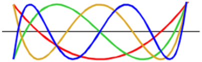
# Small ball probabilities for Gaussian processes



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*St Petersburg State University, Russia*

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Department of Mathematics

https://yulia-petrova.github.io/

Spring School "Multiplicative chaos and cascades"
TU Darmstadt, 19 February 2024

TECHNISCHE UNIVERSITÄT DARMSTADT

Surveys on the topic:

- W.V. Li and Q.Shao, 2001. Gaussian processes: inequalities, small ball probabilities and applications
- A. Nazarov and Y. Petrova, 2023. $L_2$-small ball asymptotics for Gaussian random functions: A survey
- Bibliography (by M.A. Lifshits): https://airtable.com/shrMG0nNxl9SiGxIl/tbl7Xj1mZW2VuYurm

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Actually, it can be formulated as a problem in measure theory. Let:

- $P$ — distribution of $X$ — a measure in $\mathcal{X}$, given by $P(A) = \mathbb{P}(X \in A)$
- $U := \{x \in \mathcal{X} : \ \|x\| \leqslant 1\}$ — unit ball in $\mathcal{X}$

We want to study the measure of the small balls:

$$P(\varepsilon U), \quad \text{as} \quad \varepsilon \to 0$$

# Gaussian random vectors

Gaussian random vector extends the notion of a normally distributed random variable.

**Definition**

*We call a random vector $X$, taking value in a linear topological space $\mathcal{X}$, Gaussian, if for every continuous linear functional $g \in \mathcal{X}^*$ the random variable $g(X)$ has a normal distribution.*

The distribution of a Gaussian vector is uniquely determined by:

- means of $\{g(X) : g \in \mathcal{X}^*\}$;
- covariances of $\{g(X) : g \in \mathcal{X}^*\}$.

Main example

Wiener process $W(t)$ — a random element in $C[0,1]$ or in $L^2[0,1]$:

- $\mathbb{E}W(t) \equiv 0$;
- $\operatorname{cov}(W(s), W(t)) = \min(s,t)$.

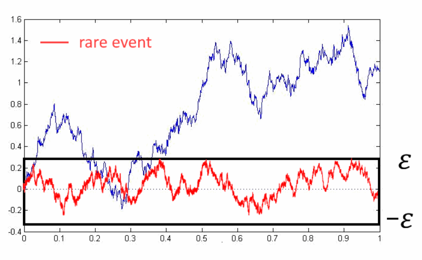
# Example

Typical answer:

$$\mathbb{P}\left(\|X\| < \varepsilon\right) \sim D \cdot \varepsilon^{C} \cdot \exp\left(-B\varepsilon^{-A}\right), \qquad \varepsilon \to 0$$

$A$, $B$ — *logarithmic* asymptotics; $A$, $B$, $C$, $D$ — *exact* asymptotics

Example: $\mathcal{X} = C[0,1]$, $X = W(t)$ — Wiener process




$$\mathbb{P}\left(\sup_{0\leqslant t\leqslant 1} |W(t)| < \varepsilon\right) \sim \frac{4}{\pi}\,\exp\left(-\frac{\pi^2}{8}\,\varepsilon^{-2}\right)$$

# Methods

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1. via metric entropy:
   - works for general classes of processes (e.g. in separable Banach spaces)
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   - J. Kuelbs, W. Li, W. Linde, T. Dunker, F. Gao, M. Lifshits, F. Aurzada, T. Kuhn, E. Belinsky, R. Blei, W. Salkeld etc

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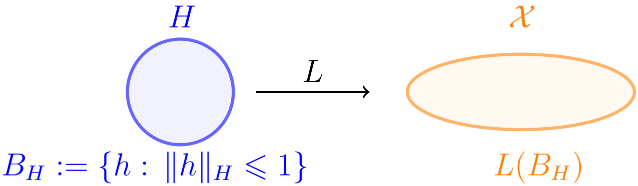

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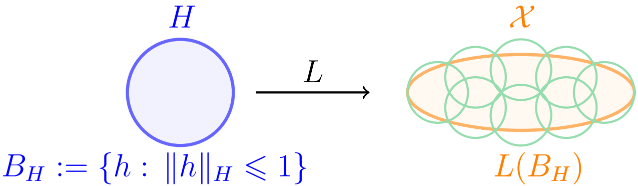


Metric entropy: $\ln N_L(\varepsilon)$, where $N_L(\varepsilon)$ — covering numbers:

$$N_L(\varepsilon) = \inf\left\{n :\ \exists\{x_j\}_{j\leqslant n}, \{Lh :\ \|h\|_H \leqslant 1\} \subset \cup_{j=1}^n B_\varepsilon(x_j)\right\}$$

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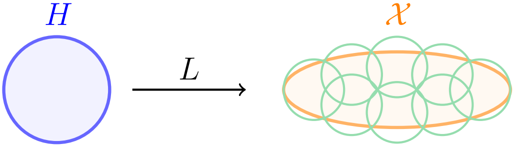


$B_H := \{h : \|h\|_H \leqslant 1\}$ $\quad$ $L(B_H)$

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$X$ — Gaussian vector in $\mathcal{X}$ $\quad\leftrightarrow\quad$ $L$ — associated linear compact operator

Example: $\ln \mathbb{P}(\|X\| < \varepsilon) \approx -\varepsilon^{-\frac{2\beta}{2-\beta}} \iff \ln N_L(\varepsilon) \approx \varepsilon^{-\beta}$, $\quad$ as $\varepsilon \to 0$

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   - St Petersburg school:
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## Karhunen-Loeve expansion (KL-expansion):

(K. Karhunen'1947, M. Loève'1948) Let $\mathcal{X} = L^2[0,1]$. Then

$$X(t) \stackrel{d}{=} \sum_{k\in\mathbb{N}} u_k(t)\,\sqrt{\mu_k}\,\xi_k$$

- $\xi_k$, $k \in \mathbb{N}$, — iid standard normal rv
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$$\mathbb{P}(\|X\|_2 < \varepsilon) = \mathbb{P}\left(\sum_{k=1}^{\infty} \mu_k \xi_k^2 < \varepsilon^2\right)$$

# Hilbert structure $\Longrightarrow$ spectral problem

**Stochastic processes**

$X(t)$, $t \in (0,1)$, —

- Gaussian process
- $\mathbb{E}X(t) \equiv 0$
- $G(s,t) = \mathbb{E}X(s)X(t)$.

**Small ball asymptotics**

$$\mathbb{P}\left(\|X\|_2 < \varepsilon\right) = \mathbb{P}\left(\sum \mu_k \xi_k^2 < \varepsilon^2\right)$$

**Spectral theory**

$\mathbb{G} : L_2[0,1] \to \mathrm{Im}(\mathbb{G})$

- integral operator of trace class
$$(\mathbb{G}u)(s) = \int_0^1 G(s,t)u(t)\,dt$$
- eigenvalues: $\sum \mu_k < \infty$

**Asymptotics of eigenvalues $\mu_k$**

Find “good” approximation to $\mu_k$

Comments:

- the whole sequence of eigenvalues $\mu_k$ is important (in contrast to large deviations where only the first eigenvalue is sufficient to know)
- If $\mathbb{G}^{-1}$ is a differential operator, then spectral theory for ODEs helps — my research interests

# Example of a general theorem (Nazarov, Petrova' 2016)

Let the eigenvalues $\mu_k$ have the asymptotics

$$\mu_k = (\vartheta(k + \delta + F(k) + O(k^{-1})))^{-d},$$

where $F(k)$ is a slowly varying function (SVF) at $\infty$.
Then for the small deviation probabilities

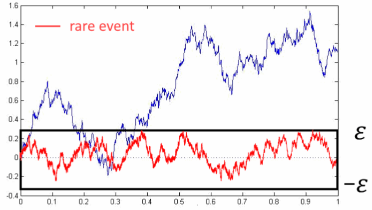
$$\mathbb{P}(\|X\|_2 < \varepsilon) \sim D \cdot \exp\left(\frac{1}{2}F_{-1}(\varepsilon^{-2})\right) \cdot \varepsilon^{C} \exp(B\varepsilon^{A}), \qquad \varepsilon \to 0,$$

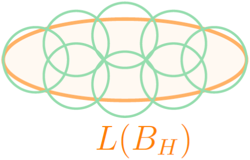
where $A = A(d)$, $B = B(d, \vartheta)$, $C = C(d, \vartheta, \delta)$, $D = D(\{\mu_k\})$:

$$A = -\frac{2}{d-1}, \qquad B = -\frac{d-1}{2}\left(\frac{\pi/d}{\vartheta \sin(\pi/d)}\right)^{\frac{d}{d-1}},$$

$$C = \frac{2-d-2\delta d}{2(d-1)}, \qquad F_{-1}(t) = \int_1^t \frac{F(x)}{x}dx \quad \text{also SVF}$$

# Vielen Dank für eure Aufmerksamkeit!




Questions? Comments?

Contact: https://yulia-petrova.github.io/
yu.pe.petrova@gmail.com

# Literature

Surveys / books :


- Site with all bibliography around small ball probabilities (collected by M. Lifshits): https://airtable.com/shrMG0nNxl9SiGxII/tbl7Xj1mZW2VuYurm
- Li, W.V. and Shao, Q.M., 2001. Gaussian processes: inequalities, small ball probabilities and applications. Handbook of Statistics, 19, pp.533-597.
- Lifshits, M., 2012. Lectures on Gaussian processes. Springer Berlin Heidelberg.
- Nazarov, A. and Petrova, Y., 2023. $L_2$-small ball asymptotics for Gaussian random functions: A survey. Probability Surveys, 20, pp.608-663.