Lecture Notes for Linear Algebra

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1 Introduction

Linear Algebra is the study of vectors and its transformation into space, and has several applications in Engineering and Computer Science. For example, matrices are utilize when applying masks in Machine Learning and Linear transformation are used for computer graphics.

Linear Algebra is often taught following this sequence:

1. Linear Equations
2. Matrix Algebra
3. Determinants
4. Vector Spaces
5. Eigenvalues and Eigenvectors
6. Orthogonality and Least Squares

Professor tend to emphasize on the algebraic component of linear algebra, but we always have to keep in mind what the structures and objects represent graphically.

Part I

Linear Equations

1.1 Overview

1.2 Solving Linear Systems

The first chapter of linear algebra focusses on how we can use the Gaussian method to solve linear systems. We want to determine if the system is compatible, and if it is, we want to verify id the solution is unique or more.

1.2.1 Matrix Notation

A common representation of linear system is with augmented matrix.

1.2.2 Types of Solutions

When we solve linear systems, we often transform it into its matrix notation. However, it can be useful to think about it in terms of vectors or plane that crosses one another.

A SEL can have different types of solutions:

1. No solution: Lines are parallel (geometrically); contradiction (algebraically)
2. Exactly one solution: Lines crosses each other; all variables are associated to one pivot
3. Infinitely many solutions: Lines are on top of each other; we have a free varaible

Definition 1.2.1 (Consistent and Inconsistent System)**.** *A system of linear equations is said to be consistent if it has one or more solutions; it is inconsistent if it has no solution.*

1.2.3 Matrix Operations

To solve a linear system, we want to use the elementary operations.

1. (Replacement) Replace one row by the sum of itself with a multiple of another row
2. (Interchange) Interchange two rows
3. (Scaling) Multiply all entries in a row by a nonzero constant

Definition 1.2.2 (Pivot)**.**

Definition 1.2.3 (Free Variables)**.**

Definition 1.2.4 (Elechon Matrix)**.** *We say we a matrix is in echelon form if:*

1.

Definition 1.2.5 (Reduced Echelon Matrix)**.**

1.2.4 Row Reduction Algorithm

1.3 Vector Equations

The second goal of this chapter is to understand vectors operations and properties. We will define them geometrically and algebrically.

1.3.1 Vector Addition

Geometrically, we can visualize vector addition as if we were adding vectors at the tip of each other.

1.3.2 Vector Properties

Since $\mathbb{R}^n$ is a field, our vector space follows the field axiom:

1.
2.
3.
4.
5.
6.
7.
8.
9.

1.3.3 Linear Combination

Linear Combination is a weighted sum of vector that allow us to generate other vectors. Later, we will see that when vectors are independant, we can generate new vector in another dimension.

Definition 1.3.1 (Linear Combination)**.**

Definition 1.3.2 (Span)**.**

1.3.4 Matrix Equation Ax=b

Definition 1.3.3 (Computation of Ax)**.** *The product of a matrix and a vector can be interpreted as TODO*

Theorem 1.3.1 (Properties of Matrix-Vector Product)**.**

Definition 1.3.4 (Homogeneous System)**.**

1.3.5 Linear Independence

Intuitively, two vectors are linearly independant if they can be expressed in terms of the other one. We care about linear independance because it will allows us to add another dimension to our basis when generating new vectors.

Definition 1.3.5 (Linear Independence)**.**

1.3.6 Linear Transfomations

Geometrically, we can interpret a matrix as a transformation ie a function that transform a vector into another one when we perform matrix multiplication. However, we rather focus on linear transformation, which are transformations where the transformed vectors keeps the same properties as it has before (addition and multiplication)

Definition 1.3.6 (Linear Transformation)**.**

Definition 1.3.7 (Types of Linear Transformation)**.** *There exists a few notable linear transformation*

1.
2.
3. *Translation*
4. *Rotation*

1.3.7 Kernel and Image

Definition 1.3.8 (Kernel)**.**

Definition 1.3.9 (Image)**.**

Part II

Matrix Algebra

1.4 Overview

1.4.1 Matrix Addition

Geometrically, we can interpret matrix addition as TODO

Theorem 1.4.1 (Properties of Matrix Addition)**.**

1.4.2 Matrix Multiplication

Matrix multiplication is often used for composition of linear transformations. Instead of performing a bunch of transformation one after the other, we can compute a single matrix that perform all the transformation in one shot.

Definition 1.4.1 (Matrix Multiplication)**.**

Theorem 1.4.2 (Properties of Matrix Multiplication)**.**

1.4.3 Transpose of a Matrix

We can interpret the transpose of a matrix as TODO

Definition 1.4.2 (Transpose of a Matrix)**.**

Theorem 1.4.3 (Transpose Properties)**.**

1.4.4 Inverse of a Matrix

Until now, we wanted to compute the matrix that represented the linear transformation to be applied onto our vector. However, if we are given the initial vector and the final vector, we can use the inverse of a matrix to find the linear transformation.

Remark. *We have the equation $Ax=b$, and we want to find the matrix A given x and b. With real, we would divide. However, with matrices, we cannot divide. The inverse matrix act similarly as a division in the real.*

Remark. *We know a matrix has an inverse if its determinant is not 0. Intuitively, the determinant measure how much the area between two vectors has increase/decrease. A determinant of zero would mean that the vector has collapse of dimension (as if we were to divide by zero)*

Definition 1.4.3 (Inverse of a Matrix)**.**

Theorem 1.4.4 (Inverse of 2x2 Matrix)**.**

Theorem 1.4.5 (Finding A^-1 with Gauss Elimination)**.** $[A|I]$ $[I|A^-1]$

Theorem 1.4.6 (The Invertible Matrix Theorem)**.**

1.4.5 Elementary Matrices

Elementary matrices are important because they help us find the inverse of a matrix. Since elementary matrices are invertible, we wil use this property when diagonalizing our matrices (later)

Definition 1.4.4 (Elementary Matrix)**.**

1.4.6 LU Factorization

The LU factorization is a technique used to solve linear systems by using the fact that triangular matrices are easier to calculate their inverse.

Theorem 1.4.7 (LU Factorization)**.**

1.4.7 Subspaces of $\mathbb{R}^n$

Subspaces are subsets of a vector space that is also a vector space. We can think of lines or planes. TODO

Definition 1.4.5 (Subspace)**.**

Definition 1.4.6 (Column Space)**.**

Definition 1.4.7 (Null Space)**.**

Theorem 1.4.8 (Basis)**.**

1.4.8 Dimension and Rank

Definition 1.4.8 (Coordinate System)**.**

Definition 1.4.9 (Dimension)**.**

Definition 1.4.10 (Rank)**.**

Theorem 1.4.9 (Rank Theorem)**.**

Theorem 1.4.10 (Basis Theorem)**.**

Theorem 1.4.11 (The Invertible Matrix Theorem)**.**

Part III

Determinants

1.5 Overview

Intuitively, the determinant of a matrix measure the ratio between the area before and after a transformation. It helps us determine wether a matrix is inversible and allow us to calculate the surface and volume of parallelograms.

Definition 1.5.1 (Determinant)**.**

Theorem 1.5.1. *If A is a triangular matrix, the det A is the product of the entries on the main diagonal of A*

1.5.1 Properties of Determinant

Theorem 1.5.2 (Row Operations)**.**

Theorem 1.5.3 (Matrix Invertible)**.** *A square matrix A is invertible if and only if $detA \neq 0$*

Theorem 1.5.4. *If A is an n x n matrix, then $A^T = detA$*

Remark. *Intuitively, we know that the determinant of a matrix and its transpose is the same because the area is the same.*

Theorem 1.5.5 (Multiplicative Property)**.** *If A and B are n x n matrices, then $detAB = (detA)(detB)$*

1.5.2 Cramer's Rule

Cramer's Rule is another method to solve linear systems. Instead of using Gaussian method with matrices, we use determinant instead.

Theorem 1.5.6 (Cramer's Rule)**.**

Theorem 1.5.7 (Inverse Formula)**.** *Let A be an invertible n x n matrix. Then*

1.5.3 Determinant as Area or Volume

Determinant are also used to calculate the area of a parallelogram.

Part IV

Vector Spaces

1.6 Overview

1.6.1 Vector Space and Subspaces

Definition 1.6.1 (Vector Space)**.**

Definition 1.6.2 (Subspace)**.**

1.6.2 Null Subspaces, Column Spaces, and Linear Transfomations

Definition 1.6.3 (Null Space)**.**

Theorem 1.6.1. *The null space of an mx n matix A is a subspace of* $\mathbb{R}^n$. *Equivalenty, the set of all solutions to a system Ax=0 of m homogeneous linear equations in n unknowns is a subspace of* $\mathbb{R}^n$.

Definition 1.6.4 (Column Space)**.**

Theorem 1.6.2. *The column space of an mx n matrix A is a subspace of* $\mathbb{R}^n$

1.6.3 Kernel and Range of Linear Transfomation

Definition 1.6.5 (Linear Transfomation)**.**

Theorem 1.6.3.

Theorem 1.6.4.

Theorem 1.6.5.

Definition 1.6.6.

Definition 1.6.7.

Part V

Eigenvalues and Eigenvectors

1.7 Overview

Part VI

Orthogonality and Least Squares

1.8 Overview