Algorithms
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Chapter 1: Algorithms with numbers

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Problem 1.1: Show that in any base $b \geq 2$, the sum of any three single-digit numbers is at most two digits long.

Maximum possible number in base b is $b-1$ e.g. 1 in binary($b=2$) and 9 in decimal($b=10$). On adding $b-1$ thrice, the result will be $3 \times (b-1)$, which is at most 2 bit long as $3 \times (b-1) < b^3$.

Problem 1.2: Show that any binary integer is at most four times as long as the corresponding decimal integer. For very large numbers, what is the ratio of these two lengths, approximately?

A number of n in base b has length $log(n)/log(b)$. So , length for binary is $log(n)/log(2)$ & for decimal is $log(n)/log(10)$. Their ratio is given by $log(10)/log(2) = 1/0.3010 > 3$ and its ceil value is 4.

Problem 1.3: A d-ary tree is a rooted tree in which each node has at most d children. Show that any d-ary tree with n nodes must have a depth of $\Omega(log(n)/log(d))$. Can you give a precise formula for the minimum depth it could possibly have?

For n child nodes and for a complete tree, minimum depth would be $log(n)/log(d)$. The sum of nodes over levels is a geometric series sum having value $(d^{k+1}-1)/(d-1)$ where k is depth of tree and this sum would be n. Hence, k would be $\Omega(log(n)/log(d))$. Precise representation would be $\lfloor log_d(n) \rfloor$.

Problem 1.4: Show that $log(n!) = \Theta(nlog(n))$.

Using hint, $n! < n^n$ & $n! > (n/2)^{n/2}$. Hence, $log(n!) < nlog(n)$ & $log(n!) < (n/2)log(n/2)$. The lower bound can also be expressed as $log(n!) < (1/2)(nlog(n) - n)$. Hence, $log(n!) = \Theta(nlog(n))$.