36-309/749 Lecture 11: Mixed Models 11/27/2012

1) **Independent Errors Assumption**
   a) "Errors" are deviations of individual observed outcomes from the population mean of all measurements that have the same levels of all of the explanatory variables.

   b) Correlation is between –1 and +1. The correlation between two random variables is equal to the square root of R^2 from simple regression. Zero is uncorrelated.

   c) The assumption of "uncorrelated errors" comes down to the idea that knowing the error (or its estimate, the residual) for one measurement tells us nothing about the error for another measurement.

   d) We often do not have any tools to check this assumption from the data; rather we *think* about the likely nature of any correlation.

   e) In repeated measures analyses and some other analyses that we have not studied, errors are *modeled* as correlated, so the assumption does not apply to these analyses. In between-subjects ANOVA, regression, logistic regression, and the chi-square test of independence, uncorrelated errors is a strong assumption. Fairly mild violations begin to alter the null sampling distributions of the test statistics resulting in incorrect p-values and confidence intervals and/or poor power. (But estimates of population means and slopes are still unbiased, i.e., on average they are correct over multiple experiments.)

2) **General Linear Mixed Models**
   a) General linear mixed models are best thought of as Normal linear models that flexibly model correlation often in the form of clustering or hierarchy. Complete or partial synonyms include hierarchical linear models (HLM), multilevel modeling, random regression models, and growth curve models.

   b) Replacing "general" by "generalized" allows non-normal outcomes (e.g., succeed vs. fail) using a more complex method. Replacing "linear" with "non-linear" allows expressions more complicated that $\beta_0+\beta_1x_1+\ldots+\beta_kx_k$, but this is not commonly needed because the x's can be transformed.

c) An intuitive hierarchical approach. (Note: Covariates are usually centered.)
Example: Repeated test scores over time are modeled as ***individual*** regression lines. U_i's represent "personal" deviations of intercepts/slopes around population averages.

i) $Y_{iT} = A_{0i} + A_{1i}\,T + \varepsilon_{iT}$, for subject *i* at each time. $\varepsilon_i \sim N(0,\sigma^2)$ and uncorrelated.

ii) $A_{0i} = \beta_0 + U_{0i} + \beta_{0A}\,CAge_i + \beta_{0M} Male_i + \beta_{0W} CWeight_i$ [where "C" means centered]

iii) $A_{1i} = \beta_1 + U_{1i} + \beta_{1W} CWeight_i$

iv) $U_{0i} \sim N(0,\tau_0^2)$, $U_{1i} \sim N(0,\tau_1^2)$, covariance$(U_{0i}, U_{1i}) = \tau_{01}$

d) The term "mixed model" derives from a mix of fixed and random effects:

i) Fixed effect factors have levels that would be the same in the next experiment. Parameter count: k-1 (e.g., intercept differences for *k* race categories).

ii) Random effect factors have levels that would be different in the next experiment, and are assumed to come from a Normal distribution with mean zero. Random intercepts and slopes are "one per upper level item". Random effects "induce" correlation. Parameter count: 1 (a variance describing the differences among all *n* subjects)

e) Both categorical and quantitative IVs are allowed. DVs are assumed to be Normally distributed quantitative variables (but discrete quantitative and categorical outcomes are allowed under "generalized linear mixed models").

f) Mixed models are generally an improvement over repeated measures analyses:

i) In addition to spherical (compound symmetry) and unstructured correlation, many other useful correlation structures are allowed.

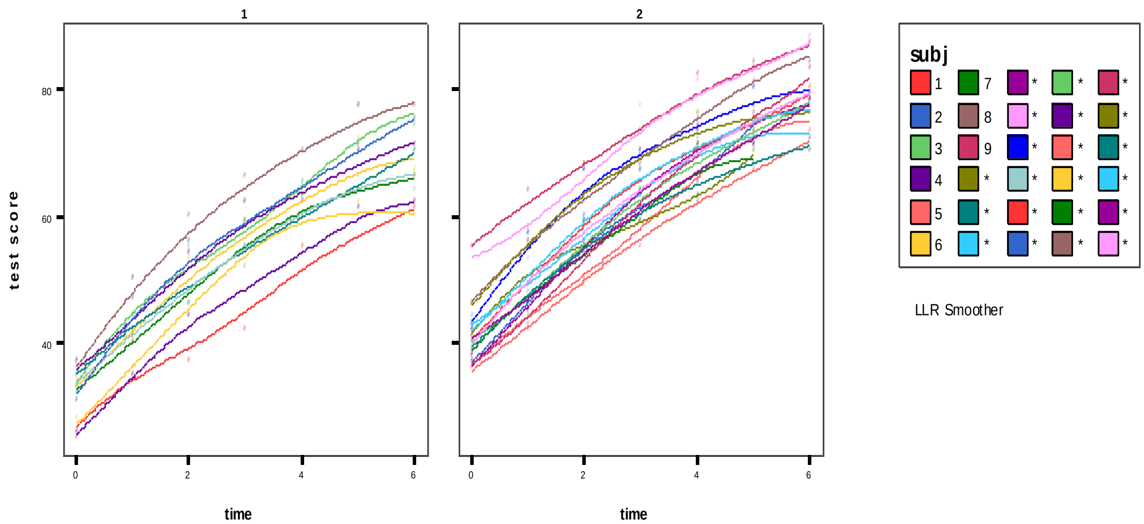
ii) Random intercepts and slopes are often scientifically meaningful.

iii) Unequally and/ inconsistently spaced data are handled correctly.

iv) Missing data is allowed and correctly handled (if missingness is not related to the value of the outcome).

g) **Example 1**: random regression or growth curve modeling for a learning task. Subjects are repeatedly tested for performance on a learned task. Subjects are randomized to one of 2 manuals giving instructions for how to learn the task quickly.

i) EDA:



Fixed quadratic model (i.e., just regression/ANCOVA):

Model Dimension(a)

		Number of Levels	Number of Parameters
Fixed Effects	Intercept	1	1
	Man	2	1
	Time	1	1
	Time2	1	1
Residual			1
Total		5	5

a Dependent Variable: test score.

Information Criteria(a)

Akaike's Information Criterion (AIC)	1202.060
Schwarz's Bayesian Criterion (BIC)	1205.301

The information criteria are displayed in smaller-is-better forms.
a Dependent Variable: test score.

<u>Absolute values are not interpretable</u>. A BIC-to-BIC difference of more than about 2 when comparing two models is clear evidence that the model with the lower BIC is better.

Estimates of Fixed Effects(b)

Parameter	Estimate	Std. Error	df	T	Sig.	95% Confidence Interval	
						Lower Bound	Upper Bound
Intercept	41.373268	.927509	189.000	44.607	.000	39.543668	43.202868
[man=1]	-8.677652	.817536	189	-10.614	.000	-10.290320	-7.064985
[man=2]	0(a)	0	.	.	.	.	.
time	10.153334	.699193	189.000	14.522	.000	8.774109	11.532560
time2	-.676040	.112806	189.000	-5.993	.000	-.898561	-.453519

a This parameter is set to zero because it is redundant.
b Dependent Variable: test score.

Estimates of Covariance Parameters(a)

Parameter	Estimate	Std. Error	Wald Z	Sig.	95% Confidence Interval	
					Lower Bound	Upper Bound
Residual	29.589380	3.043827	9.721	.000	24.186538	36.199122

a Dependent Variable: test score.

Estimated Marginal Means

manual version(b)

manual version	Mean	Std. Error	Df	95% Confidence Interval	
				Lower Bound	Upper Bound
1	32.696(a)	1.009	189.000	30.705	34.686
2	41.373(a)	.928	189.000	39.544	43.203

a Covariates appearing in the model are evaluated at the following values: time = 0, time2 = 0.
b Dependent Variable: test score.

Calculation of p-values and CI's depend on the model assumptions being true.

ii) **Random intercept model (each subject has their own intercept)**:

Model Dimension(b)

		Number of Levels	Covariance Structure	Number of Parameters	Subject Variables
Fixed Effects	Intercept	1		1	
	Man	2		1	
	Time	1		1	
	time2	1		1	
Random Effects	Intercept	1	Variance Components	1	subj
Residual				1	
Total		6		6	

b Dependent Variable: test score.

Note: "Covariance Structure" only applies when there are multiple random effects

Information Criteria(a)   [vs. prior model BIC=1205.3]

Akaike's Information Criterion (AIC)	1066.127
Schwarz's Bayesian Criterion (BIC)	1072.610

The information criteria are displayed in smaller-is-better forms.
a Dependent Variable: test score.

Estimates of Fixed Effects(b)

Parameter	Estimate	Std. Error	Df	t	Sig.	95% Confidence Interval Lower Bound	95% Confidence Interval Upper Bound
Intercept	41.446333	1.173883	38.204	35.307	.000	39.070348	43.822319
[man=1]	-8.809786	1.793315	28.120	-4.913	.000	-12.482519	-5.137052
[man=2]	0(a)	0	.	.	.	.	.
Time	10.042505	.399408	161.351	25.143	.000	9.253764	10.831246
time2	-.666276	.064444	161.351	-10.339	.000	-.793539	-.539014

a This parameter is set to zero because it is redundant.
b Dependent Variable: test score.

Estimates of Covariance Parameters(a)

Parameter		Estimate	Std. Error	Wald Z	Sig.	95% Confidence Interval Lower Bound	95% Confidence Interval Upper Bound
Residual		9.556709	1.064938	8.974	.000	7.681676	11.889422
Intercept [subject = subj]	Variance	20.866383	5.976469	3.491	.000	11.902805	36.580114

a Dependent Variable: test score.

Estimated Marginal Means

1. manual version(b)

manual version	Mean	Std. Error	Df	95% Confidence Interval Lower Bound	95% Confidence Interval Upper Bound
1	32.637(a)	1.492	33.577	29.602	35.671
2	41.446(a)	1.174	38.204	39.070	43.822

a Covariates appearing in the model are evaluated at the following values: time = 0, time2 = 0.
b Dependent Variable: test score.

iii) Conclusions: Manual 2 is better by 8.8 points (95% CI [5.1,12.5]) at each time. Subject-to-subject variability is about twice as large as within-subject variability. The random intercept is justified, so the CIs for the "manual" means and mean difference from the fixed model are wrong. For each "manual", across subjects 95% of intercepts vary roughly ±2sqrt(20.9). Consider other models to check random slope, random curvature, interaction and/or serial correlation.

iv) Equation hierarchy "collapsed": $Y_{iT} = \beta_0 + U_i + \beta_M M + \beta_T T + \beta_{T2} T^2 + \varepsilon_{iT}$ where *i* is subject number, *T* is time, β_0 is the average intercept, U_i is a random per-subject intercept deviation, β_M is the effect of manual, M is a manual indicator, T is time, ε_{iT} is residual Normally distributed error, and U_i is distributed Normally with mean 0 and variance σ_i^2 estimated at 20.9.

h) **Example 2**: hierarchical or multi-level model from the NCES "High School and Beyond" study. We will look at the relationship between the outcome math achievement test score (Math ACH) and the explanatory variable SES, taking into account the correlation of subjects within the same school.

i) Model 1: Random Intercept Only

Model Dimension(a)

		Number of Levels	Covariance Structure	Number of Parameters	Subject Variables
Fixed Effects	Intercept	1		1	
Random Effects	Intercept	1	Identity	1	School
Residual				1	
Total		2		3	

a Dependent Variable: Math ACH test score.

Information Criteria(a)

Akaike's Information Criterion (AIC)	47120.793
Schwarz's Bayesian Criterion (BIC)	47134.553

The information criteria are displayed in smaller-is-better forms.
a Dependent Variable: Math ACH test score.

Estimates of Fixed Effects(a)

Parameter	Estimate	Std. Error	Df	t	Sig.	95% Confidence Interval Lower Bound	95% Confidence Interval Upper Bound
Intercept	12.636974	.244394	156.647	51.707	.000	12.154242	13.119706

a Dependent Variable: Math ACH test score.

Estimates of Covariance Parameters(a)

Parameter		Estimate	Std. Error	Wald Z	Sig.	95% Confidence Interval Lower Bound	95% Confidence Interval Upper Bound
Residual		39.148322	.660645	59.258	.000	37.874662	40.464813
Intercept [subject = school]	Variance	8.614025	1.078804	7.985	.000	6.739122	11.010548

a Dependent Variable: Math ACH test score.

Estimated Marginal Means

Grand Mean(a)

Mean	Std. Error	df	95% Confidence Interval Lower Bound	95% Confidence Interval Upper Bound
12.637	.244	156.647	12.154	13.120

a Dependent Variable: Math ACH test score.

ii) Model 2: Add mean school SES as a fixed effect school-level variable

Model Dimension(a)

		Number of Levels	Covariance Structure	Number of Parameters	Subject Variables
Fixed Effects	Intercept	1		1	
	meanses	1		1	
Random Effects	Intercept	1	Identity	1	school
Residual				1	
Total		3		4	

a Dependent Variable: Math ACH test score.

Information Criteria(a) [vs previous BIC=47135]

Akaike's Information Criterion (AIC)	46965.285
Schwarz's Bayesian Criterion (BIC)	46979.044

The information criteria are displayed in smaller-is-better forms.
a Dependent Variable: Math ACH test score.

Estimates of Fixed Effects(a)

Parameter	Estimate	Std. Error	Df	t	Sig.	95% Confidence Interval Lower Bound	95% Confidence Interval Upper Bound
Intercept	12.649435	.149280	153.743	84.736	.000	12.354530	12.944340
meanses	5.863538	.361458	153.407	16.222	.000	5.149461	6.577616

a Dependent Variable: Math ACH test score.

(Note: meanses is in the form of a “Z-score”, so the mean is zero and the s.d. is one.)

iii) Model 3: Add student-level (relative) SES fixed effect

Model Dimension(a)

		Number of Levels	Covariance Structure	Number of Parameters	Subject Variables
Fixed Effects	Intercept	1		1	
	meanses	1		1	
	Cses	1		1	
Random Effects	Intercept	1	Identity	1	School
Residual				1	
Total		4		5	

a Dependent Variable: Math ACH test score.

Information Criteria (vs. previous BIC=46979)

Akaike's Information Criterion (AIC)	46572.580
Schwarz's Bayesian Criterion (BIC)	46586.338

Estimates of Fixed Effects(a)

Parameter	Estimate	Std. Error	Df	t	Sig.	95% Confidence Interval Lower Bound	95% Confidence Interval Upper Bound
Intercept	12.661261	.1493726	153.695	84.763	.000	12.3661734	12.9563502
meanses	5.866203	.3616896	153.363	16.219	.000	5.1516658	6.5807396
Cses	2.191165	.1086673	7021.510	20.164	.000	1.9781447	2.4041860

a Dependent Variable: Math ACH test score.

Estimates of Covariance Parameters(a)

Parameter		Estimate	Std. Error	Wald Z	Sig.	95% Confidence Interval Lower Bound	95% Confidence Interval Upper Bound
Residual		37.019064	.624777	59.252	.000	35.814555	38.264084
Intercept [subject = school]	Variance	2.692422	.405130	6.646	.000	2.004761	3.615960

a Dependent Variable: Math ACH test score.

iv) Model 4: Add student-level (relative) SES random effect

Model Dimension(a)

		Number of Levels	Covariance Structure	Number of Parameters	Subject Variables
Fixed Effects	Intercept	1		1	
	meanses	1		1	
	cses	1		1	
Random Effects	Intercept + cses	2	Diagonal	2	school
Residual				1	
Total		5		6	

a Dependent Variable: Math ACH test score.

Information Criteria (vs. previous BIC=46586)

Akaike's Information Criterion (AIC)	46564.808
Schwarz's Bayesian Criterion (BIC)	46585.446

Estimates of Covariance Parameters

Parameter		Estimate	Std. Error	Sig.	95% Confidence Interval	
					Lower Bound	Upper Bound
Residual		36.708	.626	.000	35.502	37.956
Intercept + cses [subject = school]	Var: Intercept	2.699	.405	.000	2.011	3.622
	Var: cses	.695	.281	.013	.315	1.535

Conclusions: Your score is better if you are in a high SES school, but other school-level factors remain to be discovered. It helps to have above average SES for your school, but there is little evidence that the size of this effect varies much from school to school. The largest component of variation is residual (after correcting for SES) student-to-student variation.

3) Mixed Models Summary:

Can handle missing data and unequally spaced data better than “repeated measures”

Can flexibly model correlation structure

Can model effects at different levels

Complex model selection issues