where $\gamma = \gamma_0$ corresponds to the initial configuration. Related expressions can also be derived for the relaxation that occurs during the transition era.

Given the rate of loop production (9.3.15) from a 'scaling' network, we can determine the ensuing loop distribution. We define $n(\ell, t)d\ell$ to be the number density of loops in the length range $\ell$ to $\ell + d\ell$ at the time $t$, along with the corresponding loop energy density distribution $\rho_{\rm L}(\ell, t)d\ell = \mu\ell\, n(\ell, t)d\ell$. From (9.3.15) and given dilution due to expansion, we obtain the rate of change of the loop energy density,

$$\dot{\rho}_{\rm L}(\ell, t) = -3\left(\frac{\dot{a}}{a}\right)\rho_{\rm L}(\ell, t) + g\frac{\mu}{L^4}f(\ell/L)\,. \tag{9.3.19}$$

Here, $g$ is a Lorentz factor which accounts for the fact that loops are created with a non-zero centre-of-mass kinetic energy which is lost through the redshifting of velocities. As we shall see in §9.4.4, typically we have $g = (1 - v_{\rm i}^2)^{1/2} \approx 1/\sqrt{2}$, where $v_{\rm i}$ is the initial loop velocity. The loop formation equation (9.3.19) can be easily integrated to yield, during the radiation epoch,

$$\rho_{\rm L}(\ell, t) = \frac{g\mu}{\gamma^{5/2}t^{3/2}\ell^{3/2}}\int_{\ell/\gamma t}^{\infty} dx\sqrt{x}\, f(x)\,. \tag{9.3.20}$$

At late times this has the asymptotic form

$$\rho_{\rm L}(\ell, t) = \frac{\mu\nu_{\rm r}}{(t\ell)^{3/2}}\,, \tag{9.3.21}$$

where

$$\nu_{\rm r} = g\gamma^{-5/2}\int_0^{\infty}\sqrt{x}\, f(x)dx\,. \tag{9.3.22}$$

For loops created after matter domination, (9.3.19) the corresponding solution is

$$\rho_{\rm L}(\ell, t) = \frac{\mu\nu_{\rm m}}{t^2\ell}\,, \tag{9.3.23}$$

where

$$\nu_{\rm m} = g\gamma^{-3}\int_0^{\infty} f(x)dx = gc\gamma^{-3}\,. \tag{9.3.24}$$

Note that $f(x)$ need not have the same form during the matter and radiation eras; the values of $g$, $\gamma$ and $c$ are also expected to be different.

The one-scale model based on rate equations similar to (9.3.16) and (9.3.19) was first introduced by Kibble [1985] and further developed by Bennett [1986a,b]. These models were more detailed than the above in that they used both loop creation and reconnection functions instead of the one function $f(x)$, attempting to model these by making reasonable assumptions about loop formation mechanisms. Another version of the one-scale model was also employed by Albrecht & Turok [1989] who used it to fit their numerical results. However, the weakness in this approach is that we currently do not have convincing theoretical arguments to precisely determine these functions. As we shall see, numerical simulations have also failed to determine $f(x)$, primarily because of insufficient resolution and dynamic range.

When studying the cosmological implications of strings in the following chapters, we shall use a simple model in which all loops are assumed to be created at the same relative size (9.3.12), a fixed fraction of the horizon, $\ell = \alpha t$. The loop production function $f(x)$ is then

$$f(x) = c\,\delta\left(x - \alpha/\gamma\right) . \tag{9.3.25}$$

Before determining $n(\ell, t)$ explicitly under these circumstances, we note the effect of gravitational radiation on the demise of small loops. We recall from §7.5 that loops of length $\ell$ decay at the rate $\dot{\ell} = -\Gamma G\mu$ where $\Gamma$ depends on the loop trajectory and typically $\Gamma \approx 65$. For a loop of initial length $\ell_{\mathrm{i}}$ formed at time $t_{\mathrm{i}}$, $\ell = \ell_{\mathrm{i}} - \Gamma G\mu(t - t_{\mathrm{i}})$. Eqns (9.3.21) and (9.3.23) give the loop densities in terms of the initial length $\ell_{\mathrm{i}}$. Substituting $\ell_{\mathrm{i}}$ in terms of $\ell$ and $t$ into (9.3.21) during the radiation era, assuming that $t >> t_{\mathrm{i}}$ and employing (9.3.25), we obtain

$$n(\ell, t) = \frac{\nu_{\mathrm{r}}}{t^{3/2}(\ell + \Gamma G\mu t)^{5/2}} , \qquad \ell < \alpha t , \tag{9.3.26}$$

where

$$\nu_{\mathrm{r}} = gc\sqrt{\alpha}\,\gamma^{-3} = g\sqrt{\alpha}\,\gamma_{\mathrm{r}}^{-2}\left(1 - \langle v_{\mathrm{r}}^2\rangle\right) . \tag{9.3.27}$$

For the last step in (9.3.27) we have employed the scaling condition (9.3.17). For loops created during the matter-dominated era, the corresponding distribution is

$$n(\ell, t) = \frac{\nu_{\mathrm{m}}}{t^{2}(\ell + \Gamma G\mu t)^{2}} , \qquad \ell < \alpha t , \tag{9.3.28}$$

where

$$\nu_{\mathrm{m}} = gc\gamma^{-3} = \tfrac{2}{3} g\gamma_{\mathrm{m}}^{-2}\left(1 - \langle v_{\mathrm{m}}^2\rangle\right) . \tag{9.3.29}$$

### 9.3.4 Further analytic modelling

As we shall see in §9.4, the 'one-scale' model is inadequate because it does not account for the small-scale structure that accumulates on the strings in the form of 'kinks' and 'wiggles'. This arises through string crossings and because stretching due to the expansion is insufficient to completely smooth out modes falling within the horizon. Numerical modelling has delineated the 'fractal-like' nature of this substructure but has yet to conclusively establish its 'scaling' behaviour. Although the smallest