This file is to collect various notes on our project on the higher symmetry and Seiberg duality.

1 Explicit configurations detecting anomalies

Here we describe geometries detecting $\int_{M_5} B\beta E$, $\int_{M_4} B\beta w_2$, etc. All cohomologies in this section is $\mathbb{Z}_2$-valued.

Klein bottle: We start from the Klein bottle K as a nontrivial S^1 bundle over S^1. Let us denote by a the Poincaré dual to the fiber S^1, and t the Poincaré dual to the base S^1.

We have $\beta a = ta$, since $\int_K \beta a = \int_K w_1 a$.

T^4 bundle over S^1: We now consider a T^4 bundle over S^1. We denote four directions of T^4 as 1, 2, 3 and 4, and we let the directions 1 and 3 to flip the orientation when we go around S^1. We let $a_{1,2,3,4} \in H^1(T^4)$ be the dual basis to the S^1 along four directions.

We now take $B = a_1 a_2$ and $E = a_3 a_4$. Then $\beta B = tB$ and $\beta E = tE$, and $\int B\beta E = 1$.

Realizing as $SO(3)$ bundles We now look for $SO(3)$ bundles realizing these B and E as w_2 in this T^4 bundle over S^1.

We note that an $SO(3)$ bundle over T^2 with two commuting holonomies around two directions

$$R_x = \mathrm{diag}(+1, -1, -1), \qquad R_y = \mathrm{diag}(+1, -1, -1)$$

has a nontrivial w_2, since their lift to $SU(2)$ is given by $i\sigma_x$ and $i\sigma_y$ which anticommute.

Luckily, these R_x and R_y are of order two, so we can put it over our T^4 bundle. Done.

2 Anomaly of trifundamental

Lee-kun's computation says that

$$(D\Omega^{\text{spin}})^6(B[SU(2)^3/\mathbb{Z}_2^2]) = \mathbb{Z}_2$$

generated by

$$\int w_2 \beta w_2'.$$

We would like to know if a trifundamental fermion has this anomaly.

To see this, we need to compute the eta invariant under an explicit configuration where $\int w_2\beta w_2' = 1$. Such a configuration is constructed above. Let us first find an explicit configuration of $SU(2)^{(1)} \times SU(2)^{(2)} \times SU(2)^{(3)}$ commuting up to $\mathbb{Z}_2 \times \mathbb{Z}_2$, which is generated by $(-1, -1, +1)$

and $(-1,+1,-1)$. So we just have to choose, say,

$$\text{holonomy around direction } 1 = (i\sigma_x, i\sigma_x, 1), \tag{2.1}$$

$$\text{holonomy around direction } 2 = (i\sigma_y, i\sigma_y, 1), \tag{2.2}$$

$$\text{holonomy around direction } 3 = (i\sigma_x, 1, i\sigma_x), \tag{2.3}$$

$$\text{holonomy around direction } 4 = (i\sigma_y, 1, i\sigma_y). \tag{2.4}$$

Furthermore, the spinor on our T^4 bundle over S^1 is glued around S^1 via the action of $\Gamma_1\Gamma_3$, since we flip the directions 1 and 3.

These are enough data to construct the fermion bundle over our T^4 bundle over S^1. Since the Euclidean spinor in 5d is pseudoreal, and our trifundamental is also pseudoreal, the tensor product is strictly real. The Dirac operator is therefore a real antisymmetric matrix, and the eigenvalues come in pairs $\pm\lambda$ except the zero modes. Therefore in our case the eta invariant reduces to the mod-2 index, and we just have to count the zero modes.

A Computation of $\Omega_5^{\text{spin}}\left(B\left(\frac{SO(4)\times SU(2)}{\mathbb{Z}_2}\right)\right)$

Let us consider the simplest case of the Seiberg duality and examine the anomaly consequences. For $SO(4)$ gauge theory with $N_f = 2$ flavors, fermions are charged under $\frac{SO(4)\times SU(2)}{\mathbb{Z}_2}$.

A.1 Leray-Serre SS (preparatory)

For the various input of cohomology groups, see Appendix A of our WZW paper. For the fibration

$$BSO(3) \;\to\; B\left(SO(3)\times SO(3)\right) = B\left(\frac{SO(4)}{\mathbb{Z}_2}\right) \;\to\; BSO(3) \tag{A.1}$$

one has

$$\begin{array}{c|ccccccc} \multicolumn{8}{c}{E_2^{p,q} = H^p\big(BSO(3); H^q(BSO(3);\mathbb{Z})\big)} \\ 6 & \mathbb{Z}_2 & & * & * & * & * & * \\ 5 & & & & & & & \\ 4 & \mathbb{Z} & & & * & * & & * \\ 3 & \mathbb{Z}_2 & & \mathbb{Z}_2 & \mathbb{Z}_2 & * & * & * \\ 2 & & & & & & & \\ 1 & & & & & & & \\ 0 & \mathbb{Z} & & & \mathbb{Z}_2 & \mathbb{Z} & & \mathbb{Z}_2 \\ \hline & 0 & 1 & 2 & 3 & 4 & 5 & 6 \end{array} \quad\longrightarrow\quad \begin{array}{c|c} \multicolumn{2}{c}{H^{p+q}\left(B\left(\frac{SO(4)}{\mathbb{Z}_2}\right);\mathbb{Z}\right)} \\ 6 & \mathbb{Z}_2^{\oplus 3} \\ 5 & \mathbb{Z}_2 \\ 4 & \mathbb{Z}^{\oplus 2} \\ 3 & \mathbb{Z}_2^{\oplus 2} \\ 2 & \\ 1 & \\ 0 & \mathbb{Z} \\ \hline & \end{array} \tag{A.2}$$

Here we expect non-trivial differentials to be absent (for the region of interest) from the explicit consideration of generators (since there are W_3 and W_3', there should be $(W_3)^2$, $(W_3')^2$, and W_3W_3')

or by requiring proper reproduction of the $\mathbb{Z}_2$ cohomology (which we expect to be generated by w_2, w_2', w_3, and w_3'). Then, for the fibration

$$BSU(2) \;\to\; B\left(\frac{SO(4)\times SU(2)}{\mathbb{Z}_2}\right) \;\to\; B\left(\frac{SO(4)}{\mathbb{Z}_2}\right) \tag{A.3}$$

we can further plug it into

$$E_2^{p,q} = H^p\big(B\left(\tfrac{SO(4)}{\mathbb{Z}_2}\right); H^q(BSU(2);\mathbb{Z})\big) \qquad\qquad H^{p+q}(B\left(\tfrac{SO(4)\times SU(2)}{\mathbb{Z}_2}\right);\mathbb{Z})$$

$$\begin{array}{c|ccccccc}
6 & & & & & & & \\
5 & & & & & & & \\
4 & \boxed{\mathbb{Z}} & & & * & * & * & * \\
3 & & & & & & & \\
2 & & & & & & & \\
1 & & & & & & & \\
0 & \mathbb{Z} & & & \mathbb{Z}_2^{\oplus 2} & \mathbb{Z}^{\oplus 2} & \boxed{\mathbb{Z}_2} & \mathbb{Z}_2^{\oplus 3} \\
\hline
 & 0 & 1 & 2 & 3 & 4 & 5 & 6
\end{array}
\quad\longrightarrow\quad
\begin{array}{c|c}
6 & \mathbb{Z}_2^{\oplus 3} \\
5 & \\
4 & \mathbb{Z}^{\oplus 3} \\
3 & \mathbb{Z}_2^{\oplus 2} \\
2 & \\
1 & \\
0 & \mathbb{Z} \\
\hline
 &
\end{array} \tag{A.4}$$

Taking the normalization of instanton number into account (see Ohmori-san's e-mail on 2020-08-19), the differential $\boxed{d_2 : E_{0,4} \to E_{5,0}}$ seems to be non-trivial.

So we believe the integral cohomology structure to be

d	0	1	2	3	4	5	6	$\cdots$
$H^d(B\left(\frac{SO(4)\times SU(2)}{\mathbb{Z}_2}\right);\mathbb{Z})$	$\mathbb{Z}$	0	0	$\mathbb{Z}_2^{\oplus 2}$	$\mathbb{Z}^{\oplus 3}$	0	$\mathbb{Z}_2^{\oplus 3}$	$\cdots$
generator	1	–	–	W_3	p_1	–	$(W_3)^2$	$\cdots$
				W_3'	p_1'		$(W_3')^2$	
					$2c_2$		W_3W_3'	

(A.5)

where the reduction to $\mathbb{Z}_2$ cohomology are

$$\begin{array}{rcl} W_3 & \to & w_3 \\ p_1 & \to & (w_2)^2 \end{array} \tag{A.6}$$

A.2 Atiyah-Hirzebruch SS

Having obtained (co)homology groups, one can fill in the E^2-page of the AHSS:

$$
\begin{array}{c}
E^2_{p,q} = H_p\left(B\left(\frac{SO(4)\times SU(2)}{\mathbb{Z}_2}\right);\Omega^{\text{spin}}_q\right) \\
\begin{array}{c|c|cccccc}
6 & & & & & & & \\
5 & & & & & & & \\
4 & \mathbb{Z} & & \mathbb{Z}_2^{\oplus 2} & & * & * & * \\
3 & & & & & & & \\
2 & \mathbb{Z}_2 & & \boxed{\mathbb{Z}_2^{\oplus 2}} & \boxed{\mathbb{Z}_2^{\oplus 2}} & \mathbb{Z}_2^{\oplus 3} & * & * \\
1 & \mathbb{Z}_2 & & \boxed{\mathbb{Z}_2^{\oplus 2}} & \boxed{\mathbb{Z}_2^{\oplus 2}} & \boxed{\mathbb{Z}_2^{\oplus 3}} & \boxed{\mathbb{Z}_2^{\oplus 3}} & * \\
0 & \mathbb{Z} & & \mathbb{Z}_2^{\oplus 2} & & \boxed{\mathbb{Z}^{\oplus 3}} & \boxed{\mathbb{Z}_2^{\oplus 3}} & \boxed{*} \\
\hline
 & 0 & 1 & 2 & 3 & 4 & 5 & 6
\end{array}
\end{array}
\longrightarrow
\begin{array}{c}
\widetilde{\Omega}^{\text{spin}}_{p+q}\left(B\left(\frac{SO(4)\times SU(2)}{\mathbb{Z}_2}\right)\right) \\
\begin{array}{c|c}
6 & ? \\
5 & ? \\
4 & ? \\
3 & ? \\
2 & \mathbb{Z}_2^{\oplus 2} \\
1 & \\
0 & \\
\hline
 &
\end{array}
\end{array}
\tag{A.7}
$$

Based on our belief, $\boxed{d^2 : E^2_{4,0} \to E^2_{2,1}}$ and $\boxed{d^2 : E^2_{4,1} \to E^2_{2,2}}$ should be a dual of

$$
\begin{aligned}
Sq^2 w_2 &= (w_2)^2 \\
Sq^2 w_2' &= (w_2')^2
\end{aligned}
\tag{A.8}
$$

and also $\boxed{d^2 : E^2_{5,0} \to E^2_{3,1}}$ and $\boxed{d^2 : E^2_{5,1} \to E^2_{3,2}}$ should be a dual of

$$
\begin{aligned}
Sq^2 w_3 &= w_2 w_3 \\
Sq^2 w_3' &= w_2' w_3'
\end{aligned}
\tag{A.9}
$$

and finally $\boxed{d^2 : E^2_{6,0} \to E^2_{4,1}}$ should be a dual of

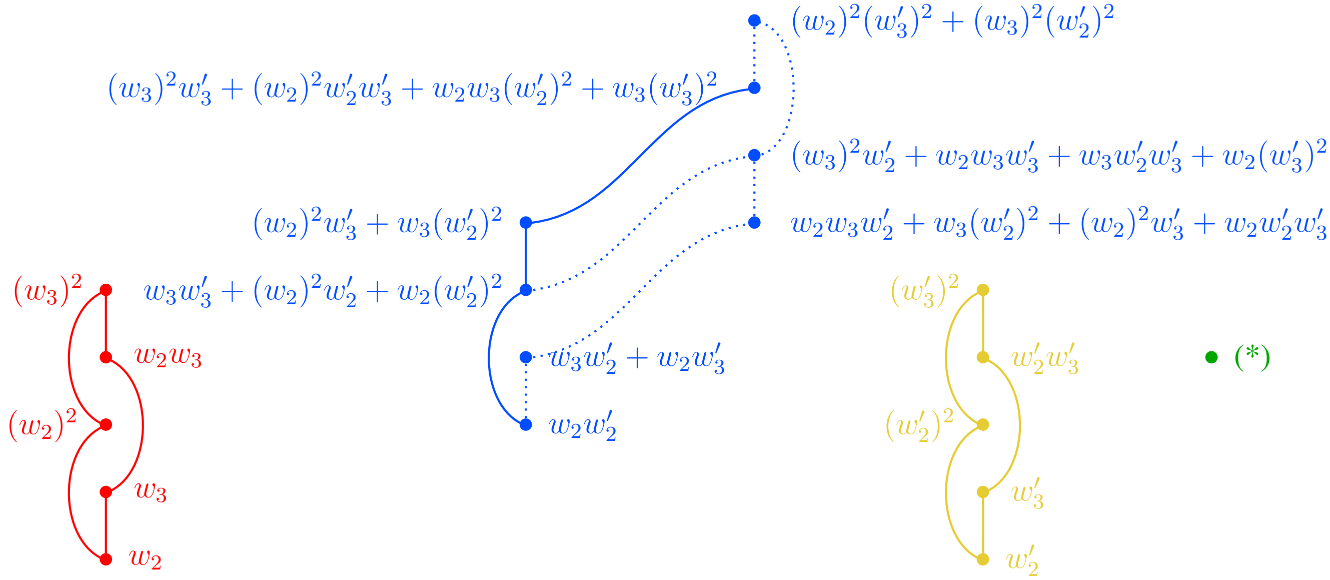
$$
Sq^2(w_2 w_2') = w_3 w_3' + (w_2)^2 w_2' + w_2 (w_2')^2 \tag{A.10}
$$

then the would-be-E_3-page is given by

$$
\begin{array}{c|c|cccccc}
6 & & & & & & & \\
5 & & & & & & & \\
4 & \mathbb{Z} & & * & & * & * & * \\
3 & & & & & & & \\
2 & \mathbb{Z}_2 & & & & * & * & * \\
1 & \mathbb{Z}_2 & & & & & * & * \\
0 & \mathbb{Z} & & \mathbb{Z}_2^{\oplus 2} & & \mathbb{Z}^{\oplus 3} & \mathbb{Z}_2 & * \\
\hline
 & 0 & 1 & 2 & 3 & 4 & 5 & 6
\end{array}
\tag{A.11}
$$

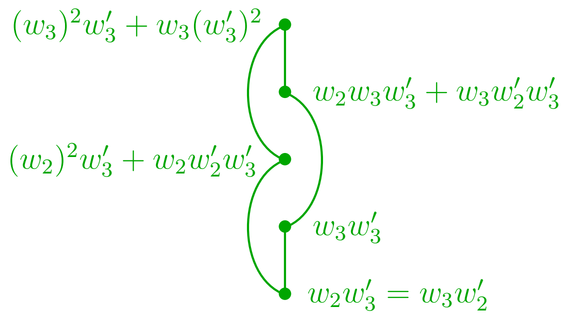
A.3 Adams SS

According to our naive guess, the module $\widetilde{H}^*(B\left(\frac{SO(4)\times SU(2)}{\mathbb{Z}_2}\right);\mathbb{Z}_2)_{\le 5}$ consists of



(A.12)

To be consistent with the AHSS computation, it seems that $w_3w_2' + w_2w_3'$ should be modded out (is it an obvious consequence of the transgression in LSSS...?) and the remaing part (*) turns out to be

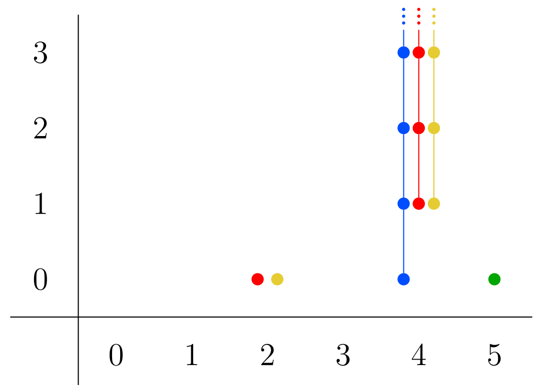


(A.13)

and therefore one concludes

$$\widetilde{H}^*(B\left(\frac{SO(4)\times SU(2)}{\mathbb{Z}_2}\right);\mathbb{Z}_2)_{\le 5} = J[2] \oplus J[2] \oplus \mathcal{A}_1/\!/\mathcal{E}_0[4] \oplus J[5]. \tag{A.14}$$

This leads to the following Adams chart:



and it indeed seems to be compatible with the AHSS computation. If the above argument (and beliefs) is correct, then the anomaly should be captured by

$$w_2 w_3' \ (= w_3 w_2'). \tag{A.15}$$

A.4 Seiberg dual

The dual theory is $SO(2)$ gauge theory with $N_f = 2$ flavors, and the fermions are charged under

$$\frac{SO(2) \times SU(2)}{\mathbb{Z}_2} = U(2) \tag{A.16}$$

Its spin bordism is known, and the relevant group turns out to be trivial:

$$\Omega_5^{\text{spin}}\big(BU(2)\big) = 0. \tag{A.17}$$

This means there is no anomaly for fermions on the dual side, and thus the $B\beta E$ anomaly cannot be canceled by fermions.