Exact small ball asymptotics in L_2-norm for some perturbations of the Brownian bridge

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We are interested in the exact L_2 small ball asymptotics

$$\mathbb{P}\left(\|\tilde{B}(t)\|_{L_2[0,1]} < \varepsilon\right) \sim ? \quad \text{as } \varepsilon \to 0.$$

- $\tilde{B}(t)$ — one-dimentional perturbation of the Brownian bridge $B(t)$:

$$\tilde{B}(t) = B(t) - \alpha\, h(t) \int_0^1 B(s)\varphi(s)\, ds, \tag{1}$$

  - $\alpha \in \mathbb{R}$ — parameter
  - $\varphi \in L_{1,loc}[0,1]$ generates a linear measurable functional on $B(t)$
  - $h(t) = \int G_B(s,t)\varphi(s)\, ds$:

$$\mathbf{q} := \mathbb{E}\langle \varphi, B\rangle^2 = \int_0^1\int_0^1 G_B(s,t)\varphi(s)\varphi(t)\, ds\, dt < +\infty, \tag{2}$$

    where $G_B(s,t) = \mathbb{E}B(s)B(t)$ is covariance.

- The covariance function of $\tilde{B}(t)$

$$G_{\tilde{B}}(s,t) = G_B(s,t) + Qh(s)h(t), \quad Q = q\alpha^2 - 2\alpha.$$

2007 — P. Deheuvels: $\varphi(t) \equiv 1$, $h(t) = t(1-t)/2$

2009 — A. Nazarov: general case $X(t)$ — Gaussian function on $O \subset \mathbb{R}^n$

Two useful facts

1. Due to the Karhunen–Loève (KL) expansion we have

$$\mathbb{P}\left(\|X(t)\|^2_{L_2[0,1]} < \varepsilon^2\right) = \mathbb{P}\Big(\sum_{k=1}^{\infty} \mu_k \xi_k^2 < \varepsilon^2\Big).$$

- ξ_k is a sequence of i.i.d. $N(0,1)$ random variables
- μ_k are eigenvalues of the integral operator with kernel $G_X(s,t)$.

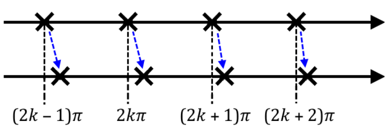
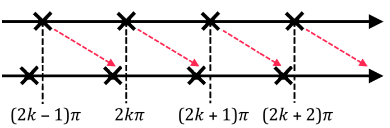
2. The Wenbo Li principle (W. Li 1992, F. Gao et al 2003)

- $X(t)$ — Gaussian process, $\mathbb{E}X(t) = 0$,
  μ_k — positive eigenvalues of integral operator with kernel $G_X(s,t)$
- $\tilde{X}(t)$ — Gaussian process, $\mathbb{E}\tilde{X}(t) = 0$,
  $\tilde{\mu}_k$ — positive eigenvalues of integral operator with kernel $G_{\tilde{X}}(s,t)$.
- If $\prod \tilde{\mu}_k/\mu_k < \infty$ then as $\varepsilon \to 0$

$$\mathbb{P}\Big\{\|X\|_{L_2[0,1]} < \varepsilon\Big\} \sim \mathbb{P}\Big\{\|\tilde{X}\|_{L_2[0,1]} < \varepsilon\Big\} \cdot \left(\prod_{k=1}^{\infty} \frac{\tilde{\mu}_k}{\mu_k}\right)^{1/2}$$

Theorem (A. Nazarov, 2009)

1. (non-critical case) If $\alpha \neq 1/\mathbf{q}$ then $\prod_{k=1}^{\infty} \frac{\mu_k}{\tilde{\mu}_k} < +\infty$.
2. (critical case) If $\alpha = 1/\mathbf{q}$, $\varphi \in L_2$ then $\prod_{k=2}^{\infty} \frac{\mu_k}{\tilde{\mu}_{k-1}} < +\infty$.

non-critical case	critical case: $\varphi \in L_2$
	

3. (intermediate case) If $\alpha = 1/\mathbf{q}, \varphi \notin L_2$ then ???

Process $\tilde{B}(t)$, $\mathbb{E}\tilde{B}(t) = 0$, with covariance function

$$G(s,t) = G_B(s,t) - h(s)h(t) = \min(s,t) - st - h(s)h(t)$$

- $-h''(t) = \varphi(t) \notin L_2$
- Additional condition: $h'(t)$ is a slowly-varying function at $t = 0$ or $t = 1$

- Such processes appear in statistics as limiting ones when building goodness-of-fit tests of ω^2-type for testing if the sample comes from some distribution when parameters are estimated from the sample.
- 2015 — A. Nazarov, Yu. Petrova:

  Kac-Kiefer-Wolfowitz processes, testing normality

$$h_1(t) = \phi(\Phi^{-1}(t)) \qquad h_2(t) = \frac{1}{\sqrt{2}} \cdot \phi(\Phi^{-1}(t)) \cdot \Phi^{-1}(t)$$

  Here $\phi(t)$, $\Phi(t)$ are the probability density and the distribution function of standard normal distrubition, respectively.
- Main tool: asymptotics of oscillating integrals with slowly-varying amplitudes such as

$$\int\limits_0^{\frac{1}{2}} F(t)\cos(\omega t)\,dt, \qquad \int\limits_0^{\frac{1}{2}}\int\limits_0^{\tau} F(t)F(\tau)\sin(\omega\tau)\cos(\omega t)\,dt\,d\tau,$$

  $\omega \to \infty$, in case $F(t)$ and $F_i(t) = tF'_{i-1}(t)$, $i \in \mathbb{N}$, are SVF.

2017 — Yu. Petrova:

- the constructed method works for limiting processes when we test a sample to come from the following distributions:

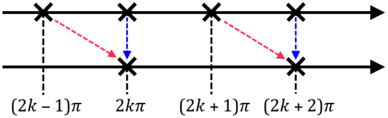
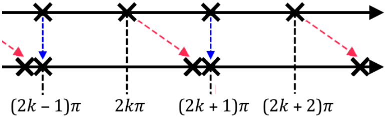
Distribution	$h_1(t)$	$h_2(t)$
Exponential	$-$	$t\ln(t)$
Laplace	$\begin{cases} s, & s \in (0,1/2] \\ 1-s, & s \in (1/2,1) \end{cases}$	$\begin{cases} s\ln(2s), & s \in (0,1/2] \\ -(1-s)\ln(2(1-s)), & s \in (1/2,1) \end{cases}$
Logistic	$\sqrt{3}t(1-t)$	$\frac{3}{\sqrt{3+\pi^2}}\, t(t-1)\ln\left(\frac{1-t}{t}\right)$
EVD	$t\ln(t)$	$C \cdot t\ln(t) \cdot \ln(-\ln(t))$

EVD= Extreme Value Distribution $\qquad$ C is a known constant

- and other distributions with exponential tails

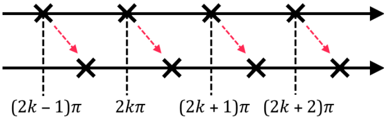
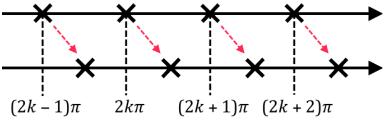
Example 1 (simple) : Laplace distribution

$$F_{lap}(x) = \begin{cases} \frac{1}{2}\exp(\beta(x-\alpha)), & x \leqslant \alpha \\ 1 - \frac{1}{2}\exp(-\beta(x-\alpha)), & x > \alpha \end{cases}$$

$h_1(t) = \begin{cases} t, & t \in (0, 1/2] \\ 1-t, & t \in (1/2, 1) \end{cases}$	$h_2(t) = \begin{cases} t\ln(2t), & t \in (0, 1/2] \\ -(1-t)\ln(2(1-t)), & t \in (1/2, 1) \end{cases}$
	
$\omega_{2k-1} = \omega_{2k} = 2\pi k$	$\omega_{2k} = 2\pi k + \pi + O\left(\frac{1}{k}\right)$ $\omega_{2k+1} = 2\pi k + \pi$

Example 2 (more complicated): Extreme Value Distribution

$$F_{evd}(x) = \exp\left(-\exp\left(-\frac{x-\alpha}{\beta}\right)\right)$$

$h_1(s) = s\ln(s)$	$h_2(s) = C \cdot s\ln(s) \cdot \ln(-\ln(s))$
	
$\omega_k = \pi k + \frac{\pi}{2} + O\left(\frac{1}{k}\right)$	$\omega_k = \pi k + \frac{\pi}{2} + F(k) + O\left(\frac{1}{\ln(k)(\ln(\ln(k)))^2}\right)$

$$F(k) = (-1)^k \cdot 2\,\mathrm{arctg}\left(\frac{1}{\ln(\ln(k))+1}\right) - \frac{1}{\ln(k)\ln(\ln(k))}$$

$\tilde{\omega}_k = \pi k + \frac{\pi}{2} + F(k)$ is a «good» approximation because $\prod \frac{\omega_k}{\tilde{\omega}_k} < +\infty$

Small ball probabilities: $\mathbb{P}\left(\|X\|_{L_2[0,1]} < \varepsilon\right) \sim$

$B(t)$		$\dfrac{2\sqrt{2}}{\sqrt{\pi}} \quad \exp\left(-\dfrac{1}{8\varepsilon^2}\right)$
Laplace h_1		$\dfrac{\sqrt{2}}{\sqrt{\pi}}\ \varepsilon^{-1} \exp\left(-\dfrac{1}{8\varepsilon^2}\right)$
Laplace h_2		$\dfrac{2\sqrt{2}}{\pi^{3/2}}\ \varepsilon^{-1} \exp\left(-\dfrac{1}{8\varepsilon^2}\right)$
EVD h_1		$\dfrac{4}{\pi^{3/2}}\ \varepsilon^{-1} \exp\left(-\dfrac{1}{8\varepsilon^2}\right)$
EVD h_2	$\dfrac{1}{\ln(\ln(\varepsilon^{-1}))}$	$C \cdot \varepsilon^{-1} \exp\left(-\dfrac{1}{8\varepsilon^2}\right)$
Normal h_1	$\sqrt{\ln(\varepsilon^{-1})}$	$C \cdot \varepsilon^{-1} \exp\left(-\dfrac{1}{8\varepsilon^2}\right)$

Thank you for your attention!