Lecture Notes for Graph Theory

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Introduction

1. Fundamentals Concepts: Graphs, Digraphs, Degrees
2. Connectivity
3. Optimization
4. Shortest Path
5. Planar Graphs
6. Coloring
7. Flow

1 Fundamentals Concepts: Graphs, Digraphs, Degrees

1.1 Why study Graph Theory

TODO

1.2 Overview

1. What is a graph
2. Terminology: walk, trail, path, circuit, cycle
3. Graph Cycle
4. Connected Vertices and Connected Graphs
5. Types of Graphs: Path Graph, Cycle Graph, Complete Graph, Complement of a graph, Bipartite Graph, Complete Bipartite Graph
6. Directed Graphs

1.3 What is a Graph

Un graphe est un "ordered pair" composé de deux éléments:

1. Vertex: ensemble des "noeuds" composants le graphe
2. Edges: ensemble de sous-ensembles qui nous dit quels "noeuds" sont reliés

Notre but est de différentier les différents types de graphes et de définir la terminlogie pour parler d'un graphe

1. Undirected Graph vs Directed Graph
2. Simple Graph
3. Order, Size
4. Adjacence

Definition 1.3.1 (Graph)**.** *A graph G is an ordered pair G=(V,E) where V is a finite set of elements and E is a set of 2 subsets of V*

Definition 1.3.2 (Directed and Undirected Graph)**.** *A directed graph, also called digraph, is a graph that has a direction associated with its edges. In other words, the subsets in the Edge set are ordered. An undirected graph is a graph whose edge subsets are not ordered. In other word, if two nodes are connected, then we can reach a to b and b from a.*

Definition 1.3.3 (Order and Size)**.**

1. *Order —V— : number of vertex in the graph*
2. *Size —E—: number of edges in the graph*

Definition 1.3.4 (Simple Graph)**.**

1. *No loop*
2. *No multiples edges*

Definition 1.3.5 (Adjacence)**.** *On peut parler d'adjacence pour les vertex et les edges.*

1. *Vertex Adjacence: 2 vertex are adjacents if they are connected by an edge*
2. *Edge Adjacence: 2 edges are adjacent if they have a vertex in between them*

1.4 Terminology

Definition 1.4.1 (Walk)**.**

1. *Walk: Sequence of adjacent vertices. We can go back on our steps: we can traverse edges and vertices several times. We say the vertices lie on the walk.*
2. *Length: Number of "steps" we make (even though we may go back and forth).*
3. *Open walk: the final vertex is not the same as where we started*
4. *Closed Walk: the end vertex is the same where we started*

Remarque. *On peut utiliser les définitions suivantes pour les trail et autres aussi:*

1. *open/closed*
2. *endpoints*
3. *length*

Definition 1.4.2 (Trail)**.** *A sequence of adjacent vertices without traversing the same edge more than once*

Definition 1.4.3 (Path)**.** *A path is a sequence of adjactent vertices, but we cannot traverse the same vertices more than once (which also means we can't traverse the same edge). Can be defined as*

1. *List of vertices:* $P = (v_1, v_2, ..., v_8)$
2. *List of alternating vertices and edges:* $P = (v_1, v_1v_2, ..., v_8)$

Habituellement, on préfère définir un chemin par une liste de vertices

Definition 1.4.4 (Circuit)**.** *TODO*

Definition 1.4.5 (Path and Cycle)**.**

1. *A Path* P_n *is a graph whose vertices can be arranged in a sequence such that the edge set is* $E = v_iv_{i+1}|i = 1, 2, ..., n-1$
2. *A Cycle* C_n *is a graph whose vertices can be arranged in a cyclic sequence such that the edge set is* $E = v_iv_{i+1}|i = 1, 2, ..., n-1 \cup v_1v_n$

Definition 1.4.6 (Degree of Path and Cycle)**.** *The degree of a path and a cycle is the number of vertex it has.*

Definition 1.4.7 (Girth)**.** *Smallest Cycle in the graph*

Definition 1.4.8 (Distance and Diameter between vertices)**.** *Soit deux noeud u et v.*

1. *Distance entre u et v: plus court chemin entre u et v*
2. *Diameter entre u et v: plus long chemin entre u et v*

Theorem 1.4.1 (Properties of Degrees in Path and Cycle)**.**

1. *A path of degree n has n nodes and (n-1) edges*
2. *A cycle of degree n has n nodes and n edges*

Proposition 1.1. *Every graph G contains a path of length n and a cycle of length at least n+1*

1.5 Connected and Disconnected Graphs

Definition 1.5.1 (Connected Graph)**.** *A graph is connected if for every pair of disinct vertices* $u, v \in V(G)$*, there is a path from u to v in G. Otherwise, we say the graph is disconnected*

Definition 1.5.2 (Connected Vertices)**.**

Definition 1.5.3 (Open and Closed Neighborhood)**.** *TODO*

1.6 Families of Graph and Special Graph

1. Complete Graph K_n: simple graph with an edge between every pair of vertices
2. Empty graph: Graph with no edges
3. Bipartite Graph: a graph whose vertex can be partitionned into two sets V_1 and V_2 such that every edges $u, v \in E$ has $u \in V_1$ and $vinV_2$
4. Complete Bipartite Graph: every node can reach all nodes in the other subset (end)
5. Star
6. Path:
7. Cycle: l'ensemble de noeud allant d'un noeud à lui-même

1.7 Bipartite Graphs

Remarque (Importance des graphes bipartites)**.** *Intuitivement, les graphes bipartites peuvent être séparés en 2 sous-ensembles dont l'image de chaque élément est mappé à l'autre set et pas sur un élément du même ensemble.*

2 Connectivity

3 Optimization

4 Shortest Path

5 Planar Graphs

6 Coloring

7 Flow

8 Ressources

8.1 Books

- Douglas B. West: Introduction to Graph Theory - Reinhard Diestel: Graph Theory

8.2 Courses

- Wrath of Math: Graph Theory Playlist - Sarada Herke: Graph Theory

8.3 Exercices