Lectures Notes from Pure Maths by Steve Warner

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July 2, 2021

Introduction

This book is an introduction to undergraduate pure mathematics. This books contains 16 chapters with exercices ranging from different levels:

1. Lesson 1 - Logic: Statements and Truth
2. Lesson 2 - Set Theory: Sets and Subsets
3. Lesson 3 - Abstract Algebra: Semigroups, Monoids and Groups
4. Lesson 4 - Number Theory: The Ring of Integers
5. Lesson 5 - Real Analysis: The Complete Ordered Field of Reals
6. Lesson 6 - Topology: The Topology of R
7. Lesson 7 - Complex Analysis: The Field of Complex Numbers
8. Lesson 8 - Linear Algebra: Vector Spaces
9. Lesson 9 - Logic: Logical Arguments
10. Lesson 10 - Set Theory: Reltions and Functions
11. Lesson 11 - Abstract ALgebra: Strucutres and Homomorphisms
12. Lesson 12 - Number Theory: Primes, GCD, and LCM
13. Lesson 13 - Real Analysis: Limits and Continuity
14. Lesson 14 - Topology: Spaces and Homeomorphisms
15. Lesson 15 - Complex Analysis: Complex Valued Functions
16. Lesson 16 - Linear Algebra: Linear Transformations

1 Lesson 1 - Logic: Statements and Truth

1.1 Overview

This section introduces the notion of statements and truth tables, which is the foundation of proofs.

1.2 Statements with words

The goal of this section is to determine wether a sentence is a statement or not.

A statement or proposition is a sentence that can be true or false, but not both simultaneously. If it expresses a single idea, we say it is an atomic statement. If we want to create a statement with more than one idea, we need to connect the atomic statement using logical connectives.

1.3 Statements with symbols

In mathematics, we want to express mathematical statement in symbols since they actract away the unecessary clutter and help us focus on the form of the statement. Consequently, we need to define a set of symbol used to define the common logical connectives. The most common are:

1. Conjunction
2. Disjunction
3. Negation
4. Implication
5. Biconditional

1.4 Truth Table

Each logical connective is associated to a truth table, which tell us the truth value of a compound statement based on the truth value of the propositional variables. Here is the truth table for the common logical connectives:

We can also use truth table to determine if two statement are equivalent. To do so, we compare their truth table.

1.5 Problem Set

1.5.1 Level 1

1.5.2 Level 2

1.5.3 Level 3

1.5.4 Level 4

1.5.5 Level 5

2 Lesson 2 - Set Theory: Sets and Subsets

2.1 Overview

This section introduce the notion of sets, subsets, union and intersection.

2.2 Why do we care about sets

Having a good intuition on sets allows us to regroup similar element and compare them together. We can show that set contains some properties so that the elements all share those properties.

2.3 Describing Sets

Definition 2.3.1 (Set)**.** *A set is a collection of objects. The set can either be finite or infinite. We can describe the set based on a common characteristic among the elements of the set using the set builder notation. We write* $\{x|P(x)\}$*, where P(x) is the common characteristic.*

Definition 2.3.2 (Axiom of Extensionality)**.** *Two sets are equivalent if they contain the same element, we write:*

$$\forall x(x \in A \leftrightarrow x \in B)$$

Definition 2.3.3 (Cardinality)**.** *The cardinality of an element is the number of different element in the set. For example, the set S=1,2,3 has the same cardinality as the set T=1,2,2,3*

Theorem 2.3.1 (Fence-Post Formula)**.** *To count the number of integers in a set, we use the fence-post formula*

$$n - m + 1$$

2.4 Subsets

Definition 2.4.1 (Subset)**.** *We say that A is a subset of B if every element of A is an element of B. We write* $A \subseteq B$

$$\forall x(x \in A \to x \in B)$$

Theorem 2.4.1. *Every set A is a subset of itself*

$$\forall x(x \in A \to x \in A)$$

Theorem 2.4.2. *The empty set is a subset of every set*

$$\forall x(x \in \emptyset \to x \in A)$$

Theorem 2.4.3 (Transitivity of sets)**.** *Let A, B, C be sets such that*

$$A \subseteq B, B \subseteq C. Then, A \subseteq C$$

Theorem 2.4.4. *There are 2^n subsets in a set*

2.5 Unions and Intersections

Definition 2.5.1 (Union)**.** *The union of the sets A and B, written $A \cup B$, is the set of elements that are in A or B (or both).*

$$\forall x(x|x \in A \vee x \in B)$$

Definition 2.5.2 (Intersection)**.** *The intersection of the sets A and B, written $A \cap B$, is the set of elements that are in A and B simultaneously.*

$$\forall x(x|x \in A \wedge x \in B)$$

Theorem 2.5.1. *If A and B are sets, then $A \subseteq A \cup B$*

Theorem 2.5.2. $B \subseteq A \iff A \cup B = A$

Theorem 2.5.3. $B \subseteq A \iff A \cap B = B$

Definition 2.5.3 (Reflexive)**.** *A relation R is reflexive if $\forall x(xRx)$*

Definition 2.5.4 (Symmetric)**.** *A relation R is symmetric if*

$$\forall x \forall y(xRy \to yRx)$$

2.6 Problem Set

2.6.1 Level 1

2.6.2 Level 2

2.6.3 Level 3

2.6.4 Level 4

2.6.5 Level 5

3 Lesson 3 - Abstract Algebra: Semigroups, Monoids and Groups

3.1 Overview

This section focus on the properties of semigroups, monoids and groups. It gives us an intuition on why some set behave a certain way while other don't.

To determine wether a set has certain properties, we often use a multiplication table.

3.2 Binary Operations and Closure

Definition 3.2.1 (Binary Operation)**.** *A binary operation on a set is a rule that combines two elements of the set to produce another element of the set*

Definition 3.2.2 (Closed)**.** *We say that the set S is closed under the partiel binary operation * if whenever $a, b \in S$, we have $a * b \in S$*

3.3 Semigroups and Associativity

Definition 3.3.1 (Associativity)**.** *Let * be a binary operation on a set. We say that * is associative in S if for all x, y, z in S, we have*

$$x * (y * z) = (x * y) * z$$

Definition 3.3.2 (Semigroup)**.** *A semigroup is a pair (S,*), where S is a set and * is an associative binary opertaion on S*

Corollary 3.3.1. *If the binary operator * is not associative in S, then the pair (S,*) is not a semigroup*

Definition 3.3.3 (Abelian or Commutative)**.** *Let * be a binary operation on a set. We say that * is abelian (or commutative) in S if for all x, y, z in S, we have*

$$x * y = y * x$$

Definition 3.3.4 (Abelian Semigroup)**.** *An abelian semigroup is a semigroup that is commutative. Therefore, it has the following properties:*

1. *Closed*
2. *Associative*
3. *Commutative*

3.4 Monoids and Identity

Definition 3.4.1 (Identity)**.** *Let (S,*) be a semigroup. An element e of S is called an identity with respect of the binary operation * if for all* $a \in S$*, we have* $a * e = e * a = a$

Definition 3.4.2 (Monoid)**.** *A monoid is a semigroup with an identity. In other word, a monoid is*

1. *Closed*
2. *Associative*
3. *Identity*

Theorem 3.4.1 (Unique Identity)**.** *Let (M, *) be a monoid with identity e. The identity element is unique*

3.5 Groups and Inverses

Definition 3.5.1 (Inverse)**.** *Let (M,*) be a monoid with identity e. An element a of M is called invertible if there is an element* $b \in M$ *such that* $a * b = b * a = e$

Definition 3.5.2 (Group)**.** *A group is a monoid in which every element is invertible. Therefore, a group follows the following properties*

1. *Closed*
2. *Associative*
3. *Identity*
4. *Inversible*

Theorem 3.5.1 (Unique Inverse)**.** *Let (G, *) be a group. Each element in G has a unique inverse*

3.6 Problem Set

3.6.1 Level 1

3.6.2 Level 2

3.6.3 Level 3

3.6.4 Level 4

3.6.5 Level 5

4 Lesson 4 - Number Theory: The Ring of Integers

4.1 Overview

The goal of this section is to familiarise ourselves with induction proofs. However, the proofs we have to prove utilize integers properties, so we have to define ring properties first.

The notion of ring utilize the concepts of closure, associativity, abelian, identity and inverses which we saw in the previous section.

We want to understand the properties of the common set $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, as it will allow us to use those properties to work with more complex objects such as vectors or complex numbers.

4.2 Ring and Distributivity

Definition 4.2.1 (Commutative Group)**.** *A commutative group is a group that follows the following properties:*

1. *Closure*
2. *Associative*
3. *Commutative*
4. *Identity*
5. *Inverse*

Lemma 4.2.1. $(\mathbb{Z}, +)$ *is a commutative group*

Definition 4.2.2 (Commutative Monoid)**.** *A commutative monoid is a monoid that follows the following properties:*

1. *Closure*
2. *Associative*
3. *Commutative*
4. *Identity*

Lemma 4.2.2. $(\mathbb{Z}, \cdot)$ *is a commutative monoid*

Definition 4.2.3 (Ring)**.** *A ring is a triple* $(R, +, \cdot)$ *where* R *is a set,* $+$ *and* $\cdot$ *are binary operations on* R *that satisfies:*

1. *(R, +) is commutative group*
2. *(R, $\cdot$) is a commutative monoid*

3. *Multiplication is distributive over addition in R. That is, for all x,y,z $\in$ R, we have*

$$x \cdot (y + z) = x \cdot y + x \cdot z \text{ and } (y + z) \cdot x = y \cdot x + z \cdot x$$

It should be noted that the properties that define a ring are called the ring axioms

Lemma 4.2.3. $(\mathbb{Z}, +, \cdot)$ *is a commutative ring*

Lemma 4.2.4. $(\mathbb{N}, +, \cdot)$ *is a ring because* $(\mathbb{N}, +)$ *is not a group. We say it is a semiring*

4.3 Divisibility

Definition 4.3.1 (Even)**.** *An integer a is called even if there is another integer b such that a = 2b*

Definition 4.3.2 (Odd)**.** *An integer a is called odd if there is another integer b such that a = 2b + 1*

Definition 4.3.3 (Sum)**.** *We define the sum of integers a and b to be a + b*

Definition 4.3.4 (Product)**.** *We define the product of integers a and b to be a · b*

Theorem 4.3.1. *The sum of two even integer is even*

Theorem 4.3.2. *The product of two integers that are each divisible by k is also divisible by k*

4.4 Induction

Definition 4.4.1 (Well Ordering Principle)**.** *The Well Ordering Principle says that every nonempty subset of natural numbers has a least element*

Theorem 4.4.1 (Principle of Mathematical Induction)**.** *Let S be a set of natural numbers such that*

1. $0 \in S$
2. *for all* $k \in \mathbb{N}, k \in S \rightarrow k + 1$. *Then,* $S = \mathbb{N}$

Lemma 4.4.2 (Standard Advanced Calculus Trick)**.** *We can add and substract the same quantities without changing the result*

4.5 Problem Set

4.5.1 Level 1

4.5.2 Level 2

4.5.3 Level 3

4.5.4 Level 4

4.5.5 Level 5

5 Lesson 5 - Real Analysis: The Complete Ordered Field of Reals

5.1 Overview

The goal of this section is to define the set of numbers. We are introduced (Q, +, $\cdot$) which is an ordered field, but we are shown that we can't generate all the numbers with that set. We need the irrational numbers.

5.2 Field

Definition 5.2.1 (Field). *A field is a triple (F, +, $\cdot$), where F is a set and + and $\cdot$ are binary operations on F satisfying:*

1. *(F, +) is a commutative group*
2. *(F, $\cdot$) is a commutative group*
3. *Multiplication is distributive over addition in F. That is, for all x,y,z $\in$ F, we have*

$$x \cdot (y + z) = x \cdot y + x \cdot z \text{ and } (y + z) \cdot x = y \cdot x + z \cdot x$$

4. *0 $\neq$ 1*

The properties that define a field are called the field axioms

Lemma 5.2.1 (Set of Natural Numbers). *The set $\mathbb{N}$ is the set of natural numbers and the structure ($\mathbb{N}$, +, $\cdot$) is a semiring*

Lemma 5.2.2 (Set of Integers). *The set $\mathbb{Z}$ is the set of integers and the structure ($\mathbb{Z}$, +, $\cdot$) is a ring*

Lemma 5.2.3 (Set of Rational Numbers). *The set $\mathbb{Q}$ is the set of rational numbers and the structure ($\mathbb{Q}$, +, $\cdot$) is a field*

Definition 5.2.2 (Substraction). *If a,b $\in$ F, we define the substraction a-b= a+(-b)*

Definition 5.2.3 (Division). *If a,b $\in$ F and b $\neq$0, we define the division $a/b = ab^-1$*

5.3 Ordered Rings and Fields

Definition 5.3.1 (Positive and Negative Elements). *If a $\in$ P, we say that a is positive and if -a $\in$ P, we say that a is negative*

Definition 5.3.2 (Ordered Ring). *We say that a ring (R,+, $\cdot$) is ordered if there is a nonempty subset P of R, called the set of positive elements of R satisfying the folowing properties*

1. *if a,b* $\in P$*, then* $a + b \in P$
2. *if a,b* $\in P$*, then* $ab \in P$
3. *if* $a \in P$*, then exactly one of the following holds:*

$$a \in P, a = 0, or - a \in P$$

Remark. *Since P is the set of positive element, the following are equivalent:*

1. $a \in P \Longleftrightarrow a \geq 0$
2. $-a \in P \Longleftrightarrow a \leq 0$
3. $a \leq b \Longleftrightarrow a < bora = b$
4. $a \geq b \Longleftrightarrow a > bora = b$

Definition 5.3.3 (Nonnegative Number)**.** *Let x be a non negative number. Then, a is positive or zero*

Theorem 5.3.1. $(\mathbb{Q}, +, \cdot)$ *is an ordered field*

Theorem 5.3.2. *Let (F,* $\leq$ *) be an ordered field. Then, for all* $x \in F$**,* $x \cdot x > 0$

Theorem 5.3.3. *Every ordered field (F,* $\leq$ *) contains a copy of the natural numbers.*

Theorem 5.3.4. *Let (F,* $\leq$ *) be an ordered field and let* $x \in F$ *with* $x > 0$*. Then,* $\frac{1}{x} > 0$

5.4 Why Isn't $\mathbb{Q}$ Enough?

Theorem 5.4.1 (Pythagorean Theorem)**.** *In a right triangle with legs of length a and b, and a hypotenuse of length c*

$$c^2 = a^2 + b^2$$

Theorem 5.4.2. *There does not exist a rational number a such that* $a^2 = 2$

Remark. *Proving a number is irrational is done by assuming the rational can be expressed as a irreductible fraction gcd(p,q)=1 and showing that the smallest divisor is not 1*

5.5 Completeness

Definition 5.5.1 (Upper Bound)**.** *Let (F,* $\leq$ *) be an ordered field and let S be a nonempty subset of F. We say that S is bounded above if there is* $M \in F$ *such that for all* $s \in S, s \leq M$*. Each number M is called an upper bound of S*

Definition 5.5.2 (Lower Bound)**.** *Let (F,* $\leq$ *) be an ordered field and let S be a nonempty subset of F. We say that S is bounded below if there is* $K \in F$ *such that for all* $s \in S, K \leq s$*. Each number K is called an lower bound of S*

Definition 5.5.3 (Bounded Set). *We say that S is bounded if it is bounded above and bounded below. Otherwise, we say that S is unbounded.*

Definition 5.5.4 (Supremum). *A least upper bound of a set S is an upper bound that is smaller than any other upper bound of S*

Definition 5.5.5 (Infimum). *A greatest lower bound of S is a lower bound that is larger than any other other lower bound of S*

Remark. *To show that the number n is a supremum/infimum, we need to show that*

1. *The set S has a upper bound/lower bound*
2. *There is no greater lower bound / lower upper bound by contradiction. We use the fact that S ¡ S' -¿ S' = S + ε, where ε ¿ 0, which contradict the fact that S' is a upper/lower bound*

Definition 5.5.6 (Completeness Property). *An ordered field (F, $\leq$) has the Completeness Property if every nonempty subset of F that is bounded above has a least upper upper bound in F. In this case, we cay that (F, $\leq$) is a complete ordered field.*

Remark. *The proof is done by contradiction. We assume that the naturals are bounded (we know that they are not because $\mathbb{N}$ is not an ordered field?) and we prove that there is a least upper bound that is not a upper bound using the compleness property.*

Corollary 5.5.1. *Every nonempty set of real numbers that is bounded below has a greatest lower bound (infimum)*

Theorem 5.5.1. *There is exactly one complete ordered field*

Theorem 5.5.2 (Archimedian Property of $\mathbb{R}$). *For every $x \in \mathbb{R}, thereisn \in \mathbb{N} suchthatn > x$*

Corollary 5.5.2. *Let $x < y$. There exist $n \in \mathbb{N}$ such that $nx > y$*

Remark. *The Archimedian property tell us that the set of natural numbers is infinite. Therefore, if we choose any arbitrary natural, we can find another natural that can be smaller or greater than it.*

Theorem 5.5.3 (Density Theorem). *If $x, y \in \mathbb{R} withx < y$ then there is $q \in \mathbb{Q}$ with $x < q < y$*

Remark. *The Density theorem tells us that we can find a rational number between any two real numbers. Intuitively, we can think that is because there is infintely many numbers in the interval, so there must be a rational number in it.*

Remark. *To prove the density theorem, we want to construct a rational number with the Archimedian and the Well Ordering Principle. TODO*

Corollary 5.5.3. *There is a real number that is not rational between any two real numbers.*

5.6 Problem Set

5.6.1 Level 1

5.6.2 Level 2

5.6.3 Level 3

5.6.4 Level 4

5.6.5 Level 5

6 Lesson 6 - Topology: The Topology of R

6.1 Overview

6.2 Intervals of Real Numbers

Definition 6.2.1 (Interval)**.** *A set I of real numbers is called an interval id any real number that lies between two numbers in I is also in I. We write:*

$$\forall x, y \in I, \forall z \in \mathbb{R}, \textit{if x is less than z and z is less than y, the z is in I}$$

Here is a list of the other types of intervals:

1. *Open Interval*
2. *Closed Interval*
3. *Half-open Interval*
4. *Infinite Open Interval*
5. *Infinit Closed Interval*

Theorem 6.2.1. *If an interval I is bounded, the there are a,b $\in \mathbb{R}$ such that one of the following holds:*

$$I = (a, b), I = (a, b],\ \textit{or}\ I = [a, b)$$

6.3 Operations on Sets

Definition 6.3.1 (Union)**.** *The union of the sets A and B, written $A \cup B$, is the set of elements that are in A or B (or both).*

$$\forall x(x|x \in A \vee x \in B)$$

Definition 6.3.2 (Intersection)**.** *The intersection of the sets A and B, written $A \cap B$, is the set of elements that are in A and B simultaneously.*

$$\forall x(x|x \in A \wedge x \in B)$$

Definition 6.3.3 (Difference)**.** *The difference A B is the set of elements that are in A and not in B*

$$A\ \ B = x|x \in A \textit{ and } x \notin B$$

Definition 6.3.4 (Symmetric Difference)**.** *The symmetric difference $A \triangle B$ is the set of elements that are in A or B, but not both*

$$A \triangle B = (A\ \ B) \cup (B\ \ A)$$

Theorem 6.3.1. *The operation of forming unions is associative*

Definition 6.3.5 (General Defintion for union and intersection)**.** *Let X be a non empty set of sets, then*

1. $\cup X = y|$ *there is* $y \in X$ *with* $y \in Y$
2. $\cap X = y|$ *forall* $y \in X, y \in Y$

6.4 Open and Closed Sets

Definition 6.4.1 (Open Set)**.** *A subset X of $\mathbb{R}$ is open if for every real number $x \in \mathbb{R}$, there is an open interval (a,b) with $x \in (a, b)$ and $(a, b) \subseteq X$*

Definition 6.4.2 (Closed Set)**.** *A subset X of $\mathbb{R}$ is said to be closed if $\mathbb{R}$ X is open*

Theorem 6.4.1. *Let $a \in \mathbb{R}$ The infinite interval (a, ∞) is an open set*

Theorem 6.4.2. *$\emptyset$ and $\mathbb{R}$ are both open sets*

Theorem 6.4.3. *A subset X of $\mathbb{R}$ is open if and only if for every real number $x \in X$, there is a positive real number c suche that $(x - c, x + c) \subseteq X$*

Theorem 6.4.4. *The union of two open sets in $\mathbb{R}$ is an open set in $\mathbb{R}$*

Theorem 6.4.5. *Let X be a set of open subsets of $\mathbb{R}$. Then UX is open*

Theorem 6.4.6. *Every open set in $\mathbb{R}$ can be expressed as a union of bounded open intervals*

Theorem 6.4.7. *The intersection of two open sets in $\mathbb{R}$ is an open set in $\mathbb{R}$*

Theorem 6.4.8. *The intersection of two closed sets in $\mathbb{R}$ is a closed set in $\mathbb{R}$*

6.5 Problem Set

6.5.1 Level 1

6.5.2 Level 2

6.5.3 Level 3

6.5.4 Level 4

6.5.5 Level 5

7 Lesson 7 - Complex Analysis: The Field of Complex Numbers

7.1 Overview

We should alway keep in mind wether we are in a field or ring when working with linear and quadratic equation.

7.2 A Limitation of the Reals

Definition 7.2.1 (Linear Equation)**.** *A linear equation has the form ax+b=0.*

Definition 7.2.2 (Quadratic Equation)**.** *A quadratic equation has the form* $a^2 + bx + c = 0$*, where* $a \neq 0$

7.3 The Complex Field

Definition 7.3.1 (Standard From of a Complex Number)**.** *The standard form of a complex number is* $a + bi$*, where a and b are real numbers. The set of complex number is* $\mathbb{C} = a + bi | a, b \in \mathbb{R}$

Definition 7.3.2 (The Complex Plane)**.** *We can visualize a complex number as a point in the Complex Plane, which has a real axis (in x) and an imaginary axis (in y). The point (0,0) is called the origin*

The Complex plane allow us to visualize a complex number as a vector. If z is a complex number such as z=a+bi, we call a the real part of z and b the imaginary part of z. We write a = Re z and b = Im z

Definition 7.3.3 (Equality)**.** *Two complex numbers are equal if and only if they have the same real and imaginary part.*

$$a + bi = c + di \iff a = c \text{ and } b = d$$

Definition 7.3.4 (Addition)**.** *We can add two complex numbers by adding their real and imaginary parts.*

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$

Definition 7.3.5 (Substraction)**.** *We can find the difference of two complex numbers by substracting their real and imaginary parts.*

$$(a + bi) - (c + di) = (a - c) + (b - d)i$$

Definition 7.3.6 (Multiplication)**.**

$$(a + bi)(c + di) = (ac - bd) + (ad + bc)i$$

Definition 7.3.7 (Division). *Let z and w be complex numbers such that $z \in \mathbb{C}$ and $w \in \mathbb{C}*$. We define the quotient $\frac{z}{w}$ by*

$$\frac{z}{w} = zw^{-1} = (a+bi) \cdot (\frac{c}{c^2+d^2} - \frac{d}{(c^2+d^2)i}) = \frac{ac+bd}{c^2+d^2} + \frac{bc-ad}{c^2+d^2}$$

Definition 7.3.8 (Conjugate). *The conjugate of the complex number z=a+bi is the complex number $\overline{z} = a - bi$*

Definition 7.3.9 (Real Number). *Let z be a complex number such that z=a+bi. If b=0, then we call z a real number.*

Definition 7.3.10 (Pure Imaginary Number). *Let z be a complex number such that z=a+bi. If a=0, then we call z a pure imaginary number*

Theorem 7.3.1. $i^2 = -1$

Theorem 7.3.2. $(\mathbb{C}, +, \cdot)$ *is a field*

Corollary 7.3.1. $(\mathbb{R}, +, \cdot)$ *is a subfield of* $(\mathbb{C}, +, \cdot)$

Theorem 7.3.3. *The field of complex numbers cannot be ordered*

7.4 Absolute Value and Distance

Definition 7.4.1 (Square Root). *If x and y are real or complex numbers such that $y = x^2$, the we call x a sqaure root of y. If x is a positive real number, then we say that x is the positive square root of y and we write $x = \sqrt(y)$*

Definition 7.4.2 (Modulus of a Complex Number). *The absolute value or the modulus of the complex number z=a+bi is the nonnegative real number*

$$|z| = \sqrt(a^2+b^2) = \sqrt((Rez)^2 + (Imz)^2)$$

Definition 7.4.3 (Distance between Complex Numbers). *The distance between the complex numbers z=a+bi and w=c+di is*

$$d(z,w) = |z-w| = \sqrt((c-a)^2 + (d-b)^2)$$

Theorem 7.4.1 (The Triangle Inequality). *For all $z, w \in \mathbb{C}, |z+w| \leq |z| + |w|$*

7.5 Basic Topology of $\mathbb{C}$

Definition 7.5.1 (Circle). *A circle in the Complex Plane is the set of all points that are at a fixed distance from a fixed point. The fixed distance is called the radius of the circle and the fixed point is called the center of the circle*

If a circle has radius of r¿0 and center c=a+bi, then any point z=x+yi on the circle must satisfy $|z-c| = r$, or equivalently, $(x-a)^2 + (y-b)^2 = r^2$

Definition 7.5.2 (Open Disk). *An open disk in $\mathbb{C}$ consists of all the points in the interior of a circle. If a is the center of the open disk and r is the radius of the open disk, then any point z inside the disk satisfies $|z-a|<r$*

Definition 7.5.3 (r-neighborhood of a). $N_r(a) = z \in \mathbb{C}||z-a|<r$ *is also called the r-neighborhood of a.*

Definition 7.5.4 (Diameter). *In $\mathbb{R}$, an r-neighborhood of a is the open interval $N_r(a) = (a-r, a+r)$ The diameter of this interval is 2r*

Definition 7.5.5 (Closed Disk). *A closed disk is the interior of a circle together with the circle itself (boundary included). If a is the center of the closed disk and r is the radius of the closed disk, the any point z inside the closed disk satisfies $|z-a| \leq r$*

Definition 7.5.6 (Punctured Open Disk). *A punctured open sidk consists of all the points in the interior of a circle except for the center of the circle. If a is the center of the punctured open disk and r is the radius of the open disk, then any point z inside the punctured disk satifies $|z-a|<r$ and $z \neq a$*

Since $z \neq a$ is equivalent to $z-a \neq 0$, then it is also equivalent to $|z-a| \neq 0$. Since —z-a— must be nonnegative, then $|z-a|>0$ or $0<|z-a|$.

Therfore, a puncture open disk with center a and radius r consists of all points z that satisfy 0¡—z-a—¡r

Definition 7.5.7 (Deleted r-neighborhood of a). $N_r^{\odot}(a) = \{z|0<|z-a|<r\}$ *is also called a deleted r-neighborhood of a*

Definition 7.5.8 (Open Subset). *A subset X of $\mathbb{C}$ is said to be open if for every complex number $z \in X$, there is an open disk D with $z \in D$ and $D \subseteq X$*

Theorem 7.5.1. *A subset X of $\mathbb{C}$ is open if and only if for every complex number $w \in X$, there is a positive real number d such that $N_d(w) \subseteq X$*

Theorem 7.5.2 (Closed Subset). *A subset X of $\mathbb{C}$ is said to be closed if the complement of $X \in \mathbb{C}$, noted $\mathbb{C}\ X$, is open*

The complement concist of all complex numbers not in X

7.6 Problem Set

7.6.1 Level 1

7.6.2 Level 2

7.6.3 Level 3

7.6.4 Level 4

7.6.5 Level 5

8 Lesson 8 - Linear Algebra: Vector Spaces

8.1 Overview

In the previous section, we looked at three structure called fields:

1. $\mathbb{Q}$: field of rational numbers
2. $\mathbb{R}$: field of real numbers
3. $\mathbb{C}$: field of complex numbers

And we also saw that $\mathbb{Q}$ is a subfield of $\mathbb{R}$, which is also a subset of $\mathbb{C}$. This means that every rational number is a real number and every real number is a complex number.

Understanding that $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ are fields is pretty neat, since they have two operations (addition and substraction) that satisifes closure, associativity, commutativity, identity, inverse, and distributive, which allows us to perform high school algebra on its elements.

Consequently, since vectors are also in these fields, we can apply the field properties on vectors.

8.2 Vector Spaces Over Fields

Definition 8.2.1 (Vector Space)**.** *A vector space over a field $\mathbb{F}$ is a set V with a binary operation + on V (called addition) and an operation called scalar multiplication satisfying:*

1. *(V,+) is a commutative group*
2. *(Closure under scalar multiplication) for all $k \in \mathbb{F}, kv \in V$*
3. *(Scalar multiplication Identity) If 1 is the multiplicative identity $\mathbb{F}$ and $v \in V$, then 1v=v*
4. *(Associativity of scalar multiplication) For all j,k $\in \mathbb{F}$ and $v \in V$, (jk)v=j(kv)*
5. *(Distributivity of 1 scalar over 2 vectors) For all $k \in \mathbb{F}$ and $v, w \in V$, k(v+w)=kv+kw*
6. *(Distributivity of 2 scalars over 1 vector) For all j,k $\in \mathbb{F}$ and $v \in V$, (j+k)v = jv + kv*

8.3 Subspaces

Definition 8.3.1 (Subspace)**.** *Let V be a vector space over a field $\mathbb{F}$. A subset U of V is called subspace of V, written $U \leq V$, if it is also a vector space with respect to the same operations of addition and scalar multiplication as they were defined in V.*

Theorem 8.3.1. *Let V be a vector space over a field $\mathbb{F}$ and let $U \subseteq V$. Then $U \leq V$ if and only if:*

1. $0 \in U$
2. *for all* $v, w \in U, v + w \in U$
3. *for all* $v \in U andk \in \mathbb{F}, kv \in U$

Theorem 8.3.2. *Let V be a vector space over a field $\mathbb{F}$ and let U and W be subspaces of V. Then $U \cap W$ is a subspace of V*

8.4 Bases

Definition 8.4.1 (Linear Combination)**.** *Let V be a vector space over a field $\mathbb{F}$, let $v, w \in V$ and $j, k \in \mathbb{F}$. The expression jv+kw is called a linear combination of vectors v and w. We call the scalars j and k weights*

Definition 8.4.2 (Span)**.** *If $v, w \in V$, where V is a vector space over a field $\mathbb{F}$, then the set of all linear combinations of v and w is called the span of v and w. Symbolically, we have $spanv, w = jv + kw|j, k \in \mathbb{F}$*

Theorem 8.4.1. *Let $V = \mathbb{R}^2 = (a, b)|a, b \in \mathbb{R}$ be the vector space over $\mathbb{R}$ with the usual definitions of addition and scalar multiplication. Then span(1,0),(0,1) $= V = \mathbb{R}^2$*

Definition 8.4.3 (Linear Independance)**.** *If $v, w \in V$, where V is avector space over a field $\mathbb{F}$, then we say that v and w are linearly independanent if neither vector is a scalar multiple of the other one. Otherwise, we say that v and w are linearly dependant.*

Theorem 8.4.2. *Let V be a vector space over a field $\mathbb{F}$ and let $v, w \in V$. Then v and w are linearly dependent if and only if there are $j, k \in \mathbb{F}$, not both 0, such that jv+kw=0*

Theorem 8.4.3. *Let $V = \mathbb{R}^n = \{(k_1, k_2, \ldots, k_n) \mid k_1, k_2, \ldots, k_n \in \mathbb{R}\}$ be the vector space over $\mathbb{R}$ with the usual definitions of addition and scalar multiplication. Then*

$$\operatorname{span}\{(1, 0, 0, \ldots, 0), (0, 1, 0, \ldots, 0), \ldots, (0, 0, 0, \ldots, 1)\} = \mathbb{R}^n$$

8.5 Problem Set

8.5.1 Level 1

8.5.2 Level 2

8.5.3 Level 3

8.5.4 Level 4

8.5.5 Level 5

9 Lesson 9 - Logic: Logical Arguments

9.1 Overview

In lesson 1, we introduce the principle of statement and logical connector to express a proposition.

9.2 Statements and Substatements

Definition 9.2.1 (Substatement)**.** *A substatement is a statement where we dropped unecessary parentheses (parenthesese that don't add clarity)*

9.3 Logical Equivalence

Definition 9.3.1 (Logical Statement)**.** *Let ϕ and ψ be statements. We say that ϕ and ψ are logically equivalent if every truth assignment of the propositional variables lead to the same truth table. We write $\phi \equiv \psi$*

Proposition (Logical Equivalence)**.** *Let p,q,r be propositional variables. Here is a list of propositional equivalence*

1. *Law of double negation:* $p \equiv \neg(\neg p)$
2. *De Morgan's laws:*
   - $\neg(p \wedge q) \equiv \neg p \vee \neg q$
   - $\neg(p \vee q) \equiv \neg p \wedge \neg q$
3. *Commutative laws:*
   - $p \wedge q \equiv q \wedge p$
   - $p \vee q \equiv q \vee p$
4. *Associative laws:*
   - $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$
   - $(p \vee q) \vee r \equiv p \vee (q \vee r)$
5. *Distributive laws:*
   - $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
   - $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
6. *Identity laws:*
   - $p \wedge \mathrm{T} \equiv p$
   - $p \wedge \mathrm{F} \equiv \mathrm{F}$
   - $p \vee \mathrm{T} \equiv \mathrm{T}$
   - $p \vee \mathrm{F} \equiv p$

7. *Negation laws:*
   - $p \wedge \neg p \equiv \mathrm{F}$
   - $p \vee \neg p \equiv \mathrm{T}$
8. *Redundancy laws:*
   - $p \wedge p \equiv p$
   - $p \vee p \equiv p$
9. *Absorption laws:*
   - $(p \vee q) \wedge p \equiv p$
   - $(p \wedge q) \vee p \equiv p$
10. *Law of the conditional:* $p \to q \equiv \neg p \vee q$
11. *Law of contrapositive:* $p \to q = \neg q \to \neg p$
12. *Biconditional:* $p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$

Definition 9.3.2 (Tautology)**.** *A statement that has truth value T for all truth assignements is called a Tautology*

Definition 9.3.3 (Contradiction)**.** *A statement that has truth value F for all truth assigments is called a contradiction*

Proposition (Law of Logical Equivalence)**.**

1. *Law of transitivity of logical equivalence: Let* ϕ, ψ, *and* τ *be statements such that* $\phi \equiv \psi$ *and* $\psi \equiv \tau$. *Then* $\phi \equiv \tau$
2. *Law of substitution of logical equivalents: Let* ϕ, ψ, *and* τ *be statements such that* $\phi \equiv \psi$ *and* ϕ *is a substatement of* τ. *Let* τ^* *be the sentence formed by replacing* ϕ *by* ψ *inside of* τ. *Then* $\tau^* \equiv \tau$
3. *Law of substitution of sentences: Let* ϕ *and* ψ *be statements such that* $\phi \equiv \psi$, *let* p *be a propositional variable, and let* τ *be a statement. Let* ϕ^* *and* ψ^* *be the sentences formed by replacing every instance of* p *with* τ *in* ϕ *and* ψ, *respectively. Then* $\phi^* \equiv \psi^*$

9.4 Validity in Sentential Logic

Definition 9.4.1 (Logical Argument)**.** *A logical argument (or proof) concists of premises (statements we are given) and conclusions (statements we are not given)*

Definition 9.4.2 (Valid Argument)**.** *A logical argument is valid if every truth assignment that makes all the premises true also makes all the conclusions true. A logical argument that is not valid is called invalid or a fallacy*

Definition 9.4.3 (Sound Argument)**.** *We say that an argument is sound if*

1. *the argument is valid*
2. *all the premises are true*

Remark. *To show that an argument is invalid, we provide a counter example*

Proposition (Fallacy of the Converse)**.** $p \to q \not\Longleftrightarrow q \to p$

Remark. *Let* $p \to q$

1. *Converse:* $p \to p$ *is not logically equivalent*
2. *Inverse:* $\neg p \to \neg q$ *is not logically equivalent*
3. *Contrapositive:* $\neg q \to \neg p$ *is logically equivalent*

Proposition (Rules of Inference)**.**

1. *Modus Ponens:* $(p \to q) \wedge p \Longrightarrow q$
2. *Modus Tollens:* $(p \to q) \wedge \neg q \Longrightarrow \neg p$
3. *Disjunctive Syllogism:* $(p \vee q) \wedge \neg p \Longrightarrow q$
4. *Hypothetical Syllogism:* $(p \to q) \wedge (q \to r) \Longrightarrow (p \to r)$
5. *Conjunctive Introduction:* $(p \wedge q) \Longrightarrow p \wedge q$
6. *Disjunctive Introduction:* $p \Longrightarrow p \vee q$
7. *Biconditional Introduction:* $(p \to q) \wedge (q \to p) \Longrightarrow (p \leftrightarrow q)$
8. *Constructive Dilemma:* $(p \to q) \wedge (r \to s) \wedge (p \vee r) \Longrightarrow (q \vee s)$
9. *Destructive Dilemma:* $(p \to q) \wedge (r \to s) \wedge (\neg q \vee \neq p) \Longrightarrow (\neq p \vee \neg r)$
10. *Disjunctive Resolution:* $(p \vee q) \wedge (\neg p \vee r) \Longrightarrow (q \vee r)$
11. *Conjunctive Elimination:* $(p \wedge q) \Longrightarrow p$
12. *Biconditional Elimination:* $(p \leftrightarrow q) \Longrightarrow (p \to q)$

9.5 Problem Set

9.5.1 Level 1

9.5.2 Level 2

9.5.3 Level 3

9.5.4 Level 4

9.5.5 Level 5

10 Lesson 10 - Set Theory: Relations and Functions

10.1 Overview

10.2 Relations

Definition 10.2.1 (unordered pair)**.** *An unordered pair is a set with 2 elements. In other words, (x,y) and (y,x) is the same*

Definition 10.2.2 (ordered pair)**.** *An ordered pair (x,y) is not the same as (x,y). We define this property as follows*

$$(x, y) = x, x, y$$

Remark (ordered k-tuple)**.**

Theorem 10.2.1. *(x,y) = (z,w)* $\longrightarrow x = z$ *and* $y = w$

Definition 10.2.3 (Cartesian Product)**.** *The cartesian product of the sets A and B, written* $A \times B$ *is the set of ordered pairs (a,b) with* $a \in A$ *and* $b \in B$

$$A \times B = (a, b)|a \in A \wedge b \in B$$

Definition 10.2.4 (Binary Relation on a set)**.** *A binary relation on a set A is a subset of* $A^2 = A \times A$*. We write*

$$R \textit{ is a binary relation on } A \ \longrightarrow R \subseteq A \times A$$

Remark (n-ary relation)**.**

Proposition (Properties of Binary Relations)**.** *We say a binary relation R on A is*

1. *(Reflexive):* $\forall a \in A, (a, a) \in R$
2. *(Symmetric):* $\forall a, b \in A, (a, b) \in R \longrightarrow (b, a) \in R$
3. *(Transitive):* $\forall a, b, c \in A, (a, b), (b, c) \in R \Longrightarrow (a, c) \in R$
4. *(Antireflexive):* $\forall a \in A, (a, a) \notin R$
5. *(antisymmetric):* $\forall a, b \in A, (a, b), (b, a) \in R \longrightarrow a = b$

Definition 10.2.5 (Partial ordering)**.** *A relation that is*

1. *Transitive*
2. *Reflexive*
3. *Antisymmetric*

is called a partial ordering

Definition 10.2.6 (Strict Partial ordering). *A relation that is*

1. *Transitive*
2. *Antireflexive*
3. *Antisymmetric*

is called a partial ordering

10.3 Equivalence Relations and Partitions

Definition 10.3.1 (Equivalence Relation). *A binary relation R on a set A is an equivalence relation if R is reflexive, symmetric and transitive*

Definition 10.3.2 (Partition of a Set). *A partition of a set S is a set of pairwise disjoint nonempty subsets of S whose union is S. Symbolically, X is a partition of S iif*

$$\forall A \in X(A \neq \emptyset \wedge AS) \wedge \forall A, B \in X(A \neq B \rightarrow A \cap B = \emptyset) \wedge \cup X = S$$

Definition 10.3.3 (Power Set). *The power set of A, written P(a), is the set consisting of all subsets of A*

$$P(A) = X|XA$$

Theorem 10.3.1. *Let P be a partition of a set S. Then, there is an equivalence relation on S for which the elements of P are the equivalence classes of . Conversely, if is an equivalence relation on a set S, then the equivalence classes of form a partition of S*

10.4 Orderings

Definition 10.4.1 (Partial Ordering). *A binary relation $\leq$ on a set A is a partial ordering on A if it is reflexive, antisymmetric and transitive on A.*

Definition 10.4.2 (Strict Partial Ordering). *A binary relation ¡ on a set A is a partial ordering on A if it is antireflexive, antisymmetric and transitive on A.*

Definition 10.4.3 (Partially Ordered Set). *A partially ordered set (or poset) is a pair $(A, \leq)$, where A is a set and $\leq$ is a partial ordering on A. A strict poset is pair (A,¡), wehre A is a set and ¡ is a strict partial ordering on A*

Definition 10.4.4 (Comparable Poset). *Let $(A, \leq)$ be a poset. We say $a, b \in A$ are comparable if $a \leq b$ or $b \leq a$ The poset satisfies the comparability condition if every pair of elements in A are comparable.*

Definition 10.4.5 (Linearly Ordered Set). *A poset that satisfies the comparability condition is called a linearly ordered set (or totally ordered set). A strict linearly ordred set (A,¡) satisfies trichotomy:*

$$\text{If } a, b \in A, \text{ then } a < b, a = b, \text{ or } b < a$$

10.5 Functions

NEXT

10.6 Equinumerosity

10.7 Problem Set

10.7.1 Level 1

10.7.2 Level 2

10.7.3 Level 3

10.7.4 Level 4

10.7.5 Level 5

11 Lesson 11 - Abstract Algebra: Structures and Homomorphisms

11.1 Overview

11.2 Problem Set

11.3 Structures and Substructures

11.4 Homomorphisms

11.5 Images and Kernels

11.6 Normal Subgroups and Ring Ideals

11.6.1 Level 1

11.6.2 Level 2

11.6.3 Level 3

11.6.4 Level 4

11.6.5 Level 5

12 Lesson 12 - Number Theory: Primes, GCD, and LCM

12.1 Overview

12.2 Prime Numbers

12.3 The Division Algorithm

12.4 GCD and LCM

12.5 Problem Set

12.5.1 Level 1

12.5.2 Level 2

12.5.3 Level 3

12.5.4 Level 4

12.5.5 Level 5

13 Lesson 13 - Real Analysis: Limits and Continuity

13.1 Overview

13.2 Strips and Rectangles

13.3 Limits and Continuity

13.4 Equivalent Definitions of Limits and Continuity

13.5 Basic Examples

13.6 Limit and Continuity Theorems

13.7 Limits Involving Infinity

13.8 One-Sided Limits

13.9 Problem Set

13.9.1 Level 1

13.9.2 Level 2

13.9.3 Level 3

13.9.4 Level 4

13.9.5 Level 5

14 Lesson 14 - Topology: Spaces and Homeomorphisms

14.1 Overview

14.2 Topological Spaces

14.3 Bases

14.4 Types of Topological Spaces

14.4.1 T-Space

Definition 14.4.1 ($T_0 - space$ (Kolmogorov space)). *A topological space (S,T) is a $T_0 - space$ if $\forall x, y \in S, x \neq y, \exists U \in T$ such that $x \in U \wedge y \notin U$ or $x \notin U \wedge y \in U$*

Definition 14.4.2 ($T_1 - space$ (Fréchet space or Tikhonoc space)). *A topological space (S,T) is a $T_1 - space$ if $\forall x, y \in S, x \neq y, \exists U, V \in T$ such that $x \in U \wedge y \notin U$ or $x \notin V \wedge y \in V$*

Definition 14.4.3 ($T_2 - space$ (Hausdorff Space)). *A topological space (S,T) is a $T_2 - space$ if $\forall x, y \in S, x \neq y, \exists U, V \in T$ with:*

1. $x \in U$
2. $y \in V$
3. $U \cap V = \emptyset$

Definition 14.4.4 ($T_3 - space$ (Regular Space)).

Definition 14.4.5 ($T_4 - space$ (Normal Space)).

Definition 14.4.6 (Separation Axioms). *T_0, T_1, T_2, T_3, T_4 are called separation axioms because they all involve "separating" points and/or closed sets from each other by open sets.*

Theorem 14.4.1 (Sufficient condition for $T_1 - space$). *A topological space (S,T) is a $T_1 - space$ iif $\forall x \in S, x$ is a closed set*

Proposition. *The standard topologies on $\mathbb{R}$ and $\mathbb{C}$ are T_0, T_1, T_2, T_3, T_4*

Proposition. *Discrete Topologies are T_0, T_1, T_2, T_3, T_4*

Proposition. *If topological space (S,T) is:*

1. *T_4, then it is T_3*
2. *T_3, then it is T_2*
3. *T_2, then it is T_1*
4. *T_1, then it is T_0*

Proposition.

14.4.2 Metric Space

14.5 Continuous Functions and Homeomorphisms

14.6 Problem Set

14.6.1 Level 1

14.6.2 Level 2

14.6.3 Level 3

14.6.4 Level 4

14.6.5 Level 5

15 Lesson 15 - Complex Analysis: Complex Valued Functions

15.1 Overview

15.2 The Unit Circle

15.3 Exponential Form of a Complex Number

15.4 Functions of a Complex Variable

15.5 Limits and Continuity

15.6 The Reimann Sphere

15.7 Problem Set

15.7.1 Level 1

15.7.2 Level 2

15.7.3 Level 3

15.7.4 Level 4

15.7.5 Level 5

16 Lesson 16 - Linear Algebra: Linear Transformations

16.1 Overview

16.2 Linear Transformations

16.3 Matrices

16.4 The Matric of a Linear Transformation

16.5 Images and Kernels

16.6 Eigenvalues and Eigenvectors

16.7 Problem Set

16.7.1 Level 1

16.7.2 Level 2

16.7.3 Level 3

16.7.4 Level 4

16.7.5 Level 5