This file is to collect various notes on our project on the higher symmetry and Seiberg duality.

1 Duality map

The duality map among $Spin$, SO_+ and SO_- was first found in [1]

$$\begin{array}{rcl} Spin(N_c) & \leftrightarrow & SO_-(N_c'), \\ SO_+(N_c) & \leftrightarrow & SO_+(N_c'), \\ SO_-(N_c) & \leftrightarrow & Spin(N_c') \end{array} \tag{1.1}$$

where $N_c' = N_f - N_c + 4$.

Razamat-Willett [2] performed a rather extensive check of this mapping by means of localization on the lens space times S^1.

Note that there is a natural action of $SL(2,\mathbb{Z})$ on the theories with $\mathbb{Z}_2$ 1-form symmetry. Requiring that this is compatible with the Seiberg duality, one finds that the mapping should in fact be

$$\begin{array}{rcl} Spin(N_c) & \leftrightarrow & T(SO_-(N_c')), \\ SO_+(N_c) & \leftrightarrow & T(SO_+(N_c')), \\ SO_-(N_c) & \leftrightarrow & T(Spin(N_c')) \end{array} \tag{1.2}$$

as discussed in [3, Sec. 6] and [4].

2 References on fermionic zero modes on monopoles

Index theorem on the monopole background: Callias [5] Bott and Seeley [6][1]

General reviews: Harvey [7, Lecture 4].

3 Explicit configurations detecting anomalies

Here we describe geometries detecting $\int_{M_5} B\beta E$, $\int_{M_4} B\beta w_2$, etc. All cohomologies in this section is $\mathbb{Z}_2$-valued.

This is to confirm that these expressions are not secretly trivial.

Klein bottle: We start from the Klein bottle K as a nontrivial S^1 bundle over S^1. Let us denote by a the Poincaré dual to the fiber S^1, and t the Poincaré dual to the base S^1.

We have $\beta a = ta$, since $\int_K \beta a = \int_K w_1 a$.

[1] These are on CMP. Papers on CMP are not open access via Springer (which is reachable by doi) but is open access at Project Euclid. I'd like a way to include links in the references appropriately.

T^4 **bundle over** S^1**:** We now consider a T^4 bundle over S^1. We denote four directions of T^4 as 1, 2, 3 and 4, and we let the directions 1 and 3 to flip the orientation when we go around S^1. We let $a_{1,2,3,4} \in H^1(T^4)$ be the dual basis to the S^1 along four directions.

We now take $B = a_1 a_2$ and $E = a_3 a_4$. Then $\beta B = tB$ and $\beta E = tE$, and $\int B\beta E = 1$.

Realizing as $SO(3)$ **bundles** We now look for $SO(3)$ bundles realizing these B and E as w_2 in this T^4 bundle over S^1.

We note that an $SO(3)$ bundle over T^2 with two commuting holonomies around two directions

$$R_x = \mathrm{diag}(+1, -1, -1), \qquad R_y = \mathrm{diag}(+1, -1, -1)$$

has a nontrivial w_2, since their lift to $SU(2)$ is given by $i\sigma_x$ and $i\sigma_y$ which anticommute.

Luckily, these R_x and R_y are of order two, so we can put it over our T^4 bundle. Done.

4 Anomaly of trifundamental

Lee-kun's computation says that

$$(D\Omega^{\text{spin}})^6(B[SU(2)^3/\mathbb{Z}_2^2]) = \mathbb{Z}_2$$

generated by

$$\int w_2 \beta w_2'.$$

We would like to know if a trifundamental fermion has this anomaly.

5 Via direct study of line operators

Let's determine the anomaly/extension of so SQCD by studying the line operators. Here I concentrate on the case of $SO(2n_c)_\pm$ with $2n_f$ flavors.

Let us first consider the case of pure $SO(2n_c)_\pm$ with n_c odd, and recall how to distinguish $\mathbb{Z}_2 \times \mathbb{Z}_2$ or $\mathbb{Z}_4$.

In either case, the group of the charges of the line operators has the structure

$$0 \to \mathbb{Z}_2 \to G \to \mathbb{Z}_2 \to 0$$

where the $\mathbb{Z}_2$ subgroup is generated by the 't Hooft line and the $\mathbb{Z}_2$ quotient is generated by the Wilson line in the vector representation. To tell whether G is $\mathbb{Z}_2 \times \mathbb{Z}_2$ or $\mathbb{Z}_4$, we simply take two 't Hooft lines. When combined, it is a charge-2 't Hooft line, which can be screened by the dynamical monopole of this charge. In the SO_+ theory, this dynamical monopole is electrically neutral, while the SO_- theory, it has the electric charge equal to the vector representation. Therefore, the charge-2 't Hooft line is equivalent to having no line operator in SO_+ theory, while it is equivalent to having a Wilson line in the vector representation in SO_-. We find that G is $\mathbb{Z}_2 \times \mathbb{Z}_2$ for SO_+ and $\mathbb{Z}_4$ for SO_-.

We learned the important lesson: the 1-form symmetry group is $\mathbb{Z}_2 \times \mathbb{Z}_2$ or $\mathbb{Z}_4$ depending on whether the electric charge of the dynamical monopole of an appropriate charge is neutral or vector.

So we would like to determine that in the SO SQCD, but the direct analysis of dynamical monopoles in this theory is hard. For this purpose, we make the following deformations which do not change the symmetry structure:

- We reduce the flavor symmetry from $su(2n_f)$ to $usp(2n_f)$.
- We add an adjoint scalar $\Phi_{[ab]}$ and the interaction $\psi^{ai}_\alpha \psi^{bj}_\beta J_{ij}\Phi_{ab}\epsilon^{\alpha\beta} + cc$, where $J_{[ij]}$ is the constant matrix for the $usp(2n_f)$ part.
- We give a vev to Φ_{ab} to break $so(2n_c)$ to $so(2)^{n_c}$.

The 't Hooft line of the original $SO(2n_c)$ theory is in the 'vector' class $(1, 0, \ldots, 0)$ under $so(2)^{n_c}$, and therefore the dynamical monopole we need to analyze has the charge $(2, 0, \ldots, 0)$.

Since the fermions are in the vector representation of $so(2n_c)$, they are charge 2 under the $so(2)$, and should give rise to fermionic zero modes transforming in

$$V_2 \otimes R_{2n_f} \tag{5.1}$$

where V_2 is the doublet of spacetime $so(3)$, R_{2n_f} is the fundamental of $usp(2n_f)$, and we need to impose the reality condition using the pseudoreality of both factors, so that there are $4n_f$ Majorana fermion in total.

I haven't analyzed the general case; let's take $n_f = 1$ and $n_f = 2$ as examples. When $n_f = 1$, there are 4 Majorana fermions. Quantizing them, we find the monopoles in

$$V_2 \otimes \mathbf{1} \oplus \mathbf{1} \otimes R_2. \tag{5.2}$$

I think we should take the spacetime scalar as indicative, (which needs to be justified) so it has the 'vector' charge under $usp(2)$ flavor symmetry. Therefore the 1-form symmetry is extended.

When $n_f = 2$, there are 8 Majorana fermions, and spatial $su(2)$ and flavor $usp(4)$ are embedded into $so(8)$ via

$$su(2) \times usp(4) \simeq so(3) \times so(5) \subset so(8). \tag{5.3}$$

Therefore, the monopoles are easily seen to be in

$$V_3 \otimes \mathbf{1} \oplus V_2 \otimes R_2 \oplus V_1 \otimes R_5. \tag{5.4}$$

Again taking the spacetime scalar part, it is in the 'adjoint' charge under the $usp(4)$ flavor symmetry. Therefore the 1-form symmetry is not extended.

So far we considered SO_+ theories. In SO_- theories, we need to add electric charges coming from the discrete theta angle. This is trivial if n_c is even, but is the 'vector' charge if n_c is odd.

A Computation of $\Omega_5^{\text{spin}}\left(B\left(\frac{SO(4)\times SU(2)}{\mathbb{Z}_2}\right)\right)$

Let us consider the simplest case of the Seiberg duality and examine the anomaly consequences. For $SO(4)$ gauge theory with $N_f = 2$ flavors, fermions are charged under $\frac{SO(4)\times SU(2)}{\mathbb{Z}_2}$.

A.1 Leray-Serre SS (preparatory)

For the various input of cohomology groups, see Appendix A of our WZW paper. For the fibration

$$BSO(3) \;\to\; B\left(SO(3)\times SO(3)\right) = B\left(\frac{SO(4)}{\mathbb{Z}_2}\right) \;\to\; BSO(3) \tag{A.1}$$

one has

$$E_2^{p,q} = H^p\big(BSO(3); H^q(BSO(3);\mathbb{Z})\big) \qquad\qquad H^{p+q}(B\left(\frac{SO(4)}{\mathbb{Z}_2}\right);\mathbb{Z})$$

$$\begin{array}{c|ccccccc}
6 & \mathbb{Z}_2 & & * & * & * & * & * \\
5 & & & & & & & \\
4 & \mathbb{Z} & & & * & * & & * \\
3 & \mathbb{Z}_2 & & \mathbb{Z}_2 & \mathbb{Z}_2 & * & * & * \\
2 & & & & & & & \\
1 & & & & & & & \\
0 & \mathbb{Z} & & & \mathbb{Z}_2 & \mathbb{Z} & & \mathbb{Z}_2 \\
\hline
 & 0 & 1 & 2 & 3 & 4 & 5 & 6
\end{array}
\quad\longrightarrow\quad
\begin{array}{c|c}
6 & \mathbb{Z}_2^{\oplus 3} \\
5 & \mathbb{Z}_2 \\
4 & \mathbb{Z}^{\oplus 2} \\
3 & \mathbb{Z}_2^{\oplus 2} \\
2 & \\
1 & \\
0 & \mathbb{Z} \\
\hline
 &
\end{array} \tag{A.2}$$

Here we expect non-trivial differentials to be absent (for the region of interest) from the explicit consideration of generators (since there are W_3 and W_3', there should be $(W_3)^2$, $(W_3')^2$, and W_3W_3') or by requiring proper reproduction of the $\mathbb{Z}_2$ cohomology (which we expect to be generated by w_2, w_2', w_3, and w_3'). Then, for the fibration

$$BSU(2) \;\to\; B\left(\frac{SO(4)\times SU(2)}{\mathbb{Z}_2}\right) \;\to\; B\left(\frac{SO(4)}{\mathbb{Z}_2}\right) \tag{A.3}$$

we can further plug it into

$$E_2^{p,q} = H^p\big(B\left(\frac{SO(4)}{\mathbb{Z}_2}\right); H^q(BSU(2);\mathbb{Z})\big) \qquad\qquad H^{p+q}(B\left(\frac{SO(4)\times SU(2)}{\mathbb{Z}_2}\right);\mathbb{Z})$$

$$\begin{array}{c|ccccccc}
6 & & & & & & & \\
5 & & & & & & & \\
4 & \boxed{\mathbb{Z}} & & & * & * & * & * \\
3 & & & & & & & \\
2 & & & & & & & \\
1 & & & & & & & \\
0 & \mathbb{Z} & & & \mathbb{Z}_2^{\oplus 2} & \mathbb{Z}^{\oplus 2} & \boxed{\mathbb{Z}_2} & \mathbb{Z}_2^{\oplus 3} \\
\hline
 & 0 & 1 & 2 & 3 & 4 & 5 & 6
\end{array}
\quad\longrightarrow\quad
\begin{array}{c|c}
6 & \mathbb{Z}_2^{\oplus 3} \\
5 & \\
4 & \mathbb{Z}^{\oplus 3} \\
3 & \mathbb{Z}_2^{\oplus 2} \\
2 & \\
1 & \\
0 & \mathbb{Z} \\
\hline
 &
\end{array} \tag{A.4}$$

Taking the normalization of instanton number into account (see Ohmori-san's e-mail on 2020-08-19), the differential $\boxed{d_2 : E_{0,4} \to E_{5,0}}$ seems to be non-trivial.

So we believe the integral cohomology structure to be

d	0	1	2	3	4	5	6	$\cdots$
$H^d(B\left(\frac{SO(4)\times SU(2)}{\mathbb{Z}_2}\right);\mathbb{Z})$	$\mathbb{Z}$	0	0	$\mathbb{Z}_2^{\oplus 2}$	$\mathbb{Z}^{\oplus 3}$	0	$\mathbb{Z}_2^{\oplus 3}$	$\cdots$
generator	1	–	–	W_3	p_1	–	$(W_3)^2$	$\cdots$
				W'_3	p'_1		$(W'_3)^2$	
					$2c_2$		$W_3W'_3$	

(A.5)

where the reduction to $\mathbb{Z}_2$ cohomology are

$$\begin{aligned} W_3 &\to w_3 \\ p_1 &\to (w_2)^2 \end{aligned} \tag{A.6}$$

A.2 Atiyah-Hirzebruch SS

Having obtained (co)homology groups, one can fill in the E^2-page of the AHSS:

$$E^2_{p,q} = H_p\left(B\left(\frac{SO(4)\times SU(2)}{\mathbb{Z}_2}\right);\Omega^{\text{spin}}_q\right) \qquad \widetilde{\Omega}^{\text{spin}}_{p+q}\left(B\left(\frac{SO(4)\times SU(2)}{\mathbb{Z}_2}\right)\right)$$

$$\begin{array}{c|c|cccccc} 6 & & & & & & & \\ 5 & & & & & & & \\ 4 & \mathbb{Z} & & \mathbb{Z}_2^{\oplus 2} & & * & * & * \\ 3 & & & & & & & \\ 2 & \mathbb{Z}_2 & & \mathbb{Z}_2^{\oplus 2} & \mathbb{Z}_2^{\oplus 2} & \mathbb{Z}_2^{\oplus 3} & * & * \\ 1 & \mathbb{Z}_2 & & \mathbb{Z}_2^{\oplus 2} & \mathbb{Z}_2^{\oplus 2} & \mathbb{Z}_2^{\oplus 3} & \mathbb{Z}_2^{\oplus 3} & * \\ 0 & \mathbb{Z} & & \mathbb{Z}_2^{\oplus 2} & & \mathbb{Z}^{\oplus 3} & \mathbb{Z}_2^{\oplus 3} & * \\ \hline & 0 & 1 & 2 & 3 & 4 & 5 & 6 \end{array} \longrightarrow \begin{array}{c|c} 6 & ? \\ 5 & ? \\ 4 & ? \\ 3 & ? \\ 2 & \mathbb{Z}_2^{\oplus 2} \\ 1 & \\ 0 & \\ \hline & \end{array} \tag{A.7}$$

Based on our belief, $\boxed{d^2 : E^2_{4,0} \to E^2_{2,1}}$ and $\boxed{d^2 : E^2_{4,1} \to E^2_{2,2}}$ should be a dual of

$$\begin{aligned} Sq^2 w_2 &= (w_2)^2 \\ Sq^2 w'_2 &= (w'_2)^2 \end{aligned} \tag{A.8}$$

and also $\boxed{d^2 : E^2_{5,0} \to E^2_{3,1}}$ and $\boxed{d^2 : E^2_{5,1} \to E^2_{3,2}}$ should be a dual of

$$\begin{aligned} Sq^2 w_3 &= w_2 w_3 \\ Sq^2 w'_3 &= w'_2 w'_3 \end{aligned} \tag{A.9}$$

and finally $\boxed{d^2 : E^2_{6,0} \to E^2_{4,1}}$ should be a dual of

$$Sq^2(w_2 w_2') \quad = \quad w_3 w_3' + (w_2)^2 w_2' + w_2 (w_2')^2 \tag{A.10}$$

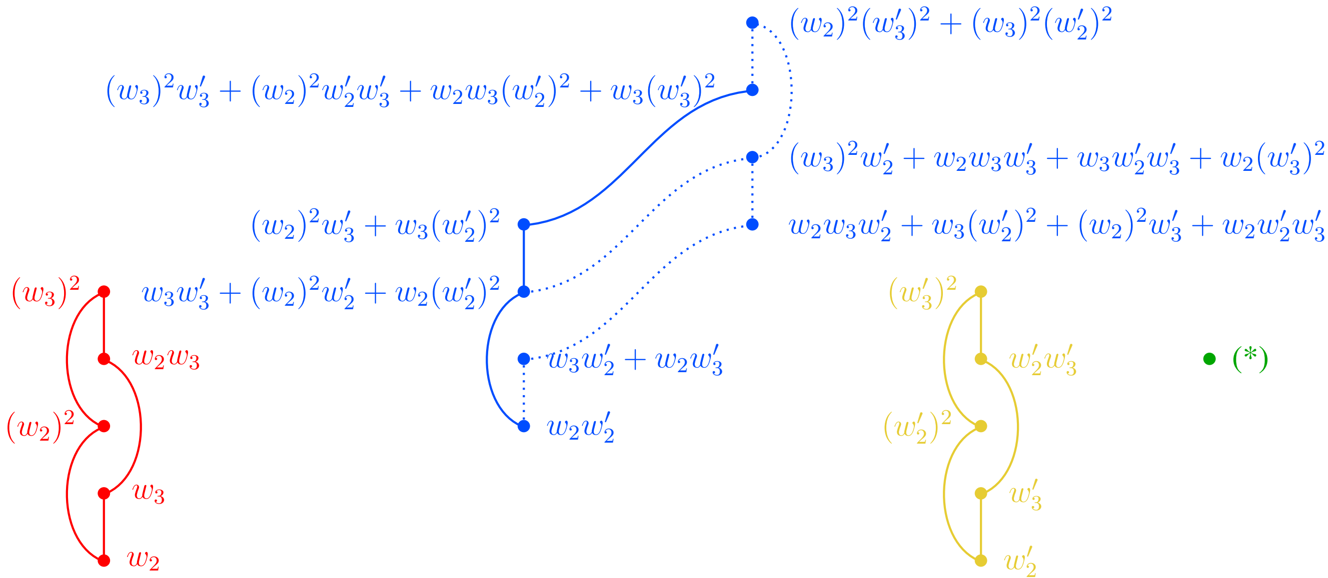
then the would-be-E_3-page is given by

6							
5							
4	$\mathbb{Z}$		$*$		$*$	$*$	$*$
3							
2	$\mathbb{Z}_2$				$*$	$*$	$*$
1	$\mathbb{Z}_2$					$*$	$*$
0	$\mathbb{Z}$		$\mathbb{Z}_2^{\oplus 2}$		$\mathbb{Z}^{\oplus 3}$	$\mathbb{Z}_2$	$*$
	0	1	2	3	4	5	6

(A.11)

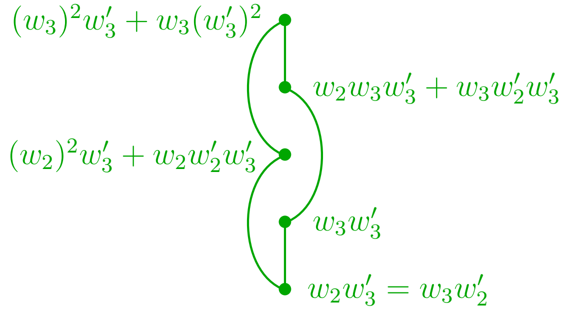
A.3 Adams SS

According to our naive guess, the module $\widetilde{H}^*(B\left(\frac{SO(4)\times SU(2)}{\mathbb{Z}_2}\right); \mathbb{Z}_2)_{\le 5}$ consists of



(A.12)

To be consistent with the AHSS computation, it seems that $w_3 w_2' + w_2 w_3'$ should be modded out (is it an obvious consequence of the transgression in LSSS...?) and the remaing part (*) turns out to be

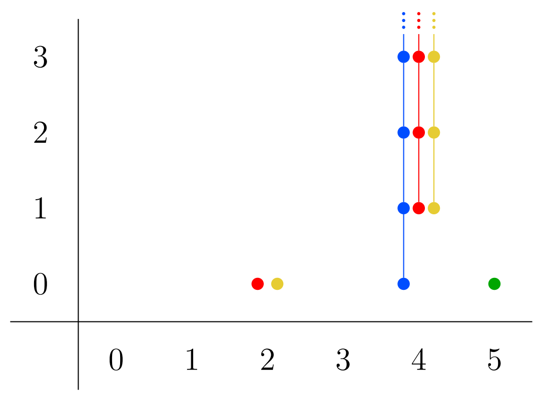


(A.13)

and therefore one concludes

$$\widetilde{H}^*(B\left(\frac{SO(4)\times SU(2)}{\mathbb{Z}_2}\right);\mathbb{Z}_2)_{\leq 5} = J[2] \oplus J[2] \oplus \mathcal{A}_1/\!/\mathcal{E}_0[4] \oplus J[5]. \tag{A.14}$$

This leads to the following Adams chart:



and it indeed seems to be compatible with the AHSS computation. If the above argument (and beliefs) is correct, then the anomaly should be captured by

$$w_2 w_3' \ (= w_3 w_2'). \tag{A.15}$$

A.4 Seiberg dual

The dual theory is $SO(2)$ gauge theory with $N_f = 2$ flavors, and the fermions are charged under

$$\frac{SO(2)\times SU(2)}{\mathbb{Z}_2} = U(2) \tag{A.16}$$

Its spin bordism is known, and the relevant group turns out to be trivial:

$$\Omega_5^{\mathrm{spin}}\big(BU(2)\big) = 0. \tag{A.17}$$

This means there is no anomaly for fermions on the dual side, and thus the $B\beta E$ anomaly cannot be canceled by fermions.

B 't Hooft-Polyakov monopole argument

For an $SU(2)$ gauge theory Higgssed by an adjoint ($\mathbf{3}$, spin-1, isovector) scalar, the gauge group is broken down to $U(1)$, and correspondingly it accommodates topological solitons (monopoles):

$$\pi_2\left(\frac{SU(2)}{U(1)}\right) = \mathbb{Z}.$$

In the presence of (additional) fermions, this monopole might acquire non-trivial charge under spacetime-Lorentz or flavor symmetries, depending on the representation of the fermions under $SU(2)$ gauge symmetry:

fermion gauge rep.	number of zero-modes	spin of zero-modes	spin of monopole
$\mathbf{2}$	1	0	0
$\mathbf{3}$	2	$\frac{1}{2}$	
$\mathbf{4}$	4	$0 \oplus 1$	$\frac{1}{2}$

The numbers of zero-modes can be computed from the Callias index theorem [5].

According to [8], there is $w_2(TM_4)\beta w_2(SU(2)_{\text{gauge}})$ anomaly for a fermion in $\mathbf{4}$ charged under

$$\frac{Spin(4)_{\text{spacetime}} \times SU(2)_{\text{gauge}}}{\mathbb{Z}_2},$$

which incarnates in the IR as an ill-definition of the effective interaction $w_2(TM)c_1(U(1)_{\text{gauge}})$, emerging after integrating out the fermion which obtained mass through Yukawa coupling. This effective interaction term should arise in order to make the monopole a fermion (*i.e.* spinor representation of $Spin(4)_{\text{spacetime}}$), but is not well-defined without a trivialization of $w_2(TM_4)$ or equivalently a spin structure.

This situation looks quite similar to our problem where the fermions are charged under

$$\frac{SO(4)_{\text{gauge}} \times SU(2)_{\text{flavor}}}{\mathbb{Z}_2}.$$

Since fermions are in the fundamental representation of the flavor symmetry, breaking $SU(2)_{\text{flavor}}$ to $U(1)_{\text{flavor}}$ gives rise to a monopole in the spinor representation this time of the $SO(4)_{\text{gauge}}$ [7]. Therefore by the same logic, one can deduce that there should be $w_2(SO(4)_{\text{gauge}})\beta w_2(SU(2)_{\text{flavor}})$ anomaly in the first place, as desired.

Also, one should be able to generalize this whole argument to the case of fermions charged under

$$\frac{SO(2n_c)_{\text{gauge}} \times SU(2n_f)_{\text{flavor}}}{\mathbb{Z}_2}$$

by breaking $SU(2n_f)_{\text{flavor}}$ to $SO(2n_f)_{\text{flavor}}$, where we have monopoles characterized by

$$\pi_2\left(\frac{SU(2n_f)}{SO(2n_f)}\right) = \begin{cases} \mathbb{Z}_2 & (n_f \geq 2) \\ \mathbb{Z} & (n_f = 1) \end{cases}.$$

C misc

C.1 $\mathbb{Z}_4$ 1-form symmetry

According to [9, Appendix C.3] and [10, Eq. (6.3)], it seems that we have

$$E^2_{p,q} = H_p\big(K(\mathbb{Z}_4,2);\Omega^{\text{spin}}_q\big) \qquad \widetilde{\Omega}^{\text{spin}}_{p+q}(K(\mathbb{Z}_4,2))$$

$q \backslash p$	0	1	2	3	4	5	6
6							
5							
4	$\mathbb{Z}$		$*$		$*$	$*$	$*$
3							
2	$\mathbb{Z}_2$		$\mathbb{Z}_2$	$\mathbb{Z}_2$	$*$	$*$	$*$
1	$\mathbb{Z}_2$		$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$*$
0	$\mathbb{Z}$		$\mathbb{Z}_4$		$\mathbb{Z}_8$	$\mathbb{Z}_2$	$*$

$\longrightarrow$

n	
6	$*$
5	
4	$\mathbb{Z}_4$
3	
2	$\mathbb{Z}_4$
1	
0	

(C.1)

The corresponding invariant in 4d is simply

$$\exp(2\pi i \frac{p}{4}\int \frac{1}{2}\mathfrak{P}(a)) \tag{C.2}$$

where $\mathfrak{P} : H^2(-,\mathbb{Z}_4) \to H^4(-,\mathbb{Z}_8)$ is the Pontryagin square, which is even mod 8 on a spin manifold.

C.2 $\mathbb{Z}_2 \times \mathbb{Z}_2$ 1-form symmetry

Exploiting the fact that $K(\mathbb{Z}_2\times\mathbb{Z}_2,2) = K(\mathbb{Z}_2,2)\times K(\mathbb{Z}_2,2)$, it seems that we have

$$E^2_{p,q} = H_p\big(K(\mathbb{Z}_2\times\mathbb{Z}_2,2);\Omega^{\text{spin}}_q\big) \qquad \widetilde{\Omega}^{\text{spin}}_{p+q}(K(\mathbb{Z}_2\times\mathbb{Z}_2,2))$$

$q \backslash p$	0	1	2	3	4	5	6
6							
5							
4	$\mathbb{Z}$		$*$		$*$	$*$	$*$
3							
2	$\mathbb{Z}_2$		$\mathbb{Z}_2^{\oplus 2}$	$\mathbb{Z}_2^{\oplus 2}$	$*$	$*$	$*$
1	$\mathbb{Z}_2$		$\mathbb{Z}_2^{\oplus 2}$	$\mathbb{Z}_2^{\oplus 2}$	$\mathbb{Z}_2^{\oplus 3}$	$\mathbb{Z}_2^{\oplus 6}$	$*$
0	$\mathbb{Z}$		$\mathbb{Z}_2^{\oplus 2}$		$\mathbb{Z}_4^{\oplus 2}\oplus\mathbb{Z}_2$	$\mathbb{Z}_2^{\oplus 3}$	$*$

$\longrightarrow$

n	
6	$*$
5	$\mathbb{Z}_2$
4	$\mathbb{Z}_2^{\oplus 3}$
3	
2	$\mathbb{Z}_2^{\oplus 2}$
1	
0	

(C.3)

The $\mathbb{Z}$ homology of $K(\mathbb{Z}_2,2)$ is again read off from [9], while the $\mathbb{Z}_2$ (co)homology is known [11] to be

$$H^*(K(\mathbb{Z}_2,2);\mathbb{Z}_2) = \mathbb{Z}_2[x_2, Sq^1x_2, Sq^2Sq^1x_2,\cdots].$$

The corresponding bordism invariants in 4d are $\mathfrak{P}(a)/2$, ab, $\mathfrak{P}(b)/2$, and the one in 5d is $a\beta b$.

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