Lecture 34

Fixed vs Random Effects

STAT 512
Spring 2011

Background Reading
KNNL: Chapter 25

Topic Overview

- Random vs. Fixed Effects
- Using Expected Mean Squares (EMS) to obtain appropriate tests in a Random or Mixed Effects Model

Fixed vs. Random Effects

- So far we have considered only ***fixed effect models*** in which the levels of each factor were fixed in advance of the experiment and we were interested in differences in response among <u>those specific levels</u>.
- A ***random effects model*** considers factors for which the factor levels are meant to be representative of a general population of possible levels.

Fixed vs. Random Effects (2)

- For a ***random effect,*** we are interested in whether that factor has a significant effect in explaining the response, but only in a general way.
- If we have both fixed and random effects, we call it a "mixed effects model".
- To include random effects in SAS, either use the MIXED procedure, or use the GLM procedure with a RANDOM statement.

Fixed vs. Random Effects (2)

- In some situations it is clear from the experiment whether an effect is fixed or random. However there are also situations in which calling an effect fixed or random depends on your point of view, and on your interpretation and understanding. So sometimes it is a personal choice. This should become more clear with some examples.

Random Effects Model

- This model is also called ANOVA II (or variance components model).
- Here is the one-way model:

$$Y_{ij} = \mu + \alpha_i + \varepsilon_{ij}$$

$$\left.\begin{aligned} \alpha_i &\sim N\left(0, \sigma_A^2\right) \\ \varepsilon_{ij} &\sim N\left(0, \sigma^2\right) \end{aligned}\right\} independent$$

$$Y_{ij} \sim N\left(\mu, \sigma_A^2 + \sigma^2\right)$$

Random Effects Model (2)

Now the cell means $\mu_i = \mu + \alpha_i$ are random variables with a common mean. The question of "are they all the same" can now be addressed by considering whether the *variance* of their distribution is zero. Of course, the estimated means will likely be at least slightly different from each other; the question is whether the difference can be explained by error variance σ^2 alone.

Two sources of variation

- Observations with the same i are <u>dependent</u> and their covariance is σ_A^2.
- The ***components of variance*** are σ_A^2 and σ^2. We want to get an idea of the relative magnitudes of these variance components.
- We often measure this by the ***intraclass correlation coefficient***:

$$\frac{\sigma_A^2}{\sigma_A^2 + \sigma^2}$$

(correlation between two obs. with the same i)

Parameters / ANOVA

- The cell means μ_{ij} are now random variables, not parameters. The important parameters are the variances σ_A^2 and σ^2
- The terms and layout of the ANOVA table are the <u>same</u> as what we used for the fixed effects model
- The expected mean squares (EMS) are <u>different</u> because of the additional random effects, so we will estimate parameters in a new way.

Parameters / ANOVA (2)

- $E(MSE) = \sigma^2$ as usual. So we use MSE to estimate σ^2
- For fixed effects, $E(MSA) = Q(A) + \sigma^2$ where Q(A) involves a l.c. of the α_i.
- For random effects it becomes $E(MSA) = n\sigma_A^2 + \sigma^2$. From this you can calculate that the estimate for σ_A^2 should be $(MSA - MSE)/n$.

Hypotheses Testing

- Our null hypothesis is that there is no effect of factor A. Under the random effects model, it takes a different form:

$$H_0 : \sigma_A^2 = 0$$

$$H_a : \sigma_A^2 \neq 0$$

- For analysis of a single factor, the test statistic is still F = MSA/MSE with (r-1) and r(n-1) df. It WILL NOT remain the same for multiple factors.

Example

- KNNL Table 25.1 (page 1036)
- SAS code: applicant.sas
- Y is the rating of a job applicant
- Factor A represents five different personnel interviewers (officers), $r = 5$ levels
- n = 4 *different* applicants were randomly chosen and interviewed by each interviewer (i.e. 20 applicants); applicant is *not* a factor since no applicant was interviewed more than once

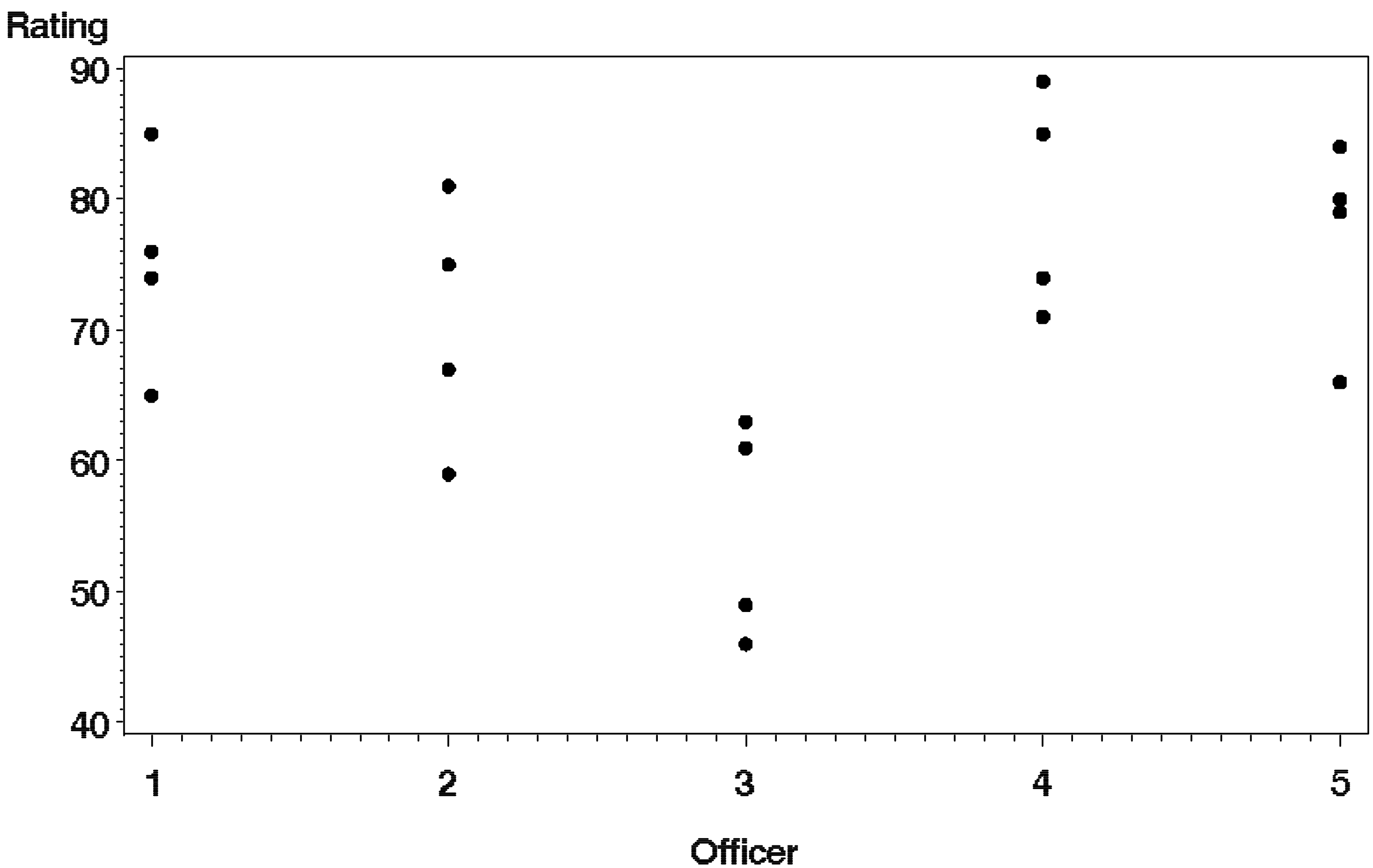
Example (2)

- The interviewers were <u>selected at random</u> from the pool of interviewers and had applicants randomly assigned.
- Here we are not so interested in the differences between the five interviewers that happened to be picked (i.e. does Joe give higher ratings than Fred, is there a difference between Ethel and Bob). Rather we are interested in quantifying and accounting for the effect of "interviewer" in general.

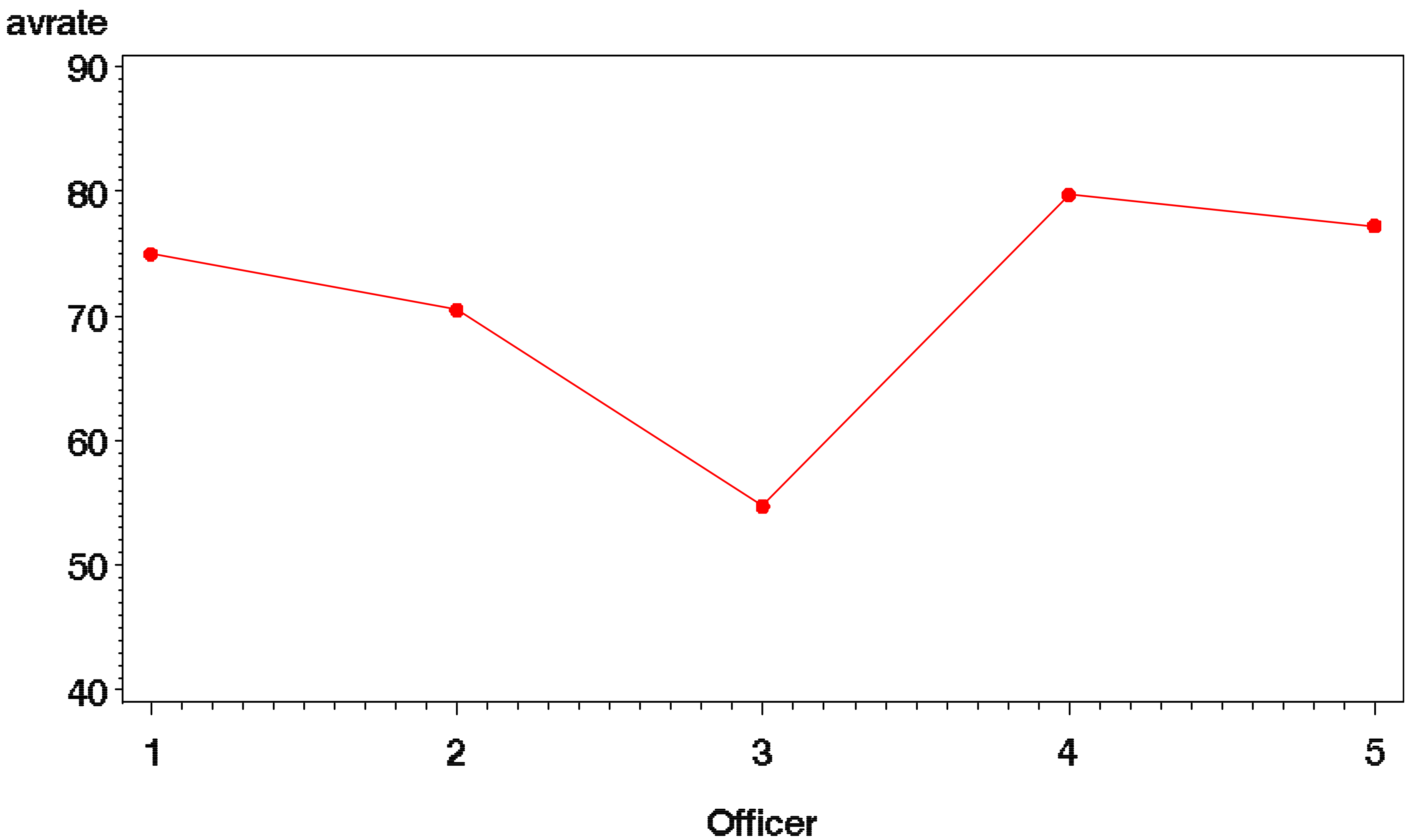
Example (3)

- There are other interviewers in the "population" and we want to make inference about them too.
- Another way to say this is that with fixed effects we are primarily interested in the ***means*** of the factor levels (and differences between them). With random effects, we are primarily interested in their ***variances***.

Plot of the Data



Plot of the means



SAS Coding

```
proc glm data=a1;
   class officer;
   model rating=officer;
   random officer /test;
```

- Random statement is used and /test will perform appropriate tests (and produce EMS)

Output

Source	DF	SS	MS	F	Pr > F
Model	4	1579.70	394.925	5.39	0.0068
Error	15	1099.25	73.283		
Total	19	2678.95			

Source	Type III Expected Mean Square
Officer	Var(Error) + 4 Var(Officer)

Source	DF	SS	MS	F	Pr > F
Officer	4	1580	395	5.39	0.0068
Error: MS(Error)	15	1099	73.3		

Output (2)

- SAS gives us the EMS (note $n = 4$ replicates): $E(MSA) = \sigma^2 + 4\sigma_A^2$
- SAS provides the appropriate test for each effect and tells you what "error term" is being used in testing. Note for this example it is as usual since there is only one factor.

Variance Components

- VARCOMP procedure can be used to obtain the variance components:

```
proc varcomp data=a1;
    class officer;
    model rating=officer;
```

- Obtain point estimates of the two variances (could construct an estimate for the ICC)

Variance Component	rating
Var(officer)	80.41042
Var(Error)	73.28333

Variance Components (2)

- SAS is providing $\hat{\sigma}^2 = 73.2833$. Note that this is simply the MSE.
- SAS also indicates $\hat{\sigma}^2_{officer} = 80.4104$. We could calculate this from the mean squares:

$$\frac{(MSA - MSE)}{n} = \frac{(394.925 - 73.283)}{4}$$

- VARCOMP procedure is somewhat limited (doesn't provide ICC or SE's)

ICC

- The estimated intraclass correlation coefficient is
$$\frac{\hat{\sigma}_A^2}{\hat{\sigma}_A^2 + \hat{\sigma}^2} = \frac{80.4104}{80.4104 + 73.2833} = 0.5232$$

- About half the variance in rating is explained by interviewer.

MIXED Procedure

- Better than GLM / VARCOMP, but also somewhat more complex to use. Advantage is that it has options specifically for mixed models

```
proc mixed data=a1 cl;
   class officer;
   model rating=;
   random officer /vcorr;
```

- Note: random effects are included ONLY in the random statement; fixed effects in the model statement. Different from GLM!

Mixed Procedure

- The `cl` option after `data=a1` asks for the confidence limits (on the variances).
- VCORR option provides the intraclass correlation coefficient.
- Have to watch out for huge amounts of output – in this case there were 5 pages – we'll just go through some of the pieces.

Output

Cov Parm	Estimate	95% CI	
officer	80.41	24.46	1499
Residual	73.28	39.99	175.5

Output from vcorr option (giving the ICC)

Row	Col1	Col2	Col3	Col4
1	1.0000	0.5232	0.5232	0.5232
2	0.5232	1.0000	0.5232	0.5232
3	0.5232	0.5232	1.0000	0.5232
4	0.5232	0.5232	0.5232	1.0000

Notes from Example

- Confidence intervals for variance components are discussed in KNNL (pgs1041-1047)
- In this example, we would like the ICC to be small, indicating that the variance due to the interviewer is small relative to the variance due to applicants. In many other examples, we may want this quantity to be large.
- What we found is that there is a significant effect of personnel officer (interviewer).

Two Random Factors

$$Y_{ijk} = \mu + \alpha_i + \beta_j + (\alpha\beta)_{ij} + \varepsilon_{ijk}$$

$$\alpha_i \sim N\left(0, \sigma_A^2\right)$$

$$\beta_j \sim N\left(0, \sigma_B^2\right)$$

$$(\alpha\beta)_{ij} \sim N\left(0, \sigma_{AB}^2\right)$$

$$\varepsilon_{ij} \sim N\left(0, \sigma^2\right)$$

$$Y_{ij} \sim N\left(\mu, \sigma_A^2 + \sigma_B^2 + \sigma_{AB}^2 + \sigma^2\right)$$

Two Random Factors (2)

- Now the component σ_μ^2 can be divided up into three components – A, B, and AB.
- There are five parameters in this model: $\mu, \sigma_A^2, \sigma_B^2, \sigma_{AB}^2, \sigma^2$.
- Again, the cell means are random variables, not parameters!!!

EMS for Two Random Factors

$$E(MSA) = \sigma^2 + bn\sigma_A^2 + n\sigma_{AB}^2$$

$$E(MSB) = \sigma^2 + an\sigma_B^2 + n\sigma_{AB}^2$$

$$E(MSAB) = \sigma^2 + n\sigma_{AB}^2$$

$$E(MSE) = \sigma^2$$

- Estimates of the variance components can be obtained from these equations or other methods.
- Notice the patterns in the EMS: (these hold for balanced data).

Patterns in EMS

- They all contain σ^2.
- For MSA, also contain any variances with A in subscript; similarly for MSB.
- The coefficient of σ^2 is one; for any other term it is the product of *n* and all letters *not* represented in the subscript. (Can also think of it as the total number of observations at each fixed level of the corresponding subscript – e.g. there are *nb* observations for each level of A)

Hypotheses Testing

- Testing based on EMS (apply null and look for ratio of 1):

$$E(MSA) = \sigma^2 + bn\sigma_A^2 + n\sigma_{AB}^2$$

$$E(MSB) = \sigma^2 + an\sigma_B^2 + n\sigma_{AB}^2$$

$$E(MSAB) = \sigma^2 + n\sigma_{AB}^2$$

$$E(MSE) = \sigma^2$$

- Test Interaction ($H_0 : \sigma_{AB}^2 = 0$) over error
- Test Main Effects ($H_0 : \sigma_A^2 = 0$ and $H_0 : \sigma_B^2 = 0$) over interaction (this is the big difference!)

Hypotheses Testing (Details)

<u>Main Effects</u>

Factor A: $H_0 : \sigma_A^2 = 0$ vs. $H_A : \sigma_A^2 \neq 0$

Test Statistic: F=MSA/MSAB – Denom is different!
DF: (a-1) in num and (a-1)(b-1) in denom

Factor B: $H_0 : \sigma_B^2 = 0$ vs. $H_A : \sigma_B^2 \neq 0$

Test Statistic: F=MSB/MSAB – Denom is different!
DF: (b-1) in num and (a-1)(b-1) in denom

Hypotheses Testing (Details)

Interaction

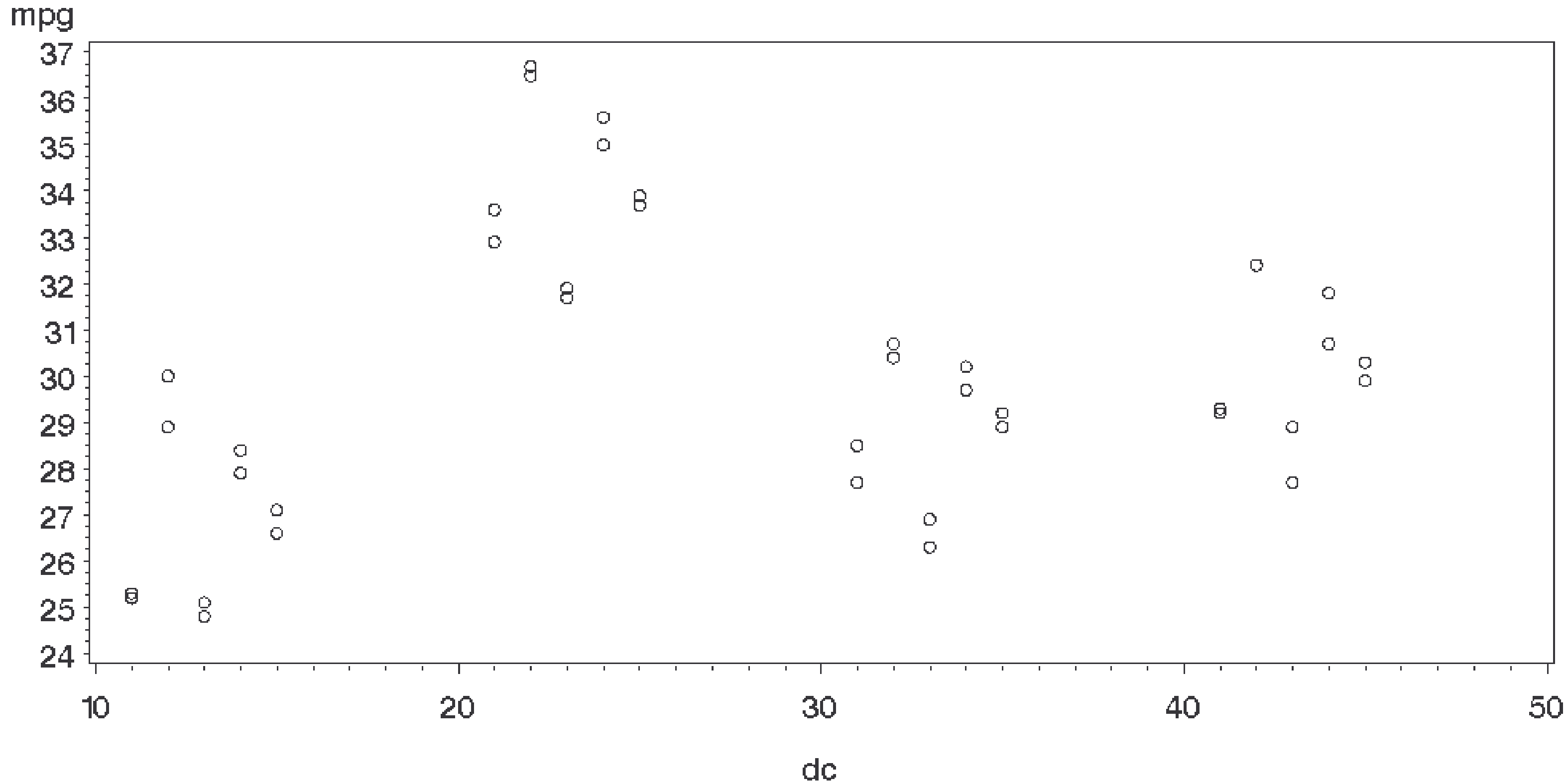
$$H_0 : \sigma^2_{AB} = 0 \text{ vs. } H_A : \sigma^2_{AB} \neq 0$$

Test Statistic: F=MSAB/MSE –
Only for interaction is Denominator the MSE
DF: (a-1)(b-1) in num and ab(n-1) in denom

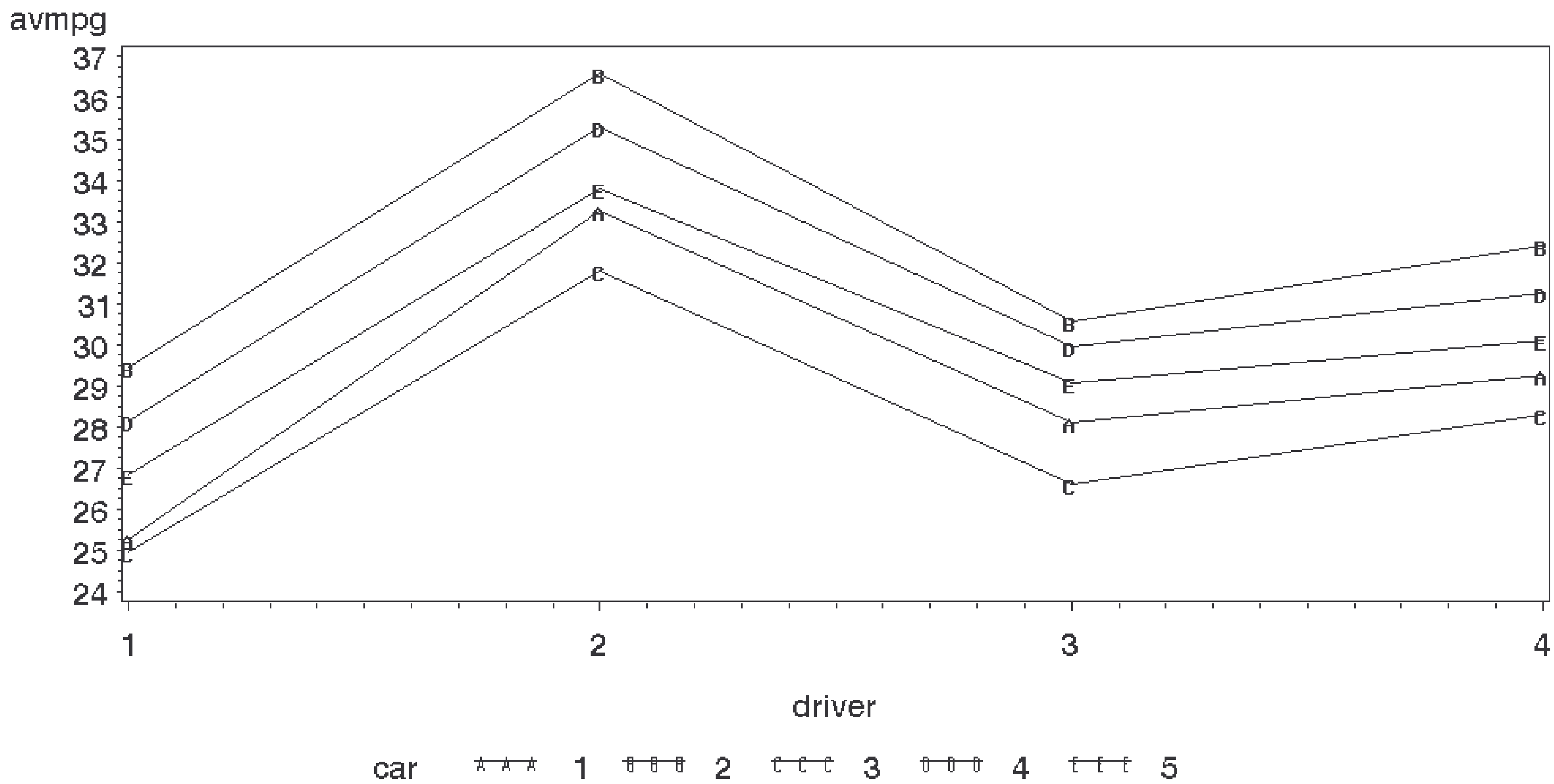
Example

- KNNL 25.15 (pg 1080)
- SAS code: mpg.sas
- Y is fuel efficiency in miles per gallon
- Factor A represents four different drivers, a=4 levels
- Factor B represents five different cars of the same model , b=5
- Each driver drove each car twice over the same 40-mile test course (n = 2)

Plot of the data



Plot of the means



SAS Coding

```
proc glm data=a1;
   class driver car;
   model mpg=driver car driver*car;
   random driver car driver*car/test;
run;
```

Output (1)

Model and error output

Source	DF	Sum of Squares	Mean Square	F Value	Pr > F
Model	19	377.4447500	19.8655132	113.03	<.0001
Error	20	3.5150000	0.1757500		
Corrected Total	39	380.9597500			

Factor effects output

Source	DF	Type I SS	Mean Square	F Value	Pr > F
driver	3	280.2847500	93.4282500	531.60	~~<.0001~~
car	4	94.7135000	23.6783750	134.73	~~<.0001~~
driver*car	12	2.4465000	0.2038750	1.16	0.3715

Random statement output

Source	Type III EMS
driver	Var(Error) + 2 Var(driver*car) + 10 Var(driver)
car	Var(Error) + 2 Var(driver*car) + 8 Var(car)
driver*car	Var(Error) + 2 Var(driver*car)

Output (2)

Note that only the interaction test is valid here: the test for interaction is MSAB/MSE, but the tests for main effects should be MSA/MSAB and MSB/MSAB which are done with the test statement, not / MSE as is done here.

Lesson: just because SAS spits out a P-value, doesn't mean it is for a meaningful test!

Output (3)

<u>Random/test output</u>

```
The GLM Procedure
Tests of Hypotheses for Random Model Analysis of Variance

Dependent Variable: mpg
Source   DF  Type III SS  Mean Square  F Value  Pr > F
driver    3   280.284750    93.428250   458.26  <.0001
car       4    94.713500    23.678375   116.14  <.0001
Error    12     2.446500     0.203875
Error: MS(driver*car)
```

This last line says the denominator of the F tests is MSAB.

```
Source             DF  Type III SS  Mean Square  F Value  Pr > F
driver*car         12     2.446500     0.203875     1.16  0.3715
Error: MS(Error)   20     3.515000     0.175750
```

For the interaction term, the denominator is MSE (which is the same test as was done above)

Output (4)

Proc varcomp

```
proc varcomp data=efficiency;
   class driver car;
   model mpg=driver car driver*car;
```

MIVQUE(0) Estimates

Variance Component	mpg
Var(driver)	9.32244
Var(car)	2.93431
Var(driver*car)	0.01406
Var(Error)	0.17575

Can use Proc Mixed to get CI for variance components.

Upcoming…

- Two-Way Mixed Model
  - One Fixed Effect
  - One Random Effect