Higher symmetries and anomalies in $\mathfrak{so}$ QCD and $\mathcal{N}=1$ duality

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We study higher symmetries and anomalies of 4d $\mathfrak{so}(2n_c)$ gauge theory with $N_f = 2n_f$ flavors. We find that they depend on the parity of n_c and n_f, on the global form of the gauge group, and the discrete theta angle. The contribution from the fermions plays a central role in our analysis. Furthermore, our conclusion applies to $\mathcal{N}=1$ supersymmetric cases as well, and we see that higher symmetries and anomalies match across the duality $\mathfrak{so}(2n_c) \leftrightarrow \mathfrak{so}(2n_f - n_c + 4)$ of Intriligator and Seiberg.

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1 Introduction and summary

Our understanding of the concept of symmetries in quantum field theories has been greatly improved in the last several years. We now have the concept of p-form symmetries acting on p-dimensional objects [GKSW14]. This concept gives a unifying point of view to both ordinary symmetries acting on point operators for $p=0$ and center symmetries of gauge theories acting on Wilson line operators for $p=1$. In addition, the 't Hooft magnetic flux [tH79] can now be thought of as a background gauge field for the 1-form center symmetry. It is also realized more recently that 0-form symmetries and 1-form symmetries can not only coexist in a direct product but also mix in a more intricate manner. They can have mixed anomalies between them. They can also combine to form a symmetry structure called 2-groups [CDI18, BCH18].

In this paper we study these issues in the case of 4d $\mathfrak{so}(N_c)$ gauge theories with N_f flavors of fermion fields in vector representation. Let us quickly recall the 0-form and 1-form symmetries these theories have.

As for the 1-form symmetry, we first need to recall that such theories come in three versions, Spin, SO_+ and SO_-, distinguished by the global form of the gauge group (Spin vs. SO) and by the choice of a discrete theta angle (SO_+ vs. SO_-) [AST13][1]. They also differ by the nontrivial line operator they possess: the Spin theory has the Wilson line W in the spinor representation, the SO_+ theory has the 't Hooft line H which is mutually non-local with respect to W, and the SO_- theory has the dyonic line $D = WH$. Furthermore, these line operators are charged under corresponding $\mathbb{Z}_2$ 1-form symmetries, which we can all electric, magnetic and dyonic 1-form symmetries.

As for the 0-form symmetry, we focus our attention on the $\mathfrak{su}(N_f)$ symmetry acting on N_f flavors of matter fields in the vector representation. There can be and definitely are other discrete symmetries, but we will not consider them in this paper for brevity.

The main question is then how the $\mathbb{Z}_2$ 1-form symmetry and the $\mathfrak{su}(N_f)$ 0-form symmetry are related.[2] We concentrate on the case when N_c and N_f are both even, $N_c = 2n_c$ and $N_f = 2n_f$.

Take for example the $Spin(2n_c)$ gauge theory with $2n_f$ flavors, when n_c is odd. Take two copies of the Wilson line W in the spinor representation. They form a Wilson line in the vector representation. This can be screened by a dynamical fermion, which was why $W^2 = 1$ as far as the 1-form symmetry charge was concerned. Now let us recall that this dynamical fermion transforms nontrivially under $-1 \in \mathrm{SU}(2n_f)$. Therefore, when we take the flavor symmetry into account, W^2 is still nontrivial. As we will recall below, formally this means that the $\mathbb{Z}_2$ 1-form symmetry extends the $\mathrm{SU}(2n_f)/\mathbb{Z}_2$ 0-form symmetry in a nontrivial manner, forming a nontrivial 2-group H

$$0 \to \mathbb{Z}_2[1] \to H \to \mathrm{SU}(2n_f)/\mathbb{Z}_2 \to 0, \tag{1.1}$$

whose Postnikov class is specified by

$$\beta v_2 \in H^3(\mathrm{SU}(2n_f)/\mathbb{Z}_2, \mathbb{Z}_2), \tag{1.2}$$

where a_2 is the generator of $H^2(\mathrm{SU}(2n_f)/\mathbb{Z}_2, \mathbb{Z}_2) = \mathbb{Z}_2$ and β is the Bockstein.

The SO_+ gauge theory is then obtained by gauging the $\mathbb{Z}_2$ 1-form symmetry [KS14]. The presence of the fermions significantly complicates the analysis. For the moment let us suppose that we have N_f scalars instead of fermions in the vector representation. Then, the argument of [Tac17] immediately applies, and we see that the $\mathbb{Z}_2$ 1-form symmetry of the $\mathrm{SO}(2n_f)$ theory and the $\mathrm{SU}(2n_f)/\mathbb{Z}_2$ 0-form flavor symmetry remains a direct product but with a mixed anomaly given by

$$2\pi i \frac{1}{2} \int B\beta v_2. \tag{1.3}$$

In the rest of the paper, we will carefully analyze how the $\mathbb{Z}_2$ 1-form symmetry and the $\mathrm{SU}(2n_f)/\mathbb{Z}_2$ 0-form symmetry are combined. The derivation will be detailed in the following, and here we simply summarize the result in Table 1. There, 'none' specifies that they remain a direct product without mixed anomaly; 'anomaly' implies that they remain a direct product but with mixed anomaly of the form (1.3); and 'extension' means that they combine into a 2-group given by (1.1) with the Postnikov class (1.2).

[1] This is when the theories are considered on spin manifolds. On more general manifolds a further distinction needs to be made [ARS19]. For simplicity we only consider spin manifolds in this paper.

[2] A partial answer was given in [HL20], but the contribution from fermions was not taken into account in that reference. Our conclusion is therefore somewhat different from theirs.

(n_c, n_f)	Spin	SO_+	SO_-
(even, even)	none	none	none
(odd, even)	extension	anomaly	extension
(even, odd)	anomaly	extension	extension
(odd, odd)	extension	extension	anomaly

Table 1: How the $\mathbb{Z}_2$ 1-form symmetry and the $\mathrm{SU}(2n_f)/\mathbb{Z}_2$ 0-form symmetry are combined in $\mathfrak{so}(2n_c)$ QCD. 'none' implies that they remain a direct product without mixed anomaly; 'anomaly' means that they remain a direct product but with mixed anomaly; and 'extension' is when they combine into a nontrivial 2-group. The orange lines show how the duality of Intriligator and Seiberg acts on this set of theories.

Our result is equally applicable in the case of $\mathcal{N}=1$ supersymmetric QCD, for which Intriligator and Seiberg found in [IS95] a duality exchanging $\mathfrak{so}(N_c)$ and $\mathfrak{so}(N_f - N_c + 4)$, which in our notation sends n_c to $n_c' = n_f - n_c + 2$. In [AST13], this duality was refined to account for the global form of the gauge group and the discrete theta angle, and it was concluded that Spin is exchanged with SO_- while SO_+ maps to itself. This mapping was checked using supersymmetric localization on $S^3/\mathbb{Z}_n \times S^1$ in [RW13]. Our analysis allows us to check this duality by comparing how the 1-form symmetry and the 0-form symmetry are combined in the dual pairs. We superimposed the action of the duality on our main Table 1. It is satisfying to see that the duality action correctly preserves the labels 'none', 'anomaly' and 'extension'.

The rest of the paper is organized as follows ...

2 2-group structure

Let us first study whether the $\mathbb{Z}_2$ 1-form symmetry and the $\mathrm{SU}(2n_f)/\mathbb{Z}_2$ flavor symmetry form a nontrivial 2-group or not. This can be found rather physically by studying the line operators.

2.1 Spin

We start by discussing the $\mathrm{Spin}(2n_c)$ gauge theories. We first recall that the center of $\mathrm{Spin}(2n_c)$ is $\mathbb{Z}_2 \times \mathbb{Z}_2$ when n_c is even and $\mathbb{Z}_4$ when n_c is odd. This corresponds to the fact that the tensor square of a spinor representation contains the identity representation when n_c is even while it contains the vector representation when n_c is odd.

We now consider the Wilson line W in the spinor representation in the $\mathrm{Spin}(2n_c)$ gauge theory with $2n_f$ fermions in the vector representation. When n_c is even, W^2 contains the identity representation, and therefore we simply have a $\mathbb{Z}_2$ 1-form symmetry independent of the $\mathfrak{su}$ flavor symmetry, and there is nothing to see here.

When n_c is odd, W^2 contains the vector representation. This can be screened by the dynamical fermion, which however carries the fundamental representation of $\mathfrak{su}(2n_f)$ flavor symmetry, and in particular transforms nontrivially under $-1 \in \mathrm{SU}(2n_f)$. In other words, the flavor Wilson line in the vector representation of $\mathrm{SU}(2n_f)$ can now be considered as the square of the gauge Wilson line in the spinor representation of $\mathrm{Spin}(2n_c)$. This means that we have the following extension of groups

$$0 \to \underbrace{\mathbb{Z}_2}_{\substack{\text{subgroup of} \\ \text{rep. of center of } \mathrm{SU}(2n_f)}} \to \mathbb{Z}_4 \to \underbrace{\mathbb{Z}_2}_{\substack{\text{group of gauge Wilson lines} \\ \text{up to screening}}} \to 0. \tag{2.1}$$

As the groups of charges of $\mathfrak{su}(2n_f)$ 0-form symmetry and $\mathbb{Z}_2$ 1-form symmetry are combined nontrivially, the 0-form symmetry group and the 1-form symmetry group are also combined nontrivially. This can be seen most clearly by considering background fields for the symmetry groups.

The fermion fields are in the vector of $\mathrm{SO}(2n_c)$ and in the fundamental of $\mathrm{SU}(2n_f)$, and therefore is a representation of $G = [\mathrm{SO}(2n_c) \times \mathrm{SU}(2n_f)]/\mathbb{Z}_2$. Given a G-bundle on a manifold X, there is an $\mathrm{SO}(2n_c)/\mathbb{Z}_2$ bundle and an $\mathrm{SU}(2n_f)/\mathbb{Z}_2$ bundle associated to it. Let us denote by $a_2, v_2 \in H^2(X, \mathbb{Z}_2)$ the obstruction classes controlling whether they lift to $\mathrm{SO}(2n_c)$ and $\mathrm{SU}(2n_f)$ respectively. Then we have $a_2 = v_2$ for a G-bundle. The flavor Wilson line in the fundamental representation is charged under $-1 \in \mathrm{SU}(2n_f)$ in the center, and v_2 can be considered as the background field for this $\mathbb{Z}_2$ 1-form center symmetry.

Now, without the flavor background, the background $E \in H^2(X, \mathbb{Z}_2)$ for the electric $\mathbb{Z}_2$ one-form symmetry of the $\mathrm{Spin}(2n_c)$ theory sets the Stiefel-Whitney class $w_2 \in H^2(X, \mathbb{Z}_2)$ of the $\mathrm{SO}(2n_c)$ gauge bundle to be $E = w_2$, which controls whether it lifts to a $\mathrm{Spin}(2n_c)$ bundle. When the flavor background v_2 is nontrivial, the obstruction class a_2 controlling the lift from $\mathrm{SO}(2n_c)/\mathbb{Z}_2$ to $\mathrm{SO}(2n_c)$ is nontrivial. In this situation when n_c is odd, w_2 can no longer be defined as a closed cochain; rather it satisfies $\delta w_2 = \beta a_2$, where β is the Bockstein operation, since together they specify the obstruction class $x_2 \in H^2(X, \mathbb{Z}_4)$ controlling the lift from $\mathrm{SO}(2n_c)/\mathbb{Z}_2 = \mathrm{Spin}(2n_c)/\mathbb{Z}_4$ to $\mathrm{Spin}(2n_c)$. As $E = w_2$ and $a_2 = v_2$, we conclude that the background fields satisfy

$$\delta E = \beta v_2. \tag{2.2}$$

This means that the $\mathbb{Z}_2$ 1-form symmetry and the $\mathrm{SU}(2n_f)/\mathbb{Z}_2$ 0-form flavor symmetry form the 2-group H fitting in the sequence

$$0 \to \mathbb{Z}_2[1] \to H \to \mathrm{SU}(2n_f)/\mathbb{Z}_2 \to 0 \tag{2.3}$$

whose Postnikov class is $\beta v_2 \in H^3(B\mathrm{SU}(2n_f)/\mathbb{Z}_2, \mathbb{Z}_2)$.

Before proceeding, we note that having the extension of groups of charges of line operators as in (2.1) is equivalent to having a nontrivial 2-group extension (2.3) whose background fields satisfy (2.2). Therefore, to find a nontrivial 2-group extension, we can simply study the group of charges of line operators, which we will carry out for $\mathrm{SO}_\pm$ gauge theories next.

2.2 $\mathrm{SO}_\pm$

We would like to study how the magnetic $\mathbb{Z}_2$ 1-form symmetry of the $\mathrm{SO}(2n_c)_+$ theory is combined with the $\mathfrak{su}(n_f)$ flavor symmetry. According to the discussions in the previous subsection, we consider what happens if we take two copies of the 't Hooft line operator H and fuse them. At the very naive level, H^2 can be screened by a dynamical monopole, but dynamical monopoles can receive flavor and gauge center charges from the fermion zero modes. To study these issues, it is useful to deform the theories to make them simpler.

For this purpose, we perform the following steps:

- We reduce the flavor symmetry from $\mathfrak{su}(2n_f)$ to $\mathfrak{usp}(2n_f)$. The fundamental representation still transforms nontrivially under $-1 \in \mathrm{USp}(2n_f)$, which is enough for our purposes.
- We add an adjoint scalar $\Phi_{[ab]}$ and the interaction $\psi^{ai}_\alpha \psi^{bj}_\beta J_{ij}\Phi_{ab}\epsilon^{\alpha\beta} + cc$, where $J_{[ij]}$ is the constant matrix for the $\mathfrak{usp}(2n_f)$ part.
- We give a vev to Φ_{ab} to break $\mathrm{SO}(2n_c)$ to $\mathrm{SO}(2)^{n_c}$.

The 't Hooft lines in the resulting $\mathrm{SO}(2)^{n_c}$ theory can be labeled by their magnetic charges $(m_1,\ldots,m_{n_c}) \in \mathbb{Z}^{n_c})$. The dynamical monopoles have the charges in the 'adjoint class', which are in the root lattice Λ of $\mathrm{SO}(2n_c)$.

The group of the magnetic charges of 't Hooft lines up to screening by the dynamical monopoles is then

$$\mathbb{Z}^{n_c}/\Lambda = \mathbb{Z}_2, \tag{2.4}$$

which agrees with the 1-form symmetry before the deformation. We now would like to study how this $\mathbb{Z}_2$ is combined with the flavor/gauge center charge $\mathbb{Z}_2$.

For this purpose we need to know slightly more details of the dynamical monopoles. The dynamical monopoles associated to the breaking of G to its Cartan were analyzed in many places, e.g. in [Wei80]. There, the following was shown. Let ϕ be the scalar vev in the real Cartan subalgebra, $\phi \in \mathfrak{h} \subset \mathfrak{g}$. This determines the simple roots α. Then you can embed the standard spherically-symmetric 't Hooft-Polyakov monopole and have a monopole solution without additional bosonic moduli.

Let us say we chose the standard ϕ such that the simple roots are $(1,-1,\ldots,0)$, $(0,1,-1,\ldots,0)$, $\ldots$, $(0,\ldots,1,-1)$ and $(0,\ldots,1,+1)$, which we call simple dynamical monopoles.

We now consider the group $\mathbb{Z}^{n_c}\times\mathbb{Z}_2$ which combine the magnetic charges $\mathbb{Z}^{n_c}$ and the flavor/gauge center charge $\mathbb{Z}_2$. What we are after is the quotient of $\mathbb{Z}^{n_c}\times\mathbb{Z}_2$ by the subgroup generated by the charges of simple dynamical monopoles, which we denote by $(1,-1,\ldots,0;q_1)$, $(0,1,-1,\ldots,0;q_2)$, $\ldots$, $(0,\ldots,1,-1;q_{n_c-1})$ and $(0,\ldots,1,+1;q_{n_c})$, respectively.

Let us determine this quotient. We do not have to determine q_1 to q_{n_c-1}. We simply use them to rewrite any charge vector $(m_1,\ldots,m_{n_c-2},m_{n_c-1},m_{n_c};q)$ into the form $(0,\ldots,0,m,m';q')$. We then have to determine q_{n-1} and q_{n_c}.

This reduces the study to the case of $\mathfrak{so}(4)\simeq\mathfrak{su}(2)_1\times\mathfrak{su}(2)_2$. The monopoles associated to the simple roots are just 't Hooft-Polyakov monopoles associated to the two factors of $\mathfrak{su}(2)$'s. The vev of the adjoint scalar in this basis can be written as (a_1,a_2), which we assume $a_1>a_2>0$.

The fermion is in the vector representation of $so(4)$. Under the monopole in $\mathfrak{su}(2)_1$, it is a doublet coupled to an adjoint vev of size a_1 with bare mass a_2, and similarly for the monopole in $\mathfrak{su}(2)_2$.

Now, the explicit analysis in [Cal78, Sec. IV] concerning the number of zero modes in the 't Hooft-Polyakov monopole says that a doublet coupled to an adjoint vev of size a with bare mass μ has a zero mode if $|a| > |m|$ and it has no zero mode if $|a| < |m|$.

With our assumption $a_1 > a_2 > 0$, this means that the monopole in $\mathfrak{su}(2)_1$ has a zero mode while the one in $\mathfrak{su}(2)_2$ does not have any. In our original basis, this means that the monopole with $(0, \ldots, 1, -1; q_{n_c-1})$ does not have any zero modes and $q_{n_c-1} = 0$, while the one with $(0, \ldots, 1, +1; q_{n_c})$ (which is in $\mathfrak{su}(2)_1$) has two zero modes per flavor.

The one-form symmetry group is obtained by dividing $\mathbb{Z}^2 \times \mathbb{Z}_2$ by the subgroup generated by $(1, -1; 0)$ and $(1, +1; q_{n_c})$. This is $\mathbb{Z}_2 \times \mathbb{Z}_2$ or $\mathbb{Z}_4$ depending on whether q_{n_c} is 0 or 1.

Let us determine q_{n_c}. We saw two zero modes per flavor; this means that there are fermionic zero modes transforming in

$$V_2 \otimes R_{2n_f} \tag{2.5}$$

where V_2 is the doublet of $\mathfrak{su}(2)_2$ (which is actually broken to $\mathfrak{u}(1)$ but keeping $\mathfrak{su}(2)_2$ representation is useful in organizing the answer), R_{2n_f} is the fundamental of $\mathfrak{usp}(2n_f)$, and we need to impose the reality condition using the pseudoreality of both factors, so that there are $4n_f$ Majorana fermion in total.

To determine the flavor/gauge center charge of the monopole, it suffices to consider the case $n_f = 1$; the general case is given by simply multiplying it by n_f. When $n_f = 1$, there are 4 Majorana fermions. Quantizing them, we find the monopoles in

$$V_2 \otimes \mathbf{1} \oplus \mathbf{1} \otimes R_2. \tag{2.6}$$

It has the 'vector' charge under $\mathfrak{usp}(2)$ flavor symmetry or is a doublet under $\mathfrak{su}(2)_2$, which corresponds to the 'vector' charge under $\mathfrak{so}(4)$ gauge symmetry. In either case, they have the flavor/gauge center charge $1 \in \{0, 1\} = \mathbb{Z}_2$. Therefore we conclude the flavor/gauge center charge q_{n_c} is simply given by n_f mod 2.

Combining the intermediate steps above, we conclude the following: for the $\mathrm{SO}(2n_c)_+$ theory, the group $\mathbb{Z}_2$ of magnetic charges of 't Hooft lines is extended by the flavor/gauge center symmetry $\mathbb{Z}_2$ to become $\mathbb{Z}_4$ when n_f is odd, while they remain separate when n_f is even.

The analysis of the $\mathrm{SO}(2n_c)_-$ theory is largely the same; the only difference is that the discrete theta angle gives an additional gauge center charge to the simple dynamical monopole with the magnetic charge $(0, 0, \ldots, 1, 1)$, so that $q_{n_c} = n_f + n_c$ mod 2. Therefore, we conclude the following: for the $\mathrm{SO}(2n_c)_-$ theory, the group $\mathbb{Z}_2$ of magnetic charges of 't Hooft lines is extended by the flavor/gauge center symmetry $\mathbb{Z}_2$ to become $\mathbb{Z}_4$ when $n_f + n_c$ is odd, while they remain separate when $n_f + n_c$ is even.

The result of the analysis is summarized in Table 2. There, 'product' means that the $\mathbb{Z}_2$ 1-form symmetry and $\mathrm{SU}(2n_f)/\mathbb{Z}_2$ flavor symmetry are kept separate and form a direct product, while 'extended' means that they form a nontrivial 2-group. We remark that the nontrivial 2-group is always given by the extension (2.3) whose background fields satisfy (2.2).

(n_c, n_f)	Spin	SO_+	SO_-
(even, even)	product	product	product
(odd, even)	extended	product	extended
(even, odd)	product	extended	extended
(odd, odd)	extended	extended	product

Table 2: How the $\mathbb{Z}_2$ 1-form symmetry and the flavor symmetry $\mathrm{SU}(N_f)/\mathbb{Z}_2$ are combined in $\mathfrak{so}(2n_c)$ QCD. 'product' means that they form a direct product, and 'extended' means that they form a nontrivial 2-group.

3 $\mathrm{SL}(2, \mathbb{Z}_2)$ action and the anomalies

In the last section we determined the 2-group structure of the $\mathfrak{so}(2n_c)$ gauge theories with $2n_f$ flavors, by studying the group of the charges of line operators. In a quantum field theory, any symmetry comes with its (possibly trivial) 't Hooft anomaly. Here we determine the anomalies of these theories using the $\mathrm{SL}(2, \mathbb{Z}_2)$ action on the set of quantum field theories (QFTs) with $\mathbb{Z}_2$ 1-form symmetry.

3.1 $\mathrm{SL}(2, \mathbb{Z}_2)$ action and $\mathfrak{so}$ gauge theories

Let us say that a four-dimensional spin QFT Q with $\mathbb{Z}_2$ 1-form symmetry is given. We denote its partition function on a manifold X by $Z_Q[E]$, where we suppress the dependence on X in the notation and $E \in H^2(X, \mathbb{Z}_2)$ is the background field for the 1-form symmetry. We then define SQ and TQ to be given by the formula

$$Z_{SQ}[B] \propto \sum_E (-1)^{\int_X B \cup E} Z_Q B, \qquad Z_{TQ}[E] = (-1)^{\int_X \frac{1}{2}\mathfrak{P}(E)} Z_Q[E]. \tag{3.1}$$

We can show that $S^2 = T^2 = 1$ and $(ST)^3 = 1$, meaning that they generate $\mathrm{SL}(2, \mathbb{Z}_2)$. This operation was considered in [GKSW14] as an analogue of the $\mathrm{SL}(2, \mathbb{Z})$ action on 3d theories with $\mathrm{U}(1)$ symmetry of [Wit03] and then further studied in [BLT20].

Importantly, $\mathrm{Spin}(2n_c)$ and $\mathrm{SO}(2n_c)_\pm$ gauge theories with $2n_f$ flavors with the same n_c and n_f form a single orbit under this $\mathrm{SL}(2, \mathbb{Z})$ action. More precisely, we need to make the distinction between $\mathrm{Spin}(2n_c)$ and $T(\mathrm{Spin}(2n_c))$ and similarly between $\mathrm{SO}(2n_c)_\pm$ and $T\mathrm{SO}(2n_c)_\pm$, respectively. Here the theories with T prepended are different from the original ones only by its coupling to the background. Then we have the following chain of actions:

$$T\mathrm{Spin} \xleftrightarrow{T} \mathrm{Spin} \xleftrightarrow{S} \mathrm{SO}_+ \xleftrightarrow{T} T\mathrm{SO}_+ \xleftrightarrow{S} T\mathrm{SO}_- \xleftrightarrow{T} \mathrm{SO}_- \xleftrightarrow{S} . \tag{3.2}$$

3.2 $\mathrm{SL}(2,\mathbb{Z}_2)$ actions with extra background

Let us now study what happens if we perform this $\mathrm{SL}(2,\mathbb{Z}_2)$ action when the $\mathbb{Z}_2$ 1-form symmetry in question is part of a larger symmetry group. So far we have been considering the effect of $\mathrm{SU}(2n_f)/\mathbb{Z}_2$ flavor symmetry, but the discussions in the last section shows that at a formal level only the background field $v_2 \in H^2(X,\mathbb{Z}_2)$ matters, which controls the lift from $\mathrm{SU}(2n_f)/\mathbb{Z}_2$ to $\mathrm{SU}(2n_f)$. Let us regard v_2 as the background field for a flavor $\mathbb{Z}_2$ 1-form symmetry.

The combined 1-form symmetry is either $\mathbb{Z}_2\times\mathbb{Z}_2$ or $\mathbb{Z}_4$, and we perform the $\mathrm{SL}(2,\mathbb{Z}_2)$ action by picking a $\mathbb{Z}_2$ subgroup. The symmetry and the anomaly of the resulting theory can be determined by a formal argument once the symmetry and the anomaly of the original theory and the action of the anomaly-free subgroup to be gauged is given [Tac17].

Let us work at the level of anomalies described at the level of cohomology, since we do not need to deal with anomalies more general than that. We consider a d-dimensional QFT with a symmetry group G with an anomaly specified by a cochain $\alpha \in C^{d+1}(G,\mathrm{U}(1))$. We pick a subgroup $H\subset G$ such that α trivializes in it, so that one can find $\mu\in C^d(H,\mathrm{U}(1))$ such that $\delta\mu = \alpha|_H$. We then gauge H, using μ as the action. What determines the symmetry and the anomaly of the gauged theory is the data (μ,α) Clearly, given $\beta\in C^d(G,\mathrm{U}(1))$, the pair (μ,α) and the pair $(\mu-\beta,\alpha-\delta\beta)$ should give the same result, since we merely added the countertem β as the action. This allows us to always choose the pair of the form $(0,\alpha')$ equivalent to a given (μ,α), by using β to be an arbitrary extension of μ from H to G. This is convenient in discussing the $\mathrm{SL}(2,\mathbb{Z}_2)$ action, since our S operation is defined in the convention that $\mu=0$.

Now, what are the possible choices of (μ,α) or $(0,\alpha')$ we need to discuss? Let us first consider $\mathbb{Z}_2\times\mathbb{Z}_2$ 1-form symmetry. As detailed in the Appendix A, the only possible anomaly for 4d spin QFTs with this symmetry is

$$\alpha = 2\pi i \tfrac{1}{2} B\beta E \tag{3.3}$$

where Y is the 5d spin manifold and $B,E\in H^2(Y,\mathbb{Z}_2)$ are the background fields. Its restriction to $\mathbb{Z}_2$ 1-form symmetry subgroup is trivially zero, and then the possible choice of μ is simply the discrete theta angle

$$\mu = 2\pi i\tfrac{1}{2}\tfrac{1}{2}\mathfrak{P}(E). \tag{3.4}$$

This μ can be extended from the $\mathbb{Z}_2$ subgroup to the entire $\mathbb{Z}_2\times\mathbb{Z}_2$ subgroup as a closed cochain. Therefore, we only have to consider pairs $(0,0)$ and $(0,\alpha)$.

Next, we consider $\mathbb{Z}_4$ 1-form symmetry. Again in the Appendix A, we show that there is no anomaly for $\mathbb{Z}_4$ 1-form symmetry. Therefore we can pick $\alpha=0$. Then the possible choice of μ for the $\mathbb{Z}_2$ 1-form subgroup is again the discrete theta angle (3.4). One difference here is that the discrete theta angle (3.4) cannot be extended as a closed cochain to the entire $\mathbb{Z}_4$ 1-form subgroup. As discussed in the Appendix B, with $\delta E = \beta a_2$, where $a_2\in H^2(X,\mathbb{Z}_4/\mathbb{Z}_2)$, one finds

$$\alpha' := \delta\mu = 2\pi i\tfrac{1}{2} a_2\beta_2 a_2. \tag{3.5}$$

Therefore, the pairs we need to consider are $(0,0)$ and $(\mu,0)\sim(0,\alpha')$.

Summarizing, we need to consider the following four choices, namely:

- For $\mathbb{Z}_2\times\mathbb{Z}_2$, the pairs $(0,0)$ and $(0,\alpha)$, which we call 'none' and 'anomaly'

- For $\mathbb{Z}_4$, the pairs $(0,0)$ and $(\mu,0)\sim(0,\alpha')$, which we call 'extended' and 'extended$_T$'.

Let us now determine how the $\mathrm{SL}(2,\mathbb{Z}_2)$ action affects these data. The case 'none' is very easy. The additional $\mathbb{Z}_2$ factor plays no role, and we find the chain of actions given by

$$\text{none} \overset{T}{\longleftrightarrow} \text{none} \overset{S}{\longleftrightarrow} \text{none} \overset{T}{\longleftrightarrow} \text{none} \overset{S}{\longleftrightarrow} \text{none} \overset{T}{\longleftrightarrow} \text{none} \overset{S}{\longleftrightarrow} . \tag{3.6}$$

Let us now consider $\mathbb{Z}_4$ 1-form symmetry and gauge its $\mathbb{Z}_2$ subgroup. The case 'extended' was analyzed in [Tac17], the resulting theory has $\mathbb{Z}_2\times\mathbb{Z}_2$ 1-form symmetry with the anomaly (3.3), which we decided to call 'anomaly'.

The case 'extended$_T$' was analyzed in [HL20], where it was shown that the gauged theory has again 'extended$_T$'. Let us quickly recall why this is the case. The gauging process involves the term

$$\exp\left(2\pi i\int_X (\frac{1}{4}\mathfrak{P}(E)+\frac{1}{2}BE)\right) \tag{3.7}$$

where E is the variable to be gauged and B is the newly introduced background field. When $\mathbb{Z}_2$ to be gauged is the $\mathbb{Z}_2$ subgroup of a $\mathbb{Z}_4$ symmetry, E is not necessarily closed, but satisfies rather the relation

$$\delta E=\beta v_2 \tag{3.8}$$

where v_2 is the background field for the quotient $\mathbb{Z}_4/\mathbb{Z}_2$ 1-form symmetry. Then the second term in (3.7) is not closed, and to even talk about the first term in (3.7), one first needs to extend the definition of the Pontryagin square $\mathfrak{P}$ to non-closed cochains.

To make the coupling (3.7) well-defined, we consider adding a background term to (3.7) so that we have

$$\exp\left(2\pi i\int_X (\frac{1}{4}\mathfrak{P}(E)+\frac{1}{2}BE+\frac{1}{4}\mathfrak{P}(B))\right)=\exp\left(2\pi i\frac{1}{4}\int_X \mathfrak{P}(E+B)\right). \tag{3.9}$$

This is perfectly well-defined if the newly-introduced background field B also satisfies

$$\delta B=\beta v_2. \tag{3.10}$$

This means that, starting from 'extended$_T$', performing S and then T, we find a theory with the data 'extended'. Therefore, simply performing S for the theory of the type 'extended$_T$', one finds 'extended$_T$'.

Combined, the preceding arguments allows us to see the chain of actions

$$\text{extended}_T \overset{T}{\longleftrightarrow} \text{extended} \overset{S}{\longleftrightarrow} \text{anomaly} \overset{T}{\longleftrightarrow} \text{anomaly} \overset{S}{\longleftrightarrow} \text{extended} \overset{T}{\longleftrightarrow} \text{extended}_T \overset{S}{\longleftrightarrow} . \tag{3.11}$$

3.3 Anomalies from $\mathrm{SL}(2,\mathbb{Z}_2)$ action

Comparing the chains of actions (3.6) and (3.11) we determined above and Table 2, we see that the anomaly is automatically determined and we obtain the result already presented in Table 1. More precisely, we should better distinguish $\mathrm{Spin}(2n_c)$ and $T\mathrm{Spin}$, etc., and we find the assignment given in Table 3. As the way we determined the symmetry structures were somewhat indirect, we confirm the symmetry structures of the Spin case in the next section in a different means.

(n_c, n_f)	TSpin	Spin	SO_+	$T\mathrm{SO}_+$	$T\mathrm{SO}_-$	SO_-
(even, even)	none	none	none	none	none	none
(odd, even)	extended$_T$	extended	anomaly	anomaly	extended	extended$_T$
(even, odd)	anomaly	anomaly	extended	extended$_T$	extended$_T$	extended
(odd, odd)	extended	extended$_T$	extended$_T$	extended	anomaly	anomaly

Table 3: The symmetry structure of $\mathfrak{so}(2n_c)$ QCD with $2n_f$ flavors, as deduced from the 2-group structures found in Sec. 2 and from the $\mathrm{SL}(2,\mathbb{Z}_2)$ action discussed in this section. The symmetry structure of the Spin case will be checked independently in the next section.

4 Fermion contribution to anomalies

So far, we first determined the 2-group structure in Sec. 2 by studying the charges of line operators, and then determined the anomalies in Sec. 3 by matching it to the action of $\mathrm{SL}(2,\mathbb{Z}_2)$. Going over the entries on the column Spin of Table 3, we find that the anomaly is trivial when (n_c, n_f) is (even,even) or (odd,even), while it is α when (n_c, n_f) is (odd,even) and α' when (odd,odd). Since the 1-form symmetry background in the Spin theory is simply the class w_2 of the $\mathrm{SO}(2n_c)$ gauge bundle, these anomalies should simply come from the anomalies of fermions under $[\mathrm{SO}(2n_c) \times \mathrm{USp}(2n_f)]/\mathbb{Z}_2$. Here we use $\mathrm{USp}(2n_f)$ instead of $\mathrm{SU}(2n_f)$, because under the latter we also have perturbative anomalies, which could potentially complicate the analysis.

More explicitly, for n_c even, the anomaly should be given by

$$\alpha = 2\pi i \frac{1}{2} w_2 \beta v_2 \tag{4.1}$$

when n_c is even, where w_2, v_2 are the classes in $H^2(X, \mathbb{Z}_2)$ controlling the lift from $\mathrm{SO}(2n_c)/\mathbb{Z}_2$ to $\mathrm{SO}(2n_c)$ and from $\mathrm{USp}(2n_f)/\mathbb{Z}_2$ to $\mathrm{USp}(2n_f)$, respectively, and for n_c odd, the anomaly cochain should be given by

$$\alpha' = 2\pi i \frac{1}{2} a_2 \beta_2 a_2 \tag{4.2}$$

where a_2 is the class in $H^2(X, \mathbb{Z}_2)$ which is the mod-2 reduction of the class in $H^2(X, \mathbb{Z}_4)$ controlling the lift from $\mathrm{SO}(2n_c)/\mathbb{Z}_2 = \mathrm{Spin}(2n_c)/\mathbb{Z}_4$ to $\mathbb{Z}_4$. We note that α' is exact, but as explained in the previous section, this still affects the gauging process.

The aim of this last section is to give a check of these anomalies from a different point of view. We will proceed as follows. We add scalar fields, adjoint under $\mathrm{USp}(2n_f)$ in the system, and break it a subgroup. We then determine the effective interaction induced by the fermion zero modes. The next step is to see what happens when the symmetry group is further changed from $\mathrm{SO}(2n_c) \times \mathrm{USp}(2n_f)$ to its $\mathbb{Z}_2$ quotient. We will see that the effective interaction will have the required anomalies.

Before proceeding, we have two remarks. First, this method was first used in [Wit95, Sec. 4] under the name of 'a curious minus sign', and recognized as determining an anomaly in [CD18, Sec. 2.4.3]. It was also used in [WWW18, Sec. 3.1 and Sec. 5.1.2]. Second, in this section we can only say that the effective interaction we find is compatible with the anomalies as found in Sec. 3, and will not be able to determine it completely. This is mostly due to the fact that the computation of the spin bordism group $\Omega_d^{\text{spin}}([\mathrm{SO}(2n_c)\times\mathrm{USp}(2n_f)]/\mathbb{Z}_2)$ which governs the anomaly is quite hard, because even the integral cohomology of the group in question is hard to obtain. In only a couple of cases we can say more, as we comment along the way.

4.1 Effective interaction

We break $\mathrm{USp}(2n_f)$ down to $\mathrm{U}(n_f)$, using a scalar field, such that the fundamental representation of $\mathrm{USp}(2n_f)$ splits to the fundamental plus the antifundamental of $\mathrm{U}(n_f)$. The monopole charge is given by the first Chern class c_1 of the low-energy $\mathrm{U}(n_f)$.

Take a standard 't Hooft-Polyakov monopole associated to $\mathrm{U}(1)\subset\mathrm{USp}(2)$ and embed it into $\mathrm{U}(n_f)\subset\mathrm{USp}(2n_f)$. The fermion zero modes form a vector representation of $\mathrm{SO}(2n_c)$, whose quantization leads to the spinor representation. This means that there is an effective interaction

$$2\pi i\frac{1}{2}\int_X w_2c_1 \tag{4.3}$$

where w_2 is the Stiefel-Whitney class of the $\mathrm{SO}(2n_c)$ bundle and c_1 is the first Chern class of $\mathrm{U}(n_f)$.

4.2 Anomalies

We now take the $\mathbb{Z}_2$ quotient, changing the symmetry group from $\mathrm{SO}(2n_c)\times\mathrm{USp}(2n_f)$ to $[\mathrm{SO}(2n_c)\times\mathrm{USp}(2n_f)]/\mathbb{Z}_2$. Note that $\pi_1(U(n_f)/\mathbb{Z}_2)$ is $\mathbb{Z}\times\mathbb{Z}_2$ or $\mathbb{Z}$ depending on whether n_f is even or odd.

Furthermore, when n_f is even, $c_1(\mathrm{U}(n_f))=c_1(\mathrm{U}(n_f)/\mathbb{Z}_2)$ and $v_2(\mathrm{U}(n_f)/\mathbb{Z}_2)=v_2(\mathrm{USp}(2n_f))$. When n_f is odd, $c_1(\mathrm{U}(n_f)/\mathbb{Z}_2)=2c_1(\mathrm{U}(n_f))$ when the latter is well-defined and $v_2(\mathrm{U}(n_f)/\mathbb{Z}_2)$ is the mod-2 reduction of $c_1(\mathrm{U}(n_f)/\mathbb{Z}_2)$. We now compute the anomaly cochains in the four cases separately:

(n_c,n_f)**=(even,even):** $w_2(\mathrm{SO}(2n_c))$ and $c_1(\mathrm{U}(n_f))$ can be generalized without problem, and therefore

$$\delta(\frac{1}{2}w_2(\mathrm{SO}(2n_c))c_1(\mathrm{U}(n_f)))=0. \tag{4.4}$$

(n_c,n_f)**=(odd,even):** $w_2(\mathrm{SO}(2n_c))$ needs to be upgraded to a $\mathbb{Z}_4$-valued cochain $a_2(\mathrm{SO}(2n_c)/\mathbb{Z}_2)$. The original interaction is then

$$\frac{1}{4}a_2(\mathrm{SO}(2n_c)/\mathbb{Z}_2)c_1(\mathrm{U}(n_f)) \tag{4.5}$$

which is closed without problem, and therefore its δ is zero.

(n_c, n_f)**=(even,odd):** Here we need to replace $c_1(\mathrm{U}(n_f))$ by $c_1(\mathrm{U}(n_f)/\mathbb{Z}_2)/2$. The effective interaction is then

$$\frac{1}{4}w_2(\mathrm{SO}(2n_c)/\mathbb{Z}_2)c_1(\mathrm{U}(n_f)/\mathbb{Z}_2) \tag{4.6}$$

whose δ is

$$\begin{aligned}\delta(\frac{1}{4}w_2(\mathrm{SO}(2n_c)/\mathbb{Z}_2)c_1(\mathrm{U}(n_f)/\mathbb{Z}_2)) &= \frac{1}{2}(\frac{1}{2}\delta w_2(\mathrm{SO}(2n_c))c_1(\mathrm{U}(n_f)/\mathbb{Z}_2)) &(4.7)\\ &= \frac{1}{2}(\beta w_2(\mathrm{SO}(2n_c)))c_1(\mathrm{U}(n_f)/\mathbb{Z}_2) &(4.8)\end{aligned}$$

which is the pull-back of the anomaly cochain

$$\frac{1}{2}(\beta w_2(\mathrm{SO}(2n_c)))v_2(\mathrm{USp}(2n_f)/\mathbb{Z}_2). \tag{4.9}$$

When $n_c = 2$ and $n_f = 1$, we can confirm this is indeed the entire anomaly, since we can compute $\mathrm{Hom}(\Omega_5^{\text{spin}}([\mathrm{SO}(4)\times\mathrm{USp}(2)]/\mathbb{Z}_2), \mathrm{U}(1))$ and show that this is the only nontrivial element there. For details, see Appendix A.3.

(n_c, n_f)**=(odd,odd):** Now we make the replacement on both sides and the effective interaction is

$$\frac{1}{8}a_2(\mathrm{SO}(2n_c)/\mathbb{Z}_2)c_1(\mathrm{U}(n_f)/\mathbb{Z}_2) \tag{4.10}$$

whose δ is

$$\begin{aligned}\delta(\frac{1}{8}a_2(\mathrm{SO}(2n_c)/\mathbb{Z}_2)c_1(\mathrm{U}(n_f)/\mathbb{Z}_2)) &= \frac{1}{2}(\frac{1}{4}\delta a_2(\mathrm{SO}(2n_c)/\mathbb{Z}_2)c_1(\mathrm{U}(n_f)/\mathbb{Z}_2)) &(4.11)\\ &= \frac{1}{2}(\beta_2 a_2(\mathrm{SO}(2n_c)/\mathbb{Z}_2))c_1(\mathrm{U}(n_f)/\mathbb{Z}_2) &(4.12)\end{aligned}$$

which is the pull-back of

$$\frac{1}{2}(\beta_2 a_2(\mathrm{SO}(2n_c)))v_2(\mathrm{USp}(2n_f)). \tag{4.13}$$

We now recall that the symmetry we are considering now is $[\mathrm{SO}(2n_c)\times\mathrm{USp}(2n_f)]/\mathbb{Z}_2$, and therefore there is a single degree-2 obstruction cochain which equals both a_2 and v_2, and therefore the anomaly cochain is

$$\frac{1}{2}(\beta_2 a_2(\mathrm{SO}(2n_c)))a_2(\mathrm{USp}(2n_f)). \tag{4.14}$$

This is what we wanted to show. Recall that this anomaly cochain is exact as we repeatedly mentioned. This is consistent with our computation of the bordism group in Appendix A.4, where we show that $\Omega_5^{\text{spin}}([\mathrm{SO}(2n_c)\times\mathrm{USp}(2n_f)]/\mathbb{Z}_2 = 0$ when n_c is odd.

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A Bordism group computations

A.1 $\mathbb{Z}_4$ 1-form symmetry

According to [Cle02, Appendix C.3] and [WW18, Eq. (6.3)], it seems that we have

$$
\begin{array}{c|c:cccccc}
\multicolumn{8}{c}{E^2_{p,q} = H_p\big(K(\mathbb{Z}_4,2);\Omega^{\text{spin}}_q\big)} \\
6 & & & & & & & \\
5 & & & & & & & \\
4 & \mathbb{Z} & & * & & * & * & * \\
3 & & & & & & & \\
2 & \mathbb{Z}_2 & & \mathbb{Z}_2 & \mathbb{Z}_2 & * & * & * \\
1 & \mathbb{Z}_2 & & \mathbb{Z}_2 & \mathbb{Z}_2 & \mathbb{Z}_2 & \mathbb{Z}_2 & * \\
0 & \mathbb{Z} & & \mathbb{Z}_4 & & \mathbb{Z}_8 & \mathbb{Z}_2 & * \\
\hline
 & 0 & 1 & 2 & 3 & 4 & 5 & 6
\end{array}
\quad \longrightarrow \quad
\begin{array}{c|c}
\multicolumn{2}{c}{\widetilde{\Omega}^{\text{spin}}_{p+q}(K(\mathbb{Z}_4,2))} \\
6 & * \\
5 & \\
4 & \mathbb{Z}_4 \\
3 & \\
2 & \mathbb{Z}_4 \\
1 & \\
0 & \\
\hline
\end{array}
\tag{A.1}
$$

The corresponding invariant in 4d is simply

$$\exp(2\pi i \frac{p}{4} \int \frac{1}{2}\mathfrak{P}(a)) \tag{A.2}$$

where $\mathfrak{P} : H^2(-,\mathbb{Z}_4) \to H^4(-,\mathbb{Z}_8)$ is the Pontryagin square, which is even mod 8 on a spin manifold.

A.2 $\mathbb{Z}_2 \times \mathbb{Z}_2$ 1-form symmetry

Exploiting the fact that $K(\mathbb{Z}_2 \times \mathbb{Z}_2, 2) = K(\mathbb{Z}_2, 2) \times K(\mathbb{Z}_2, 2)$, it seems that we have

$$E^2_{p,q} = H_p\big(K(\mathbb{Z}_2 \times \mathbb{Z}_2, 2); \Omega^{\text{spin}}_q\big) \qquad\qquad \widetilde{\Omega}^{\text{spin}}_{p+q}(K(\mathbb{Z}_2 \times \mathbb{Z}_2, 2))$$

$q \backslash p$	0	1	2	3	4	5	6
6							
5							
4	$\mathbb{Z}$		$*$		$*$	$*$	$*$
3							
2	$\mathbb{Z}_2$		$\mathbb{Z}_2^{\oplus 2}$	$\mathbb{Z}_2^{\oplus 2}$	$*$	$*$	$*$
1	$\mathbb{Z}_2$		$\mathbb{Z}_2^{\oplus 2}$	$\mathbb{Z}_2^{\oplus 2}$	$\mathbb{Z}_2^{\oplus 3}$	$\mathbb{Z}_2^{\oplus 6}$	$*$
0	$\mathbb{Z}$		$\mathbb{Z}_2^{\oplus 2}$		$\mathbb{Z}_4^{\oplus 2} \oplus \mathbb{Z}_2$	$\mathbb{Z}_2^{\oplus 3}$	$*$

$\longrightarrow$

n	
6	$*$
5	$\mathbb{Z}_2$
4	$\mathbb{Z}_2^{\oplus 3}$
3	
2	$\mathbb{Z}_2^{\oplus 2}$
1	
0	

(A.3)

The $\mathbb{Z}$ homology of $K(\mathbb{Z}_2, 2)$ is again read off from [Cle02], while the $\mathbb{Z}_2$ (co)homology is known [Ser53] to be

$$H^*(K(\mathbb{Z}_2, 2); \mathbb{Z}_2) = \mathbb{Z}_2[x_2, Sq^1 x_2, Sq^2 Sq^1 x_2, \cdots].$$

The corresponding bordism invariants in 4d are $\mathfrak{P}(a)/2$, ab, $\mathfrak{P}(b)/2$, and the one in 5d is $a\beta b$.

A.3 $[\mathrm{SO}(4) \times \mathrm{SU}(2)]/\mathbb{Z}_2$

A.3.1 Leray-Serre SS (preparatory)

For the various input of cohomology groups, see Appendix A of our WZW paper. For the fibration

$$BSO(3) \;\to\; B\left(SO(3) \times SO(3)\right) = B\left(\frac{SO(4)}{\mathbb{Z}_2}\right) \;\to\; BSO(3) \tag{A.4}$$

one has

$$E_2^{p,q} = H^p\big(BSO(3); H^q(BSO(3); \mathbb{Z})\big) \qquad\qquad H^{p+q}\left(B\left(\frac{SO(4)}{\mathbb{Z}_2}\right); \mathbb{Z}\right)$$

$q \backslash p$	0	1	2	3	4	5	6
6	$\mathbb{Z}_2$		$*$	$*$	$*$	$*$	$*$
5							
4	$\mathbb{Z}$			$*$	$*$		$*$
3	$\mathbb{Z}_2$		$\mathbb{Z}_2$	$\mathbb{Z}_2$	$*$	$*$	$*$
2							
1							
0	$\mathbb{Z}$			$\mathbb{Z}_2$	$\mathbb{Z}$		$\mathbb{Z}_2$

$\longrightarrow$

n	
6	$\mathbb{Z}_2^{\oplus 3}$
5	$\mathbb{Z}_2$
4	$\mathbb{Z}^{\oplus 2}$
3	$\mathbb{Z}_2^{\oplus 2}$
2	
1	
0	$\mathbb{Z}$

(A.5)

Here we expect non-trivial differentials to be absent (for the region of interest) from the explicit consideration of generators (since there are W_3 and W_3', there should be $(W_3)^2$, $(W_3')^2$, and $W_3 W_3'$)

or by requiring proper reproduction of the $\mathbb{Z}_2$ cohomology (which we expect to be generated by w_2, w_2', w_3, and w_3'). Then, for the fibration

$$BSU(2) \;\to\; B\left(\frac{SO(4)\times SU(2)}{\mathbb{Z}_2}\right) \;\to\; B\left(\frac{SO(4)}{\mathbb{Z}_2}\right) \tag{A.6}$$

we can further plug it into

$$\begin{array}{ccc}
E_2^{p,q} = H^p\left(B\left(\frac{SO(4)}{\mathbb{Z}_2}\right); H^q(BSU(2);\mathbb{Z})\right) & & H^{p+q}\left(B\left(\frac{SO(4)\times SU(2)}{\mathbb{Z}_2}\right);\mathbb{Z}\right) \\
\begin{array}{c|ccccccc}
6 & & & & & & & \\
5 & & & & & & & \\
4 & \boxed{\mathbb{Z}} & & & * & * & * & * \\
3 & & & & & & & \\
2 & & & & & & & \\
1 & & & & & & & \\
0 & \mathbb{Z} & & & \mathbb{Z}_2^{\oplus 2} & \mathbb{Z}^{\oplus 2} & \boxed{\mathbb{Z}_2} & \mathbb{Z}_2^{\oplus 3} \\
\hline
 & 0 & 1 & 2 & 3 & 4 & 5 & 6
\end{array}
& \longrightarrow &
\begin{array}{c|c}
6 & \mathbb{Z}_2^{\oplus 3} \\
5 & \\
4 & \mathbb{Z}^{\oplus 3} \\
3 & \mathbb{Z}_2^{\oplus 2} \\
2 & \\
1 & \\
0 & \mathbb{Z} \\
\hline
 &
\end{array}
\end{array} \tag{A.7}$$

Taking the normalization of instanton number into account (see Ohmori-san's e-mail on 2020-08-19), the differential $\boxed{d_2 : E_{0,4} \to E_{5,0}}$ seems to be non-trivial.

So we believe the integral cohomology structure to be

$$\begin{array}{c|cccccccc}
d & 0 & 1 & 2 & 3 & 4 & 5 & 6 & \cdots \\
\hline
H^d\left(B\left(\frac{SO(4)\times SU(2)}{\mathbb{Z}_2}\right);\mathbb{Z}\right) & \mathbb{Z} & 0 & 0 & \mathbb{Z}_2^{\oplus 2} & \mathbb{Z}^{\oplus 3} & 0 & \mathbb{Z}_2^{\oplus 3} & \cdots \\
\hline
\text{generator} & 1 & - & - & W_3 & p_1 & - & (W_3)^2 & \cdots \\
 & & & & W_3' & p_1' & & (W_3')^2 & \\
 & & & & & 2c_2 & & W_3 W_3' &
\end{array} \tag{A.8}$$

where the reduction to $\mathbb{Z}_2$ cohomology are

$$\begin{array}{ccc}
W_3 & \to & w_3 \\
p_1 & \to & (w_2)^2
\end{array} \tag{A.9}$$

A.3.2 Atiyah-Hirzebruch SS

Having obtained (co)homology groups, one can fill in the E^2-page of the AHSS:

$$E^2_{p,q} = H_p\left(B\left(\frac{SO(4)\times SU(2)}{\mathbb{Z}_2}\right); \Omega^{\text{spin}}_q\right)$$

$$\begin{array}{c|c:cccccc}
6 & & & & & & & \\
5 & & & & & & & \\
4 & \mathbb{Z} & & \mathbb{Z}_2^{\oplus 2} & & * & * & * \\
3 & & & & & & & \\
2 & \mathbb{Z}_2 & & \mathbb{Z}_2^{\oplus 2} & \mathbb{Z}_2^{\oplus 2} & \mathbb{Z}_2^{\oplus 3} & * & * \\
1 & \mathbb{Z}_2 & & \mathbb{Z}_2^{\oplus 2} & \mathbb{Z}_2^{\oplus 2} & \mathbb{Z}_2^{\oplus 3} & \mathbb{Z}_2^{\oplus 3} & * \\
0 & \mathbb{Z} & & \mathbb{Z}_2^{\oplus 2} & & \mathbb{Z}^{\oplus 3} & \mathbb{Z}_2^{\oplus 3} & * \\
\hline
 & 0 & 1 & 2 & 3 & 4 & 5 & 6
\end{array} \tag{A.10}$$

Based on our belief, $\boxed{d^2 : E^2_{4,0} \to E^2_{2,1}}$ and $\boxed{d^2 : E^2_{4,1} \to E^2_{2,2}}$ should be a dual of

$$\begin{aligned} Sq^2 w_2 &= (w_2)^2 \\ Sq^2 w'_2 &= (w'_2)^2 \end{aligned} \tag{A.11}$$

and also $\boxed{d^2 : E^2_{5,0} \to E^2_{3,1}}$ and $\boxed{d^2 : E^2_{5,1} \to E^2_{3,2}}$ should be a dual of

$$\begin{aligned} Sq^2 w_3 &= w_2 w_3 \\ Sq^2 w'_3 &= w'_2 w'_3 \end{aligned} \tag{A.12}$$

and finally $\boxed{d^2 : E^2_{6,0} \to E^2_{4,1}}$ should be a dual of

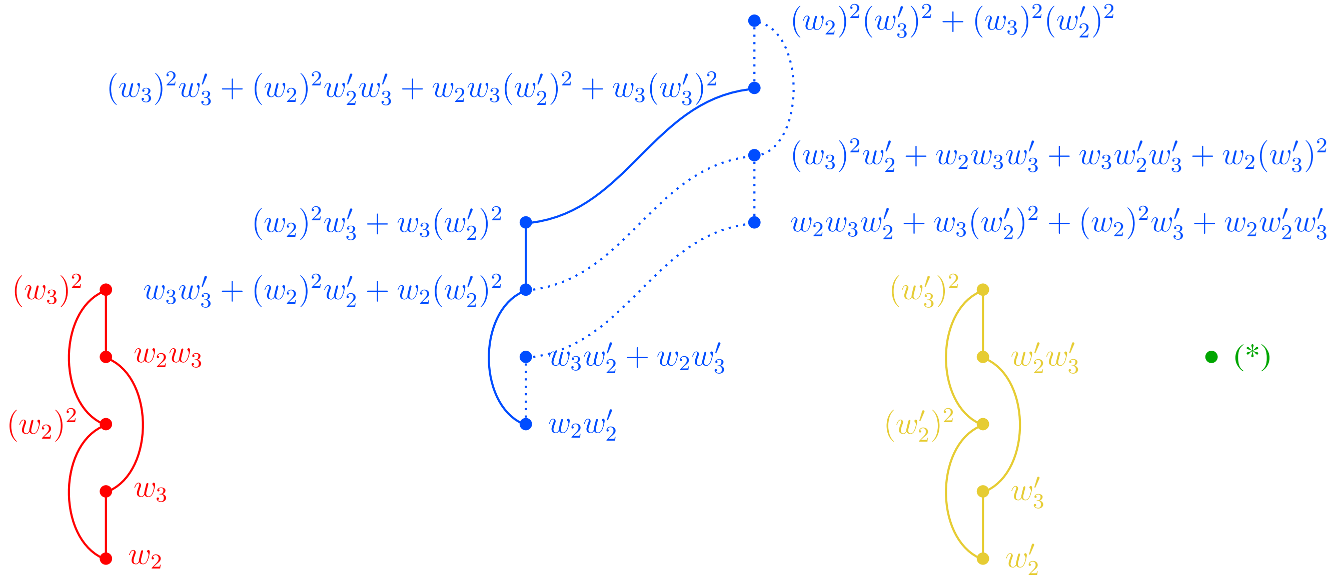
$$Sq^2(w_2 w'_2) = w_3 w'_3 + (w_2)^2 w'_2 + w_2 (w'_2)^2 \tag{A.13}$$

then the would-be-E_3-page is given by

$$\begin{array}{c|c:cccccc}
6 & & & & & & & \\
5 & & & & & & & \\
4 & \mathbb{Z} & & * & & * & * & * \\
3 & & & & & & & \\
2 & \mathbb{Z}_2 & & & & * & * & * \\
1 & \mathbb{Z}_2 & & & & & * & * \\
0 & \mathbb{Z} & & \mathbb{Z}_2^{\oplus 2} & & \mathbb{Z}^{\oplus 3} & \mathbb{Z}_2 & * \\
\hline
 & 0 & 1 & 2 & 3 & 4 & 5 & 6
\end{array} \tag{A.14}$$

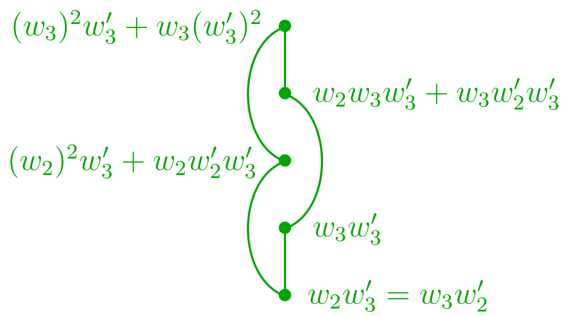
A.3.3 Adams SS

According to our naive guess, the module $\widetilde{H}^*(B\left(\frac{SO(4)\times SU(2)}{\mathbb{Z}_2}\right);\mathbb{Z}_2)_{\leq 5}$ consists of



(A.15)

To be consistent with the AHSS computation, it seems that $w_3w_2'+w_2w_3'$ should be modded out (is it an obvious consequence of the transgression in LSSS...?) and the remaing part (*) turns out to be

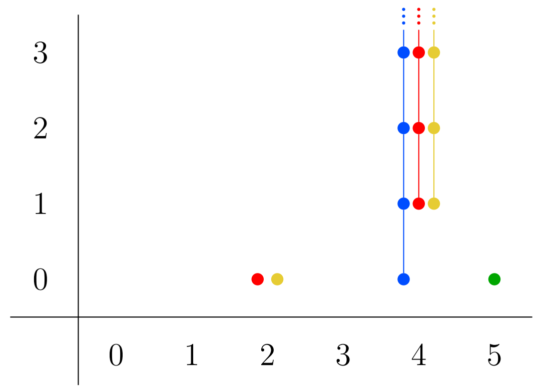


(A.16)

and therefore one concludes

$$\widetilde{H}^*(B\left(\frac{SO(4)\times SU(2)}{\mathbb{Z}_2}\right);\mathbb{Z}_2)_{\leq 5}=J[2]\oplus J[2]\oplus \mathcal{A}_1/\!\!/\mathcal{E}_0[4]\oplus J[5]. \tag{A.17}$$

This leads to the following Adams chart:



and it indeed seems to be compatible with the AHSS computation. If the above argument (and beliefs) is correct, then the anomaly should be captured by

$$w_2 w_3' \; (= w_3 w_2'). \tag{A.18}$$

A.4 $[SO(4n_c'+2)\times SU(2n_f)]/\mathbb{Z}_2$

A.4.1 Cohomology of $BPSO(4n_c'+2)$

Recalling that the $\mathbb{Z}_2$ cohomology read off from [KM75] allowed us to determine the $\mathbb{Z}$ cohomology by using Bockstein SS (see `wzw-memo.pdf`, copy later), we had

d	0	1	2	3	4	5	6	$\cdots$
$H^d(BPSO(4n_c'+2);\mathbb{Z})$	$\mathbb{Z}$	0	0	$\mathbb{Z}_4$	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\cdots$
$H^d(BPSO(4n_c'+2);\mathbb{Z}_2)$	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2^{\oplus 2}$	$\cdots$
generator ($\mathbb{Z}_2$)	1	$-$	a_2	$y'(1)$	$(a_2)^2$	$y'(2)$	$(a_2)^3$ $y'(1)^2$	$\cdots$

(A.19)

and if we assume the suspension $\bar{\theta} : H^*(PSO(4n_c'+2);\mathbb{Z}_2) \to H^*(BPSO(4n_c'+2);\mathbb{Z}_2)$ to commute with Steenrod squares (which seems to be true according to Nishimoto-san's notes), then these elements are supposed to be related as

$$\begin{array}{rcl} \beta_2 a_2 & = & y'(1) \\ Sq^2 y'(1) & = & y'(2) \\ Sq^1 y'(2) & = & y'(1)^2 \end{array}$$

where β_f is a higher Bockstein operator (reduced to $\mathbb{Z}_2$ cohomology as a whole), whose image corresponds to $\mathbb{Z}_{2^f}$ in integral cohomology.

A.4.2 Leray-Serre SS (preparatory)

For the fibration

$$BSU(2n_f) \;\to\; B\left(\frac{SO(4n_c'+2)\times SU(2n_f)}{\mathbb{Z}_2}\right) \;\to\; BPSO(4n_c'+2) \tag{A.20}$$

we have

$$
E_2^{p,q} = H^p\big(BPSO(4n_c'+2); H^q(BSU(2n_f);\mathbb{Z})\big) \qquad\qquad H^{p+q}\Big(B\left(\tfrac{SO(4n_c'+2)\times SU(2n_f)}{\mathbb{Z}_2}\right);\mathbb{Z}\Big)
$$

$$
\begin{array}{c|ccccccc}
6 & \mathbb{Z} & & & * & * & & * \\
5 & & & & & & & \\
4 & \mathbb{Z} & & & * & * & & * \\
3 & & & & & & & \\
2 & & & & & & & \\
1 & & & & & & & \\
0 & \mathbb{Z} & & & \mathbb{Z}_4 & \mathbb{Z} & & \mathbb{Z}_2 \\
\hline
 & 0 & 1 & 2 & 3 & 4 & 5 & 6
\end{array}
\quad \longrightarrow \quad
\begin{array}{c|c}
6 & \mathbb{Z}\oplus\mathbb{Z}_2 \\
5 & \\
4 & \mathbb{Z}^{\oplus 2} \\
3 & \mathbb{Z}_4 \\
2 & \\
1 & \\
0 & \mathbb{Z} \\
\hline
 &
\end{array}
\tag{A.21}
$$

A.4.3 Atiyah-Hirzebruch SS

Having obtained (co)homology groups, one can fill in the E^2-page of the AHSS:

$$
E^2_{p,q} = H_p\Big(B\left(\tfrac{SO(4n_c'+2)\times SU(2n_f)}{\mathbb{Z}_2}\right);\Omega_q^{\text{spin}}\Big)
$$

$$
\begin{array}{c|c:cccccc}
6 & & & & & & & \\
5 & & & & & & & \\
4 & \mathbb{Z} & & * & & * & * & * \\
3 & & & & & & & \\
2 & \mathbb{Z}_2 & & \boxed{\mathbb{Z}_2} & \boxed{\mathbb{Z}_2} & * & * & * \\
1 & \mathbb{Z}_2 & & \boxed{\mathbb{Z}_2} & \boxed{\mathbb{Z}_2} & \boxed{\mathbb{Z}_2^{\oplus 2}} & \boxed{*} & * \\
0 & \mathbb{Z} & & \mathbb{Z}_4 & & \boxed{\mathbb{Z}^{\oplus 2}} & \boxed{\mathbb{Z}_2} & \boxed{*} \\
\hline
 & 0 & 1 & 2 & 3 & 4 & 5 & 6
\end{array}
\tag{A.22}
$$

Based on our belief, $\boxed{d^2 : E^2_{4,0} \to E^2_{2,1}}$ and $\boxed{d^2 : E^2_{4,1} \to E^2_{2,2}}$ should be a dual of

$$
Sq^2 a_2 \;=\; (a_2)^2 \tag{A.23}
$$

and also $\boxed{d^2 : E^2_{5,0} \to E^2_{3,1}}$ and $\boxed{d^2 : E^2_{5,1} \to E^2_{3,2}}$ should be a dual of

$$
Sq^2 y'(1) \;=\; y'(2) \tag{A.24}
$$

and finally $\boxed{d^2 : E^2_{6,0} \to E^2_{4,1}}$ should be a dual of

$$
Sq^2 c_2 \;=\; c_3 \tag{A.25}
$$

then the would-be-E_3-page is given by

$$\begin{array}{c|c:cccccc}
6 & & & & & & & \\
5 & & & & & & & \\
4 & \mathbb{Z} & & * & & * & * & * \\
3 & & & & & & & \\
2 & \mathbb{Z}_2 & & & & * & * & * \\
1 & \mathbb{Z}_2 & & & & & * & * \\
0 & \mathbb{Z} & & \mathbb{Z}_4 & & \mathbb{Z}^{\oplus 2} & & * \\
\hline
 & 0 & 1 & 2 & 3 & 4 & 5 & 6
\end{array} \tag{A.26}$$

B Pontrjagin square for non-closed cochains

By definition, the variation of the Pontrjagin square $\mathfrak{P} : H^\bullet(-;\mathbb{Z}_{2^m}) \to H^{2\bullet}(-;\mathbb{Z}_{2^{m+1}})$ term is

$$\begin{aligned}
\delta\left(\frac{1}{2^{m+1}}\mathfrak{P}(x)\right) &= \frac{1}{2^{m+1}} \cdot \delta\Big(\widetilde{x} \cup \widetilde{x} - \widetilde{x} \cup_1 \delta\widetilde{x}\Big) \\
&= \frac{1}{2^{m+1}} \cdot \Big[\Big(\delta\widetilde{x} \cup \widetilde{x} + \widetilde{x} \cup \delta\widetilde{x}\Big) - \Big(\widetilde{x} \cup \delta\widetilde{x} - \delta\widetilde{x} \cup \widetilde{x} + \delta\widetilde{x} \cup_1 \delta\widetilde{x}\Big)\Big] \\
&= \frac{1}{2^{m+1}} \cdot \Big[2 \cdot \delta\widetilde{x} \cup \widetilde{x} - \delta\widetilde{x} \cup_1 \delta\widetilde{x}\Big],
\end{aligned} \tag{B.1}$$

and thus if x were a $\mathbb{Z}_{2^m}$-cocycle, its integral lift $\widetilde{x} \in C^\bullet(-;\mathbb{Z})$ would be a cocycle mod 2^m *i.e.* $\delta\widetilde{x} = 0 \pmod{2^m}$, and the right hand side would be 0 mod 1, which then means that $\mathfrak{P}(x)$ would be a $\mathbb{Z}_{2^{m+1}}$-cocycle as desired. However, when x is not a cocycle but merely a cochain, $\mathfrak{P}(x)$ is also not a cocycle and it is not clear whether this term is well-defined in the first place.

This problem arises when we consider $SO(2n_c)$ QCD, but it turns out that this can be saved somewhat miraculously as follows. First, a short exact sequence

$$0 \to \mathbb{Z}_{2^f} \xrightarrow{\times 2^f} \mathbb{Z}_{2^{2f}} \xrightarrow{p} \mathbb{Z}_{2^f} \to 0$$

buys us cohomology operations called the higher Bockstein $\beta_f : H^\bullet(-;\mathbb{Z}_{2^f}) \to H^{\bullet+1}(-;\mathbb{Z}_{2^f})$, and for the element $y \in C^\bullet(-;\mathbb{Z}_{2^f})$ one has

$$\delta(p^* y) = 2^f \beta_f(y) \in C^\bullet(-;\mathbb{Z}_{2^{2f}}).$$

Now, let us consider the case of odd n_c. Here, the cochain $w_2(c) \in C^2(SO(4n_c'+2) \times SU(2n_f);\mathbb{Z}_2)$ can be thought of as a mod-2 reduction of $\widetilde{w}_2(c) \in C^2(SO(4n_c'+2) \times SU(2n_f);\mathbb{Z}_4)$. Dividing the $SO \times SU$ by $\mathbb{Z}_2$, these cochains become non-closed

$$\delta\widetilde{w}_2(c) = 2\beta v_2(c)$$

where $v_2(c) = a_2 \in C^2\left(\frac{SO(4n_c'+2)\times SU(2n_f)}{\mathbb{Z}_2};\mathbb{Z}_2\right)$. Since this implies $\delta\widetilde{w}_2(c) = 0$ or 2 mod 4 and furthermore $\delta(2\widetilde{w}_2(c)) = 0$ mod 4, one can safely define the Pontrjagin square $\frac{1}{8}\mathfrak{P}\big(2\widetilde{w}_2(c)\big)$.[3] Note

[3] Be careful that this factor 2 here is not the usual map sending $\{0,1\} = \mathbb{Z}_2$ to $\{0,2\} \subset \mathbb{Z}_4$. This time we are *really* multiplying by 2.

that this can naively be regarded as $2 \cdot \frac{1}{4}\mathfrak{P}\big(w_2(c)\big)$ at the integral cochain level. While the second term in the last line of (B.1) together with the overall factor takes value in $\frac{(2^m)^2}{2^{m+1}}\mathbb{Z} = 2^{m-1}\mathbb{Z} = 2\mathbb{Z}$ and dividing by two does not cause any trouble, the first term does as it takes value in $\frac{2\cdot 2^m}{2^{m+1}}\mathbb{Z} = \mathbb{Z}$.

Let us take a closer look at the latter. The long exact sequence of cochain groups implies that $2\widetilde{w}_2(c)$ can be replaced by $\widetilde{v}_2(c) = \widetilde{a}_2$, which is a cocycle mod 4. Then the term of interest is

$$\frac{1}{8} \cdot 2 \cdot 4\beta_2\widetilde{a}_2 \cup \widetilde{a}_2.$$

Therefore, the anomalous variation of the Pontrjagin square term seems to result in

$$\delta\left(\frac{1}{4}\mathfrak{P}(w_2(c))\right) = \frac{1}{2}a_2\beta_2 a_2.$$

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