This file is to collect various notes on our project on the higher symmetry and Seiberg duality.

1 Duality map

The duality map among $Spin$, SO_+ and SO_- was first found in [1]

$$\begin{array}{rcl} Spin(N_c) & \leftrightarrow & SO_-(N_c'), \\ SO_+(N_c) & \leftrightarrow & SO_+(N_c'), \\ SO_-(N_c) & \leftrightarrow & Spin(N_c') \end{array} \tag{1.1}$$

where $N_c' = N_f - N_c + 4$.

Razamat-Willett [2] performed a rather extensive check of this mapping by means of localization on the lens space times S^1.

Note that there is a natural action of $SL(2,\mathbb{Z})$ on the theories with $\mathbb{Z}_2$ 1-form symmetry. Requiring that this is compatible with the Seiberg duality, one finds that the mapping should in fact be

$$\begin{array}{rcl} Spin(N_c) & \leftrightarrow & T(SO_-(N_c')), \\ SO_+(N_c) & \leftrightarrow & T(SO_+(N_c')), \\ SO_-(N_c) & \leftrightarrow & T(Spin(N_c')) \end{array} \tag{1.2}$$

as discussed in [3, Sec. 6] and [4].

2 References on fermionic zero modes on monopoles

Index theorem on the monopole background: Callias [5] Bott and Seeley [6][1]

General reviews: Harvey [7, Lecture 4].

3 Explicit configurations detecting anomalies

Here we describe geometries detecting $\int_{M_5} B\beta E$, $\int_{M_4} B\beta w_2$, etc. All cohomologies in this section is $\mathbb{Z}_2$-valued.

This is to confirm that these expressions are not secretly trivial.

Klein bottle: We start from the Klein bottle K as a nontrivial S^1 bundle over S^1. Let us denote by a the Poincaré dual to the fiber S^1, and t the Poincaré dual to the base S^1.

We have $\beta a = ta$, since $\int_K \beta a = \int_K w_1 a$.

[1] These are on CMP. Papers on CMP are not open access via Springer (which is reachable by doi) but is open access at Project Euclid. I'd like a way to include links in the references appropriately.

T^4 bundle over S^1: We now consider a T^4 bundle over S^1. We denote four directions of T^4 as 1, 2, 3 and 4, and we let the directions 1 and 3 to flip the orientation when we go around S^1. We let $a_{1,2,3,4} \in H^1(T^4)$ be the dual basis to the S^1 along four directions.

We now take $B = a_1 a_2$ and $E = a_3 a_4$. Then $\beta B = tB$ and $\beta E = tE$, and $\int B\beta E = 1$.

Realizing as $SO(3)$ bundles We now look for $SO(3)$ bundles realizing these B and E as w_2 in this T^4 bundle over S^1.

We note that an $SO(3)$ bundle over T^2 with two commuting holonomies around two directions

$$R_x = \mathrm{diag}(+1, -1, -1), \qquad R_y = \mathrm{diag}(+1, -1, -1)$$

has a nontrivial w_2, since their lift to $SU(2)$ is given by $i\sigma_x$ and $i\sigma_y$ which anticommute.

Luckily, these R_x and R_y are of order two, so we can put it over our T^4 bundle. Done.

4 Anomaly of trifundamental

Lee-kun's computation says that

$$(D\Omega^{\text{spin}})^6(B[SU(2)^3/\mathbb{Z}_2^2]) = \mathbb{Z}_2$$

generated by

$$\int w_2 \beta w_2'.$$

We would like to know if a trifundamental fermion has this anomaly.

5 Via direct study of line operators

Let's determine the anomaly/extension of so SQCD by studying the line operators. Here I concentrate on the case of $SO(2n_c)_\pm$ with $2n_f$ flavors.

Let us first consider the case of pure $SO(2n_c)_\pm$ with n_c odd, and recall how to distinguish $\mathbb{Z}_2 \times \mathbb{Z}_2$ or $\mathbb{Z}_4$.

In either case, the group of the charges of the line operators has the structure

$$0 \to \mathbb{Z}_2 \to G \to \mathbb{Z}_2 \to 0$$

where the $\mathbb{Z}_2$ subgroup is generated by the 't Hooft line and the $\mathbb{Z}_2$ quotient is generated by the Wilson line in the vector representation. To tell whether G is $\mathbb{Z}_2 \times \mathbb{Z}_2$ or $\mathbb{Z}_4$, we simply take two 't Hooft lines. When combined, it is a charge-2 't Hooft line, which can be screened by the dynamical monopole of this charge. In the SO_+ theory, this dynamical monopole is electrically neutral, while the SO_- theory, it has the electric charge equal to the vector representation. Therefore, the charge-2 't Hooft line is equivalent to having no line operator in SO_+ theory, while it is equivalent to having a Wilson line in the vector representation in SO_-. We find that G is $\mathbb{Z}_2 \times \mathbb{Z}_2$ for SO_+ and $\mathbb{Z}_4$ for SO_-.

We learned the important lesson: the 1-form symmetry group is $\mathbb{Z}_2 \times \mathbb{Z}_2$ or $\mathbb{Z}_4$ depending on whether the electric charge of the dynamical monopole of an appropriate charge is neutral or vector.

So we would like to determine that in the SO SQCD, but the direct analysis of dynamical monopoles in this theory is hard. For this purpose, we make the following deformations which do not change the symmetry structure:

- We reduce the flavor symmetry from $su(2n_f)$ to $usp(2n_f)$.
- We add an adjoint scalar $\Phi_{[ab]}$ and the interaction $\psi^{ai}_\alpha \psi^{bj}_\beta J_{ij} \Phi_{ab} \epsilon^{\alpha\beta} + cc$, where $J_{[ij]}$ is the constant matrix for the $usp(2n_f)$ part.
- We give a vev to Φ_{ab} to break $so(2n_c)$ to $so(2)^{n_c}$.

The 't Hooft line of the original $SO(2n_c)$ theory is in the 'vector' class $(1, 0, \ldots, 0)$ under $so(2)^{n_c}$, and therefore the dynamical monopole we need to analyze has the charge in the 'adjoint' class.

The monopoles associated to the breaking of G to its Cartan were analyzed in many places, e.g. in [8]. There, the following was shown. Let ϕ be the scalar vev in the real Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$. This determines the simple roots α. Then you can embed the standard spherically-symmetric 't Hooft-Polyakov monopole and have a monopole solution without additional bosonic moduli.

Let us say we chose the standard ϕ such that the simple roots are $(1, -1, \ldots, 0)$, $(0, 1, -1, \ldots, 0)$, $\ldots$, $(0, \ldots, 1, -1)$ and $(0, \ldots, 1, +1)$. They generate the sublattice Λ of $\mathbb{Z}^n$. The group of the 't Hooft line operators up to the screening by the dynamical monopoles is then

$$\mathbb{Z}^n/\Lambda = \mathbb{Z}_2. \tag{5.1}$$

What we want to do is take into account the center charge $\mathbb{Z}_2$ of $USp(2n_f)$. For this, we consider $\mathbb{Z}^n \times \mathbb{Z}_2$, and divide it by the subgroup generated by the charges of the monopoles associated to the simple roots, which we denote by $(1, -1, \ldots, 0; q_1)$, $(0, 1, -1, \ldots, 0; q_2)$, $\ldots$, $(0, \ldots, 1, -1; q_{n-1})$ and $(0, \ldots, 1, +1; q_n)$.

We do not have to determine q_1 to q_{n-1}. We simply use them to rewrite any charge vector

$$(m_1, \ldots, m_{n-2}, m_{n-1}, m_n; q) \tag{5.2}$$

into

$$(0, \ldots, 0, m, m'; q') \tag{5.3}$$

We then have to determine q_{n-1} and q_n.

This reduces the study to the case of $so(4) \simeq su(2)_1 \times su(2)_2$. The monopoles associated to the simple roots are just 't Hooft-Polyakov monopoles associated to the two factors of $su(2)$'s. The vev of the adjoint scalar in this basis can be written as (a_1, a_2), which we assume $a_1 > a_2 > 0$.

The fermion is in the vector representation of $so(4)$. Under the monopole in $su(2)_1$, it is a doublet coupled to an adjoint vev of size a_1 with bare mass a_2, and similarly for the monopole in $su(2)_2$.

Now, the explicit analysis in [5, Sec. IV] concerning the number of zero modes in the 't Hooft-Polyakov monopole says that a doublet coupled to an adjoint vev of size a with bare mass μ has a zero mode if $|a| > |m|$ and it has no zero mode if $|a| < |m|$.

With our assumption $a_1 > a_2 > 0$, this means that the monopole in $su(2)_1$ has a zero mode while the one in $su(2)_2$ doesn't have any. In our original basis, this means that the monopole with $(0, \ldots, 1, -1; q_{n-1})$ does not have any zero mode and $q_{n-1} = 0$, while the one with $(0, \ldots, 1, +1; q_n)$ (which is in $su(2)_1$) has two zero modes per flavor.

The one-form symmetry group is obtained by dividing $\mathbb{Z}^2 \times \mathbb{Z}_2$ by the subgroup generated by $(1, -1; 0)$ and $(1, +1; q_n)$. This is $\mathbb{Z}_2 \times \mathbb{Z}_2$ or $\mathbb{Z}_4$ depending on whether q_n is 0 or 1.

Let us determine q_n. We saw two zero modes per flavor; this means that there are fermionic zero modes transforming in

$$V_2 \otimes R_{2n_f} \tag{5.4}$$

where V_2 is the doublet of $su(2)_2$ (which is actually broken to $u(1)$ but keeping $su(2)_2$ representation is useful in organizing the answer), R_{2n_f} is the fundamental of $usp(2n_f)$, and we need to impose the reality condition using the pseudoreality of both factors, so that there are $4n_f$ Majorana fermion in total.

I haven't analyzed the general case; let's take $n_f = 1$ and $n_f = 2$ as examples. When $n_f = 1$, there are 4 Majorana fermions. Quantizing them, we find the monopoles in

$$V_2 \otimes \mathbf{1} \oplus \mathbf{1} \otimes R_2. \tag{5.5}$$

It has the 'vector' charge under $usp(2)$ flavor symmetry or is a doublet under $su(2)_2$. In either case, the -1 in $SO(2n_c) \times USp(2n_f)$ acts nontrivially, so $q_n = 1$. and the 1-form symmetry is extended.

When $n_f = 2$, there are 8 Majorana fermions, and $su(2)_2$ and flavor $usp(4)$ are embedded into $so(8)$ via

$$su(2) \times usp(4) \simeq so(3) \times so(5) \subset so(8). \tag{5.6}$$

Therefore, the monopoles are easily seen to be in

$$V_3 \otimes \mathbf{1} \oplus V_2 \otimes R_2 \oplus V_1 \otimes R_5. \tag{5.7}$$

The charge is in the 'adjoint' charge under the $usp(4)$ flavor symmetry, and therefore $q_n = 0$. Therefore the 1-form symmetry is not extended.

So far we considered SO_+ theories. In SO_- theories, we need to add electric charges coming from the discrete theta angle. This is trivial if n_c is even, but is the 'vector' charge if n_c is odd.

A Computation of $\Omega_5^{\text{spin}}\left(B\left(\frac{SO(4)\times SU(2)}{\mathbb{Z}_2}\right)\right)$

Let us consider the simplest case of the Seiberg duality and examine the anomaly consequences. For $SO(4)$ gauge theory with $N_f = 2$ flavors, fermions are charged under $\frac{SO(4)\times SU(2)}{\mathbb{Z}_2}$.

A.1 Leray-Serre SS (preparatory)

For the various input of cohomology groups, see Appendix A of our WZW paper. For the fibration

$$BSO(3) \;\to\; B\left(SO(3)\times SO(3)\right) = B\left(\frac{SO(4)}{\mathbb{Z}_2}\right) \;\to\; BSO(3) \tag{A.1}$$

one has

$$E_2^{p,q} = H^p\big(BSO(3); H^q(BSO(3);\mathbb{Z})\big) \qquad\qquad H^{p+q}\left(B\left(\frac{SO(4)}{\mathbb{Z}_2}\right);\mathbb{Z}\right)$$

$q \backslash p$	0	1	2	3	4	5	6
6	$\mathbb{Z}_2$		$*$	$*$	$*$	$*$	$*$
5							
4	$\mathbb{Z}$			$*$	$*$		$*$
3	$\mathbb{Z}_2$		$\mathbb{Z}_2$	$\mathbb{Z}_2$	$*$	$*$	$*$
2							
1							
0	$\mathbb{Z}$			$\mathbb{Z}_2$	$\mathbb{Z}$		$\mathbb{Z}_2$

$\longrightarrow$

n	H^n
6	$\mathbb{Z}_2^{\oplus 3}$
5	$\mathbb{Z}_2$
4	$\mathbb{Z}^{\oplus 2}$
3	$\mathbb{Z}_2^{\oplus 2}$
2	
1	
0	$\mathbb{Z}$

(A.2)

Here we expect non-trivial differentials to be absent (for the region of interest) from the explicit consideration of generators (since there are W_3 and W_3', there should be $(W_3)^2$, $(W_3')^2$, and W_3W_3') or by requiring proper reproduction of the $\mathbb{Z}_2$ cohomology (which we expect to be generated by w_2, w_2', w_3, and w_3'). Then, for the fibration

$$BSU(2) \;\to\; B\left(\frac{SO(4)\times SU(2)}{\mathbb{Z}_2}\right) \;\to\; B\left(\frac{SO(4)}{\mathbb{Z}_2}\right) \tag{A.3}$$

we can further plug it into

$$E_2^{p,q} = H^p\left(B\left(\frac{SO(4)}{\mathbb{Z}_2}\right); H^q(BSU(2);\mathbb{Z})\right) \qquad\qquad H^{p+q}\left(B\left(\frac{SO(4)\times SU(2)}{\mathbb{Z}_2}\right);\mathbb{Z}\right)$$

$q \backslash p$	0	1	2	3	4	5	6
6							
5							
4	$\boxed{\mathbb{Z}}$			$*$	$*$	$*$	$*$
3							
2							
1							
0	$\mathbb{Z}$			$\mathbb{Z}_2^{\oplus 2}$	$\mathbb{Z}^{\oplus 2}$	$\boxed{\mathbb{Z}_2}$	$\mathbb{Z}_2^{\oplus 3}$

$\longrightarrow$

n	H^n
6	$\mathbb{Z}_2^{\oplus 3}$
5	
4	$\mathbb{Z}^{\oplus 3}$
3	$\mathbb{Z}_2^{\oplus 2}$
2	
1	
0	$\mathbb{Z}$

(A.4)

Taking the normalization of instanton number into account (see Ohmori-san's e-mail on 2020-08-19), the differential $\boxed{d_2 : E_{0,4} \to E_{5,0}}$ seems to be non-trivial.

So we believe the integral cohomology structure to be

$$
\begin{array}{c|cccccccc}
d & 0 & 1 & 2 & 3 & 4 & 5 & 6 & \cdots \\
\hline
H^d(B\left(\frac{SO(4)\times SU(2)}{\mathbb{Z}_2}\right);\mathbb{Z}) & \mathbb{Z} & 0 & 0 & \mathbb{Z}_2^{\oplus 2} & \mathbb{Z}^{\oplus 3} & 0 & \mathbb{Z}_2^{\oplus 3} & \cdots \\
\hline
\text{generator} & 1 & - & - & W_3 & p_1 & - & (W_3)^2 & \cdots \\
 & & & & W_3' & p_1' & & (W_3')^2 & \\
 & & & & & 2c_2 & & W_3 W_3' &
\end{array}
\tag{A.5}
$$

where the reduction to $\mathbb{Z}_2$ cohomology are

$$
\begin{array}{rcl}
W_3 & \to & w_3 \\
p_1 & \to & (w_2)^2
\end{array}
\tag{A.6}
$$

A.2 Atiyah-Hirzebruch SS

Having obtained (co)homology groups, one can fill in the E^2-page of the AHSS:

$$
E^2_{p,q} = H_p(B\left(\frac{SO(4)\times SU(2)}{\mathbb{Z}_2}\right);\Omega_q^{\text{spin}})
$$

$$
\begin{array}{c|c|cccccc}
6 & & & & & & & \\
5 & & & & & & & \\
4 & \mathbb{Z} & & \mathbb{Z}_2^{\oplus 2} & & * & * & * \\
3 & & & & & & & \\
2 & \mathbb{Z}_2 & & \boxed{\mathbb{Z}_2^{\oplus 2}} & \boxed{\mathbb{Z}_2^{\oplus 2}} & \mathbb{Z}_2^{\oplus 3} & * & * \\
1 & \mathbb{Z}_2 & & \boxed{\mathbb{Z}_2^{\oplus 2}} & \boxed{\mathbb{Z}_2^{\oplus 2}} & \boxed{\mathbb{Z}_2^{\oplus 3}} & \boxed{\mathbb{Z}_2^{\oplus 3}} & * \\
0 & \mathbb{Z} & & \mathbb{Z}_2^{\oplus 2} & & \boxed{\mathbb{Z}^{\oplus 3}} & \boxed{\mathbb{Z}_2^{\oplus 3}} & \boxed{*} \\
\hline
 & 0 & 1 & 2 & 3 & 4 & 5 & 6
\end{array}
\tag{A.7}
$$

Based on our belief, $\boxed{d^2 : E^2_{4,0} \to E^2_{2,1}}$ and $\boxed{d^2 : E^2_{4,1} \to E^2_{2,2}}$ should be a dual of

$$
\begin{array}{rcl}
Sq^2 w_2 & = & (w_2)^2 \\
Sq^2 w_2' & = & (w_2')^2
\end{array}
\tag{A.8}
$$

and also $\boxed{d^2 : E^2_{5,0} \to E^2_{3,1}}$ and $\boxed{d^2 : E^2_{5,1} \to E^2_{3,2}}$ should be a dual of

$$
\begin{array}{rcl}
Sq^2 w_3 & = & w_2 w_3 \\
Sq^2 w_3' & = & w_2' w_3'
\end{array}
\tag{A.9}
$$

and finally $\boxed{d^2 : E^2_{6,0} \to E^2_{4,1}}$ should be a dual of

$$
Sq^2(w_2 w_2') = w_3 w_3' + (w_2)^2 w_2' + w_2 (w_2')^2 \tag{A.10}
$$

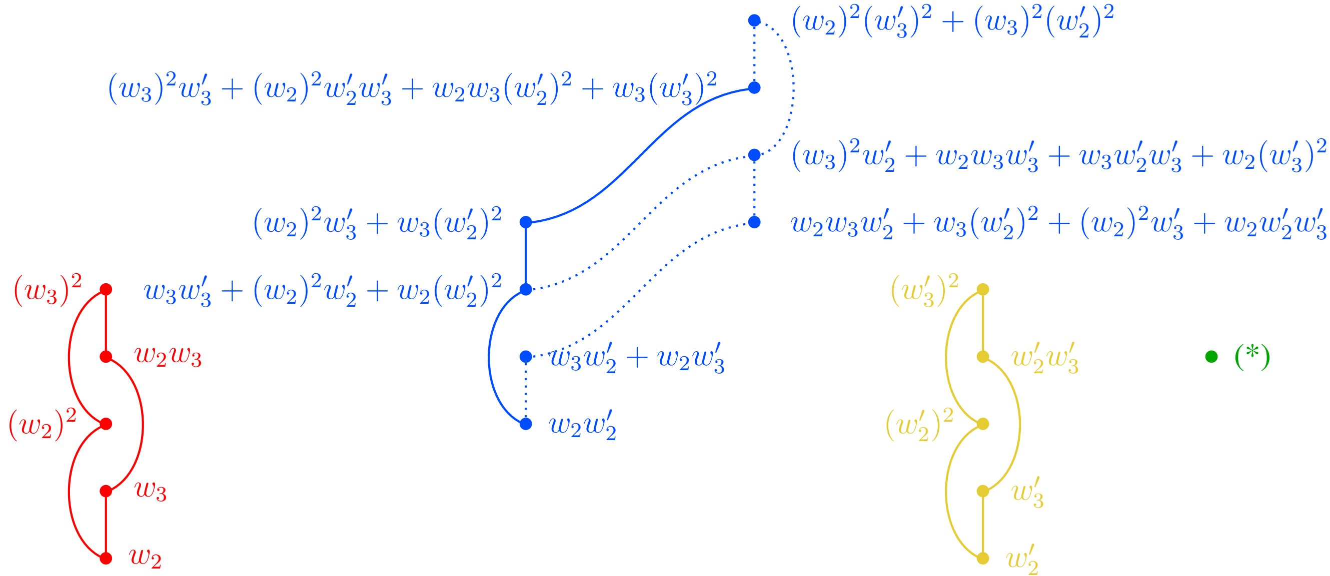
then the would-be-E_3-page is given by

6							
5							
4	$\mathbb{Z}$		$*$		$*$	$*$	$*$
3							
2	$\mathbb{Z}_2$				$*$	$*$	$*$
1	$\mathbb{Z}_2$					$*$	$*$
0	$\mathbb{Z}$		$\mathbb{Z}_2^{\oplus 2}$		$\mathbb{Z}^{\oplus 3}$	$\mathbb{Z}_2$	$*$
	0	1	2	3	4	5	6

(A.11)

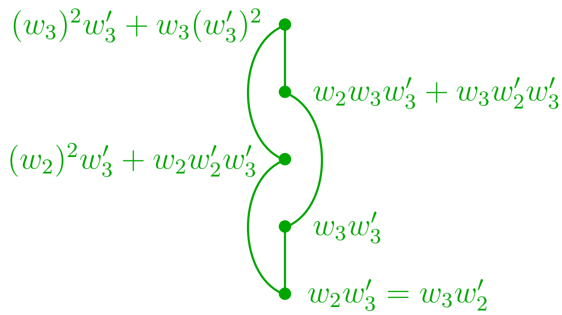
A.3 Adams SS

According to our naive guess, the module $\widetilde{H}^*(B\left(\frac{SO(4)\times SU(2)}{\mathbb{Z}_2}\right);\mathbb{Z}_2)_{\leq 5}$ consists of



(A.12)

To be consistent with the AHSS computation, it seems that $w_3w_2' + w_2w_3'$ should be modded out (is it an obvious consequence of the transgression in LSSS...?) and the remaing part $(*)$ turns out to be

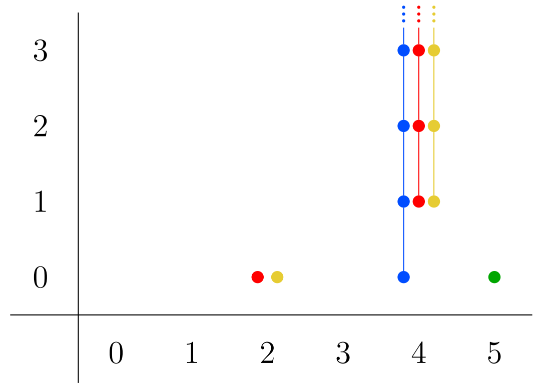


(A.13)

and therefore one concludes

$$\widetilde{H}^*(B\left(\tfrac{SO(4)\times SU(2)}{\mathbb{Z}_2}\right);\mathbb{Z}_2)_{\leq 5} = J[2]\oplus J[2]\oplus \mathcal{A}_1/\!\!/\mathcal{E}_0[4]\oplus J[5]. \tag{A.14}$$

This leads to the following Adams chart:



and it indeed seems to be compatible with the AHSS computation. If the above argument (and beliefs) is correct, then the anomaly should be captured by

$$w_2 w_3' \; (= w_3 w_2'). \tag{A.15}$$

A.4 Seiberg dual

The dual theory is $SO(2)$ gauge theory with $N_f = 2$ flavors, and the fermions are charged under

$$\frac{SO(2)\times SU(2)}{\mathbb{Z}_2} = U(2) \tag{A.16}$$

Its spin bordism is known, and the relevant group turns out to be trivial:

$$\Omega_5^{\text{spin}}\big(BU(2)\big) = 0. \tag{A.17}$$

This means there is no anomaly for fermions on the dual side, and thus the $B\beta E$ anomaly cannot be canceled by fermions.

B Computation of $\Omega_5^{\text{spin}}\big(B\big(\frac{SO(4n_c'+2)\times SU(2n_f)}{\mathbb{Z}_2}\big)\big)$

B.1 Cohomology of $BPSO(4n_c'+2)$

Recalling that the $\mathbb{Z}_2$ cohomology read off from [9] allowed us to determine the $\mathbb{Z}$ cohomology by using Bockstein SS (see `wzw-memo.pdf`, copy later), we had

$$
\begin{array}{c|cccccccc}
d & 0 & 1 & 2 & 3 & 4 & 5 & 6 & \cdots \\
\hline
H^d(BPSO(4n_c'+2);\mathbb{Z}) & \mathbb{Z} & 0 & 0 & \mathbb{Z}_4 & \mathbb{Z} & 0 & \mathbb{Z}_2 & \cdots \\
H^d(BPSO(4n_c'+2);\mathbb{Z}_2) & \mathbb{Z}_2 & 0 & \mathbb{Z}_2 & \mathbb{Z}_2 & \mathbb{Z}_2 & \mathbb{Z}_2 & \mathbb{Z}_2^{\oplus 2} & \cdots \\
\hline
\text{generator } (\mathbb{Z}_2) & 1 & - & a_2 & y'(1) & (a_2)^2 & y'(2) & (a_2)^3 & \cdots \\
 & & & & & & & y'(1)^2 &
\end{array}
\tag{B.1}
$$

and if we assume the suspension $\bar{\theta} : H^*(PSO(4n_c'+2);\mathbb{Z}_2) \to H^*(BPSO(4n_c'+2);\mathbb{Z}_2)$ to commute with Steenrod squares (which seems to be true according to Nishimoto-san's notes), then these elements are supposed to be related as

$$
\begin{array}{rcl}
\beta_2 a_2 & = & y'(1) \\
Sq^2 y'(1) & = & y'(2) \\
Sq^1 y'(2) & = & y'(1)^2
\end{array}
$$

where β_f is a higher Bockstein operator (reduced to $\mathbb{Z}_2$ cohomology as a whole), whose image corresponds to $\mathbb{Z}_{2^f}$ in integral cohomology.

B.2 Leray-Serre SS (preparatory)

For the fibration

$$
BSU(2n_f) \to B\left(\frac{SO(4n_c'+2)\times SU(2n_f)}{\mathbb{Z}_2}\right) \to BPSO(4n_c'+2) \tag{B.2}
$$

we have

$$
E_2^{p,q} = H^p\big(BPSO(4n_c'+2); H^q(BSU(2n_f);\mathbb{Z})\big) \qquad\qquad H^{p+q}\left(B\left(\frac{SO(4n_c'+2)\times SU(2n_f)}{\mathbb{Z}_2}\right);\mathbb{Z}\right)
$$

$$
\begin{array}{c|ccccccc}
6 & \mathbb{Z} & & & * & * & & * \\
5 & & & & & & & \\
4 & \mathbb{Z} & & & * & * & & * \\
3 & & & & & & & \\
2 & & & & & & & \\
1 & & & & & & & \\
0 & \mathbb{Z} & & & \mathbb{Z}_4 & \mathbb{Z} & & \mathbb{Z}_2 \\
\hline
 & 0 & 1 & 2 & 3 & 4 & 5 & 6
\end{array}
\qquad \longrightarrow \qquad
\begin{array}{c|c}
6 & \mathbb{Z}\oplus\mathbb{Z}_2 \\
5 & \\
4 & \mathbb{Z}^{\oplus 2} \\
3 & \mathbb{Z}_4 \\
2 & \\
1 & \\
0 & \mathbb{Z} \\
\hline
 &
\end{array}
\tag{B.3}
$$

B.3 Atiyah-Hirzebruch SS

Having obtained (co)homology groups, one can fill in the E^2-page of the AHSS:

$$E^2_{p,q} = H_p\Big(B\left(\tfrac{SO(4n_c'+2)\times SU(2n_f)}{\mathbb{Z}_2}\right);\Omega^{\text{spin}}_q\Big)$$

$$\begin{array}{c|c:cccccc}
6 & & & & & & & \\
5 & & & & & & & \\
4 & \mathbb{Z} & & * & & * & * & * \\
3 & & & & & & & \\
2 & \mathbb{Z}_2 & & \mathbb{Z}_2 & \mathbb{Z}_2 & * & * & * \\
1 & \mathbb{Z}_2 & & \mathbb{Z}_2 & \mathbb{Z}_2 & \mathbb{Z}_2^{\oplus 2} & * & * \\
0 & \mathbb{Z} & & \mathbb{Z}_4 & & \mathbb{Z}^{\oplus 2} & \mathbb{Z}_2 & * \\
\hline
 & 0 & 1 & 2 & 3 & 4 & 5 & 6
\end{array} \tag{B.4}$$

Based on our belief, $\boxed{d^2: E^2_{4,0} \to E^2_{2,1}}$ and $\boxed{d^2: E^2_{4,1} \to E^2_{2,2}}$ should be a dual of

$$Sq^2 a_2 \;=\; (a_2)^2 \tag{B.5}$$

and also $\boxed{d^2: E^2_{5,0} \to E^2_{3,1}}$ and $\boxed{d^2: E^2_{5,1} \to E^2_{3,2}}$ should be a dual of

$$Sq^2 y'(1) \;=\; y'(2) \tag{B.6}$$

and finally $\boxed{d^2: E^2_{6,0} \to E^2_{4,1}}$ should be a dual of

$$Sq^2 c_2 \;=\; c_3 \tag{B.7}$$

then the would-be-E_3-page is given by

$$\begin{array}{c|c:cccccc}
6 & & & & & & & \\
5 & & & & & & & \\
4 & \mathbb{Z} & & * & & * & * & * \\
3 & & & & & & & \\
2 & \mathbb{Z}_2 & & & & * & * & * \\
1 & \mathbb{Z}_2 & & & & & * & * \\
0 & \mathbb{Z} & & \mathbb{Z}_4 & & \mathbb{Z}^{\oplus 2} & & * \\
\hline
 & 0 & 1 & 2 & 3 & 4 & 5 & 6
\end{array} \tag{B.8}$$

C Pontrjagin square for non-closed cochains

By definition, the variation of the Pontrjagin square $\mathfrak{P}: H^\bullet(-;\mathbb{Z}_{2^m}) \to H^{2\bullet}(-;\mathbb{Z}_{2^{m+1}})$ term is

$$\begin{aligned}
\delta\left(\frac{1}{2^{m+1}}\mathfrak{P}(x)\right) &= \frac{1}{2^{m+1}}\cdot\delta\Big(\widetilde{x}\cup\widetilde{x}-\widetilde{x}\cup_1\delta\widetilde{x}\Big) \\
&= \frac{1}{2^{m+1}}\cdot\Big[\Big(\delta\widetilde{x}\cup\widetilde{x}+\widetilde{x}\cup\delta\widetilde{x}\Big)-\Big(\widetilde{x}\cup\delta\widetilde{x}-\delta\widetilde{x}\cup\widetilde{x}+\delta\widetilde{x}\cup_1\delta\widetilde{x}\Big)\Big] \\
&= \frac{1}{2^{m+1}}\cdot\Big[2\cdot\delta\widetilde{x}\cup\widetilde{x}-\delta\widetilde{x}\cup_1\delta\widetilde{x}\Big],
\end{aligned} \tag{C.1}$$

and thus if x were a $\mathbb{Z}_{2^m}$-cocycle, its integral lift $\widetilde{x}\in C^\bullet(-;\mathbb{Z})$ would be a cocycle mod 2^m *i.e.* $\delta\widetilde{x}=0$ (mod 2^m), and the right hand side would be 0 mod 1, which then means that $\mathfrak{P}(x)$ would be a $\mathbb{Z}_{2^{m+1}}$-cocycle as desired. However, when x is not a cocycle but merely a cochain, $\mathfrak{P}(x)$ is also not a cocycle and it is not clear whether this term is well-defined in the first place.

This problem arises when we consider $SO(2n_c)$ QCD, but it turns out that this can be saved somewhat miraculously as follows. First, a short exact sequence

$$0\to\mathbb{Z}_{2^f}\xrightarrow{\times 2^f}\mathbb{Z}_{2^{2f}}\xrightarrow{p}\mathbb{Z}_{2^f}\to 0$$

buys us cohomology operations called the higher Bockstein $\beta_f: H^\bullet(-;\mathbb{Z}_{2^f})\to H^{\bullet+1}(-;\mathbb{Z}_{2^f})$, and for the element $y\in C^\bullet(-;\mathbb{Z}_{2^f})$ one has

$$\delta(p^*y)=2^f\beta_f(y)\in C^\bullet(-;\mathbb{Z}_{2^{2f}}).$$

Now, let us consider the case of odd n_c. Here, the cochain $w_2(c)\in C^2(SO(4n_c'+2)\times SU(2n_f);\mathbb{Z}_2)$ can be thought of as a mod-2 reduction of $\widetilde{w}_2(c)\in C^2(SO(4n_c'+2)\times SU(2n_f);\mathbb{Z}_4)$. Dividing the $SO\times SU$ by $\mathbb{Z}_2$, these cochains become non-closed

$$\delta\widetilde{w}_2(c)=2\beta v_2(c)$$

where $v_2(c)=a_2\in C^2\left(\frac{SO(4n_c'+2)\times SU(2n_f)}{\mathbb{Z}_2};\mathbb{Z}_2\right)$. Since this implies $\delta\widetilde{w}_2(c)=0$ or 2 mod 4 and furthermore $\delta(2\widetilde{w}_2(c))=0$ mod 4, one can safely define the Pontrjagin square $\frac{1}{8}\mathfrak{P}\big(2\widetilde{w}_2(c)\big)$.[2] Note that this can naively be regarded as $2\cdot\frac{1}{4}\mathfrak{P}\big(w_2(c)\big)$ at the integral cochain level. While the second term in the last line of (C.1) together with the overall factor takes value in $\frac{(2^m)^2}{2^{m+1}}\mathbb{Z}=2^{m-1}\mathbb{Z}=2\mathbb{Z}$ and dividing by two does not cause any trouble, the first term does as it takes value in $\frac{2\cdot 2^m}{2^{m+1}}\mathbb{Z}=\mathbb{Z}$.

Let us take a closer look at the latter. The long exact sequence of cochain groups implies that $2\widetilde{w}_2(c)$ can be replaced by $\widetilde{v}_2(c)=\widetilde{a}_2$, which is a cocycle mod 4. Then the term of interest is

$$\frac{1}{8}\cdot 2\cdot 4\beta_2\widetilde{a}_2\cup\widetilde{a}_2.$$

Therefore, the anomalous variation of the Pontrjagin square term seems to result in

$$\delta\left(\frac{1}{4}\mathfrak{P}(w_2(c))\right)=\frac{1}{2}a_2\beta_2 a_2.$$

[2] Be careful that this factor 2 here is not the usual map sending $\{0,1\}=\mathbb{Z}_2$ to $\{0,2\}\subset\mathbb{Z}_4$. This time we are *really* multiplying by 2.

D 't Hooft-Polyakov monopole argument

D.1 Basics

For an $SU(2)$ gauge theory Higgssed by an adjoint ($\mathbf{3}$, spin-1, isovector) scalar, the gauge group is broken down to $U(1)$, and correspondingly it accommodates topological solitons (monopoles):

$$\pi_2\left(\frac{SU(2)}{U(1)}\right) = \mathbb{Z}.$$

In the presence of (additional) fermions, this monopole might acquire non-trivial charge under spacetime-Lorentz or flavor symmetries, depending on the representation of the fermions under $SU(2)$ gauge symmetry:

fermion gauge rep.	number of zero-modes	spin of zero-modes	spin of monopole
$\mathbf{2}$	1	0	0
$\mathbf{3}$	2	$\frac{1}{2}$	
$\mathbf{4}$	4	$0 \oplus 1$	$\frac{1}{2}$

The numbers of zero-modes can be computed from the Callias index theorem [5].

According to [10], there is $w_2(TM_4)\beta w_2(SU(2)_{\text{gauge}})$ anomaly for a fermion in $\mathbf{4}$ charged under

$$\frac{Spin(4)_{\text{spacetime}} \times SU(2)_{\text{gauge}}}{\mathbb{Z}_2},$$

which incarnates in the IR as an ill-definition of the effective interaction $w_2(TM)c_1(U(1)_{\text{gauge}})$, emerging after integrating out the fermion which obtained mass through Yukawa coupling. This effective interaction term should arise in order to make the monopole a fermion (*i.e.* spinor representation of $Spin(4)_{\text{spacetime}}$), but is not well-defined without a trivialization of $w_2(TM_4)$ or equivalently a spin structure.

This situation looks quite similar to our problem where the fermions are charged under

$$\frac{SO(4)_{\text{gauge}} \times SU(2)_{\text{flavor}}}{\mathbb{Z}_2}.$$

Since fermions are in the fundamental representation of the flavor symmetry, breaking $SU(2)_{\text{flavor}}$ to $U(1)_{\text{flavor}}$ gives rise to a monopole in the spinor representation this time of the $SO(4)_{\text{gauge}}$ [7]. Therefore by the same logic, one can deduce that there should be $w_2(SO(4)_{\text{gauge}})\beta w_2(SU(2)_{\text{flavor}})$ anomaly in the first place, as desired.

D.2 Generalization

D.2.1 Breaking pattern 1

Also, one should be able to generalize this whole argument to the case of fermions charged under

$$\frac{SO(2n_c)_{\text{gauge}} \times SU(2n_f)_{\text{flavor}}}{\mathbb{Z}_2}$$

by breaking $SU(2n_f)_{\text{flavor}}$ to $SO(2n_f)_{\text{flavor}}$, where we have monopoles characterized by

$$\pi_2\left(\frac{SU(2n_f)}{SO(2n_f)}\right) = \left\{ \begin{array}{cc} \mathbb{Z}_2 & (n_f \geq 2) \\ \mathbb{Z} & (n_f = 1) \end{array} \right. .$$

D.2.2 Breaking pattern 2

We can also consider $SO(2n_c) \times USp(2n_f)$ breaking to $SO(2n_c) \times U(n_f)$ where the $2n_f$ dimensional irrep of $USp(2n_f)$ becomes a fundamental plus an antifundamental of $U(n_f)$.

The induced effective interaction is

$$\frac{1}{2}w_2(SO(2n_c))c_1(U(n_f)). \tag{D.1}$$

(In the following we consider cochains valued in $\mathbb{R}/\mathbb{Z}$.)

We now take the $\mathbb{Z}_2$ quotient. Note that $\pi_1(U(n_f)/\mathbb{Z}_2)$ is $\mathbb{Z}\times\mathbb{Z}_2$ or $\mathbb{Z}$ depending on whether n_f is even or odd. Furthermore, when n_f is even, $c_1(U(n_f)) = c_1(U(n_f)/\mathbb{Z}_2)$ and $v_2(U(n_f)/\mathbb{Z}_2) = v_2(USp(2n_f))$/ When n_f is odd, $c_1(U(n_f)/\mathbb{Z}_2) = 2c_1(U(n_f))$ when the latter is well-defined and $v_2(U(n_f)/\mathbb{Z}_2)$ is the mod-2 reduction of $c_1(U(n_f)/\mathbb{Z}_2)$.

We now compute the anomaly cochains in the four cases separately:

(n_c, n_f)**=(even,even):** $w_2(SO(2n_c))$ and $c_1(U(n_f))$ can be generalized without problem, and therefore

$$\delta(\frac{1}{2}w_2(SO(2n_c))c_1(U(n_f))) = 0. \tag{D.2}$$

(n_c, n_f)**=(odd,even):** $w_2(SO(2n_c))$ needs to be upgraded to a $\mathbb{Z}_4$-valued cochain $a_2(SO(2n_c)/\mathbb{Z}_2)$. The original interaction is then

$$\frac{1}{4}a_2(SO(2n_c)/\mathbb{Z}_2)c_1(U(n_f)) \tag{D.3}$$

which is closed without problem.

(n_c, n_f)**=(even,odd):** Here we need to replace $c_1(U(n_f))$ by $c_1(U(n_f)/\mathbb{Z}_2)/2$. The effective interaction is then

$$\frac{1}{4}w_2(SO(2n_c)/\mathbb{Z}_2)c_1(U(n_f)/\mathbb{Z}_2) \tag{D.4}$$

whose δ is

$$\delta(\frac{1}{4}w_2(SO(2n_c)/\mathbb{Z}_2)c_1(U(n_f)/\mathbb{Z}_2)) = \frac{1}{2}(\frac{1}{2}\delta w_2(SO(2n_c))c_1(U(n_f)/\mathbb{Z}_2)) \tag{D.5}$$

$$= \frac{1}{2}(\beta w_2(SO(2n_c)))c_1(U(n_f)/\mathbb{Z}_2) \tag{D.6}$$

which is the pull-back of the anomaly cochain

$$\frac{1}{2}(\beta w_2(SO(2n_c)))v_2(USp(2n_f)/\mathbb{Z}_2). \tag{D.7}$$

(n_c, n_f)**=(odd,odd):** Now we make the replacement on both sides and the effective interaction is

$$\frac{1}{8}a_2(SO(2n_c)/\mathbb{Z}_2)c_1(U(n_f)/\mathbb{Z}_2) \tag{D.8}$$

whose δ is

$$\delta(\frac{1}{8}a_2(SO(2n_c)/\mathbb{Z}_2)c_1(U(n_f)/\mathbb{Z}_2)) = \frac{1}{2}(\frac{1}{4}\delta a_2(SO(2n_c)/\mathbb{Z}_2)c_1(U(n_f)/\mathbb{Z}_2)) \tag{D.9}$$

$$= \frac{1}{2}(\beta_2 a_2(SO(2n_c)/\mathbb{Z}_2))c_1(U(n_f)/\mathbb{Z}_2) \tag{D.10}$$

which is the pull-back of

$$\frac{1}{2}(\beta_2 a_2(SO(2n_c)))v_2(USp(2n_f)), \tag{D.11}$$

which is more or less what we need, since $a_2(SO(2n_c) = v_2(USp(2n_f)/\mathbb{Z}_2)$ up to exact terms.

E misc

E.1 $\mathbb{Z}_4$ 1-form symmetry

According to [11, Appendix C.3] and [12, Eq. (6.3)], it seems that we have

$$
\begin{array}{c|ccccccc}
\multicolumn{8}{c}{E^2_{p,q} = H_p\big(K(\mathbb{Z}_4,2);\Omega^{\text{spin}}_q\big)} \\
6 & & & & & & & \\
5 & & & & & & & \\
4 & \mathbb{Z} & & * & & * & * & * \\
3 & & & & & & & \\
2 & \mathbb{Z}_2 & & \boxed{\mathbb{Z}_2} & \boxed{\mathbb{Z}_2} & * & * & * \\
1 & \mathbb{Z}_2 & & \boxed{\mathbb{Z}_2} & \boxed{\mathbb{Z}_2} & \boxed{\mathbb{Z}_2} & \boxed{\mathbb{Z}_2} & * \\
0 & \mathbb{Z} & & \mathbb{Z}_4 & & \boxed{\mathbb{Z}_8} & \boxed{\mathbb{Z}_2} & * \\
\hline
 & 0 & 1 & 2 & 3 & 4 & 5 & 6
\end{array}
\longrightarrow
\begin{array}{c|c}
\multicolumn{2}{c}{\widetilde{\Omega}^{\text{spin}}_{p+q}(K(\mathbb{Z}_4,2))} \\
6 & * \\
5 & \\
4 & \mathbb{Z}_4 \\
3 & \\
2 & \mathbb{Z}_4 \\
1 & \\
0 & \\
\hline
\end{array}
\tag{E.1}
$$

The corresponding invariant in 4d is simply

$$\exp(2\pi i \frac{p}{4} \int \frac{1}{2}\mathfrak{P}(a)) \tag{E.2}$$

where $\mathfrak{P} : H^2(-,\mathbb{Z}_4) \to H^4(-,\mathbb{Z}_8)$ is the Pontryagin square, which is even mod 8 on a spin manifold.

E.2 $\mathbb{Z}_2 \times \mathbb{Z}_2$ 1-form symmetry

Exploiting the fact that $K(\mathbb{Z}_2 \times \mathbb{Z}_2, 2) = K(\mathbb{Z}_2, 2) \times K(\mathbb{Z}_2, 2)$, it seems that we have

$$
\begin{array}{c|ccccccc}
\multicolumn{8}{c}{E^2_{p,q} = H_p\big(K(\mathbb{Z}_2\times\mathbb{Z}_2,2);\Omega^{\text{spin}}_q\big)} \\
6 & & & & & & & \\
5 & & & & & & & \\
4 & \mathbb{Z} & & * & & * & * & * \\
3 & & & & & & & \\
2 & \mathbb{Z}_2 & & \boxed{\mathbb{Z}_2^{\oplus 2}} & \boxed{\mathbb{Z}_2^{\oplus 2}} & * & * & * \\
1 & \mathbb{Z}_2 & & \boxed{\mathbb{Z}_2^{\oplus 2}} & \boxed{\mathbb{Z}_2^{\oplus 2}} & \boxed{\mathbb{Z}_2^{\oplus 3}} & \boxed{\mathbb{Z}_2^{\oplus 6}} & * \\
0 & \mathbb{Z} & & \mathbb{Z}_2^{\oplus 2} & & \boxed{\mathbb{Z}_4^{\oplus 2} \oplus \mathbb{Z}_2} & \boxed{\mathbb{Z}_2^{\oplus 3}} & \boxed{*} \\
\hline
 & 0 & 1 & 2 & 3 & 4 & 5 & 6
\end{array}
\longrightarrow
\begin{array}{c|c}
\multicolumn{2}{c}{\widetilde{\Omega}^{\text{spin}}_{p+q}(K(\mathbb{Z}_2\times\mathbb{Z}_2,2))} \\
6 & * \\
5 & \mathbb{Z}_2 \\
4 & \mathbb{Z}_2^{\oplus 3} \\
3 & \\
2 & \mathbb{Z}_2^{\oplus 2} \\
1 & \\
0 & \\
\hline
\end{array}
\tag{E.3}
$$

The $\mathbb{Z}$ homology of $K(\mathbb{Z}_2, 2)$ is again read off from [11], while the $\mathbb{Z}_2$ (co)homology is known [13] to be

$$H^*(K(\mathbb{Z}_2, 2); \mathbb{Z}_2) = \mathbb{Z}_2[x_2, Sq^1 x_2, Sq^2 Sq^1 x_2, \cdots].$$

The corresponding bordism invariants in 4d are $\mathfrak{P}(a)/2$, ab, $\mathfrak{P}(b)/2$, and the one in 5d is $a\beta b$.

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