Lectures Notes for Linear Algebra - Nathaniel Johnston

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Introduction

Lecture Notes from Nathaniel Johnston

Why Study Linear Algebra?

1 Overview of Linear Algebra

Part I

Linear Algebra

1 Overview

1. Introduction to Vectors: Length, Dot Product, Linear Combination (Lecture 1 to 7)
2. Matrices: Addition, Scalar Multiplication, Matrix Multiplication, transpose (Lecture 8 to 13)
3. Linear Tranformation (Lecture 14 to 16)
4. System of Linear Equation (Lecture 17 to 21)
5. Inverse of a Matrix (Lecture 22 to 25)
6. Subspaces and Basis (Lecture 26 to 33)
7. Determinants (Lecture 34 to 36)
8. Eigenvalues and Eigenvectors (Lecture 37 to)
9. Diagonalization

2 Vectors in 2D

1. Vector Representation: coordinates, direction, length
2. Vector Manipulation: addition, multiplication by scalar
3. Geometric Interpretation of vector manipulation
4. Parallelogram Rule

2.1 problem

1. add and PPS with vectors

3 Vectors in Higher Dimensions

1. Properties of Vector Operations

3.1 problem

1. Prove properties of vector operations
2. Apply vector properties to simplify vector

4 Linear Combinations

1. Linear Combinations
2. Standard Basis Vector $(e_1, e_2, ...), e_1 = (1, 0, 0, ..., 0)$

4.1 problem

1. Determine if vector is a linear combination of sets of vector or not
2. Write vector as linear combination of standard basis vector

5 The Dot Product

1. Dot Product
2. Properties of Dot Product

5.1 problem

1. Perform dot product on two vectors
2. Prove properties of dot product
3. Use properties of dot product

6 The Length of a Vector

1. Length of a Vector
2. Properties of Vector Length
3. Unit Vector

6.1 problem

1. Calculate vector length
2. Prove properties of vector length
3. Use properties of vector length
4. Calculate Unit Vector using normalization

7 The Cauchy Schwarz and Triangle Inequalities

1. Cauchy-Schwarz Inegality: $|v \cdot w| \leq ||v||||w||$
2. Triangle Inequlity

Remarque (Cauchy-Schwarz). *The Cauchy-Schwarz Inequality gives us a relation between the dot product (wether v and w are in same direction) and the length of the vector*

Remarque (Triangle Inequality). *The triangle inequality tells us that if we go in a straight line, we walk a shorter distance than if we don't*

7.1 problem

1. Proof for Cauchy-Schwarz
2. Proof for Triangle Inequality using Cauchy-Schwarz
3. Use the Cauchy-Schwarz Inequality to prove that no vector v and w produce dot product of 7

8 The Angle Between Vectors

1. Angle between vectors: $v \cdot w = ||v||||w|| \cos(\theta)$
2. Orthogonality: $v \cdot w = 0$
3. The zero vector is always orthogonal to any vector

Remarque (Why we can compute arccos). *To find the angle, we can use arccos because we know that* $-1 \leq \frac{v \cdot w}{||v||||w||} \leq 1$ *(Cauchy-Schwarz)*

8.1 problem

1. Proof for Angle between vectors
2. Find Angle between vectors
3. Algebraic Idea between Orthogonality using angle between vector
4. Determine if two vectors are orthogonal

9 Matrix Notation, Addition, and Scalar Multiplication

1. Matrix Addition
2. Properties of Matrix Addition
3. Scalar Multiplication
4. Properties of Scalar Multiplication

9.1 problem

1. Compute addition and scalar multiplication on matrices
2. Prove properties of matrix addition and multiplication
3. Use Matrix addition and scalar multiplication to compute

10 The Mechanics of Matrix Multiplication

1. Matrix Multiplication: row x columns
2. Properties of Matrix Multiplication
3. Why matrix multiplication is not commutative
4. Identity Matrix and Zero Matrix

10.1 problem

1. Compute the prduct of two matrices
2. Prove the properties of matrix multiplication
3. Multiplication by identity matrix and zero matrix

11 The Transpose of a Matrix

1. Transpose of Matrix: swap i and j
2. Properties of Transpose of Matrix
3. $(AB)^T = B^T A^T$

11.1 problem

1. Find transpose of matrix
2. Prove properties of transpose matrix

12 Powers of a Matrix

1. Matrix power: $A^k = AA..A$
2. Properties of Matrix Power

12.1 problem

1. Compute Matrix powers
2. Prove Properties of Matrix powers

13 Block Matrices

1. Block Matrices: partition of matrices for big matrice
2. Theorem: Matrix Vector Multiplication is a linear combination
3. Theorem: Matrix multiplication can be performed column-wise

13.1 problem

1. Perform Matrix Multiplication using Block Matrices
2. Proof for theorem using block multiplication

14 Introduction to Linear Transformations

1. Algebraic Definition of Linear Transformation
2. Geometric Definition of Linear Transformation
3. Linear Transformation using Standard Basis
4. Theorem: Every Linear Transformation T is completely determined by the vectors $T(e_1), ..., T(e_n)$

14.1 problem

1. Determine if Matrix is a linear transformation algebraicly and geometrically

15 The Standard Matrix of a Linear Transformation

1. Theorem: Standard Matrix of a Linear Transformation: all linear transformation can be written as Ax=b
2. Standard Matrix of T: apply transformation to basis $[T] = [T(e_1)...T(e_n)]$

15.1 problem

1. Proof for Standard Matrix of Linear Transformation
2. Find standard matrix of linear transformation
3. Verification for standard matrix of linear transformation: Ax=b

16 A Catalog of Linear Transformations

1. Linear Tranformation of Zero and Identity Matrix
2. Diagonal Transformations/matrices
3. Projection onto a line
4. Rotations

16.1 problem

17 Composition of Linear Transformations

1. Theorem: Composition of Linear Transformation
    - Multiplication of Linear Tranformation is also linear
    - Computing composition of linear transformation as matrix multiplication

17.1 problem

1. Proof of Composition of Linear Transformation
2. Computer Composition of Linear transformation using matrix multiplication

18 Introduction to Systems of Linear Equations

18.1 problem

19 Trichotomy for Linear Systems

19.1 problem

20 Solving Linear Systems (Part 1)

20.1 problem

21 Solving Linear Systems (Part 2: Row Echelon Form)

21.1 problem

22 Solving Linear Systems (Part 3: Zero and Infinitely Many Solutions)

22.1 problem

23 Elementary Matrices

Intuition. *Building blocks of Matrices, like prime number are for integers*

1. Theorem: row operations can be represented as matrices (from identity matrices)
2. Elementary Matrices: can be obtained from identity matrice via single row colum
3. Theorem: Row Reduction: $[A|I]$ $[R|E]$, where R=EA

23.1 problem

24 Introduction to the Inverse of a Matrix

Intuition. *We can use elementary matrices to find the inverse of a matrix. We can undo our linear transformation*

1. Inverse of a Matrix: $A^-1A = AA^-1 = I$
2. Theorem: The inverse of a Matrix is unique
3. Properties of Matrix Inverses
4. How to know if Matrix is inversible

5. Theorem: Caracterization of Invertible Matrices

24.1 problem

1. Proof that inverse of a matrix is unique
2. Show that A is the inverse of B
3. Proof of Properties of Matrix Inverses
4. Zero Matrix and Projection Matrix are not inversible (we cannot undo ie we cannot find the original vector)
5. Determine if matrix is invertible or not: we can use row operations to get identity matrix

25 Computing the Inverse of a Matrix

1. Theorem: Computing Inverses: $[A|I]$ $[I|A^{-}1]$

25.1 problem

1. Determine if Matrix is invertible and find its inverse if it exists

26 One Sided Inverses and a Formula for 2x2 Inverses

1. Theorem: One-Sided Matrix Inverse: if AB=I or BA=I, then A is invertible and $A^{-}1 = B$
2. Theorem: Inverse of 2x2 Matrix:

Intuition. *On dit que c'est one-sided parce qu'on a qu'à vérifier l'inégalité d'un seul bord pour dire que A et B sont leurs inverses respectifs*

26.1 problem

1. Proof for One-Sided Matrix
2. Proof for 2x2 inverse matrices
3. Determine if 2x2 matrix is invertible and calculate its inverse if it exist

27 Subspaces

1. Definition of a Subspace

27.1 problem

1. Determine if set of vectors is a subspace of $\mathbb{R}^n$

28 The Range and Null Space of a Matrix

1. Range of Matrix: Ax
2. Null Space: $Ax = 0$
3. Theorem: Range and Null space are subspace

28.1 problem

1. Find the range and the null space of the matrix

29 The Span of a Set of Vectors

1. Span:
2. Theorem: The span is the subspace of $\mathbb{R}^n$
3. Theorem: Range equals the span of columns
4. Theorem: Spanning Sets and Invertible Matrices

29.1 problem

1. Show that set of vector span subspace
2. Proof
3. Proof

30 Linear Dependence and Independence

Intuition. *Some vectors in the span are redundant*

1. Linear Dependence and Independance
2. Theorem:
3. Theorem: Independance and Invertible Matrices: If A is invertible, then its row and column are linearly independant

30.1 problem

1. Determine if vectors are linearly dependant or not
2. Proofs

31 Bases of Subspaces

1. Defintion of Bases

31.1 problem

1. Determine if sets of vectors form a basis or not

32 The Dimension of a Subspace

1. Theorem: Uniqueness of Size of Bases: Every basis of S has the same number of vectors
2. Dimension of a Subspace

32.1 problem

33 The Rank of a Matrix

Intuition. *How much information we have after transformation*

1. Rank of the Matrix
2. Theorem: Charaterization of Rank

33.1 problem

34 The Nullity of a Matrix

34.1 problem

35 Introduction to the Determinant (Geometrically)

35.1 problem

36 Computing the Determinant (via Gaussian Elimination)

36.1 problem

37 Explicit Formulas for the Determinant

37.1 problem

38 Introduction to Eigenvalues and Eigenvectors

38.1 problem

39 Complex Numbers and Complex Eigenvalues

39.1 problem

40 The Characteristic Polynomial and Multiplicity

40.1 problem

41 Diagonalization and Large Matrix Powers

41.1 problem

42 Matrices with Distinct Eigenvalues are Diagonalizable

42.1 problem

43 The Fibonacci Sequence via Diagonalization

43.1 problem

44 Arbitrary Matrix Powers via Diagonalization

44.1 problem

45 Matrix Functions via Diagonalization

45.1 problem

Part II