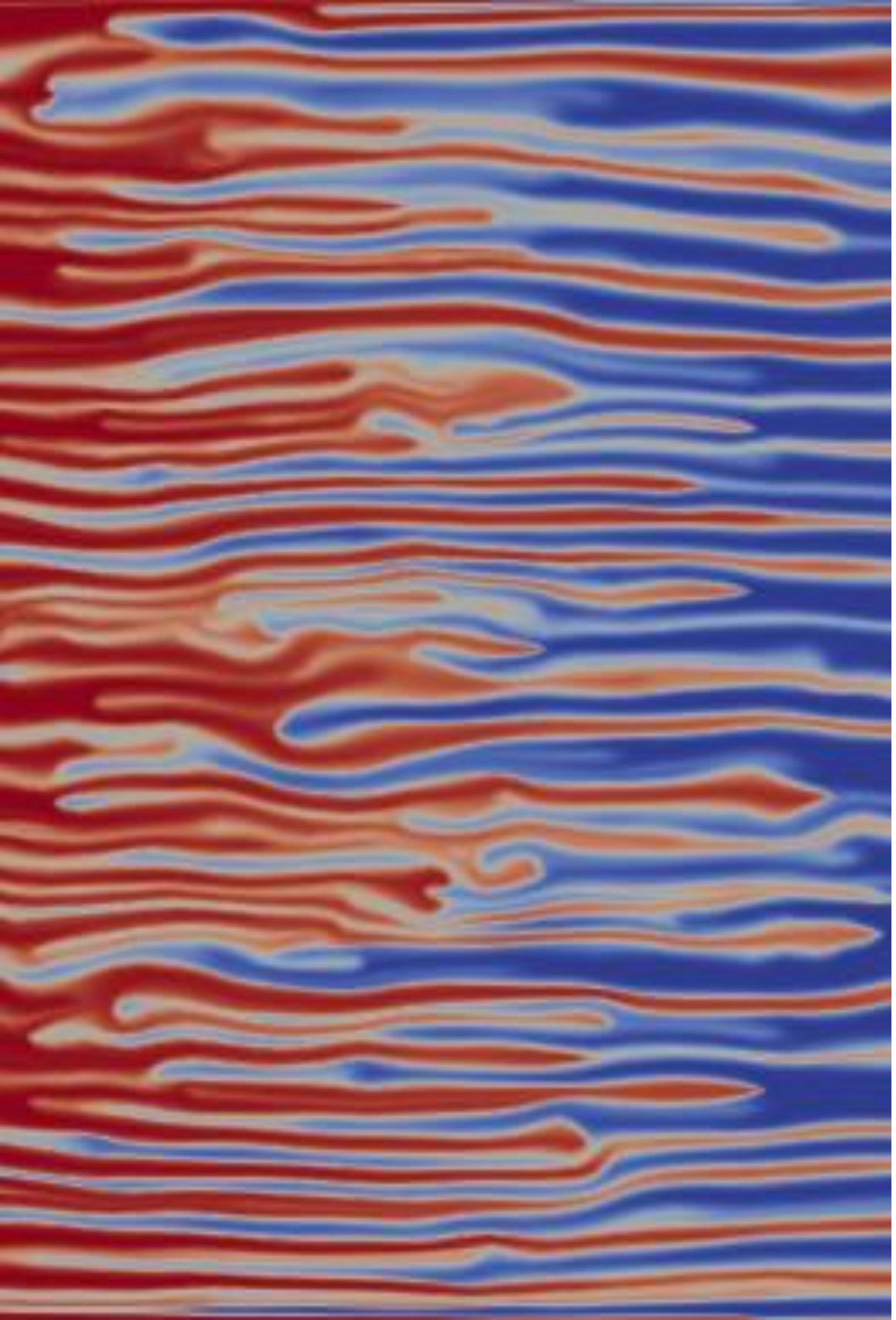

Travelling waves: dynamical perspective

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22 September 2023

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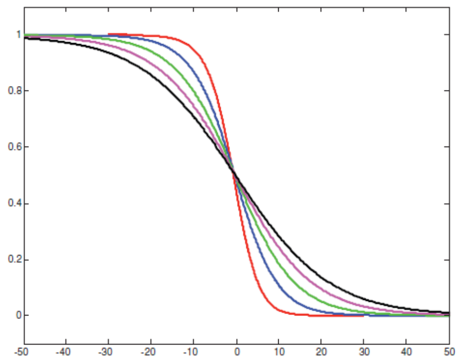
Based on work in progress with:

- Sergey Tikhomirov (PUC-Rio)
- Yalchin Efendiev (Texas A&M)

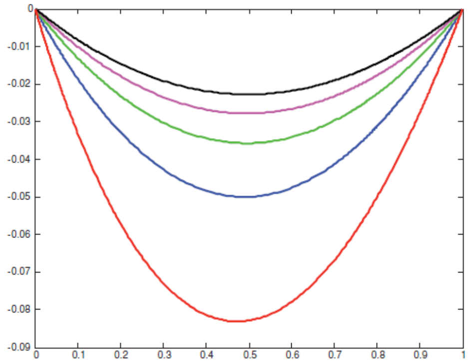
AIM: show how dynamical systems can help PDEs

1. Travelling waves in PDEs $\leftrightarrow$ Heteroclinic orbits in Dynamical Systems

Population dynamics
(spreading of animals)

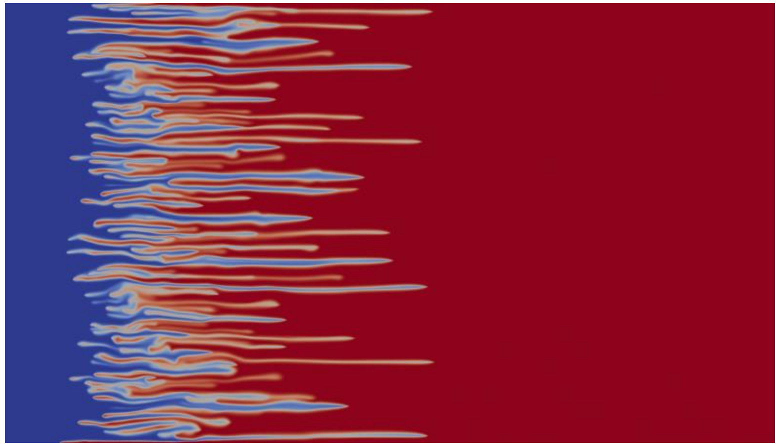


$\leftrightarrow$



Miscible displacement in porous media

2. Viscous / gravitational fingering: ``toy '' model



- Formulation (PDEs)  : a vida está russa (=complicada)
- Proof (…tem o gosto dinâmico!!!)

Work in progress with Sergey Tikhomirov (PUC-Rio) and Yalchin Efendiev (Texas A&M)

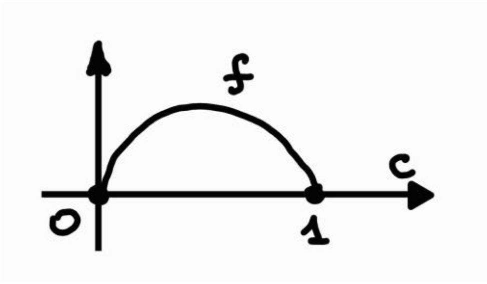
Example 1: population dynamics

- spreading of animals
- $c(x,t) \in [0,1]$ – density of population of mosquitoes
- propagation due to reproduction and diffusion

Aedes aegypti
(yellow fever mosquito)

Reaction-diffusion equation:
$$c_t = \Delta c + f(c)$$

1. Reproduction:
$$c_t = f(c)$$



2. Diffusion:
$$c_t = \Delta c$$

Aim: understand the behaviour of $c(x,t)$ as $t \to \infty$

Fisher-KPP equation (for $x \in \mathbb{R}$)

"Invasion" occurs!

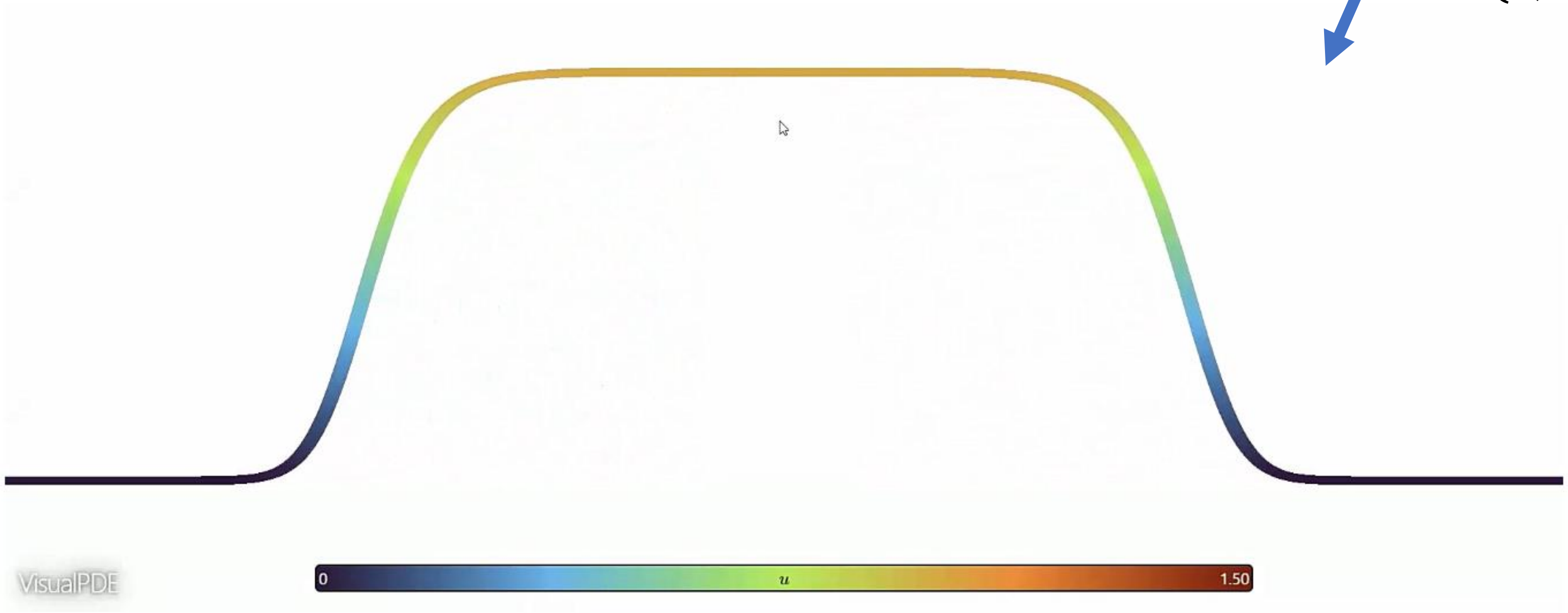
$$c_t = c_{xx} + c(1 - c)$$

Question: how to find a speed v of propagation?

Travelling Wave Solution

$$c(t,x) = c(x - vt)$$

$$c(-\infty) = 1$$
$$c(+\infty) = 0$$



Credit from https://visualpde.com/
It is a fun - enjoy!

1. Fisher, R.A., 1937. The wave of advance of advantageous genes. Annals of eugenics, 7(4), pp.355-369.
2. A.N. Kolmogorov, I.G. Petrovskii and N.S. Piskunov, Etude de l'équation de la diffusion avec croissance de la quantité de matière et son application à un problème biologique, Bulletin Universite d'Etat a Moscou, Serie Internationale, section A 1, 1937, 1–26.

Travelling wave solution $\leftrightarrow$ Heteroclinic orbit

1-dim Fisher KPP equation:

$$c_t = c_{xx} + c(1-c)$$

Travelling Wave Solution

$$c(t,x) = c(x - vt)$$

with
$$c(-\infty) = 1$$
$$c(+\infty) = 0$$

$$-vc' = c'' + c(1-c)$$

$$\begin{cases} c' = w \\ w' = -vw - c(1-c) \end{cases}$$

$$\left.\begin{aligned} (c,w)(-\infty) &= (1{,}0) \\ (c,w)(+\infty) &= (0{,}0) \end{aligned}\right\} \text{Fixed points}$$

Take-home message 1:

FINDING TRAVELLING WAVE SOLUTIONS FOR PDE $\longleftrightarrow$ (sometimes) FINDING HETEROCLINIC (HOMO-) ORBITS IN DYNAMICAL SYSTEM

… looking for heteroclinic orbits …

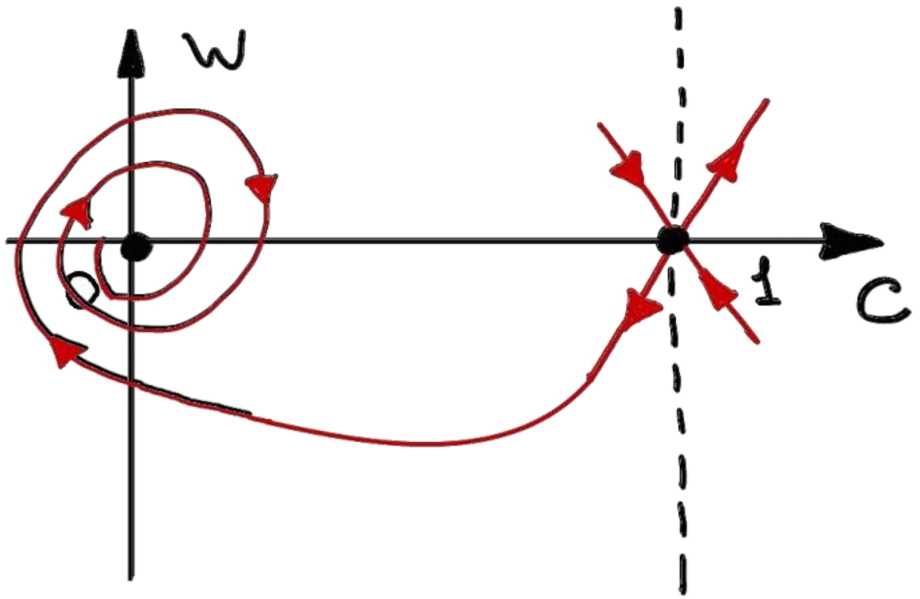
$c \in [0,1]$ – population density

$v \in \mathbb{R}$ – speed

$$\begin{cases} c' = w \\ w' = -vw - c(1-c) \end{cases}$$

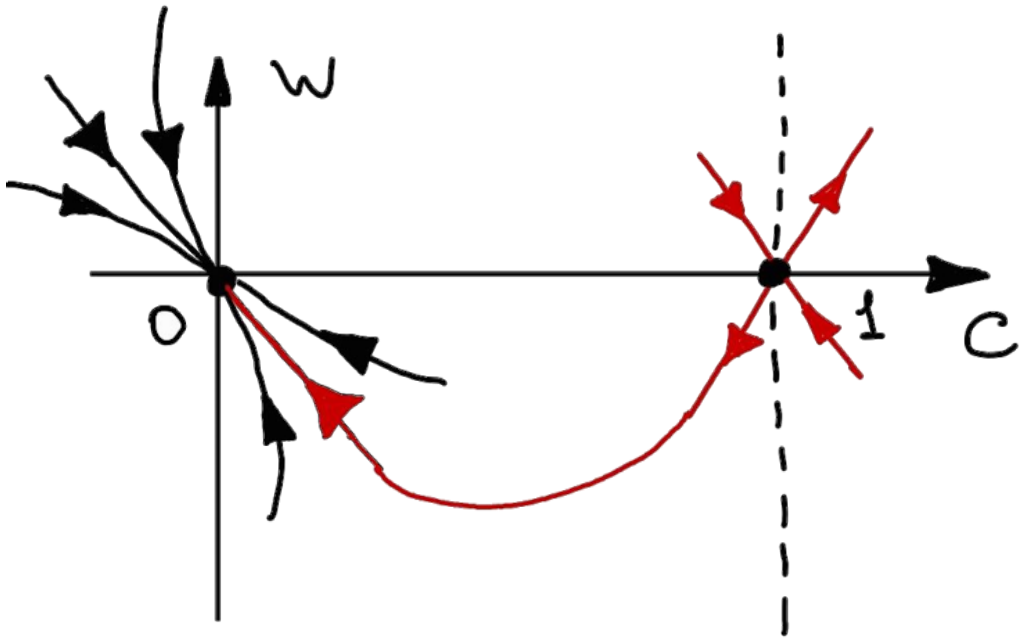
$(c,w)(-\infty) = (1,0)$ – saddle point

$(c,w)(+\infty) = (0,0)$



$v \in (0,2)$

No orbit with restriction $c \in [0,1]$



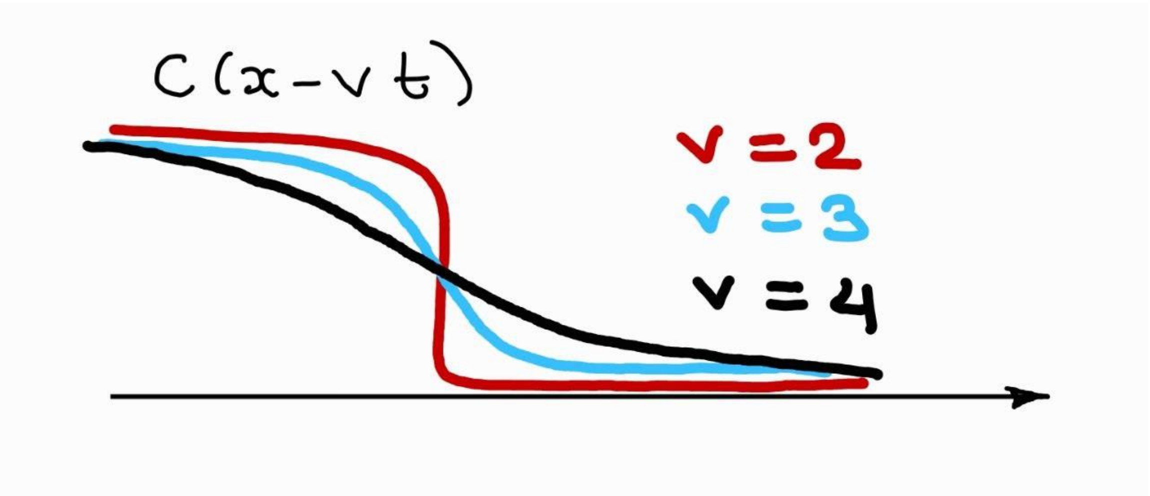
$v \in [2,+\infty)$

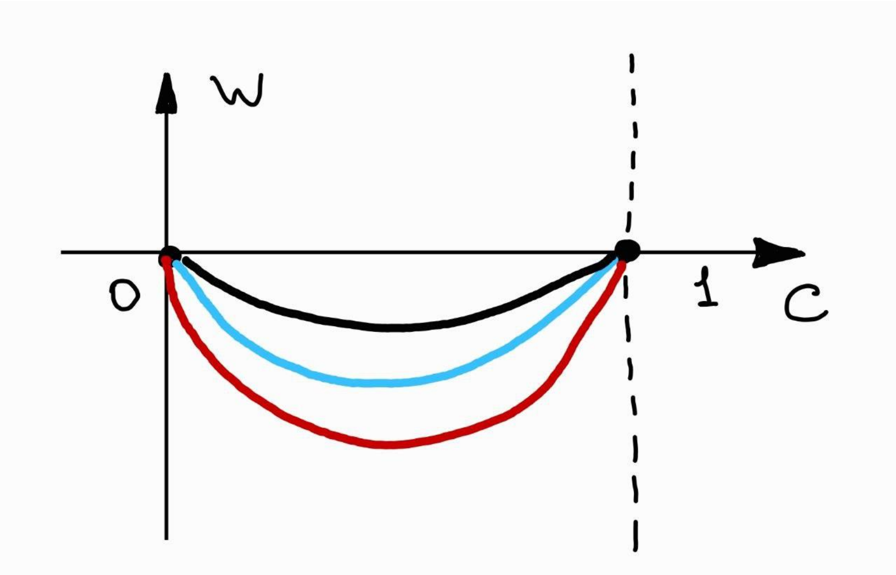
For any $v \in [2,+\infty)$ there exists an orbit

Travelling wave solution (TW) $\leftrightarrow$ Heteroclinic orbit

- There exists a family of TW parametrized by speed $v \in [2, \infty)$

Questions?





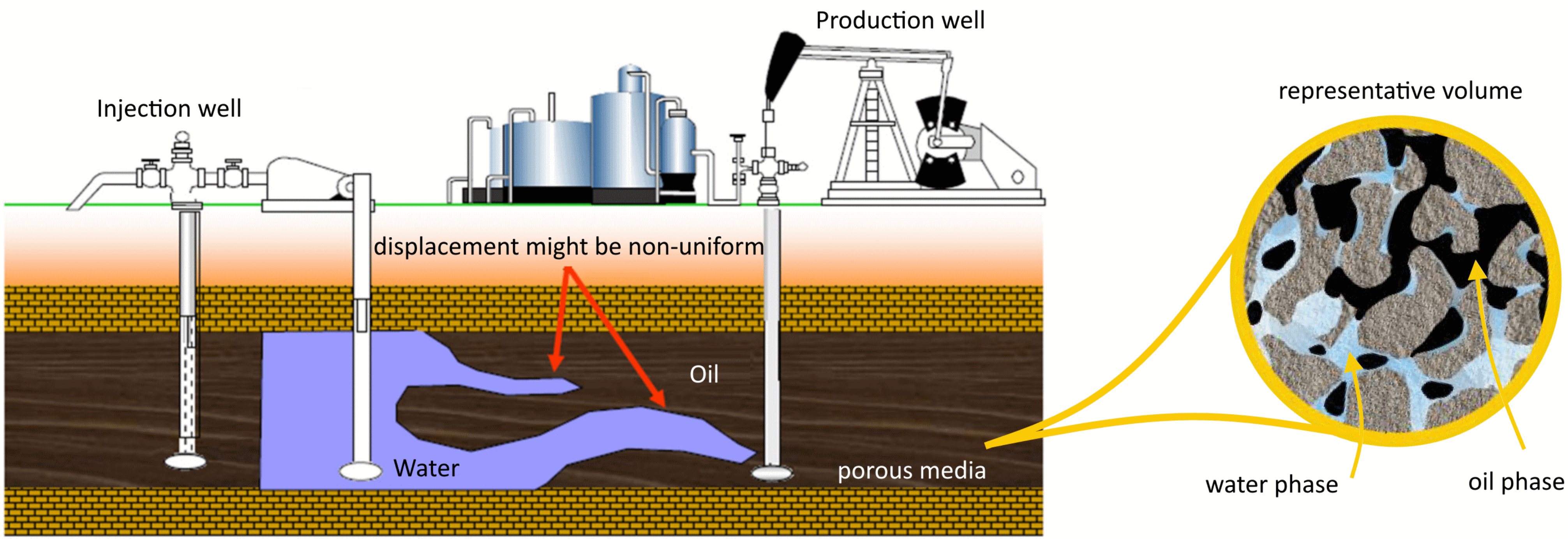
- If initial data has compact support, then the limiting TW has speed $v = 2$ (the minimal speed)
  Proof of convergence of solution to the TW is a PDE stuff…

Take-home message 1:

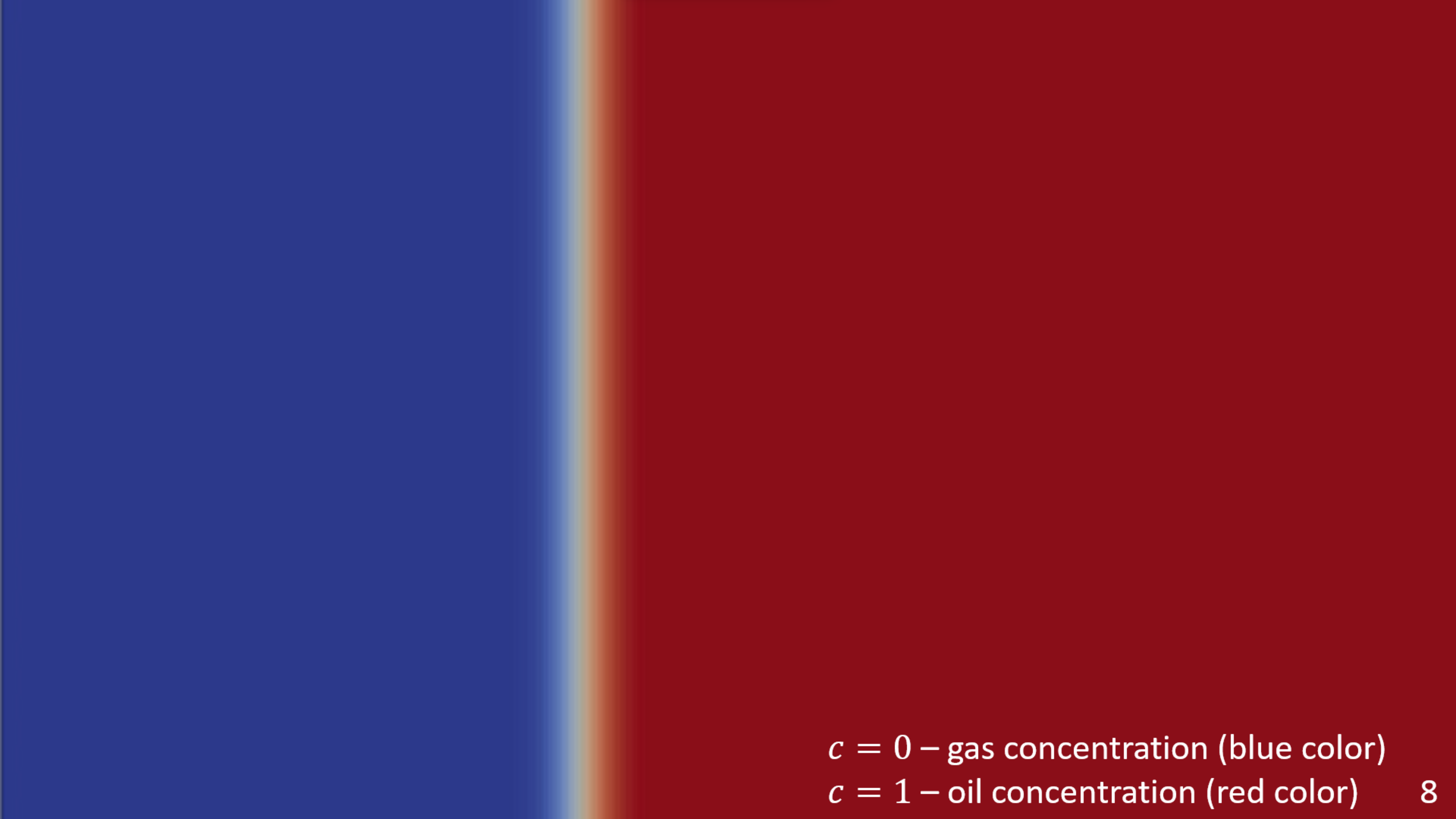
FINDING TRAVELLING WAVE SOLUTIONS FOR PDES $\Leftrightarrow$ FINDING HOMO-/HETEROCLINIC ORBITS IN DYNAMICAL SYSTEMS

Example 2: motivation from oil recovery

We are interested in the mathematical model of oil recovery

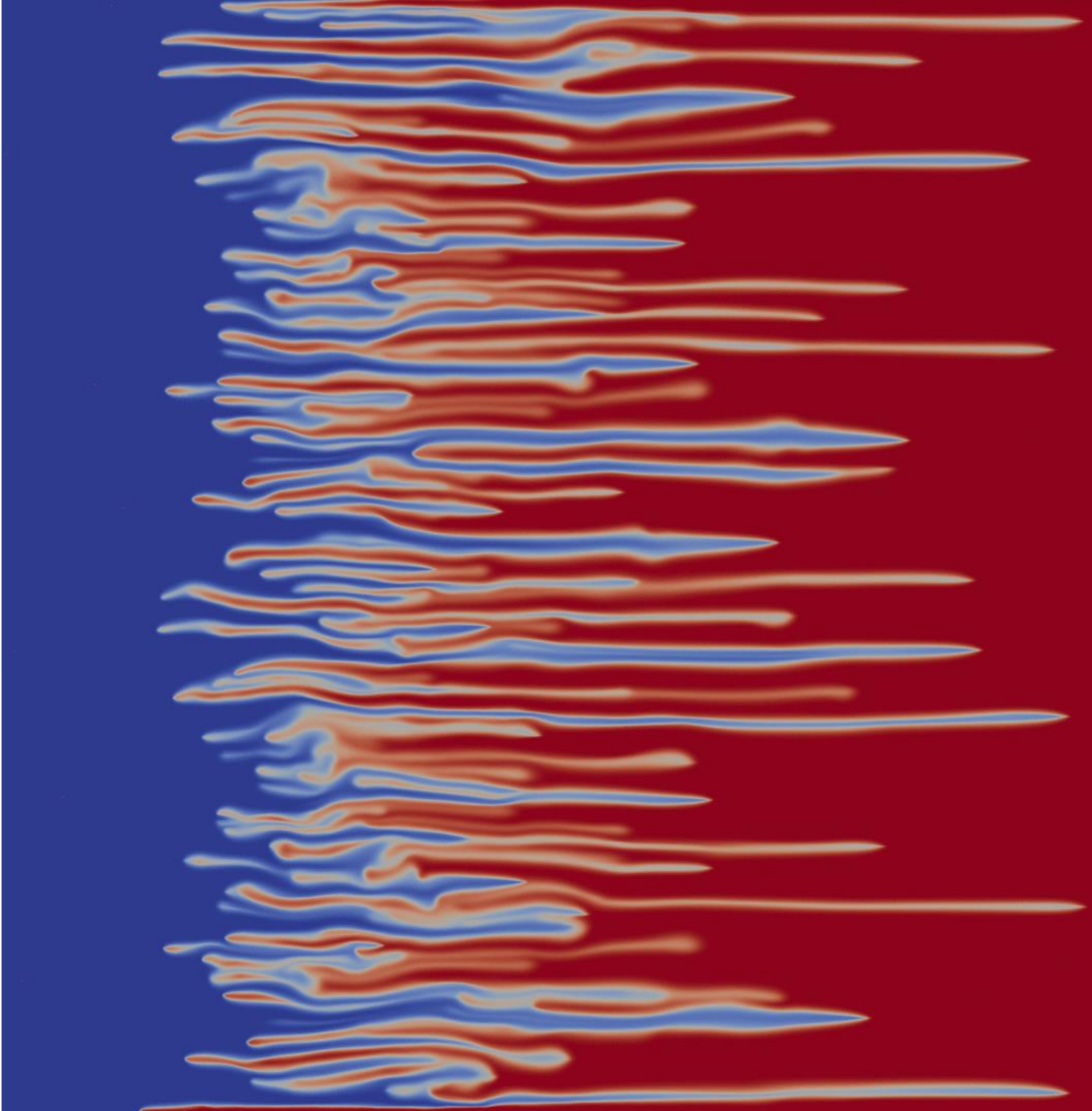


- Porous media (averaged models of flow)
- Unknown variables: $c(x,t)$ – the averaged oil concentration in representative volume
  $(1-c)$ – the average water (gas) concentration in small volume
- Relatively small speeds (0.3 meters per day): Navier-Stokes → Darcy's law



$c = 0$ – gas concentration (blue color)

$c = 1$ – oil concentration (red color)



“Miscible displacement in porous media”
Credit: Pavlov Dmitrii, St. Petersburg State University

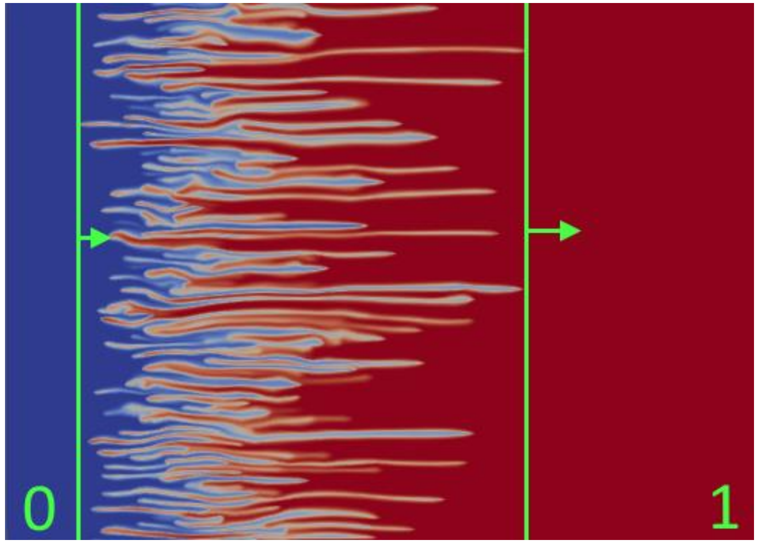
Viscous fingering phenomenon

$c = 0$ – gas concentration (blue color)

$c = 1$ – oil concentration (red color)

Linear growth of the mixing zone

- many laboratory and numerical experiments show *linear growth of the mixing zone* *

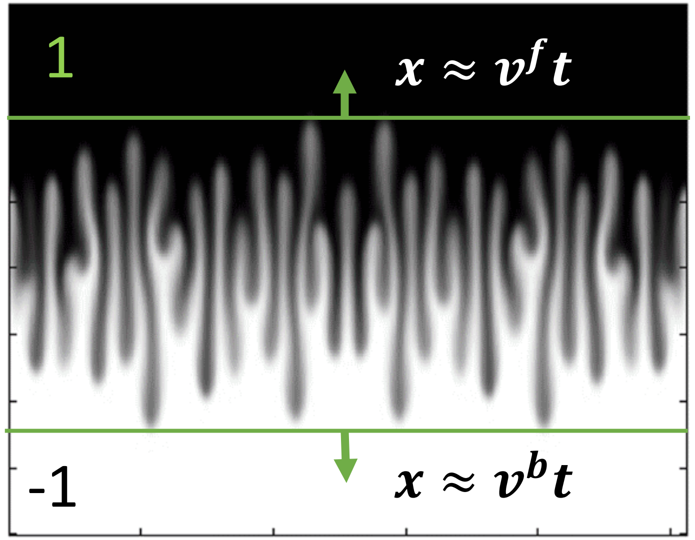


PDEs ("black box"):

$$c_t + div(uc) = \Delta c$$
$$div(u) = 0$$
$$u = -\nabla p - (0, c)$$

- c – concentration
- u – velocity
- p – pressure

Gravitational fingering



Question: how to find speeds v^b and v^f of propagation?

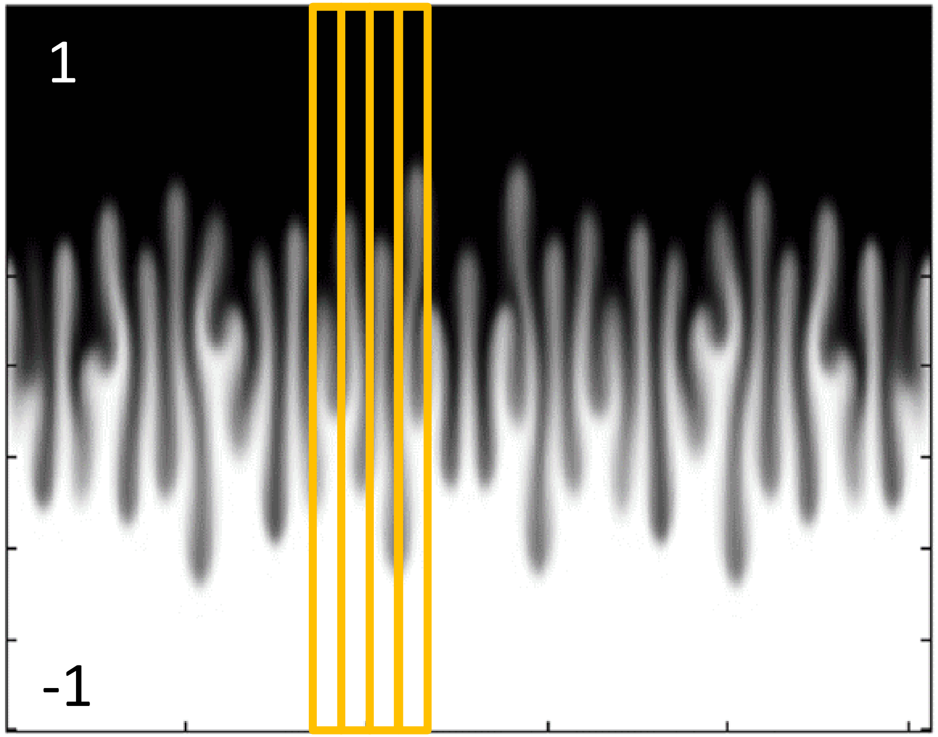
Open problem…

* *Nijjer J., Hewitt D., and Neufeld J. The dynamics of miscible viscous fingering from onset to shutdown. Journal of Fluid Mechanics 837 (2018): 520-545.*

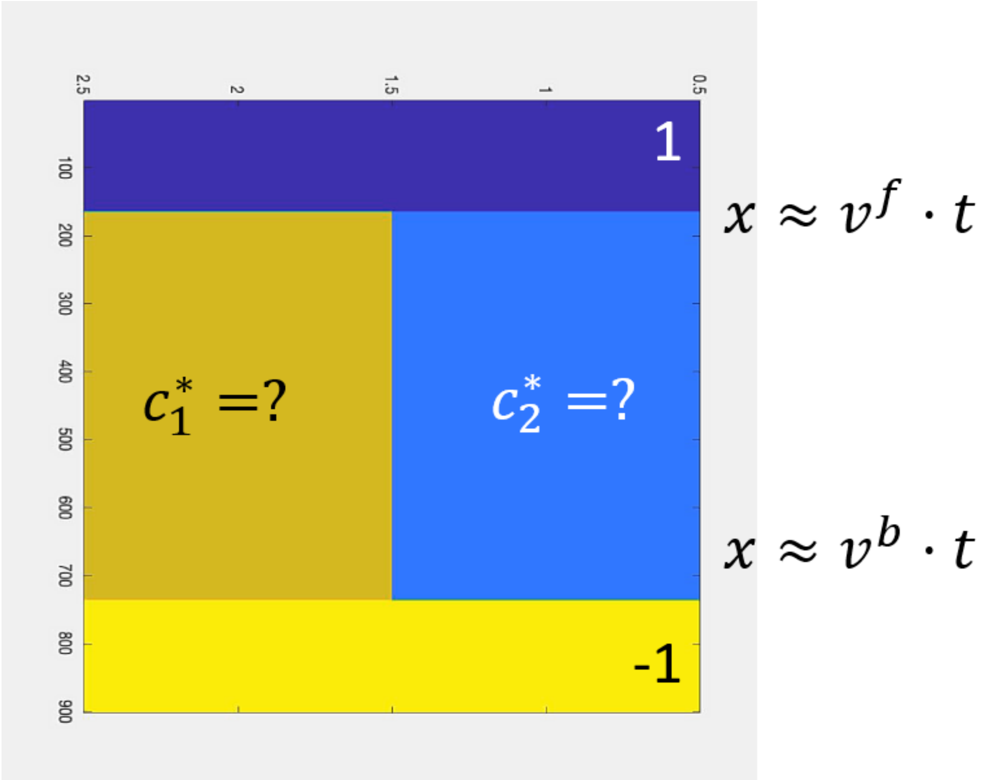
* *Bakharev, F., Enin, A., Groman, A., Kalyuzhnyuk, A., Matveenko, S., **Petrova, Y.**, Starkov, I. and **Tikhomirov, S.**, 2022. Velocity of viscous fingers in miscible displacement: Comparison with analytical models. Journal of Computational and Applied Mathematics, 402, p.113808.*

“Toy” model of gravitational fingering

- Discretize in horizontal direction
- Take n tubes, $n = 2,3,4, \ldots$



- For simplicity, $n = 2$
- What does numerical simulation tell us?



As we can observe, there are travelling wave solutions!

Can we rigorously prove their existence?

First, we need to formulate a mathematical model

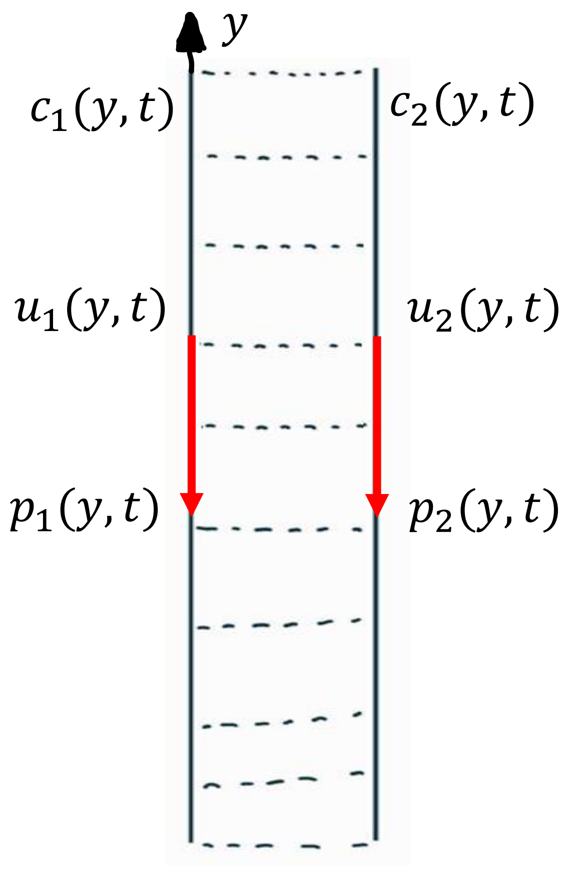
Two-tubes model (with gravity)

1. Original equation on c:

$$c_t + div(uc) - \Delta c = 0$$

Two-tubes equations on c:

$$\partial_t c_1 + \partial_y(u_1 c_1) - \partial_{yy} c_1 = 0$$
$$\partial_t c_2 + \partial_y(u_2 c_2) - \partial_{yy} c_2 = 0$$

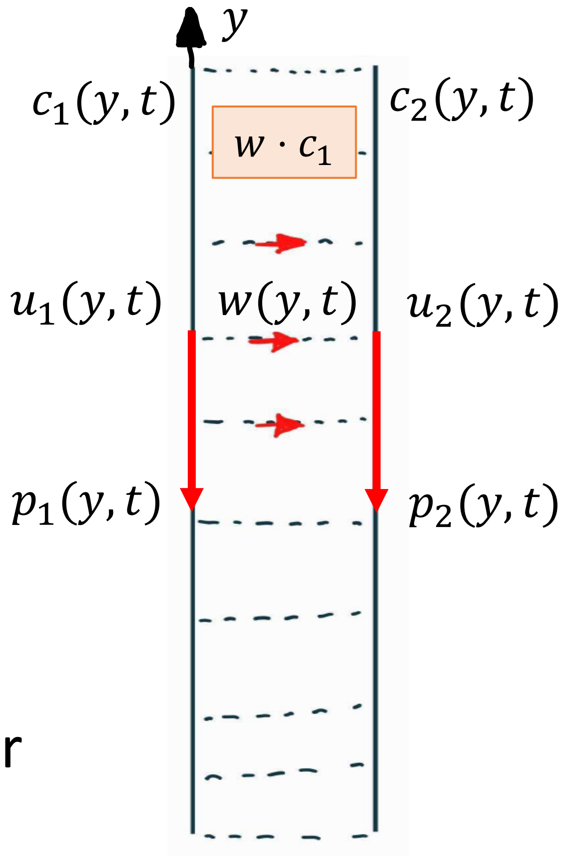


Two-tubes model (with gravity)

1. Original equation on c: $\boxed{c_t + div(uc) - \Delta c = 0}$

   Two-tubes equations on c:

$$\partial_t c_1 + \partial_y(u_1 c_1) - \partial_{yy} c_1 = \boxed{-w \cdot c_1}$$
$$\partial_t c_2 + \partial_y(u_2 c_2) - \partial_{yy} c_2 = \boxed{+w \cdot c_1}$$



2. Original equation on p: $\boxed{u = -\nabla p - (0, c)}$

   Two-tubes equations on p:

$$u_1 = -\partial_y p_1 - c_1$$
$$u_2 = -\partial_y p_2 - c_2$$
$$w = \frac{p_2 - p_1}{l}$$

l - parameter

3. Original equation on u: $\boxed{div(u) = 0}$

   Two-tubes equations on u:

$$w = \partial_y u_1$$

Initial condition:

$$c_{1,2}(y,0) = -1, y < 0$$
$$c_{1,2}(y,0) = +1, y > 0$$

Two-tubes model (with gravity)

1. Original equation on c: $\boxed{c_t + div(uc) - \Delta c = 0}$

   Two-tubes equations on c:

$$\partial_t c_1 + \partial_y(u_1 c_1) - \partial_{yy} c_1 = \boxed{-B}$$
$$\partial_t c_2 + \partial_y(u_2 c_2) - \partial_{yy} c_2 = \boxed{+B}$$

2. Original equation on p: $\boxed{u = -\nabla p - (0, c)}$

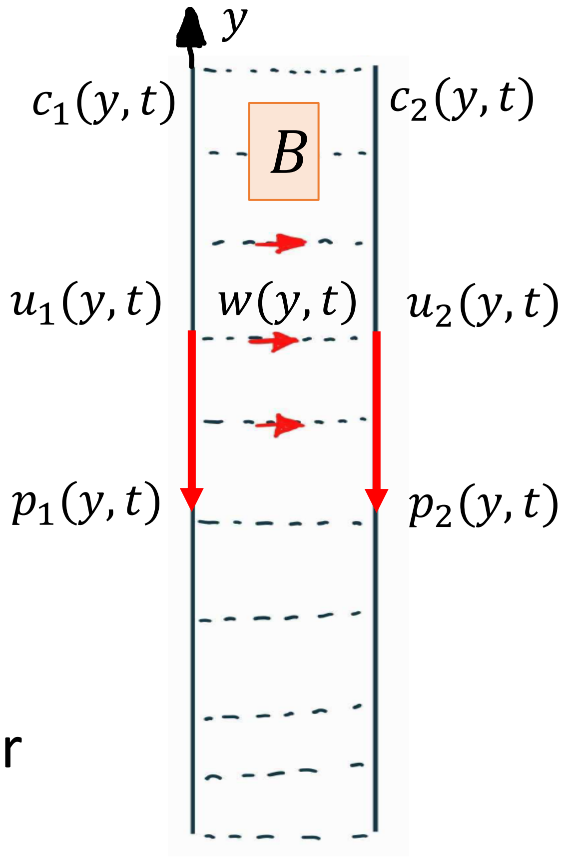
   Two-tubes equations on p:

$$u_1 = -\partial_y p_1 - c_1$$
$$u_2 = -\partial_y p_2 - c_2$$
$$w = \frac{p_2 - p_1}{l}$$

3. Original equation on u: $\boxed{div(u) = 0}$

   Two-tubes equations on u:

$$w = \partial_y u_1$$



l - parameter

$$\boxed{B} = \begin{cases} -w \cdot c_1, & w < 0, \\ +w \cdot c_2, & w > 0 \end{cases}$$

Main result

A prova tem o "gosto dinâmico"!!

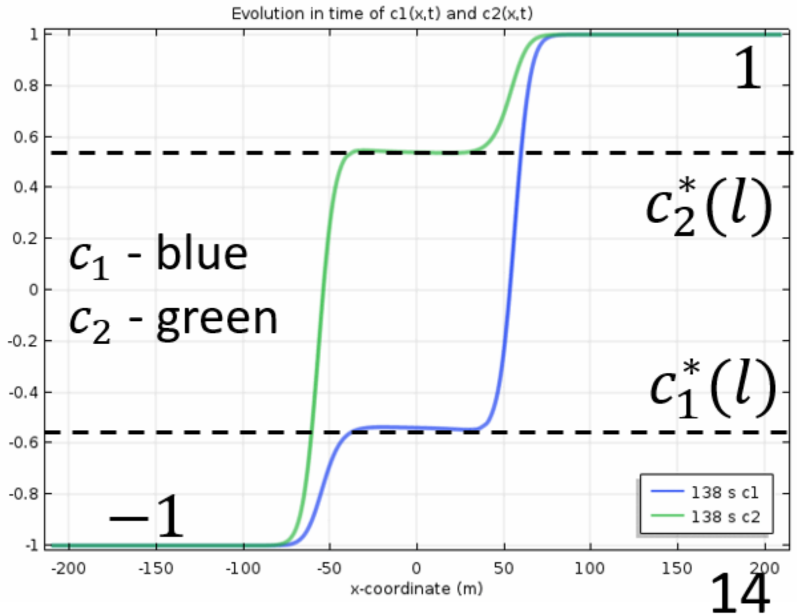
$$
(*)\begin{cases}
\partial_t c_1 + \partial_y(u_1 c_1) - \partial_{yy} c_1 = -B \\
\partial_t c_2 + \partial_y(u_2 c_2) - \partial_{yy} c_2 = B \\
u_1 = -\partial_y p_1 - c_1 \\
u_2 = -\partial_y p_2 - c_2 \\
\partial_y u_1 = -\partial_y u_2 = \dfrac{p_2 - p_1}{l}
\end{cases}
$$

$$
B = \begin{cases}
-\partial_y u_1 \cdot c_1, & \partial_y u_1 < 0, \\
+\partial_y u_2 \cdot c_2, & \partial_y u_1 > 0
\end{cases}
$$

Remark: $\lim_{l\to 0} c_1^*(l) = -0.5$ $\quad \lim_{l\to 0} v^b(l) = -0.25$

$\lim_{l\to 0} c_2^*(l) = +0.5$ $\quad \lim_{l\to 0} v^f(l) = +0.25$

As $t \to \infty$ we observe:



Theorem (Efendiev, P., Tikhomirov, 2023+)

Consider a two-tube model with gravity $(*)$.

Then for all *$l > 0$ sufficiently small* there exists $c_1^*(l), c_2^*(l)$ such that there exist two travelling waves (TW):

TW1 with speed $v^b(l)$: $\quad (-1,-1) \quad \to \quad (c_1^*,(l)\ c_2^*(l))$

TW2 with speed $v^f(l)$: $\quad (c_1^*,(l)\ c_2^*(l)) \quad \to \quad (1,1).$

… a vida está russa (…ou ruça?) …

Travelling wave ansatz with fixed v:

$$c_1(t,y) = c_1(y - vt)$$
$$c_2(t,y) = c_2(y - vt)$$

$$u_1(t,y) = u_1(y - vt)$$
$$u_2(t,y) = u_2\,(y - vt)$$

$$p_1(t,y) = p_1(y - vt)$$
$$p_2(t,y) = p_2(y - ct)$$

With condition at $+\infty$:

$$c_1(+\infty) = 1$$
$$c_2(+\infty) = 1$$
$$u_1(+\infty) = 0$$
$$u_2(+\infty) = 0$$
$$(p_1 - p_2)(+\infty) = 0$$

System of ODEs in $\mathbb{R}^6$:

$$\boxed{\begin{cases} \dot{X} = F_v(X,Y) \\ \varepsilon \cdot \dot{Y} = AY - BX \end{cases}}$$

Here:

- $X = \begin{pmatrix} c_1 \\ c_2 \\ \partial_\xi c_1 \\ \partial_\xi c_2 \end{pmatrix} \in \mathbb{R}^4, \quad Y = \begin{pmatrix} u_1 \\ p_1 - p_2 \end{pmatrix} \in \mathbb{R}^2$
- $A = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}, \qquad B \in M^{2\times 4}, \qquad \varepsilon = \sqrt{l} \ll 1$

Aim:

find a heteroclinic orbit $\big(X(\xi), Y(\xi)\big), \xi \in \mathbb{R}$ such that $\big(X(+\infty), Y(+\infty)\big) =$ given point.

… a vida está russa (…ou ruça?) …

Travelling wave ansatz with fixed v:

$$c_1(t,y) = c_1(y - vt)$$
$$c_2(t,y) = c_2(y - vt)$$

$$u_1(t,y) = u_1(y - vt)$$
$$u_2(t,y) = u_2\,(y - vt)$$

$$p_1(t,y) = p_1(y - vt)$$
$$p_2(t,y) = p_2(y - ct)$$

With condition at $+\infty$:

$$c_1(+\infty) = 1$$
$$c_2(+\infty) = 1$$
$$u_1(+\infty) = 0$$
$$u_2(+\infty) = 0$$
$$(p_1 - p_2)(+\infty) = 0$$

System of ODEs in $\mathbb{R}^6$:

$$\boxed{\begin{cases} \dot{X} = F_v(X,Y) \\ \varepsilon \cdot \dot{Y} = AY - BX \end{cases}}$$

Here:

- $X = \begin{pmatrix} c_1 \\ c_2 \\ \partial_\xi c_1 \\ \partial_\xi c_2 \end{pmatrix} \in \mathbb{R}^4, \quad Y = \begin{pmatrix} u_1 \\ p_1 - p_2 \end{pmatrix} \in \mathbb{R}^2$
- $A = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}, \qquad B \in M^{2\times 4}, \qquad \varepsilon = \sqrt{l} \ll 1$

Observation:

for $\varepsilon \to 0$ this system has a special ``slow-fast'' structure. Let me say several words on what is called geometric singular perturbation theory (GSPT)

Simple example

Slow system

$$\begin{cases} \dot{x} &= -x \\ \varepsilon \cdot \dot{y} &= x^2 - y \end{cases}$$

$$t = \varepsilon \cdot s$$

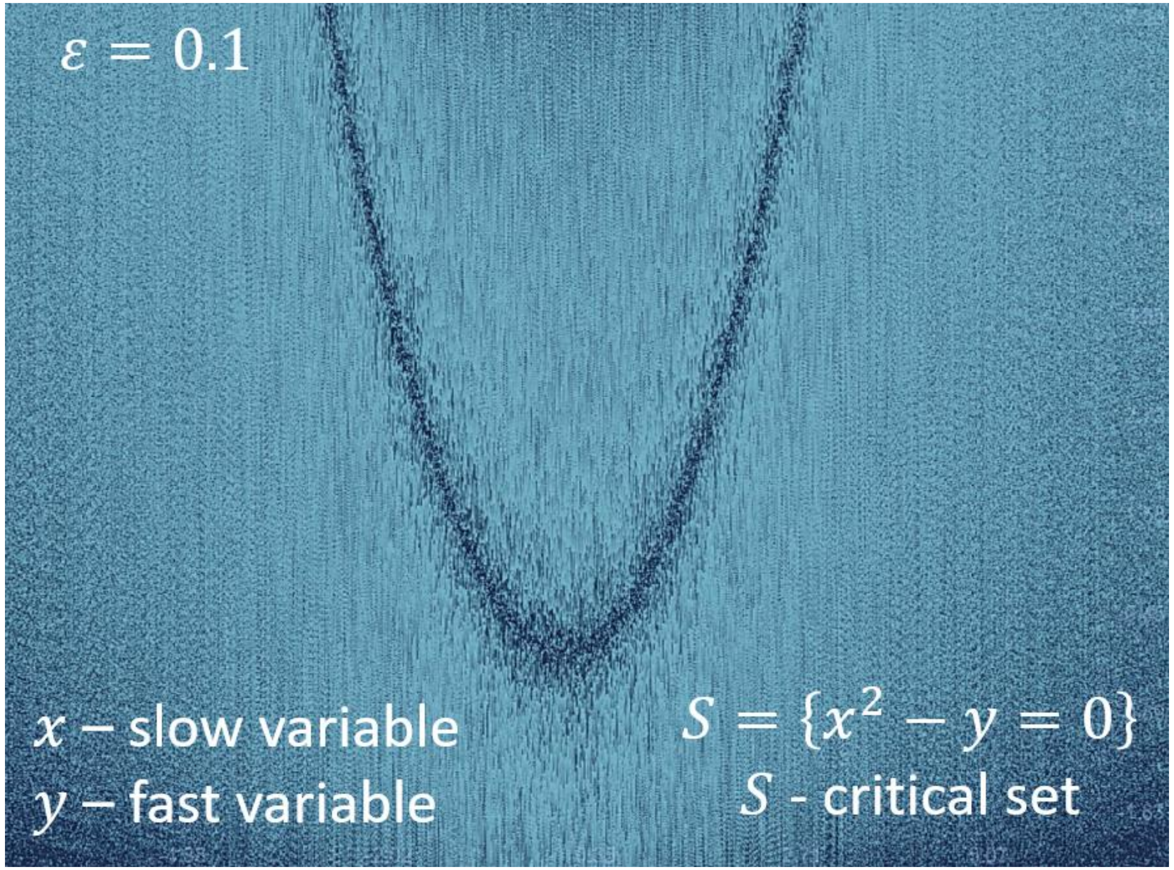
$$0 < \varepsilon \ll 1$$

Fast system

$$\begin{cases} x' &= \varepsilon \cdot (-x) \\ y' &= \quad x^2 - y \end{cases}$$

Formally

$\varepsilon \to 0$



Formally

$\varepsilon \to 0$

Reduced slow system

$$\begin{cases} \dot{x} &= -x \\ 0 &= x^2 - y \end{cases}$$

Reduced fast system

$$\begin{cases} x' &= 0 \\ y' &= x^2 - y \end{cases}$$

Geometric singular perturbation theory (GSPT)

Slow system (t – slow time)

$$\begin{cases} \dot{X} = F(X,Y,\varepsilon) \\ \varepsilon \cdot \dot{Y} = G(X,Y,\varepsilon) \end{cases}$$

$$t = \varepsilon \cdot s$$

$$0 < \varepsilon \ll 1$$

Fast system (s – fast time)

$$\begin{cases} X' = \varepsilon \cdot F(X,Y,\varepsilon) \\ Y' = G(X,Y,\varepsilon) \end{cases}$$

Formally $\varepsilon \to 0$

Reduced slow system

$$\begin{cases} \dot{X} = F(X,Y,0) \\ 0 = G(X,Y,0) \end{cases}$$

Formally $\varepsilon \to 0$

Reduced fast system

$$\begin{cases} X' = 0 \\ Y' = G(X,Y,0) \end{cases}$$

$S = \{G(X,Y,0) = 0\}$ – critical set

empty or consists of isolated points
(regular perturbation problem)

contains a differentiable manifold
(singular perturbation problem)

Normally hyperbolic manifolds ("fast-slow" version)

$$\begin{cases} X' &= \varepsilon \cdot F(X,Y,\varepsilon) \\ Y' &= G(X,Y,\varepsilon) \end{cases}$$

$(X,Y) \in \mathbb{R}^m \times \mathbb{R}^n$, $F(X,Y,\varepsilon), G(X,Y,\varepsilon)$ – smooth

$S = \{(X,Y) \in \mathbb{R}^{m+n}: \ G(X,Y,0) = 0\}$ – critical manifold

Definition: A smooth compact manifold $S_0 \subset S$ is called *normally hyperbolic* if the $n \times n$ matrix $DG_Y(X,Y,0)$ is hyperbolic for all $(X,Y) \in S_0$.

In particular, S_0 is called:

- attracting, if all eigenvalues of $DG_y(X,Y,0)$ have negative real part
- repelling, if all eigenvalues of $DG_y(X,Y,0)$ have positive real part
- of saddle-type, if it is neither attracting nor repelling

Take-home message 2:

Normal hyperbolicity of critical manifold $\Rightarrow$ ``nice'' perturbation

Fenichel, N., 1979. Geometric singular perturbation theory for ordinary differential equations. Journal of differential equations, 31(1), pp.53-98.

Fenichel’s theorem (``fast-slow’’ version)

Let S_0 be a compact normally hyperbolic submanifold (possibly with boundary) of the critical manifold S of the system

$$\begin{cases} X' = \varepsilon \cdot F\,(X,Y,\varepsilon) \\ Y' = G(X,Y,\varepsilon) \end{cases}$$

and that $F, G \in C^r$ $(r < \infty)$.

Then for $\varepsilon > 0$ sufficiently small, the following hold:

(F1) There exists a locally invariant manifold S_ε diffeomorphic to S_0.

(F2) S_ε has Hausdorff distance $O(\varepsilon)$ from S_0 (as $\varepsilon \to 0$).

(F3) The flow on S_ε converges to the flow of the reduced slow system (as $\varepsilon \to 0$).

(F4) S_ε is C^r-smooth and normally hyperbolic

Remark: S_ε may be not unique

Local invariance means that trajectories can enter or leave S_ε only through its boundaries.

Travelling wave dynamical system: GSPT

$$\begin{cases} \dot{X} = F_v(X,Y) \\ \varepsilon \cdot \dot{Y} = AY - BX \end{cases}$$

- $X \in \mathbb{R}^4, Y \in \mathbb{R}^2$

We have:

- Critical manifold:

$$S = \{(X,Y): \quad Y = A^{-1}BX\}, \qquad \dim S = 4$$

- $K \subset S$ (compact) is normally hyperbolic as the matrix

$$A = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix} \quad \text{has eigenvalues } \lambda_\pm = \pm\sqrt{2}$$

Thus, by Fenichel's theorem:

- For any compact submanifold $K \subset S$ there exists a locally invariant manifold $K_\varepsilon \subset \mathbb{R}^6$

$$K_\varepsilon = \{(X,Y): \quad Y = A^{-1}BX + \varepsilon h(X,\varepsilon)\} \qquad \text{for some smooth function } h$$

Result: 6-dim system on (X,Y) $\Rightarrow$ 4-dim system on X on K_ε:

$$\dot{X} = F_v(X, A^{-1}BX + \varepsilon h(X,\varepsilon))$$

Somos pé-quente! (=Temos sorte!)

We have a perturbation problem ($X \in \mathbb{R}^4$):

$\varepsilon > 0$: $$\boxed{\dot{X} = F_v(X, A^{-1}BX + \varepsilon h(X, \varepsilon))}$$

$\varepsilon = 0$: $$\boxed{\dot{X} = F_v(X, A^{-1}BX)}$$

$$\begin{cases} \dot{a} = r \\ \dot{b} = s \\ \dot{r} = -vr - \dfrac{a}{2}(s - r) \\ \dot{s} = -vs - ra \end{cases}$$

… we can find all heteroclinic orbits explicitly when $\varepsilon = 0$!...

To prove the existence of heteroclinic orbits for $\varepsilon > 0$,

você precisa combinar com os russos (não hoje)….

Nunca esqueça de combinar com os russos!

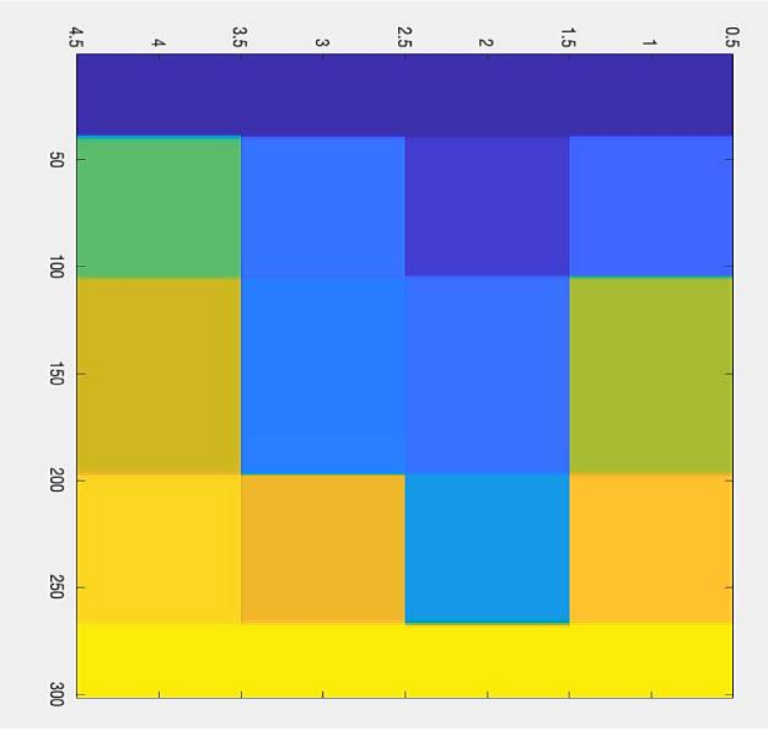
What do we NOT know and would be great to know?

1. Does there exist a heteroclinic orbit for this system for arbitrary l?

$$
\begin{cases}
\dot{a} = r \\
\dot{b} = s \\
\dot{r} = -vr + u_1 s - \dfrac{aq}{l} \\
\dot{s} = -vs + u_1 r + \dfrac{aq}{l} \\
u_1 = \dfrac{q}{l} \\
\dot{q} = 2u_1 + a
\end{cases}
$$

We have just discussed this system for $l \to 0$

2. Can we generalize the theorem on existence of travelling wave solutions (cascades) for arbitrary number n of tubes?



Example for $n = 4$: we observe 4 travelling waves

3. What about studying travelling waves for original 2-dim model? (…infinite dimensional dynamical systems…)

References Muito obrigada pela atenção!

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Muito obrigada pela atenção!

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