Define $f(\mathbf{\Sigma}) := L_\alpha(\mathbf{\Sigma}) + \lambda \|\mathbf{\Sigma}\|_{1,\text{off}}$, and consider the following optimization problem:

$$\max_{f(\widehat{\mathbf{\Sigma}}^+) \le f(\mathbf{\Sigma}) \le f(\widehat{\mathbf{\Sigma}}^+) + \tau d} \|\mathbf{\Sigma} - \mathbf{\Sigma}^*\|_\infty + \frac{4}{\alpha}\left(\tau d + f(\widehat{\mathbf{\Sigma}}^+) - f(\mathbf{\Sigma})\right) \tag{1}$$

where $\widehat{\mathbf{\Sigma}}^+ \in \arg\min_{\mathbf{\Sigma} \succeq \mathbf{0}} f(\mathbf{\Sigma})$ is the positive semi-definite estimator.

Lemma 1. *Let $\mathbf{\Sigma}$ be an optimal solution to (1) with $f(\mathbf{\Sigma}) < f(\widehat{\mathbf{\Sigma}}^+) + \tau d$, then $\|\mathbf{\Sigma} - \mathbf{\Sigma}^*\|_\infty \lesssim \alpha$ with high probability.*

Proof: Let $(k,l) \in \arg\max_{k',l'} |\Sigma_{k'l'} - \Sigma^*_{k'l'}|$, i.e. $|\Sigma_{kl} - \Sigma^*_{kl}| = \|\mathbf{\Sigma} - \mathbf{\Sigma}^*\|_\infty$. For the sake of contradictory, assume that $\Delta_{kl} := \Sigma_{kl} - \Sigma^*_{kl} > 3\alpha/2$. We can easily see that

$$\partial f_{kl}(\Sigma_{kl}) = \text{Conv}\left\{1 \pm \lambda - \frac{4}{\alpha} \cdot \frac{1}{n}\sum_{i=1}^n \rho'_\alpha(\Delta_{kl} + \Sigma^*_{kl} - x_{ik}x_{il})\right\} \tag{2}$$

Similar to our argument in Proposition 5, we have

$$(1/n)\sum_{i=1}^n 1\left(|\Sigma^*_{kl} - x_{ik}x_{il}| \le \alpha/2\right) > 2/3$$

with high probability. Let $\mathcal{G}_{kl} := \{i \in [n] : |\Sigma^*_{kl} - x_{ik}x_{il}| \le \alpha/2\}$.

$$\begin{aligned}&\frac{1}{n}\sum_{i=1}^n \rho'_\alpha(\Delta_{kl} + \Sigma^*_{kl} - x_{ik}x_{il})\\ &= \frac{1}{n}\sum_{i \in \mathcal{G}_{kl}} \rho'_\alpha(\Delta_{kl} + \Sigma^*_{kl} - x_{ik}x_{il}) + \frac{1}{n}\sum_{i \notin \mathcal{G}_{kl}} \rho'_\alpha(\Delta_{kl} + \Sigma^*_{kl} - x_{ik}x_{il})\\ &> 2\alpha/3 - \alpha/3 = \alpha/3\end{aligned}$$

where the last inequality follows because for $i \in \mathcal{G}_{kl}$, $\Delta_{kl} + \Sigma^*_{kl} - x_{ik}x_{il} \ge 3\alpha/2 - \alpha/2 = \alpha$. Combining this with (2), we have

$$\partial f_{kl}(\Sigma_{kl}) < 1 + \lambda - \frac{4}{\alpha} \cdot \alpha/3 = \lambda - 1/3 < 0. \tag{3}$$

The last inequality follows with $\lambda \asymp \sqrt{\log d/n}$ given $n \gtrsim \log d$, which contradicts $0 \in \partial f_{kl}$, i.e. the optimality condition. Similarly, the case for $\Delta_{kl} := \Sigma_{kl} - \Sigma^*_{kl} < -3\alpha/2$ also leads to contradictory. ∎

Lemma 2. *Any optimal solution $\mathbf{\Sigma}$ to (1) with $f(\mathbf{\Sigma}) = f(\widehat{\mathbf{\Sigma}}^+) + \tau d$ satisfies $\|\mathbf{\Sigma} - \mathbf{\Sigma}^*\|_\infty \lesssim \alpha$ with high probability.*

Proof: Let $(k,l) \in \arg\max_{k',l'} |\Sigma_{k'l'} - \Sigma^*_{k'l'}|$, i.e. $|\Sigma_{kl} - \Sigma^*_{kl}| = \|\mathbf{\Sigma} - \mathbf{\Sigma}^*\|_\infty$. For the sake of contradictory, assume that $\Delta_{kl} := \Sigma_{kl} - \Sigma^*_{kl} > 3\alpha/2$. Let

$$\partial f^-_{kl}(\Sigma_{kl}) := \lim_{h \to 0^+} \frac{f_{kl}(\Sigma_{kl} - h) - f_{kl}(\Sigma_{kl})}{h}. \tag{4}$$

Then

$$\partial f^-_{kl}(\Sigma_{kl}) = -1 \pm \lambda + \frac{4}{\alpha} \cdot \frac{1}{n}\sum_{i=1}^n \rho'_\alpha(\Delta_{kl} + \Sigma^*_{kl} - x_{ik}x_{il}) \tag{5}$$

and given $\mathbf{\Sigma}$ is an optimal solution to (1), we must have $f(\mathbf{\Sigma} - h\boldsymbol{E_{kl}}) \le f(\mathbf{\Sigma})$ for positive and sufficiently small h. Hence (4) implies $\partial f^-_{kl}(\Sigma_{kl}) \le 0$. However, with the same argument as in Lemma 1, $\partial f^-_{kl}(\Sigma_{kl}) > 1/3 - \lambda > 0$, which is a contradistory. ∎

In the following Theorem, we combine Lemma 1 and Lemma 2 to conclude that $\left\|\widetilde{\mathbf{\Sigma}} - \mathbf{\Sigma}^*\right\|_\infty \lesssim \alpha$.

Theorem 3. $\left\|\widetilde{\mathbf{\Sigma}} - \mathbf{\Sigma}^*\right\|_\infty \le 3\alpha/2 + 4\tau d/\alpha$ *with high probability. By taking $0 < \tau \lesssim \alpha^2/d$, we have* $\left\|\widetilde{\mathbf{\Sigma}} - \mathbf{\Sigma}^*\right\|_\infty \lesssim \alpha$.

Proof: Recall that

$$\widehat{\mathbf{\Sigma}}^+ \in \arg\min_{\mathbf{\Sigma} \succeq \mathbf{0}} f(\mathbf{\Sigma}) \tag{6}$$

and that

$$\widetilde{\mathbf{\Sigma}} \in \arg\min_{\mathbf{\Sigma} \succ \mathbf{0}} f(\mathbf{\Sigma}) - \tau \log\det \Sigma. \tag{7}$$

We can view (7) as the log-barrier relaxation of (6), and $\widetilde{\mathbf{\Sigma}}$ as a point on the central path towards $\widehat{\mathbf{\Sigma}}^+$. Then it follows from the convergence of central path that

$$f(\widetilde{\mathbf{\Sigma}}) \le f(\widehat{\mathbf{\Sigma}}^+) + \tau d. \tag{8}$$

Hence, by combining this with Lemma 1 and Lemma 2,

$$\begin{aligned}\left\|\widetilde{\mathbf{\Sigma}} - \mathbf{\Sigma}^*\right\|_\infty + \frac{4}{\alpha}\left(\tau d + f(\widehat{\mathbf{\Sigma}}^+) - f(\widetilde{\mathbf{\Sigma}})\right) &\le 3\alpha/2 + \frac{4}{\alpha}\left(\tau d + f(\widehat{\mathbf{\Sigma}}^+) - \right.\\ &\le 3\alpha/2 + 4\tau d/\alpha.\end{aligned}$$

where $\mathbf{\Sigma}_\tau$ is any optimal solution to (1), and the last inequality follows from $f(\widehat{\mathbf{\Sigma}}^+) \le f(\mathbf{\Sigma}_\tau)$. Finally,

$$\left\|\widetilde{\mathbf{\Sigma}} - \mathbf{\Sigma}^*\right\|_\infty \le \left\|\widetilde{\mathbf{\Sigma}} - \mathbf{\Sigma}^*\right\|_\infty + \frac{4}{\alpha}\left(\tau d + f(\widehat{\mathbf{\Sigma}}^+) - f(\widetilde{\mathbf{\Sigma}})\right)$$

because of (8). ∎

With $\left\|\widetilde{\mathbf{\Sigma}} - \mathbf{\Sigma}^*\right\|_\infty \lesssim \alpha$, the rest of the proof follows immediately, and $\widetilde{\mathbf{\Sigma}}$ enjoys the same convergence rate as $\widehat{\mathbf{\Sigma}}^+$, which is minimax optimal.