Anomaly of electromagnetic duality of the Maxwell theory

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based on my work with

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$$
\begin{aligned}
\operatorname{div}\vec{B} &= 0, & \operatorname{rot}\vec{E} + \partial_t\vec{B} &= \vec{0},\\
\operatorname{div}\vec{E} &= 0, & \operatorname{rot}\vec{B} - \partial_t\vec{E} &= \vec{0}.
\end{aligned}
$$

Very important.

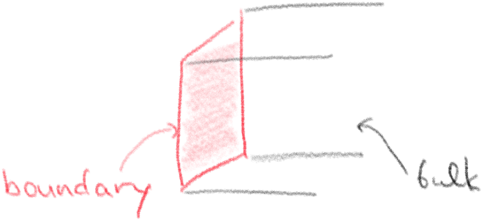
$$\mathbf{div}\,\vec{B} = 0, \qquad \mathbf{rot}\,\vec{E} + \partial_t \vec{B} = \vec{0},$$
$$\mathbf{div}\,\vec{E} = 0, \qquad \mathbf{rot}\,\vec{B} - \partial_t \vec{E} = \vec{0}.$$

Not invariant under the Galilei transformation!

$$\downarrow$$

It's rather invariant under the Lorentz transformation!

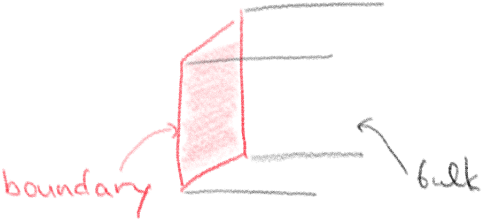
Anomalies and **topological phases** are closely related.



Consider the prototypical case of the integer quantum Hall effect:

- **1+1**d boundary hosts **gapless chiral fermion**
- **2+1**d bulk described by **Chern-Simons**

Anomalies and **topological phases** are closely related.



Today I consider instead

- **3+1**d boundary hosts **the Maxwell theory**
- **4+1**d bulk described by **a cousin of Chern-Simons**

Electromagnetic duality and Dirac quantization condition

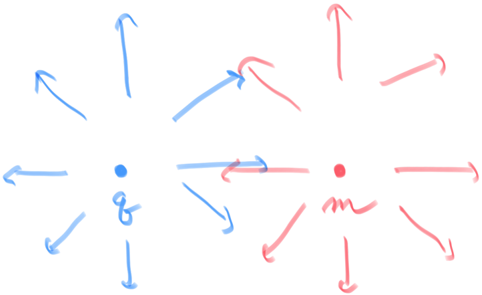
$$\mathbf{div}\,\vec{B} = 0, \qquad \mathbf{rot}\,\vec{E} + \partial_t \vec{B} = \vec{0},$$
$$\mathbf{div}\,\vec{E} = 0, \qquad \mathbf{rot}\,\vec{B} - \partial_t \vec{E} = \vec{0}.$$

Electromagnetic duality: $\vec{E} \leftrightarrow \vec{B}$

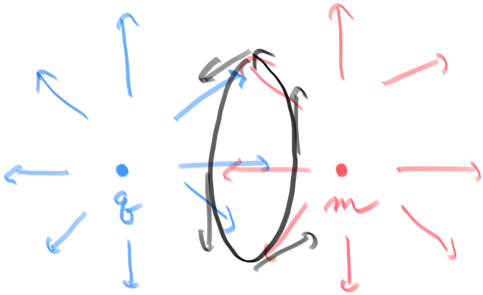
$$\mathrm{div}\,\vec{B} = 0, \qquad \mathrm{rot}\,\vec{E} + \partial_t \vec{B} = \vec{0},$$
$$\mathrm{div}\,\vec{E} = 0, \qquad \mathrm{rot}\,\vec{B} - \partial_t \vec{E} = \vec{0}.$$

$$S : (\vec{E}, \vec{B}) \mapsto (-\vec{B}, \vec{E})$$
$$S^2 : (\vec{E}, \vec{B}) \mapsto -(\vec{E}, \vec{B})$$

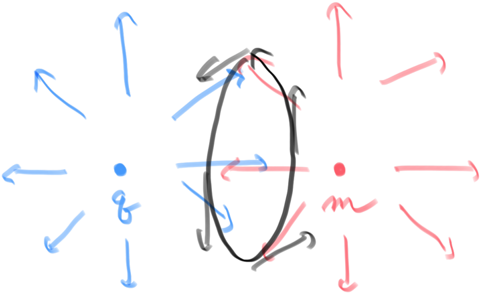
Although called duality, doing twice is still nontrivial.



Consider electric charge q and magnetic charge m.



Poynting vector $\vec{\boldsymbol{E}} \times \vec{\boldsymbol{B}}$ gives
the angular momentum $\propto \boldsymbol{qm}$.

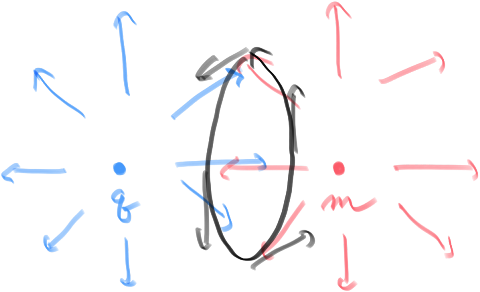


We fix the units so that

$$\text{anglular momentum} = \frac{\hbar}{2} qm.$$

QM demands qm is an integer.

This is the **Dirac quantization condition** (1931).



We haven't chosen the units of charges either.

$$\text{anglular momentum} = \frac{\hbar}{2} qm.$$

Choose a dimensionless unit such that the smallest charges are $q = 1$ and $m = 1$.

$$(q, m) \rightleftarrows (q', m')$$

We can also consider particles having both electric and magnetic charges.

$$\text{angular momentum} = \frac{\hbar}{2}(qm' - mq') = \frac{\hbar}{2}\det\begin{pmatrix} q & q' \\ m & m' \end{pmatrix}$$

The transformation

$$\begin{pmatrix} \vec{E} \\ \vec{B} \end{pmatrix} \to \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \vec{E} \\ \vec{B} \end{pmatrix}$$

acts on (q, m) which are integers.

So a, b, c, d are also integers.

$\det\begin{pmatrix} q & q' \\ m & m' \end{pmatrix}$ needs to be conserved, so $\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = 1$。

$$\left.\begin{array}{c} a, b, c, d \text{ are integers} \\ \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = 1 \end{array}\right\} \Longleftrightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})$$

In particular

$$S : \begin{pmatrix} \vec{E} \\ \vec{B} \end{pmatrix} \mapsto \begin{pmatrix} -\vec{B} \\ +\vec{E} \end{pmatrix}$$

comes from

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Recall that bosons and fermions have angular momenta

boson	$0\hbar$		$\hbar$		$2\hbar$		$\cdots$
fermion		$\frac{1}{2}\hbar$		$\frac{3}{2}\hbar$		$\frac{5}{2}\hbar$	$\cdots$

So we have

$$b + b = b; \qquad b + f = f; \qquad f + f = b$$

Let's imagine a world where neutral particles are all bosons.

Then two possibilities:

elec. charge	$\cdots$	-3	-2	-1	0	$+1$	$+2$	$+3$	$\cdots$
b or f ?		b	b	b	b	b	b	b	

or

elec. charge	$\cdots$	-3	-2	-1	0	$+1$	$+2$	$+3$	$\cdots$
b or f ?		f	b	f	b	f	b	f	

Similarly, there are two possibilities:

mag. charge	$\cdots$	-3	-2	-1	0	$+1$	$+2$	$+3$	$\cdots$
b or f ?		b	b	b	b	b	b	b	

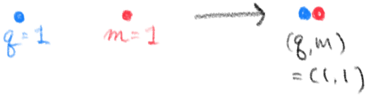
or

mag. charge	$\cdots$	-3	-2	-1	0	$+1$	$+2$	$+3$	$\cdots$
b or f ?		f	b	f	b	f	b	f	

In a world where neutral particles are all bosons, there are $\mathbf{2 \times 2 = 4}$ slightly different types of Maxwell theories, depending on

- whether particles with elec. charge $\boldsymbol{q \equiv 1 \mod 2}$ is $\boldsymbol{b}$ or $\boldsymbol{f}$
- whether particles with mag. charge $\boldsymbol{m \equiv 1 \mod 2}$ is $\boldsymbol{b}$ or $\boldsymbol{f}$

How about $\boldsymbol{b}/\boldsymbol{f}$ of particles with both elec. and mag. charges?



Need to recall that there is an additional angular momentum from $\frac{\hbar}{2}qm = \frac{\hbar}{2}$.
Then, depending on (q, m), we have

$(1,0)$	$+$	$(0,1)$	$\to$	$(1,1)$
b	$+$	b	$\to$	f
b	$+$	f	$\to$	b
f	$+$	b	$\to$	b
f	$+$	f	$\to$	f

$(1,0)$	$+$	$(0,1)$	$\to$	$(1,1)$
b	$+$	b	$\to$	f
b	$+$	f	$\to$	b
f	$+$	b	$\to$	b
f	$+$	f	$\to$	f

$SL(2,\mathbb{Z})$ permutes $(q,m) \equiv (1,0)$, $(0,1)$, $(1,1)$.

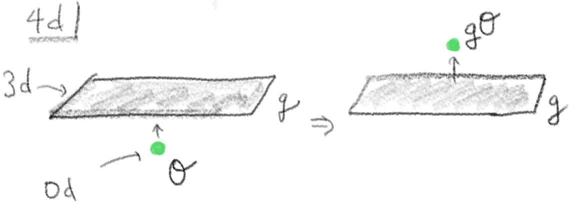
The last one is invariant under $SL(2,\mathbb{Z})$.

(Sometimes called the all fermion electrodynamics.)

The first three permuted by $SL(2,\mathbb{Z})$. Not invariant under $SL(2,\mathbb{Z})$.

Generalized symmetry and the Maxwell theory

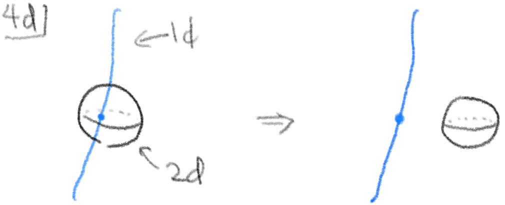
charged particle in 3d   charged particle in 4d



Symmetry $g \in G$ in 4d can be visualized as

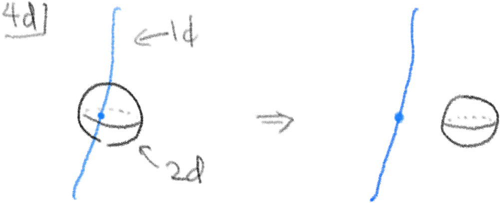
an operator $\mathcal{O}$ crossing a 3d wall.

Take $G = \mathbb{Z}_2$. If $\mathcal{O}$ is odd, gets multiplied by -1 when crossing the wall.

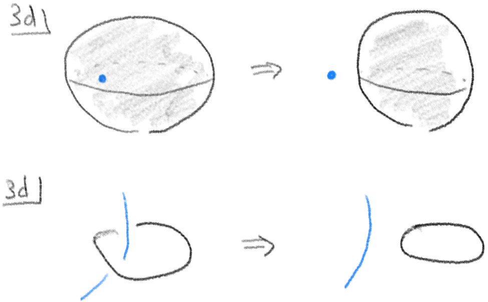


Can consider “symmetry” acting on a line operator, rather than a point operator $\mathcal{O}$.

Captured b a 1d world-line crossing a 2d wall.

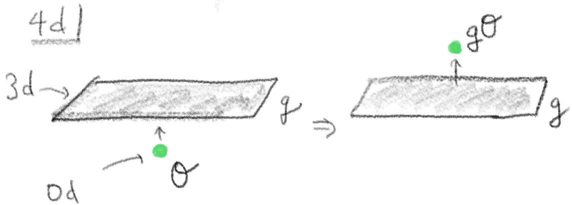


looks differently depending on how to project it:

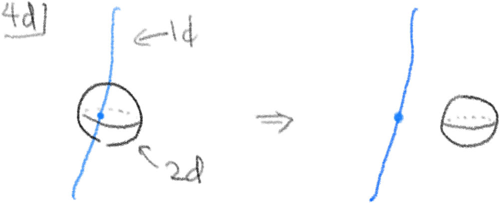


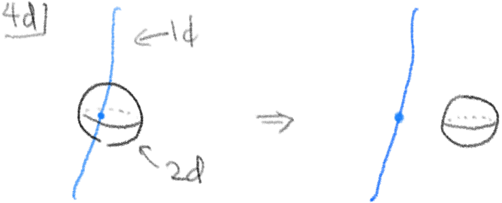
Call a "symmetry" acting on $\boldsymbol{p}$-dim'l objects a $\boldsymbol{p}$-symmetry.

0-symmetry



1-symmetry

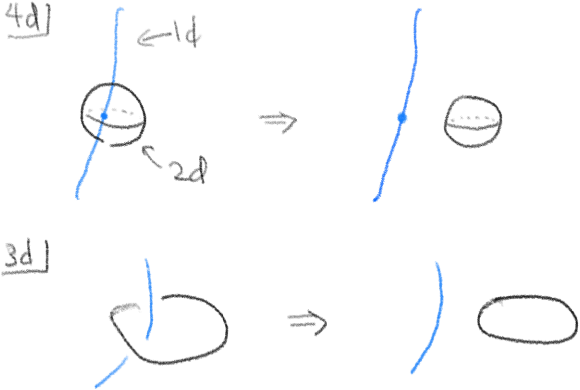




Maxwell theory has the following two $\mathbb{Z}_2$ **1-symmetry**:

- Electric $\mathbb{Z}_2$ **1-symmetry**:
  Multiplied by $(-1)^q$ when crossing a worldline of elec. charge q,
- magnetic $\mathbb{Z}_2$ **1-symmetry**:
  Multiplied by $(-1)^m$ when crossing a worldline of mag. charge m

Note that the exp. value of a world-line of elec. charge q is the Aharanov-Bohm phase



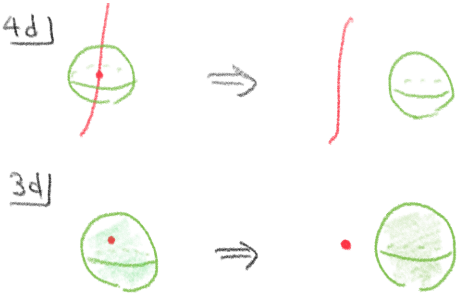
Multiplied by $(-1)^q$ when crossing a worldline of elec. charge q.

Multiplied by $(-1)^q$ when crossing a worldline of elec. charge q.

$$\exp(2\pi i q \int_C \vec{A}\cdot d\vec{x}) \mapsto \exp(2\pi i q \int_{C'} \vec{A}\cdot d\vec{x})(-1)^q$$

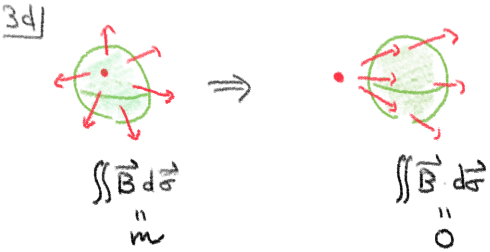
This means that the black wall realizing the electric $\mathbb{Z}_2$ **1-symmetry** has half the flux of the magnetic quantum

$$\vec{B} = \oint \vec{A}\cdot d\vec{x} = \int_C \vec{A}\cdot d\vec{x} - \int_{C'} \vec{A}\cdot d\vec{x} = \pm\frac{1}{2}$$



Multiplied by $(-1)^m$ when crossing a worldline of mag. charge m.

Multiplied by $(-1)^m$ when crossing a worldline of mag. charge m.



This means that the green wall realizing the magnetic $\mathbb{Z}_2$ **1-symmetry** has the factor

$$\exp(\pi i \iint \vec{B} \cdot d\vec{\sigma})$$

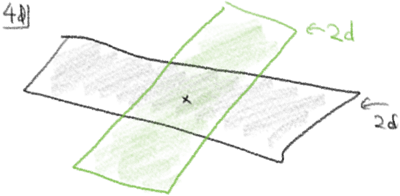
Wall for elec. $\mathbb{Z}_2$ **1-symmetry**

$$\vec{B} = \pm\frac{1}{2}$$

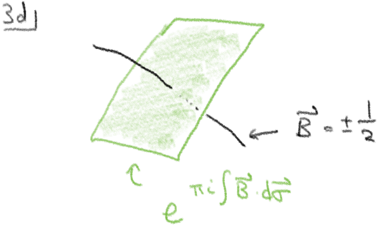
Wall for mag. $\mathbb{Z}_2$ **1-symmetry**

$$\exp(\pi i \iint \vec{B}\cdot d\vec{\sigma})$$

Problematic if both are inserted at the same time, since two 2d surfaces intersects at points in 4d.



If depicted in one lower dimension,



You can't tell if the phase is which of

$$e^{\pm\pi i/2} = \pm i$$

An **anomaly** is when the phase of the partition function and the expectation values become ambiguous in the Quantum Field Theory.

Originally found concerning the $U(1)$ symmetry of chiral fermions in the late 60s.

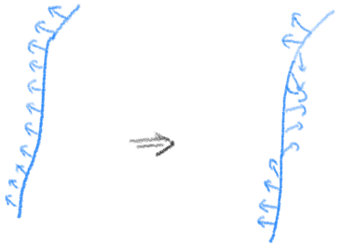
What I described so far is the mixed anomaly of elec. $\mathbb{Z}_2$ **1-symmetry** and mag. $\mathbb{Z}_2$ **1-symmetry**.

Global Gravitational Anomaly of the Maxwell theory

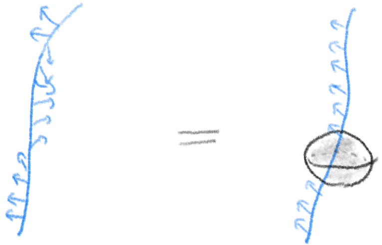
Let's combine the two stories together.

In 3+1d, rotating $\mathbf{720^\circ}$ is trivial, while rotating $\mathbf{360^\circ}$ is nontrivial.

Something is a fermion if $\mathbf{360^\circ}$ gives you $\mathbf{-1}$:



Particles with elec. charge $q = 1$ are fermions
$\leftrightarrow$ twisting by 360° is equal to surrounding by a wall for elec. $\mathbb{Z}_2$ **1-symmetry**



Particles with mag. charge $m = 1$ are fermions
$\leftrightarrow$ twisting by $360°$ is equal to surrounding by a wall for mag. $\mathbb{Z}_2$ **1-symmetry**

You can't say you aren't interested in twisting particles!

When 3+1d spacetime is nontrivial, the pasting together coordinate patches can produce twisting by $\mathbf{360^\circ}$.

This is measured by the **Stiefel-Whitney class** $\boldsymbol{w_2}$, introduced in the 1940s.

They are walls of $\mathbb{Z}_\mathbf{2}$ **1-sym** representing $\mathbf{350^\circ}$ twists, intrinsically present in 3+1d spacetime $\boldsymbol{M_4}$.

Now recall the four versions of the Maxwell theory:

(q,m)	$(1,0)$	$(0,1)$	$(1,1)$
	b	b	f
	b	f	b
	f	b	b
	f	f	f

$SL(2,\mathbb{Z})$ exchanges $(q,m) \equiv (1,0)$, $(0,1)$, $(1,1)$.

The last one is invariant under $SL(2,\mathbb{Z})$.

The first three are permuted by $SL(2,\mathbb{Z})$.

(q,m)	$(1,0)$	$(0,1)$	$(1,1)$
	b	b	f
	b	f	b
	f	b	b
	f	f	f

The first three $SL(2,\mathbb{Z})$ non-invariant have no issues:
You never insert both walls for elec. $\mathbb{Z}_2$ **1-symmetry**
and walls for mag. $\mathbb{Z}_2$ **1-symmetry**.

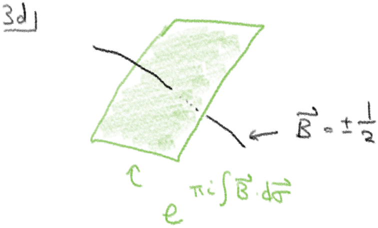
The last $SL(2,\mathbb{Z})$ invariant one has a problem:
Need to insert both walls for elec. $\mathbb{Z}_2$ **1-symmetry**
and walls for mag. $\mathbb{Z}_2$ **1-symmetry**.

The partition function can be ambiguous by ± 1.

(q,m)	$(1,0)$	$(0,1)$	$(1,1)$
	f	f	f

For example, on the complex projective space $\mathbb{CP}^2$,
to account for 360° twists,
we need to put both walls for elec. $\mathbb{Z}_2$ **1-sym.** and walls for mag. $\mathbb{Z}_2$ **1-sym** at $w_2 = \mathbb{CP}^1 \subset \mathbb{CP}^2$.

Two $\mathbb{CP}^1$ within $\mathbb{CP}^2$ intersect at one point:



producing a ± 1 ambiguity.

For example, a global coordinate transformation $[z : w] \mapsto [\bar{z} : \bar{w}]$ of $\mathbb{CP}^2$ flips the sign of the partition function.

Summarizing, the $\boldsymbol{SL(2,\mathbb{Z})}$-invariant Maxwell theory with

(q,m)	$(1,0)$	$(0,1)$	$(1,1)$
	f	f	f

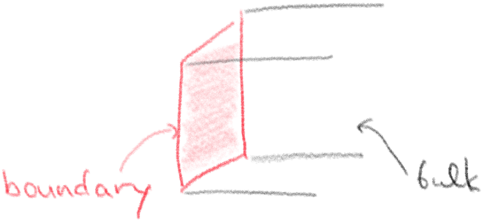
has a **subtle breaking of general covariance**.

This is an example of a **global gravitational anomaly**.

[Wang-Wen-Witten 1810.00844]

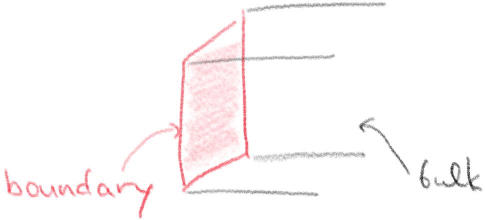
Maxwell theory as living on the boundary

Anomalies and **topological phases** are closely related.



The phase ambiguity of the boundary partition function is canceled against the phase ambiguity of the bulk partition function.

Anomalies and **topological phases** are closely related.



Consider the prototypical case of the integer quantum Hall effect: details

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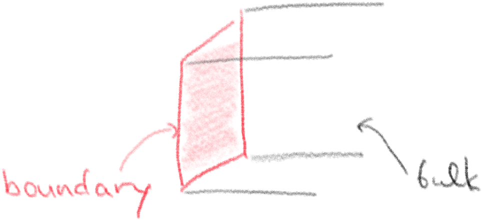


Today I consider instead

- **3+1**d boundary hosts **the Maxwell theory**
- **4+1**d bulk described by **a cousin of Chern-Simons**

For elec. $\mathbb{Z}_2$ **1-symmetry** and mag. $\mathbb{Z}_2$ **1-symmetry**, we introduce background fields

$$V_{\mathrm{e}}, V_{\mathrm{m}} \in H^2(M_5, \mathbb{Z}_2).$$

The bulk action is

$$\exp(\pi i \int_{M_5} V_{\mathrm{e}} \mathrm{Sq}^1 V_{\mathrm{m}})$$

where Sq^1 is one of the **Steenrod squaring operation** (which is introduced in 1940s in algebraic topology)

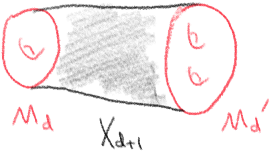
For the anomaly under general covariance, we take both $\boldsymbol{V_{\mathrm{e}}, V_{\mathrm{m}}}$ to be the **Stiefel-Whitney class w_2**. Then the bulk action is

$$\begin{aligned}\exp\left(\pi i \int_{M_5} V_{\mathrm{e}} \mathrm{Sq}^1 V_{\mathrm{m}}\right) &= \exp\left(\pi i \int_{M_5} w_2 \mathrm{Sq}^1 w_2\right) \\ &= \exp\left(\pi i \int_{M_5} w_2 w_3\right).\end{aligned}$$

known as the **de Rham invariant**.

(This invariant was introduced in 1931. The expression in terms of Stiefel-Whitney class came slightly later)

When two oriented manifolds M_d, M_d' are connected by X_{d+1} ,



they are called bordant: $M_d \sim M_d'$. The equivalence class if the bordism class. In particular, if $M_d \sim \varnothing$ if

There are only two oriented bordism classes in 5d, and distinguished by de Rham invariant:

$$\exp(\pi i \int_{M_5} w_2 w_3) = \pm 1$$

You can shrink M_5 if $+1$; you can't if -1.

These are results of algebraic topology in the 1950s and 1960s. Chern-Simons was introduced in the mid-1970s, so these are older stories.

In mathematical physics we use various subfields of math. But **we haven't used much algebraic topology**.

It's interesting that we started to use **algebraic topology from 50 years ago** relatively recently, and that this trend started from mathematical condensed-matter theory.

The relation to our work

So far I only talked about known stories, on which our work is based.

Recently with Chang-Tse Hsieh (謝長澤) and Kazuya Yonekura (米倉和也), we determined the anomaly of the $SL(2,\mathbb{Z})$ electromagnetic duality of the Maxwell theory.

Thanks to the bulk-boundary correspondence, all have to do is to analyze the 4+1d bulk Chern-Simons-like theory. It's somewhat complicated but it can be done.

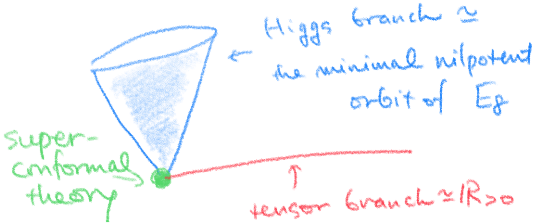
We found that the anomaly is $\mathbf{56}$ times that of a fermion.

Why $\mathbf{56}$?

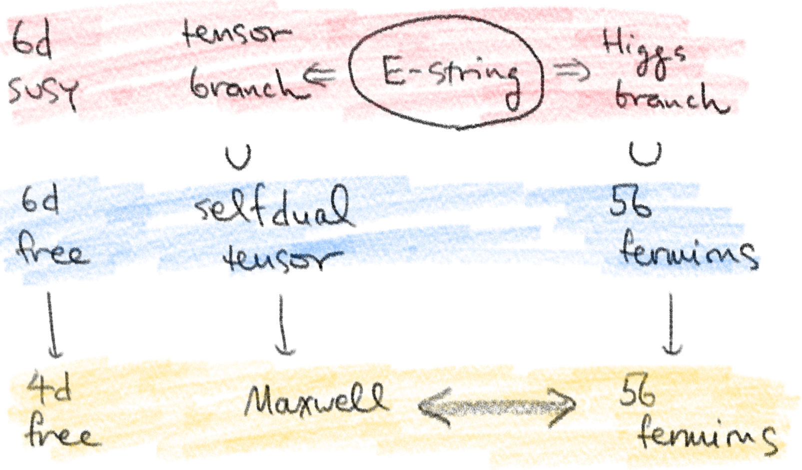
It's the dimension of the smallest nontrivial representation of E_7, one of the exceptional Lie group.

The point is that the 3+1d Maxwell theory is a $\boldsymbol{T^2}$ comapctification of a 5+1d self-dual tensor theory.

This self-dual tensor theory can be embedded into the tensor branch of the E-string theory, which is a superconformal field theory with $\boldsymbol{E_8}$ symmetry. The E-string theory also has the Higgs branch, to which we can go continuously.



On the Higgs branch, $\boldsymbol{E_8}$ is broken to $\boldsymbol{E_7}$, leaving $\mathbf{56}$ fermions.


6d susy
tensor branch
E-string
Higgs branch
6d free
selfdual tensor
56 fermions
4d free
Maxwell
56 fermions

The E-string theory is the smallest nontrivial 5+1d superconformal theory, and Kazuya and I were studying this from a totally different motivation for about six years.

Somehow it turned out to be 'useful' to understand a subtle feature of 3+1d Maxwell theory. This pleasantly surprised me.

We already have a short letter [1905.08943]. We're preparing a longer version to fill in the detail. Please have a look if you're interested.

On the chiral fermions on the boundary of the Chern-Simons theory

Consider the electromagnetic $U(1)$ symmetry. The gauge transformation is

$$A_\mu \to A_\mu + \frac{1}{2\pi i} e^{-2\pi i\chi} \partial_\mu e^{2\pi i\chi} = A_\mu + \partial_\mu \chi$$

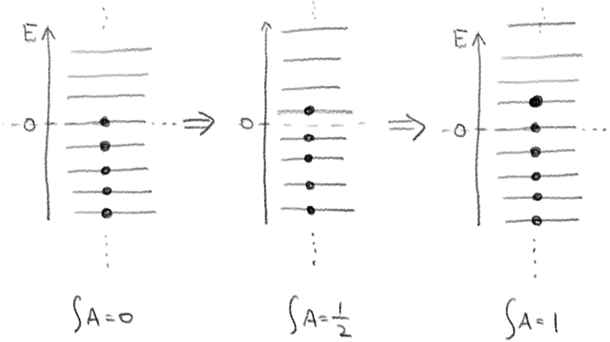
so we have

$$\oint A_t dt \to \oint A_t dt + \oint \partial_t \chi dt$$

Note that $\chi \sim \chi + 1$ so we need to identify

$$\oint A_t dt \sim \oint A_t dt + 1.$$

Consider a $\mathbf{1+1}$d fermion on a circle:



Energy levels of one particle states are

$$\boldsymbol{E \propto n + \oint A_t dt}$$

The negative energy states are filled; this is the Dirac sea.

A careful regularization shows that the Dirac sea has the electric charge

$$q = \oint A_t dt.$$

The partition function is

$$Z = \mathrm{tr}\, e^{-\beta H} e^{iq \oint A_t dt}$$

which varies under the gauge transformation

$$\oint A_t dt \to \oint A_t dt + \oint \partial_t \chi dt$$

as

$$Z \to Z e^{iq \oint \partial_t \chi dt} = Z \exp(i \oint \partial_t \chi dt \oint A_t dt).$$

The phase of the partition function is ambiguous!

The bulk effective action is the Chern-Simons term $e^{iS_{\text{CS}}}$ where

$$S_{\text{CS}} = \int_{M_3} AdA \propto \int_{M_3} \epsilon^{\mu\nu\rho} A_\mu \partial_\rho A_\sigma$$

The gauge transformation

$$A \mapsto A + d\chi$$

changes it by

$$\delta \int_{M_3} AdA = \int_{M_3} d\chi dA = \int_{\partial M_3} (d\chi) A$$

which becomes

$$= \oint \partial_t \chi dt \oint A_x dx$$

in our situation, cancelling the phase ambiguity of the 1+1d fermion on the boundary.

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