Looking for exact mixing velocities in miscible displacement: two-tube approach

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Based on discussions with Sergey Tikhomirov, Yalchin Efendiev, Aleksandr Enin, Fedor Bakharev, Dmitry Pavlov, Sergey Matveenko, Nikita Rastegaev etc

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organized by Gabriel Lame Chair Jean-Michel Roquejoffre

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Story: our experiments with Hele-Shaw cell

Beginning of the project with GazpromNeft

But...

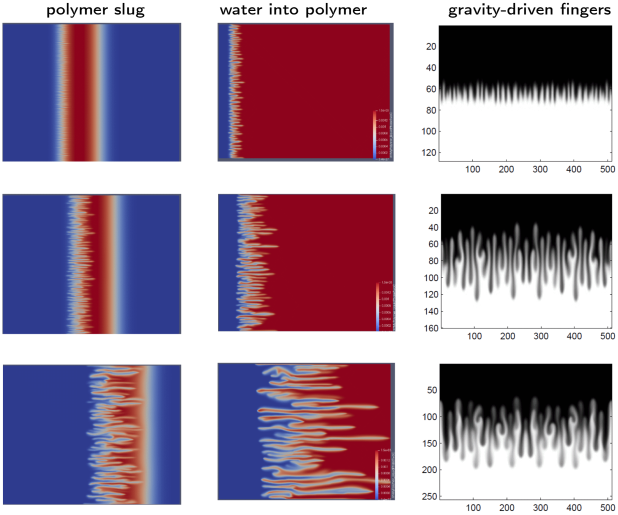
- we investigate the displacement of fluids in porous medium
- “similar” instabilities occur

Story: our numerical experiments in porous media

Video of fingers formation

Motivation and main question

- water flooding;
- chemical flooding
- cause problems for oil recovery



Main question of interest:
rigorous bounds on velocities of mixing zone propagation

Peaceman vs TFE model

Basic equations:

$$\text{(conservation of species)} \qquad c_t + \vec{u} \cdot \nabla c = \varepsilon \cdot \Delta c,$$

$$\text{(incompressibility condition)} \qquad \mathrm{div}(\vec{u}) = 0.$$

Peaceman model:	TFE model:
$\vec{u} = -m(c)\nabla p$	$\vec{u} = (u^1, u^2), \qquad u^1 = -\frac{m(c)}{\overline{m}(x,t)}$ $\overline{m}(x,t)$ is the average mobility over the transverse direction to the flow
• strong non-locality due to the presence of pressure	• weaker non-locality (depends only on vertical line, not the whole space)
• ¿no rigorous result?	• Felix Otto (2006) rigorously proved the linear bounds on velocities

Hypothesis: models show similar behavior when $p(x, y, t) \approx p(x, t)$ (“moderate” times)

TFE estimates

- $c \in [0, 1]$, thus
$$\frac{m(c)}{m(0)} \leqslant |u^1| = \frac{m(c)}{\overline{m}(x,t)} \leqslant \frac{m(c)}{m(1)}$$
- Consider two 1-dimensional equations:
$$c_t^{\max} + \frac{m(c^{\max})}{m(0)} c_x^{\max} = \varepsilon c_{xx}^{\max}$$
$$c_t^{\min} + \frac{m(c^{\min})}{m(0)} c_x^{\min} = \varepsilon c_{xx}^{\min}$$
Their solutions are super and subsolutions for 2-dim TFE model.
- In fact, they correspond to travelling waves connecting states 0 and 1 with velocities
$$v^f = \frac{\int_0^1 m(c)\, dc}{m(1)} \qquad v^b = \frac{\int_0^1 m(c)\, dc}{m(0)}$$

What is known for Peaceman model?

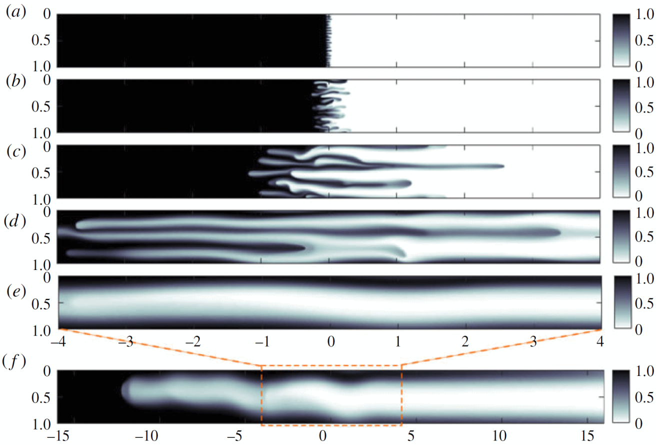
Detailed numerical analysis by Nijjer, Hewit, Neufield (Cambridge), 2018

The dynamics of miscible viscous fingering from onset to shutdown

$h(t)$ — mixing zone width

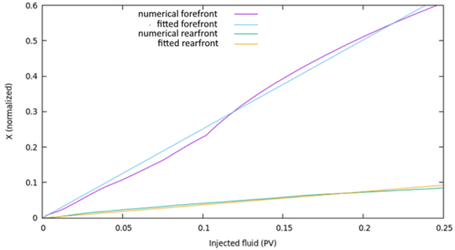
Three flow regimes:

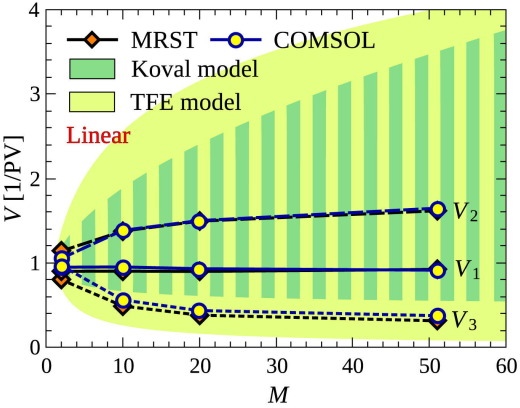
1. early-time
   $h \sim C_1 \sqrt{t}$
2. intermediate-time
   $h \sim C_2 t$
3. late-time
   $h \sim C_3$



Peaceman model: numerical experiments

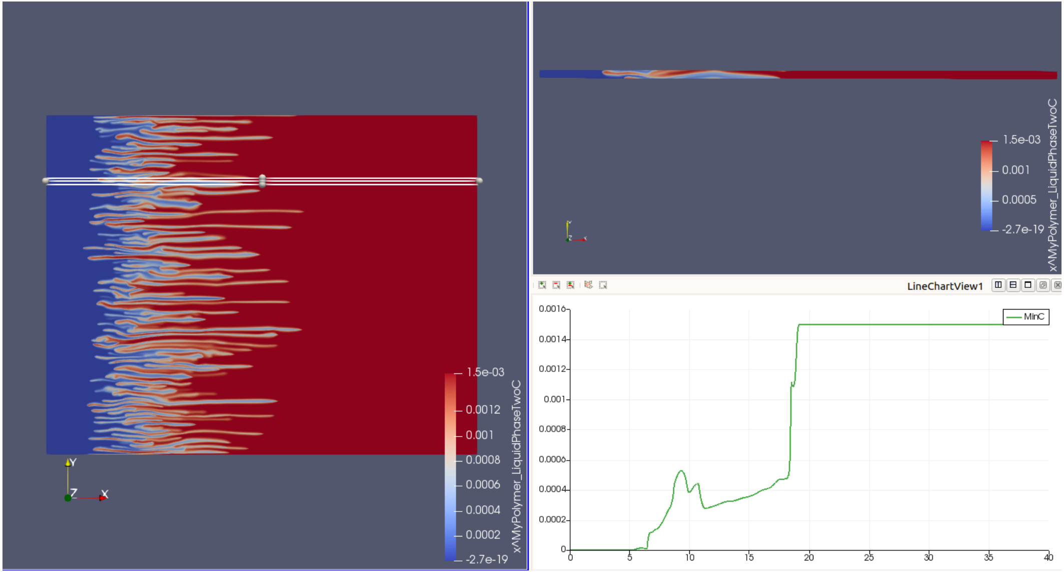
- linear growth of the "fastest finger"
- comparison to TFE model
- TFE too pessimistic



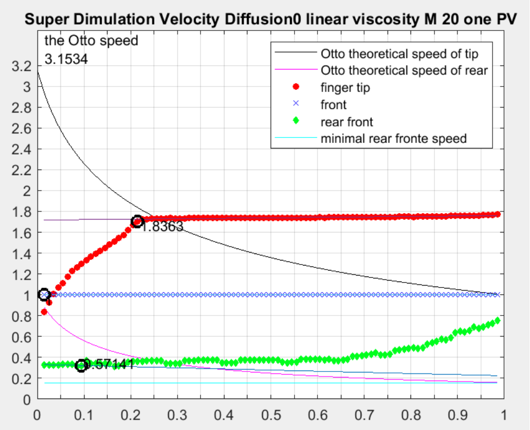


Experiments by Dmitry Pavlov (DuMuX), Ivan Starkov (COMSOL) and Sergey Matveenko (MRST)

Peaceman model: numerical experiments (show video)



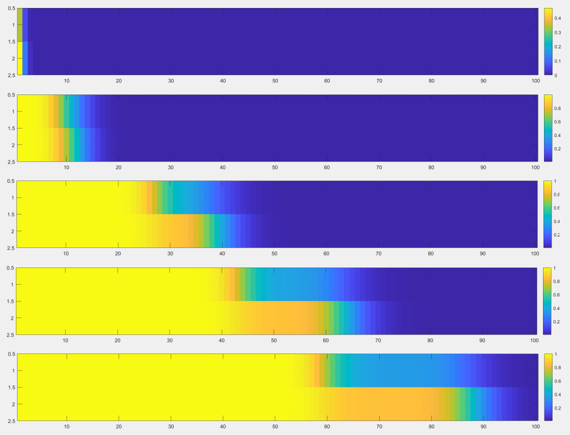
Peaceman model: velocities at the tip of the finger



Idea: looks like a travelling wave connecting some intermediate concentration c^* and 1

Numerical experiments: 2 tubes “speaking” to each other

Finite-volume scheme, simulations by Yalchin Efendiev

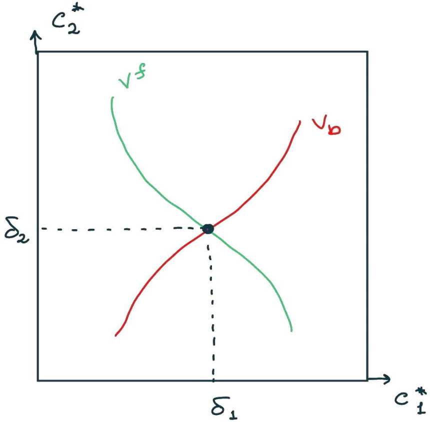


Result of experiments — seems that there are two travelling waves running at different speeds and exists some intermediate concentration c^*

Two-tube approach: general scheme

Aim: to find intermediate concentrations δ_1 and δ_2 such that there exists:

- a travelling wave between $(0,0)$ and (δ_1, δ_2) with some velocity v^b
- a travelling wave between (δ_1, δ_2) and $(1,1)$ with some velocity v^f



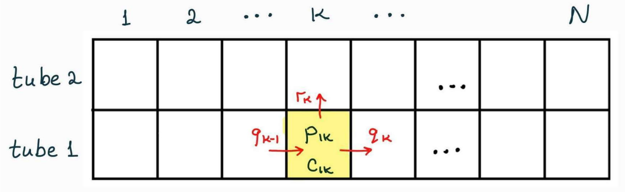
Two-tube approach: discrete and continuous settings

Discrete case

- system of 2 ODEs and 1 algebraic equation
- Travelling wave solutions: system of ODEs with delay

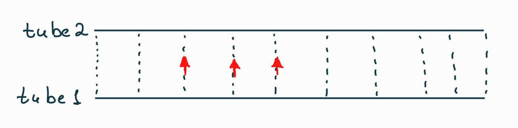
Unknowns:

- $c_{1k}(t), c_{2k}(t)$ — concentrations
- p_{1k}, p_{2k} — pressures
- $q_k(t), r_k(t)$ — velocities



Continuous case

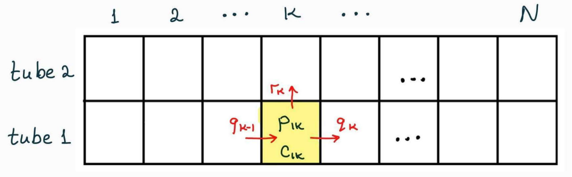
- two coupled conservation laws
- system of 4 ODEs for travelling wave solutions



Two-tube approach: discrete setting

Unknowns:

- $c_{1k}(t), c_{2k}(t)$ — concentrations
- p_{1k}, p_{2k} — pressures
- $q_k(t), r_k(t)$ — velocities



(pretending to be physicists — all terms up to $\pm$)

(conservation of species)

$$\frac{d}{dt} c_{1k} = -q_k c_{1k} + q_{k-1} c_{1,k-1} - r_k c_{1k}$$

$$\frac{d}{dt} c_{2k} = -(2 - q_k) c_{2k} + (2 - q_{k-1}) c_{2,k-1} + r_k c_{1k}$$

(incompressibility condition)

$$q_k + r_k - q_{k-1} = 0$$

(Darcy's law)

$$q_k = -\frac{p_{1k} - p_{1,k+1}}{\mu(c_{1k, c_{1,k+1}})}$$

(pressure law)

$$(p_{2k} - p_{1k}) + (p_{1k} - p_{1,k-1}) + \\ +(p_{1,k-1} - p_{2,k-1}) + (p_{2,k-1} - p_{2k}) = 0$$

Looking for travelling wave solutions in discrete setting

Looking for travelling waves solutions:

$$c_{1k}(t) = f_1(k - v^b t) = f_1(\xi), \qquad c_{2k}(t) = f_2(k - v^b t) = f_2(\xi), \qquad q_k = g(k - v^b t).$$

We get system of ODEs (backward-forward)

$$\begin{aligned}
-v^b f_1' &= F_1(f_1(\xi), f_1(\xi - 1), g(\xi), g(\xi - 1)),\\
-v^b f_2' &= F_2(f_2(\xi), f_2(\xi - 1), g(\xi), g(\xi - 1)),\\
0 &= F_3(g(\xi), g(\xi - 1), g(\xi + 1), f_{1,2}(\xi), f_{1,2}(\xi - 1), f_{1,2}(\xi + 1))
\end{aligned}$$

...too difficult....

Two-tube approach: derivation continuous from discrete

Two coupled conservation laws

$$\begin{aligned} \frac{\partial c_1}{\partial t} + q\frac{\partial c_1}{\partial x} &= 0, \\ \frac{\partial c_2}{\partial t} + (2-q)\frac{\partial c_1}{\partial x} - (c_2 - c_1)\frac{\partial q}{\partial x} &= 0. \end{aligned} \tag{1}$$

For TFE model velocity u is defined as follows

$$q = \frac{2\mu(c_2, c_2)}{\mu(c_1, c_1) + \mu(c_2, c_2)}$$

In general, $q = G(c_1, c_2)$.
Here $\mu(c_1, c_2)$ is the viscosity function — some mean between $\mu(c_1)$ and $\mu(c_2)$.

¿Problem? The system (1) is not in the divergence form usual conservation laws

$$U_t + (F(U))_x = 0$$

Looking for solution being two travelling waves

Add simplest diffusion terms:

$$\frac{\partial c_1}{\partial t} + q\frac{\partial c_1}{\partial x} = \varepsilon_1 \frac{\partial^2 c_1}{\partial x^2},$$

$$\frac{\partial c_2}{\partial t} + (2-q)\frac{\partial c_1}{\partial x} - (c_2 - c_1)\frac{\partial q}{\partial x} = \varepsilon_2 \frac{\partial^2 c_2}{\partial x^2}.$$

We are looking for travelling wave solutions for the back front:

$$c_1(x,t) = f_1(x - v^b t) = f_1(\xi), \qquad c_2(x,t) = f_2(x - v^b t) = f_2(\xi).$$

We get (because of non-divergence form we cannot easily get rid of 1 derivative!):

$$-v_b \frac{\partial f_1}{\partial \xi} + G(f_1, f_2) \cdot \frac{\partial f_1}{\partial \xi} = \varepsilon_1 \frac{\partial^2 f_1}{\partial \xi^2},$$

$$-v_b \frac{\partial f_2}{\partial \xi} + (2 - G(f_1, f_2)) \cdot \frac{\partial f_2}{\partial \xi} - (f_2 - f_1)\left(G_1' \frac{\partial f_1}{\partial \xi} + G_2' \frac{\partial f_2}{\partial \xi}\right) = \varepsilon_2 \frac{\partial^2 f_2}{\partial \xi^2}.$$

Dynamical system for a travelling wave solution

Let us introduce $g_1 = \frac{\partial f_1}{\partial \xi}$ and $g_2 = \frac{\partial f_2}{\partial \xi}$. Using functions g_1 and g_2 we get a system of 4 ODEs:

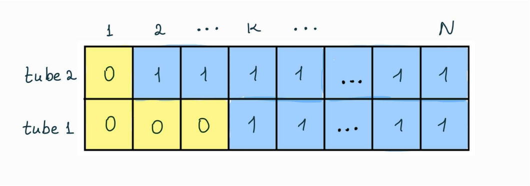
$$
\begin{aligned}
\frac{\partial f_1}{\partial \xi} &= g_1, \\
\frac{\partial g_1}{\partial \xi} &= \varepsilon_1^{-1}\big(-v_b + G(f_1, f_2)\big)g_1, \\
\frac{\partial f_2}{\partial \xi} &= g_2, \\
\frac{\partial g_2}{\partial \xi} &= \varepsilon_2^{-1}(-v_b + 2 - G(f_1, f_2) - (f_2 - f_1)G_2') \cdot g_2 - \varepsilon_2^{-1} G_1'(f_2 - f_1) \cdot g_1.
\end{aligned}
$$

Question: how to determine (δ_1, δ_2) for any v^b such that there exists a trajectory

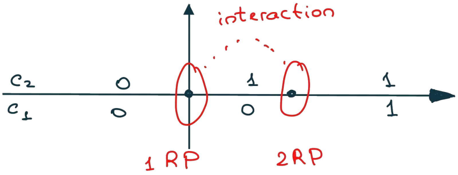
$$
\begin{array}{llll}
f_1(-\infty) = 0, & f_2(-\infty) = 0, & f_1(\infty) = \delta_1, & f_2(\infty) = \delta_2, \\
g_1(-\infty) = 0, & g_2(-\infty) = 0, & g_1(\infty) = 0, & g_2(\infty) = 0?
\end{array}
$$

Two coupled conservation laws: another idea

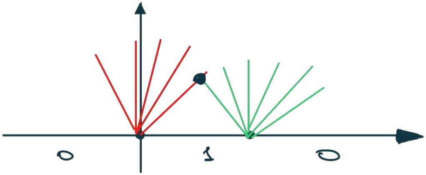
Consider two Riemann problems that interact with each other!



	1	2	...	k	...			N
tube 2	0	1	1	1	1	...	1	1
tube 1	0	0	0	1	1	...	1	1



(hahaha — welcome back to slugs — very similar)



Thank you for your attention!

Any comments? Ideas? Questions?

List of things to do (besides checking all arithmetic and $\pm$):

1. careful numerical modelling of the 2-dimensional TFE model and comparison to Peaceman model. Are they really close up to some time?
2. experiment with two-tube TFE numerical model: are there really two travelling waves? Analyze the pairs of intermediate concentrations (δ_1, δ_2) for different viscosity functions (viscosity ratios).
3. analysis of dynamical system of 4-th order for a travelling wave solution: determine functions $v^b \to (\delta_1, \delta_2)$ and $v^f \to (\delta_1, \delta_2)$
4. try approach with two interacting Riemann problems
5. ...

Any volunteer?