

Introduction
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Review on conservation laws
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Main result
oooooooo

Proof
ooooooooo

Summarize
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On the impact of diffusion ratio on vanishing viscosity solutions of Riemann problems for chemical flooding models

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Centro de Projetos
e Inovação IMPA

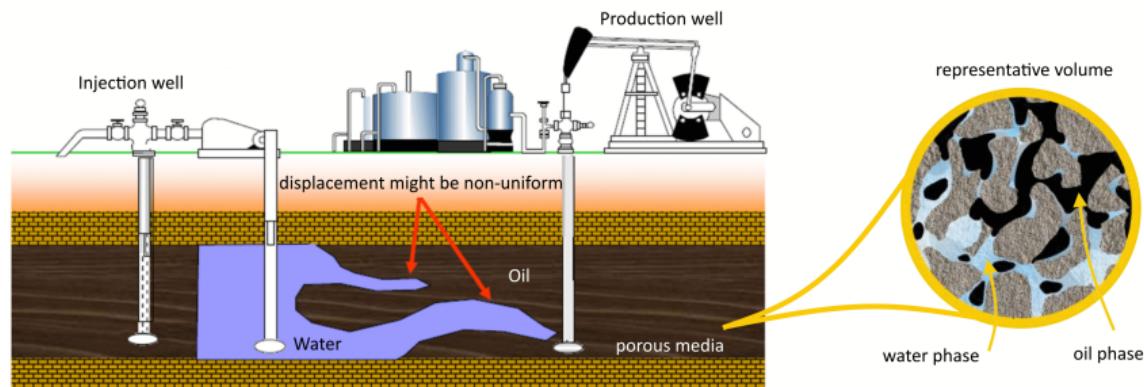
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Analysis and PDEs seminar at IMPA

Joint work with Fedor Bakharev, Aleksandr Enin and Nikita Rastegaev: arXiv:2111.15001

Motivation

We are interested in the mathematical model of oil recovery. Some features:

- Porous media (averaged models of flow)
- Unknown variables: $s(t, x)$ — the averaged water saturation in small volume
 $1 - s$ — the average oil saturation in representative volume
- Relatively small speeds (≈ 1 meter per day): Navier-Stokes \rightarrow Darcy's law
- Multiphase flow: oil, water, gas.
- Applications to EOR (enhanced oil recovery) methods: chemical, thermal, gas etc

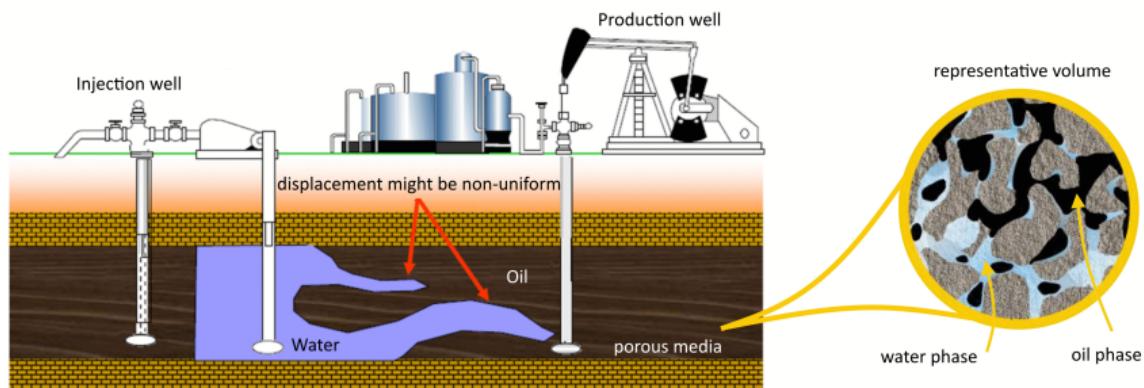


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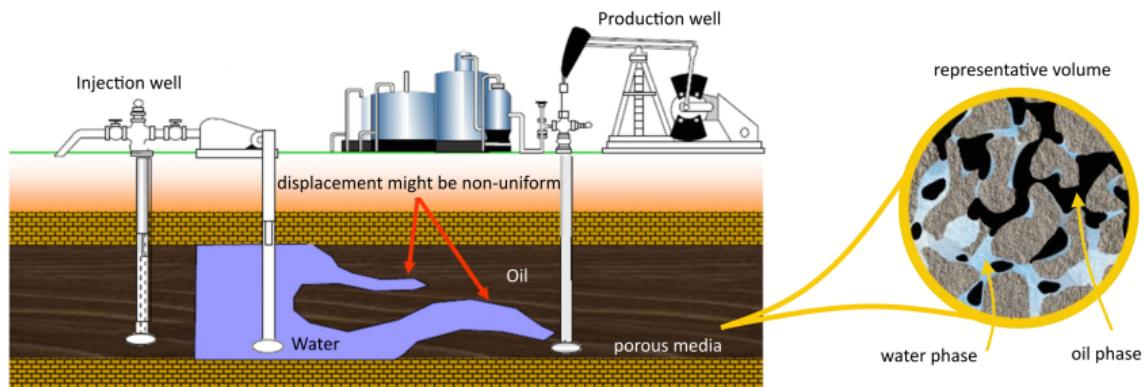


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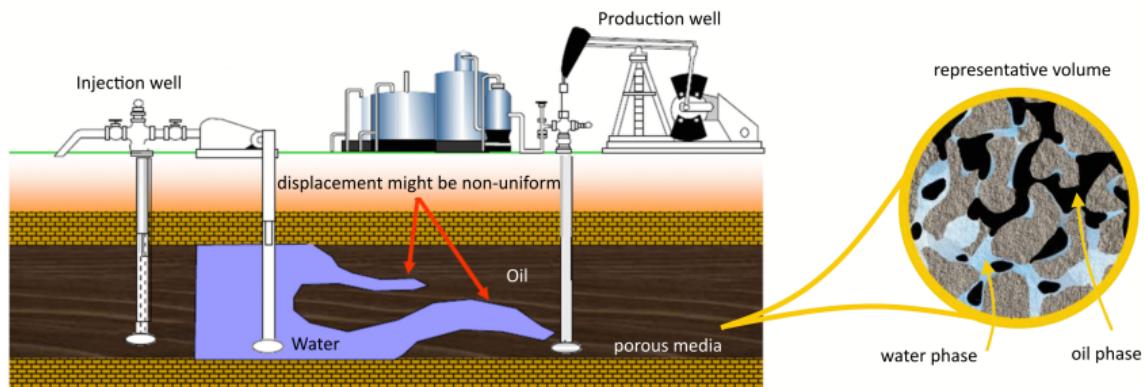


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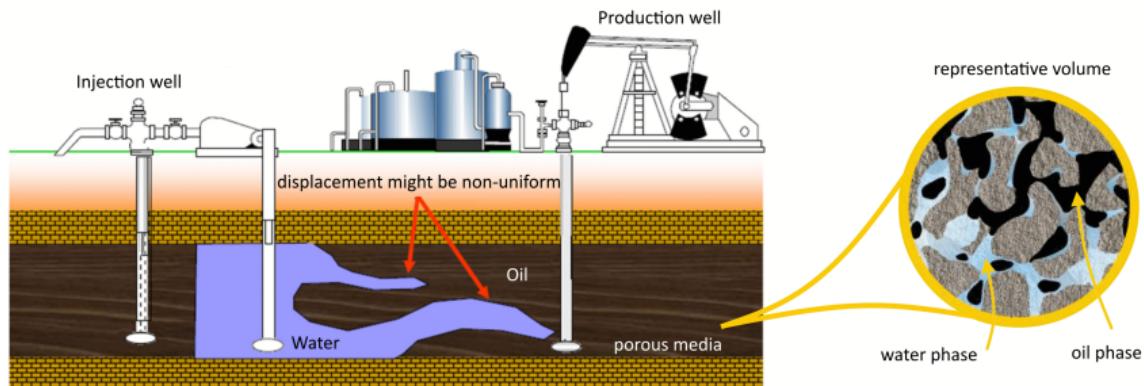


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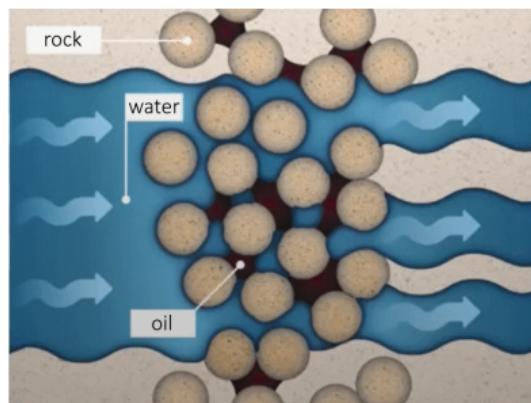
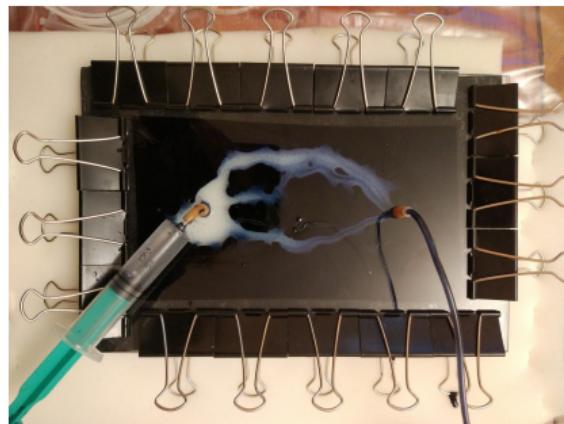
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Problems: macroscopic and microscopic sweep efficiency



- happens due to very viscous oil or inhomogeneous media

- local entrapment of oil in pores due to high capillary pressure

Possible solution

- Inject gas (CO_2 , natural) to decrease the oil viscosity
- Add **chemicals (polymer)** to increase the water viscosity
- Add **chemicals (surfactant)** that reduce the surface tension etc

Fundamental research: two main directions

1-dim in spatial variable

- Stable displacement



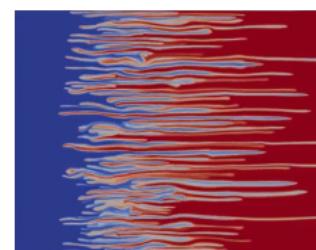
- main question: find an exact solution to a Riemann problem
- **hyperbolic conservation laws**

$$\begin{aligned} s_t + f(s, c)_x &= 0, \\ (cs + a(c))_t + (cf(s, c))_x &= 0. \end{aligned}$$

Example: chemical flooding model

2-dim (or 3-dim) in spatial variable

- Unstable displacement



- source of instability: water and oil/polymer have different viscosities
- **viscous fingering phenomenon**

$$\begin{aligned} c_t + u \cdot \nabla c &= \varepsilon \Delta c, \\ \operatorname{div}(u) &= 0, \\ u &= -\nabla p / \mu(c). \end{aligned}$$

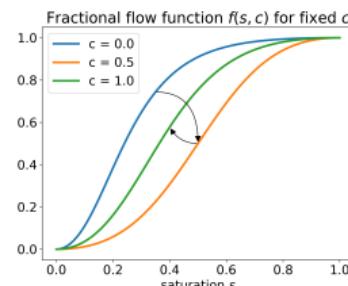
Example: Peaceman model

Problem statement

Chemical flooding can be described as the system of conservation laws ($x \in \mathbb{R}, t > 0$):

$$\begin{aligned} s_t + f(s, c)_x &= 0, && \text{(conservation of water)} \\ (cs + a(c))_t + (cf(s, c))_x &= 0. && \text{(conservation of chemical)} \end{aligned} \tag{1}$$

- $s = s(x, t)$ — water phase saturation;
- $f(s, c)$ — fractional flow function (usually *S*-shaped);
- $c = c(x, t)$ — concentration of a chemical agent in water;
- $a(c)$ — adsorption of a chemical agent on a rock (usually increasing, concave).



Initial data:

$$(s, c)|_{t=0} = \begin{cases} (1, 1), & \text{if } x \leq 0, \\ (0, 0), & \text{if } x > 0, \end{cases} \tag{2}$$

Aim:

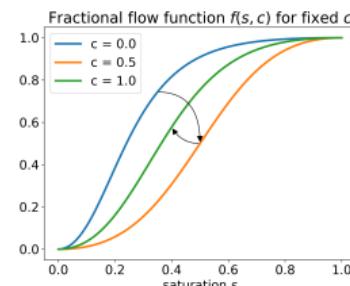
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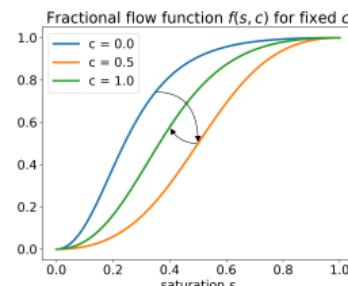
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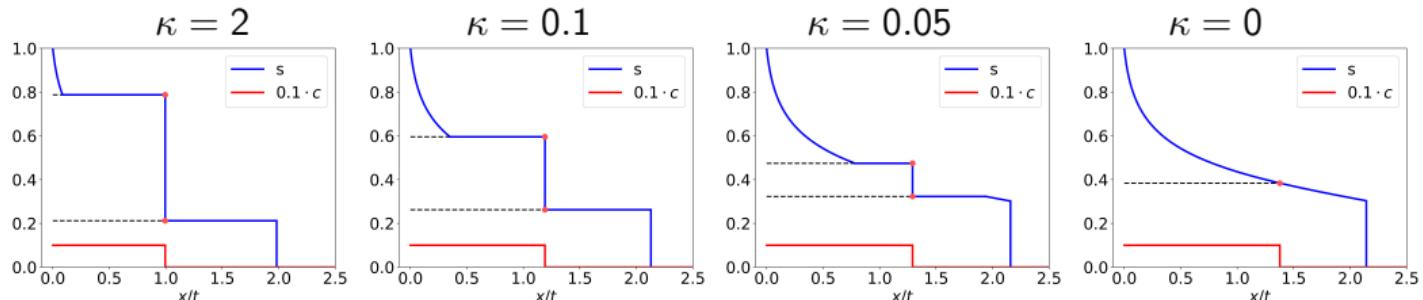
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Preview of results

- weak solutions \Rightarrow non-uniqueness of solutions to a Riemann problem
- use vanishing viscosity criterion — add small diffusion/capillary terms

$$\begin{aligned}s_t + f(s, c)_x &= \varepsilon_c (A(s, c)s_x)_x, \\ (cs + a(c))_t + (cf(s, c))_x &= \varepsilon_c (cA(s, c)s_x)_x + \varepsilon_d c_{xx}.\end{aligned}$$

- $f(s, c)$ monotone in $c \Rightarrow$ uniqueness of vanishing viscosity solution (1988)
- Main idea of the talk:** $f(s, c)$ non-monotone in $c \Rightarrow$ exist multiple vanishing viscosity solutions, depending on ratio $\kappa = \varepsilon_d / \varepsilon_c$



Hyperbolic systems of conservation laws

$$G(u)_t + F(u)_x = 0 \quad (3)$$

Here

- $G(u)$ — accumulation function (conserved quantities)
- $F(u)$ — flux function (flux of conserved quantities)

Simplest example: wave equation

$$y_{tt} - c^2 y_{xx} = 0 \quad (\text{J. d'Alambert, 1750})$$

can be rewritten as a system of two first-order equations on the state-vector $u = \begin{pmatrix} y_x \\ y_t \end{pmatrix}$

$$u_t + Du_x = 0, \quad \text{with} \quad D = \begin{pmatrix} 0 & -1 \\ -c^2 & 0 \end{pmatrix}$$

- eigenvalues $\lambda_1 = c$ and $\lambda_2 = -c$ are real, the system is hyperbolic. Solution contain two wave modes that propagate at the velocities λ_1 and λ_2 .

Hyperbolic systems of conservation laws

$$u_t + \left(\frac{u^2}{2} \right)_x = 0 \quad (\text{Burger's equation, 1948})$$

- non-linearity implies wave speed $\lambda(u) = u$ depends on state u
- So the wave can spread (rarefaction wave) or concentrate (shock wave)



$$u_t + (f(u))_x = 0$$

(Buckley-Leverett equation)

- Typical f is an S-shaped function and solution is rarefaction wave + shock
- Gelfand, Oleinik

Riemann problem (1858)

- Riemann solved the initial-value problem with data having a **single jump**

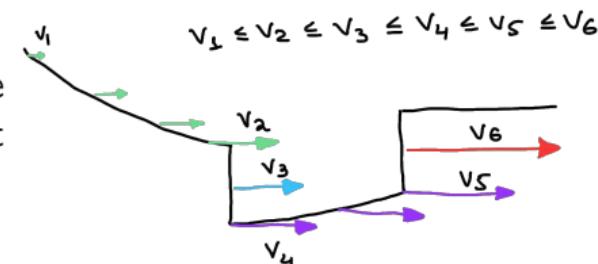
$$u|_{t=0} = \begin{cases} u^L, & x \leq 0; \\ u^R, & x > 0. \end{cases}$$

- took advantage of the **scale invariance** of the equations and the data:

$$u(\alpha x, \alpha t) = u(x, t) \quad \text{for all } \alpha > 0$$

- solution to a Riemann problem is important because:
 - it appears in a long-term behavior of Cauchy problem
 - helps to prove the existence of solutions to Cauchy problem (Glimm's method)
 - helps to construct numerical solution (Godunov method)

Any solution to a Riemann problem consists of a sequence of rarefaction or shock waves (and constant states) that are **compatible by speeds**



Shock waves: RH condition and admissibility criteria

- discontinuous solutions are defined in the sense of distributions (**weak form**)
- for a shock wave from u^- to u^+ moving with velocity v , the weak condition amounts to the following **Rankine-Hugoniot** condition (RH)

$$-v G(u^-) + F(u^-) = -v G(u^+) + F(u^+) \quad (\text{RH})$$

- RH means conservation: what flows into left side flows out of the right side
- Problems from the perspectives of both mathematics and physics:
 - if all RH solutions are allowed, a Riemann problem has multiple solutions
 - some RH solutions violate physical principles
- **Vanishing viscosity criteria:** consider a diffusive system of conservation laws

$$G(u)_t + F(u)_x = \varepsilon [B(u) u_x]_x, \quad \varepsilon \rightarrow 0$$

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Traveling wave solutions of diffusive system (Hopf, 1948)

- $u(x, t) = \hat{u}(\xi)$ with $\xi := x - v t$ for a fixed shock velocity v
- reduction to first-order system of ordinary differential equations:

$$\varepsilon B(\hat{u}) \hat{u}_\xi = -v [G(\hat{u}) - G(u^-)] + F(\hat{u}) - F(u^-)$$

- u^- and u^+ are fixed points and we look for an orbit connecting them

$$\hat{u}(-\infty) = u^-, \quad \hat{u}(+\infty) = u^+$$

- diffusive terms cause a shock wave to have a thin, smooth internal structure as a result of balancing nonlinear focusing and diffusive spreading
- traveling wave solution approaches the jump discontinuity in L^1 as $\varepsilon \rightarrow 0^+$

Historical review: zero adsorption

$$\begin{aligned} s_t + f(s, c)_x &= 0, \\ (sc)_t + (cf(s, c))_x &= 0. \end{aligned} \quad (s, c)|_{t=0} = \begin{cases} (s_L, c_L), & \text{if } x \leq 0, \\ (s_R, c_R), & \text{if } x > 0, \end{cases} \quad (4)$$

Can be rewritten in a form:

$$\begin{aligned} s_t + (sg(s, b))_x &= 0, & b = sc &\text{ — total amount of chemicals} \\ b_t + (bg(s, b))_x &= 0. & g = f/s &\text{ — new flux function} \end{aligned}$$

- 1980 — elasticity theory (Keyfitz, Kranzer): $g(s, b) = \tilde{g}(s^2 + b^2)$.
- 1980 — polymer flooding (Isaacson, Glimm, Temple): $f(s, c)$ monotone in c
- many-many generalisations (delta shocks etc)

Interesting properties:

- has coordinate system of Riemann invariants
- rarefaction and shock curves coincide

Problems:

- non-strictly hyperbolic system
- has contact discontinuities

Work in progress with D. Marchesin, B. Plohr: justification of Isaacson-Glimm admissibility criterion & construction of a solution for the case when $f(s, c)$ is non-monotone in c

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Historical review: non-zero adsorption

$$\begin{aligned} s_t + f(s, c)_x &= 0, \\ (sc + a(c))_t + (cf(s, c))_x &= 0. \end{aligned} \quad (s, c)|_{t=0} = \begin{cases} (s_L, c_L), & \text{if } x \leq 0, \\ (s_R, c_R), & \text{if } x > 0, \end{cases} \quad (5)$$

Proposition (Johansen-Winther, 1988 (JW))

The solution $c(x, t)$ is monotone in x for any $t > 0$. If $a(c)$ is concave, then

- is constant if $c_L = c_R$;
- contains several rarefaction waves if $c_L < c_R$;
- contains exactly one shock wave if $c_L > c_R$.

Historical review:

- 1988 — JW, $f(s, c)$ monotone in c . Found a unique vanishing viscosity solution.

When $f(s, c)$ non-monotone in c , multiple vanishing viscosity solutions are possible.

- 2017 — W. Shen, gave some examples in Lagrangian coordinates. See also
- 1986 — Entov, Kerimov, non-rigorous consideration of the non-monotone case.

Reduced problem

$$\begin{aligned} s_t + f(s, c)_x &= 0, \\ (sc + a(c))_t + (cf(s, c))_x &= 0. \end{aligned} \quad (s, c)|_{t=0} = \begin{cases} (1, 1), & \text{if } x \leq 0, \\ (0, 0), & \text{if } x > 0, \end{cases} \quad (6)$$

Proposition (Johansen-Winther, 1988 (JW))

There exists $u^- = (s^-, 1)$ and $u^+ = (s^+, 0)$ such that the solution to a Riemann problem has the following structure:

$$(1, 1) \xrightarrow{c=1} u^- \xrightarrow{c \text{ jumps from 1 to 0}} u^+ \xrightarrow{c=0} (0, 0). \quad (7)$$

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Dissipative system

To define a shock wave between u^- and u^+ we consider dissipative system:

$$\begin{aligned}s_t + f(s, c)_x &= \varepsilon_c (A(s, c)s_x)_x, \\(cs + \alpha)_t + (cf(s, c))_x &= \varepsilon_c (cA(s, c)s_x)_x + \varepsilon_d c_{xx}, \\ \alpha_t &= \varepsilon_r^{-1} (a(c) - \alpha).\end{aligned}$$

- ε_c — dimensionless capillary pressure
- ε_d — dimensionless diffusion term
- ε_r — dimensionless relaxation time
- $A(s, c)$ — capillary pressure term
- $\alpha = \alpha(x, t)$ — dynamic adsorption

We consider two particular cases:

Capillarity and Diffusion

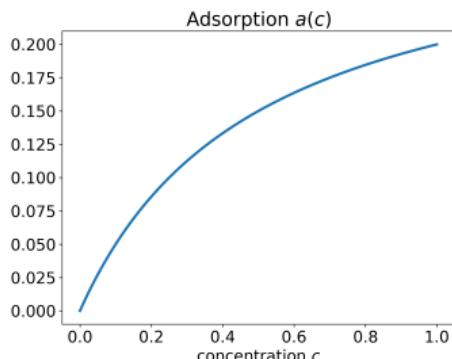
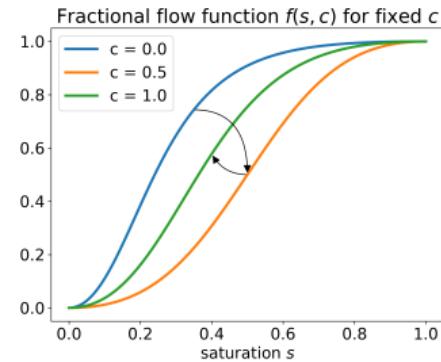
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Capillarity and Dynamic Adsorption

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Restrictions on f and a

- (F1) $f \in C^2([0, 1]^2); f(0, c) = 0; f(1, c) = 1;$
- (F2) $f_s(s, c) > 0$ for $s \in (0, 1)$, $c \in [0, 1]$;
 $f_s(0, c) = f_s(1, c) = 0$;
- (F3) f is S-shaped in s ;
- (F4) f is non-monotone in c :
 $\forall s \in (0, 1) \exists c^*(s) \in (0, 1)$:
 - $f_c(s, c) < 0$ for $0 < s < 1$, $0 < c < c^*(s)$;
 - $f_c(s, c) > 0$ for $0 < s < 1$, $c^*(s) < c < 1$;



- (A) A is bounded from zero and infinity;
- (a) $a \in C^2$, $a(0) = 0$, a is strictly increasing and concave.

NB: assumptions are just the simplest possible ones. Could be generalised.

Travelling wave dynamical system

$$\begin{aligned}s_t + f(s, c)_x &= \varepsilon_c(A(s, c)s_x)_x, \\ (cs + a(c))_t + (cf(s, c))_x &= \varepsilon_c(cA(s, c)s_x)_x + \varepsilon_d c_{xx}.\end{aligned}$$

Searching for travelling wave solutions $s = s(\xi)$, $c = c(\xi)$, $\xi := \varepsilon_c^{-1}(x - vt)$ with boundary conditions

$$s(\pm\infty) = s^\pm, \quad c(-\infty) = 1, \quad c(+\infty) = 0,$$

we arrive at

$$\begin{aligned}A(s, c)s_\xi &= f(s, c) - v(s + d_1), \\ \kappa c_\xi &= v(d_1 c - d_2 - a(c)).\end{aligned}\tag{8}$$

- Parameters: $\kappa = \varepsilon_d/\varepsilon_c$ and v .
- Note that u^\pm are fixed points of dynamical system (8);
- We are only interested in the trajectories connecting two saddle points (or saddle-nodes) due to compatibility of speeds in

$$(1, 1) \rightarrow u^- \xrightarrow{\text{c-shock}} u^+ \rightarrow (0, 0).$$

Main result

Consider a dynamical system under assumptions (F1)–(F4), (A), (a):

$$\begin{aligned} A(s, c)s_\xi &= f(s, c) - v(s + d_1), \\ \kappa c_\xi &= v(d_1 c - d_2 - a(c)). \end{aligned}$$

Theorem (Bakharev, Enin, P., Rastegaev, 2021, arxiv:2111.15001)

There exist $0 < v_{\min} < v_{\max} < \infty$, such that for every $\kappa = \varepsilon_d/\varepsilon_c \in (0, +\infty)$, there exist unique

- points $s^-(\kappa) \in [0, 1]$ and $s^+(\kappa) \in [0, 1]$;
- velocity $v(\kappa) \in [v_{\min}, v_{\max}]$,

such that there exists a travelling wave, connecting two saddle points $u^-(\kappa) = (s^-(\kappa), 1)$ and $u^+(\kappa) = (s^+(\kappa), 0)$ with velocity $v(\kappa)$. Moreover, $v(\kappa)$ is monotone and continuous; $v(\kappa) \rightarrow v_{\min}$ as $\kappa \rightarrow \infty$; $v(\kappa) \rightarrow v_{\max}$ as $\kappa \rightarrow 0$.

Scheme of proof

The Theorem can be divided into simpler statements:

- $\forall v \in [v_{\min}, v_{\max}] \quad \exists! \kappa(v)$: there is a saddle-to-saddle travelling wave with $\kappa(v)$.
- $\kappa(v)$ is continuous.
- $\nexists v_1 \neq v_2 : \kappa(v_1) = \kappa(v_2)$, thus $\kappa(v)$ is monotone.
- $\kappa(v) \rightarrow \infty$ as $v \rightarrow v_{\min}$.
- $\kappa(v) \rightarrow \kappa_{\text{crit}} \geq 0$ as $v \rightarrow v_{\max}$.
- When $\kappa < \kappa_{\text{crit}}$ and $v = v_{\max}$ there is a saddle to saddle-node travelling wave

$\kappa(v)$ is monotone and continuous thus there exists an inverse function satisfying the Theorem.

Phase portrait, fixed points, isoclines

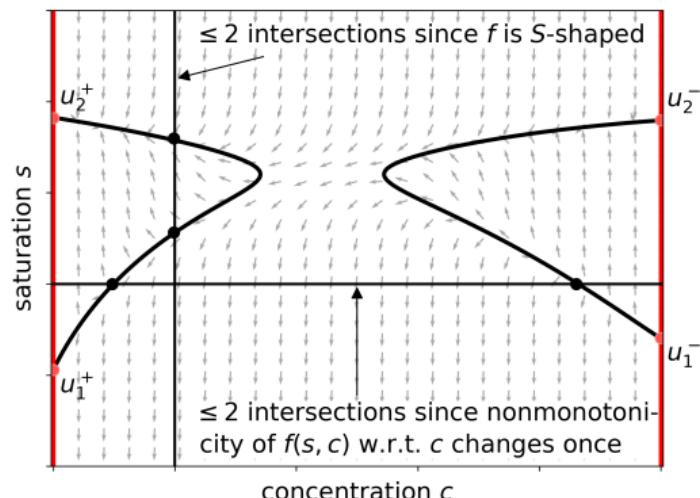
We consider travelling wave dynamical system:

$$\begin{aligned} A(s, c)s_\xi &= f(s, c) - v(s + d_1), \\ \kappa c_\xi &= v(d_1c - d_2 - a(c)), \end{aligned}$$

Isoclines:

red lines are $d_1c - d_2 - a(c) = 0$,
black lines are $f(s, c) - v(s + d_1) = 0$.

Fixed points:
 u_1^+ and u_2^- — saddle points;
 u_2^+ — attractor; u_1^- — repeller



Aim:

find all pairs (κ, v) for which there exists a trajectory
from saddle point u_2^- to saddle point u_1^+

Phase portraits types. Main and intermediate classes

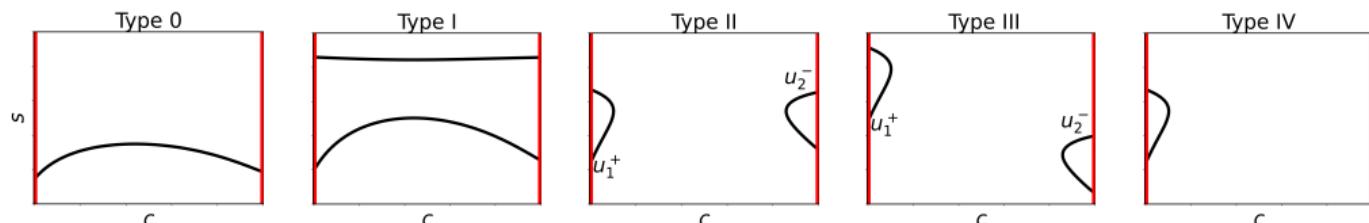


Figure 1: Five wide classes of phase portraits

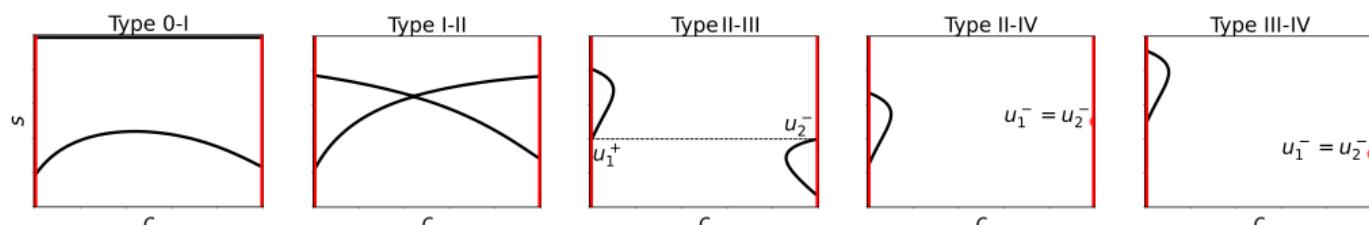


Figure 2: Intermediate types of phase portraits, appearing under the assumptions (F1)–(F4)

- Only Type II phase portrait has saddle-to-saddle connections.
- Type I-II corresponds to v_{\min} .
- Type II-III or Type II-IV correspond to v_{\max} .

Phase portraits: monotone dependence on v

black lines $f(s, c) = v(s + d_1)$

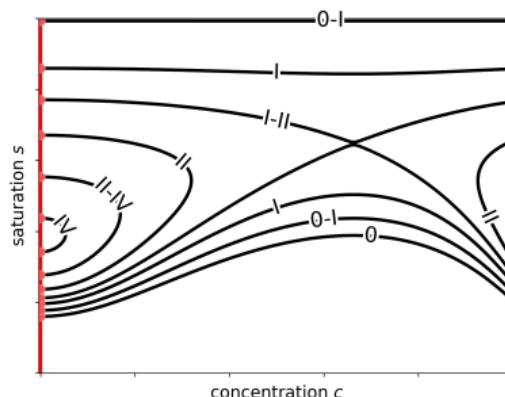
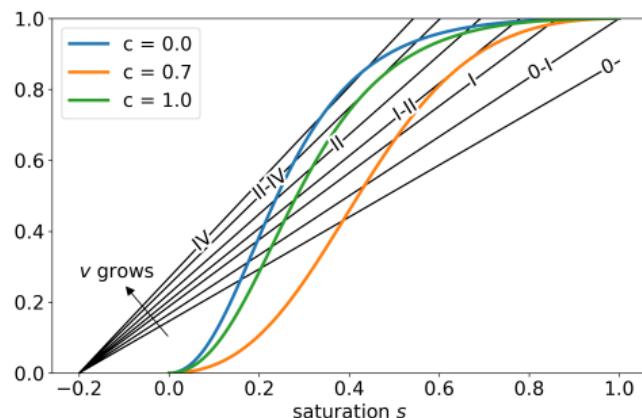


Figure 3: Phase portrait evolution as v grows: Type 0 → Type I → Type II → Type IV

Phase portraits: bad cases

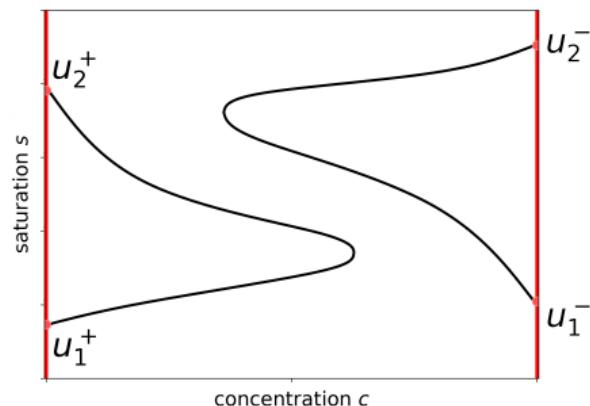


Figure 4: If f is not S-shaped.

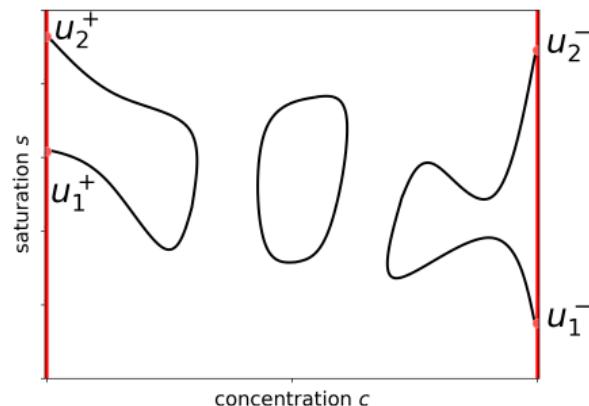


Figure 5: If non-monotonicity is more complex.

We believe that the similar result is true without conditions (F3)–(F4).

Type II portrait: for every v there exist κ

$$\frac{s_\xi}{c_\xi} = \kappa \cdot \frac{v^{-1}f(s, c) - (s + d_1)}{A(s, c)(d_1 c - d_2 - a(c))}$$

Used property: continuous and monotonous dependence of trajectories on κ .

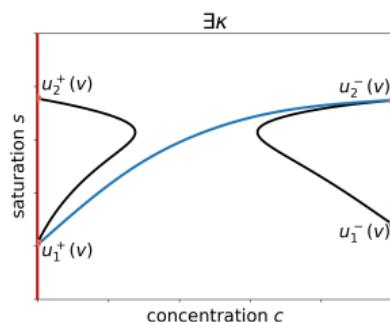
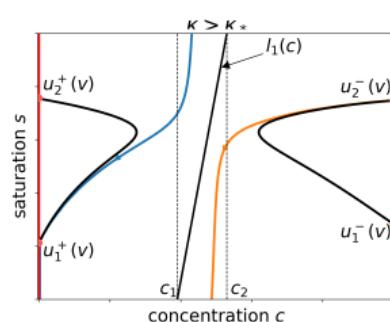
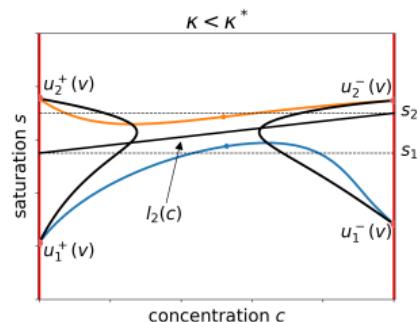


Figure 6: $\kappa \ll 1$.

Figure 7: $\kappa \gg 1$.

Figure 8: $\exists \kappa$.

Type II portrait: $\kappa(v)$ is unique for every $v \in (v_{\min}, v_{\max})$.

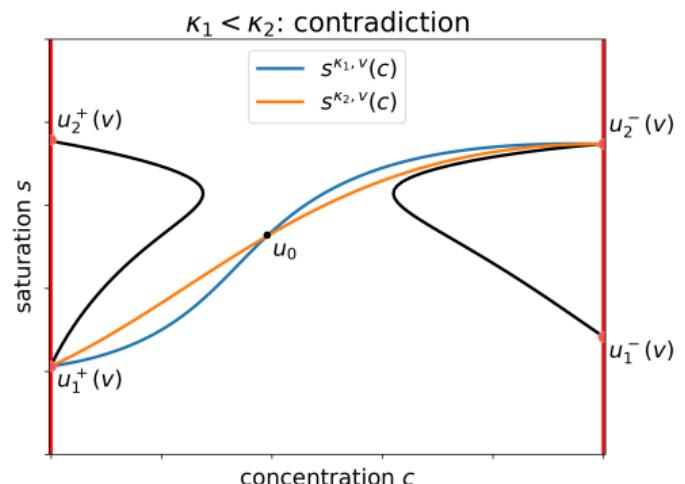
If there are $\kappa_1 < \kappa_2$ for one v , then the corresponding trajectories must intersect, which leads to a contradiction.

The slope

$$\frac{s_\xi}{c_\xi} = \kappa \cdot \frac{v^{-1}f(s, c) - (s + d_1)}{A(s, c)(d_1c - d_2 - a(c))}$$

is positive at the intersection point (s, c) , so it strictly increases when κ increases.

NB: this property might be lost for more complex phase portraits.

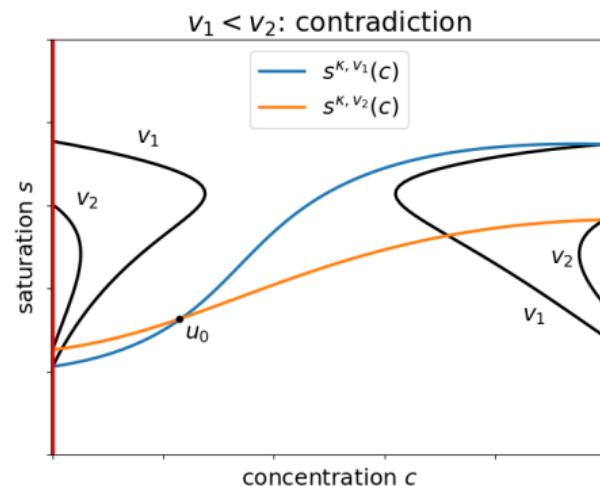


Type II portrait: monotonicity of $\kappa(v)$

If $\kappa(v_1) = \kappa(v_2)$ for $v_1 < v_2$, then the corresponding trajectories must intersect, which leads to a contradiction. The slope

$$\frac{s_\xi}{c_\xi} = \kappa \cdot \frac{v^{-1}f(s, c) - (s + d_1)}{A(s, c)(d_1c - d_2 - a(c))}$$

is positive at the intersection point (s, c) , so it strictly increases when v increases.



Solution construction algorithm

1. From κ we calculate $v(\kappa)$ (binary search).
2. From v we determine $s^-(v)$ and $s^+(v)$ via Rankine-Hugoniot condition.
3. Construct waves $(1, 1) \rightarrow (s^-(v), 1)$ and $(s(v), 0) \rightarrow (0, 0)$.

Example: “boomerang” model:

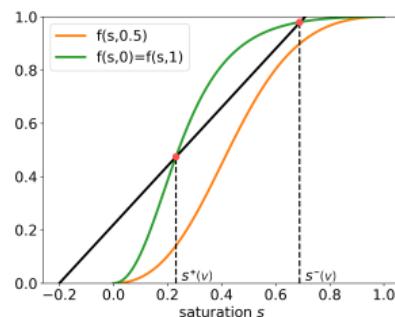


Figure 9: Flux functions

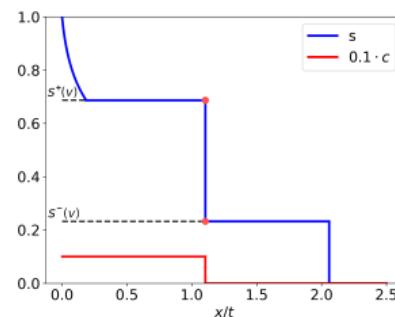


Figure 10: Solution s and c

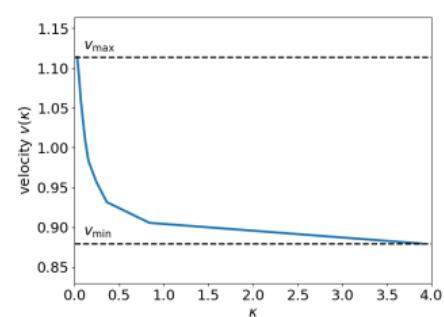


Figure 11: Function $v(\kappa)$

Possible directions for future research

- System with three small parameters $\varepsilon_c, \varepsilon_d, \varepsilon_r \rightarrow 0$
- General classes of f and a
- Construct solution to any Riemann problem
- Asymptotic stability as $t \rightarrow \infty$:
is it true that a solution of a Cauchy problem with some «good» initial data tends to a solution of a Riemann problem?

Muito obrigada pela atenção!

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Literature

Own works:

- F. Bakharev, A. Enin, Yu. Petrova, N. Rastegaev, Impact of dissipation ratio on vanishing viscosity solutions of the Riemann problem for chemical flooding model. arXiv:2111.15001.

Other works:

- Johansen, T. and Winther, R., 1988. The solution of the Riemann problem for a hyperbolic system of conservation laws modeling polymer flooding. SIAM journal on mathematical analysis, 19(3), pp.541-566.
- Shen, W., 2017. On the uniqueness of vanishing viscosity solutions for Riemann problems for polymer flooding. Nonlinear Differential Equations and Applications NoDEA, 24(4), pp.1-25.
- Entov, V.M. and Kerimov, Z.A., 1986. Displacement of oil by an active solution with a nonmonotonic effect on the flow distribution function. Fluid Dynamics, 21(1), pp.64-70.