

Additional questions for chapter 4

1. *A stock price is currently \$ 100. Over the next two six-month periods it is expected to go up by 10% or go down by 10%. The risk-free interest rate is 8% per annum with continuous compounding.*
 - (i) *What is the value of a one-year European call option with a strike price of \$ 100.*
 - (ii) *What is the value of a one-year European put option with a strike price of \$ 100.*
 - (iii) *Verify that the European call and the European put satisfy put-call parity.*

Solution:

Parameters are $u = 0.1, d = -0.1, 1 + r = e^{0.5 \times 0.08}$. So the risk-neutral probability is $p^* = 0.7$. After evaluation of the options at the terminal nodes we use the risk-neutral valuation to get (i)

$$\pi_C(0) = e^{-2(0.5 \times 0.08)} [0.7^2 \times 21 + 2 \times 0.7(1 - 0.7) \times 0 + (1 - 0.7)^2 \times 0] = 9.61$$

and (ii)

$$\pi_P(0) = e^{-2(0.5 \times 0.08)} [0.7^2 \times 0 + 2 \times 0.7(1 - 0.7) \times 1 + (1 - 0.7)^2 \times 19] = 1.92$$

(iii) For put-call parity one has to verify $S - \pi_C + \pi_P = Ke^{-r}$, here :

$$100 - 9.61 + 1.92 = 100e^{-0.08}.$$

2. Assume a standard 3-period CRR binomial model. The price of the stock is currently \$100. The risk-free interest rate with continuous compounding is 6% per annum. Over the next three 4 month periods, the stock is expected to go up by 8% or go down by 7% in each period.

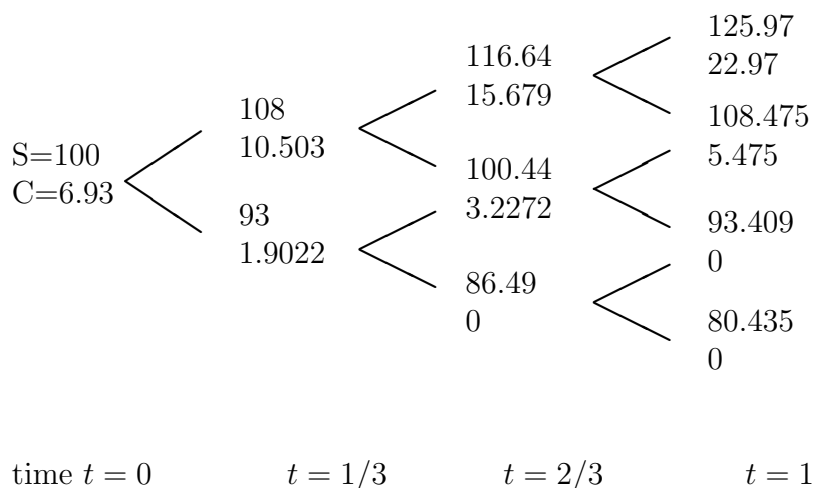
- (a) What is the value of a one-year European call with strike price \$103?
 (b) What is the value of a one-year European put with strike price \$103?
 (c) Verify the Put-Call parity for the European call and the European put.

Solution:

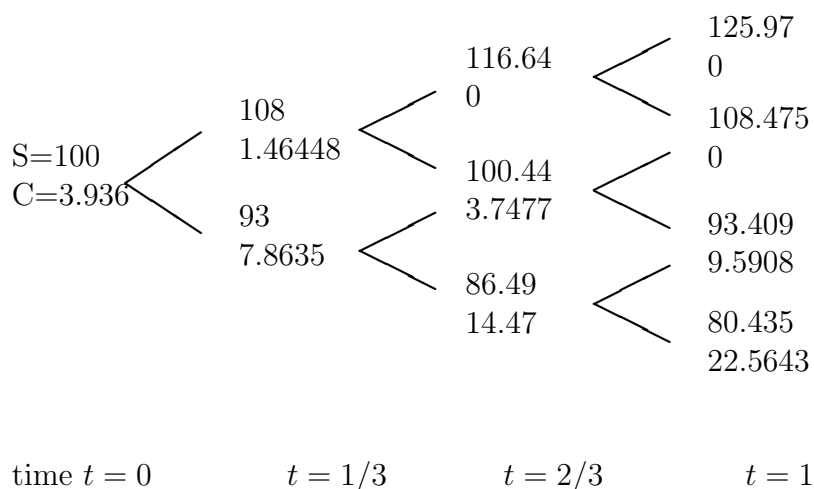
We first calculate the Martingale probability in the tree. We get

$$p = \frac{r - d}{u - d} = \frac{e^{0.06/3} - 1 + 0.07}{0.08 + 0.07} = 0.6013423$$

- (a) The tree for the call option looks as follows:



- (b) The tree for the put option is:



- (c) The Put-Call parity holds:

$$C - P = 6.9342 - 3.936 = 2.9982 = 100 - 103e^{-0.06} = 100 - 97.0017 = S - Ke^{-rT}.$$

3. Consider a 3-period Cox-Ross-Rubinstein model. The annual interest rate is $r = 0.05$ (discrete), $u = 0.1$ and $d = -0.1$. The initial price of the stock is $S(0) = 100$. The time horizon is $T = 3$ years.

(a) Calculate the risk-neutral probability and the stock prices at each node in the binomial tree (correct up to 2 decimal places after the decimal point).

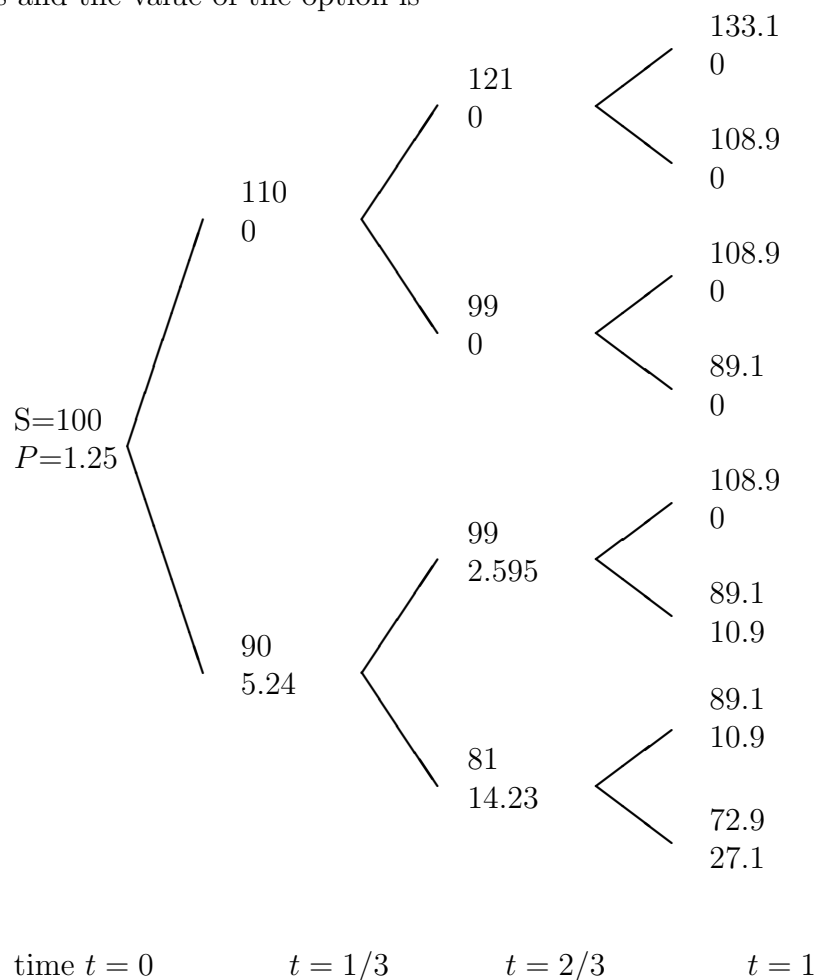
(b) Calculate the value of the European option with payoff

$$P(T) = \begin{cases} \sup_{0 \leq t \leq T} S_t - S_T & S_t < 110 \quad \forall t \\ 0 & \text{otherwise} \end{cases}$$

(c) Find a replicating portfolio for the above option for the first trading period.

Solution:

(a) For the risk-neutral probability we get $p = \frac{r-d}{u-d} = \frac{3}{4}$. The tree with the stock prices and the value of the option is



(b) The replicating portfolio can be found by solving the equations

$$\begin{aligned} 1.05 \cdot \varphi_1 + 110 \cdot \varphi_2 &= 0 \\ 1.05 \cdot \varphi_1 + 90 \cdot \varphi_2 &= 5.24 \end{aligned}$$

As solution we get $\varphi_1 = 27.45$ and $\varphi_2 = -0.262$.

4. Construct a three period binomial tree using the parameters $r = 0.1$ (discrete, per period), $u = 0.15$, $d = -0.05$ and $S_0 = 100$.

(a) Find the price of a European Put P with strike 105 and maturity date $T = 3$.

(b) Find the price of the knock in Call option C with knock in level $H = 110$, strike $K = 90$ and maturity date $T = 3$, i.e.

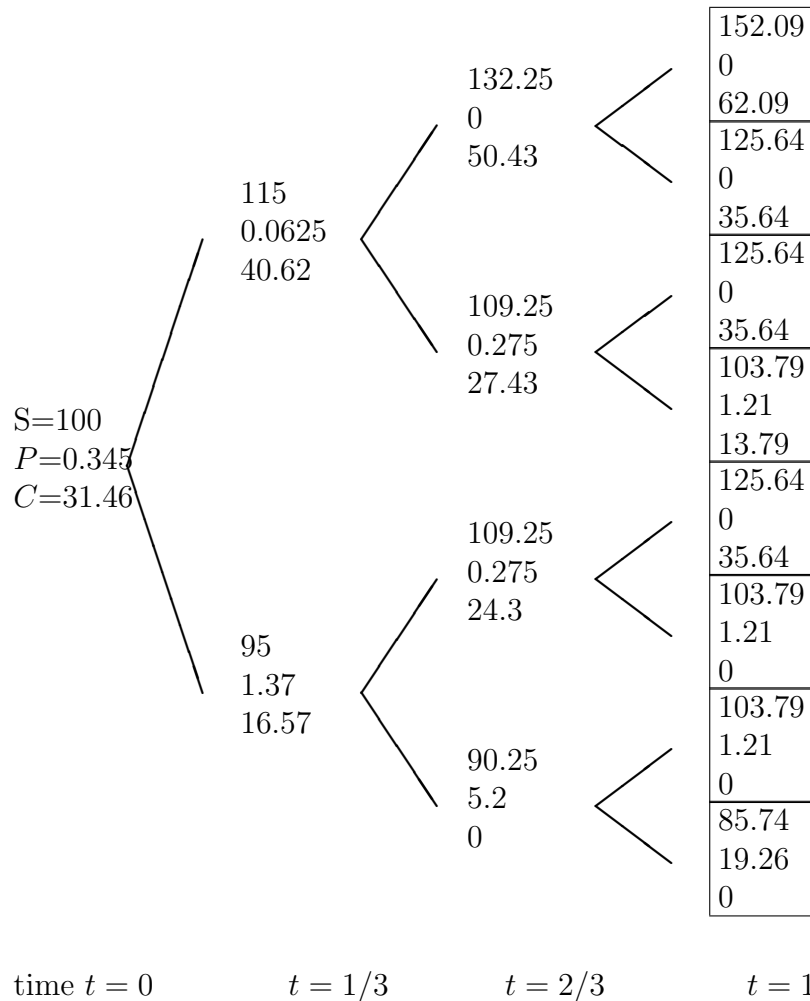
$$C = \begin{cases} (S(T) - 90)^+ & \exists t : S_t > H = 110 \\ 0 & S_t \leq H = 110 \forall t. \end{cases}$$

Solution:

The risk neutral probability is

$$p = \frac{r - d}{u - d} = \frac{0.1 + 0.05}{0.15 + 0.05} = \frac{3}{4}$$

We first set up a tree with the stock price movements, then compute the values of the two options:



5. Assume a 3-period Cox-Ross-Rubinstein model. The annual interest rate with continuous compounding is $r = 0.06$. The volatility of the stock is $\sigma = 0.2$ with a price of $S(0) = 100$. Furthermore, there exists an American Put with maturity date $T = 1$ und strike $K = 90$.

- (a) Calculate the risk-neutral probability and the stock prices at each node in the binomial tree (correct up to 2 decimal places after the decimal point).
 (b) Calculate the value of the American Put for all nodes in the tree.
 (c) What is the optimal stopping time? Justify your answer.

Solution:

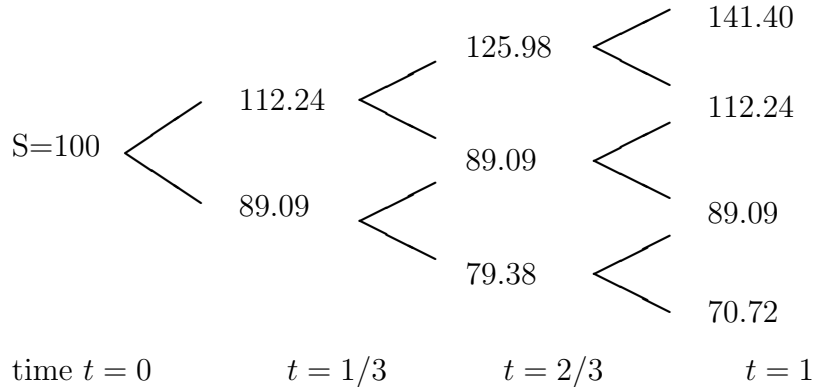
- (a) The parameter-values are

$$\Delta = \frac{1}{3}, \quad 1+r_d = e^{r\Delta} = 1.0202, \quad 1+u = e^{\sigma\sqrt{\Delta}} = 1.1224, \quad 1+d = e^{-\sigma\sqrt{\Delta}} = 0.8909.$$

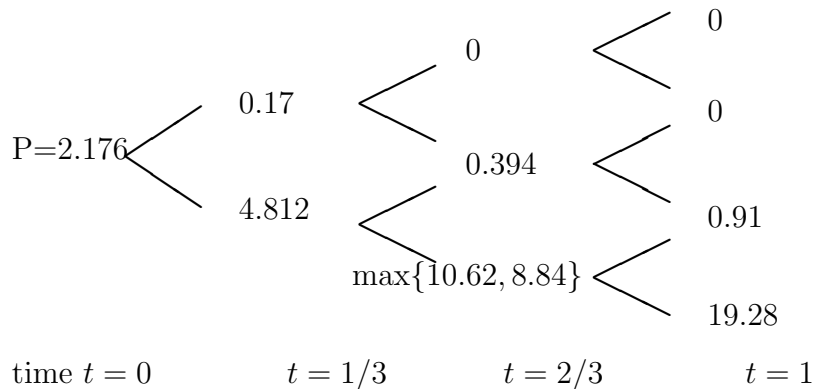
For the risk-neutral probability we get

$$p^* = \frac{r_d - d}{u - d} = 0.5584.$$

The tree with the stock prices is



- (b) The prices for the american Put are



- (c) Let $\Omega = \{u, d\}^3$. The optimal exercise date is

$$\tau(\omega) = \begin{cases} n = 2 & \omega \in \{ddu, ddd\} \\ n = 3 & \text{otherwise.} \end{cases}$$

For $\omega \in \{(ddu), (ddd)\}$, we have $\frac{1}{1+r_d} \mathbb{E}[p^* f_{32} + (1-p^*) f_{33}] < (K - S_0(1+d)^2)^+$. Here f_{ij} denotes the price of the claim in period i with j -down movements.

6. Assume that we have a three period CRR model with initial stock price $S = \$150$, interest rate $r = 0.05$ and volatility $\sigma = 0.2$.

(a) What is the value of an American Put with strike \$150, which matures in 6 months?

(b) What is the value of an American Call with strike \$150, which matures in 6 months?

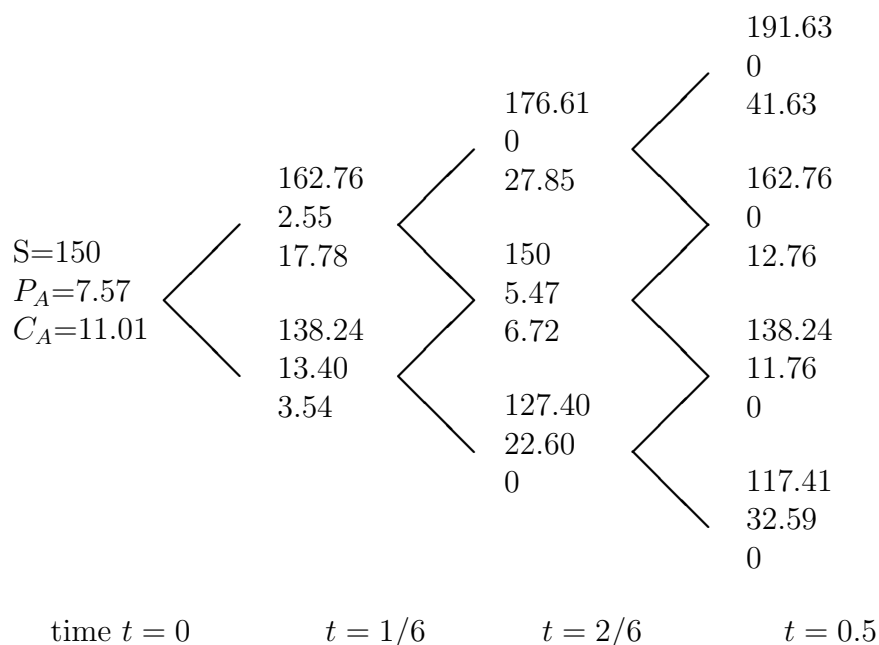
(c) Verify that the following inequalities hold:

$$S - K \leq C_A - P_A \leq S - Ke^{-rT}$$

Solution:

The martingale probability is $p = 0.5308$ with $u = 0.085$, $d = -0.0784$ and $r = 0.0084$.

(a), (b) For the American Put and Call we get:



(c) We have $0 \leq 11.01 - 7.57 = 3.44 \leq 3.7035 = 150 - 150 \cdot e^{-0.025}$

7. Show that a security market is arbitrage-free with respect to Φ iff it is arbitrage-free with respect to Φ_a . Here Φ is the set of all self-financing trading strategies and Φ_a is the set of all admissible strategies, that means all $\varphi \in \Phi$ with $V_\varphi(t) \geq 0 \quad t = 0, \dots, T$.

Solution:

First note, if $\varphi \in \Phi_a$ is an arbitrage strategy, then it is by definition of Φ_a also a strategy in Φ . We now have to show that if we have an arbitrage strategy $\varphi \in \Phi$, then there exists an arbitrage strategy $\psi \in \Phi_a$.

Assume that $\varphi \in \Phi$ is an arbitrage strategy. Then we have $V_\varphi(0) = 0$, $P(V_\varphi(T) \geq 0) = 1$ and $P(V_\varphi(T) > 0) > 0$. We have to distinguish between two cases:

Case 1: $V_\varphi(t) \geq 0 \quad t = 0, \dots, T$. Then $\varphi \in \Phi_a$ and we found the admissible arbitrage strategy.

Case 2: $\exists t^*, A \in \mathcal{F}_{t^*}$ with $V_\varphi(t^*, \omega) < 0 \quad \forall \omega \in A$ and $V_\varphi(t) \geq 0 \quad t > t^*$. Then define a new strategy ψ . Set $\psi(u, \omega) = 0 \quad \forall \omega \in A^c \quad \forall u$. Furthermore $\psi(u, \omega) = 0 \quad \omega \in A$ and $u \leq t^*$. For the remaining possibilities set

$$\psi_0(u, \omega) = \varphi_0(u, \omega) - \frac{V_\varphi(t^*, \omega)}{S_0(t^*, \omega)} \quad \forall \omega \in A \quad u > t^*$$

and

$$\psi_i(u, \omega) = \varphi_i(u, \omega) \quad \forall \omega \in A \quad i = 1, \dots, d \quad u > t^*$$

We have to show that this strategy is self-financing and admissible. For $\omega \in A^c$ we clearly have no problem. There is nothing to show for $\omega \in A, u \leq t^*$. ψ is also clearly self-financing for $u > t^* + 1$ as it just replicates the other strategy there. We have to show that $\psi(t^*)S(t^*) = \psi(t^* + 1)S(t^*)$. For $\omega \in A$ we have

$$\psi_0(t^* + 1)S_0(t^*) = \varphi_0(t^* + 1)S_0(t^*) - V_\varphi(t^*) \quad \text{and} \quad \psi_i(t^* + 1) = \varphi_i(t^* + 1)$$

Thus we get

$$\psi(t^* + 1)S(t^*) = \mathbb{1}_A(\varphi(t^* + 1)S(t^*) - V_\varphi(t^*)) = \mathbb{1}_A(\varphi(t^*)S(t^*) - V_\varphi(t^*)) = 0 = \psi(t^*)S(t^*)$$

It remains to show that ψ is admissible and an arbitrage opportunity. We get

$$V_\psi(t) = 0 \quad t \leq t^*$$

and

$$V_\psi(t) = \mathbb{1}_A \left(\varphi(t)S(t) - V_\varphi(t^*) \frac{S_0(t)}{S_0(t^*)} \right) = \mathbb{1}_A \left(V_\varphi(t) - V_\varphi(t^*) \frac{S_0(t)}{S_0(t^*)} \right) \geq 0$$

and > 0 on A for $t = T$ because $V_\varphi(t^*) < 0$. We also have that $V_\psi(t) = 0 \quad \forall t \leq t^*$. Therefore ψ is admissible and an arbitrage opportunity.

8. (a) State the Black-Scholes formula for an European Call and Put. (Hint: The Put-Call parity $C - P = S - Ke^{-r(T-t)}$ might be useful)
- (b) Replicate the European straddle with payoff $D(T) = |S(T) - K|$ using standard European options.
- (c) What is the Black-Scholes price of the straddle?
- (d) What is the Δ of the straddle? How much does the value of the straddle approximately change if the stock price changes from S_t to $S_t + \varepsilon$? (Hint: The Δ of the Call is $N(d_1)$)

Solution:

- (a) The Black-Scholes formula for an European Call and Put is

$$C(t) = S(t)N(d_1) - Ke^{-r(T-t)}N(d_2)$$

$$P(t) = Ke^{-r(T-t)}N(-d_2) - S(t)N(-d_1)$$

where

$$d_1 = \frac{\log(S/K) + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$$

$$d_2 = d_1 - \sigma\sqrt{T-t}.$$

- (b) We can replicate the straddle $D(T) = |S(T) - K|$ by buying one call and one put, both with strike K .
- (c) The Black-Scholes price of the straddle is

$$D(t) = C(t) + P(t) = S(t)N(d_1) - Ke^{-r(T-t)}N(d_2) + Ke^{-r(T-t)}N(-d_2) - S(t)N(-d_1) =$$

$$= S(t)(2N(d_1) - 1) - Ke^{-r(T-t)}(2N(d_2) - 1).$$

- (d) The Delta of the straddle is

$$\Delta_D = \Delta_C + \Delta_P = N(d_1) - N(-d_1) = 2N(d_1) - 1.$$

When the stock price changes from S_t to $S_t + \varepsilon$, then the price of the straddle changes about $\varepsilon(2N(d_1) - 1)$.

9. Consider a financial market in which the Black-Scholes formula for a European call option holds. The risk-free interest rate (cont. compounding) is r . The underlying stock has value S with volatility σ . For a European call with strike K and maturity T , show that the following relations hold:

$$\begin{aligned}\Delta &= \frac{\partial C}{\partial S} = N(d_1) \\ \Gamma &= \frac{\partial C}{\partial S^2} = \frac{N'(d_1)}{S\sigma\sqrt{T-t}} \\ \Theta &= \frac{\partial C}{\partial t} = -\frac{SN'(d_1)\sigma}{2\sqrt{T-t}} - rKe^{-r(T-t)}N(d_2) \\ \rho &= \frac{\partial C}{\partial r} = K(T-t)e^{-r(T-t)}N(d_2) \\ \nu &= \frac{\partial C}{\partial \sigma} = SN'(d_1)\sqrt{T-t}\end{aligned}$$

Show that the call satisfies the partial differential equation

$$\frac{\partial C}{\partial t} + rS\frac{\partial C}{\partial S} + \frac{1}{2}\sigma^2S^2\frac{\partial^2 C}{\partial S^2} - rC = 0.$$

Solution:

We first show that $SN'(d_1) = Ke^{-r(T-t)}N'(d_2)$:

$$\begin{aligned}SN'(d_1) - Ke^{-r(T-t)}N'(d_2) &= \frac{1}{\sqrt{2\pi}} \left(Se^{-d_1^2/2} - Ke^{-r(T-t)}e^{-d_2^2/2} \right) = \\ &= \frac{1}{\sqrt{2\pi}} \left(Se^{-d_1^2/2} - Ke^{-r(T-t)}e^{-d_1^2/2 + d_1\sigma\sqrt{T-t} - \sigma^2(T-t)/2} \right) = \\ &= \frac{1}{\sqrt{2\pi}} \left(Se^{-d_1^2/2} - Se^{-d_1^2/2} \frac{K}{S} e^{-r(T-t)} e^{d_1\sigma\sqrt{T-t} - \sigma^2(T-t)/2} \right) = \\ &= \frac{1}{\sqrt{2\pi}} \left(Se^{-d_1^2/2} - Se^{-d_1^2/2} \frac{K}{S} e^{-r(T-t)} e^{\log(S/K) + (r + \sigma^2/2)(T-t) - \sigma^2(T-t)/2} \right) = \\ &= \frac{1}{\sqrt{2\pi}} \left(Se^{-d_1^2/2} - Se^{-d_1^2/2} \right) = 0\end{aligned}$$

Now we calculate the Greeks:

(a)

$$\begin{aligned}\Delta &= \frac{\partial C}{\partial S} = N(d_1) + SN'(d_1)\frac{\partial d_1}{\partial S} - Ke^{-r(T-t)}N'(d_2)\frac{\partial d_2}{\partial S} = \\ &= N(d_1) + SN'(d_1)\left(\frac{\partial d_1}{\partial S} - \frac{\partial d_2}{\partial S}\right) = \\ &= N(d_1)\end{aligned}$$

(b)

$$\Gamma = \frac{\partial C}{\partial S^2} = \frac{\partial \Delta}{\partial S} = N'(d_1)\frac{\partial d_1}{\partial S} = \frac{N'(d_1)}{S\sigma\sqrt{T-t}}$$

(c)

$$\begin{aligned}
\Theta &= \frac{\partial C}{\partial t} = SN'(d_1) \frac{\partial d_1}{\partial t} - Ke^{-r(T-t)} N'(d_2) \frac{\partial d_2}{\partial t} - Kre^{-r(T-t)} N(d_2) = \\
&= SN'(d_1) \left(\frac{\partial d_1}{\partial t} - \frac{\partial d_2}{\partial t} \right) - Kre^{-r(T-t)} N(d_2) = \\
&= -SN'(d_1) \frac{\sigma}{2\sqrt{T-t}} - rKe^{-r(T-t)} N(d_2)
\end{aligned}$$

(d)

$$\begin{aligned}
\rho &= \frac{\partial C}{\partial r} = SN'(d_1) \frac{\partial d_1}{\partial r} - Ke^{-r(T-t)} N'(d_2) \frac{\partial d_2}{\partial r} + K(T-t)e^{-r(T-t)} N(d_2) = \\
&= SN'(d_1) \left(\frac{\partial d_1}{\partial r} - \frac{\partial d_2}{\partial r} \right) + K(T-t)e^{-r(T-t)} N(d_2) = \\
&= K(T-t)e^{-r(T-t)} N(d_2)
\end{aligned}$$

(e)

$$\begin{aligned}
\nu &= \frac{\partial C}{\partial \sigma} = SN'(d_1) \frac{\partial d_1}{\partial \sigma} - Ke^{-r(T-t)} N'(d_2) \frac{\partial d_2}{\partial \sigma} = \\
&= SN'(d_1) \left(\frac{\partial d_1}{\partial \sigma} - \frac{\partial d_2}{\partial \sigma} \right) = \\
&= SN'(d_1) \sqrt{T-t}.
\end{aligned}$$

The partial differential equation holds because:

$$\begin{aligned}
&\frac{\partial C}{\partial t} + rS \frac{\partial C}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} - rC = \\
&= -SN'(d_1) \frac{\sigma}{2\sqrt{T-t}} - rKe^{-r(T-t)} N(d_2) + \\
&+ rSN(d_1) + \\
&+ \frac{1}{2} \sigma^2 S^2 \frac{N'(d_1)}{S\sigma\sqrt{T-t}} + \\
&+ rC = \\
&= r(SN(d_1) - Ke^{-r(T-t)} N(d_2) - C) = 0.
\end{aligned}$$

10. Prove the following limit relations used in the proof of Proposition 3.5.1, assuming that $k_n \rightarrow \infty$ ($n \rightarrow \infty$):

$$\lim_{n \rightarrow \infty} \hat{p}_n = \frac{1}{2}, \quad \lim_{n \rightarrow \infty} k_n(1 - 2\hat{p}_n)\sqrt{\Delta_n} = -T \left(\frac{r}{\sigma} + \frac{\sigma}{2} \right)$$

Solution:

We have the following definitions for the variables:

$$\begin{aligned} \Delta_n &= \frac{T}{k_n} \\ u_n &= e^{\sigma\sqrt{\Delta_n}} - 1 \\ d_n &= e^{-\sigma\sqrt{\Delta_n}} - 1 \\ r_n &= e^{r\Delta_n} - 1 \\ p_n^* &= \frac{r_n - d_n}{u_n - d_n} \\ \hat{p}_n &= p_n^* \frac{1 + u_n}{1 + r_n} \end{aligned}$$

Then, we get for the first limit relation:

$$\lim_{n \rightarrow \infty} \hat{p}_n = \lim_{n \rightarrow \infty} p_n^* \underbrace{\lim_{n \rightarrow \infty} \frac{1 + u_n}{1 + r_n}}_{\rightarrow 1(n \rightarrow \infty)} = \lim_{n \rightarrow \infty} \frac{e^{r\Delta_n} - e^{-\sigma\sqrt{\Delta_n}}}{e^{\sigma\sqrt{\Delta_n}} - e^{-\sigma\sqrt{\Delta_n}}} = \frac{1}{2}.$$

In order to show the last equality, it suffices to show that

$$\lim_{x \rightarrow 0_+} \frac{e^{rx^2} - e^{-\sigma x}}{e^{\sigma x} - e^{-\sigma x}} = \frac{1}{2}.$$

as $\sqrt{\Delta_n} \rightarrow 0_+(n \rightarrow \infty)$.

By L'Hospital we get

$$\lim_{x \rightarrow 0_+} \frac{e^{rx^2} - e^{-\sigma x}}{e^{\sigma x} - e^{-\sigma x}} = \lim_{x \rightarrow 0_+} \frac{2xre^{rx^2} + \sigma e^{-\sigma x}}{\sigma e^{\sigma x} + \sigma e^{-\sigma x}} = \frac{1}{2}.$$

For the second limit relation we get:

$$\begin{aligned} \lim_{n \rightarrow \infty} k_n(1 - 2\hat{p}_n)\sqrt{\Delta_n} &= \lim_{n \rightarrow \infty} \sqrt{Tk_n} \left(1 - 2 \frac{e^{r\Delta_n} - e^{-\sigma\sqrt{\Delta_n}}}{e^{\sigma\sqrt{\Delta_n}} - e^{-\sigma\sqrt{\Delta_n}}} \frac{e^{-r\Delta_n}}{e^{-\sigma\sqrt{\Delta_n}}} \right) = \\ &= \lim_{n \rightarrow \infty} \sqrt{T} \frac{1 - e^{-2\sigma\sqrt{\Delta_n}} - 2 + 2e^{-\sigma\sqrt{\Delta_n} - r\Delta_n}}{\frac{1}{\sqrt{k_n}} (1 - e^{-2\sigma\sqrt{\Delta_n}})} \\ &= \sqrt{T} \left(-\sqrt{T} \left(\frac{\sigma}{2} + \frac{r}{\sigma} \right) \right) = -T \left(\frac{\sigma}{2} + \frac{r}{\sigma} \right). \end{aligned}$$

For the second to last equation it suffices to show that:

$$\lim_{x \rightarrow 0_+} \frac{-e^{-2\sigma x} - 1 + 2e^{-\sigma x - rx^2}}{\frac{x}{\sqrt{T}} (1 - e^{-2\sigma x})} = -\sqrt{T} \left(\frac{\sigma}{2} + \frac{r}{\sigma} \right)$$

as $\sqrt{\Delta_n} \rightarrow 0_+ (n \rightarrow \infty)$.

We are using L'Hospital twice and get:

$$\begin{aligned} \lim_{x \rightarrow 0_+} \frac{-e^{-2\sigma x} - 1 + 2e^{-\sigma x - rx^2}}{\frac{x}{\sqrt{T}}(1 - e^{-2\sigma x})} &= \\ &= \lim_{x \rightarrow 0_+} \sqrt{T} \frac{2\sigma e^{-2\sigma x} - 2(\sigma + 2rx)e^{-\sigma x - rx^2}}{(1 - e^{-2\sigma x}) + 2x\sigma e^{-2\sigma x}} = \\ &= \lim_{x \rightarrow 0_+} \sqrt{T} \frac{-4\sigma^2 e^{-2\sigma x} + 2(\sigma + 2rx)^2 e^{-\sigma x - rx^2} - 4r e^{-\sigma x - rx^2}}{2\sigma e^{-2\sigma x} + 2\sigma e^{-2\sigma x} - 4x\sigma^2 e^{-2\sigma x}} = \\ &= \sqrt{T} \frac{-4\sigma^2 + 2\sigma^2 - 4r}{4\sigma} = -\sqrt{T} \left(\frac{\sigma}{2} + \frac{r}{\sigma} \right). \end{aligned}$$



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Risk-Neutral Valuation

Pricing and Hedging of Financial Derivatives

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