Assignment - 01

M-3

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CSE 'A' sec

(1) S.T $w = f(z) = z + e^z$ is analytic and hence find $\frac{dw}{dz}$

> $w = z + e^z$ by data

$u + iv = (x + iy) + e^{(x+iy)}$

$= (x+iy) + e^x e^{iy}$

$= (x+iy) + e^x(\cos y + i\sin y) = (x + e^x\cos y) + i(y + e^x \sin y)$

$u = x + e^x \cos y \qquad v = y + e^x \sin y$

$u_x = 1 + e^x \cos y \qquad v_x = e^x \sin y$

$u_y = -e^x \sin y \qquad v_y = 1 + e^x \cos y$

we observe that C-R Eqns in the cartesian form $u_x = v_y$ & $v_x = -u_y$ are satisfied. Thus, $w = z + e^z$ is analytic.

$\therefore$ Also we have $\frac{dw}{dz} = f'(z) = u_x + i v_x$

i.e $\frac{dw}{dz} = (1 + e^x \cos y) + i(e^x \sin y)$

$= 1 + e^x(\cos y + i \sin y) = 1 + e^x e^{iy} = 1 + e^{(x+iy)}$

Since $z = x + iy$, $\frac{dw}{dz} = 1 + e^z$ //.

(2) Derive Cauchy-Reimann equation in cartesian form.

> Statement:- The necessary conditions that the function $w = f(z) = u(x,y) + iv(x,y)$ may be analytic at any point $z = x + iy$ is that, there exists four conditions first order Partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ and satisfy the equation.

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$. These are known as C-R equations. $u_x = u_y$ & $v_x = -u_y$.

Proof: Let $f(z)$ be analytic at a point $z=x+iy$ & hence by the ~~defn~~ r. defn. $f'(z) = \lim_{\delta z \to 0} \frac{f(z+\delta z) - f(z)}{\delta z}$ exists and unique.

In the cartesian form $f(z) = u(x,y) + iv(x,y)$ & Let δz be the increment in z corresponding to the increment $\delta x, \delta y$ in x, y.

$$f'(z) = \lim_{\delta z \to 0} \frac{[u(x+\delta x, y+\delta y) + iv(x+\delta x, y+\delta y)] - [u(x,y) + iv(x,y)]}{\delta z}$$

$$f'(z) = \lim_{\delta z \to 0} \frac{[u(x+\delta x, y+\delta y) - u(x,y)]}{\delta z} + i \lim_{\delta z \to 0} \frac{[v(x+\delta x, y+\delta y) - v(x,y)]}{\delta z} \longrightarrow (1)$$

Now, $\delta z = (z+\delta z) - z$, where $z = x+iy$.

$\therefore \delta z = [(x+\delta x) + i(y+\delta y)] - [x+iy]$

i.e $\delta z = \delta x + i\delta y$.

Since δz tends to zero, we have the following two possibilities.

Case(i): Let $\delta y = 0$, So that $\delta z = \delta x$ & $\delta z \to 0$ imply $\delta x \to 0$. Now Eqn (1) $\Rightarrow$

$$f'(z) = \lim_{\delta x \to 0} \frac{u(x+\delta x, y) - u(x,y)}{\delta x} + i \lim_{\delta x \to 0} \frac{v(x+\delta x, y) - v(x,y)}{\delta x}$$

These limits form the basic defn are the partial derivatives of u and v w.r.t 'x'.

$$\therefore f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \longrightarrow (2)$$

Case(ii): Let $\delta x = 0$ So that $\delta z = i\delta y$ & $\delta z \to 0$ imply $i\delta y \to 0$ (or) $\delta y \to 0$ Now, Eqn (1) $\Rightarrow$

$$f'(z) = \lim_{\delta y \to 0} \frac{u(x, y+\delta y) - u(x,y)}{i\delta y} + i \lim_{\delta y \to 0} \frac{v(x, y+\delta y) - v(x,y)}{i\delta y}$$

But $\frac{1}{i} = \frac{i}{i^2} = \frac{i}{-1} = -i$ & Hence we have,

$$f'(z) = \lim_{\delta y \to 0} -i \frac{u(x, y+\delta y) - u(x,y)}{\delta y} + \lim_{\delta y \to 0} \frac{v(x,y+\delta y) - v(x,y)}{\delta y}$$

$$f'(z) = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

$$\therefore f'(z) = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} \longrightarrow (3)$$

Equating RHS of Eqn (1) & (3), we have,

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$

Now equating ~~real~~ real and imaginary part,

we get,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \& \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

Thus, we have established C-R equations: $u_x = v_y$ & $v_x = -u_y$

These are the necessary conditions in the cartesian form for the complex valued funⁿ.

$f(z) = u + iv$ to be analytic.

(3) Find the analytic function $f(z) = u + iv$ given, $v = \left[r - \frac{1}{r}\right]\sin\theta$, $r \neq 0$.

By data $v = \left(r - \frac{1}{r}\right)\sin\theta$ — (1)

differentiate w.r.t 'r'

$$v_r = \left(1 - \left(-\frac{1}{r^2}\right)\right)\sin\theta$$

$$v_r = \left(1 + \frac{1}{r^2}\right)\sin\theta \quad \dots (2)$$

differentiate Eqn (1) w.r.t θ.

$$v_\theta = \left(r - \frac{1}{r}\right)\cos\theta \longrightarrow (3)$$

$$\text{W.K.T } f'(z) = e^{-i\theta}\left(\frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{\partial v}{\partial r}\right) \longrightarrow (4)$$

Substitute eqn (2) & (3) in (4)

$$f'(z) = e^{-i\theta}\left\{\frac{1}{r}\left[\left(r - \frac{1}{r}\right)\cos\theta\right] + i\left[\left(1 + \frac{1}{r^2}\right)\sin\theta\right]\right\}$$

Put $r = z$, $\theta = 0$ in above eqn we get.

$$f'(z) = e^{-i(0)}\left\{\frac{1}{z}\left[\left(z - \frac{1}{z}\right)\cos 0\right] + i\left[\left(1 + \frac{1}{z^2}\right)\cancel{\sin(0)}\right]\right\}$$

$$= \frac{1}{z}\left[z - \frac{1}{z}\right]$$

$$f'(z) = 1 - \frac{1}{z^2} //.$$

Integrating on both sides,

$$\int f'(z) = \int\left(1 - \frac{1}{z^2}\right)dz + c$$

$$f(z) = z - \left(-\frac{1}{z}\right) + c$$

$$f(z) = z + \frac{1}{z} + c //.$$

(4) If $f(z)$ is analytic, S.T $\left[\left(\frac{\partial^2}{\partial x^2}\right) + \left(\frac{\partial^2}{\partial y^2}\right)\right] |f(z)|^2 = 4|f'(z)|^2$

7 Given, $f(z) = u(x,y) + i v(x,y) = u + iv$ is regular

$$\Rightarrow f'(z) = u_x + i v_x$$

$$\text{& W.K.T } f(z) = \sqrt{u^2 + v^2}$$

$$f(|z|^2) = u^2 + v^2 = \phi \longrightarrow (1)$$

$$f'(z) = \sqrt{u_x^2 + v_x^2} \longrightarrow (2)$$

$$\text{LHS} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \cdot f(|z|^2)$$

$$= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\phi$$

$$LHS: \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2}$$

$$\therefore \frac{\partial \phi}{\partial x} = 2uu_x + 2vv_x$$

$$\therefore \frac{\partial^2 \phi}{\partial x^2} = 2(u_x u_x + u u_{xx}) + 2(v_x v_x + v.v_{xx})$$

$$\Rightarrow \frac{\partial^2 \phi}{\partial x^2} = 2(u_x^2 + u u_{xx} + v_x^2 + v.v_{xx}) \longrightarrow (3)$$

$$\therefore \frac{\partial \phi}{\partial y} = 2u u_y + 2v.v_y$$

$$\Rightarrow \frac{\partial^2 \phi}{\partial y^2} = 2(u_y u_y + u u_{yy}) + 2(v_y v_y + v.v_{yy})$$

$$\Rightarrow \frac{\partial^2 \phi}{\partial y^2} = 2(u_y^2 + u.u_{yy} + v_y^2 + v.v_{yy}) \longrightarrow (4)$$

$$LHS = 2(u_x^2 + u.u_{xx} + v_x^2 + v.v_{xx} + u_y^2 + u.u_{yy} + v_y^2 + v.v_{yy})$$

$$= 2[u_x^2 + v_x^2 + u_y^2 + v_y^2 + u(u_{xx} + u_{yy}) + v(v_{xx} + v_{yy})]$$

$$= 2[u_x^2 + v_x^2 + u_y^2 + v_y^2 + u(0) + v(0)]$$

$$\cancel{= 2[u_x^2 + v_x^2 + u_y^2 + v_y^2]}$$

$$= 2[u_x^2 + v_x^2 + (-v_x)^2 + (u_x)^2]$$

$$= 2[u_x^2 + v_x^2 + v_x^2 + u_x^2]$$

$$= 2[2(u_x^2 + v_x^2)]$$

$$= 4(u_x^2 + v_x^2)$$

$$= 4(|f'(z)|^2)$$

$$= RHS$$

(5) S.T the function $u = \sin x \cosh y + 2\cos x \sinh y + x^2 - y^2 + 4xy$ is harmonic. Also determine the corresponding analytic function.

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Given, $u = \sin x \cosh y + 2\cos x \sinh y + x^2 - y^2 + 4xy$.

$u_x = \cos x \cosh y - 2\sin x \sinh y + 2x + 4y$

$u_y = \sin x \sinh y + 2\cos x \cosh y - 2y + 4x$

$u_{xx} = -\sin x \cosh y - 2\cos x \sinh y + 2$ —— (1)

$u_{yy} = \sin x \cosh y + 2\cos x \sinh y - 2$ —— (2)

$\therefore$ (1) + (2) $\Rightarrow u_{xx} + u_{yy} = 0$

$\Rightarrow u$ is harmonic.

Consider $f'(z) = u_x + iv_x$ & $v_x = -u_y$

$\Rightarrow f'(z) = u_x - iu_y$

$\Rightarrow f'(z) = [\cos x \cosh y - 2\sin x \sinh y + 2x + 4y]$
$- i[\sin x \sinh y + 2\cos x \cosh y - 2y + 4x]$

Put $x = z$ & $y = 0$

$\Rightarrow f'(z) = (\cos z + 2z) - i(2\cos z + 4z)$

integrating w.r.t 'z'

$\Rightarrow f(z) = \int \{(\cos z + 2z) - i(2\cos z + 4z)\} dz + c$

$= (\sin z + z^2) - i(2\sin z + 2z^2) + c$

$= (1-2i)\sin z + (1-2i)z^2 + c$

$f(z) = (1-2i)(\sin z + z^2) + c$ // Ans