Higher symmetries and anomalies in $\mathfrak{so}$ QCD and $\mathcal{N}$=1 duality

Yasunori Lee[1], Kantaro Ohmori[2], and Yuji Tachikawa[1]

[1] Kavli Institute for the Physics and Mathematics of the Universe (WPI), University of Tokyo, Kashiwa, Chiba 277-8583, Japan

[2] Department of Physics, Faculty of Science, University of Tokyo, Bunkyo, Tokyo 113-0033, Japan

We study higher symmetries and anomalies of 4d $\mathfrak{so}(2n_c)$ gauge theory with $N_f = 2n_f$ flavors. We find that they depend on the parity of n_c and n_f, on the global form of the gauge group, and the discrete theta angle. The contribution from the fermions plays a central role in our analysis. Furthermore, our conclusion applies to $\mathcal{N}$=1 supersymmetric cases as well, and we see that higher symmetries and anomalies match across the duality $\mathfrak{so}(2n_c) \leftrightarrow \mathfrak{so}(2n_f - n_c + 4)$ of Intriligator and Seiberg.

Contents

1 Introduction and summary

Our understanding of the concept of symmetries in quantum field theories has been greatly improved in the last several years. We now have the concept of p-form symmetries acting on p-dimensional objects [GKSW14]. This concept gives a unifying point of view to both ordinary symmetries acting on point operators for $p = 0$ and center symmetries of gauge theories acting on Wilson line operators for $p = 1$. In addition, the 't Hooft magnetic flux [tH79] can now be thought of as a background gauge field for the 1-form center symmetry. It is also realized more recently that 0-form symmetries and 1-form symmetries can not only coexist in a direct product but also mix in a more intricate manner. They can have mixed anomalies between them. They can also combine to form a symmetry structure called 2-groups [CDI18, BCH18].

In this paper we study these issues in the case of 4d $\mathfrak{so}(N_c)$ gauge theories with N_f flavors of fermion fields in vector representation. Let us quickly recall the 0-form and 1-form symmetries these theories have.

As for the 1-form symmetry, we first need to recall that such theories come in three versions, Spin, SO_+ and SO_-, distinguished by the global form of the gauge group (Spin vs. SO) and by the choice of a discrete theta angle (SO_+ vs. SO_-) [AST13][1]. They also differ by the nontrivial line operator they possess: the Spin theory has the Wilson line W in the spinor representation, the SO_+ theory has the 't Hooft line H which is mutually non-local with respect to W, and the SO_- theory has the dyonic line $D = WH$. Furthermore, these line operators are charged under corresponding $\mathbb{Z}_2$ 1-form symmetries, which we can all electric, magnetic and dyonic 1-form symmetries.

As for the 0-form symmetry, we focus our attention on the $\mathfrak{su}(N_f)$ symmetry acting on N_f flavors of matter fields in the vector representation. There can be and definitely are other discrete symmetries, but we will not consider them in this paper for brevity.

The main question is then how the $\mathbb{Z}_2$ 1-form symmetry and the $\mathfrak{su}(N_f)$ 0-form symmetry are related.[2] We concentrate on the case when N_c and N_f are both even, $N_c = 2n_c$ and $N_f = 2n_f$.

Take for example the $Spin(2n_c)$ gauge theory with $2n_f$ flavors, when n_c is odd. Take two copies of the Wilson line W in the spinor representation. They form a Wilson line in the vector representation. This can be screened by a dynamical fermion, which was why $W^2 = 1$ as far as the 1-form symmetry charge was concerned. Now let us recall that this dynamical fermion

[1] This is when the theories are considered on spin manifolds. On more general manifolds a further distinction needs to be made [ARS19]. For simplicity we only consider spin manifolds in this paper.

[2] A partial answer was given in [HL20], but the contribution from fermions was not taken into account in that reference. Our conclusion is therefore somewhat different from theirs.

(n_c, n_f)	Spin	SO_+	SO_-
(even, even)	none	none	none
(odd, even)	extension	anomaly	extension
(even, odd)	anomaly	extension	extension
(odd, odd)	extension	extension	anomaly

Table 1: How the $\mathbb{Z}_2$ 1-form symmetry and the $\mathrm{SU}(2n_f)/\mathbb{Z}_2$ 0-form symmetry are combined in $\mathfrak{so}(2n_c)$ QCD. 'none' implies that they remain a direct product without mixed anomaly; 'anomaly' means that they remain a direct product but with mixed anomaly; and 'extension' is when they combine into a nontrivial 2-group. The orange lines show how the duality of Intriligator and Seiberg acts on this set of theories.

transforms nontrivially under $-1 \in \mathrm{SU}(2n_f)$. Therefore, when we take the flavor symmetry into account, W^2 is still nontrivial. As we will recall below, formally this means that the $\mathbb{Z}_2$ 1-form symmetry extends the $\mathrm{SU}(2n_f)/\mathbb{Z}_2$ 0-form symmetry in a nontrivial manner, forming a nontrivial 2-group H

$$0 \to \mathbb{Z}_2[1] \to H \to \mathrm{SU}(2n_f)/\mathbb{Z}_2 \to 0, \tag{1.1}$$

whose Postnikov class is specified by

$$\beta v_2 \in H^3(\mathrm{SU}(2n_f)/\mathbb{Z}_2, \mathbb{Z}_2), \tag{1.2}$$

where a_2 is the generator of $H^2(\mathrm{SU}(2n_f)/\mathbb{Z}_2, \mathbb{Z}_2) = \mathbb{Z}_2$ and β is the Bockstein.

The SO_+ gauge theory is then obtained by gauging the $\mathbb{Z}_2$ 1-form symmetry [KS14]. The presence of the fermions significantly complicates the analysis. For the moment let us suppose that we have N_f scalars instead of fermions in the vector representation. Then, the argument of [Tac17] immediately applies, and we see that the $\mathbb{Z}_2$ 1-form symmetry of the $\mathrm{SO}(2n_f)$ theory and the $\mathrm{SU}(2n_f)/\mathbb{Z}_2$ 0-form flavor symmetry remains a direct product but with a mixed anomaly given by

$$2\pi i \frac{1}{2} \int B \beta v_2. \tag{1.3}$$

In the rest of the paper, we will carefully analyze how the $\mathbb{Z}_2$ 1-form symmetry and the $\mathrm{SU}(2n_f)/\mathbb{Z}_2$ 0-form symmetry are combined. The derivation will be detailed in the following, and here we simply summarize the result in Table 1. There, 'none' specifies that they remain a direct product without mixed anomaly; 'anomaly' implies that they remain a direct product but with mixed anomaly of the form (1.3); and 'extension' means that they combine into a 2-group given by (1.1) with the Postnikov class (1.2).

Our result is equally applicable in the case of $\mathcal{N}$=1 supersymmetric QCD, for which Intriligator and Seiberg found in [IS95] a duality exchanging $\mathfrak{so}(N_c)$ and $\mathfrak{so}(N_f - N_c + 4)$, which in our notation sends n_c to $n_c' = n_f - n_c + 2$. In [AST13], this duality was refined to account for the global form of the gauge group and the discrete theta angle, and it was concluded that Spin is exchanged with SO_- while SO_+ maps to itself. This mapping was checked using supersymmetric localization on $S^3/\mathbb{Z}_n \times S^1$ in [RW13]. Our analysis allows us to check this duality by comparing how the 1-form symmetry and the 0-form symmetry are combined in the dual pairs. We superimposed the action of the duality on our main Table 1. It is satisfying to see that the duality action correctly preserves the labels 'none', 'anomaly' and 'extension'.

The rest of the paper is organized as follows ...

2 2-group structure

Let us first study whether the $\mathbb{Z}_2$ 1-form symmetry and the $\mathrm{SU}(2n_f)/\mathbb{Z}_2$ flavor symmetry form a nontrivial 2-group or not. This can be found rather physically by studying the line operators.

2.1 Spin

We start by discussing the $\mathrm{Spin}(2n_c)$ gauge theories. We first recall that the center of $\mathrm{Spin}(2n_c)$ is $\mathbb{Z}_2 \times \mathbb{Z}_2$ when n_c is even and $\mathbb{Z}_4$ when n_c is odd. This corresponds to the fact that the tensor square of a spinor representation contains the identity representation when n_c is even while it contains the vector representation when n_c is odd.

We now consider the Wilson line W in the spinor representation in the $\mathrm{Spin}(2n_c)$ gauge theory with $2n_f$ fermions in the vector representation. When n_c is even, W^2 contains the identity representation, and therefore we simply have a $\mathbb{Z}_2$ 1-form symmetry independent of the $\mathfrak{su}$ flavor symmetry, and there is nothing to see here.

When n_c is odd, W^2 contains the vector representation. This can be screened by the dynamical fermion, which however carries the fundamental representation of $\mathfrak{su}(2n_f)$ flavor symmetry, and in particular transforms nontrivially under $-1 \in \mathrm{SU}(2n_f)$. In other words, the flavor Wilson line in the vector representation of $\mathrm{SU}(2n_f)$ can now be considered as the square of the gauge Wilson line in the spinor representation of $\mathrm{Spin}(2n_c)$. This means that we have the following extension of groups

$$0 \to \underbrace{\mathbb{Z}_2}_{\substack{\text{subgroup of} \\ \text{rep. of center of } \mathrm{SU}(2n_f)}} \to \mathbb{Z}_4 \to \underbrace{\mathbb{Z}_2}_{\substack{\text{group of gauge Wilson lines} \\ \text{up to screening}}} \to 0. \tag{2.1}$$

As the groups of charges of $\mathfrak{su}(2n_f)$ 0-form symmetry and $\mathbb{Z}_2$ 1-form symmetry are combined nontrivially, the 0-form symmetry group and the 1-form symmetry group are also combined nontrivially. This can be seen most clearly by considering background fields for the symmetry groups.

The fermion fields are in the vector of $\mathrm{SO}(2n_c)$ and in the fundamental of $\mathrm{SU}(2n_f)$, and therefore is a representation of $G = [\mathrm{SO}(2n_c) \times \mathrm{SU}(2n_f)]/\mathbb{Z}_2$. Given a G-bundle on a manifold X, there is an $\mathrm{SO}(2n_c)/\mathbb{Z}_2$ bundle and an $\mathrm{SU}(2n_f)/\mathbb{Z}_2$ bundle associated to it. Let us denote by $a_2, v_2 \in H^2(X, \mathbb{Z}_2)$ the obstruction classes controlling whether they lift to $\mathrm{SO}(2n_c)$ and $\mathrm{SU}(2n_f)$ respectively. Then we have $a_2 = v_2$ for a G-bundle. The flavor Wilson line in the fundamental representation is charged under $-1 \in \mathrm{SU}(2n_f)$ in the center, and v_2 can be considered as the background field for this $\mathbb{Z}_2$ 1-form center symmetry.

Now, without the flavor background, the background $E \in H^2(X, \mathbb{Z}_2)$ for the electric $\mathbb{Z}_2$ one-form symmetry of the $\mathrm{Spin}(2n_c)$ theory sets the Stiefel-Whitney class $w_2 \in H^2(X, \mathbb{Z}_2)$ of the $\mathrm{SO}(2n_c)$ gauge bundle to be $E = w_2$, which controls whether it lifts to a $\mathrm{Spin}(2n_c)$ bundle. When the flavor background v_2 is nontrivial, the obstruction class a_2 controlling the lift from $\mathrm{SO}(2n_c)/\mathbb{Z}_2$ to $\mathrm{SO}(2n_c)$ is nontrivial. In this situation when n_c is odd, w_2 can no longer be defined as a closed cochain; rather it satisfies $\delta w_2 = \beta a_2$, where β is the Bockstein operation, since together they specify the obstruction class $x_2 \in H^2(X, \mathbb{Z}_4)$ controlling the lift from $\mathrm{SO}(2n_c)/\mathbb{Z}_2 = \mathrm{Spin}(2n_c)/\mathbb{Z}_4$ to $\mathrm{Spin}(2n_c)$. As $E = w_2$ and $a_2 = v_2$, we conclude that the background fields satisfy

$$\delta E = \beta v_2. \tag{2.2}$$

This means that the $\mathbb{Z}_2$ 1-form symmetry and the $\mathrm{SU}(2n_f)/\mathbb{Z}_2$ 0-form flavor symmetry form the 2-group H fitting in the sequence

$$0 \to \mathbb{Z}_2[1] \to H \to \mathrm{SU}(2n_f)/\mathbb{Z}_2 \to 0 \tag{2.3}$$

whose Postnikov class is $\beta v_2 \in H^3(B\mathrm{SU}(2n_f)/\mathbb{Z}_2, \mathbb{Z}_2)$.

Before proceeding, we note that having the extension of groups of charges of line operators as in (2.1) is equivalent to having a nontrivial 2-group extension (2.3) whose background fields satisfy (2.2). Therefore, to find a nontrivial 2-group extension, we can simply study the group of charges of line operators, which we will carry out for $\mathrm{SO}_\pm$ gauge theories next.

2.2 $\mathrm{SO}_\pm$

We would like to study how the magnetic $\mathbb{Z}_2$ 1-form symmetry of the $\mathrm{SO}(2n_c)_+$ theory is combined with the $\mathfrak{su}(n_f)$ flavor symmetry. According to the discussions in the previous subsection, we consider what happens if we take two copies of the 't Hooft line operator H and fuse them. At the very naive level, H^2 can be screened by a dynamical monopole, but dynamical monopoles can receive flavor and gauge center charges from the fermion zero modes. To study these issues, it is useful to deform the theories to make them simpler.

For this purpose, we perform the following steps:

- We reduce the flavor symmetry from $\mathfrak{su}(2n_f)$ to $\mathfrak{usp}(2n_f)$. The fundamental representation still transforms nontrivially under $-1 \in \mathrm{USp}(2n_f)$, which is enough for our purposes.
- We add an adjoint scalar $\Phi_{[ab]}$ and the interaction $\psi^{ai}_\alpha \psi^{bj}_\beta J_{ij} \Phi_{ab} \epsilon^{\alpha\beta} + cc$, where $J_{[ij]}$ is the constant matrix for the $\mathfrak{usp}(2n_f)$ part.
- We give a vev to Φ_{ab} to break $\mathrm{SO}(2n_c)$ to $\mathrm{SO}(2)^{n_c}$.

The 't Hooft lines in the resulting $\mathrm{SO}(2)^{n_c}$ theory can be labeled by their magnetic charges $(m_1, \ldots, m_{n_c}) \in \mathbb{Z}^{n_c})$. The dynamical monopoles have the charges in the 'adjoint class', which are in the root lattice Λ of $\mathrm{SO}(2n_c)$.

The group of the magnetic charges of 't Hooft lines up to screening by the dynamical monopoles is then

$$\mathbb{Z}^{n_c}/\Lambda = \mathbb{Z}_2, \tag{2.4}$$

which agrees with the 1-form symmetry before the deformation. We now would like to study how this $\mathbb{Z}_2$ is combined with the flavor/gauge center charge $\mathbb{Z}_2$.

For this purpose we need to know slightly more details of the dynamical monopoles. The dynamical monopoles associated to the breaking of G to its Cartan were analyzed in many places, e.g. in [Wei80]. There, the following was shown. Let ϕ be the scalar vev in the real Cartan subalgebra, $\phi \in \mathfrak{h} \subset \mathfrak{g}$. This determines the simple roots α. Then you can embed the standard spherically-symmetric 't Hooft-Polyakov monopole and have a monopole solution without additional bosonic moduli.

Let us say we chose the standard ϕ such that the simple roots are $(1, -1, \ldots, 0)$, $(0, 1, -1, \ldots, 0)$, $\ldots$, $(0, \ldots, 1, -1)$ and $(0, \ldots, 1, +1)$, which we call simple dynamical monopoles.

We now consider the group $\mathbb{Z}^{n_c} \times \mathbb{Z}_2$ which combine the magnetic charges $\mathbb{Z}^{n_c}$ and the flavor/gauge center charge $\mathbb{Z}_2$. What we are after is the quotient of $\mathbb{Z}^{n_c} \times \mathbb{Z}_2$ by the subgroup generated by the charges of simple dynamical monopoles, which we denote by $(1, -1, \ldots, 0; q_1)$, $(0, 1, -1, \ldots, 0; q_2)$, $\ldots$, $(0, \ldots, 1, -1; q_{n_c-1})$ and $(0, \ldots, 1, +1; q_{n_c})$, respectively.

Let us determine this quotient. We do not have to determine q_1 to q_{n_c-1}. We simply use them to rewrite any charge vector $(m_1, \ldots, m_{n_c-2}, m_{n_c-1}, m_{n_c}; q)$ into the form $(0, \ldots, 0, m, m'; q')$. We then have to determine q_{n-1} and q_{n_c}.

This reduces the study to the case of $\mathfrak{so}(4) \simeq \mathfrak{su}(2)_1 \times \mathfrak{su}(2)_2$. The monopoles associated to the simple roots are just 't Hooft-Polyakov monopoles associated to the two factors of $\mathfrak{su}(2)$'s. The vev of the adjoint scalar in this basis can be written as (a_1, a_2), which we assume $a_1 > a_2 > 0$. The fermion is in the vector representation of $so(4)$. Under the monopole in $\mathfrak{su}(2)_1$, it is a doublet coupled to an adjoint vev of size a_1 with bare mass a_2, and similarly for the monopole in $\mathfrak{su}(2)_2$.

Now, the explicit analysis in [Cal78, Sec. IV] concerning the number of zero modes in the 't Hooft-Polyakov monopole says that a doublet coupled to an adjoint vev of size a with bare mass μ has a zero mode if $|a| > |m|$ and it has no zero mode if $|a| < |m|$.

With our assumption $a_1 > a_2 > 0$, this means that the monopole in $\mathfrak{su}(2)_1$ has a zero mode while the one in $\mathfrak{su}(2)_2$ does not have any. In our original basis, this means that the monopole with $(0, \ldots, 1, -1; q_{n_c-1})$ does not have any zero modes and $q_{n_c-1} = 0$, while the one with $(0, \ldots, 1, +1; q_{n_c})$ (which is in $\mathfrak{su}(2)_1$) has two zero modes per flavor.

The one-form symmetry group is obtained by dividing $\mathbb{Z}^2 \times \mathbb{Z}_2$ by the subgroup generated by $(1, -1; 0)$ and $(1, +1; q_{n_c})$. This is $\mathbb{Z}_2 \times \mathbb{Z}_2$ or $\mathbb{Z}_4$ depending on whether q_{n_c} is 0 or 1.

Let us determine q_{n_c}. We saw two zero modes per flavor; this means that there are fermionic zero modes transforming in

$$V_2 \otimes R_{2n_f} \tag{2.5}$$

where V_2 is the doublet of $\mathfrak{su}(2)_2$ (which is actually broken to $\mathfrak{u}(1)$ but keeping $\mathfrak{su}(2)_2$ representation is useful in organizing the answer), R_{2n_f} is the fundamental of $\mathfrak{usp}(2n_f)$, and we need to impose the reality condition using the pseudoreality of both factors, so that there are $4n_f$ Majorana fermion in total.

(n_c, n_f)	Spin	SO_+	SO_-
(even, even)	product	product	product
(odd, even)	extended	product	extended
(even, odd)	product	extended	extended
(odd, odd)	extended	extended	product

Table 2: How the $\mathbb{Z}_2$ 1-form symmetry and the flavor symmetry $\mathrm{SU}(N_f)/\mathbb{Z}_2$ are combined in $\mathfrak{so}(2n_c)$ QCD. 'product' means that they form a direct product, and 'extended' means that they form a nontrivial 2-group.

To determine the flavor/gauge center charge of the monopole, it suffices to consider the case $n_f = 1$; the general case is given by simply multiplying it by n_f. When $n_f = 1$, there are 4 Majorana fermions. Quantizing them, we find the monopoles in

$$V_2 \otimes \mathbf{1} \oplus \mathbf{1} \otimes R_2. \tag{2.6}$$

It has the 'vector' charge under $\mathfrak{usp}(2)$ flavor symmetry or is a doublet under $\mathfrak{su}(2)_2$, which corresponds to the 'vector' charge under $\mathfrak{so}(4)$ gauge symmetry. In either case, they have the flavor/gauge center charge $1 \in \{0, 1\} = \mathbb{Z}_2$. Therefore we conclude the flavor/gauge center charge q_{n_c} is simply given by n_f mod 2.

Combining the intermediate steps above, we conclude the following: for the $\mathrm{SO}(2n_c)_+$ theory, the group $\mathbb{Z}_2$ of magnetic charges of 't Hooft lines is extended by the flavor/gauge center symmetry $\mathbb{Z}_2$ to become $\mathbb{Z}_4$ when n_f is odd, while they remain separate when n_f is even.

The analysis of the $\mathrm{SO}(2n_c)_-$ theory is largely the same; the only difference is that the discrete theta angle gives an additional gauge center charge to the simple dynamical monopole with the magnetic charge $(0, 0, \ldots, 1, 1)$, so that $q_{n_c} = n_f + n_c$ mod 2. Therefore, we conclude the following: for the $\mathrm{SO}(2n_c)_-$ theory, the group $\mathbb{Z}_2$ of magnetic charges of 't Hooft lines is extended by the flavor/gauge center symmetry $\mathbb{Z}_2$ to become $\mathbb{Z}_4$ when $n_f + n_c$ is odd, while they remain separate when $n_f + n_c$ is even.

The result of the analysis is summarized in Table 2. There, 'product' means that the $\mathbb{Z}_2$ 1-form symmetry and $\mathrm{SU}(2n_f)/\mathbb{Z}_2$ flavor symmetry are kept separate and form a direct product, while 'extended' means that they form a nontrivial 2-group. We remark that the nontrivial 2-group is always given by the extension (2.3) whose background fields satisfy (2.2).

References

[ARS19] J. Ang, K. Roumpedakis, and S. Seifnashri, *Line Operators of Gauge Theories on Non-Spin Manifolds*, JHEP **04** (2020) 087, arXiv:1911.00589 [hep-th].

[AST13] O. Aharony, N. Seiberg, and Y. Tachikawa, *Reading Between the Lines of Four-Dimensional Gauge Theories*, JHEP **08** (2013) 115, arXiv:1305.0318 [hep-th].

[BCH18] F. Benini, C. Córdova, and P.-S. Hsin, *On 2-Group Global Symmetries and their Anomalies*, JHEP **03** (2019) 118, arXiv:1803.09336 [hep-th].

[Cal78] C. Callias, *Index Theorems on Open Spaces*, Commun. Math. Phys. **62** (1978) 213–234.

[CDI18] C. Córdova, T. T. Dumitrescu, and K. Intriligator, *Exploring 2-Group Global Symmetries*, JHEP **02** (2019) 184, arXiv:1802.04790 [hep-th].

[GKSW14] D. Gaiotto, A. Kapustin, N. Seiberg, and B. Willett, *Generalized Global Symmetries*, JHEP **02** (2015) 172, arXiv:1412.5148 [hep-th].

[HL20] P.-S. Hsin and H. T. Lam, *Discrete Theta Angles, Symmetries and Anomalies*, SciPost Phys. **10** (2021) 032, arXiv:2007.05915 [hep-th].

[IS95] K. A. Intriligator and N. Seiberg, *Duality, Monopoles, Dyons, Confinement and Oblique Confinement in Supersymmetric SO(N_c) Gauge Theories*, Nucl. Phys. B **444** (1995) 125–160, arXiv:hep-th/9503179.

[KS14] A. Kapustin and N. Seiberg, *Coupling a QFT to a TQFT and Duality*, JHEP **04** (2014) 001, arXiv:1401.0740 [hep-th].

[RW13] S. S. Razamat and B. Willett, *Global Properties of Supersymmetric Theories and the Lens Space*, Commun. Math. Phys. **334** (2015) 661–696, arXiv:1307.4381 [hep-th].

[Tac17] Y. Tachikawa, *On Gauging Finite Subgroups*, SciPost Phys. **8** (2020) 015, arXiv:1712.09542 [hep-th].

[tH79] G. 't Hooft, *A Property of Electric and Magnetic Flux in Nonabelian Gauge Theories*, Nucl. Phys. B **153** (1979) 141–160.

[Wei80] E. J. Weinberg, *Fundamental Monopoles and Multi-Monopole Solutions for Arbitrary Simple Gauge Groups*, Nucl. Phys. B **167** (1980) 500–524.