Exact L_2-small ball asymptotics for detrended Green Gaussian processes

Problem statement

We look for the asymptotics of eigenvalues of the problem

$$(-1)^p u^{(2p)}(t) = \lambda u(t) + \mathcal{P}_{n-2p}(t), \quad t \in [0,1] \tag{1}$$

$$\int_0^1 t^i u(t)\,dt = 0, \quad i = 0 \dots n-1, \tag{2}$$

where $n, p \in \mathbf{N}$, $n > 2p$, and $\mathcal{P}_{n-2p}(t)$ is a polynomial of degree less than $(n-2p)$ with unknown coefficients.

Theorem (asymptotics of eigenvalues)

$$\lambda_k = \left(\pi k + \frac{2n-p-1}{2} + O\left(\frac{1}{k}\right)\right)^{2p}, \quad \text{as } k \to \infty. \tag{3}$$

The equivalent problem

$$(-1)^p y^{(2n)}(t) = \lambda y^{(2n-2p)}(t), \quad t \in [0,1] \tag{4}$$

$$y^{(j)}(0) = y^{(j)}(1) = 0, \quad j = 0 \dots n-1. \tag{5}$$

NB: the principle eigenvalue of (4)-(5) gives the sharp constant in the embedding theorem $\overset{\circ}{W}{}_2^n(0,1) \hookrightarrow \overset{\circ}{W}{}_2^{n-p}(0,1)$.
The equivalence can be seen by putting $u(t) = y^{(n)}(t)$.

History (M. Janet, [1])

Problem (4)-(5) was solved for $n \in \mathbb{Z}_+$ and $p = 1$ in 1931. For arbitrary p the answer was only formulated without proof and in implicit terms.

Application to small ball asymptotics

We apply the asymptotic formula (3) to calculate sharp L_2-small ball asymptotics as $\varepsilon \to 0$ of $\mathbb{P}\{\|X_n(t)\|_{L_2[0,1]} < \varepsilon\}$ for Gaussian process

$$X_n(t) := X(t) - \sum_{i=0}^{n-1} a_i t^i,$$

where a_i are determined by relations

$$\int_0^1 t^i X_n(t)\,dt = 0, \;\; i = 0 \dots n-1.$$

Here $X(t)$, $t \in [0,1]$, is a Gaussian process, $\mathbb{E}X = 0$, covariance function $G(s,t) = \mathbb{E}X(s)X(t)$ is the Green function for a BVP:

$$Lu := (-1)^p u^{(2p)} = \lambda u \;+\; \text{some boundary conditions.}$$

In case $n > 2p$ the process X_n does not depend on the original boundary conditions.

$n = 1$, $p = 1$	centered Brownian bridge and motion	2005 — E. Orsingher, Ya. Nikitin 2006 — P. Deheuvels [2]
$n = 2$, $p = 1$	detrended Brownian motion	2012 — X. Ai, W. Li [3]
$\forall n \geqslant 3$, $p = 1$	n-th order detrended Brownian motion	2014 — X. Ai, W. Li [4]
$\forall n, p$, $n > 2p$	n-th order detrended Gaussian process $X(t)$	2016 — Yu. Petrova [5]

So we get ($\gamma = \frac{1-2np+p^2}{2p-1}$), $\varepsilon \to 0$:

$$\mathbb{P}\Big\{\|X_n\|_{L_2[0,1]} < \varepsilon\Big\} \sim C\varepsilon^\gamma \exp\left(-\frac{2p-1}{2(2p\sin(\frac{\pi}{2p}))^{\frac{2p}{2p-1}}}\varepsilon^{-\frac{2}{2p-1}}\right).$$

Proof of theorem

Step 1: Odd solutions

WLOG we can assume that the eigenfunction is odd or even (wrt $t = \frac{1}{2}$). If $y(t)$ is an even solution of the eq. (4):

$$(-1)^p y^{(2n)}(t) - \lambda y^{(2n-2p)}(t) = 0,$$

then

$$(-1)^p (y')^{(2n-2)}(t) - \lambda (y')^{(2n-2-2p)}(t) = C$$

and the constant $C = 0$, as the left hand side is odd.

$\boxed{\lambda \text{ of even solution of (4) with parameters } (n,p)} \;=\; \boxed{\lambda \text{ of odd solution of (4) with parameters } (n-1,p)}$

So we can restrict ourselves only to odd solutions.

Step 2: Determinant

Every odd solution of the equation (4) is of the form:

$$\begin{aligned} y = &a_0 \sin \xi_0(2t-1) + \dots + a_{p-1} \sin \xi_{p-1}(2t-1) + \\ &+ a_p(2t-1) + \dots + a_{n-1}(2t-1)^{2n-2p-1}, \end{aligned}$$

here $\xi_k = \frac{1}{2}|\lambda|^{\frac{1}{2p}} e^{\frac{ik\pi}{p}}$, $k = 0 \dots p-1$.
Substituting $y(t)$ into the boundary conditions (5), we get the equation $\Delta_{n,p}(\lambda) = 0$, where $\Delta_{n,p}(\lambda)$ is some determinant.

Step 3: Equation on determinant

$\Delta_{n,p}$ as a function of $\xi_0, \dots, \xi_{p-1}$ satisfies such an equation:

$$\frac{\partial^p}{\partial \xi_0 \dots \partial \xi_{p-1}} \Delta_{n,p} = C \cdot \xi_0 \cdot \ldots \cdot \xi_{p-1} \cdot \Delta_{n-1,p}. \tag{6}$$

Step 4: Asymptotics of eigenvalues

$$\Delta_{p,p} = C \begin{vmatrix} \xi_0^{1/2} \mathcal{J}_{1/2}(\xi_0) & \dots & \xi_{p-1}^{1/2} \mathcal{J}_{1/2}(\xi_{p-1}) \\ \xi_0^{3/2} \mathcal{J}_{3/2}(\xi_0) & \dots & \xi_{p-1}^{3/2} \mathcal{J}_{3/2}(\xi_{p-1}) \\ \dots & \dots & \dots \\ \xi_0^{(2p-1)/2} \mathcal{J}_{(2p-1)/2}(\xi_0) & \dots & \xi_{p-1}^{(2p-1)/2} \mathcal{J}_{(2p-1)/2}(\xi_{p-1}) \end{vmatrix}$$

Here $\mathcal{J}_k(x)$ are Bessel functions of the first kind. Using relation (6) we get the following representation for $\Delta_{n,p}$

$$\begin{vmatrix} \xi_0^{(2n-2p+1)/2} \mathcal{J}_{(2n-2p+1)/2}(\xi_0) & \dots & \xi_{p-1}^{(2n-2p+1)/2} \mathcal{J}_{(2n-2p+1)/2}(\xi_{p-1}) \\ \xi_0^{(2n-2p+3)/2} \mathcal{J}_{(2n-2p+3)/2}(\xi_0) & \dots & \xi_{p-1}^{(2n-2p+3)/2} \mathcal{J}_{(2n-2p+3)/2}(\xi_{p-1}) \\ \dots & \dots & \dots \\ \xi_0^{(2n-1)/2} \mathcal{J}_{(2n-1)/2}(\xi_0) & \dots & \xi_{p-1}^{(2n-1)/2} \mathcal{J}_{(2n-1)/2}(\xi_{p-1}) \end{vmatrix}$$

The final equation will be of the form

$$\Delta_{n,p}(\lambda) \cdot \Delta_{n-1,p}(\lambda) = 0.$$

Using asymptotics of Bessel functions we get (3).

References

[1] Janet M. Sur le minimum du rapport de certaines intégrales // Comptes Rendus de l'Académie des Sciences Paris. — 1931. — Vol. **193**. — P. 977–979.

[2] Deheuvels P. A Karhunen–Loève expansion for a mean-centered Brownian bridge // Statistics & Probability Letters. — 2007. — Vol. **77**, no. **12**. — P. 1190–1200.

[3] Ai X., Li W., Liu G. Karhunen–Loève expansions for the detrended Brownian motion // Statistics & Probability Letters. — 2012. — Vol. **82**, no. **7**. — P. 1235–1241.

[4] Ai X., Li W. Karhunen–Loève expansions for the m-th order detrended Brownian motion // Science China Mathematics. — 2014. — Vol. **57**, no. **10** — P. 2043–2052.

[5] Petrova Yu. Spectral asymptotics for problems with integral constraints// Mathematical Notes. — 2017. — Vol. **102**, no. **3** — P. 369-377.

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