This file is to collect various notes on our project on the higher symmetry and Seiberg duality.

1 Explicit configurations detecting anomalies

Here we describe geometries detecting $\int_{M_5} B\beta E$, $\int_{M_4} B\beta w_2$, etc. All cohomologies in this section is $\mathbb{Z}_2$-valued.

Klein bottle: We start from the Klein bottle K as a nontrivial S^1 bundle over S^1. Let us denote by a the Poincaré dual to the fiber S^1, and t the Poincaré dual to the base S^1.

We have $\beta a = ta$, since $\int_K \beta a = \int_K w_1 a$.

T^4 bundle over S^1: We now consider a T^4 bundle over S^1. We denote four directions of T^4 as 1, 2, 3 and 4, and we let the directions 1 and 3 to flip the orientation when we go around S^1. We let $a_{1,2,3,4} \in H^1(T^4)$ be the dual basis to the S^1 along four directions.

We now take $B = a_1 a_2$ and $E = a_3 a_4$. Then $\beta B = tB$ and $\beta E = tE$, and $\int B\beta E = 1$.

Realizing as $SO(3)$ bundles We now look for $SO(3)$ bundles realizing these B and E as w_2 in this T^4 bundle over S^1.

We note that an $SO(3)$ bundle over T^2 with two commuting holonomies around two directions

$$R_x = \mathrm{diag}(+1, -1, -1), \qquad R_y = \mathrm{diag}(+1, -1, -1)$$

has a nontrivial w_2, since their lift to $SU(2)$ is given by $i\sigma_x$ and $i\sigma_y$ which anticommute.

Luckily, these R_x and R_y are of order two, so we can put it over our T^4 bundle. Done.

2 Anomaly of trifundamental

Lee-kun's computation says that

$$(D\Omega^{\text{spin}})^6(B[SU(2)^3/\mathbb{Z}_2^2]) = \mathbb{Z}_2$$

generated by

$$\int w_2 \beta w_2'.$$

We would like to know if a trifundamental fermion has this anomaly.

To see this, we need to compute the eta invariant under an explicit configuration where $\int w_2 \beta w_2' = 1$. Such a configuration is constructed above. Let us first find an explicit configuration of $SU(2)^{(1)} \times SU(2)^{(2)} \times SU(2)^{(3)}$ commuting up to $\mathbb{Z}_2 \times \mathbb{Z}_2$, which is generated by $(-1, -1, +1)$

and $(-1,+1,-1)$. So we just have to choose, say,

$$\text{holonomy around direction } 1 = (i\sigma_x, i\sigma_x, 1), \tag{2.1}$$
$$\text{holonomy around direction } 2 = (i\sigma_y, i\sigma_y, 1), \tag{2.2}$$
$$\text{holonomy around direction } 3 = (i\sigma_x, 1, i\sigma_x), \tag{2.3}$$
$$\text{holonomy around direction } 4 = (i\sigma_y, 1, i\sigma_y). \tag{2.4}$$

Furthermore, the spinor on our T^4 bundle over S^1 is glued around S^1 via the action of $\Gamma_1\Gamma_3$, since we flip the directions 1 and 3.

These are enough data to construct the fermion bundle over our T^4 bundle over S^1. Since the Euclidean spinor in 5d is pseudoreal, and our trifundamental is also pseudoreal, the tensor product is strictly real. The Dirac operator is therefore a real antisymmetric matrix, and the eigenvalues come in pairs $\pm\lambda$ except the zero modes. Therefore in our case the eta invariant reduces to the mod-2 index, and we just have to count the zero modes.