Matching higher symmetries
across Intriligator-Seiberg duality

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We study higher symmetries and anomalies of 4d $\mathfrak{so}(2n_c)$ gauge theory with $N_f = 2n_f$ flavors. We find that they depend on the parity of n_c and n_f, the global form of the gauge group, and the discrete theta angle. The contribution from the fermions plays a central role in our analysis. Furthermore, our conclusion applies to $\mathcal{N}=1$ supersymmetric cases as well, and we see that higher symmetries and anomalies match across the duality between $\mathfrak{so}(2n_c) \leftrightarrow \mathfrak{so}(2n_f - 2n_c + 4)$, originally found by Intriligator and Seiberg.

Contents

1 Introduction and summary

Our understanding of the concept of symmetries in quantum field theories (QFT) has been greatly improved in the last several years. We now have the concept of p-form symmetries acting on p-dimensional objects [GKSW14]. This concept gives a point of view which unifies both ordinary symmetries acting on point operators for $p = 0$ and center symmetries of gauge theories acting on Wilson line operators for $p = 1$. In addition, the 't Hooft magnetic flux [tH79] can now be thought of as a background gauge field for the 1-form center symmetry. It is also realized more recently that 0-form symmetries and 1-form symmetries can not only coexist in a direct product but also mix in a more intricate manner. They can have mixed anomalies between them, or they can also combine to form a *higher* symmetry structure called 2-groups [CDI18, BCH18].

In this paper, we study these issues in the case of 4d $\mathfrak{so}$ quantum chromodynamics (QCD), i.e. $\mathfrak{so}(N_c)$ gauge theories with N_f flavors of fermion fields in the vector representation. Let us quickly recall the 0-form and 1-form symmetries these theories have.

As for the 1-form symmetry, we first need to recall that such theories come in three versions, Spin, SO_+ and SO_-, distinguished firstly by the global form of the gauge group (Spin vs. SO)

and further by the choice of the discrete theta angle (SO_+ vs. SO_-) [AST13][1]. They also differ by the nontrivial line operator they possess: the Spin theory has the Wilson line W in the spinor representation, the SO_+ theory has the 't Hooft line H which is mutually non-local with respect to W, and the SO_- theory has the dyonic line $D = WH$. Furthermore, these line operators are charged under corresponding $\mathbb{Z}_2$ 1-form symmetries, which we respectively call electric, magnetic and dyonic 1-form symmetries.

As for the 0-form symmetry, we focus our attention on the $\mathfrak{su}(N_f)$ symmetry acting on N_f flavors of matter fields in the vector representation. There can be and definitely are other discrete symmetries, but we will not consider them in this paper for brevity.

The main question is then how the $\mathbb{Z}_2$ 1-form symmetry and the $\mathfrak{su}(N_f)$ 0-form symmetry are related.[2] We concentrate on the case when N_c and N_f are both even: $N_c = 2n_c$ and $N_f = 2n_f$. We introduce three possible behaviors, which we call 'extension', 'anomaly' and 'none'.

The 'extension': Take for example the $\mathrm{Spin}(2n_c)$ gauge theory with $2n_f$ flavors. When n_c is odd, two copies of the Wilson line W in the spinor representation form a Wilson line in the vector representation. This can be screened by a dynamical fermion, which was why $W^2 = 1$ as far as the 1-form symmetry charge was concerned. Now let us recall that this dynamical fermion transforms nontrivially under $-1 \in \mathrm{SU}(2n_f)$. Therefore, when we further take the flavor symmetry into account, W^2 is still nontrivial. As we will detail in this paper, this means that the $\mathbb{Z}_2$ 1-form symmetry extends the $\mathrm{SU}(2n_f)/\mathbb{Z}_2$ 0-form symmetry in a nontrivial manner, forming a 2-group H

$$0 \to \mathbb{Z}_2[1] \to H \to \mathrm{SU}(2n_f)/\mathbb{Z}_2 \to 0, \tag{1.1}$$

whose extension class is specified by

$$\beta a_2 \in H^3(B(\mathrm{SU}(2n_f)/\mathbb{Z}_2); \mathbb{Z}_2), \tag{1.2}$$

where a_2 is the generator of $H^2(B(\mathrm{SU}(2n_f)/\mathbb{Z}_2); \mathbb{Z}_2) = \mathbb{Z}_2$ [KM75] and β is the Bockstein homomorphism.

The 'anomaly': The SO_+ gauge theory is obtained by gauging the $\mathbb{Z}_2$ 1-form symmetry of the Spin gauge theory [KS14]. The presence of the fermions significantly complicates the analysis. For the moment let us suppose that we have N_f scalars instead of fermions in the vector representation. Then, the argument of [Tac17] immediately applies, and we see that the $\mathbb{Z}_2$ 1-form symmetry of the $\mathrm{SO}(2n_f)$ theory and the $\mathrm{SU}(2n_f)/\mathbb{Z}_2$ 0-form flavor symmetry remains a direct product but with a mixed anomaly given by

$$2\pi i \cdot \frac{1}{2} \int B\beta a_2. \tag{1.3}$$

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[1]This is when the theories are considered on spin manifolds. For non-spin manifolds, a further distinction needs to be made [ARS19]. For simplicity, we only consider spin manifolds in this paper.

[2]A partial answer was given in [HL20], but the contribution from fermions was not taken into account in that reference. Our conclusion is therefore somewhat different from theirs.

(n_c, n_f)	Spin	SO_+	SO_-
(even, even)	none	none	none
(odd, even)	extension	anomaly	extension
(even, odd)	anomaly	extension	extension
(odd, odd)	extension	extension	anomaly

Table 1: How the $\mathbb{Z}_2$ 1-form symmetry and the $\mathrm{SU}(2n_f)/\mathbb{Z}_2$ 0-form symmetry are combined in $\mathfrak{so}(2n_c)$ QCD. 'none' implies that they remain a direct product without mixed anomaly; 'anomaly' means that they remain a direct product but with mixed anomaly; and 'extension' is when they combine into a nontrivial 2-group. The orange lines show how the duality of Intriligator and Seiberg acts on this set of theories.

In the rest of the paper, we will carefully analyze how the $\mathbb{Z}_2$ 1-form symmetry and the $\mathrm{SU}(2n_f)/\mathbb{Z}_2$ 0-form symmetry are combined. The derivation will be detailed in the following, and here we simply summarize the result in Table 1. There, 'none' specifies that they remain a direct product without mixed anomaly; 'anomaly' implies that they remain a direct product but with mixed anomaly of the form (1.3); and 'extension' means that they combine into a 2-group given by (1.1) with the extension class (1.2).

Our result is equally applicable in the case of $\mathcal{N}{=}1$ supersymmetric QCD, since they are connected to the non-supersymmetric QCD by a continuous deformation preserving all the symmetries we care about. Now, let us recall that Intriligator and Seiberg found in [IS95] a duality exchanging $\mathfrak{so}(N_c)$ and $\mathfrak{so}(N_f - N_c + 4)$, which in our notation sends n_c to $n'_c = n_f - n_c + 2$. In [AST13], this duality was refined to account for the global form of the gauge group and the discrete theta angle, and it was concluded that Spin is exchanged with SO_- while SO_+ maps to itself. This mapping was given a further check e.g. by using supersymmetric localization on $S^3/\mathbb{Z}_n \times S^1$ in [RW13]. Our analysis allows us to check this duality by comparing how the 1-form symmetry and the 0-form symmetry are combined in the dual pairs. We superimposed the action of the duality on our main Table 1. It is satisfying to see that the duality action correctly preserves the behaviors 'none', 'anomaly' and 'extension'. In the last couple of years, the study of higher symmetries and their anomalies of supersymmetric theories has seen some activity[3], but mostly from the point of view of string theory or M-theory. The authors hope that this paper paves a way toward a more field-theoretical analysis of these matters.

The rest of the paper is organized as follows. In Sec. 2, we determine exactly when the electric/magnetic/dyonic $\mathbb{Z}_2$ 1-form symmetries and the $\mathrm{SU}(2n_f)/\mathbb{Z}_2$ flavor 0-form symmetry form a nontrivial 2-group, by examining the charges of line operators in each theory. In Sec. 3, we exploit the $\mathrm{SL}(2, \mathbb{Z}_2)$ actions on theories with $\mathbb{Z}_2$ 1-form symmetries, including our $\mathfrak{so}(2n_f)$ QCDs. This will allow us to determine the 't Hooft anomalies they possess. Combining the results with those

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[3]See e.g. [TZ19, EKSNW19, MSNW20, ADZGEH20, ADL20, CDLZ20, GHP20, DZGEH20, DZO20, BSN20, BHSN21a, ABOSN21, CDLZ21, AvBGSN21, BHSN21b, BJ21, BOL21, CHTZ21].

name	$G/\mathbb{Z}_n$	G	$\mathbb{Z}_n$	comments
w_2	$\mathrm{SO}(2n_c)$	$\mathrm{Spin}(2n_c)$	$\mathbb{Z}_2$	
v_2	$\mathrm{SO}(2n_c)/\mathbb{Z}_2$	$\mathrm{SO}(2n_c)$	$\mathbb{Z}_2$	
x_2	$\mathrm{SO}(2n_c)/\mathbb{Z}_2$	$\mathrm{Spin}(2n_c)$	$\mathbb{Z}_4$	$(n_c : \mathrm{odd})$
a_2	$\mathrm{SU}(2n_f)/\mathbb{Z}_2$	$\mathrm{SU}(2n_f)$	$\mathbb{Z}_2$	
a_2	$\mathrm{USp}(2n_f)/\mathbb{Z}_2$	$\mathrm{USp}(2n_f)$	$\mathbb{Z}_2$	

Table 2: Our names for the obstruction classes $\in H^2(X, \mathbb{Z}_n)$ controlling whether a $G/\mathbb{Z}_n$ bundle on X lifts to a G bundle.

obtained in Sec. 2, one can completely determine the structures of symmetries and anomalies of $\mathfrak{so}(2n_f)$ QCDs, and can further confirm that they are indeed compatible with the Intriligator-Seiberg duality. Although the result itself is satisfactory, the analysis leading to it is somewhat ad-hoc, so we partially complement it with a more direct computation of fermion anomalies in Sec. 4. Finally, the two appendices provide technical details of the mathematical facts used in the main part; in Appendix A, we compute relevant bordism groups capturing the fermion anomalies associated to various symmetries; and in Appendix B, we describe some subtleties concerning the Pontrjagin square.

Before proceeding, we list the obstruction classes we frequently encounter in this paper. In general, given a group G, a subgroup $\mathbb{Z}_n$ in the center of G, and a $G/\mathbb{Z}_n$ bundle on a manifold X, there is a obstruction class $\in H^2(X; \mathbb{Z}_n)$ controlling whether this bundle lifts to G. For $G = \mathrm{Spin}(N_c)$ and $G/\mathbb{Z}_2 = \mathrm{SO}(N_c)$ this is the familiar second Stiefel-Whitney class w_2. The classes we use are listed in Table 2.

2 2-group structure

Let us first study whether the $\mathbb{Z}_2$ 1-form symmetry and the $\mathrm{SU}(2n_f)/\mathbb{Z}_2$ 0-form flavor symmetry form a nontrivial 2-group or not. This can be found rather physically by studying the line operators.

2.1 Spin

We start by discussing the $\mathrm{Spin}(2n_c)$ gauge theories. The results presented in this section was originally found in [HL20, Sec. 4.4].

First, recall that the center of $\mathrm{Spin}(2n_c)$ is $\mathbb{Z}_2 \times \mathbb{Z}_2$ or $\mathbb{Z}_4$ depending on whether n_c is even or odd. This corresponds to the fact that the tensor square of a spinor representation contains the identity representation when n_c is even while it contains the vector representation when n_c is odd.

We now consider the Wilson line W in the spinor representation in the $\mathrm{Spin}(2n_c)$ gauge theory with $2n_f$ fermions in the vector representation. When n_c is even, W^2 contains the identity representation, and therefore we simply have a $\mathbb{Z}_2$ 1-form symmetry independent of the flavor symmetry, and there is nothing to see here.

When n_c is odd, W^2 contains the vector representation. This can be screened by the dynamical fermion, which however carries the fundamental representation of $\mathfrak{su}(2n_f)$ flavor symmetry, and in particular transforms nontrivially under $-1 \in \mathrm{SU}(2n_f)$. In other words, the flavor Wilson line in the vector representation of $\mathrm{SU}(2n_f)$ can now be considered as the square of the gauge Wilson line in the spinor representation of $\mathrm{Spin}(2n_c)$. This means that we have the following extension of groups

$$0 \to \underbrace{\mathbb{Z}_2}_{\substack{\text{group of} \\ \text{charges under} \\ \{\pm 1\} \in \mathrm{SU}(2n_f)}} \to \mathbb{Z}_4 \to \underbrace{\mathbb{Z}_2}_{\substack{\text{group of} \\ \text{gauge Wilson lines} \\ \text{up to screening}}} \to 0. \tag{2.1}$$

As the groups of charges of $\mathfrak{su}(2n_f)$ 0-form symmetry and $\mathbb{Z}_2$ 1-form symmetry are combined nontrivially, the symmetry groups themselves are also combined nontrivially. This can be seen most clearly by considering their background fields.

The fermion fields are simultaneously in the vector representation of $\mathrm{SO}(2n_c)$ and the fundamental representation of $\mathrm{SU}(2n_f)$, and therefore are in a representation of $G = [\mathrm{SO}(2n_c) \times \mathrm{SU}(2n_f)]/\mathbb{Z}_2$. Given a G-bundle on a manifold X, there is an $\mathrm{SO}(2n_c)/\mathbb{Z}_2$ bundle and an $\mathrm{SU}(2n_f)/\mathbb{Z}_2$ bundle associated to it. Let us denote by $v_2, a_2 \in H^2(X; \mathbb{Z}_2)$ the obstruction classes controlling whether they lift to $\mathrm{SO}(2n_c)$ and $\mathrm{SU}(2n_f)$ respectively. Then we have $v_2 = a_2$ for a G-bundle. The flavor Wilson line in the fundamental representation is charged under $-1 \in \mathrm{SU}(2n_f)$ in the center, and a_2 can be considered as the background field for this $\mathbb{Z}_2$ 1-form center symmetry.

Now, without the flavor background, the background $E \in H^2(X; \mathbb{Z}_2)$ for the electric $\mathbb{Z}_2$ 1-form symmetry of the $\mathrm{Spin}(2n_c)$ theory sets the Stiefel-Whitney class $w_2 \in H^2(X; \mathbb{Z}_2)$ of the $\mathrm{SO}(2n_c)$ gauge bundle to be $E = w_2$, which controls whether it lifts to a $\mathrm{Spin}(2n_c)$ bundle. When the flavor background a_2 is nontrivial, the obstruction class v_2 controlling the lift from $\mathrm{SO}(2n_c)/\mathbb{Z}_2$ to $\mathrm{SO}(2n_c)$ is nontrivial. In this situation when n_c is odd, w_2 can no longer be defined as a closed cochain; rather it satisfies $\delta w_2 = \beta v_2$, where β is the Bockstein operation, since together they specify the obstruction class $x_2 \in H^2(X; \mathbb{Z}_4)$ controlling the lift from $\mathrm{SO}(2n_c)/\mathbb{Z}_2 = \mathrm{Spin}(2n_c)/\mathbb{Z}_4$ to $\mathrm{Spin}(2n_c)$. As $E = w_2$ and $v_2 = a_2$, we conclude that the background field satisfies

$$\delta E = \beta a_2. \tag{2.2}$$

In general, a 2-group H combining a 1-form symmetry A and a 0-form symmetry G of the form

$$0 \to A[1] \to H \to G \to 0 \tag{2.3}$$

with the extension class $\alpha \in H^3(BG, A)$ is defined as a symmetry whose background field is given by a pair of a degree-2 cochain $E \in C^2(X, A)$ and a background G field $g : X \to BG$ such that $\delta E = g^*(\alpha)$. Here $A[1]$ means the Abelian group A regarded as a 1-form symmetry, and we drop the pull-back symbol g^* when its presence is clear from the context. In our case, we see that the $\mathbb{Z}_2$ 1-form symmetry and the $\mathrm{SU}(2n_f)/\mathbb{Z}_2$ 0-form flavor symmetry form the 2-group H fitting in the sequence

$$0 \to \mathbb{Z}_2[1] \to H \to \mathrm{SU}(2n_f)/\mathbb{Z}_2 \to 0 \tag{2.4}$$

with the extension class being $\beta a_2 \in H^3(B(\mathrm{SU}(2n_f)/\mathbb{Z}_2); \mathbb{Z}_2)$.

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Note that having the extension of groups of charges of line operators as in (2.1) is equivalent to having a nontrivial 2-group extension (2.4) whose background field satisfies (2.2). Therefore, to find a nontrivial 2-group extension, we can simply study the group of charges of line operators, which we will carry out for $\mathrm{SO}_\pm$ gauge theories next.

2.2 $\mathrm{SO}_\pm$

We would like to study how the magnetic $\mathbb{Z}_2$ 1-form symmetry of the $\mathrm{SO}(2n_c)_+$ theory is combined with the $\mathfrak{su}(2n_f)$ flavor symmetry. In accord with the discussions in the previous subsection, we consider what happens when we take two copies of the 't Hooft line operator H and fuse them. At the very naive level, H^2 can be screened by dynamical monopoles, but dynamical monopoles can receive flavor/gauge center charges from the fermion zero modes.

Deformations: To study these issues, it is useful to deform the theory and make it simpler by performing the following steps:

- Reduce the flavor symmetry from $\mathfrak{su}(2n_f)$ to $\mathfrak{usp}(2n_f)$. The fundamental representation still transforms nontrivially under $-1 \in \mathrm{USp}(2n_f)$, which is enough for our purposes.

- Add an adjoint scalar $\Phi_{[ab]}$ and the interaction $\psi_\alpha^{ai}\psi_\beta^{bj} J_{ij}\Phi_{ab}\epsilon^{\alpha\beta} + c.c.$. Here a, b and i, j are vector indices of $\mathfrak{so}(2n_c)$ and $\mathfrak{usp}(2n_f)$, α, β are the spinor indices, and $J_{[ij]}$ is the constant matrix for the $\mathfrak{usp}(2n_f)$.

- Give a generic vacuum expectation value (vev) to Φ_{ab} to break $\mathrm{SO}(2n_c)$ to $\mathrm{SO}(2)^{n_c}$.

The 't Hooft lines in the resulting $\mathrm{SO}(2)^{n_c}$ theory can be labeled by their magnetic charges $(m_1, \ldots, m_{n_c}) \in \mathbb{Z}^{n_c}$. The dynamical monopoles have the charges in the 'adjoint class', which are in the root lattice Λ of $\mathrm{SO}(2n_c)$. Then, the group of the magnetic charges of 't Hooft lines up to screening by the dynamical monopoles is

$$\mathbb{Z}^{n_c}/\Lambda = \mathbb{Z}_2, \tag{2.5}$$

which agrees with the 1-form symmetry before the deformation. We now would like to study how this $\mathbb{Z}_2$ is combined with the flavor/gauge center $\mathbb{Z}_2$ charge.

Reduction to the $\mathfrak{so}(4)$ case: For this purpose we need to know slightly more details of the dynamical monopoles. The dynamical monopoles associated to the breaking of a gauge group to its Cartan were analyzed in many places, e.g. in [Wei80]. There, the following was shown. Let ϕ be the scalar vev in the real Cartan subalgebra, $\phi \in \mathfrak{h} \subset \mathfrak{g}$. This determines the simple roots α. Then you can embed the standard spherically-symmetric 't Hooft-Polyakov monopole using the $\mathfrak{su}(2)$ subalgebra associated to α, and have a monopole solution without additional bosonic moduli.

Let us say we chose the standard ϕ such that the simple roots are

$$(1, -1, \ldots, 0), \ (0, 1, -1, \ldots, 0), \ldots, (0, \ldots, 1, -1), \ (0, \ldots, 1, +1) \in \mathbb{Z}^{n_c}, \tag{2.6}$$

which we call simple dynamical monopoles. Now, consider the group $\mathbb{Z}^{n_c} \times \mathbb{Z}_2$ which combines the magnetic charges in $\mathbb{Z}^{n_c}$ and the flavor/gauge center charge $q \in \mathbb{Z}_2$. What we are after is the quotient of $\mathbb{Z}^{n_c} \times \mathbb{Z}_2$ by the subgroup generated by the charges of simple dynamical monopoles, which we denote respectively by

$$(1, -1, \ldots, 0; q_1), \ (0, 1, -1, \ldots, 0; q_2), \ldots, (0, \ldots, 1, -1; q_{n_c-1}), \ (0, \ldots, 1, +1; q_{n_c}). \quad (2.7)$$

To determine this quotient, we do not have to determine the all q_i's; we simply use $q_{i=1,\ldots,n_c-2}$ to rewrite any charge vector $(m_1, \ldots, m_{n_c-2}, m_{n_c-1}, m_{n_c}; q)$ into the form $(0, \ldots, 0, m, m'; q')$. Then, only q_{n_c-1} and q_{n_c} need to be determined. This reduces the study to the case of $n_c = 2$ and $\mathfrak{so}(2n_c) \simeq \mathfrak{su}(2)_1 \times \mathfrak{su}(2)_2$, where the monopoles associated to the simple roots are just 't Hooft-Polyakov monopoles associated to the two factors of $\mathfrak{su}(2)$'s.

Analysis of the $\mathfrak{so}(4)$ case: The vev of the adjoint scalar in this basis can be written as (a_1, a_2), which we assume to be $a_1 > a_2 > 0$. Here, the fermion is in the vector representation of $\mathfrak{so}(4)$. Under the monopole in $\mathfrak{su}(2)_1$, it is a doublet coupled to an adjoint vev of size a_1 with bare mass a_2, and similarly for the monopole in $\mathfrak{su}(2)_2$.

Now, the explicit analysis in [Cal78, Sec. IV] concerning the number of zero modes in the 't Hooft-Polyakov monopole says that a doublet fermion coupled to an adjoint vev of size a with bare mass μ has a zero mode if $|a| > |\mu|$ and has no zero modes if $|a| < |\mu|$. With our assumption $a_1 > a_2 > 0$, this means that the monopole in $\mathfrak{su}(2)_1$ has a zero mode, while the monopole in $\mathfrak{su}(2)_2$ does not. In our original basis, this means that the monopole with $(0, \ldots, 1, -1; q_{n_c-1})$ does not produce any zero modes and $q_{n_c-1} = 0$, while the monopole with $(0, \ldots, 1, +1; q_{n_c})$ has two zero modes per flavor. The 1-form symmetry group is obtained by dividing $\mathbb{Z}^2 \times \mathbb{Z}_2$ by the subgroup generated by $(1, -1; 0)$ and $(1, +1; q_{n_c})$. This is $\mathbb{Z}_2 \times \mathbb{Z}_2$ or $\mathbb{Z}_4$ depending on whether q_{n_c} is 0 or 1.

Let us determine q_{n_c}, the center charge of the monopole in $\mathfrak{su}(2)_1$. We saw that there are two zero modes per flavor; this means that there are fermionic zero modes transforming in

$$V_2 \otimes R_{2n_f} \quad (2.8)$$

where V_2 is the doublet of $\mathfrak{su}(2)_2$[4], R_{2n_f} is the fundamental of $\mathfrak{usp}(2n_f)$, and we need to impose the reality condition using the pseudo-reality of both factors, so that there are $4n_f$ Majorana fermion in total.

To determine the flavor/gauge center charge q_{n_c} of the monopole, it suffices to consider the case $n_f = 1$; the general case is given simply by multiplying it by n_f. When $n_f = 1$, there are 4 Majorana fermions. Quantizing them, we find the monopoles in

$$(V_2 \otimes \mathbf{1}) \oplus (\mathbf{1} \otimes R_2). \quad (2.9)$$

It has the 'vector' charge under $\mathfrak{usp}(2) \simeq \mathfrak{su}(2)$ flavor symmetry or is a doublet under $\mathfrak{su}(2)_2$, which corresponds to the 'vector' charge under $\mathfrak{so}(4)$ gauge symmetry. In either case, they have the flavor/gauge center charge $1 \in \{0, 1\} = \mathbb{Z}_2$. Therefore we conclude the flavor/gauge center charge q_{n_c} is simply given by $n_f \bmod 2$.

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[4]It is actually broken to $\mathfrak{u}(1)$, but keeping $\mathfrak{su}(2)_2$ representation is useful in organizing the answer.

(n_c, n_f)	Spin	SO_+	SO_-
(even, even)	product	product	product
(odd, even)	extended	product	extended
(even, odd)	product	extended	extended
(odd, odd)	extended	extended	product

Table 3: How the $\mathbb{Z}_2$ 1-form symmetry and the flavor symmetry $SU(N_f)/\mathbb{Z}_2$ are combined in $\mathfrak{so}(2n_c)$ QCD. 'product' means that they form a direct product, and 'extended' means that they form a nontrivial 2-group.

Summary: Combining the intermediate steps we described above, we conclude the following: for the $SO(2n_c)_+$ theory, the group $\mathbb{Z}_2$ of magnetic charges of 't Hooft lines is extended by the flavor/gauge center symmetry $\mathbb{Z}_2$ to become $\mathbb{Z}_4$ when n_f is odd, while they remain separate when n_f is even.

The analysis of the $SO(2n_c)_-$ theory is largely the same; the only difference is that the discrete theta angle gives an additional gauge center charge to the simple dynamical monopole with the magnetic charge $(0, 0, \ldots, 1, +1)$, so that $q_{n_c} = n_f + n_c$ mod 2. Therefore, we conclude the following: for the $SO(2n_c)_-$ theory, the group $\mathbb{Z}_2$ of magnetic charges of 't Hooft lines is extended by the flavor/gauge center symmetry $\mathbb{Z}_2$ to become $\mathbb{Z}_4$ when $n_f + n_c$ is odd, while they remain separate when $n_f + n_c$ is even.

The result of the analysis is summarized in Table 3. There, 'product' means that the $\mathbb{Z}_2$ 1-form symmetry and $SU(2n_f)/\mathbb{Z}_2$ flavor symmetry are kept separate and form a direct product, while 'extended' means that they form a nontrivial 2-group. We remark that the nontrivial 2-group is always given by the extension (2.4) whose background fields satisfy (2.2).

3 $SL(2, \mathbb{Z}_2)$ **action and the anomalies**

In the last section we determined the 2-group structure of the $\mathfrak{so}(2n_c)$ gauge theories with $2n_f$ flavors, by studying the group of the charges of line operators. Here we determine the anomalies of these symmetries, utilizing the $SL(2, \mathbb{Z}_2)$ action on the set of QFTs with $\mathbb{Z}_2$ 1-form symmetry.

3.1 $SL(2, \mathbb{Z}_2)$ **action and $\mathfrak{so}$ gauge theories**

Let us say that we are given a four-dimensional spin QFT Q with $\mathbb{Z}_2$ 1-form symmetry. We denote its partition function on a manifold X by $Z_Q[E]$, where we suppress the dependence on X in the notation and $E \in H^2(X; \mathbb{Z}_2)$ is the background field for the $\mathbb{Z}_2$ 1-form symmetry. We then define SQ and TQ to be QFTs with partition functions given by the formula

$$Z_{SQ}[B] \propto \sum_E (-1)^{\int_X B \cup E} Z_Q[E], \qquad Z_{TQ}[E] = (-1)^{\int_X \frac{1}{2} \mathfrak{P}(E)} Z_Q[E]. \qquad (3.1)$$

We can show that $S^2 = T^2 = 1$ and $(ST)^3 = 1$, meaning that they generate $\mathrm{SL}(2, \mathbb{Z}_2)$. This operation was introduced in [GKSW14] as an analogue of the $\mathrm{SL}(2, \mathbb{Z})$ action on 3d theories with $\mathrm{U}(1)$ symmetry of [Wit03] and then further studied in [BLT20].

Importantly, $\mathrm{Spin}(2n_c)$ and $\mathrm{SO}(2n_c)_\pm$ gauge theories with $2n_f$ flavors with the same n_c and n_f form a single orbit under this $\mathrm{SL}(2, \mathbb{Z})$ action. More precisely, we need to make the distinction between $\mathrm{Spin}(2n_c)$ and $T(\mathrm{Spin}(2n_c))$ and similarly between $\mathrm{SO}(2n_c)_\pm$ and $T(\mathrm{SO}(2n_c)_\pm)$, respectively. Here the theories with T prepended are different from the original ones only by its discrete theta coupling to the background. Then we have the following chain of actions:

$$T(\mathrm{Spin}) \overset{T}{\longleftrightarrow} \mathrm{Spin} \overset{S}{\longleftrightarrow} \mathrm{SO}_+ \overset{T}{\longleftrightarrow} T(\mathrm{SO}_+) \overset{S}{\longleftrightarrow} T(\mathrm{SO}_-) \overset{T}{\longleftrightarrow} \mathrm{SO}_-.$$
$$\underset{S}{\underline{\hspace{8cm}}} \tag{3.2}$$

3.2 $\mathrm{SL}(2, \mathbb{Z}_2)$ actions with extra background

Let us now study what happens if we perform this $\mathrm{SL}(2, \mathbb{Z}_2)$ action when the $\mathbb{Z}_2$ 1-form symmetry in question is part of a larger symmetry group. So far we have been considering the effect of $\mathrm{SU}(2n_f)/\mathbb{Z}_2$ flavor symmetry, but the discussions in the last section shows that, at a formal level, only the background field $a_2 \in H^2(X; \mathbb{Z}_2)$ matters, which controls the lift from $\mathrm{SU}(2n_f)/\mathbb{Z}_2$ to $\mathrm{SU}(2n_f)$. Let us regard a_2 as the background field for a flavor $\mathbb{Z}_2$ 1-form symmetry.

The combined 1-form symmetry is either $\mathbb{Z}_2 \times \mathbb{Z}_2$ or $\mathbb{Z}_4$, and we perform the $\mathrm{SL}(2, \mathbb{Z}_2)$ action by picking a $\mathbb{Z}_2$ subgroup. The symmetry and the anomaly of the resulting theory can be determined by a formal argument independent of the dynamics of the theory, once the symmetry and the anomaly of the original theory and the action of the anomaly-free subgroup to be gauged are given, as discussed in [Tac17].

Let us work at the level of anomalies described by cohomology, since we do not need to deal with more general anomalies described by bordism. We consider a d-dimensional QFT with a symmetry group G with an anomaly specified by a cochain $\alpha \in C^{d+1}(G; \mathrm{U}(1))$. We pick a subgroup $H \subset G$ such that α trivializes in it, so that one can find $\mu \in C^d(H; \mathrm{U}(1))$ such that $\delta\mu = \alpha|_H$. We then gauge H, using μ as the action. What determines the symmetry and the anomaly of the gauged theory is the data (μ, α). Clearly, given $\nu \in C^d(G; \mathrm{U}(1))$, the pair (μ, α) and the pair $(\mu - \nu|_H, \alpha - \delta\nu)$ should give the same result, since we merely added the counterterm $\nu|_H$ as the action. This allows us to always choose the pair of the form $(0, \alpha')$ equivalent to a given (μ, α), by taking ν to be an arbitrary lift of μ from H to G.[5] This is convenient in discussing the $\mathrm{SL}(2, \mathbb{Z}_2)$ action, since our S operation is defined in the convention that $\mu = 0$.

The four choices: Now, what are the possible choices of $(\mu, \alpha) \sim (0, \alpha')$ we need to discuss? Let us first consider $\mathbb{Z}_2 \times \mathbb{Z}_2$ 1-form symmetry. As detailed in the Appendix A, the only possible

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[5]At this stage, the residual identifications $(0, \alpha') \sim (0, \alpha'')$ are of the form $\alpha'' = \alpha' + \delta\nu$, where $\nu \in C^d(G; \mathrm{U}(1))$ is required to satisfy $\nu|_H = 0$. Their equivalence classes form the relative cohomology group $H^{d+1}(G, H; \mathrm{U}(1))$. It might be interesting to study anomalies taking values in the relative cohomology groups.

anomaly for 4d spin QFTs with this symmetry is

$$\alpha = \frac{1}{2}B\beta E \tag{3.3}$$

where $B, E \in H^2(Y; \mathbb{Z}_2)$ are the background fields on the bulk 5d spin manifold Y, and we use $\mathbb{Q}/\mathbb{Z}$-valued cochains to describe the anomaly. Its restriction to $\mathbb{Z}_2$ 1-form symmetry subgroup is trivial i.e. $\alpha|_{H=\mathbb{Z}_2} = 0$, and thus the possible choice of μ is simply the discrete theta angle

$$\mu = \frac{1}{4}\mathfrak{P}(E), \tag{3.4}$$

where $\mathfrak{P}$ is the Pontrjagin square. This μ can be lifted from the $\mathbb{Z}_2$ subgroup to the entire $\mathbb{Z}_2 \times \mathbb{Z}_2$ group as a closed cochain, and therefore does not affect the gauging process. Therefore, we only have to consider pairs $(0,0)$ and $(0,\alpha)$.

Next, we consider $\mathbb{Z}_4$ 1-form symmetry. In the Appendix A, we show that there is no anomaly for $\mathbb{Z}_4$ 1-form symmetry. Therefore we can pick $\alpha = 0$. Then the only possible choice of μ for the $\mathbb{Z}_2$ 1-form subgroup is again the discrete theta angle (3.4). One difference here is that the discrete theta angle (3.4) cannot be lifted as a closed cochain to the entire $\mathbb{Z}_4$ 1-form subgroup. As discussed in the Appendix B, with $\delta E = \beta a_2$, where $a_2 \in H^2(X; \mathbb{Z}_4/\mathbb{Z}_2)$, one finds

$$\alpha' := \delta\mu = \frac{1}{2}a_2\beta_2 a_2, \tag{3.5}$$

where β_2 is the Bockstein associated to the sequence $0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}_4 \to 0$, whose action on a_2 can be defined since a_2 lifts to a $\mathbb{Z}_4$-valued cochain. Summarizing, the pairs we need to consider for the $\mathbb{Z}_4$ one-form symmetry are $(0,0)$ and $(\mu, 0) \sim (0, \alpha')$.

Summarizing, we need to consider the following four choices, namely:

- For $\mathbb{Z}_2 \times \mathbb{Z}_2$, the pairs $(0,0)$ and $(0,\alpha)$, which we call 'none' and 'anomaly'

- For $\mathbb{Z}_4$, the pairs $(0,0)$ and $(\mu, 0) \sim (0, \alpha')$, which we call 'extended' and 'extended$_T$'.

$\mathrm{SL}(2, \mathbb{Z}_2)$ **action on the four choices:** Let us now determine how the $\mathrm{SL}(2, \mathbb{Z}_2)$ action affects these data. The case 'none' is very easy. The additional $\mathbb{Z}_2$ factor plays no role, and we find the chain of actions given by

$$\text{none} \xleftrightarrow{T} \text{none} \xleftrightarrow{S} \text{none} \xleftrightarrow{T} \text{none} \xleftrightarrow{S} \text{none} \xleftrightarrow{T} \text{none}. \tag{3.6}$$

Let us now consider $\mathbb{Z}_4$ 1-form symmetry and gauge its $\mathbb{Z}_2$ subgroup. The case 'extended' was analyzed in [Tac17], and the resulting theory has $\mathbb{Z}_2 \times \mathbb{Z}_2$ 1-form symmetry with the anomaly (3.3), which we decided to call 'anomaly'.

The case 'extended$_T$' was analyzed in [HL20], where it was shown that the gauged theory has again 'extended$_T$'. Let us quickly recall why this is the case. The gauging process involves the term

$$\exp\left[2\pi i \int_X \left(\frac{1}{4}\mathfrak{P}(E) + \frac{1}{2}BE\right)\right] \tag{3.7}$$

where E is the variable to be gauged and B is the newly introduced background field. When $\mathbb{Z}_2$ to be gauged is the $\mathbb{Z}_2$ subgroup of a $\mathbb{Z}_4$ symmetry, E is not necessarily closed, but rather satisfies the relation

$$\delta E = \beta a_2 \tag{3.8}$$

where a_2 is the background field for the quotient $\mathbb{Z}_4/\mathbb{Z}_2$ 1-form symmetry. Then the second term in (3.7) is not closed, and to even talk about the first term in (3.7), one first needs to extend the definition of the Pontryagin square $\mathfrak{P}$ to non-closed cochains.

To make the coupling (3.7) well-defined, we consider adding a counterterm depending solely on the newly introduced field B to (3.7) so that we have

$$\exp\left[2\pi i \int_X \left(\frac{1}{4}\mathfrak{P}(E) + \frac{1}{2}BE + \frac{1}{4}\mathfrak{P}(B)\right)\right] = \exp\left[2\pi i \int_X \frac{1}{4}\mathfrak{P}(E+B)\right]. \tag{3.9}$$

This is perfectly well-defined and has no problem at all with the lifting, if the newly-introduced background field B also satisfies

$$\delta B = \beta a_2. \tag{3.10}$$

This means that, starting from 'extended$_T$', performing S and then T, we find a theory with the data 'extended'. Therefore, simply performing S for the theory of the type 'extended$_T$', one finds 'extended$_T$'.

Combined, the preceding arguments allow us to see the chain of actions

$$\text{extended}_T \xleftrightarrow{T} \text{extended} \xleftrightarrow{S} \text{anomaly} \xleftrightarrow{T} \text{anomaly} \xleftrightarrow{S} \text{extended} \xleftrightarrow{T} \text{extended}_T.$$
$$\underset{S}{\underline{\hspace{10cm}}} \tag{3.11}$$

3.3 Anomalies from $\mathrm{SL}(2,\mathbb{Z}_2)$ action

Let us now combine our result in Table 3, which summarizes our knowledge whether the 0-form part and the 1-form part form a nontrivial 2-group, and the $SL(2,\mathbb{Z}_2)$ action on the four choices, (3.6) and (3.11), we determined above. We first need to double each column of Table 3, since we need to distinguish Spin from $T\mathrm{Spin}$ and $\mathrm{SO}_\pm$ from $T\mathrm{SO}_\pm$. The entry 'product' in Table 3 corresponds to either 'none' or 'anomaly', and the entry 'extended' there corresponds to either 'extension' or 'extended$_T$'. We now demand that the $SL(2,\mathbb{Z}_2)$ action (3.2) on $\mathfrak{so}$ QCDT to be compatible with the $SL(2,\mathbb{Z}_2)$ action on the labels, (3.6) and (3.11). The only consistent assignment is given in Table 4. As the way we determined the symmetry structures were somewhat indirect, we confirm the symmetry structures of the Spin case in the next section in a different means.

We can also use this Table 4 to give a further check of the Intriligator-Seiberg duality, which is known to act as follows:

$$\begin{aligned}
\mathrm{Spin}(2n_c) &\leftrightarrow T(\mathrm{SO}_-(2n_f - 2n_c + 4)), \\
\mathrm{SO}_+(2n_c) &\leftrightarrow T(\mathrm{SO}_+(2n_f - 2n_c + 4)), \\
\mathrm{SO}_-(2n_c) &\leftrightarrow T(\mathrm{Spin}(2n_f - 2n_c + 4)).
\end{aligned} \tag{3.12}$$

(n_c, n_f)	$T(\text{Spin})$	Spin	SO_+	$T(\text{SO}_+)$	$T(\text{SO}_-)$	SO_-
(even, even)	none	none	none	none	none	none
(odd, even)	extended$_T$	extended	anomaly	anomaly	extended	extended$_T$
(even, odd)	anomaly	anomaly	extended	extended$_T$	extended$_T$	extended
(odd, odd)	extended	extended$_T$	extended$_T$	extended	anomaly	anomaly

Table 4: The symmetry structure of $\mathfrak{so}(2n_c)$ QCD with $2n_f$ flavors, as deduced from the 2-group structures found in Sec. 2 and from the $\text{SL}(2, \mathbb{Z}_2)$ action discussed in this section. The symmetry structure of the Spin case will be checked independently in the next section.

We see that the symmetry structure is indeed preserved across the duality. This duality action taking the finer distinction of Q and TQ into account was first discussed in [GKSW14, Sec. 6] and further discussed in [BLT20].

4 Fermion contribution to anomalies

So far, we first determined the 2-group structure in Sec. 2 by studying the charges of line operators, and then determined the anomalies in Sec. 3 by matching it to the action of $\text{SL}(2, \mathbb{Z}_2)$. Going over the entries on the column Spin of Table 4, we find that the anomaly is trivial when (n_c, n_f) is (even,even) or (odd,even), while it is α given in (3.3) when (n_c, n_f) is (odd,even) and α' given in (3.5) when (odd,odd). Since the 1-form symmetry background in the Spin theory is simply the Stiefel-Whitney class w_2 of the $\text{SO}(2n_c)$ gauge bundle, these anomalies should simply come from the anomalies of fermions charged under $[\text{SO}(2n_c) \times \text{USp}(2n_f)]/\mathbb{Z}_2$. Here we use $\text{USp}(2n_f)$ instead of $\text{SU}(2n_f)$, because under the latter we also have perturbative anomalies, which would complicate the analysis.

For even n_c, the anomaly should be given by

$$\alpha = \frac{1}{2} w_2 \beta a_2 \tag{4.1}$$

where $w_2, a_2 \in H^2(X; \mathbb{Z}_2)$ controls the lift from $\text{SO}(2n_c)$ to $\text{Spin}(2n_c)$ and from $\text{USp}(2n_f)/\mathbb{Z}_2$ to $\text{USp}(2n_f)$, respectively. For odd n_c, the anomaly cochain should be given by

$$\alpha' = \frac{1}{2} v_2 \beta_2 v_2 \tag{4.2}$$

where v_2 is the class in $H^2(X; \mathbb{Z}_2)$ which is the mod-2 reduction of the class $x_2 \in H^2(X; \mathbb{Z}_4)$ controlling the lift from $\text{SO}(2n_c)/\mathbb{Z}_2 = \text{Spin}(2n_c)/\mathbb{Z}_4$ to $\text{Spin}(2n_c)$. We note that α' is exact, but as explained in the previous section, this still affects the gauging process.

The aim of this last section is to give a check of these anomalies from a different point of view. We will proceed as follows. Starting from the theory where the fermions are charged under

$\mathrm{SO}(2n_c) \times \mathrm{USp}(2n_f)$, we add scalar fields which are adjoint under $\mathrm{USp}(2n_f)$ in the system, and break it down to a subgroup. We then determine the effective interaction induced by the fermion zero modes. The next step is to see what happens when the symmetry group is changed from $\mathrm{SO}(2n_c) \times \mathrm{USp}(2n_f)$ to its $\mathbb{Z}_2$ quotient; we will see that the effective interaction will have the required anomalies.

Before proceeding, we have two remarks. First, this method was first used in [Wit95, Sec. 4] to understand 'a curious minus sign' appearing in the topologically-twisted Seiberg-Witten theory, which was more recently recognized as determining an anomaly in [CD18, Sec. 2.4.3]. It was also used in [WWW18, Sec. 3.1 and Sec. 5.1.2] to relate the 'new' $\mathrm{SU}(2)$ anomaly with the effective interaction in the $\mathrm{U}(1)$ theory. Second, in this section we can only say that the effective interaction we find is compatible with the anomalies as found in Sec. 3, and will not be able to determine it completely. This is mostly due to the fact that the computation of the spin bordism group $\Omega_d^{\mathrm{spin}}([\mathrm{SO}(2n_c) \times \mathrm{USp}(2n_f)]/\mathbb{Z}_2)$ which governs the anomaly is quite hard, because even the integral cohomology of the group in question is hard to obtain. Only in a couple of cases we can say more, as we comment along the way.

4.1 Effective interaction

We break $\mathrm{USp}(2n_f)$ down to $\mathrm{U}(n_f)$ using a scalar field, such that the fundamental representation of $\mathrm{USp}(2n_f)$ splits into the fundamental plus the anti-fundamental representation of $\mathrm{U}(n_f)$. The monopole charge is given by the first Chern class c_1 of the low-energy $\mathrm{U}(n_f)$ flavor symmetry.

Take a standard 't Hooft-Polyakov monopole associated to $\mathrm{U}(1) \subset \mathrm{USp}(2)$ and embed it into $\mathrm{U}(n_f) \subset \mathrm{USp}(2n_f)$. The fermion zero modes form a vector representation of $\mathrm{SO}(2n_c)$, whose quantization leads to the spinor representation.

As first discussed in [Tho14] and also used in [WWW18, Sec. 3.1], this means that there is an effective interaction

$$\frac{1}{2} \int_X w_2 c_1. \tag{4.3}$$

One way to understand it is as follows.

We started from a system which has $\mathrm{SO}(2n_c)$ symmetry, but the spinor representation is only a projective representation of this symmetry. This is an anomaly at the core of the monopole, which needs to flow in from the bulk. Indeed, take the spacetime to be $X = \mathbb{R}_{\geq 0} \times \mathbb{R}_t \times S^2$ around the monopole. Reducing the bulk term (4.3) on S^2 with $\int_{S^2} c_1 = 1$, we have the effective interaction $\int_Y w_2$ on the half-space $Y = \mathbb{R}_{\geq 0} \times \mathbb{R}_t$, and the monopole lives on the boundary. Therefore, the degree of freedom on the boundary is in the projective representation characterized by $w_2 \in H^2(\mathrm{SO}(2n_c), \mathbb{Z}_2)$.

4.2 Anomalies

We now take the $\mathbb{Z}_2$ quotient, changing the symmetry group from $\mathrm{SO}(2n_c) \times \mathrm{USp}(2n_f)$ to $[\mathrm{SO}(2n_c) \times \mathrm{USp}(2n_f)]/\mathbb{Z}_2$. Note that $\pi_1(\mathrm{U}(n_f)/\mathbb{Z}_2) = \mathbb{Z} \times \mathbb{Z}_2$ or $\mathbb{Z}$ depending on whether n_f is even or odd.

We denote by a_2 the obstruction class to lift a $\mathrm{U}(n_f)/\mathbb{Z}_2$ bundle to a $\mathrm{U}(n_f)$ bundle. This implies the following:

- When n_f is even, $c_1(\mathrm{U}(n_f)) = c_1(\mathrm{U}(n_f)/\mathbb{Z}_2)$ and $a_2(\mathrm{U}(n_f)/\mathbb{Z}_2) = a_2(\mathrm{USp}(2n_f))$.

- When n_f is odd, $c_1(\mathrm{U}(n_f)/\mathbb{Z}_2) = 2c_1(\mathrm{U}(n_f))$ when the latter is well-defined. More generally, $a_2(\mathrm{U}(n_f)/\mathbb{Z}_2)$ is the mod-2 reduction of $c_1(\mathrm{U}(n_f)/\mathbb{Z}_2)$.

We now compute the anomaly cochains in the four cases separately:

(n_c, n_f)=**(even,even):** $w_2(\mathrm{SO}(2n_c))$ and $c_1(\mathrm{U}(n_f))$ can be generalized to closed cochains of $\mathrm{SO}(2n_c)/\mathbb{Z}_2$ and of $\mathrm{U}(n_f)/\mathbb{Z}_2$ without any problem, and therefore

$$\delta\left(\frac{1}{2}w_2(\mathrm{SO}(2n_c))c_1(\mathrm{U}(n_f))\right) = 0. \tag{4.4}$$

(n_c, n_f)=**(odd,even):** $w_2(\mathrm{SO}(2n_c))$ needs to be upgraded to a $\mathbb{Z}_4$-valued cochain $x_2(\mathrm{SO}(2n_c)/\mathbb{Z}_2)$. The original interaction is then

$$\frac{1}{4}x_2(\mathrm{SO}(2n_c)/\mathbb{Z}_2)c_1(\mathrm{U}(n_f)) \tag{4.5}$$

which is closed without problem, and therefore taking δ results in zero.

(n_c, n_f)=**(even,odd):** Here we need to replace $c_1(\mathrm{U}(n_f))$ by $c_1(\mathrm{U}(n_f)/\mathbb{Z}_2)/2$. The effective interaction is then

$$\frac{1}{4}w_2(\mathrm{SO}(2n_c)/\mathbb{Z}_2)c_1(\mathrm{U}(n_f)/\mathbb{Z}_2) \tag{4.6}$$

and

$$\delta\left(\frac{1}{4}w_2(\mathrm{SO}(2n_c)/\mathbb{Z}_2)c_1(\mathrm{U}(n_f)/\mathbb{Z}_2)\right) = \frac{1}{2}\left(\frac{1}{2}\delta w_2(\mathrm{SO}(2n_c))c_1(\mathrm{U}(n_f)/\mathbb{Z}_2)\right) \tag{4.7}$$

$$= \frac{1}{2}\left(\beta w_2(\mathrm{SO}(2n_c))\right)c_1(\mathrm{U}(n_f)/\mathbb{Z}_2) \tag{4.8}$$

which is a pull-back of the anomaly cochain

$$\frac{1}{2}\left(\beta w_2(\mathrm{SO}(2n_c))\right)a_2(\mathrm{USp}(2n_f)/\mathbb{Z}_2). \tag{4.9}$$

When $n_c = 2$ and $n_f = 1$, we can confirm that this is indeed the entire anomaly, since we can compute $\mathrm{Hom}(\Omega_5^{\mathrm{spin}}([\mathrm{SO}(4) \times \mathrm{USp}(2)]/\mathbb{Z}_2), \mathrm{U}(1))$ and show that this is the only nontrivial element there. For details, see Appendix A.3.

(n_c, n_f)=**(odd,odd):**   Now we make the replacement on both sides and therefore the effective interaction is

$$\frac{1}{8} x_2(\mathrm{SO}(2n_c)/\mathbb{Z}_2) c_1(\mathrm{U}(n_f)/\mathbb{Z}_2) \tag{4.10}$$

and

$$\delta\left(\frac{1}{8} x_2(\mathrm{SO}(2n_c)/\mathbb{Z}_2) c_1(\mathrm{U}(n_f)/\mathbb{Z}_2)\right) = \frac{1}{2}\left(\frac{1}{4}\delta v_2(\mathrm{SO}(2n_c)/\mathbb{Z}_2) c_1(\mathrm{U}(n_f)/\mathbb{Z}_2)\right) \tag{4.11}$$

$$= \frac{1}{2}\Big(\beta_2 v_2(\mathrm{SO}(2n_c)/\mathbb{Z}_2)\Big) c_1(\mathrm{U}(n_f)/\mathbb{Z}_2) \tag{4.12}$$

which is the pull-back of

$$\frac{1}{2}\Big(\beta_2 v_2(\mathrm{SO}(2n_c))\Big) a_2(\mathrm{USp}(2n_f)). \tag{4.13}$$

We now recall that the symmetry we are considering now is $[\mathrm{SO}(2n_c) \times \mathrm{USp}(2n_f)]/\mathbb{Z}_2$, and therefore there is a single degree-2 obstruction cochain which equals both v_2 and a_2, and therefore the anomaly cochain is

$$\frac{1}{2} v_2 \beta_2 v_2. \tag{4.14}$$

This is what we wanted to show. Recall that this anomaly cochain is exact as we have repeatedly mentioned. This is consistent with our computation of the bordism group in Appendix A.4, where we show that $\Omega_5^{\mathrm{spin}}([\mathrm{SO}(2n_c) \times \mathrm{USp}(2n_f)]/\mathbb{Z}_2 = 0$ when n_c is odd.

Acknowledgments

A Bordism group computations

A.1 $\mathbb{Z}_4$ 1-form symmetry

According to [Cle02, Appendix C.3] and [WW18, Eq. (6.3)], we have

$$E^2_{p,q} = H_p\big(K(\mathbb{Z}_4, 2); \Omega^{\mathrm{spin}}_q\big) \qquad\qquad \widetilde{\Omega}^{\mathrm{spin}}_{p+q}(K(\mathbb{Z}_4, 2))$$

$$\longrightarrow \tag{A.1}$$

Therefore there is no anomaly in 4d, and the bordism invariants in 4d are given simply by

$$\exp\!\left(2\pi i \frac{p}{4} \int \frac{1}{2}\mathfrak{P}(a)\right) \tag{A.2}$$

where $\mathfrak{P} : H^2(-; \mathbb{Z}_4) \to H^4(-; \mathbb{Z}_8)$ is the Pontryagin square, which is even mod 8 on a spin manifold.

A.2 $\mathbb{Z}_2 \times \mathbb{Z}_2$ 1-form symmetry

Exploiting the fact that $K(\mathbb{Z}_2 \times \mathbb{Z}_2, 2) = K(\mathbb{Z}_2, 2) \times K(\mathbb{Z}_2, 2)$, we have

$$E^2_{p,q} = H_p\big(K(\mathbb{Z}_2 \times \mathbb{Z}_2, 2); \Omega^{\mathrm{spin}}_q\big) \qquad\qquad \widetilde{\Omega}^{\mathrm{spin}}_{p+q}(K(\mathbb{Z}_2 \times \mathbb{Z}_2, 2))$$

$$\longrightarrow \tag{A.3}$$

The $\mathbb{Z}$ homology of $K(\mathbb{Z}_2, 2)$ is again read off from [Cle02], while the $\mathbb{Z}_2$ (co)homology is known [Ser53] to be

$$H^*(K(\mathbb{Z}_2, 2); \mathbb{Z}_2) = \mathbb{Z}_2[x_2, Sq^1 x_2, Sq^2 Sq^1 x_2, \cdots].$$

The corresponding bordism invariants in 4d are $\mathfrak{P}(a)/2$, ab, $\mathfrak{P}(b)/2$, and the one in 5d is $a\beta b$.

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A.3 $[SO(4) \times SU(2)]/\mathbb{Z}_2$

A.3.1 Leray-Serre SS (preparatory)

For the various input of cohomology groups, see Appendix A of our WZW paper [LOT20]. For the fibration

$$BSO(3) \; \to \; B\left(SO(3) \times SO(3)\right) = B\left(\tfrac{SO(4)}{\mathbb{Z}_2}\right) \; \to \; BSO(3) \tag{A.4}$$

one has

$$E_2^{p,q} = H^p\big(BSO(3); H^q(BSO(3); \mathbb{Z})\big) \qquad H^{p+q}(B\left(\tfrac{SO(4)}{\mathbb{Z}_2}\right); \mathbb{Z})$$

Here we expect non-trivial differentials to be absent (for the region of interest) from the explicit consideration of generators (since there are W_3 and W_3', there should be $(W_3)^2$, $(W_3')^2$, and $W_3 W_3'$) or by requiring proper reproduction of the $\mathbb{Z}_2$ cohomology (which we expect to be generated by w_2, w_2', w_3, and w_3'). Then, for the fibration

$$BSU(2) \; \to \; B\left(\tfrac{SO(4) \times SU(2)}{\mathbb{Z}_2}\right) \; \to \; B\left(\tfrac{SO(4)}{\mathbb{Z}_2}\right) \tag{A.6}$$

we can further plug it into

$$E_2^{p,q} = H^p\big(B\left(\tfrac{SO(4)}{\mathbb{Z}_2}\right); H^q(BSU(2); \mathbb{Z})\big) \qquad H^{p+q}(B\left(\tfrac{SO(4) \times SU(2)}{\mathbb{Z}_2}\right); \mathbb{Z})$$

Taking the normalization of instanton number into account (see Ohmori-san's e-mail on 2020-08-19), the differential $\boxed{d_2 : E_{0,4} \to E_{5,0}}$ seems to be non-trivial.

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So we believe the integral cohomology structure to be

d	0	1	2	3	4	5	6	$\cdots$
$H^d\left(B\left(\frac{\mathrm{SO}(4)\times\mathrm{SU}(2)}{\mathbb{Z}_2}\right);\mathbb{Z}\right)$	$\mathbb{Z}$	0	0	$\mathbb{Z}_2^{\oplus 2}$	$\mathbb{Z}^{\oplus 3}$	0	$\mathbb{Z}_2^{\oplus 3}$	$\cdots$
generator	1	$-$	$-$	W_3	p_1	$-$	$(W_3)^2$	$\cdots$
				W_3'	p_1'		$(W_3')^2$	
					$2c_2$		$W_3 W_3'$	

$$(A.8)$$

where the reduction to $\mathbb{Z}_2$ cohomology are

$$
\begin{aligned}
W_3 &\to w_3 \\
p_1 &\to (w_2)^2
\end{aligned}
\tag{A.9}
$$

A.3.2 Atiyah-Hirzebruch SS

Having obtained (co)homology groups, one can fill in the E^2-page of the AHSS:

$$E_{p,q}^2 = H_p\left(B\left(\frac{\mathrm{SO}(4)\times\mathrm{SU}(2)}{\mathbb{Z}_2}\right);\Omega_q^{\mathrm{spin}}\right)$$

$$(A.10)$$

Based on our belief, $\boxed{d^2 : E_{4,0}^2 \to E_{2,1}^2}$ and $d^2 : E_{4,1}^2 \to E_{2,2}^2$ should be a dual of

$$
\begin{aligned}
Sq^2 w_2 &= (w_2)^2 \\
Sq^2 w_2' &= (w_2')^2
\end{aligned}
\tag{A.11}
$$

and also $\boxed{d^2 : E_{5,0}^2 \to E_{3,1}^2}$ and $d^2 : E_{5,1}^2 \to E_{3,2}^2$ should be a dual of

$$
\begin{aligned}
Sq^2 w_3 &= w_2 w_3 \\
Sq^2 w_3' &= w_2' w_3'
\end{aligned}
\tag{A.12}
$$

and finally $\boxed{d^2 : E_{6,0}^2 \to E_{4,1}^2}$ should be a dual of

$$
Sq^2(w_2 w_2') = w_3 w_3' + (w_2)^2 w_2' + w_2 (w_2')^2
\tag{A.13}
$$

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then the would-be-E_3-page is given by

$$(A.14)$$

A.3.3 Adams SS

According to our naive guess, the module $\widetilde{H}^*(B\left(\frac{\text{SO}(4)\times\text{SU}(2)}{\mathbb{Z}_2}\right);\mathbb{Z}_2)_{\leq 5}$ consists of
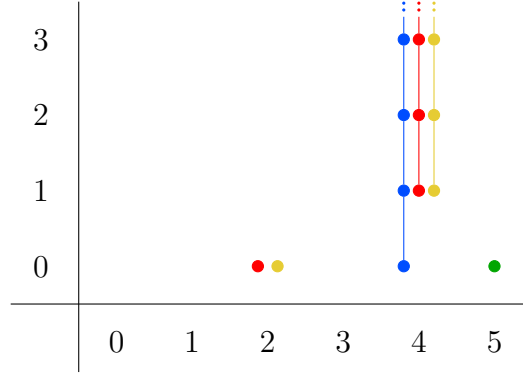

$$(A.15)$$

To be consistent with the AHSS computation, it seems that $w_3w_2' + w_2w_3'$ should be modded out (is it an obvious consequence of the transgression in LSSS...?) and the remaing part (*) turns out to be

$$(A.16)$$

and therefore one concludes

$$\widetilde{H}^*(B\left(\tfrac{\mathrm{SO}(4)\times\mathrm{SU}(2)}{\mathbb{Z}_2}\right);\mathbb{Z}_2)_{\leq 5} = J[2] \oplus J[2] \oplus \mathcal{A}_1/\!\!/\mathcal{E}_0[4] \oplus J[5].\tag{A.17}$$

This leads to the following Adams chart:

and it indeed seems to be compatible with the AHSS computation. If the above argument (and beliefs) is correct, then the anomaly should be captured by

$$w_2 w_3' \; (= w_3 w_2').\tag{A.18}$$

A.4 $[\mathrm{SO}(4n_c' + 2) \times \mathrm{SU}(2n_f)]/\mathbb{Z}_2$

A.4.1   Cohomology of $BSO(4n_c' + 2)/\mathbb{Z}_2$

Recalling that the $\mathbb{Z}_2$ cohomology read off from [KM75] allowed us to determine the $\mathbb{Z}$ cohomology by using Bockstein SS (see `wzw-memo.pdf`, copy later), we had

d	0	1	2	3	4	5	6	$\cdots$
$H^d(BPSO(4n_c'+2);\mathbb{Z})$	$\mathbb{Z}$	0	0	$\mathbb{Z}_4$	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\cdots$
$H^d(BPSO(4n_c'+2);\mathbb{Z}_2)$	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2^{\oplus 2}$	$\cdots$
generator $(\mathbb{Z}_2)$	1	$-$	v_2	$y'(1)$	$(v_2)^2$	$y'(2)$	$(v_2)^3$ $y'(1)^2$	$\cdots$

$$\tag{A.19}$$

and if we assume the suspension $\bar{\theta} : H^*(\mathrm{SO}(4n_c' + 2)/\mathbb{Z}_2; \mathbb{Z}_2) \to H^*(BSO(4n_c' + 2)/\mathbb{Z}_2; \mathbb{Z}_2)$ to commute with Steenrod squares (which seems to be true according to Nishimoto-san's notes), then these elements are supposed to be related as

$$\begin{aligned}
\beta_2 v_2 &= y'(1) \\
Sq^2 y'(1) &= y'(2) \\
Sq^1 y'(2) &= y'(1)^2
\end{aligned}$$

where β_f is a higher Bockstein operator (reduced to $\mathbb{Z}_2$ cohomology as a whole), whose image corresponds to $\mathbb{Z}_{2^f}$ in integral cohomology.

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A.4.2 Leray-Serre SS (preparatory)

For the fibration

$$BSU(2n_f) \ \to \ B\left(\frac{\mathrm{SO}(4n'_c+2)\times\mathrm{SU}(2n_f)}{\mathbb{Z}_2}\right) \ \to \ BSO(4n'_c+2)/\mathbb{Z}_2 \tag{A.20}$$

we have

$$E_2^{p,q} = H^p\big(BSO(4n'_c+2)/\mathbb{Z}_2; H^q(BSU(2n_f);\mathbb{Z})\big) \qquad H^{p+q}\big(B\left(\tfrac{\mathrm{SO}(4n'_c+2)\times\mathrm{SU}(2n_f)}{\mathbb{Z}_2}\right);\mathbb{Z}\big)$$

6	$\mathbb{Z}$		*	*		*			6	$\mathbb{Z}\oplus\mathbb{Z}_2$
5									5	
4	$\mathbb{Z}$		*	*		*			4	$\mathbb{Z}^{\oplus 2}$
3									3	$\mathbb{Z}_4$
2									2	
1									1	
0	$\mathbb{Z}$			$\mathbb{Z}_4$	$\mathbb{Z}$		$\mathbb{Z}_2$		0	$\mathbb{Z}$
	0	1	2	3	4	5	6			

$$\longrightarrow$$

$$\tag{A.21}$$

A.4.3 Atiyah-Hirzebruch SS

Having obtained (co)homology groups, one can fill in the E^2-page of the AHSS:

$$E^2_{p,q} = H_p\big(B\left(\tfrac{\mathrm{SO}(4n'_c+2)\times\mathrm{SU}(2n_f)}{\mathbb{Z}_2}\right);\Omega_q^{\mathrm{spin}}\big)$$

6							
5							
4	$\mathbb{Z}$		*		*	*	*
3							
2	$\mathbb{Z}_2$		$\mathbb{Z}_2$	$\mathbb{Z}_2$	*	*	*
1	$\mathbb{Z}_2$		$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2^{\oplus 2}$	*	*
0	$\mathbb{Z}$		$\mathbb{Z}_4$		$\mathbb{Z}^{\oplus 2}$	$\mathbb{Z}_2$	*
	0	1	2	3	4	5	6

$$\tag{A.22}$$

Based on our belief, $\boxed{d^2 : E^2_{4,0} \to E^2_{2,1}}$ and $\overline{d^2 : E^2_{4,1} \to E^2_{2,2}}$ should be a dual of

$$Sq^2 v_2 \ = \ (v_2)^2 \tag{A.23}$$

and also $\boxed{d^2 : E^2_{5,0} \to E^2_{3,1}}$ and $\overline{d^2 : E^2_{5,1} \to E^2_{3,2}}$ should be a dual of

$$Sq^2 y'(1) \ = \ y'(2) \tag{A.24}$$

and finally $\boxed{d^2 : E_{6,0}^2 \to E_{4,1}^2}$ should be a dual of

$$Sq^2 c_2 \;=\; c_3 \tag{A.25}$$

then the would-be-E_3-page is given by

$$\tag{A.26}$$

B Pontrjagin square for non-closed cochains

We take the definition of the Pontrjagin square for an element $x \in C^\bullet(-; \mathbb{Z}_2)$ to be

$$\mathfrak{P}(x) := \widetilde{x} \cup \widetilde{x} - \widetilde{x} \cup_1 \delta\widetilde{x} \tag{B.1}$$

where $\widetilde{x}$ is an integral cochain which reduces to x. The variation of the Pontrjagin square is then given by

$$
\begin{aligned}
\delta\left(\frac{1}{2^{m+1}}\mathfrak{P}(x)\right) &= \frac{1}{2^{m+1}} \cdot \delta\left(\widetilde{x} \cup \widetilde{x} - \widetilde{x} \cup_1 \delta\widetilde{x}\right) \\
&= \frac{1}{2^{m+1}} \cdot \left[\left(\delta\widetilde{x} \cup \widetilde{x} + \widetilde{x} \cup \delta\widetilde{x}\right) - \left(\widetilde{x} \cup \delta\widetilde{x} - \delta\widetilde{x} \cup \widetilde{x} + \delta\widetilde{x} \cup_1 \delta\widetilde{x}\right)\right] \\
&= \frac{1}{2^{m+1}} \cdot \left[2 \cdot \delta\widetilde{x} \cup \widetilde{x} - \delta\widetilde{x} \cup_1 \delta\widetilde{x}\right].
\end{aligned}
\tag{B.2}
$$

If x is a $\mathbb{Z}_{2^m}$-cocycle, $\widetilde{x} \in C^\bullet(-; \mathbb{Z})$ is a cocycle mod 2^m i.e. $\delta\widetilde{x} = 0 \pmod{2^m}$, and the right hand side is 0 mod 1, which then means that $\mathfrak{P}(x)$ is a $\mathbb{Z}_{2^{m+1}}$-cocycle as desired, defining a cohomology operation $\mathfrak{P} : H^\bullet(-; \mathbb{Z}_{2^m}) \to H^{2\bullet}(-; \mathbb{Z}_{2^{m+1}})$.

However, when x is not a cocycle but merely a cochain, $\mathfrak{P}(x)$ is also not a cocycle. For the purpose of this paper, we only need to consider the case when

$$\delta x = \beta y \tag{B.3}$$

for a cocycle $y \in Z^2(-; \mathbb{Z}_2)$. In this case, x and y combine to define a cocycle $z \in Z^2(-; \mathbb{Z}_4)$, such that $z = y$ modulo 2 and $\widetilde{z} = 2\widetilde{x}$ when $y = 0$. This motivates us to consider the term

$$\frac{1}{4} \cdot \frac{1}{4}\mathfrak{P}(\widetilde{z}) \tag{B.4}$$

as a replacement for

$$\frac{1}{4}\mathfrak{P}(\widetilde{x}). \tag{B.5}$$

Using (B.2), we find

$$\delta\left(\frac{1}{4}\cdot\frac{1}{4}\mathfrak{P}(\widetilde{z})\right) = \frac{1}{2}(\frac{1}{4}\delta\widetilde{z})\cup\widetilde{z} = \frac{1}{2}\beta_2 y\cup y \tag{B.6}$$

where equalities are all modulo 1.

Here, β_2 is the higher Bockstein operation defined for cocycles $y\in H^2(-;\mathbb{Z}_2)$ known to be liftable to $z\in H^2(-;\mathbb{Z}_4)$, by the formula

$$\beta_2 y := \frac{1}{4}\delta\widetilde{z} \tag{B.7}$$

where $\widetilde{z}$ is a lift to an integral cochain of z.

In general, the short exact sequence

$$0\to\mathbb{Z}\xrightarrow{\times 2^f}\mathbb{Z}\xrightarrow{p}\mathbb{Z}_{2^f}\to 0$$

buys us a cohomology operation called the higher Bockstein $\beta_f : H^\bullet(-;\mathbb{Z}_{2^f})\to H^{\bullet+1}(-;\mathbb{Z})$, defined for the element $z\in Z^\bullet(-;\mathbb{Z}_{2^f})$ by

$$\beta_f(z) := \frac{1}{2^f}\delta(\widetilde{z})\in C^\bullet(-;\mathbb{Z}).$$

where $\widetilde{z}$ is a lift of z as an integral cochain. Then, $\beta_{n+1}y$ for $y\in H^2(-;\mathbb{Z}_2)$ is defined only for those y satisfying $\beta_n y=0$, and is the simplest example of a higher cohomology operation.

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