Two-period model

Equations

$$\frac{c_2}{c_1} = [\beta(r+\theta)]^{\frac{1}{\sigma}} + \theta \quad (c6)$$

$$\left(\frac{l_1}{l_2}\right)^{\frac{1}{\epsilon}} = \beta R \frac{w_1}{w_2} = \beta(1+r)\frac{w_1}{w_2} \quad (c7)$$

$$\psi l_2^{\frac{1}{\epsilon}}(c_2 - \theta c_1)^{\sigma} = w_2 \quad (c8)$$

Assumptions

$$\beta(r+\theta) = 1 \Rightarrow c_2 = (1+\theta)c_1 \quad \frac{w_1}{w_2} = \beta R \Rightarrow l_1 = l_2$$

Shock

$$R - 1 = r \quad \hat{r} = \hat{A} \quad w_2 = \hat{A}$$

Condition for $\hat{c_1} > 0$

By linearizing (c6), (c7), (c8):

$$\hat{c_2} = \hat{c_1} + \left(\frac{1}{\sigma}\right)\frac{c_2 - \theta c_1}{c_2} \cdot \frac{r}{R+\theta}\hat{A} \quad (c9)$$

$$\hat{l}_2 - \frac{\epsilon}{1+r}\hat{A} = \hat{l}_1 \quad (c10)$$

$$\frac{1}{\epsilon}\hat{l}_2 + \frac{\sigma c_2}{c_2 - \theta c_1}\hat{c_2} - \frac{\sigma\theta c_1}{c_2 - \theta c_1}\hat{c_1} = \hat{A} \quad (c11)$$

Linearizing the life-time budget (c1)

$$c_1\hat{c_1} + \frac{c_2}{R}(\hat{c_2} - \frac{r}{R}\hat{A}) = w_1 l_1 \hat{l}_1 + \frac{w_2 l_2}{R}(\hat{l}_2 + \frac{1}{R}\hat{A}) \quad (c12)$$

Substituting (c10), (c11) into (c12) to eliminate $\hat{l}_1, \hat{l}_2$

$$\left[c_1 - (w_1 l_1 + \frac{w_2 l_2}{R})\epsilon\frac{\sigma\theta c_1}{c_2 - \theta c_1}\right]c_1 + \left[(\frac{w_1 l_1 \epsilon}{R} - \frac{w_2 l_2}{R^2}) - (w_1 l_1 + \frac{w_2 l_2}{R})\epsilon - \frac{c_2 r}{R^2}\right]\hat{A} + \left[\frac{c_2}{R} + (w_1 l_1 + \frac{w_2 l_2}{R})\epsilon\frac{\sigma c_2}{c_2 - \theta c_1}\right]\hat{c_2} = 0 \quad (c13)$$

Substituting (c9) into (c13) to eliminate $\hat{c_2}$

$$(1 + \frac{1+\theta}{R})(\sigma\epsilon + 1)\hat{c_1} = \left(\frac{(1+\theta)r}{R^2} + \frac{1}{R^2} + \frac{\epsilon}{R} - \frac{R+\theta}{R}\epsilon(\frac{1}{R} - 1) - \frac{r}{\sigma(R+\theta)}[\frac{1}{R} + (1 + \frac{1+\theta}{R})\sigma\epsilon]\right)\hat{A} \quad (c14)$$

$\hat{c_1} > 0 \Rightarrow$

$$\sigma > \frac{1 - \beta\theta}{1 + \theta + (1+\beta)\theta\epsilon + \frac{1+\epsilon}{r}} \equiv \underline{\sigma} \quad (c15)$$

In the case where $\hat{R} = \hat{A}$,

$$\underline{\sigma} = \frac{1 - \beta\theta}{1 + \theta + (1+\beta)\theta\epsilon} \quad (*)$$

We have one more term $(\frac{1+\epsilon}{r})$ in the new denominator

$$\frac{\partial \underline{\sigma}}{\partial \epsilon} = \frac{-(1-\beta\theta)}{[1+\theta+(1+\beta)\theta\epsilon+\frac{1+\epsilon}{r}]^2}\left[(1+\beta)\theta+\frac{1}{r}\right] \quad (c16)$$

Condition for $\hat{l_1} > 0$

Substituting (c9), (c10) into (c12) to eliminate $\hat{c}_2$

$$c\hat{c}_1 + \frac{c_2}{R}\left(\hat{c}_1 + \frac{c_2-\theta c_1}{\sigma c_2}\frac{r}{R+\theta}\hat{A} - \frac{r}{R}\hat{A}\right) = w_1 l_1 \hat{l}_1 + \frac{w_2 l_2}{R}\left[\hat{l}_1 + \frac{\epsilon}{R}\hat{A} + \frac{1}{R}\hat{A}\right] \quad (c17)$$

Substituting (c9) into (c11) to eliminate $\hat{C}_2$

$$\hat{c}_1 = \frac{1}{\sigma}\left[\left(\frac{1+\theta}{R+\theta} - \frac{1}{R}\right)\hat{A} - \frac{1}{\epsilon}\hat{l}_1\right] \quad (c18)$$

Substituting (c18) into (c17) to eliminate $\hat{C}_1$

$$\hat{l}_1 = \frac{\hat{A}}{(1+\frac{1}{\sigma\epsilon})(1+\frac{1+\theta}{R})}\frac{1}{R^2}\left[\frac{(1+\theta)r}{\sigma} - [(1+\theta)r+(1+\epsilon)]\right] \quad (c19)$$

$\hat{l}_1 > 0 \Rightarrow$

$$\sigma < \frac{1}{1+\frac{1+\epsilon}{(1+\theta)r}} \equiv \bar{\sigma} \quad (c20)$$

In the case where $\hat{R} = \hat{A}$,

$$\bar{\sigma} = 1 \quad (*)$$

We have one more term $(\frac{1+\epsilon}{(1+\theta)r})$ in the new denominator

Condition for $\hat{k} > 0$

$$\frac{\theta}{R}\hat{k} = \left[\left(\frac{R+\theta}{R} + \frac{1}{\sigma\epsilon}\right)\frac{1}{(1+\frac{1}{\sigma\epsilon})(1+\frac{1+\theta}{R})}\frac{1}{R^2}\left[\frac{(1+\theta)r}{\sigma} - [(1+\theta)r+(1+\epsilon)]\right] - \frac{\theta r}{R(R+\theta)\sigma}\right]\hat{A} \quad (c21)$$

Denote

$$F(\sigma) = -[(1+\theta)r+(1+\epsilon)]\frac{R+\theta}{R}\sigma^2 - \left[\frac{(1+\theta)r}{\epsilon} + \frac{1}{\epsilon} + 1 + \theta r(1+\beta) - \frac{r}{\beta R}(1+\theta)\right]\sigma + \frac{r\beta R}{\epsilon} \quad (c23)$$

In the case where $\hat{R} = \hat{A}$,

$$F(\sigma) = -\frac{(1+\theta)(R+\theta)^2}{R}\sigma^2 - \left[(1+\theta)(R+\theta)\frac{1}{\epsilon} - \frac{(1+\theta)(R+\theta)^2}{R} + \theta(R+1+\theta)\right]\sigma + \frac{R}{\epsilon} \quad (*)$$

$$\hat{k} > 0 \Rightarrow F(\sigma) > 0$$

$$F(0) = \frac{r\beta R}{\epsilon} > 0 \quad F(\bar{\sigma}) = -\theta r^2\frac{(1+\beta)(1+\theta)}{(1+\theta)r+(1+\epsilon)} - \frac{r}{\epsilon}(1+\theta-\beta R) < 0$$

Denote σ_0 as the larger root of $F(\sigma) = 0$, then

$$\sigma_0 < \bar{\sigma}$$

For a positive TFP news shock to cause simultaneous increase in current consumption, hours worked, and investment, the value of σ should be :

$$\underline{\sigma} < \sigma < \sigma_0$$

$$\left.\frac{\partial F}{\partial \sigma}\right|_{\sigma=\sigma_0} < 0$$

$$\left.\frac{\partial F}{\partial \epsilon}\right|_{\sigma=\sigma_0} = \frac{r\beta R}{\epsilon[(1+\theta)r+(1+\epsilon)]} + \frac{\epsilon^2 r(\beta R + \frac{\theta}{R} + \frac{\theta^2}{R}) + (R+\theta r-\epsilon)[(1+\theta)r+(1+\epsilon)]}{\epsilon^2[(1+\theta)r+(1+\epsilon)]}\sigma_0 \quad (c26)$$

Cause $\epsilon \in [0,1]$, $\left.\frac{\partial F}{\partial \epsilon}\right|_{\sigma=\sigma_0} > 0$

Thus

$$\frac{d\sigma_0}{d\epsilon} = -\left.\frac{\frac{\partial F}{\partial \epsilon}}{\frac{\partial F}{\partial \sigma}}\right|_{\sigma=\sigma_0} > 0 \quad (c27)$$