

Lectures Notes from Pure Maths by Steve Warner

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Introduction

This book is an introduction to undergraduate pure mathematics. This books contains 16 chapters with exercices ranging from different levels:

1. Lesson 1 - Logic: Statements and Truth
2. Lesson 2 - Set Theory: Sets and Subsets
3. Lesson 3 - Abstract Algebra: Semigroups, Monoids and Groups
4. Lesson 4 - Number Theory: The Ting of Integers
5. Lesson 5 - Real Analysis: The Complete Ordered Field of Reals
6. Lesson 6 - Topology: The Topology of \mathbb{R}
7. Lesson 7 - Complex Analysis: The Field of Complex Numbers
8. Lesson 8 - Linear Algebra: Vector Spaces
9. Lesson 9 - Logic: Logical Arguments
10. Lesson 10 - Set Theory: Reltions and Functions
11. Lesson 11 - Abstract ALgebra: Strucutres and Homomorphisms
12. Lesson 12 - Number Theory: Primes, GCD, and LCM
13. Lesson 13 - Real Analysis: Limits and Continuity
14. Lesson 14 - Topology: Spaces and Homeomorphisms
15. Lesson 15 - Complex Analysis: Complex Valued Functions
16. Lesson 16 - Linear Algebra: Linear Transformations

1 Lesson 1 - Logic: Statements and Truth

1.1 Overview

This section introduces the notion of statements and truth tables.

1.2 Statements with words

The goal of this section is to determine whether a sentence is a statement or not.

A statement or proposition is a sentence that can be true or false, but not both simultaneously. If it expresses a single idea, we say it is an atomic statement. If we want to create a statement with more than one idea, we need to connect the atomic statement using logical connectives.

1.3 Statements with symbols

In mathematics, we want to express mathematical statement in symbols since they attract away the unnecessary clutter and help us focus on the form of the statement. Consequently, we need to define a set of symbols used to define the common logical connectives. The most common are:

1. Conjunction
2. Disjunction
3. Negation
4. Implication
5. Biconditional

1.4 Truth Table

Each logical connective is associated to a truth table, which tell us the truth value of a compound statement based on the truth value of the propositional variables. Here is the truth table for the common logical connectives:

We can also use truth table to determine if two statements are equivalent. To do so, we compare their truth table.

1.5 Problem Set

1.5.1 Level 1

1.5.2 Level 2

1.5.3 Level 3

1.5.4 Level 4

1.5.5 Level 5

2 Lesson 2 - Set Theory: Sets and Subsets

2.1 Overview

This section introduce the notion of sets, subsets, union and intersection.

2.2 Describing Sets

Definition 2.2.1 (Set) *A set is a collection of objects. The set can either be finite or infinite. We can describe the set based on a common characteristic among the elements of the set using the set builder notation. We write $\{x|P(x)\}$, where $P(x)$ is the common characteristic.*

Definition 2.2.2 (Axiom of Extensionality) *Two sets are equivalent if they contain the same element, we write:*

$$\forall x(x \in A \leftrightarrow x \in B)$$

Definition 2.2.3 (Cardinality) *The cardinality of an element is the number of different element in the set. For example, the set $S=1,2,3$ has the same cardinality as the set $T=1,2,2,3$*

Theorem 2.2.1 (Fence-Post Formula) *To count the number of integers in a set, we use the fence-post formula*

$$n - m + 1$$

2.3 Subsets

Definition 2.3.1 (Subset) *We say that A is a subset of B if every element of A is an element of B . We write $A \subseteq B$*

$$\forall x(x \in A \rightarrow x \in B)$$

Theorem 2.3.1 *Every set A is a subset of itself*

$$\forall x(x \in A \rightarrow x \in A)$$

Theorem 2.3.2 *The empty set is a subset of every set*

$$\forall x(x \in \emptyset \rightarrow x \in A)$$

Theorem 2.3.3 (Transitivity of sets) *Let A , B , C be sets such that*

$$A \subseteq B, B \subseteq C. \text{ Then, } A \subseteq C$$

Theorem 2.3.4 *There are 2^n subsets in a set*

2.4 Unions and Intersections

Definition 2.4.1 (Union) *The union of the sets A and B , written $A \cup B$, is the set of elements that are in A or B (or both).*

$$\forall x(x|x \in A \vee x \in B)$$

Definition 2.4.2 (Intersection) *The intersection of the sets A and B , written $A \cap B$, is the set of elements that are in A and B simultaneously.*

$$\forall x(x|x \in A \wedge x \in B)$$

Theorem 2.4.1 *If A and B are sets, then $A \subseteq A \cup B$*

Theorem 2.4.2 $B \subseteq A \iff A \cup B = A$

Theorem 2.4.3 $B \subseteq A \iff A \cap B = B$

Definition 2.4.3 (Reflexive) *A relation R is reflexive if $\forall x(xRx)$*

Definition 2.4.4 (Symmetric) *A relation R is symmetric if*

$$\forall x \forall y (xRy \rightarrow yRx)$$

2.5 Problem Set

2.5.1 Level 1

2.5.2 Level 2

2.5.3 Level 3

2.5.4 Level 4

2.5.5 Level 5

3 Lesson 3 - Abstract Algebra: Semigroups, Monoids and Groups

3.1 Overview

This section focus on the properties of semigroups, monoids and groups. It gives us an intuition on why some set behave a certain way while other don't.

To determine wether a set has certain properties, we often use a multiplication table.

3.2 Binary Operations and Closure

Definition 3.2.1 (Binary Operation) *A binary operation on a set is a rule that combines two elements of the set to produce another element of the set*

Definition 3.2.2 (Closed) *We say that the set S is closed under the partial binary operation $*$ if whenever $a, b \in S$, we have $a * b \in S$*

3.3 Semigroups and Associativity

Definition 3.3.1 (Associativity) *Let $*$ be a binary operation on a set. We say that $*$ is associative in S if for all x, y, z in S , we have*

$$x * (y * z) = (x * y) * z$$

Definition 3.3.2 (Semigroup) *A semigroup is a pair $(S, *)$, where S is a set and $*$ is an associative binary operation on S*

Corollary 3.3.1 *If the binary operator $*$ is not associative in S , then the pair $(S, *)$ is not a semigroup*

Definition 3.3.3 (Abelian or Commutative) *Let $*$ be a binary operation on a set. We say that $*$ is abelian (or commutative) in S if for all x, y, z in S , we have*

$$x * y = y * x$$

Definition 3.3.4 (Abelian Semigroup) *An abelian semigroup is a semigroup that is commutative. Therefore, it has the following properties:*

1. Closed
2. Associative
3. Commutative

3.4 Monoids and Identity

Definition 3.4.1 (Identity) *Let $(S, *)$ be a semigroup. An element e of S is called an identity with respect of the binary operation $*$ if for all $a \in S$, we have $a * e = e * a = a$*

Definition 3.4.2 (Monoid) *A monoid is a semigroup with an identity. In other word, a monoid is*

1. *Closed*
2. *Associative*
3. *Identity*

Theorem 3.4.1 (Unique Identity) *Let $(M, *)$ be a monoid with identity e . The identity element is unique*

3.5 Groups and Inverses

Definition 3.5.1 (Inverse) *Let $(M, *)$ be a monoid with identity e . An element a of M is called invertible if there is an element $b \in M$ such that $a * b = b * a = e$*

Definition 3.5.2 (Group) *A group is a monoid in which every element is invertible. Therefore, a group follows the following properties*

1. *Closed*
2. *Associative*
3. *Identity*
4. *Inversible*

Theorem 3.5.1 (Unique Inverse) *Let $(G, *)$ be a group. Each element in G has a unique inverse*

3.6 Problem Set

3.6.1 Level 1

3.6.2 Level 2

3.6.3 Level 3

3.6.4 Level 4

3.6.5 Level 5

4 Lesson 4 - Number Theory: The Ring of Integers

4.1 Overview

The goal of this section is to familiarise ourselves with induction proofs. However, the proofs we have to prove utilize integers properties, so we have to define ring properties first.

The notion of ring utilize the concepts of closure, associativity, abelian, identity and inverses which we saw in the previous section.

4.2 Ring and Distributivity

Definition 4.2.1 (Commutative Group) *A commutative group is a group that follows the following properties:*

1. Closure
2. Associative
3. Commutative
4. Identity
5. Inverse

Lemma 4.2.1 $(\mathbb{Z}, +)$ is a commutative group

Definition 4.2.2 (Commutative Monoid) *A commutative monoid is a monoid that follows the following properties:*

1. Closure
2. Associative
3. Commutative
4. Identity

Lemma 4.2.2 (\mathbb{Z}, \cdot) is a commutative monoid

Definition 4.2.3 (Ring) *A ring is a triple $(R, +, \cdot)$ where R is a set, $+$ and \cdot are binary operations on R that satisfies:*

1. $(R, +)$ is commutative group
2. (R, \cdot) is a commutative monoid
3. Multiplication is distributive over addition in R . That is, for all $x, y, z \in R$, we have

$$x \cdot (y + z) = x \cdot y + x \cdot z \text{ and } (y + z) \cdot x = y \cdot x + z \cdot x$$

It should be noted that the properties that define a ring are called the ring axioms

Lemma 4.2.3 $(\mathbb{Z}, +, \cdot)$ is a commutative ring

Lemma 4.2.4 $(\mathbb{N}, +, \cdot)$ is a ring because $(\mathbb{N}, +)$ is not a group. We say it is a semiring

4.3 Divisibility

Definition 4.3.1 (Even) An integer a is called even if there is another integer b such that $a = 2b$

Definition 4.3.2 (Odd) An integer a is called odd if there is another integer b such that $a = 2b + 1$

Definition 4.3.3 (Sum) We define the sum of integers a and b to be $a + b$

Definition 4.3.4 (Product) We define the product of integers a and b to be $a \cdot b$

Theorem 4.3.1 The sum of two even integer is even

Theorem 4.3.2 The product of two integers that are each divisible by k is also divisible by k

4.4 Induction

Definition 4.4.1 (Well Ordering Principle) The Well Ordering Principle says that every nonempty subset of natural numbers has a least element

Theorem 4.4.1 (Principle of Mathematical Induction) Let S be a set of natural numbers such that

1. $0 \in S$
2. for all $k \in \mathbb{N}, k \in S \rightarrow k + 1$. Then, $S = \mathbb{N}$

Lemma 4.4.2 (Standard Advanced Calculus Trick) We can add and subtract the same quantities without changing the result

4.5 Problem Set

4.5.1 Level 1

4.5.2 Level 2

4.5.3 Level 3

4.5.4 Level 4

4.5.5 Level 5

5 Lesson 5 - Real Analysis: The Complete Ordered Field of Reals

5.1 Overview

The goal of this section is TODO

5.2 Field

Definition 5.2.1 (Field) A field is a triple $(F, +, \cdot)$, where F is a set and $+$ and \cdot are binary operations on F satisfying:

1. $(F, +)$ is a commutative group
2. (F, \cdot) is a commutative group
3. Multiplication is distributive over addition in F . That is, for all $x, y, z \in F$, we have

$$x \cdot (y + z) = x \cdot y + x \cdot z \text{ and } (y + z) \cdot x = y \cdot x + z \cdot x$$

4. $0 \neq 1$

The properties that define a field are called the field axioms

Lemma 5.2.1 (Set of Natural Numbers) The set \mathbb{N} is the set of natural numbers and the structure $(\mathbb{N}, +, \cdot)$ is a semiring

Lemma 5.2.2 (Set of Integers) The set \mathbb{Z} is the set of integers and the structure $(\mathbb{Z}, +, \cdot)$ is a ring

Lemma 5.2.3 (Set of Rational Numbers) The set \mathbb{Q} is the set of rational numbers and the structure $(\mathbb{Q}, +, \cdot)$ is a field

Definition 5.2.2 (Subtraction) If $a, b \in F$, we define the subtraction $a - b = a + (-b)$

Definition 5.2.3 (Division) If $a, b \in F$ and $b \neq 0$, we define the division $a/b = ab^{-1}$

5.3 Ordered Rings and Fields

Definition 5.3.1 (Positive and Negative Elements) If $a \in P$, we say that a is positive and if $-a \in P$, we say that a is negative

Definition 5.3.2 (Ordered Ring) We say that a ring $(R, +, \cdot)$ is ordered if there is a nonempty subset P of R , called the set of positive elements of R satisfying the following properties

1. if $a, b \in P$, then $a + b \in P$
2. if $a, b \in P$, then $ab \in P$

3. if $a \in P$, then exactly one of the following holds:

$$a \in P, a = 0, \text{ or } -a \in P$$

Theorem 5.3.1 $(\mathbb{Q}, +, \cdot)$ is an ordered field

Theorem 5.3.2 Let (F, \leq) be an ordered field. Then, for all $x \in F^*$, $x \cdot x > 0$

Theorem 5.3.3 Every ordered field (F, \leq) contains a copy of the natural numbers.

Theorem 5.3.4 Let (F, \leq) be an ordered field and let $x \in F$ with $x > 0$. Then, $\frac{1}{x} > 0$

5.4 Why Isn't \mathbb{Q} Enough?

Theorem 5.4.1 (Pythagorean Theorem) In a right triangle with legs of length a and b , and a hypotenuse of length c

$$c^2 = a^2 + b^2$$

Theorem 5.4.2 There does not exist a rational number a such that $a^2 = 2$

5.5 Completeness

Definition 5.5.1 (Upper Bound) Let (F, \leq) be an ordered field and let S be a nonempty subset of F . We say that S is bounded above if there is $M \in F$ such that for all $s \in S$, $s \leq M$. Each number M is called an upper bound of S

Definition 5.5.2 (Lower Bound) Let (F, \leq) be an ordered field and let S be a nonempty subset of F . We say that S is bounded below if there is $K \in F$ such that for all $s \in S$, $K \leq s$. Each number K is called a lower bound of S

Definition 5.5.3 (Bounded Set) We say that S is bounded if it is bounded above and bounded below. Otherwise, we say that S is unbounded.

Definition 5.5.4 (Supremum) A least upper bound of a set S is an upper bound that is smaller than any other upper bound of S

Definition 5.5.5 (Infimum) A greatest lower bound of S is a lower bound that is larger than any other lower bound of S

Definition 5.5.6 (Completeness Property) An ordered field (F, \leq) has the Completeness Property if every nonempty subset of F that is bounded above has a least upper bound in F . In this case, we say that (F, \leq) is a complete ordered field.

Theorem 5.5.1 There is exactly one complete ordered field

Theorem 5.5.2 (Archimedean Property of \mathbb{R}) For every $x \in \mathbb{R}$, there is $n \in \mathbb{N}$ such that $n > x$

Theorem 5.5.3 (Density Theorem) If $x, y \in \mathbb{R}$ with $x < y$ then there is $q \in \mathbb{Q}$ with $x < q < y$

5.6 Problem Set

5.6.1 Level 1

5.6.2 Level 2

5.6.3 Level 3

5.6.4 Level 4

5.6.5 Level 5

6 Lesson 6 - Topology: The Topology of \mathbb{R}

6.1 Overview

6.2 Intervals of Real Numbers

Definition 6.2.1 (Interval) A set I of real numbers is called an interval if any real number that lies between two numbers in I is also in I . We write:

$$\forall x, y \in I, \forall z \in \mathbb{R}, \text{ if } x \text{ is less than } z \text{ and } z \text{ is less than } y, \text{ then } z \text{ is in } I$$

Here is a list of the other types of intervals:

1. Open Interval
2. Closed Interval
3. Half-open Interval
4. Infinite Open Interval
5. Infinite Closed Interval

Theorem 6.2.1 If an interval I is bounded, then there are $a, b \in \mathbb{R}$ such that one of the following holds:

$$I = (a, b), I = (a, b], \text{ or } I = [a, b)$$

6.3 Operations on Sets

Definition 6.3.1 (Union) The union of the sets A and B , written $A \cup B$, is the set of elements that are in A or B (or both).

$$\forall x (x \in A \vee x \in B)$$

Definition 6.3.2 (Intersection) The intersection of the sets A and B , written $A \cap B$, is the set of elements that are in A and B simultaneously.

$$\forall x (x \in A \wedge x \in B)$$

Definition 6.3.3 (Difference) The difference $A \setminus B$ is the set of elements that are in A and not in B .

$$A \setminus B = \{x \mid x \in A \text{ and } x \notin B\}$$

Definition 6.3.4 (Symmetric Difference) The symmetric difference $A \triangle B$ is the set of elements that are in A or B , but not both.

$$A \triangle B = (A \setminus B) \cup (B \setminus A)$$

Theorem 6.3.1 The operation of forming unions is associative

6.4 Open and Closed Sets

Definition 6.4.1 (Open Set) A subset X of \mathbb{R} is open if for every real number $x \in \mathbb{R}$, there is an open interval (a, b) with $x \in (a, b)$ and $(a, b) \subseteq X$

Definition 6.4.2 (Closed Set)

Theorem 6.4.1 Let $a \in \mathbb{R}$ The infinite interval (a, ∞) is an open set

Theorem 6.4.2 \emptyset and \mathbb{R} are both open sets

Theorem 6.4.3 A subset X of \mathbb{R} is open if and only if for every real number $x \in X$, there is a positive real number c such that $(x - c, x + c) \subseteq X$

Theorem 6.4.4 The union of two open sets in \mathbb{R} is an open set in \mathbb{R}

Theorem 6.4.5 Let X be a set of open subsets of \mathbb{R} . Then $\bigcup X$ is open

Theorem 6.4.6 Every open set in \mathbb{R} can be expressed as a union of bounded open intervals

Theorem 6.4.7 The intersection of two open sets in \mathbb{R} is an open set in \mathbb{R}

Theorem 6.4.8

Theorem 6.4.9

6.5

6.6 Problem Set

6.6.1 Level 1

6.6.2 Level 2

6.6.3 Level 3

6.6.4 Level 4

6.6.5 Level 5

7 Lesson 7 - Complex Analysis: The Field of Complex Numbers

7.1 Overview

7.2 Problem Set

7.2.1 Level 1

7.2.2 Level 2

7.2.3 Level 3

7.2.4 Level 4

7.2.5 Level 5

8 Lesson 8 - Linear Algebra: Vector Spaces

8.1 Overview

8.2 Problem Set

8.2.1 Level 1

8.2.2 Level 2

8.2.3 Level 3

8.2.4 Level 4

8.2.5 Level 5

9 Lesson 9 - Logic: Logical Arguments

9.1 Overview

9.2 Problem Set

9.2.1 Level 1

9.2.2 Level 2

9.2.3 Level 3

9.2.4 Level 4

9.2.5 Level 5

10 Lesson 10 - Set Theory: Relations and Functions

10.1 Overview

10.2 Problem Set

10.2.1 Level 1

10.2.2 Level 2

10.2.3 Level 3

10.2.4 Level 4

10.2.5 Level 5

11 Lesson 11 - Abstract Algebra: Structures and Homomorphisms

11.1 Overview

11.2 Problem Set

11.2.1 Level 1

11.2.2 Level 2

11.2.3 Level 3

11.2.4 Level 4

11.2.5 Level 5

12 Lesson 12 - Number Theory: Primes, GCD, and LCM

12.1 Overview

12.2 Problem Set

12.2.1 Level 1

12.2.2 Level 2

12.2.3 Level 3

12.2.4 Level 4

12.2.5 Level 5

13 Lesson 13 - Real Analysis: Limits and Continuity

13.1 Overview

13.2 Problem Set

13.2.1 Level 1

13.2.2 Level 2

13.2.3 Level 3

13.2.4 Level 4

13.2.5 Level 5

14 Lesson 14 - Topology: Spaces and Homeomorphisms

14.1 Overview

14.2 Problem Set

14.2.1 Level 1

14.2.2 Level 2

14.2.3 Level 3

14.2.4 Level 4

14.2.5 Level 5

15 Lesson 15 - Complex Analysis: Complex Valued Functions

15.1 Overview

15.2 Problem Set

15.2.1 Level 1

15.2.2 Level 2

15.2.3 Level 3

15.2.4 Level 4

15.2.5 Level 5

16 Lesson 16 - Linear Algebra: Linear Transformations

16.1 Overview

16.2 Problem Set

16.2.1 Level 1

16.2.2 Level 2

16.2.3 Level 3

16.2.4 Level 4

16.2.5 Level 5