1, u_2, \ldots, u_n) + (v_1, v_2, \ldots, v_n)) + (w_1, w_2, \ldots, w_n) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$.

(b) $\mathbf{u} + (-\mathbf{u}) = (u_1, u_2, \ldots, u_n) + (-1)(u_1, u_2, \ldots, u_n)$
$= (u_1, u_2, \ldots, u_n) + (-u_1, -u_2, \ldots, -u_n) = (u_1-u_1, u_2-u_2, \ldots, u_n-u_n)$
$= (0,0, \ldots,0) = \mathbf{0}$.

(c) $(c+d)\mathbf{u} = (c+d)(u_1, u_2, \ldots, u_n) = ((c+d)u_1, (c+d)u_2, \ldots, (c+d)u_n)$
$= (cu_1+du_1, cu_2+du_2, \ldots, cu_n+du_n)$
$= (cu_1, cu_2, \ldots, cu_n) + (du_1, du_2, \ldots, du_n)$
$= c(u_1, u_2, \ldots, u_n) + d(u_1, u_2, \ldots, u_n) = c\mathbf{u} + d\mathbf{u}$.

(d) $1\mathbf{u} = 1(u_1, u_2, \ldots, u_n) = (1xu_1, 1xu_2, \ldots, 1xu_n) = (u_1, u_2, \ldots, u_n) = \mathbf{u}$.

Exercise Set 1.4

1. (a) Let W be the subset of vectors of the form (a,3a). Let **u**=(a,3a), **v**=(b,3b) and k be a scalar. Then **u**+**v**=((a+b), 3(a+b)) and k**u**=(ka,3ka).
The second component of both **u**+**v** and k**u** is 3 times the first component. Thus W is closed under addition and scalar multiplication - it is a subspace.

(b) Let W be the subset of vectors of the form (a,-a). Let **u**=(a,-a), **v**=(b,-b) and k be a scalar. Then **u**+**v**=((a+b), -(a+b)) and k**u**=(ka,-ka).
The second component of both **u**+**v** and k**u** is minus the first component. Thus W is

closed under addition and scalar multiplication - it is a subspace.

(c) Let W be the subset of vectors of the form (a,0). Let **u**=(a,0), **v**=(b,0) and k be a scalar. Then **u**+**v**=((a+b), 0) and k**u**=(ka,0).
The second component of both **u**+**v** and k**u** is zero. Thus W is closed under addition and scalar multiplication - it is a subspace.

(d) Let W be the subset of vectors of the form (2a,3a). Let **u**=(2a,3a), **v**=(2b,3b) and k be a scalar. Then **u**+**v**=(2(a+b), 3(a+b)) and k**u**=(2ka,3ka).
The second component of both **u**+**v** and k**u** is 3/2 times the first component. Thus W is closed under addition and scalar multiplication - it is a subspace.

2. (a) Let W be the subset of vectors of the form (a,b,b). Let **u**=(a,b,b), **v**=(c,d,d) and k be a scalar. Then **u**+**v**=(a+c, b+d,b+d) and k**u**=(ka,kb,kb).
The second and third components of **u**+**v** are the same; so are those of k**u**. Thus W is closed under addition and scalar multiplication - it is a subspace.

(b) Let W be the subset of vectors of the form (a,-a,b). Let **u**=(a,-a,b), **v**=(c,-c,d) and k be a scalar. Then **u**+**v**=(a+c,-(a+c),b+d) and k**u**=(ka,-ka,kb).
The second component of **u**+**v** equals minus the first component; and same for k**u**. Thus W is closed under addition and scalar multiplication - it is a subspace.

(c) Let W be the subset of vectors of the form (a,2a,-a). Let **u**=(a,2a,-a), **v**=(b,2b,-b) and k be a scalar. Then **u**+**v**=(a+b,2(a+b),-(a+b)) and k**u**=(ka,2ka,-ka).
The second component of **u**+**v** is twice the first, and the third component is minus the first; and same for k**u**. Thus W is closed under addition and scalar multiplication - it is a subspace.

(d) Let W be the subset of vectors of the form (a,a,b,b). Let **u**=(a,a,b,b), **v**=(c,c,d,d) and k be a scalar. Then **u**+**v**=(a+c,a+c,b+d,b+d) and k**u**=(ka,ka,kb,kb).
The 1^{st} and 2^{nd} components of **u**+**v** are the same, so are 3^{rd} and 4^{th}; same for k**u**. Thus W is closed under addition and scalar multiplication - it is a subspace.

3. (a) Let W be the subset of vectors of the form (a,b,2a+3b). Let **u**=(a,b,2a+3b), **v**=(c,d,2c+3d) and k be a scalar. Then **u**+**v**=(a+c, b+d,2(a+c)+3(b+d)) and k**u**=(ka,kb,2ka+3kb).The third component of **u**+**v** is twice the first plus three times the second; same for k**u**. Thus W is closed under addition and scalar multiplication - it is a subspace.

(b) Let W be the subset of vectors of the form (a,b,3). Let **u**=(a,b,3), **v**=(c,d,3) and k be a scalar. Then **u**+**v**=(a+c,b+d,6). The third component is not 6. Thus **u**+**v** is not in W. W is not a subspace. Let us check closure under scalar multiplication. k**u**=(ka,kb,3k). Thus unless k=1, k**u** is not in W. W is not closed under scalar multiplication either.

(c) Let W be the subset of vectors of the form (a,a+2,b). Let **u**=(a,a+2,b), **v**=(c,c+2,d) and k be a scalar. Then **u**+**v**=(a+c,a+c+4,b+d). The second component is not the first plus 2. Thus **u**+**v** is not in W. W is not a subspace. Let us check closure under scalar multiplication. k**u**=(ka,ka+2k,kb). Thus unless k=1 k**u** is not in W. W is not closed under scalar multiplication either.

(d) Let W be the subset of vectors of the form (a,-a,0). Let **u**=(a,-a,0), **v**=(b,-b,0) and k be a scalar. Then **u**+**v**=(a+b,-(a+b),0) and k**u**=(ka,-ka,0).The second component of **u**+**v** is minus the first and the last component is zero; same for k**u**. Thus W is closed under addition and scalar multiplication - it is a subspace.

4. $\begin{bmatrix}1 & 3 & 1 & 0\\ 0 & 1 & 1 & 0\\ 2 & 7 & 3 & 0\end{bmatrix} \approx \begin{bmatrix}1 & 3 & 1 & 0\\ 0 & 1 & 1 & 0\\ 0 & 1 & 1 & 0\end{bmatrix} \approx \begin{bmatrix}1 & 0 & -2 & 0\\ 0 & 1 & 1 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}$. General solution (2r,-r,r).

Let W be the subset of vectors of the form (2r,-r,r). Let **u**=(2r,-r,r), **v**=(2s,-s,s) and k be a scalar. Then **u**+**v**=(2(r+s),-(r+s),r+s) and k**u**=(2kr, -kr, kr).The first component of **u**+**v** is twice the last component, and the second component is minus the last component; same for k**u**. Thus W is closed under addition and scalar multiplication - it is a subspace. Line defined by vector (2, -1, 1).

5. $\begin{bmatrix}1 & 1 & -7 & 0\\ 0 & 1 & -4 & 0\\ 1 & 0 & -3 & 0\end{bmatrix} \approx \begin{bmatrix}1 & 1 & -7 & 0\\ 0 & 1 & -4 & 0\\ 0 & -1 & 4 & 0\end{bmatrix} \approx \begin{bmatrix}1 & 0 & -3 & 0\\ 0 & 1 & -4 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}$. General solution (3r,4r,r).

Let W be the subset of vectors of the form (3r,4r,r). Let **u**=(3r,4r,r), **v**=(3s,4s,s) and k be a scalar. Then **u**+**v**=(3(r+s),4(r+s),r+s) and k**u**=(3kr, 4kr, kr).The first component of **u**+**v** is three times the last component, and the second component is four times the last component; same for k**u**. Thus W is closed under addition and scalar multiplication - it is a subspace. It is a line defined by vector (3,4,1).

6. $\begin{bmatrix}1 & -2 & 3 & 0\\ -1 & 2 & 1 & 0\\ 1 & -2 & 4 & 0\end{bmatrix} \approx \begin{bmatrix}1 & -2 & 3 & 0\\ 0 & 0 & 4 & 0\\ 0 & 0 & 1 & 0\end{bmatrix} \approx \begin{bmatrix}1 & -2 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}$. General solution (2r, r, 0).

Let W be the subset of vectors of the form (2r,r,0). Let **u**=(2r,r,0), **v**=(2s,s,0) and k be a scalar. Then **u**+**v**=(2(r+s),r+s,0) and k**u**=(2kr, kr, 0).The first component of **u**+**v** is twice the second component, and the last component is zero; same for k**u**. Thus W is closed under addition and scalar multiplication - it is a subspace. Line defined by vector (2, 1, 0).

7. $\begin{bmatrix} 1 & 2 & -1 & 0 \\ 1 & 3 & 1 & 0 \\ 3 & 7 & -1 & 0 \end{bmatrix} \approx \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & 0 \end{bmatrix} \approx \begin{bmatrix} 1 & 0 & -5 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. General solution (5r,-2r,r).

Let W be the subset of vectors of the form (5r,-2r,r). Let **u**=(5r,-2r,r), **v**=(5s,-2s,s) and k be a scalar. Then **u**+**v**=(5(r+s),-2(r+s),r+s) and k**u**=(5kr, -2kr, kr).The first component of **u**+**v** is five times the last component, and the second component is minus two times the last component; same for k**u**. Thus W is closed under addition and scalar multiplication - it is a subspace. Line defined by vector (5, -2, 1).

8. General solution is (2r-2s, r -3s, r, s). Let **u**=(2r-2s, r -3s, r, s) and **v**=(2p-2q, p-3q, p, q). Then **u**+**v**=(2r-2s+2p-2q, r-3s+p-3q, r+p, s+q) = (2(r+p)-2(s+q), (r+p)-3(s+q), r+p, s+q). The first component is twice the third minus twice the fourth. The second component is the third minus three times the fourth - the required form. Thus space is closed under addition. Let k be a scalar. Then k**u**= (2kr -2ks, kr -3ks, kr, ks) – the required form. Thus the set of solutions is closed under addition and scalar multiplication; it is a subspace.

9. General solution is (3r + s, -r - 4s, r, s). (a) Two specific solutions: r=1, s=1, **u**(4, -5, 1, 1); r=-1, s=2, **v**(-1, -7, -1, 2). (b) Other solutions: **u**+**v**=(3, -12, 0, 3) and say, -2**u**=(-8, 10, -2, -2), 4**v**=(-4, -28, -4, 8). (-2**u**)+(4**v**)=(-12, -18, -6, 6). (c) Are solutions for r=0,s=3; r=-2,s=-2; r=-4,s=8; r=-6,s=6 respectively.

10. General solution is (2r, 3r, r). (a) Two specific solutions: r=1 gives **u**(2, 3, 1); r=2 gives **v**(4, 6, 2). (b) Other solutions: **u**+**v** = (6, 9, 3); 4u = (8, 12, 4); -**v** = (-4, -6, - 2); **u** – **v** = (-2, -3, -1). Are solutions for r = 3. r = 4, r = -2, r = -1 respectively.

11. General solution is (2r - s, -3r - 2s r, s). (a) Two specific solutions: r=1, s=1, **u**(1, -5, 1, 1); r=2, s=-1, **v**(5, -4, 2, -1). (b) Other solutions: **u**+**v**=(6, -9, 3, 0). and say 2**u**=(2, -10, 2, 2), 3**v**=(15, -12, 6, -3). (2**u**)+(3**v**)=(17, -22, 8, -1). (c) Are solutions for r=3,s=0; r=2,s=2; r=6,s=-3; r=8,s=-1 respectively.

Exercise Set 1.5

1. Standard basis for $\mathbf{R}^2$: {(1, 0), (0, 1)}. (a) Let (a, b) be an arbitrary vector in $\mathbf{R}^2$. We can write (a, b) = a(1, 0) + b(0, 1). Thus vectors (1, 0) and (0, 1) span $\mathbf{R}^2$. (b) Let us examine the identity p(1, 0) + q(0, 1) = (0, 0). This gives (p, 0) + (0, q) = 0, (p, q) = (0, 0). Thus p=0 and q=0. The vectors are linearly independent.

2. Standard basis for $\mathbf{R}^4$: {(1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1)}.
(a) Let (a,b,c,d) be an arbitrary vector in $\mathbf{R}^4$. We can write
(a,b,c,d) = a(1,0,0,0) +b(0,1,0,0) +c(0,0,1,0) +d(0,0,0,1). Thus vectors in basis span $\mathbf{R}^4$.

(b) Let us examine the identity p(1,0,0,0)+q(0,1,0,0)+r(0,0,1,0)+s(0,0,0,1) = (0,0,0,0). This gives (p,q,r,s) = 0. Thus p=0, q=0, r=0, s=0. The vectors are linearly independent.

3. (a) (a, a, b) + (c, c, d) = (a+c, a+c, b+d). 1st components same. Closed under addition. k(a, a, b) = (ka, ka, kb). 1st components same. Closed under scalar multiplication. Subspace. (a, a, b) = a(1, 1, 0) + b(0, 0, 1). Vectors (1, 1, 0) and (0, 0, 1) span space and are linearly independent. {(1, 1, 0), (0, 0, 1)} is a basis. Dimension is 2.

   (b) (a, 2a, b) + (c, 2c, d) = (a+c, 2(a+c), b+d). 2nd component is twice first. Closed under addition. k(a, 2a, b) = (ka, 2ka, kb). 2nd component is twice 1st. Closed under scalar multiplication. Subspace. (a, 2a, b) = a(1, 2, 0) + b(0, 0, 1). Vectors (1, 2, 0) and (0, 0, 1) span the space and are linearly independent. {(1, 2, 0, (0, 0, 1)} is a basis. Dimension is 2.

   (c) (a, 2a, 4a) + (b, 2b, 4b) = (a+b, 2a+2b, 4a+4b) =(a+b, 2(a+b), 4(a+b)). 2nd component is twice 1st, 3rd component four times 1st. Closed under addition. k(a, 2a, 4a) = (ka, 2ka, 4ka). 2nd component is twice 1st, 3rd component four times 1st. Closed under scalar multiplication. Subspace. (a, 2a, 4a) = a(1, 2, 4). {(1, 2, 4)} is a basis. Dimension is 1. Space is a line defined by the vector (1, 2, 4).

   (d) (a, -a, 0) + (b, -b, 0) = (a+b, -a-b, 0) = (a+b, -(a+b), 0). 2nd component is negative of 1st. Closed under addition. k(a, -a, 0) = (ka, -ka, 0). 2nd component is negative first. Closed under scalar multiplication. Subspace. (a, -a, 0) = a(1, -1, 0). {(1, -1, 0)} is a basis. Dimension is 1. Space is line defined by the vector (1, -1, 0).

4. (a) (a, b, a) + (c, d, c) = (a+c, b+d, a+c). 3rd component same as 1st. Closed under addition. k(a, b, a) = (ka, kb, ka). 3rd component same as 1st. Closed under scalar multiplication. Subspace. (a, b, a) = a(1, 0, 1) + b(0, 1, 0). {(1, 0, 1), (0, 1, 0)} is a basis. Dimension is 2.

   (b) (a, b, 0) + (c, d, 0) = (a+c, b+d, 0). Last component is zero. Closed under addition. k(a, b, 0) = (ka, kb, 0). Last component is zero. Closed under scalar multiplication. Subspace. (a, b, 0) = a(1, 0, 0) + b(0, 1, 0). {(1, 0, 0), (0, 1, 0)} is a basis. Dimension is 2. It is xy-plane.

   (c) (a, b, 2) + (c, d, 2) = (a+c, b+d, 4). Last component not 2. Not closed under addition. Not a subspace.

   (d) (a, a, a+3) + (b, b, b+3) = (a+b, a+b, a+b+6). Last component is not 1st plus 3. Not closed under addition. Not a subspace.

5. (a) a(1, 2, 3) + b(1, 2, 3) = (a+b)(1, 2, 3). Sum is a scalar multiple of (1, 2, 3). Closed under addition. ka(1, 2, 3) = (ka)(1, 2, 3). It is a scalar multiple of (1, 2, 3). Closed under scalar multiplication. Subspace of $\mathbf{R}^3$. Basis {(1, 2, 3)}. Dimension is 1. Space is line defined by vector (1, 2, 3).

(b) (a, 0, 0) + (b, 0, 0) = (a+b, 0, 0). Last two components zero. Closed under addition. k(a, 0, 0) = (ka, 0, 0). Last two components zero. Closed under scalar multiplication. Subspace of $\mathbf{R}^3$. Basis {(1, 0, 0)}. Dimension is 1. Space is the x-axis.

(c) (a, 2a) + (b, 2b) = (a+b, 2a+2b) = (a+b, 2(a+b)). 2nd component is twice 1st. Closed under addition. k(a, 2a) = (ka, 2ka). 2nd component is twice 1st. Closed under scalar multiplication. Subspace. Basis {(1, 2)}. Dimension is 1. Space is line defined by vector (1, 2) in $\mathbf{R}^2$.

(d) (a, b, c, 1) + (d, e, f, 1) = (a+d, b+e, c+f, 2). Last component is not 1. Not closed under addition. Not a subspace of $\mathbf{R}^4$.

6. (a) True: Arbitrary vector can be expressed as a linear combination of (1, 0) and (0, 1). (a, b) = a(1, 0) + b(0, 1).

(b) True: (a, b) = a(1, 0) + b(0, 1) + 0(1, 1). (Some of the scalars can be zero).

(c) True: p(1, 0) + q(0, 1) = (0, 0) has the unique solution p=0, q=0.

(d) False: Consider the identity p(1, 0) + q(0, 1) + r(0, 2) = (0, 0). Does this have the unique solution p=0, q=0, r=0? No, can have 0(1, 0) + 2(0, 1) - 1(0, 2) = (0, 0). Vectors are not linearly independent - we say that they are linearly dependent.

(e) True: (x, y) = x(1, 0) - y(0, -1). Thus (1, 0) and (0, -1) span $\mathbf{R}^2$. Further, p(1, 0)+q(0, -1)=(0, 0) has the unique solution p=0, q=0. Vectors are linearly independent.

(f) True: (x, y) = $\frac{x}{2}$(2, 0) + $\frac{y}{3}$(0, 3). Thus (2, 0) and (0, 3) span $\mathbf{R}^2$. Further, p(2, 0) + q(0, 3) = (0, 0) ⌠ (2p, 3q) = (0, 0), has the unique solution p=0, q=0. Vectors linearly independent.

7. (a) True: (1, 0, 0) and (0, 1, 0) span the subset of vectors of the form (a, b, 0). Further, p(1, 0, 0) + q(0, 1, 0) = (0, 0, 0) has the unique solution p=0, q=0. Vectors are linearly independent. Subspace is 2D since 2 base vectors. The subspace is the xy plane.

(b) True: The vector (1, 0, 0) spans the subset of vectors of the form a(1, 0, 0). Further p(1, 0, 0)=(0,0,0) has unique solution p=0. Subspace is line defined by vector (1, 0, 0). One vector in basis, thus 1D.

(c) True: Can write (a, 2a, b) in the form (a, 2a, b) = a(1, 2, 0) + b(0, 0, 1).

(d) True: Can write (a, b, 2a-b) in the form (a, b, 2a-b) = a(1, 0, 2) + b(0, 1, -1).

(e) False: 1(1, 0, 0) + 1(0, 1, 0) -1(1, 1, 0) = (0, 0, 0). Thus vectors not linearly independent.

(f) False: $\mathbf{R}^2$ is not a subset of $\mathbf{R}^3$. e.g., (1, 2) is an element of $\mathbf{R}^2$, but not of $\mathbf{R}^3$.

8. (a) Let (x, y) be an arbitrary vector in $\mathbf{R}^2$. Then (x, y) = x(1, 0) + y(0, 1). Thus (1, 0), (0, 1) span $\mathbf{R}^2$. Notice that both vectors are needed to span $\mathbf{R}^2$ - we cannot just use one of them. Further, a(1, 0) + b(0, 1) = (0, 0) => (a, b) = (0, 0) => a=0, and b=0. Thus (1, 0) and (0, 1) are linearly independent. They form a basis for $\mathbf{R}^2$.

(b) (x, y) can be expressed (x, y) = x(1, 0) + 3y(0, 1) - y(0, 2), or (x, y) = x(1, 0) + 5y(0, 1) - 2y(0, 2); there are many ways. Thus (1, 0), (0, 1), (0, 2) spans $\mathbf{R}^2$. But it is not an efficient spanning set. The vector (0, 2) is not really needed.

(c) 0(1, 0) + 2(0, 1) - (0, 2) = (0, 0). Thus {(1, 0), (0, 1), (0, 2)} is linearly dependent. It is not a basis.

9. (a) Let (x, y) be an arbitrary vector in $\mathbf{R}^2$. Then (x, y) = x(1, 0) + y(0, 1). Thus (1, 0), (0, 1) span $\mathbf{R}^2$. Notice that both vectors are needed to span $\mathbf{R}^2$ - we cannot just use one of them. Further, a(1, 0) + b(0, 1) = (0, 0) => (a, b) = (0, 0) => a=0, and b=0. Thus (1, 0) and (0, 1) are linearly independent. They form a basis for $\mathbf{R}^2$.

(b) (x, y) can be expressed (x, y) = x(1, 0) + y(0, 1) + 0(1, 1); or (x, y) = (x+y)(1, 0) + 2y(0, 1) - y(1,1) - there are many ways. Thus (1, 0), (0, 1), (1, 1) span $\mathbf{R}^2$. But it is not an efficient spanning set. The vector (1, 1) is not really needed.

(c) 1(1, 0) + 1(0, 1) - 1(1, 1) = (0, 0). Thus {(1, 0), (0, 1), (1, 1)} is linearly dependent. It is not a basis.

10. (a) Let (x, y, z) be an arbitrary vector in $\mathbf{R}^3$. Then (x, y, z) = x(1, 0, 0) + y(0, 1, 0) + z(0, 0, 1). Thus (1, 0, 0), (0, 1, 0), (0, 0, 1) span $\mathbf{R}^3$. Notice that all three vectors are needed to span $\mathbf{R}^3$ - we cannot just use two of them. Further, a(1, 0, 0) + b(0, 1, 0) + c(0, 0, 1) = (0, 0, 0) => (a, b, c) = (0, 0, 0) => a=0, b=0, and c=0. Thus (1, 0, 0) and (0, 1, 0, (0, 0, 1) are linearly independent. They form a basis for $\mathbf{R}^3$.

(b) (x, y, z) can be expressed (x, y, z) = x(1, 0, 0) + y(0, 1, 0) + z(0, 0, 1) + 0(0, 1, 1); or (x, y, z) = x(1, 0, 0) + $\frac{(y-z)}{2}$(0, 1, 0) + $\frac{(-y+z)}{2}$(0, 0, 1) + $\frac{(y+z)}{2}$ (0, 1, 1).
- there are many ways. Thus (1, 0, 0), (0, 1, 0), (0, 0, 1), (0, 1, 1) span $\mathbf{R}^2$. But it is not an efficient spanning set. The vector (0, 1, 1) is not really needed.

(c) 1(1, 0, 0) + 1(0, 1, 0) + 1(0, 0, 1) - 0(0, 1, 1) = (0, 0, 0). Thus (1, 0, 0), (0, 1, 0), (0, 0, 1), (0, 1, 1) are linearly dependent. Do not form a basis.

11. (a) Vectors (1, 0, 0), (0, 1, 0), (0, 0, 1), and (1, 1, 1) span $\mathbf{R}^3$ but are not linearly independent.
(b) Vectors (1, 0, 0), (0, 1, 0) are linearly independent but do not span $\mathbf{R}^3$.

12. Separate the variables in the general solution, (2r - 2s, r - 3s, r, s)= r(2, 1, 1, 0) + s(-2, -3, 0, 1).
Vectors (2, 1, 1, 0) and (-2, 3, 0, 1) thus span W.
Also, identity p(2, 1, 1, 0) + q(-2, 3, 0, 1) = (0, 0, 0, 0) leads to p=0, q=0. The two vectors are thus linearly independent. Set {(2, 1, 1, 0), (-2, 3, 0, 1)} is therefore a basis for W.
Dimension of W is 2. Solutions form a plane through the origin in $\mathbf{R}^4$.

13. Separate the variables in the general solution, (3r + s, -r - 4s, r, s)= r(3, -1, 1, 0)+s(1, -4, 0, 1).
Vectors (3, -1, 1, 0) and (1, -4, 0, 1) thus span W.
Also, identity p(3, -1, 1, 0) + q(1, -4, 0, 1) = (0, 0, 0, 0) leads to p=0, q=0. The two vectors are thus linearly independent. Set {(3, -1, 1, 0), (1, -4, 0, 1)} is therefore a basis for W.
Dimension of W is 2. Solutions form a plane through the origin in $\mathbf{R}^4$.

14. (2r, 3r, r) = r(2, 3, 1). The set of solutions form the line through the origin in $\mathbf{R}^3$ defined by the vector (2, 3, 1). Set {(2, 3, 1)} is a basis. Dimension of W is 1.

15. Separate the variables in the solution, (2r - s, -3r - 2s, r, s)= r(2, -3, 1, 0) + s(-1, -2, 0, 1).
Vectors (2, -3, 1, 0) and (-1, -2, 0, 1) thus span W.
Also, identity p(2, -3, 1, 0) + q(-1, -2, 0, 1) = (0, 0, 0, 0) leads to p=0, q=0. The two vectors are thus linearly independent. The set {(2, -3, 1, 0), (-1, -2, 0, 1)} is therefore a basis for W.
Dimension of W is 2. Solutions form a plane through the origin in $\mathbf{R}^4$.

16. (2r, -r, 4r, r) = r(2, -1, 4, 1). The set of solutions form the line through the origin in $\mathbf{R}^4$ defined by the vector (2,-1,4,1). Set {(2, -1, 4)} is a basis. Dimension of W is 1.

17. Separate the variables in the solution, $(3r-s, r, s) = r(3, 1, 0) + s(-1, 0, 1)$.
Vectors $(3, 1, 0)$ and $(-1, 0, 1)$ thus span W.
Also, identity $p(3, 1, 0) + q(-1, 0, 1) = (0, 0, 0)$ leads to $p=0$, $q=0$. The two vectors are thus linearly independent. The set $\{(3, 1, 0), (-1, 0, 1)\}$ is therefore a basis for W.
Dimension of W is 2. Solutions form a plane through the origin in $\mathbf{R}^3$.

18. Separate the variables in the solution, $(2r + t, 3r - 2s, r, s, t) = r(2, 3, 1, 0, 0) + s(0, -2, 0, 1, 0) + t(1, 0, 0, 0, 1)$. Vectors $(2, 3, 1, 0, 0)$, $(0, -2, 0, 1, 0)$, $(1, 0, 0, 0, 1)$ thus span W. The identity $p(2, 3, 1, 0, 0) + q(0, -2, 0, 1, 0) + h(1, 0, 0, 0, 1) = (0, 0, 0, 0, 0)$ leads to $p = 0$, $q = 0$, $h = 0$. The vectors are thus linearly independent. The set $\{(2, 3, 1, 0, 0), (0, -2, 0, 1, 0), (1, 0, 0, 0, 1)\}$ is a basis for W. The dimension of W is 3. Solutions form a three-dimensional subspace of the five-dimensional space $\mathbf{R}^5$.

19. (a) $a(1, 0, 2) + b(1, 1, 0) + c(5, 3, 6)=\mathbf{0}$ gives $a+b+5c=0$, $b+3c=0$, $2a+6c=0$.
Unique solution $a=0$, $b=0$, $c=0$. Thus linearly independent.

(b) $a(1,1,1) + b(2, -1, 1) + c(3, -3, 0)=\mathbf{0}$ gives $a + 2b + 3c = 0$, $a - b -3c = 0$, $a + b = 0$.
Unique solution $a=0$, $b=0$, $c=0$. Thus linearly independent.

(c) $a(1, -1, 1) +b(2, 1, 0)+c(4, -1, 2)=\mathbf{0}$ gives $a+2b+4c=0$, $-a+b-c=0$, $a+2c=0$.
Many solutions, $a=-2r$, $b=-r$, $c=r$, where r is a real number. Thus linearly dependent.

(d) $a(1, 2, 1)+b(-2, 1, 3)+c(-1, 8, 9)=\mathbf{0}$ gives $a-2b-c=0$, $2a+b+8c=0$, $a+3b+9c=0$.
Many solutions, $a=-3r$, $b=-2r$, $c=r$. Thus linearly dependent.

(e) $a(-2,0,3) + b(5,2,1) + c(10,6,9)=0$ gives $-2a+5b+10c=0$, $2b+6c=0$, $3a+b+9c=0$.
Unique solution $a=0$, $b=0$, $c=0$. Thus linearly independent.

(f) $a(3,4,1)+b(2,1,0)+c(9,7,1)=\mathbf{0}$ gives $3a+2b+9c=0$, $4a+b+7c=0$, $a+c=0$.
Many solutions, $a=-r$, $b=-3r$, $c=r$. Thus linearly dependent.

Exercise Set 1.6

1. (a) $(2,1)\cdot(3,4) = 2\text{x}3 + 1\text{x}4 = 6 + 4 = 10$

(b) $(1,-4)\cdot(3,0) = 1x3 + -4x0 = 3$

(c) $(2,0)\cdot(0,-1) = 2x0 + 0x-1= 0$

(d) $(5,-2)\cdot(-3,-4) = 5x-3 + -2x-4 = -15 + 8 = -7$

2. (a) $(1,2,3)\cdot(4,1,0) = 1x4 + 2x1 + 3x0 = 4 + 2 + 0 = 6$

(b) $(3,4,-2)\cdot(5,1,-1) = 3x5 + 4x1 + -2x-1 = 15 + 4 + 2 = 21$

(c) $(7,1,-2)\cdot(3,-5,8) = 7x3 + 1x-5 + -2x8 = 21 - 5 - 16 = 0$

(d) $(3, 2, 0).(5, -2, 8) = 3x5 + 2x-2 + 0x8 = 15 - 4 + 0 = 11$

3. (a) $(5,1)\cdot(2,-3) = 5x2 + 1x-3 = 10 - 3 = 7$

(b) $(-3,1,5)\cdot(2,0,4) = -3x2 + 1x0 + 5x4 = -6 + 0 + 20 = 14$

(c) $(7,1,2,-4)\cdot(3,0,-1,5) = 7x3 + 1x0 + 2x-1 + -4x5 = 21 + 0 - 2 -20 = -1$

(d) $(2,3,-4,1,6)\cdot(-3,1,-4,5,-1) = 2x-3 + 3x1 + -4x-4 + 1x5 + 6x-1$
$= -6 + 3 + 16 + 5 - 6 = 12$

(e) $(1,2,3,0,0,0)\cdot(0,0,0,-2,-4,9) = 1x0 + 2x0 + 3x0 + 0x-2 + 0x-4 + 0x9 = 0$

4. (a) $\begin{bmatrix} 1 \\ 3 \end{bmatrix} . \begin{bmatrix} -2 \\ 5 \end{bmatrix} = 1x-2 + 3x5 = -2 + 15 = 13$

(b) $\begin{bmatrix} 5 \\ 0 \end{bmatrix} . \begin{bmatrix} 4 \\ -6 \end{bmatrix} = 5x4 + 0x-6 = 20 +0 = 20$

(c) $\begin{bmatrix} 2 \\ 0 \\ -5 \end{bmatrix} . \begin{bmatrix} 3 \\ 6 \\ -4 \end{bmatrix} = 2x3 + 0x6 + -5x-4 = 6 + 0 + 20 = 26$

(d) $\begin{bmatrix} 1 \\ 3 \\ -7 \end{bmatrix} . \begin{bmatrix} -2 \\ 8 \\ -3 \end{bmatrix} = 1x-2 + 3x8 + -7x-3 = -2 + 24 + 21 = 43$

5. (a) $\|(1, 2)\| = \sqrt{1^2 + 2^2} = \sqrt{5}$

(b) $\|(3, -4)\| = \sqrt{3^2 + (-4)^2} = \sqrt{25} = 5$

(c) $\|(4, 0)\| = \sqrt{4^2 + 0^2} = \sqrt{16} = 4$

(d) $\|(-3, 1)\| = \sqrt{(-3)^2 + 1^2} = \sqrt{10}$

(e) $\|(0, 27)\| = \sqrt{0^2 + 27^2} = 27$

6. (a) $\|(1,3,-1)\| = \sqrt{1^2 + 3^2 + (-1)^2} = \sqrt{11}$

(b) $\|(3,0,4)\| = \sqrt{3^2 + 0^2 + 4^2} = \sqrt{25} = 5$

(c) $\|(5,1,1)\| = \sqrt{5^2 + 1^2 + 1^2} = \sqrt{27} = 3\sqrt{3}$

(d) $\|(0,5,0)\| = \sqrt{0^2 + 5^2 + 0^2} = \sqrt{25} = 5$

(e) $\|(7,-2,-3)\| = \sqrt{7^2 + (-2)^2 + (-3)^2} = \sqrt{62}$

7. (a) $\|(5,2)\| = \sqrt{5^2 + 2^2} = \sqrt{29}$

(b) $\|(-4,2,3)\| = \sqrt{(-4)^2 + 2^2 + 3^2} = \sqrt{29}$

(c) $\|(1,2,3,4)\| = \sqrt{1^2 + 2^2 + 3^2 + 4^2} = \sqrt{30}$

(d) $\|(4,-2,1,3)\| = \sqrt{4^2 + (-2)^2 + 1^2 + 3^2} = \sqrt{30}$

(e) $\|(-3,0,1,4,2)\| = \sqrt{(-3)^2 + 0^2 + 1^2 + 4^2 + 2^2} = \sqrt{30}$

(f) $\|(0,0,0,7,0,0)\| = \sqrt{0^2 + 0^2 + 0^2 + 7^2 + 0^2 + 0^2} = \sqrt{49} = 7$

8. (a) $\left\| \begin{bmatrix} 3 \\ 4 \end{bmatrix} \right\| = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$

(b) $\left\| \begin{bmatrix} 2 \\ -7 \end{bmatrix} \right\| = \sqrt{2^2 + (-7)^2} = \sqrt{53}$

(c) $\left\|\begin{bmatrix}1\\2\\3\end{bmatrix}\right\| = \sqrt{1^2+2^2+3^2} = \sqrt{14}$

(d) $\left\|\begin{bmatrix}-2\\0\\5\end{bmatrix}\right\| = \sqrt{(-2)^2+0^2+5^2} = \sqrt{29}$

(e) $\left\|\begin{bmatrix}2\\3\\5\\9\end{bmatrix}\right\| = \sqrt{2^2+3^2+5^2+9^2} = \sqrt{119}$

9. (a) $\dfrac{(1,3)}{\|(1,3)\|} = \left(\dfrac{1}{\sqrt{10}}, \dfrac{3}{\sqrt{10}}\right)$

(b) $\dfrac{(2,-4)}{\|(2,-4)\|} = \left(\dfrac{2}{2\sqrt{5}}, \dfrac{-4}{2\sqrt{5}}\right) = \left(\dfrac{1}{\sqrt{5}}, \dfrac{-2}{\sqrt{5}}\right)$

(c) $\dfrac{(1,2,3)}{\|(1,2,3)\|} = \left(\dfrac{1}{\sqrt{14}}, \dfrac{2}{\sqrt{14}}, \dfrac{3}{\sqrt{14}}\right)$

(d) $\dfrac{(-2,4,0)}{\|(-2,4,0)\|} = \left(\dfrac{-2}{\sqrt{20}}, \dfrac{4}{\sqrt{20}}, 0\right) = \left(\dfrac{-1}{\sqrt{5}}, \dfrac{2}{\sqrt{5}}, 0\right)$

(e) $\dfrac{(0,5,0)}{\|(0,5,0)\|} = (0,1,0)$

10. (a) $\dfrac{(4,2)}{\|(4,2)\|} = \left(\dfrac{4}{2\sqrt{5}}, \dfrac{2}{2\sqrt{5}}\right) = \left(\dfrac{2}{\sqrt{5}}, \dfrac{1}{\sqrt{5}}\right)$

(b) $\dfrac{(4,1,1)}{\|(4,1,1)\|} = \left(\dfrac{4}{3\sqrt{2}}, \dfrac{1}{3\sqrt{2}}, \dfrac{1}{3\sqrt{2}}\right)$

(c) $\dfrac{(7,2,0,1)}{\|(7,2,0,1)\|} = \left(\dfrac{7}{3\sqrt{6}}, \dfrac{2}{3\sqrt{6}}, 0, \dfrac{1}{3\sqrt{6}}\right)$

(d) $\dfrac{(3,-1,1,2)}{\|(3,-1,1,2)\|} = \left(\dfrac{3}{\sqrt{15}}, \dfrac{-1}{\sqrt{15}}, \dfrac{1}{\sqrt{15}}, \dfrac{2}{\sqrt{15}}\right)$

(e) $\dfrac{(0,0,0,7,0,0)}{\|(0,0,0,7,0,0)\|} = (0,0,0,1,0,0)$

11. (a) $\begin{bmatrix}4\\3\end{bmatrix} / \left\|\begin{bmatrix}4\\3\end{bmatrix}\right\| = \begin{bmatrix}4/5\\3/5\end{bmatrix}$ (b) $\begin{bmatrix}1\\-3\end{bmatrix} / \left\|\begin{bmatrix}1\\-3\end{bmatrix}\right\| = \begin{bmatrix}1/\sqrt{10}\\-3/\sqrt{10}\end{bmatrix}$

(c) $\begin{bmatrix}3\\4\\0\end{bmatrix} / \left\|\begin{bmatrix}3\\4\\0\end{bmatrix}\right\| = \begin{bmatrix}3/5\\4/5\\0\end{bmatrix}$ (d) $\begin{bmatrix}-1\\2\\-5\end{bmatrix} / \left\|\begin{bmatrix}-1\\2\\-5\end{bmatrix}\right\| = \begin{bmatrix}-1/\sqrt{30}\\2/\sqrt{30}\\-5/\sqrt{30}\end{bmatrix}$

(e) $\begin{bmatrix}3\\0\\1\\8\end{bmatrix} / \left\|\begin{bmatrix}3\\0\\1\\8\end{bmatrix}\right\| = \begin{bmatrix}3/\sqrt{74}\\0\\1/\sqrt{74}\\8/\sqrt{74}\end{bmatrix}$

12. (a) $\cos\theta = \dfrac{(-1,1)\Diamond(0,1)}{\|(-1,1)\|\,\|(0,1)\|} = \dfrac{1}{\sqrt{2}}$. $\theta = \dfrac{\pi}{4} = 45°$

(b) $\cos\theta = \dfrac{(2,0)\cdot(1,\sqrt{3})}{\|(2,0)\|\,\|(1,\sqrt{3})\|} = \dfrac{2}{4} = \dfrac{1}{2}$. $\theta = \dfrac{\pi}{3} = 60°$

(c) $\cos\theta = \dfrac{(2,3)\cdot(3,-2)}{\|(2,3)\|\,\|(3,-2)\|} = 0$. $\theta = \dfrac{\pi}{2} = 90°$

(d) $\cos\theta = \dfrac{(5,2)\cdot(-5,-2)}{\|(5,2)\|\,\|(-5,-2)\|} = \dfrac{-29}{29} = -1$. $\theta = \pi = 180°$

13. (a) $\cos\theta = \dfrac{(4,-1)\cdot(2,3)}{\|(4,-1)\|\,\|(2,3)\|} = \dfrac{5}{\sqrt{17}\sqrt{13}}$ $(\theta=70.3462^0)$

(b) $\cos\theta = \dfrac{(3,-1,2)\cdot(4,1,1)}{\|(3,-1,2)\|\,\|(4,1,1)\|} = \dfrac{13}{\sqrt{14}\sqrt{18}} = \dfrac{13}{6\sqrt{7}}$ $(\theta=35.0229^0)$

(c) $\cos\theta = \dfrac{(2,-1,0)\cdot(5,3,1)}{\|(2,-1,0)\|\,\|(5,3,1)\|} = \dfrac{7}{\sqrt{5}\sqrt{35}} = \dfrac{7}{5\sqrt{7}} = \dfrac{\sqrt{7}}{5}$ $(\theta = 58.0519^0)$

(d) $\cos\theta = \dfrac{(7,1,0,0)\cdot(3,2,1,0)}{\|(7,1,0,0)\|\,\|(3,2,1,0)\|} = \dfrac{23}{\sqrt{50}\sqrt{14}} = \dfrac{23}{10\sqrt{7}}$ $(\theta=29.6205^0)$

(e) $\cos\theta = \dfrac{(1,2,-1,3,1)\cdot(2,0,1,0,4)}{\|(1,2,-1,3,1)\|\,\|(2,0,1,0,4)\|} = \dfrac{5}{4\sqrt{21}}$ ($\theta=74.1707^0$)

14. (a) $\cos\theta = \dfrac{\begin{bmatrix}1\\2\end{bmatrix}\cdot\begin{bmatrix}-1\\4\end{bmatrix}}{\left\|\begin{bmatrix}1\\2\end{bmatrix}\right\|\left\|\begin{bmatrix}-1\\4\end{bmatrix}\right\|} = \dfrac{7}{\sqrt{5}\sqrt{17}}$ ($\theta=40.6013^0$)

(b) $\cos\theta = \dfrac{\begin{bmatrix}5\\1\end{bmatrix}\cdot\begin{bmatrix}0\\-3\end{bmatrix}}{\left\|\begin{bmatrix}5\\1\end{bmatrix}\right\|\left\|\begin{bmatrix}0\\-3\end{bmatrix}\right\|} = \dfrac{-3}{\sqrt{26}\sqrt{9}} = \dfrac{-1}{\sqrt{26}}$ ($\theta=101.3099^0$)

(c) $\cos\theta = \dfrac{\begin{bmatrix}1\\-3\\0\end{bmatrix}\cdot\begin{bmatrix}2\\5\\-1\end{bmatrix}}{\left\|\begin{bmatrix}1\\-3\\0\end{bmatrix}\right\|\left\|\begin{bmatrix}2\\5\\-1\end{bmatrix}\right\|} = \dfrac{-13}{\sqrt{10}\sqrt{30}} = \dfrac{-13}{10\sqrt{3}}$ ($\theta=138.6385^0$)

(d) $\cos\theta = \dfrac{\begin{bmatrix}-2\\3\\-4\end{bmatrix}\cdot\begin{bmatrix}2\\5\\-1\end{bmatrix}}{\left\|\begin{bmatrix}-2\\3\\-4\end{bmatrix}\right\|\left\|\begin{bmatrix}2\\5\\-1\end{bmatrix}\right\|} = \dfrac{15}{\sqrt{29}\sqrt{30}} = \dfrac{\sqrt{15}}{\sqrt{29}\sqrt{2}}$ ($\theta=59.4329^0$)

15. (a) $(1,3)\cdot(3,-1) = 1\text{x}3 + 3\text{x}-1 = 0$, thus the vectors are orthogonal.

(b) $(-2,4)\cdot(4,2) = -2\text{x}4 + 4\text{x}2 = 0$, thus the vectors are orthogonal.

(c) $(3,0)\cdot(0,-2) = 3\text{x}0 + 0\text{x}-2= 0$, thus the vectors are orthogonal.

(d) $(7,-1)\cdot(1,7) = 7\text{x}1 + -1\text{x}7= 0$, thus the vectors are orthogonal.

16. (a) $(3,-5)\cdot(5,3) = 3\text{x}5 + -5\text{x}3 = 0$, thus the vectors are orthogonal.

(b) $(1,2,-3)\cdot(4,1,2) = 1\text{x}4 + 2\text{x}1 + -3\text{x}2 = 0$, thus the vectors are orthogonal.

(c) (7,1,0)·(2,-14,3) = 7x2 + 1x-14 + 0x3 = 0, thus the vectors are orthogonal.

(d) (5,1,0,2)·(-3,7,9,4) = 5x-3 + 1x7 + 0x9 + 2x4 = 0, thus the vectors are orthogonal.

(e) (1,-1,2,-5,9)·(4,7,4,1,0) = 1x4 + -1x7 + 2x4 + -5x1+ 9x0 = 0, thus the vectors are orthogonal.

17. (a) $\begin{bmatrix} 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} -6 \\ 3 \end{bmatrix}$ = 1x-6 + 2x3 = 0, thus the vectors are orthogonal.

(b) $\begin{bmatrix} 5 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 10 \end{bmatrix}$ = 5x4 + -2x10 = 0, thus the vectors are orthogonal.

(c) $\begin{bmatrix} 4 \\ -1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 8 \\ -1 \end{bmatrix}$ = 4x2 + -1x8 + 0x-1 = 0, thus the vectors are orthogonal.

(d) $\begin{bmatrix} -2 \\ 3 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 6 \\ -7 \end{bmatrix}$ = -2x2 + 3x6 + 2x-7 = 0, thus the vectors are orthogonal.

18. (a) If (a,b) is orthogonal to (1,3), then (a,b)·(1,3) = a + 3b = 0, so a = -3b. Thus any vector of the form (-3b,b) is orthogonal to (1,3).

(b) If (a,b) is orthogonal to (7,-1), then (a,b)·(7,-1) = 7a - b = 0, so b = 7a. Thus any vector of the form (a,7a) is orthogonal to (7,-1).

(c) If (a,b) is orthogonal to (-4,-1), then (a,b)·(-4,-1) = -4a - b = 0, so b = -4a. Thus any vector of the form (a,-4a) is orthogonal to (-4,-1).

(d) If (a,b) is orthogonal to (-3,0), then (a,b)·(-3,0) = -3a = 0, so a = 0. Thus any vector of the form (0,b) is orthogonal to (-3,0).

19. (a) If (a,b) is orthogonal to (5,-1), then (a,b)·(5,-1) = 5a - b = 0, so b = 5a. Thus any vector of the form (a,5a) is orthogonal to (5,-1).

(b) If (a,b,c) is orthogonal to (1,-2,3), then (a,b,c)·(1,-2,3) = a - 2b +3c = 0, so a = 2b-3c. Thus any vector of the form (2b-3c,b,c) is orthogonal to (1,-2,3).

(c) If (a,b,c) is orthogonal to (5,1,-1), then (a,b,c)·(5,1,-1) = 5a + b - c = 0, so c = 5a+b. Thus any vector of the form (a,b,5a+b) is orthogonal to (5,1,-1).

(d) If (a,b,c,d) is orthogonal to (5,0,1,1), then $(a,b,c,d)\cdot(5,0,1,1) = 5a + c + d = 0$, so $d = -5a-c$. Thus any vector of the form $(a,b,c,-5a-c)$ is orthogonal to (5,0,1,1).

(e) If (a,b,c,d) is orthogonal to (6,-1,2,3), then $(a,b,c,d)\cdot(6,-1,2,3) = 6a - b + 2c + 3d = 0$, so $b = 6a+2c+3d$. Thus any vector of the form $(a,6a+2c+3d, c,d)$ is orthogonal to (6,-1,2,3).

(f) If (a,b,c,d,e) is orthogonal to (0,-2,3,1,5), then $(a,b,c,d,e)\cdot(0,-2,3,1,5) = -2b +3c + d + 5e = 0$, so $d = 2b-3c-5e$. Thus any vector of the form $(a,b,c,2b-3c-5e,e)$ is orthogonal to (0,-2,3,1,5).

20. If (a,b,c) is orthogonal to both (1,2,-1) and (3,1,0), then $(a,b,c)\cdot(1,2,-1) = a + 2b - c = 0$ and $(a,b,c)\cdot(3,1,0) = 3a + b = 0$. These equations yield the solution $b = -3a$ and $c = -5a$, so any vector of the form $(a,-3a,-5a)$ is orthogonal to both (1,2,-1) and (3,1,0).

21. Let (a,b,c) be in W. Then (a,b,c) is orthogonal to (-1,1,1). $(a,b,c)\cdot(-1,1,1)=0$, $-a+b+c=0$, $c=a-b$. W consists of vectors of the form (a,b,a-b). Separate the variables. $(a,b,a-b)=a(1,0,1)+b(0,1,-1)$. (1,0,1), (0,1,-1) span W. Vectors are also linearly independent. {(1,0,1),(0,1,-1)} is a basis for W. The dimension of W is 2. It is a plane spanned by (1,0,1) and (0,1,-1).

22. Let (a,b,c) be in W. Then (a,b,c) is orthogonal to (-3,4,1). $(a,b,c)\cdot(-3,4,1)=0$, $-3a+4b+c=0$, $c=3a-4b$. W consists of vectors of the form (a,b,3a-4b). Separate the variables. $(a,b,3a-4b)=a(1,0,3)+b(0,1,-4)$. {(1,0,3), ((0,1,-4)} is a basis for W. The dimension of W is 2. It is a plane spanned by (1,0,3) and (0,1,-4).

23. Let (a,b,c) be in W. Then (a,b,c) is orthogonal to (1,-2,5). $(a,b,c)\cdot(1,-2,5)=0$, $a-2b+5c=0$, $a=2b-5c$. W consists of vectors of the form (2b-5c,b,c). Separate the variables. $(2b-5c,b,c)=b(2,1,0)+c(-5,0,1)$. {(2,1,0), ((-5,0,1)} is a basis for W. The dimension of W is 2. It is a plane spanned by (2,1,0) and (-5,0,1).

24. Let (a,b,c,d) be in W. Then (a,b,c,d) is orthogonal to (1,-3,7,4).$(a,b,c,d)\cdot(1,-3,7,4)=0$, $a-3b+7c+4d=0$, $a=3b-7c-4d$. W consists of vectors of the form (3b-7c-d,b,c,d). Separate the variables. $(3b-7c-4d,b,c,d)=b(3,1,0,0)+c(-7,0,1,0) +d(-4,0,0,1)$. {(3,1,0,0), (-7,0,1,0), (-4,0,0,1)} is a basis for W. The dimension of W is 3.

25. (a) $d = \sqrt{(6-2)^2+(5-2)^2} = 5$. (b) $d = \sqrt{(3+4)^2+(1-0)^2} = \sqrt{50} = 5\sqrt{2}$.

(c) $d = \sqrt{(7-2)^2+(-3-2)^2} = \sqrt{50} = 5\sqrt{2}$. (d) $d = \sqrt{(1-5)^2+(-3-1)^2} = 4\sqrt{2}$.

26. (a) $d = \sqrt{(4-2)^2+(1+3)^2} = \sqrt{20} = 2\sqrt{5}$. (b) $d = \sqrt{(1-2)^2+(2-1)^2+(3-0)^2} = \sqrt{11}$.

(c) $d = \sqrt{(-3-4)^2+(1+1)^2+(2-1)^2} = \sqrt{54} = 3\sqrt{6}$.

(d) $d^2 = (5-2)^2 + (1-0)^2 + (0-1)^2 + (0-3)^2 = 20$, so $d = \sqrt{20} = 2\sqrt{5}$.

(e) $d^2 = (-3-2)^2 + (1-1)^2 + (1-4)^2 + (0-1)^2 + (2+1)^2 = 44$, so $d = \sqrt{44} = 2\sqrt{11}$.

27. (a)
$$\begin{aligned}(\mathbf{u}+\mathbf{v})\cdot\mathbf{w} &= (u_1+v_1)w_1 + (u_2+v_2)w_2 + \dots + (u_n+v_n)w_n\\ &= u_1w_1 + v_1w_1 + u_2w_2 + v_2w_2 + \dots + u_nw_n + v_nw_n\\ &= u_1w_1 + u_2w_2 + \dots + u_nw_n + v_1w_1 + v_2w_2 + \dots + v_nw_n = \mathbf{u}\cdot\mathbf{w} + \mathbf{v}\cdot\mathbf{w}.\end{aligned}$$

(b) $c\mathbf{u}\cdot\mathbf{v} = cu_1v_1 + cu_2v_2 + \dots + cu_nv_n = c(u_1v_1 + u_2v_2 + \dots + u_nv_n) = c(\mathbf{u}\cdot\mathbf{v})$, and $cu_1v_1 + cu_2v_2 + \dots + cu_nv_n = u_1cv_1 + u_2cv_2 + \dots + u_ncv_n = \mathbf{u}\cdot c\mathbf{v}$.

28. **u** is a positive scalar multiple of **v** so it has the same direction as **v**. The magnitude of **u** is

$$||\mathbf{u}|| = \frac{1}{||\mathbf{v}||}\sqrt{(v_1)^2+(v_2)^2+\dots+(v_n)^2} = \frac{||\mathbf{v}||}{||\mathbf{v}||} = 1,$$ so **u** is a unit vector.

29. **u** and **v** are orthogonal if and only if the cosine of the angle θ between them is zero.

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{||\mathbf{u}||\,||\mathbf{v}||} = 0 \text{ if and only if } \mathbf{u}\cdot\mathbf{v} = 0.$$

30. If $\mathbf{u}\cdot\mathbf{v} = \mathbf{u}\cdot\mathbf{w}$ then $\mathbf{u}\cdot(\mathbf{v}-\mathbf{w}) = 0$ for all vectors **u** in U. Since $\mathbf{v}-\mathbf{w}$ is a vector in U this means that $(\mathbf{v}-\mathbf{w})\cdot(\mathbf{v}-\mathbf{w}) = 0$. Therefore $\mathbf{v}-\mathbf{w} = \mathbf{0}$, so $\mathbf{v} = \mathbf{w}$.

31.
$$\begin{aligned}\mathbf{u}\cdot(a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \dots + a_n\mathbf{v}_n) &= \mathbf{u}\cdot(a_1\mathbf{v}_1) + \mathbf{u}\cdot(a_2\mathbf{v}_2 + \dots + a_n\mathbf{v}_n)\\ &= \mathbf{u}\cdot(a_1\mathbf{v}_1) + \mathbf{u}\cdot(a_2\mathbf{v}_2) + \mathbf{u}\cdot(a_3\mathbf{v}_3 + \dots + a_n\mathbf{v}_n) = \dots = \mathbf{u}\cdot(a_1\mathbf{v}_1) + \mathbf{u}\cdot(a_2\mathbf{v}_2) + \dots\\ &\quad + \mathbf{u}\cdot(a_n\mathbf{v}_n)\\ &= a_1(\mathbf{u}\cdot\mathbf{v}_1) + a_2(\mathbf{u}\cdot\mathbf{v}_2) + \dots + a_n(\mathbf{u}\cdot\mathbf{v}_n) = a_1\,\mathbf{u}\cdot\mathbf{v}_1 + a_2\,\mathbf{u}\cdot\mathbf{v}_2 + \dots + a_n\,\mathbf{u}\cdot\mathbf{v}_n.\end{aligned}$$

32. (a) vector (b) not valid (c) not valid (d) scalar

(e) not valid (f) scalar (g) not valid

(h) not valid

33. $||c(3,0,4)|| = \sqrt{3cx3c+4cx4c} = |c|\sqrt{9+16} = 5|c| = 15$, so $|c| = 3$ and $c = \pm 3$.

34. $||\mathbf{u}+\mathbf{v}||^2 = (u_1 +v_1)^2 + (u_2 +v_2)^2 + . . . + (u_n +v_n)^2$

$= u_1^2 + 2u_1 v_1 + v_1^2 + u_2^2 + 2u_2 v_2 + v_2^2 + . . . + u_n^2 + 2u_n v_n + v_n^2$

$= u_1^2 + u_2^2 + . . . + u_n^2 + 2u_1 v_1 + 2u_2 v_2 + . . . + 2u_n v_n + v_1^2 + v_2^2 + . . . + v_n^2$

$= u_1^2 + u_2^2 + . . . + u_n^2 + 2(u_1 v_1 + u_2 v_2 + . . . + u_n v_n) + v_1^2 + v_2^2 + . . . + v_n^2$

$= ||\mathbf{u}||^2 + 2(\mathbf{u}\cdot\mathbf{v}) + ||\mathbf{v}||^2 = ||\mathbf{u}||^2 + ||\mathbf{v}||^2$ if and only if $\mathbf{u}\cdot\mathbf{v} = 0$, i.e., if and only if $\mathbf{u}$ and $\mathbf{v}$ are orthogonal.

35. $(a,b)\cdot(-b,a) = a \times -b + b \times a = 0$, so $(-b,a)$ is orthogonal to (a,b).

36. $(\mathbf{u} + \mathbf{v})\cdot(\mathbf{u} - \mathbf{v}) = (u_1 + v_1)(u_1 - v_1) + (u_2 + v_2)(u_2 - v_2) + . . . + (u_n + v_n)(u_n - v_n)$
$= u_1^2 - v_1^2 + u_2^2 - v_2^2 + . . . + u_n^2 - v_n^2$
$= u_1^2 + u_2^2 + . . . + u_n^2 - v_1^2 - v_2^2 - . . . - v_n^2 = ||\mathbf{u}|| - ||\mathbf{v}||$.

Thus $||\mathbf{u}|| - ||\mathbf{v}||=0$ if and only if $(\mathbf{u} + \mathbf{v})\cdot(\mathbf{u} - \mathbf{v}) = 0$. That is $||\mathbf{u}|| = ||\mathbf{v}||$ if and only if $\mathbf{u} + \mathbf{v}$ and $\mathbf{u} - \mathbf{v}$ are orthogonal.

37. (a) $||\mathbf{u}||^2 = u_1^2 +u_2^2 + . . . +u_n^2 \geq 0$, so $||\mathbf{u}|| \geq 0$.

(b) $||\mathbf{u}|| = 0$ if and only if $u_1^2 +u_2^2 + . . . +u_n^2 = 0$ if and only if $u_1 = u_2 = . . . = u_n = 0$.

(c) $||c\mathbf{u}||^2 = (cu_1)^2 +(cu_2)^2 + . . . +(cu_n)^2 = c^2 (u_1^2+u_2^2+ . . . +u_n^2) = c^2 ||\mathbf{u}||^2$,
so $||c\mathbf{u}|| = |c|\ ||\mathbf{u}||$.

38. (a) $||\mathbf{u}|| = |u_1| + |u_2| + \ldots + |u_n| \geq 0$ since each term is equal to or greater than zero.

$|u_1| + |u_2| + \ldots + |u_n| = 0$ if and only if each term is zero.

$||c\mathbf{u}|| = |cu_1| + |cu_2| + \ldots + |cu_n| = |c||u_1| + |c||u_2| + \ldots + |c||u_n|$

$= |c|(|u_1| + |u_2| + \ldots + |u_n|) = |c|\ ||\mathbf{u}||.$

$||(1,2)|| = |1| + |2| = 3$, $||(-3,4)|| = |-3| + |4| = 7$, $||(1,2,-5)|| = |1| + |2| + |-5| = 8$,

and $||(0,-2,7)|| = |0| + |-2| + |7| = 9$.

(b) $||\mathbf{u}|| = \max_{i=1,\ldots,n} |u_i| \geq 0$ since the absolute value of any number is equal to or greater than zero.

$\max_{i=1,\ldots,n} |u_i| = 0$ if and only if all $|u_i| = 0$.

$||c\mathbf{u}|| = \max_{i=1,\ldots,n} |cu_i| = |c| \max_{i=1,\ldots,n} |u_i| = |c|\ ||\mathbf{u}||.$

For (a), $||(1,2)|| = |1| + |2| = 3$, $||(-3,4)|| = |-3| + |4| = 7$, $||(1,2,-5)|| = |1| + |2| + |-5| = 8$,

and $||(0,-2,7)|| = |0| + |-2| + |7| = 9$.

For (b), $||(1,2)|| = |2| = 2$, $||(-3,4)|| = |4| = 4$, $||(1,2,-5)|| = |-5| = 5$, and $||(0,-2,7)|| = |7| = 7$.

39. (a) $d(\mathbf{x},\mathbf{y}) = ||\mathbf{x}-\mathbf{y}|| \geq 0$.

(b) $d(\mathbf{x},\mathbf{y}) = ||\mathbf{x}-\mathbf{y}|| = 0$ if and only if $\mathbf{x}-\mathbf{y} = \mathbf{0}$ if and only if $\mathbf{x} = \mathbf{y}$.

(c) $d(\mathbf{x},\mathbf{z}) = ||\mathbf{x}-\mathbf{y}+\mathbf{y}-\mathbf{z}|| \leq ||\mathbf{x}-\mathbf{y}|| + ||\mathbf{y}-\mathbf{z}|| = d(\mathbf{x},\mathbf{y}) + d(\mathbf{y},\mathbf{z})$, from the triangle inequality.

<u>Exercise Set 1.7</u>

Exercises 1, 2, and 3 can be solved simultaneously since the coefficient matrices are the same for all three.