Option pricing. Girsanov theorem and B&S models

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1 Introduction

In 1973, Black and Scholes, with the assistance of Merton, developed one of the most significant models in finance. This model, which earned them two Nobel Prizes, revolutionized the valuation of options. The derivation of the Black-Scholes formula, a partial differential equation (PDE), fundamentally changed how market participants assessed option prices.

Similar to the binomial model, the Black-Scholes model is based on the concept of a replicating portfolio, which consists partly of units of the underlying asset and partly of risk-free bonds. The main difference between the binomial model and the Black & Scholes model is that, in this case, the hypothesis says that returns are distributed among infinite states of nature according to a normal statistical law. The Black and Scholes model represents the limit in the continuum of the binomial model [1]. The goal is to find a formula for pricing European call or put options by using a replicating strategy. This strategy involves replicating the option's payoff with a combination of stocks and bonds, or, as in our case, the risk-neutral approach. To achieve this, a stochastic differential equation is initially used to describe the stock price dynamics:

$$\frac{dS_t}{S_t} = \mu\, dt + \sigma\, dW_t \tag{1}$$

Then the model describes the percentual variation of the stock price over a period (dt) as the sum of two contributions: the deterministic contribution, equal to μ multiplied with the period used, and the stochastic contribution, even to constant sigma (σ) multiplied with the increase between t and $t + dt$ of a geometric Brownian motion (one of the assumptions)[2].

2 Assumptions

The model is based on many assumptions to create a perfect market:

- The stock doesn't pay dividends during the maturity;
- The stock price (S) movements follow the Geometric Brownian Motion where sigma and volatility are known and both are constant. Thanks to this assumption, it is possible to add the empirical characteristics of the stock prices as the Markov process: percentage returns between successive periods are independent and the volatility of the percentage returns does not depend on the stock;

[1] https://www.bankpedia.org/termine.php?c_id=21155

[2] https://fermat.dima.unige.it/didattica/matematica/docpdf/Note%20CORSI/lez_4.pdf

- There is no arbitrage;
- The risk-free interest rate (r) is constant in time [$dS(t) = rS(t)\,dt$];
- There are no transaction costs and the bonds sold are divisible;
- It is possible to sell short without any penalty;
- The exchange of bonds occurs continuously.

Among these assumptions, some are essential, for example, the second and the third, while others are not, for example, the first (we will discuss that later). If the market follows these assumptions, it is possible, thanks to this model, to calculate the value and risk characteristics of an options position.

3 The Girsanov Theorem

3.1 Transformation of Measure and Girsanov theorem (A discrete example)

Suppose you toss coins where the probabilities of tossing heads and tails are different. You bet 1 Euro each time, $(\eta_n)_{n\geq 1}$ is an i.i.d. random variable sequence that satisfies:

$$P\left(\eta_n = 1\right) = p, P(\eta_n = -1) = 1 - p, 0 < p \neq \frac{1}{2} < 1. \tag{2}$$

Set $F_n = \delta(\eta_1 \ldots \eta_n)$, $\quad \varepsilon_n = \sum_{k=1}^{n} \eta_k$. $(\varepsilon_n, F_n)_{n\geq 1}$ is not a martingale and the game is unfair. In Figure 1, the probability of going up more than going down is higher.

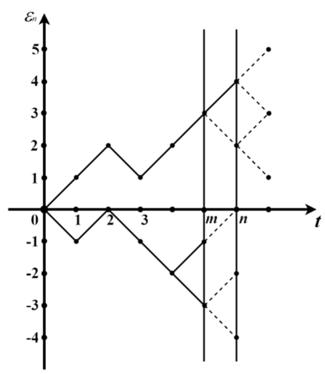

Figure 1:

If you want to make the game fair, make sure that the results of tossing a coin are equal. From a mathematical point of view, it is to reweight "winning" and "losing" under $\mathbb{Q}$, or it can be understood as a measurement transformation.

$$\mathbb{Q}\left(\eta_n = 1\right) = \frac{1}{2p} P\left(\eta_n = 1\right), \mathbb{Q}\left(\eta_n = -1\right) = \frac{1}{2(1-p)} P(\eta_n = -1) = \frac{1}{2}. \tag{3}$$

Now, we define the weight process $Z_n = \left(\frac{P(\eta_n=1)}{P(\eta_n=-1)}\right)^{\eta_n}$, using the concept of Girsanov transformation. With this concept, under the new probability measure $\mathbb{Q}$, the probability of getting heads and tails will be adjusted accordingly:

$$P^{\mathbb{Q}}(\eta_n = 1) = \frac{P(\eta_n = 1)}{Z_n}, P^{\mathbb{Q}}(\eta_n = -1) = \frac{P(\eta_n = -1)}{Z_n} \tag{4}$$

Then, the expected value of each toss under $\mathbb{Q}$ becomes: $E^{\mathbb{Q}}(\varepsilon_n) = E^P(\varepsilon_n Z_n) = 0$. By using the weight process Z_n defined above, we have made the game fair by reweighting the outcomes of the coin tosses under the new probability measure $\mathbb{Q}$. Here $Z = \frac{d\mathbb{Q}}{dP}$ called Radon-Nikodym derivative, such that for any measurable set A, we have:

$$\mathbb{Q}(A) = \int_A \frac{d\mathbb{Q}}{dP} dP \tag{5}$$

3.2 The role played by the Girsanov theorem

Girsanov's theorem points out that the drift term of Brownian dW_t^P under the measure $\mathbb{P}$ is converted into the Brownian motion dW_t^Q under the new measure $\mathbb{Q}$, so that the drift of the new Brownian motion item is zero. Simply, the actual price of the underlying asset has a certain trend and does not meet the assumptions of the B-S option derivation formula. Measurement transformation is performed through the theorem to satisfy the them.

$$dX_t = \left(\mu + \frac{\sigma^2}{2}\right) dt + \sigma dW_t^P, X_t = \ln \frac{S_t}{S_0} \tag{6}$$

Under the risk-neutral measure, the drift rate of asset prices is equal to the risk-free interest rate, so $dW_t^Q = dW_t^P - r dt$. Then,

$$dX_t = \left(\mu + \frac{\sigma^2}{2}\right) dt + \sigma \left(dW_t^{\mathbb{Q}} + \nu dt\right) \tag{7}$$

$$dX_t = r dt + \sigma dW_t^P \tag{8}$$

3.3 Pricing under a risk-neutral measure

Under the risk-neutral measure, the value of a discounted derivative security is a martingale, a measure in which the expected return on all traded assets in the long run is equal to the risk-free rate, r.

$$E\left(\frac{dS}{S}\right) = r \tag{9}$$

For the BS option pricing formula, by assuming that investors are risk neutral, the expected return rate on an option is equal to the risk-free rate. In this way, option pricing can be simplified to the expected return on a portfolio in a risk-neutral world.

4 Proof of the risk-neutral approach formula

In our proof we consider a European Call option, because as mentioned in the assumptions it is essential not to pay the dividends during the maturity, with a strike price K and with expiry date T. Then the price at the end T will be equal to S_T.

One of the assumptions tells us that the movement of the stock follows a Geometric Brownian Motion (GBM). We can say the stock S follows the GBM if it satisfies the linear Stochastic Differential Equation (SDE Linear):

$$dS(t) = \mu S(t)\, dt + \sigma S(t)\, dW^{\mathbb{P}}(t) \tag{10}$$

This SDE linear shows us how the infinitesimal increase in the price of the stock S, which is a stochastic variable, is determined in the first part of the right side, by a deterministic factor, in our case the drift coefficient μ, and in the second part of the right side, by the volatility coefficient σ. The first part explains the expected rate of return, while the second part measures the volatility associated with the stock price. If this hypothesis is satisfied, the stock price S follows the Brownian motion and then will be:

$$S(t) = S_0\, e^{W(t)} \tag{11}$$

Where $S_0 > 0$ and $W(t)$ is a Brownian motion with drift μ and variance θ^2. Furthermore, we know $W(t) \sim \mathcal{N}(\mu t, \theta^2 t)$, we can also write $W(t) = \mu t + \theta\sqrt{t Z}$ where $t \in [0, T]$ and $Z \sim \mathcal{N}(0, 1)$. It is essential that the movement of the stock price is exactly Geometric Brownian motion; this was understood by Bachelier, who noted during his studies that if the stock price had followed a simple Brownian motion, some results would have been negative and thus insignificant. Furthermore, it is the logarithm of the ratio that has the normal distribution, meaning the price dynamic is described in relative variations and not in absolute variations.

We consider F_t (our standard filtration for W). The standard filtration is essential because it ensures that only the information present in the market influences the option price; this is one of the assumptions. We can note that the conditional expected value of the stock S at time $u + t$ (where $u, t \in [0, T]$) is conditioned up to time u, and in fact is:

$$E_{\mathcal{P}}[S_{u+t} \mid \mathcal{F}_u] = S_u e^{\mu t + \frac{t\sigma^2}{2}} \tag{12}$$

Thanks to the Girsanov theorem we can pass from a probability $\mathbb{P}$ to $\mathbb{Q}$, that is an equivalent measurement in a neutral world. Then we have:

$$E_Q[S_{u+t} \mid \mathcal{F}_u] = S_u e^{rt} \tag{13}$$

Doing this transformation, the equilibrium price of the European call option becomes the expected value of the discount payoff:

$$f_t = E_Q[e^{-r(T-t)} f_T] = e^{-r(T-t)} E_Q[f_T] \quad 0 \le t < T \tag{14}$$

Where $f_T = (S_T - K^+) = \max(0, S_T - K)$ because we consider a call option. As just mentioned, we consider S_T the price of the stock at the maturity and then now it is equal to:

$$S_T = S_t e^{(W_{T-t})} = S_t e^(r - \frac{\sigma^2}{2})(T - t) + \sigma\sqrt{(T - tZ)} \tag{15}$$

If

$$S_T > K \Leftrightarrow e^{((r-\frac{\sigma^2}{2})(T-t)+\sigma\sqrt{(T-tZ)})} > \frac{K}{S_t} \Leftrightarrow Z > \frac{\ln\left(\frac{K}{S_t}\right) - (r-\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{(T-t)}} \tag{16}$$

To arrive at the end, we must add and subtract $\sigma^2(T-t)$ in the numerator.
Obtaining in this way: $Z > \sigma\sqrt{(T-t)} - d_1$ where $d_1 = \frac{((r+\frac{\sigma^2}{2})(T-t)+\ln(S_t/K))}{\sigma\sqrt{(T-t)}}$.
Now we can continue by saying that:

$$\begin{aligned} f_t = e^{-r(T-t)}\mathbb{E}_Q\left[\max(0, S_T - K)\right] = e^{-r(T-t)}\mathbb{E}_Q\left[I_A(S_T - K)\right] = \\ e^{-r(T-t)}\mathbb{E}_Q\left[I_A S_T\right] - e^{-r(T-t)}K\mathbb{E}_Q\left[I_A\right] \end{aligned} \tag{17}$$

Here we state $A = \{S_T > K\} \in \mathcal{F}$ and I_A characteristic function of A, and this is defined as $I_A(w) = 1$ if "w" belongs to A, otherwise as $I_A(w) = 0$ if "W" dies not belong to A.

$$\begin{aligned} E_Q[I_A] = Q(S_T > K) = Q(Z > \sigma\sqrt{(T-t)} - d_1) \\ = 1 - \Phi(\sigma\sqrt{(T-t)} - d_1) = \Phi(d_1 - \sigma\sqrt{(T-t)}) \end{aligned} \tag{18}$$

"Φ" is the cumulative distribution function of the normal distribution. Setting $c = \sigma\sqrt{(T-t)} - d_1$ we can say that the expected value (in $\mathbb{Q}$) of the stock at the maturity (in I_A) is equal to:

$$\begin{aligned} E_Q[I_A S_T] = \frac{1}{\sqrt{2\pi}}\int_c^{+\infty} S_t e^{(r-\frac{\sigma^2}{2})(T-t)+\sigma x\sqrt{(T-t)}} e^{-\frac{x^2}{2}}\,dx \\ = E_Q[S_T|A]Q_A = \frac{1}{\sqrt{2\pi}} S_t e^{r(T-t)} \int_c^{+\infty} e^{-\left(x-\sigma\sqrt{(T-t)}\right)^2/2}\,dx \end{aligned} \tag{19}$$

$E_Q[I_A S_T]$ represents the weighted average of all possible future values of the Stock price at maturity, considering the market situation "A" and that, in the Black-Scholes formula, is essential because it helps us to calculate the fair price of an option and to evaluate the option portfolio. Now if we set $y = x - \sigma\sqrt{(T-t)}$ to simplify the equation, we can arrive to say that $E_Q[I_A S_T]$ is equal to the discounted cash flow of the European call option and all is influenced by the cumulative distribution function of d_1:

$$E_Q[I_A S_T] = S_t e^{r(T-t)} \frac{1}{\sqrt{2\pi}} \int_{-d_1}^{+\infty} e^{-\frac{y^2}{2}}\,dy = S_t e^{r(T-t)} Q(Z > d_1) = e^{r(T-t)} S_t \Phi(d_1) \tag{20}$$

Therefore, combining the previous equations we obtain that the value of the call at time t is a difference between the maximum profit that the buyer would obtain if he exercised the option immediately and the value of the strike price influenced by the cost of money over time, because we must think that the call option can be exercised but also not, both multiplicate with the respective probabilities.

$$\begin{aligned} f_t = e^{-r(T-t)} e^{r(T-t)} S_t \Phi(d_1) - e^{-r(T-t)} K\Phi(d_1 - \sigma\sqrt{T-t}) \\ = S_t\Phi(d_1) - e^{-r(T-t)} K\Phi(d_1 - \sigma\sqrt{T-t}) \end{aligned} \tag{21}$$

The terms $N(d_1)$ and $N(d_1 - \sigma\sqrt{(T-t)})$ are the probability of the option expiring in the money under the equivalent exponential martingale probability measure respectively for the stock and the for the risk-free asset or also called risk-neutral probability measure.

In the end the Black-Scholes formula is:

$$C_t = C(S_t, T, K, \sigma, r) = S_t\Phi(d_1) - e^{-r(T-t)}K\Phi(d_1 - \sigma\sqrt{(T-t)}) \quad \text{with} \quad 0 \leq t < T \tag{22}$$

Where:

$$d_1 = \frac{(r + \frac{\sigma^2}{2})(T-t) + \ln\left(\frac{S_t}{K}\right)}{\sigma\sqrt{(T-t)}} \tag{23}$$

5 Comparison between the replicating strategy and the risk evaluation formula

The two methods are almost identical. The replicating strategy is based on the replication of the entire portfolio in order to price the option. It's very useful for the European, but we should continuously rehedge the portfolio. So we have to take in account the Delta Hedge to avoid issues and big costs. This approach was firstly introduced by Merton and it uses the Ito's lemma to derive the PDE. The portfolio is said to be self-financing. Instead, the risk evaluation formula is easier to get, but it's generally used for exotic options. Since the final result is equal, preferring one rather than the other is a general choice based on the disposal parameters.

6 Dividends

Incorporating Dividends in the Black-Scholes Model

The assumption of non-dividends can be relaxed without changing the entire Black-Scholes model. The first to introduce dividends into the model was Merton, who created his own version by modifying certain features.

If dividends are paid, the value of the stock will drop, which should be reflected in the model by reducing the future expected value of the stock. This adjustment impacts the option pricing: call options become less valuable, while put options become more valuable as expected dividend payments increase.

We consider continuous dividend payments in this framework, indicated by the dividend yield $q(S_t)$ with $q < 1$.

Consider a portfolio with shares in the listed companies in the FTSE100 index. It is likely that the investor will receive small dividend payments throughout the year.

Then, the partial differential equation (PDE) is modeled to include dividends and will be equal to:

$$\begin{aligned} \text{Call option price:} \quad & C = S_0 e^{-qT} N(d_1) - K e^{-rT} N(d_2), \\ \text{Put option price:} \quad & P = K e^{-rT} N(-d_2) - S_0 e^{-qT} N(-d_1). \end{aligned}$$

Where:

$$
\begin{aligned}
d_1 &= \frac{\ln\left(\frac{S_0}{K}\right) + \left(r - q + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}, \\
d_2 &= d_1 - \sigma\sqrt{T}.
\end{aligned}
$$

In this framework q is the continuous dividend yield This is defined as the modified Black-Scholes model.

The adjustments have two effects. First, the value takes into account the expected drop in asset value. Second, the interest rate is subtracted from the dividend yield because it explains the lower carrying cost from holding the asset in the replicating portfolio.

Since we receive $q(S_t)$ and we hold $-\Delta$ of the underlying, the portfolio will be equal to:

7 Limitations

Despite the wide usage of BSM, we cannot rely on it to price the European options. There is a contrast in the real-world that makes it almost unrealistic. There are different limitations in its application.

7.1 Brownian motion

First the main limit is due to the choice of the geometric brownian motion. Stock prices are typically modeled as log-normally distributed and assumed to be independently and identically distributed (i.i.d.), leading to the assumption that log-returns are uncorrelated. However, this assumption does not align with empirical observations (reality). They are linked to the efficient market hypothesis, because the price reflects all the available information in the market. One of the most crucial stylized facts in finance is that absolute returns or squared returns exhibit autocorrelation and long-range dependence, indicating that past values influence future values, contradicting market efficiency. That's because the past values do not have predictive power. Fractional Brownian motion offers a better description of these phenomena, as it allows for modeling the regularity of functions using the Hurst exponent, which quantifies the smoothness or roughness in the data. Contrary to the i.i.d. assumption and the normal distribution approximation, real-world price data exhibit fat tails and high peaks, suggesting a higher probability of extreme positive and negative outliers. This type of distribution, known as leptokurtic (or heavy-tailed), more accurately reflects the observed market behavior.

7.2 Interest rates

The model assumes that interest rates are constant and known, but it's unrealistic and that's why a model should consider how to predict them.

7.3 Ignorance of transaction costs, perfect liquidity, constant trading

BS does not consider transaction costs or trading barriers. More crucially is the assumption of perfect trading, meaning that the investors can buy or sell any amount of stocks at any time. As we already know, the investors have to pay bid-ask or brokerage fees to get the option, bringing out the difference between the price got by the formula and the real price on the market.

7.4 Volatility changes

In this model the volatility is assumed constant over the life of the option, but in reality it is not at all in this way. Market has fluctuations, especially during the crisis periods and investors who only rely on BSM should face to losses. Then, the market is inefficient and the BSM does not capture that. Additionally, the model should include also the real events that happen with the option, such as the flexibility to delay, the abandonment of projects based on market conditions or uncertainties, the intermittency of the volatility or the correlation between the volatility and the trading volume. So, this is not the perfect model to apply to make an investment decision! In other words, the option price from BS model does not coincide with the real option price. For that reason, we should use the implied volatility that is the unique value that substituted into the BS formula, it will give the right option price.[3] But even though we have found the way to adjust the BS formula, the implied volatility is not sufficient.[4] A good performance is given by the stochastic volatility model, that takes in account different factors and not only the dynamic of the stock price. In conclusion, the volatility is still an unsolved problem in financial world.

8 Conclusions

Even though the BSM is the first model that absolved the aim to price the European option, it is based on simple and limited assumptions that make it less useful for complex market and options. That's why researchers are working for more realistic models that could capture the dynamics of the model in a proper way to finally end up to rigorous investors decisions. In order to complete the explanation, we want to list the advanced models like the Garch model, the jump model, the stochastic volatility models and the Heston model, that seems an extension of the BSM [5].

[3]See the implied volatility and the volatility estimation.

[4]It always misses a piece, although with the implied volatility and the strikes we are able to explain the volatility smile surface, it's not yet a satisfactory value. It doesn't include the regularity patterns and it is always connected to the real market price, lacking of predictive power.

[5]As we've said above, it is a multi-factor model that describes the volatility like a stochastic dynamic as the BSM did for the stock prices.

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