Assignment - 03

M4 (18MAT41)

Shahul Hameed. S
1KN18CS097
CSE 'A' sec 4th Sem.

1) Discuss the transformation of $w = z + 1/z$

> $w = z + 1/z \Rightarrow \frac{dw}{dz} = 1 - \frac{1}{z^2} \Rightarrow \frac{dw}{dz} = 0, \; z = \pm 1$

$\therefore$ transformation $w = z + \frac{1}{z}$ is not conformal at $z = \pm 1$ and is analytic at every other point of the z-plane

Let $z = re^{i\theta}$ and $w = u + iv$

$w = z + \frac{1}{z}$ $\quad u + iv = re^{i\theta} + \frac{1}{re^{i\theta}}$

$u + iv = re^{i\theta} + \frac{1}{r} e^{-i\theta}$

$u + iv = r[\cos\theta + i\sin\theta] + \frac{1}{r}[\cos\theta - i\sin\theta] = r\cos\theta + i\sin\theta + \frac{1}{r\cos\theta} - \frac{i1}{r}\sin\theta$

$= r\cos\theta + \frac{1}{r}\cos\theta + ir\sin\theta - i\frac{1}{r}\sin\theta$

$u + iv = \left(r + \frac{1}{r}\right)\cos\theta + i\left(r - \frac{1}{r}\right)\sin\theta$

Separating the real and imaginary terms,

$u = \left(r + \frac{1}{r}\right)\cos\theta \quad v = \left(r - \frac{1}{r}\right)\sin\theta \quad \longleftarrow (1)$

Case I :- we have to eliminate θ in eqn (1)

$u = \left(r + \frac{1}{r}\right)\cos\theta \quad v = \left(r - \frac{1}{r}\right)\sin\theta$

$$\cos\theta = \frac{u}{\left(r+\frac{1}{r}\right)} \quad \text{--- (2)} \qquad \sin\theta = \frac{v}{\left(r-\frac{1}{r}\right)} \quad \text{--- (3)}$$

Add and square Eq^n (2) and (3)

$$\frac{u^2}{\left(r+\frac{1}{r}\right)^2} + \frac{v^2}{\left(r-\frac{1}{r}\right)^2} = \cancel{\cos^2\theta + \sin^2\theta} \rightarrow 1$$

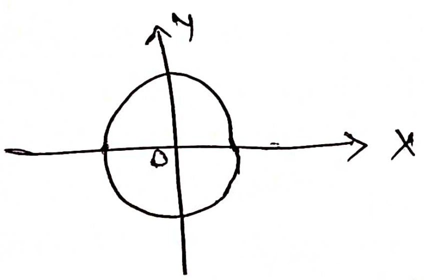
$$\frac{u^2}{\left(r+\frac{1}{r}\right)^2} + \frac{v^2}{\left(r-\frac{1}{r}\right)^2} = 1 \quad \text{--- (4)}$$

Let $r = k_1$, where k_1 is a constant. This represents a circle centred at origin in the z plane.

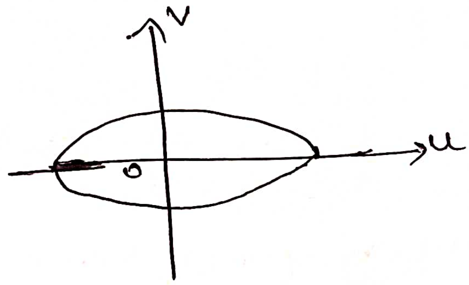
Then Eq^n (4) becomes,

$$\frac{u^2}{\left(k_1+\frac{1}{k_1}\right)^2} + \frac{v^2}{\left(k_1-\frac{1}{k_1}\right)^2} = 1$$

This represents an ellipse having centre at origin in the w plane.



Z-plane.



W-plane

Case 2: Eliminating r in Eq^n (1)

$$u = \left(r+\frac{1}{r}\right)\cos\theta \qquad v = \left(r-\frac{1}{r}\right)\sin\theta$$

$$\frac{u}{\cos\theta} = \left(r+\frac{1}{r}\right) \quad \text{--- (5)} \qquad \frac{v}{\sin\theta} = \left(r-\frac{1}{r}\right) \quad \text{--- (6)}$$

Eq^n $(5^2) - (6^2)$

$$\frac{u^2}{\cos^2\theta} - \frac{v^2}{\sin^2\theta} = \left(r+\frac{1}{r}\right)^2 - \left(r-\frac{1}{r}\right)^2$$

$$= 4r\left(\frac{1}{r}\right)$$

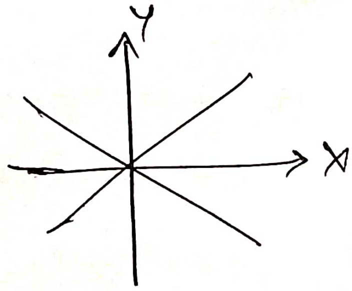
$$= 4$$

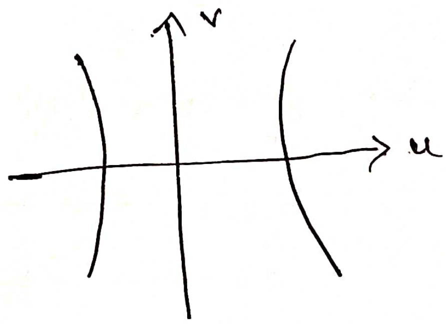
$$\therefore \frac{u^2}{\cos^2\theta} - \frac{v^2}{\sin^2\theta} = 4 \quad \text{--- (7)}$$

Let $\theta = k_2$ where k_2 a constant. This represents a radial lines in the z-plane.

Thus eqn (7) $\Rightarrow \frac{u^2}{\cos^2 k_2} - \frac{v^2}{\sin^2 k_2} = 4$

This represents a hyperbola having centre at origin in the w-plane.





Thus the transformation $w = z + \frac{1}{z}$ transforms circle with centre origin to ellipse having centre at origin and the radial lines to hyperbola having centre at origin.

(9) Find the bilinear transformation which maps the points $-1, 0, 1$ from z plane onto the points $0, i, 3i$ into w plane.

$w = \frac{az+b}{cz+d}$ If $z_1 = -1$ $w_1 = 0 \Rightarrow 0 = \frac{-a+b}{-c+d}$

$$-a+b=0 \quad \text{—— (1)}$$

$z_2=0 \qquad w_2=i \quad \Rightarrow \quad i=\frac{b}{d}$

$$id=b$$

$$id-b=0 \quad \text{—— (2)}$$

$z_3=1 \qquad w_3=3i \quad \Rightarrow \quad 3i=\frac{a+b}{c+d}$

$$3i(c+d)=a+b$$

$$3ic+3id=a+b$$

$$a+b-3ic-3id=0 \quad \text{—— (3)}$$

Add Eq^n (1) and (3)

$$-a+b+0c+0d=0$$

$$a+b-3ic-3id=0$$

$$2b-3ic-3id=0 \quad \text{—— (4)}$$

Consider eq^n (2) and (4) and write in the form of

$$-b+0c+id=0$$

$$2b-3ic-3id=0$$

Apply cross multiplication rule.

$$\frac{b}{\begin{vmatrix} 0 & i \\ -3i & -3i \end{vmatrix}} = \frac{-c}{\begin{vmatrix} -1 & i \\ 2 & -3i \end{vmatrix}} = \frac{d}{\begin{vmatrix} -1 & 0 \\ 2 & -3i \end{vmatrix}} = k$$

$$\frac{b}{-3}=k \qquad \frac{-c}{i}=k \qquad \frac{d}{3i}=k$$

$$b=-3k \qquad c=-ik \qquad d=3ik$$

Eq^n (3) becomes

$$a-3-3i(-i)-3i(3i)=0$$

$$a-3+3i^2-9i^2=0$$

$$a-3-6i^2=0$$

$$a-3-6(-1)=0$$

$$a-3+6=0$$

$$a+3=0$$

$$\boxed{a=-3}$$

$$w=\frac{-3z-3}{-iz+3i}=\frac{\cancel{-}(3z+3)}{\cancel{-}(iz-3i)}$$

$$\boxed{w=\frac{3z+3}{iz-3i}}$$

(3) The probability distribution of a finite random variable X is given by the following table.

x_i	-2	-1	0	1	2	3
$P(x_i)$	0.1	k	0.2	2k	0.3	k

Determine the value of K and find the mean, variance and standard deviation.

> The probability distribution is valid if $P(x)\geq 0$ and $\Sigma p(x)=1$

$$0.1+k+0.2+2k+0.3+k=1$$

$$0.6+4k=1$$

$$4k=1-0.6$$

$$k=\frac{0.4}{4}=0.1 \qquad \boxed{k=0.1}$$

∴ The probability distribution is given by,

x_i	-2	-1	0	1	2	3
$p(x_i)$	0.1	0.1	0.2	0.2	0.3	0.1

Mean $(\mu)=\Sigma x_i\, p(x_i)$

$= x_i\, p(x_1) + x_2\, p(x_2) + x_3\, p(x_3) + x_4\, p(x_4)$
$+ x_5\, p(x_5) + x_6\, p(x_6)$

$= -2\times 0.1 - 1(0.1) + 0\times 0.2 + 1\times 0.2 + 2\times 0.3 + 3\times 0.1$

Mean $(\mu) = -0.2 - 0.1 + 0.2 + 0.6 + 0.3 = 0.8$

Variance $(V) = \Sigma (x_i - \mu)^2 p(x_i)$

$= (x_1-\mu)^2 p(x_1) + (x_2-\mu)^2 p(x_2) + (x_3-\mu)^2 p(x_3)$
$+ (x_4-\mu)^2 p(x_4) + (x_5-\mu)^2 p(x_5) + (x_6-\mu)^2 p(x_6)$

variance $(V) = (-2-0.8)^2 (0.1) + (-1-0.8)^2 (0.1) + (0-0.8)^2 (0.2)$
$+ (1-0.8)^2 (0.2) + (2-0.8)^2 (0.3) + (3-0.8)^2 (0.1)$

$= 0.784 + 0.324 + 0.128 + 8\times 10^{-3} + 0.432 + 0.484$

$\boxed{V = 2.16}$

Standard deviation $SD = \sqrt{V} = \sqrt{2.16} = 1.4696$ //.

(4) The probability that a bomb dropped hits the target is 0.2 find the probability that out of 6 bombs dropped (i) Exactly 2 will hit the target. (ii) at least 2 will hit the target.

> (i) probability is given by

probability $= p(r=2)$ put $r=2$

probability $= {}^6C_2 \left(\frac{1}{5}\right)^2 \left(\frac{4}{5}\right)^4 = \frac{6!}{(6-2)!\,2!} \times \left(\frac{1}{5}\right)^2 \times \left(\frac{4}{5}\right)^4$

$= \frac{3\times 5\times 4^4}{5^2\times 5^4} = \frac{3\times 4^4}{5^5} = \frac{768}{3125} = 0.24576$ //.

(iii) The required probability is.

$\sum_{k=2}^{6} {}^6C_k (p)^k (1-p)^{6-k}$ where $p = 0.2 = 1/5$

$= 1/5^6 [{}^6C_2 4^4 + {}^6C_3 4^3 + {}^6C_4 4^2 + {}^6C_5 4^1 + {}^6C_6 4^0]$

$= 1/5^6 [15(256) + 20(64) + 15(16) + 6(4) + 1]$

$= \frac{5385}{5^6} = 0.345\%.$

(i) Exactly 2 will hit the target $= 0.24576\%$.

(ii) Atleast 2 will hit the target $= 0.345\%$.

(5) The joint distribution of two random variables X and Y is as follows.

X\Y	-4	2	7
1	1/8	1/4	1/8
5	1/4	1/8	1/8

Compute the following

(i) $E(X)$ & $E(Y)$ (ii) $E(XY)$ (iii) σ_x & σ_y (iv) $f(x,y)$

> Given :- $x_1 = 1$ and $x_2 = 5$

$y_1 = -4$, $y_2 = 2$, $y_3 = 7$

joint probability distribution

X\Y	-4	2	7	$f(x_i)$
1	1/8	1/4	1/8	1/2
5	1/4	1/8	1/8	1/2
$g(y_j)$	3/8	3/8	1/4	1

distribution of x distribution of Y

x_i	1	5
$f(x_i)$	$1/2$	$1/2$

y_j	-4	2	7
$g(y_j)$	$3/8$	$3/8$	$1/4$

(i) $E(X) = \Sigma x_i \cdot f(x_i)$

$= (1 \times 1/2) + (5 \times \frac{1}{2})$

$E(X) = 3$

$E(Y) = \Sigma y_j g(y_j)$

$= (4 \times \frac{3}{8}) + (2 \times \frac{3}{8}) + (7 \times \frac{1}{4})$

$E(Y) = 1$

(ii) $E(XY) = \Sigma x_i y_j J_{ij}$

$= (1 \times -4 \times \frac{1}{8}) + (1 \times 2 \times \frac{1}{4}) + (1 \times 7 \times \frac{1}{8}) + (5 \times -4 \times \frac{1}{4}) + (5 \times 2 \times \frac{1}{8})$

$+ (5 \times 7 \times \frac{1}{8})$

$= 3/2$ //.

(iii) $\sigma_x^2 = E(x^2) - \mu_x^2$

$E(x^2) = \Sigma x_i^2 f(x_i)$

$= (1)^2 (1/2) + (5)^2 (1/2)$

$= 1/2 + 25/2 = 13$

$E(x^2) = 13$

$\mu_x^2 = 3^2 = 9$

$\sigma_x^2 = 4$

$\sigma_x = \sqrt{4}$

$\boxed{\sigma_x = 2}$

$\sigma_y^2 = E(Y^2) - \mu_y^2$

$E(Y^2) = \sum y_j^2 \, g(y_j)$

$= (-4)^2 \,(3/8) + 2^2 \,(3/8) + (7)^2 \,(1/4)$

$= \frac{79}{4}$

$\mu_y^2 = 1^2 = 1$

$\sigma_y^2 = E(Y^2) - \mu_y^2$

$= \frac{79}{4} - 1$

$\sigma_y^2 = \frac{75}{4}$

$\sigma_y = \sqrt{75/4} = 4.33$

(iv) $cov(X, Y)$

$cov(X,Y) = E(XY) - \mu_x \mu_y$

$= 3/2 - 3(1)$

$= \frac{3}{2} - 3 = -\frac{3}{2}$ or $= -1.5$

(v) $\rho(X,Y) = \frac{cov(X,Y)}{\sigma_x \sigma_y} = \frac{-3/2}{2 \times 4.33}$

$\rho(X,Y) = -0.1732$ //.

(6) Determine (i) marginal distribution (ii) covariance b/w the discrete random variable X & Y, using the joint probability distribution.

X \ Y	3	4	5
2	1/6	1/6	1/6
5	1/12	1/12	1/12
7	1/12	1/12	1/12

? given, $x_1 = 2$ $x_2 = 5$ $x_3 = 7$ $y_2 = 3$ $y_2 = 4$ $y_3 = 5$

The joint distribution table is.

$x \backslash y$	3	4	5	$f(x_i)$
2	1/6	1/6	1/6	1/2
5	1/12	1/12	1/12	1/4
7	1/12	1/12	1/12	1/4
$g(y_j)$	1/3	1/3	1/3	1

(i) marginal distribution

Distribution of x

x	2	5	7
$f(x_i)$	1/2	1/4	1/4

Distribution of y

y	3	4	5
$g(y_j)$	1/3	1/3	1/3

(ii) $cor(x,y) = ?$

W.K.T $cov(x,y) = E(xy) - \mu_x \mu_y$

$E(x) = x_i f(x_i)$

$= \left(2 \times \frac{1}{2}\right) + \left(5 \times \frac{1}{4}\right) + \left(7 \times \frac{1}{4}\right)$

$E(x) = 4 = \mu_x$

$E(y) = y_j g(y_j)$

$= \left(3 \times \frac{1}{3}\right) + \left(4 \times \frac{1}{3}\right) + \left(5 \times \frac{1}{3}\right)$

$E(y) = 4 = \mu_y$

$E(xy) = \Sigma x_i y_j J_{ij}$

$= \left(2 \times 3 \times \frac{1}{6}\right) + \left(2 \times 4 \times \frac{1}{6}\right) + \left(2 \times 5 \times \frac{1}{6}\right) + \left(5 \times 3 \times \frac{1}{12}\right) + \left(5 \times 4 \times \frac{1}{12}\right) +$

$\left(5 \times 5 \times \frac{1}{12}\right) + \left(7 \times 3 \times 1/12\right) + \left(7 \times 4 \times \frac{1}{12}\right) + \left(7 \times 5 \times 1/12\right)$

$E(xy) = 16$

$\therefore cor(x,y) = E(xy) - \mu_x \mu_y$

$= 16 - 4 \times 4 = 16 - 16$

$cov(x,y) = 0$ //.