Problem Set 4 Solution

Instructor : Yujung Hwang *

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Question 1.

(a)

$$
\begin{aligned}
E\left[\Delta u_{it}(\Delta u_{it-1}+\Delta u_{it}+\Delta u_{it+1})\right] &= E\left[(\zeta_{it}+m_{it})(\zeta_{it-1}+\zeta_{it}+\zeta_{it+1}-m_{it-1}+m_{it+1})\right]=\sigma_\zeta^2 \\
E\left[\Delta u_{it}\Delta u_{it-1}\right] &= E\left[(\zeta_{it}+m_{it}-m_{it-1})(\zeta_{it-1}+m_{it-1}-m_{it-2})\right]=-\sigma_m^2
\end{aligned}
$$

(b) Note that α_{it} is a random variable.

$$
\begin{aligned}
E\left[\Delta u_{it}|L_{it}=1,L_{it-1}=1\right] &= \sigma_{\zeta\eta}E\left[\frac{\phi(\alpha_{it})}{1-\Phi(\alpha_{it})}\right] \\
E\left[\Delta u_{it}(\Delta u_{it-1}+\Delta u_{it}+\Delta u_{it+1})|L_{it-2}=1,L_{it-1}=1,L_{it}=1,L_{it+1}=1\right] &= \sigma_\zeta^2+\sigma_{\zeta\eta}^2E\left[\frac{\phi(\alpha_{it})}{1-\Phi(\alpha_{it})}\alpha_{it}\right] \\
E\left[\Delta u_{it}^2|L_{it}=1,L_{it-1}=1\right] &= \sigma_\zeta^2+\sigma_{\zeta\eta}^2E\left[\frac{\phi(\alpha_{it})}{1-\Phi(\alpha_{it})}\alpha_{it}\right]+2\sigma_m^2
\end{aligned}
$$

To see this,

$$
\begin{aligned}
E\left[\Delta u_{it}|L_{it}=1,L_{it-1}=1\right] &= E\left[E\left[\zeta_{it}+m_{it}-m_{it-1}|\eta_{it}>\alpha_{it},\eta_{it-1}>\alpha_{it-1}\right]\right] \\
&= E\left[E\left[\zeta_{it}|\eta_{it}>\alpha_{it}\right]\right] \\
&= \sigma_{\zeta\eta}E\left[\frac{\phi(\alpha_{it})}{1-\Phi(\alpha_{it})}\right] \; (\because \text{truncated bivariate normal distribution formula})
\end{aligned}
$$

$$
\begin{aligned}
&E\left[\Delta u_{it}(\Delta u_{it-1}+\Delta u_{it}+\Delta u_{it+1})|L_{it-2}=1,L_{it-1}=1,L_{it}=1,L_{it+1}=1\right] \\
&= E\left[E\left[\zeta_{it}^2|\eta_{it}>\alpha_{it}\right]\right] \\
&= \sigma_\zeta^2E\left[E\left[\tilde{\zeta}_{it}^2|\eta_{it}>\alpha_{it}\right]\right] \quad (\because \tilde{\zeta}_{it}\equiv\zeta_{it}/\sigma_\zeta) \\
&= \sigma_\zeta^2E\left[E\left[E\left[\tilde{\zeta}_{it}^2|\eta_{it}\right]|\eta_{it}>\alpha_{it}\right]\right] \\
&= \sigma_\zeta^2E\left[E\left[Var(\tilde{\zeta}_{it}|\eta_{it})+E\left[\tilde{\zeta}_{it}|\eta_{it}\right]^2|\eta_{it}>\alpha_{it}\right]\right] \\
&= \sigma_\zeta^2E\left[E\left[1-\rho_{\zeta\eta}^2+\rho_{\zeta\eta}^2\eta_{it}^2|\eta_{it}>\alpha_{it}\right]\right] \\
&= \sigma_\zeta^2\left(1-\rho_{\zeta\eta}^2+\rho_{\zeta\eta}^2E\left[E\left[\eta_{it}^2|\eta_{it}>\alpha_{it}\right]\right]\right)
\end{aligned}
$$

*yujungghwang@gmail.com

$$
\begin{aligned}
&= \sigma_\zeta^2(1-\rho_{\zeta\eta}^2+\rho_{\zeta\eta}^2(E\left[\alpha_{it}\frac{\phi(\alpha_{it})}{1-\Phi(\alpha_{it})}\right]+1))\\
&= \sigma_\zeta^2(1+\rho_{\zeta\eta}^2 E\left[\alpha_{it}\frac{\phi(\alpha_{it})}{1-\Phi(\alpha_{it})}\right])
\end{aligned}
$$

$$
\begin{aligned}
&E\left[\Delta u_{it}^2|L_{it}=1,L_{it-1}=1\right]\\
&= E\left[E\left[(\zeta_{it}+m_{it}-m_{it-1})^2|\eta_{it}>\alpha_{it},\eta_{it-1}>\alpha_{it-1}\right]\right]\\
&= E\left[E\left[\zeta_{it}^2|\eta_{it}>\alpha_{it}\right]\right]+2\sigma_m^2\\
&= \sigma_\zeta^2(1+\rho_{\zeta\eta}^2 E\left[\alpha_{it}\frac{\phi(\alpha_{it})}{1-\Phi(\alpha_{it})}\right])+2\sigma_m^2
\end{aligned}
$$

Question 2.

(a)

```
workeq <- glm(work ~ age + age2 + factor(educ) + noveliv, family = binomial(link = "probit"), data
# (a)
alphahat <- -cbind(rep(1,nobs),data$age,data$age^2,(data$educ=="College_Graduate"),
                   (data$educ=="Postgraduate"),(data$educ=="Some_College"),
                   data$noveliv)%*%workeq$coefficients
lambdahat <- dnorm(alphahat)/(1-pnorm(alphahat))
```

(b)

$$\log w_{it} - \log w_{it-1} = \Delta Z_{it}'\beta + \zeta_{it} + m_{it} + m_{it-1} \tag{0.1}$$

```
reg1  <- lm(d1lnwage ~ d1age2 + lambdahat)
# compute residual and find uhat
duhat <- d1lnwage - reg1$coefficients[1] - reg1$coefficients[2]*d1age2
```

(c)

```
> paramest
[1] 0.03119798 0.03159257 0.01941181
> stderr
[1] 0.0010169239 0.0125732414 0.0006523193
```