This file is to collect various notes on our project on the higher symmetry and Seiberg duality.

1 Duality map

The duality map among $Spin$, SO_+ and SO_- was first found in [1]

$$\begin{array}{rcl} Spin(N_c) & \leftrightarrow & SO_-(N_c'), \\ SO_+(N_c) & \leftrightarrow & SO_+(N_c'), \\ SO_-(N_c) & \leftrightarrow & Spin(N_c') \end{array} \tag{1.1}$$

where $N_c' = N_f - N_c + 4$.

Razamat-Willett [2] performed a rather extensive check of this mapping by means of localization on the lens space times S^1.

Note that there is a natural action of $SL(2,\mathbb{Z})$ on the theories with $\mathbb{Z}_2$ 1-form symmetry. Requiring that this is compatible with the Seiberg duality, one finds that the mapping should in fact be

$$\begin{array}{rcl} Spin(N_c) & \leftrightarrow & T(SO_-(N_c')), \\ SO_+(N_c) & \leftrightarrow & T(SO_+(N_c')), \\ SO_-(N_c) & \leftrightarrow & T(Spin(N_c')) \end{array} \tag{1.2}$$

as discussed in [3, Sec. 6] and [4].

2 References on fermionic zero modes on monopoles

Index theorem on the monopole background: Callias [5] Bott and Seeley [6][1]

General reviews: Harvey [7, Lecture 4].

3 Explicit configurations detecting anomalies

Here we describe geometries detecting $\int_{M_5} B\beta E$, $\int_{M_4} B\beta w_2$, etc. All cohomologies in this section is $\mathbb{Z}_2$-valued.

This is to confirm that these expressions are not secretly trivial.

Klein bottle: We start from the Klein bottle K as a nontrivial S^1 bundle over S^1. Let us denote by a the Poincaré dual to the fiber S^1, and t the Poincaré dual to the base S^1.

We have $\beta a = ta$, since $\int_K \beta a = \int_K w_1 a$.

[1] These are on CMP. Papers on CMP are not open access via Springer (which is reachable by doi) but is open access at Project Euclid. I'd like a way to include links in the references appropriately.

T^4 bundle over S^1: We now consider a T^4 bundle over S^1. We denote four directions of T^4 as 1, 2, 3 and 4, and we let the directions 1 and 3 to flip the orientation when we go around S^1. We let $a_{1,2,3,4} \in H^1(T^4)$ be the dual basis to the S^1 along four directions.

We now take $B = a_1 a_2$ and $E = a_3 a_4$. Then $\beta B = tB$ and $\beta E = tE$, and $\int B \beta E = 1$.

Realizing as $SO(3)$ bundles We now look for $SO(3)$ bundles realizing these B and E as w_2 in this T^4 bundle over S^1.

We note that an $SO(3)$ bundle over T^2 with two commuting holonomies around two directions

$$R_x = \mathrm{diag}(+1, -1, -1), \qquad R_y = \mathrm{diag}(+1, -1, -1)$$

has a nontrivial w_2, since their lift to $SU(2)$ is given by $i\sigma_x$ and $i\sigma_y$ which anticommute.

Luckily, these R_x and R_y are of order two, so we can put it over our T^4 bundle. Done.

4 Anomaly of trifundamental

Lee-kun's computation says that

$$(D\Omega^{\text{spin}})^6(B[SU(2)^3/\mathbb{Z}_2^2]) = \mathbb{Z}_2$$

generated by

$$\int w_2 \beta w_2'.$$

We would like to know if a trifundamental fermion has this anomaly.

A Computation of $\Omega_5^{\text{spin}}\big(B\big(\frac{SO(4)\times SU(2)}{\mathbb{Z}_2}\big)\big)$

Let us consider the simplest case of the Seiberg duality and examine the anomaly consequences. For $SO(4)$ gauge theory with $N_f = 2$ flavors, fermions are charged under $\frac{SO(4)\times SU(2)}{\mathbb{Z}_2}$.

A.1 Leray-Serre SS (preparatory)

For the various input of cohomology groups, see Appendix A of our WZW paper. For the fibration

$$BSO(3) \;\to\; B\left(SO(3)\times SO(3)\right) = B\left(\frac{SO(4)}{\mathbb{Z}_2}\right) \;\to\; BSO(3) \tag{A.1}$$

one has

$$
\begin{array}{c|ccccccc}
\multicolumn{8}{c}{E_2^{p,q} = H^p\big(BSO(3); H^q(BSO(3);\mathbb{Z})\big)} \\
6 & \mathbb{Z}_2 & & * & * & * & * & * \\
5 & & & & & & & \\
4 & \mathbb{Z} & & & * & * & & * \\
3 & \mathbb{Z}_2 & & \mathbb{Z}_2 & \mathbb{Z}_2 & * & * & * \\
2 & & & & & & & \\
1 & & & & & & & \\
0 & \mathbb{Z} & & & \mathbb{Z}_2 & \mathbb{Z} & & \mathbb{Z}_2 \\
\hline
 & 0 & 1 & 2 & 3 & 4 & 5 & 6
\end{array}
\longrightarrow
\begin{array}{c|c}
\multicolumn{2}{c}{H^{p+q}(B\left(\frac{SO(4)}{\mathbb{Z}_2}\right);\mathbb{Z})} \\
6 & \mathbb{Z}_2^{\oplus 3} \\
5 & \mathbb{Z}_2 \\
4 & \mathbb{Z}^{\oplus 2} \\
3 & \mathbb{Z}_2^{\oplus 2} \\
2 & \\
1 & \\
0 & \mathbb{Z} \\
\hline
 &
\end{array}
\tag{A.2}
$$

Here we expect non-trivial differentials to be absent (for the region of interest) from the explicit consideration of generators (since there are W_3 and W_3', there should be $(W_3)^2$, $(W_3')^2$, and W_3W_3') or by requiring proper reproduction of the $\mathbb{Z}_2$ cohomology (which we expect to be generated by w_2, w_2', w_3, and w_3'). Then, for the fibration

$$
BSU(2) \to B\left(\frac{SO(4)\times SU(2)}{\mathbb{Z}_2}\right) \to B\left(\frac{SO(4)}{\mathbb{Z}_2}\right)
\tag{A.3}
$$

we can further plug it into

$$
\begin{array}{c|ccccccc}
\multicolumn{8}{c}{E_2^{p,q} = H^p\big(B\left(\frac{SO(4)}{\mathbb{Z}_2}\right); H^q(BSU(2);\mathbb{Z})\big)} \\
6 & & & & & & & \\
5 & & & & & & & \\
4 & \boxed{\mathbb{Z}} & & & * & * & * & * \\
3 & & & & & & & \\
2 & & & & & & & \\
1 & & & & & & & \\
0 & \mathbb{Z} & & & \mathbb{Z}_2^{\oplus 2} & \mathbb{Z}^{\oplus 2} & \boxed{\mathbb{Z}_2} & \mathbb{Z}_2^{\oplus 3} \\
\hline
 & 0 & 1 & 2 & 3 & 4 & 5 & 6
\end{array}
\longrightarrow
\begin{array}{c|c}
\multicolumn{2}{c}{H^{p+q}(B\left(\frac{SO(4)\times SU(2)}{\mathbb{Z}_2}\right);\mathbb{Z})} \\
6 & \mathbb{Z}_2^{\oplus 3} \\
5 & \\
4 & \mathbb{Z}^{\oplus 3} \\
3 & \mathbb{Z}_2^{\oplus 2} \\
2 & \\
1 & \\
0 & \mathbb{Z} \\
\hline
 &
\end{array}
\tag{A.4}
$$

Taking the normalization of instanton number into account (see Ohmori-san's e-mail on 2020-08-19), the differential $\boxed{d_2 : E_{0,4} \to E_{5,0}}$ seems to be non-trivial.

So we believe the integral cohomology structure to be

d	0	1	2	3	4	5	6	$\cdots$
$H^d(B\left(\frac{SO(4)\times SU(2)}{\mathbb{Z}_2}\right);\mathbb{Z})$	$\mathbb{Z}$	0	0	$\mathbb{Z}_2^{\oplus 2}$	$\mathbb{Z}^{\oplus 3}$	0	$\mathbb{Z}_2^{\oplus 3}$	$\cdots$
generator	1	–	–	W_3	p_1	–	$(W_3)^2$	$\cdots$
				W_3'	p_1'		$(W_3')^2$	
					$2c_2$		W_3W_3'	

(A.5)

where the reduction to $\mathbb{Z}_2$ cohomology are

$$\begin{array}{ccc} W_3 & \to & w_3 \\ p_1 & \to & (w_2)^2 \end{array} \tag{A.6}$$

A.2 Atiyah-Hirzebruch SS

Having obtained (co)homology groups, one can fill in the E^2-page of the AHSS:

$$E^2_{p,q} = H_p\big(B\left(\tfrac{SO(4)\times SU(2)}{\mathbb{Z}_2}\right); \Omega^{\text{spin}}_q\big) \qquad\qquad \widetilde{\Omega}^{\text{spin}}_{p+q}\big(B\left(\tfrac{SO(4)\times SU(2)}{\mathbb{Z}_2}\right)\big)$$

$$\begin{array}{c|c:cccccc}
6 & & & & & & & \\
5 & & & & & & & \\
4 & \mathbb{Z} & & \mathbb{Z}_2^{\oplus 2} & & * & * & * \\
3 & & & & & & & \\
2 & \mathbb{Z}_2 & & \boxed{\mathbb{Z}_2^{\oplus 2}} & \boxed{\mathbb{Z}_2^{\oplus 2}} & \mathbb{Z}_2^{\oplus 3} & * & * \\
1 & \mathbb{Z}_2 & & \boxed{\mathbb{Z}_2^{\oplus 2}} & \boxed{\mathbb{Z}_2^{\oplus 2}} & \boxed{\mathbb{Z}_2^{\oplus 3}} & \boxed{\mathbb{Z}_2^{\oplus 3}} & * \\
0 & \mathbb{Z} & & \mathbb{Z}_2^{\oplus 2} & & \boxed{\mathbb{Z}_2^{\oplus 3}} & \boxed{\mathbb{Z}_2^{\oplus 3}} & \boxed{*} \\
\hline
 & 0 & 1 & 2 & 3 & 4 & 5 & 6
\end{array}
\quad\longrightarrow\quad
\begin{array}{c|c}
6 & ? \\
5 & ? \\
4 & ? \\
3 & ? \\
2 & \mathbb{Z}_2^{\oplus 2} \\
1 & \\
0 & \\
\hline
\end{array} \tag{A.7}$$

Based on our belief, $\boxed{d^2 : E^2_{4,0} \to E^2_{2,1}}$ and $\boxed{d^2 : E^2_{4,1} \to E^2_{2,2}}$ should be a dual of

$$\begin{array}{ccc} Sq^2 w_2 & = & (w_2)^2 \\ Sq^2 w_2' & = & (w_2')^2 \end{array} \tag{A.8}$$

and also $\boxed{d^2 : E^2_{5,0} \to E^2_{3,1}}$ and $\boxed{d^2 : E^2_{5,1} \to E^2_{3,2}}$ should be a dual of

$$\begin{array}{ccc} Sq^2 w_3 & = & w_2 w_3 \\ Sq^2 w_3' & = & w_2' w_3' \end{array} \tag{A.9}$$

and finally $\boxed{d^2 : E^2_{6,0} \to E^2_{4,1}}$ should be a dual of

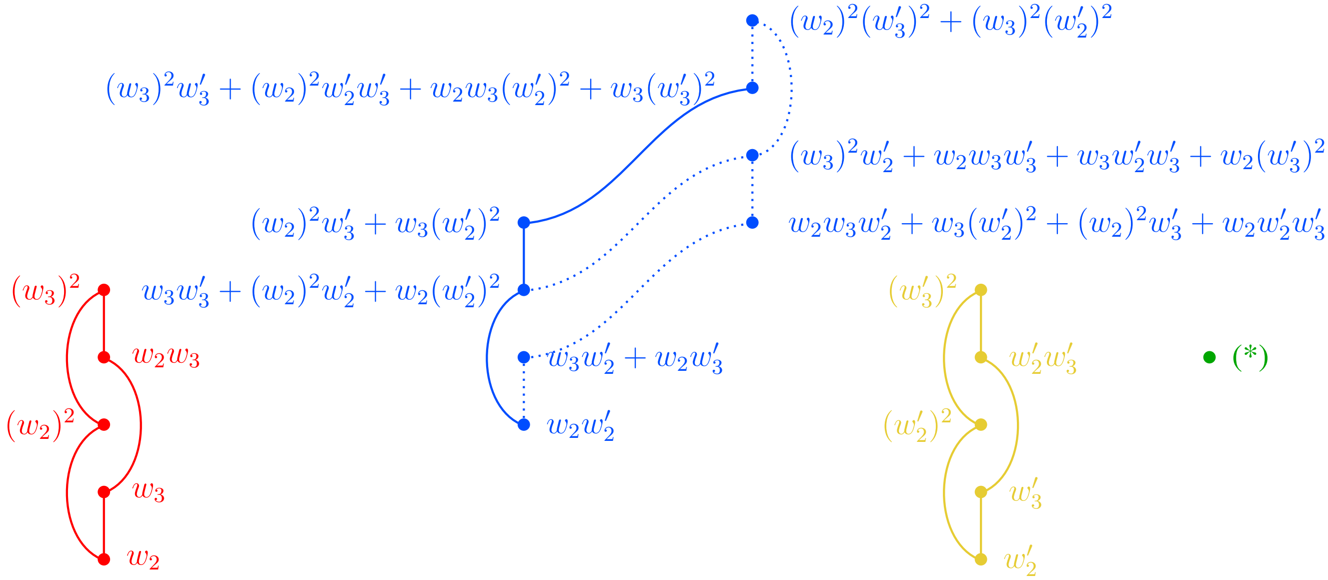
$$Sq^2(w_2 w_2') \quad = \quad w_3 w_3' + (w_2)^2 w_2' + w_2 (w_2')^2 \tag{A.10}$$

then the would-be-E_3-page is given by

$$\begin{array}{c|c:cccccc}
6 & & & & & & & \\
5 & & & & & & & \\
4 & \mathbb{Z} & & * & & * & * & * \\
3 & & & & & & & \\
2 & \mathbb{Z}_2 & & & & * & * & * \\
1 & \mathbb{Z}_2 & & & & & * & * \\
0 & \mathbb{Z} & & \mathbb{Z}_2^{\oplus 2} & & \mathbb{Z}^{\oplus 3} & \mathbb{Z}_2 & * \\
\hline
 & 0 & 1 & 2 & 3 & 4 & 5 & 6
\end{array} \tag{A.11}$$

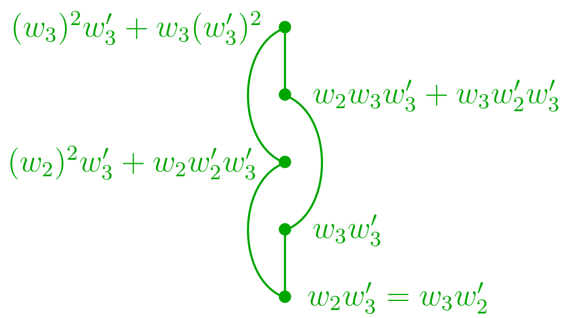
A.3 Adams SS

According to our naive guess, the module $\widetilde{H}^*(B\left(\frac{SO(4)\times SU(2)}{\mathbb{Z}_2}\right);\mathbb{Z}_2)_{\leq 5}$ consists of



(A.12)

To be consistent with the AHSS computation, it seems that $w_3w_2' + w_2w_3'$ should be modded out (is it an obvious consequence of the transgression in LSSS...?) and the remaing part (*) turns out to be

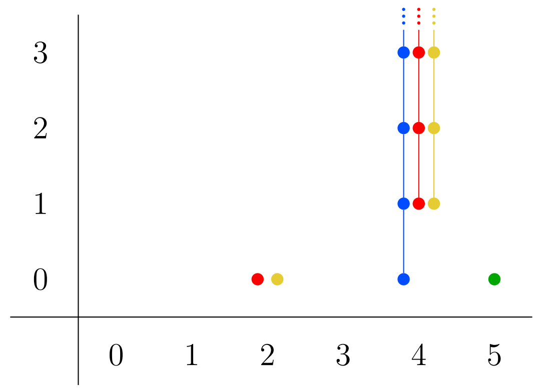


(A.13)

and therefore one concludes

$$\widetilde{H}^*(B\left(\frac{SO(4)\times SU(2)}{\mathbb{Z}_2}\right);\mathbb{Z}_2)_{\leq 5} = J[2] \oplus J[2] \oplus \mathcal{A}_1/\!/\mathcal{E}_0[4] \oplus J[5]. \tag{A.14}$$

This leads to the following Adams chart:



and it indeed seems to be compatible with the AHSS computation. If the above argument (and beliefs) is correct, then the anomaly should be captured by

$$w_2 w_3' \, (= w_3 w_2'). \tag{A.15}$$

A.4 Seiberg dual

The dual theory is $SO(2)$ gauge theory with $N_f = 2$ flavors, and the fermions are charged under

$$\frac{SO(2) \times SU(2)}{\mathbb{Z}_2} = U(2) \tag{A.16}$$

Its spin bordism is known, and the relevant group turns out to be trivial:

$$\Omega_5^{\mathrm{spin}}\big(BU(2)\big) = 0. \tag{A.17}$$

This means there is no anomaly for fermions on the dual side, and thus the $B\beta E$ anomaly cannot be canceled by fermions.

B 't Hooft-Polyakov monopole argument

For an $SU(2)$ gauge theory Higgssed by an adjoint ($\mathbf{3}$, spin-1, isovector) scalar, the gauge group is broken down to $U(1)$, and correspondingly it accommodates topological solitons (monopoles):

$$\pi_2\left(\frac{SU(2)}{U(1)}\right) = \mathbb{Z}.$$

In the presence of (additional) fermions, this monopole might acquire non-trivial charge under spacetime-Lorentz or flavor symmetries, depending on the representation of the fermions under $SU(2)$ gauge symmetry:

fermion gauge rep.	number of zero-modes	spin of zero-modes	spin of monopole
$\mathbf{2}$	1	0	0
$\mathbf{3}$	2	$\frac{1}{2}$	
$\mathbf{4}$	4	$0 \oplus 1$	$\frac{1}{2}$

The numbers of zero-modes can be computed from the Callias index theorem [5].

According to [8], there is $w_2(TM_4)\beta w_2(SU(2)_{\text{gauge}})$ anomaly for a fermion in $\mathbf{4}$ charged under

$$\frac{Spin(4)_{\text{spacetime}} \times SU(2)_{\text{gauge}}}{\mathbb{Z}_2},$$

which incarnates in the IR as an ill-definition of the effective interaction $w_2(TM)c_1(U(1)_{\text{gauge}})$, emerging after integrating out the fermion which obtained mass through Yukawa coupling. This effective interaction term should arise in order to make the monopole a fermion (*i.e.* spinor representation of $Spin(4)_{\text{spacetime}}$), but is not well-defined without a trivialization of $w_2(TM_4)$ or equivalently a spin structure.

This situation looks quite similar to our problem where the fermions are charged under

$$\frac{SO(4)_{\text{gauge}} \times SU(2)_{\text{flavor}}}{\mathbb{Z}_2}.$$

Since fermions are in the fundamental representation of the flavor symmetry, breaking $SU(2)_{\text{flavor}}$ to $U(1)_{\text{flavor}}$ gives rise to a monopole in the spinor representation this time of the $SO(4)_{\text{gauge}}$ [7]. Therefore by the same logic, one can deduce that there should be $w_2(SO(4)_{\text{gauge}})\beta w_2(SU(2)_{\text{flavor}})$ anomaly in the first place, as desired.

Also, one should be able to generalize this whole argument to the case of fermions charged under

$$\frac{SO(2n_c)_{\text{gauge}} \times SU(2n_f)_{\text{flavor}}}{\mathbb{Z}_2}$$

by breaking $SU(2n_f)_{\text{flavor}}$ to $SO(2n_f)_{\text{flavor}}$, where we have monopoles characterized by

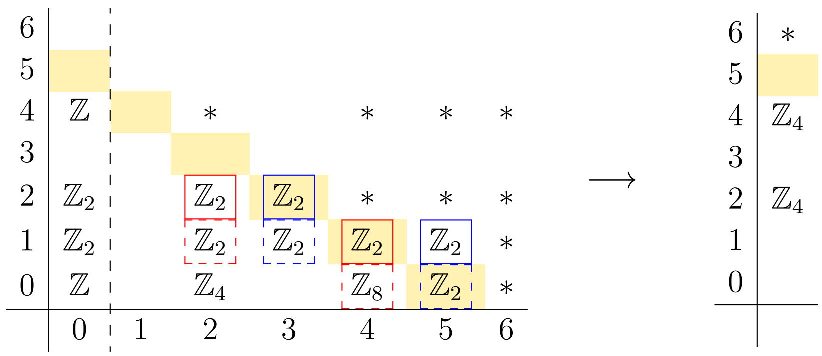
$$\pi_2\left(\frac{SU(2n_f)}{SO(2n_f)}\right) = \begin{cases} \mathbb{Z}_2 & (n_f \geq 2) \\ \mathbb{Z} & (n_f = 1) \end{cases}.$$

C misc

C.1 $\mathbb{Z}_4$ 1-form symmetry

According to [9, Appendix C.3] and [10, Eq. (6.3)], it seems that we have

$$E^2_{p,q} = H_p\big(K(\mathbb{Z}_4,2);\Omega^{\text{spin}}_q\big) \qquad\qquad \widetilde{\Omega}^{\text{spin}}_{p+q}(K(\mathbb{Z}_4,2)) \tag{C.1}$$



The corresponding invariant in 4d is simply

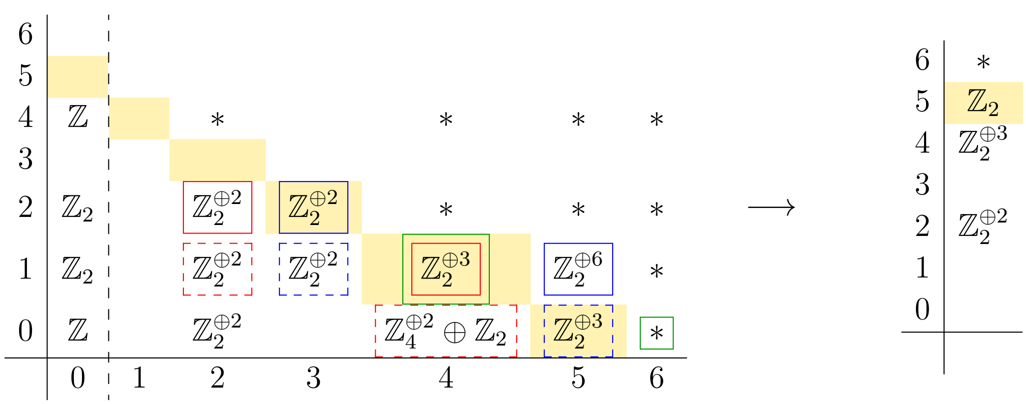
$$\exp(2\pi i\frac{p}{4}\int\frac{1}{2}\mathfrak{P}(a)) \tag{C.2}$$

where $\mathfrak{P}: H^2(-,\mathbb{Z}_4)\to H^4(-,\mathbb{Z}_8)$ is the Pontryagin square, which is even mod 8 on a spin manifold.

C.2 $\mathbb{Z}_2\times\mathbb{Z}_2$ 1-form symmetry

Exploiting the fact that $K(\mathbb{Z}_2\times\mathbb{Z}_2,2) = K(\mathbb{Z}_2,2)\times K(\mathbb{Z}_2,2)$, it seems that we have

$$E^2_{p,q} = H_p\big(K(\mathbb{Z}_2\times\mathbb{Z}_2,2);\Omega^{\text{spin}}_q\big) \qquad\qquad \widetilde{\Omega}^{\text{spin}}_{p+q}(K(\mathbb{Z}_2\times\mathbb{Z}_2,2)) \tag{C.3}$$



The $\mathbb{Z}$ homology of $K(\mathbb{Z}_2,2)$ is again read off from [9], while the $\mathbb{Z}_2$ (co)homology is known [11] to be

$$H^*(K(\mathbb{Z}_2,2);\mathbb{Z}_2) = \mathbb{Z}_2[x_2, Sq^1x_2, Sq^2Sq^1x_2,\cdots].$$

The corresponding bordism invariants in 4d are $\mathfrak{P}(a)/2$, ab, $\mathfrak{P}(b)/2$, and the one in 5d is $a\beta b$.

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