# Two tubes model of miscible displacement: travelling waves and normal hyperbolicity


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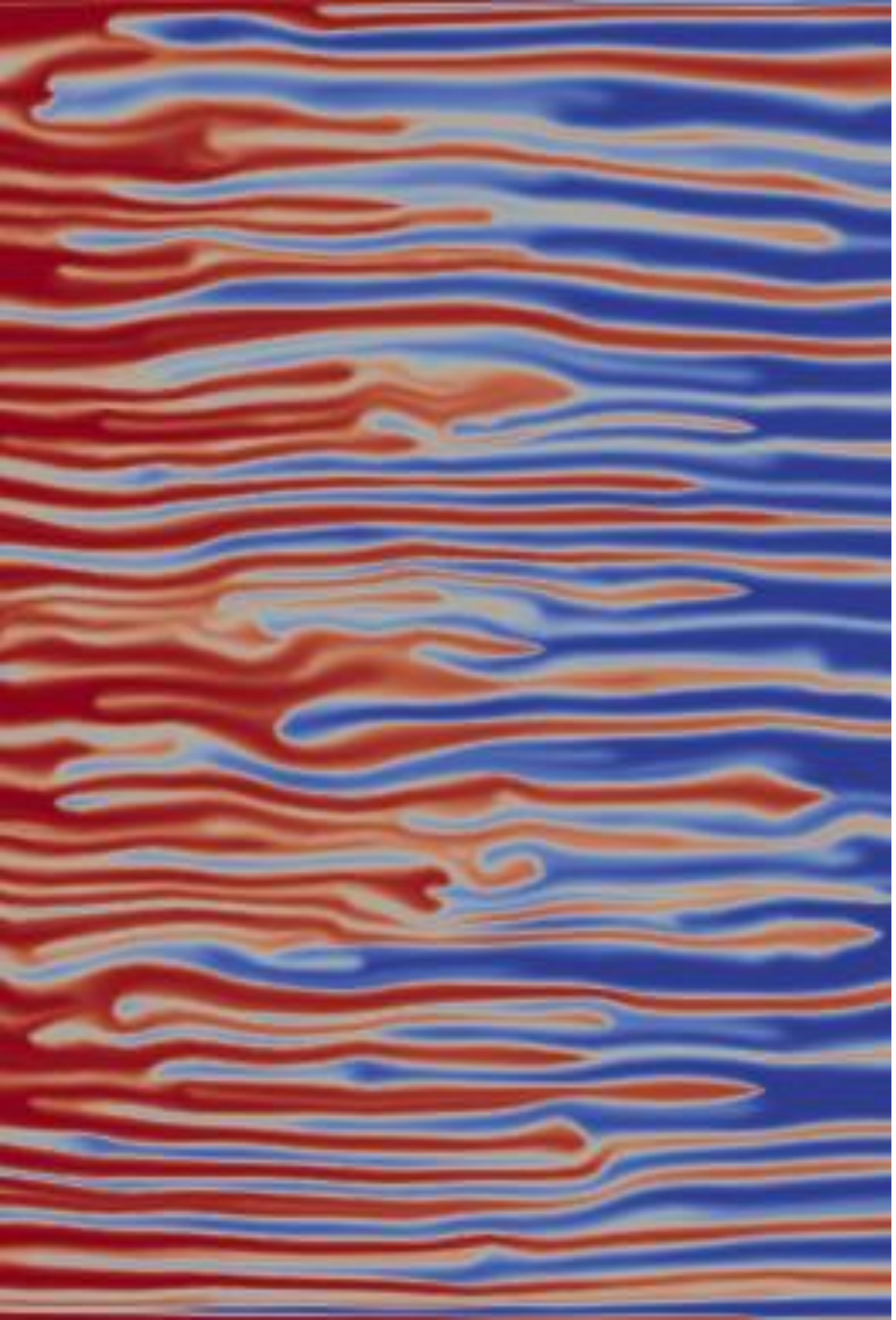


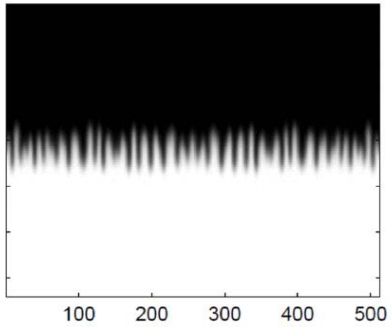
# Outline

1. General phenomenon
   - Viscous fingers
   - Gravitational fingers
2. Motivation of the statement of the problem
   - Why we believe that our setting is important
3. Theorem and Conjectures
4. Further questions

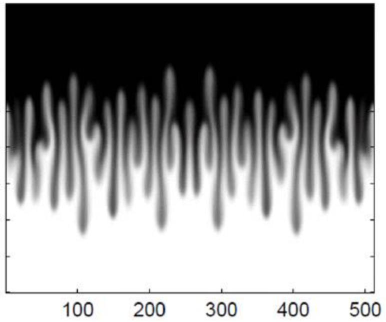
# Two settings

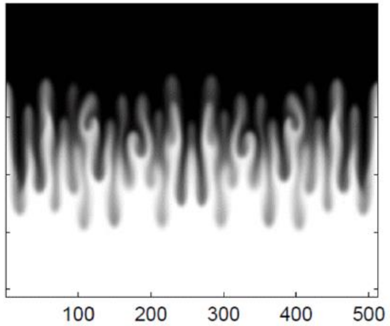
## 1. Gravity-driven fingers

$$c_t + u \cdot \nabla c = \varepsilon\, \Delta c$$
$$div\, u = 0$$
$$u = -\nabla p - (0, c)$$




- $c$ – concentrations of heavy spices transport equation
  $$c \in [-1, 1]$$
- $u$ – velocity of fluid incompressible fluid
- $p$ – pressure.
  Velocity is defined by Darcy law and gravitation

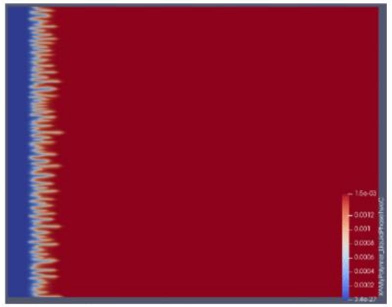


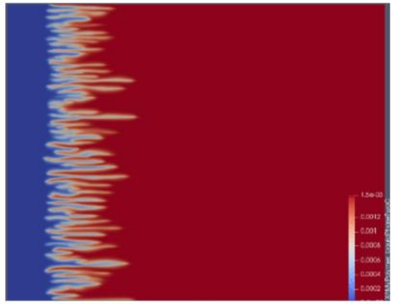

We have some theorems.

## 2. Viscosity-driven fingers

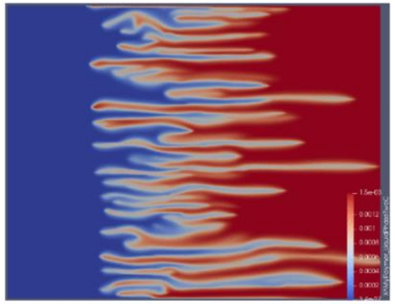
$$c_t + u \cdot \nabla c = \varepsilon\, \Delta c$$
$$div\, u = 0$$
$$u = -k \cdot m(c) \nabla p$$



- $c$ – concentrations of viscous spices transport equation
  $$c \in [0, 1]$$
- $u$ – velocity of fluid incompressible fluid
- $p$ – pressure.
  Velocity is defined by Darcy law and mobility of liquid $m(c)$.
  $m(c)$ – decreasing function.
  $$m(c) = \mathrm{e}^{-\mathrm{a}c}$$



We did a lot of numerical simulations.
Motivation of statement of the problem.



# Transverse Flow Equilibrium Model. Gravity-driven fingers

- Let us consider Peaceman model with an extra assumption

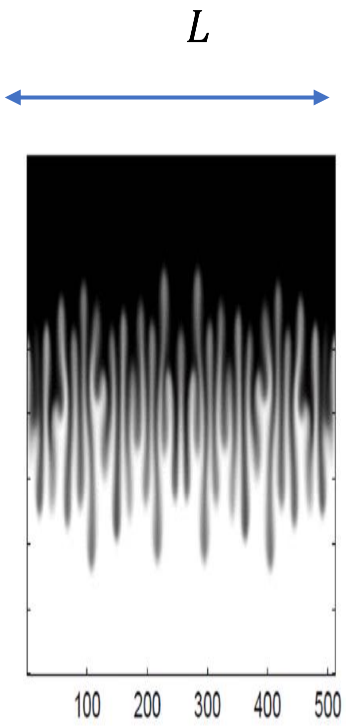
$$p(x,y) \sim p(y), \qquad p_y(x,y) \sim p_y(y)$$

**Peaceman model**

$$c_t + u \cdot \nabla c = \varepsilon \, \Delta c$$
$$div\, u = 0$$
$$u = -\nabla p - (0, c)$$

**TFE model**

$$c_t + u \cdot \nabla c = \varepsilon \, \Delta c$$
$$div\, u = 0$$
$$u = (u^x, u^y)$$
$$u^y = \bar{c} - c$$
$$\bar{c}(y) = \frac{1}{L} \int c(x,y)\, dx$$




- TFE model has no pressure
- It is a closed system of equations
- TFE model contradicts to assumptions which deduces it

# Comparison theorem for TFE model

- TFE model

$$c_t + u \cdot \nabla c = \varepsilon \, \Delta c$$
$$div\, u = 0$$
$$u = (u^x, u^y)$$
$$u^y = \bar{c} - c$$

Consider 1d equations (Burgers equation)

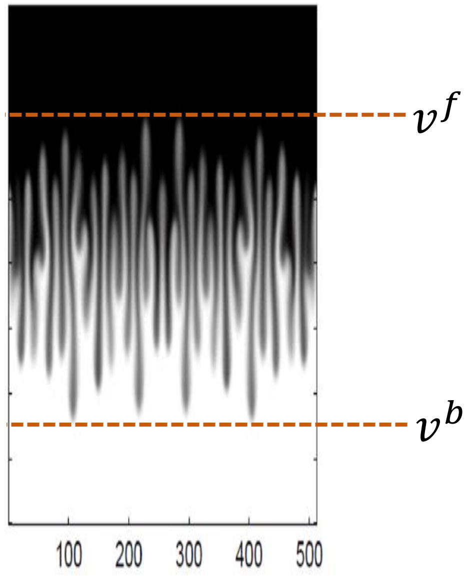
$$c_t^{max} + (1 - c^{max}) c_y^{max} = \varepsilon \, (c^{max})_{yy}$$

$$c_t^{min} + (-1 - c^{min}) \cdot c_y^{min} = \varepsilon \left(c^{min}\right)_{yy}$$

**Comparison theorem (Otto-Menon, 2005)**

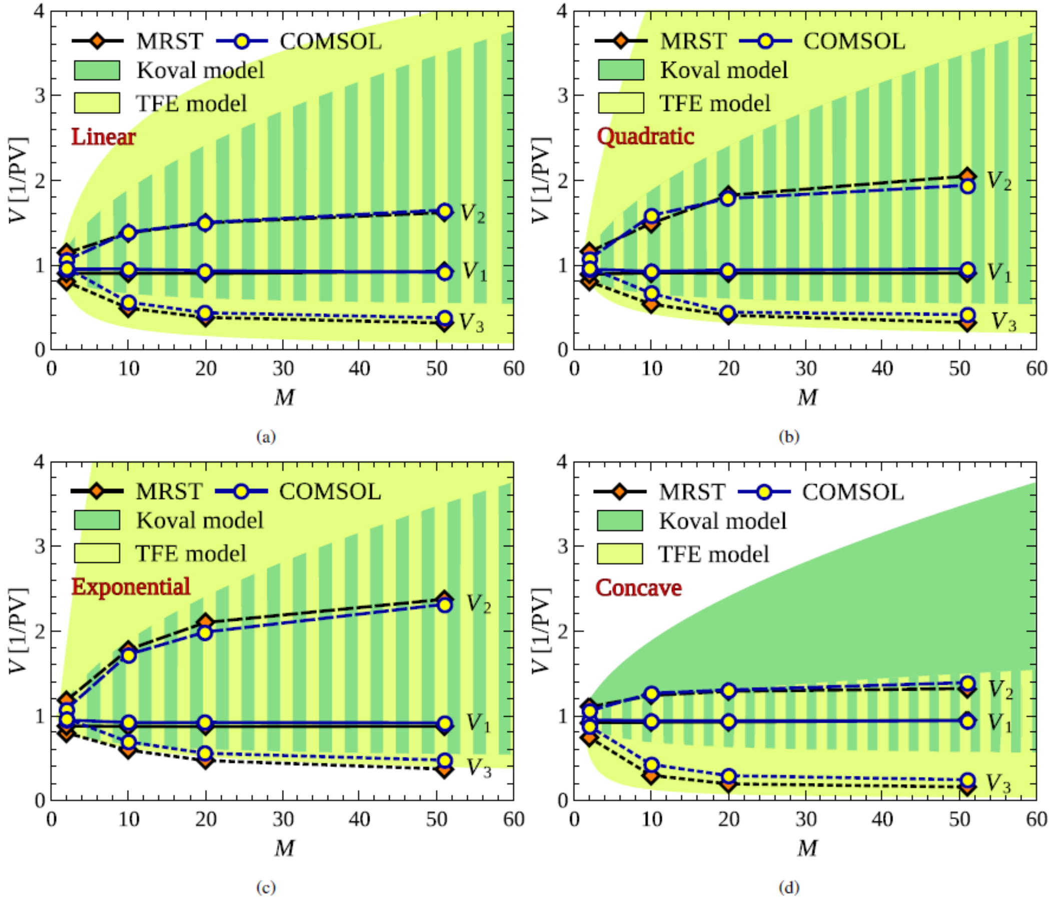
- If $c(0,x,y) < c^{max}(0,y)$ then $c(t,x,y) \leq c^{max}(t,y)$
- If $c(0,x,y) > c^{min}(0,y)$ then $c(t,x,y) \geq c^{min}(t,y)$

1. It gives upper bound for the faster finger
$$v^f \leq 1$$
2. It gives upper bound for the back front
$$v^b \geq -1$$
3. Estimate is sharp if
    1. There is no transverse flow
    2. Drop of concentration on a finger tip is -1 -> +1
4. Numerics shows that estimate is far from sharp
5. We want to get better estimate

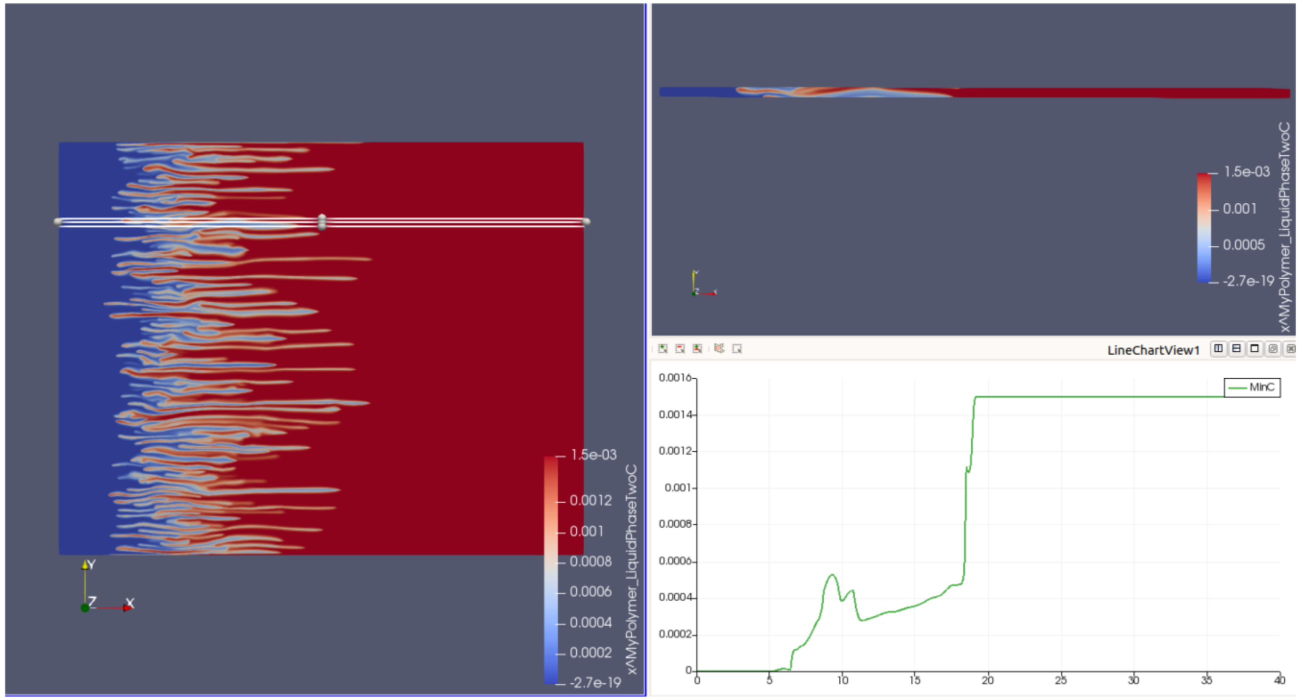


# Numerics for viscous fingers

F. Bakharev, A. Enin, A. Groman, A. Kalyuzhnyuk,
S. Matveenko, **Yu. Petrova**, I. Starkov, **S. Tikhomirov**
*“Velocity of viscous fingers in miscible displacement: Comparison with analytical models”*
Journal of Computational and Applied Mathematics, 2022




Possible mechanism: intermediate concentration



# Two-tubes model

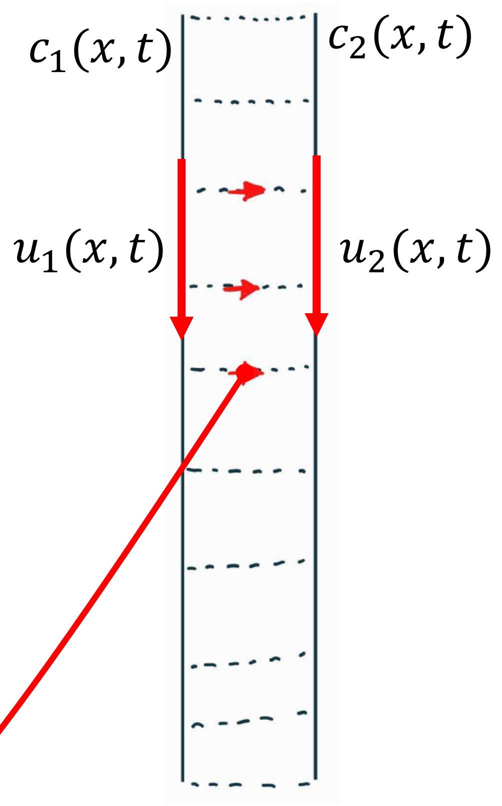
Original equations

$$c_t + u \cdot \nabla c = \varepsilon \, \Delta c$$
$$div\, u = 0$$

Inclusion of transverse flow

$$\partial_t c_1 + \partial_x (u_1 c_1) - \varepsilon \partial_{xx} c_1 = -(-1)^{1,2} \partial_x u_{1,2} \cdot c_{1,2},$$
$$\partial_t c_2 + \partial_x (u_2 c_2) - \varepsilon \partial_{xx} c_2 = (-1)^{1,2} \partial_x u_{1,2} \cdot c_{1,2},$$

$$(-1)^{1,2} \partial_x u_{1,2} \cdot c_{1,2} = \begin{cases} -\partial_x u_1 \cdot c_1, & \partial_x u_1 < 0, \\ +\partial_x u_2 \cdot c_2, & \partial_x u_1 \geq 0. \end{cases}$$

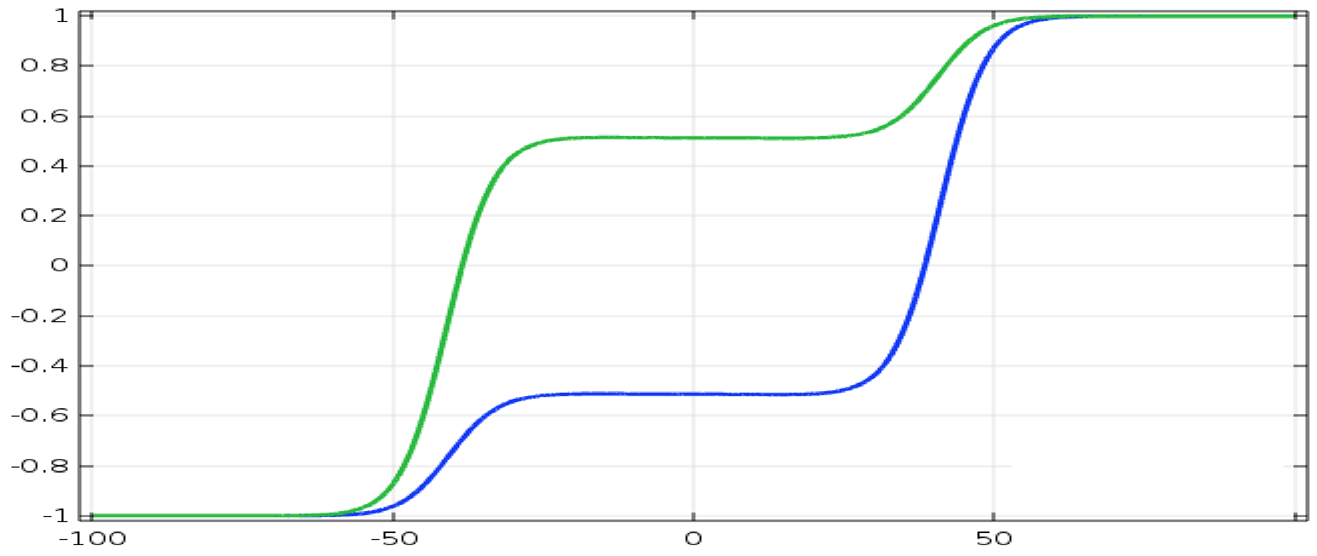


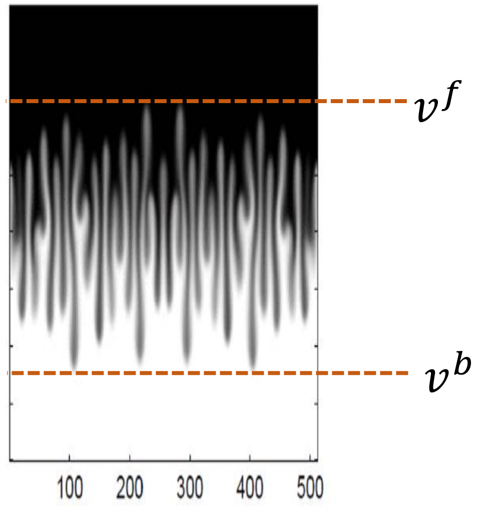

Model for velocities is different for Peaceman and TFE:

- TFE: $u = \bar{c} - c, \quad u_1 = \frac{c_1 + c_2}{2} - c_1, \quad u_2 = \frac{c_1 + c_2}{2} - c_2$
- Peaceman: we need to introduce pressure, we will do this later

# Two-tubes theorem.

Numerics for gravitation




**Theorem (Efendiev, P., T.)**

There exists unique (up to swap) $c_1^*, c_2^*$ such that TFE two-tubes system has travelling waves

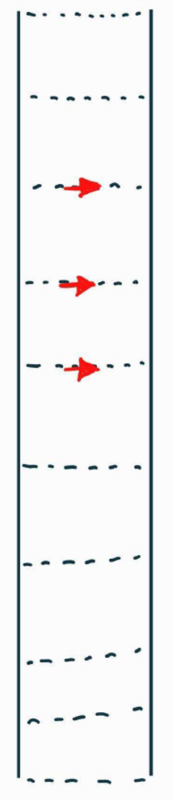
$$(-1,-1) \to (c_1^*, c_2^*) \to (1,1)$$

Moreover

$$c_1^* = -\frac{1}{2}, c_2^* = \frac{1}{2},$$
$$v^b = -\frac{1}{4}, v^f = \frac{1}{4}.$$

Including in the system cross-flow automatically creates intermediate concentration

# Travelling waves. Equations.



**Original system:**

$$\partial_t c_1 + \partial_x (u_1 c_1) - \varepsilon \partial_{xx} c_1 = -(-1)^{1,2} \partial_x u_{1,2} \cdot c_{1,2},$$

$$\partial_t c_2 + \partial_x (u_2 c_2) - \varepsilon \partial_{xx} c_2 = (-1)^{1,2} \partial_x u_{1,2} \cdot c_{1,2},$$

$$(-1)^{1,2} \partial_x u_{1,2} \cdot c_{1,2} = \begin{cases} \boxed{-\partial_x u_1 \cdot c_1, \quad \partial_x u_1 < 0,} \\ +\partial_x u_2 \cdot c_2, \quad \partial_x u_1 \geq 0. \end{cases}$$

**Travelling wave ansatz:**

$\xi = x - vt, \qquad c_{1,2}(x,t) = c_{1,2}(\xi),$

$c_{1,2}(\pm\infty) = c^{\pm}_{1,2}$

**4d system:**

$$\dot{c}_1 = g_1,$$

$$\dot{g}_1 = g_1(u_1 - v),$$

$$\dot{c}_2 = g_2,$$

$$\dot{g}_2 = (u_2 - v) g_2 + (c_1 - c_2)\dot{u}_1.$$

**Conservation laws – 3d dynamical system:**

$$\dot{c}_1 = g_1,$$

$$\dot{g}_1 = g_1(u_1 - v),$$

$$\dot{c}_2 = -v(c_1 + c_2 - c_1^+ - c_2^+) + (u_1 c_1 + u_2 c_2 - u_1^+ c_1^+ - u_2^+ c_2^+) - g_1.$$

**Connection between $c^{\pm}_{1,2}$ and $v$:**
**(Rankine-Hugoniot condition)**

$$v[c_1 + c_2]\Big|_{-\infty}^{+\infty} = [u_1 c_1 + u_2 c_2]\Big|_{-\infty}^{+\infty}.$$

**TFE velocity model:**

$$u_1 = \frac{c_1 + c_2}{2} - c_1, \qquad u_2 = \frac{c_1 + c_2}{2} - c_2$$

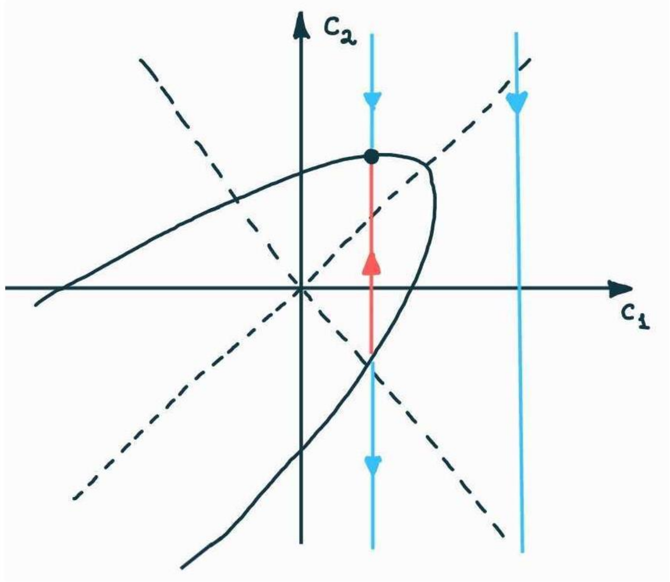
# Travelling waves. Phase portrait.

Substitute $u_{1,2}$

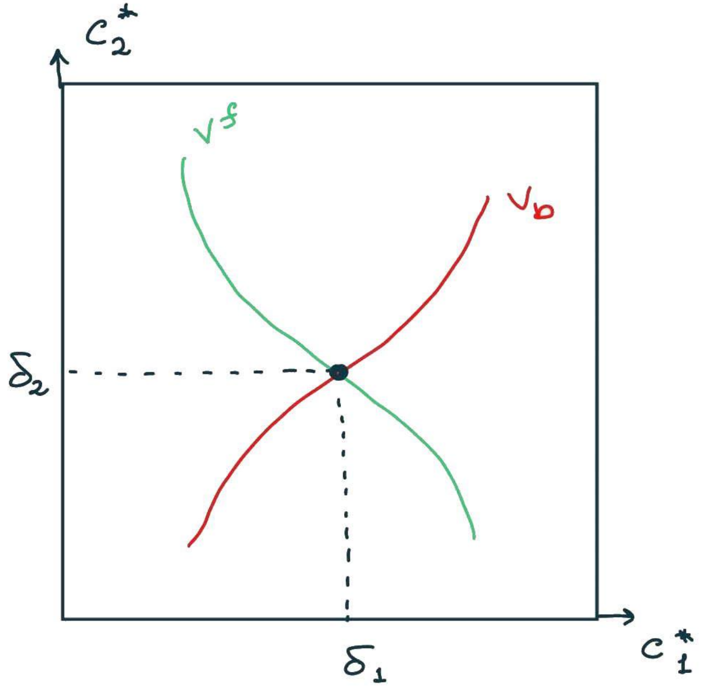
$$\dot{c}_1 = g_1,$$
$$\dot{g}_1 = g_1\left(\frac{c_2 - c_1}{2} - v\right),$$
$$\dot{c}_2 = -v(c_1 + c_2 - c_1^+ - c_2^+) - \frac{1}{2}((c_1 - c_2)^2 - (c_1^+ - c_2^+)^2) - g_1.$$

$$v(c_2^- + c_1^- - c_2^+ - c_1^+) = -\frac{1}{2}((c_1^- - c_2^-)^2 - (c_1^+ - c_2^+)^2).$$

Phase portrait




- For each $v$ expected a travelling wave
- This generates a curve of possible $c_1, c_2$
- We apply this procedure for travelling wave to (+1, +1) and from (-1, -1)



# Two tubes. Invariant surface.

Equations

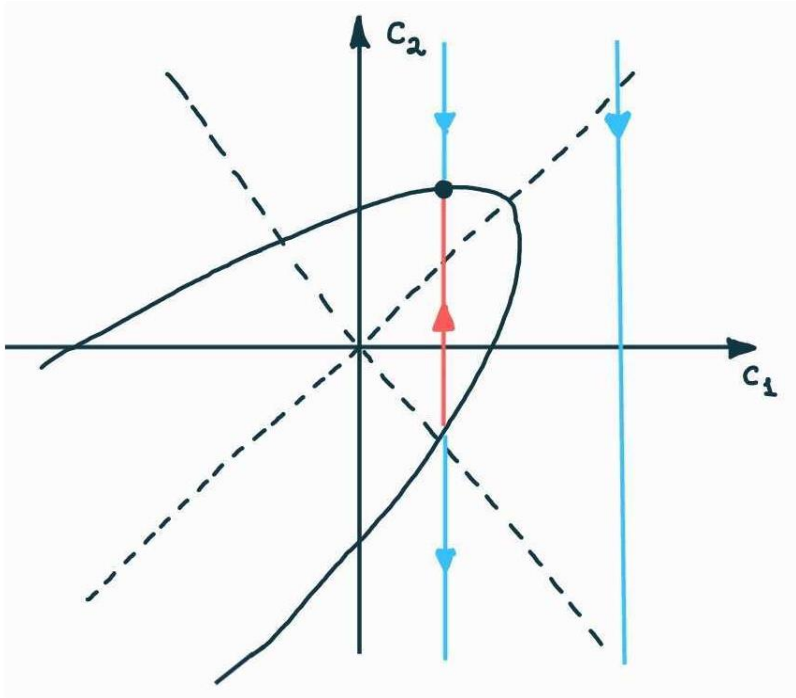
$$\dot{c}_1 = g_1,$$

$$\dot{g}_1 = g_1 \left( \frac{c_2 - c_1}{2} - v \right),$$

$$\dot{c}_2 = -v(c_1 + c_2 - c_1^+ - c_2^+) - \frac{1}{2}((c_1 - c_2)^2 - (c_1^+ - c_2^+)^2) - g_1.$$

Travelling wave speed

$$v(c_2^- + c_1^- - c_2^+ - c_1^+) = -\frac{1}{2}((c_1^- - c_2^-)^2 - (c_1^+ - c_2^+)^2).$$




There exists 2dim invariant surface

$$g_1 = \tfrac{3}{4}(-v(c_2 + c_1 - c_2^+ - c_1^+) - \tfrac{1}{2}((c_1 - c_2)^2 - (c_1^+ - c_2^+)^2)),$$

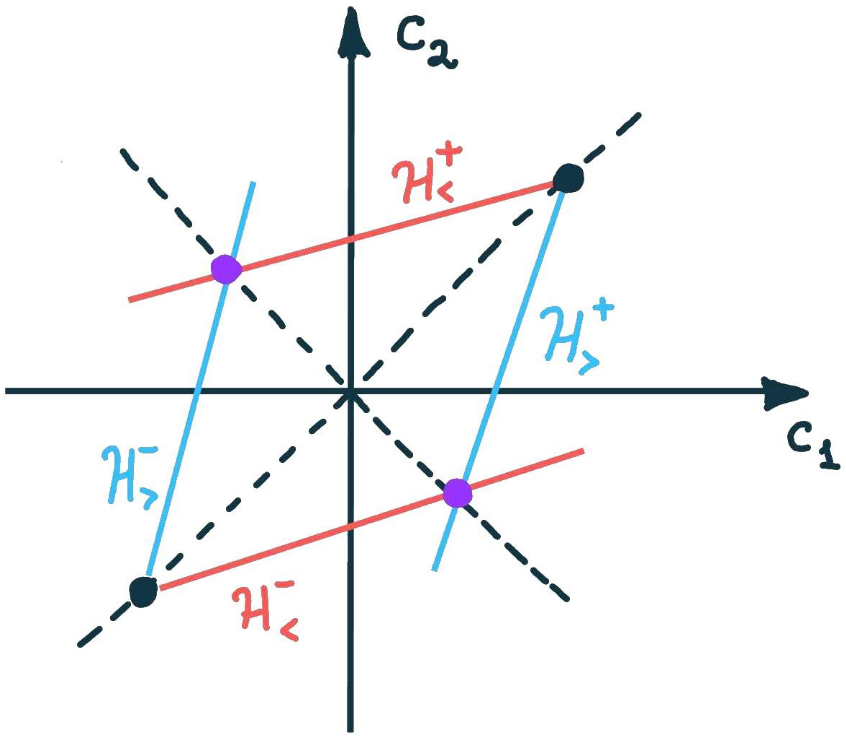
On all (for any $c_{1,2}^+$) heteroclinic holds

$$3(c_2 - c_2^+) = c_1 - c_1^+,$$

We have solved our “heteroclinic” problem analytically

# Finally answer.

Admissible curves on the plane




**Speed and concentration**

$$v^b = -\frac{1}{4}$$
$$v^f = \frac{1}{4}$$
$$c_1^* = -1/2$$
$$c_2^* = 1/2$$

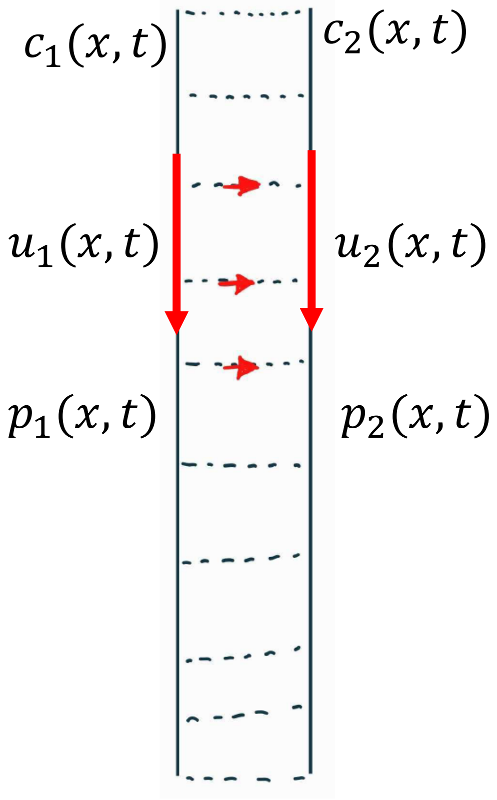
# Two-tubes model. Peaceman.

## Original equations

$$c_t + u \cdot \nabla c = \varepsilon \, \Delta c$$
$$div \, u = 0$$

$$\partial_t c_1 + \partial_x (u_1 c_1) - \varepsilon \partial_{xx} c_1 = -(-1)^{1,2} \partial_x u_{1,2} \cdot c_{1,2},$$
$$\partial_t c_2 + \partial_x (u_2 c_2) - \varepsilon \partial_{xx} c_2 = (-1)^{1,2} \partial_x u_{1,2} \cdot c_{1,2},$$

$$(-1)^{1,2} \partial_x u_{1,2} \cdot c_{1,2} = \begin{cases} -\partial_x u_1 \cdot c_1, & \partial_x u_1 < 0, \\ +\partial_x u_2 \cdot c_2, & \partial_x u_1 \geq 0. \end{cases}$$




## Velocity model for Peaceman: add $p_1$ and $p_2$

(Darcy's law in each tube) $\quad u_1 = -\partial_x p_1 - c_1, \quad u_2 = -\partial_x p_2 - c_2,$

(Darcy's law between tubes) $\quad \partial_x u_1 = (p_2 - p_1)/l, \quad \partial_x u_2 = -(p_2 - p_1)/l.$

$$q = p_2 - p_1$$

# Travelling wave for Peaceman.

## Equations for travelling waves for Peaceman

$$\boxed{\begin{aligned}\dot c_1 &= g_1,\\ \dot g_1 &= g_1(u_1 - v),\\ \dot c_2 &= -v(c_1 + c_2 - c_1^+ - c_2^+) + u_1(c_1 - c_2) - u_1^+(c_1^+ - c_2^+) - g_1,\end{aligned}}$$

“Pressure part”

$$q = p_2 - p_1$$

$$\dot q = u_2 - u_1 + c_2 - c_1,$$

$$\dot u_1 = q/l,$$

$$\dot u_2 = -q/l.$$

Proper rescaling

$$q/\sqrt{l} \to q \qquad \sqrt{l} = \delta$$

$$\boxed{\begin{aligned}\delta\,\dot q &= -2u_1 + c_2 - c_1,\\ \delta\,\dot u_1 &= q.\end{aligned}}$$

## Equations for TFE

$$\dot c_1 = g_1,$$

$$\dot g_1 = g_1(u_1 - v),$$

$$\dot c_2 = -v(c_1 + c_2 - c_1^+ - c_2^+) + u_1(c_1 - c_2) - u_1^+(c_1^+ - c_2^+) - g_1,$$

$$u_1 = \frac{c_1 + c_2}{2} - c_1 = \frac{c_2 - c_1}{2},$$

Corresponds to formal limit $\delta \to 0$

Statement (based on normal hyperbolicity):
For small enough $\delta$ the “Peaceman system” has invariant 3-dimensional manifold with dynamics close to “TFE system ”

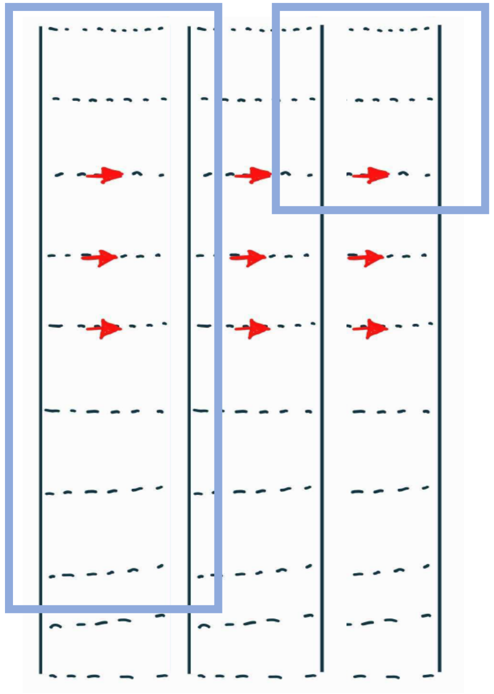
Conjecture 1 (in progress)
For $l \to 0$ we have $c_{1,2}^*(l, v^b) \to c_{1,2}^*(v^b)$

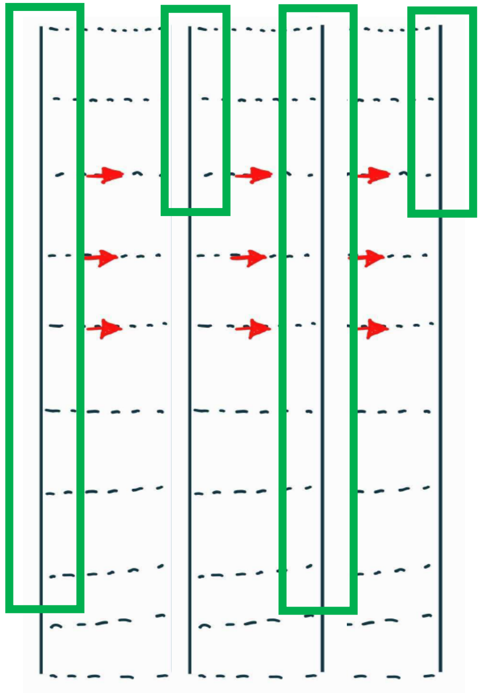
Conjecture 2 (in progress)
For $l \to 0$ we have $c_{1,2}^*(l) \to c_{1,2}^*$

# What’s next? **Thank you very much for your attention**

1. Otto-Menon suggested that
   after time t fingers have length $\sim \sqrt{t}$
   What is the mechanism of merging of fingers?

   4-tubes model
   What is more stable:
   - Two thin fingers?
   - One thick finger





2. TFE as a limit of Peaceman when $\frac{k_y}{k_x} \to \infty$

**Peaceman model**

$$c_t + u \cdot \nabla c = \varepsilon \, \Delta c$$
$$div \, u = 0$$
$$u = -\begin{pmatrix} k_x & 0 \\ 0 & k_y \end{pmatrix} \nabla p - (0, c)$$

**TFE model**

$$c_t + u \cdot \nabla c = \varepsilon \, \Delta c$$
$$div \, u = 0$$
$$u = (u^x, u^y)$$
$$u^y = \bar{c} - c$$

# Additional slides

# Multiple cascade of travelling waves.