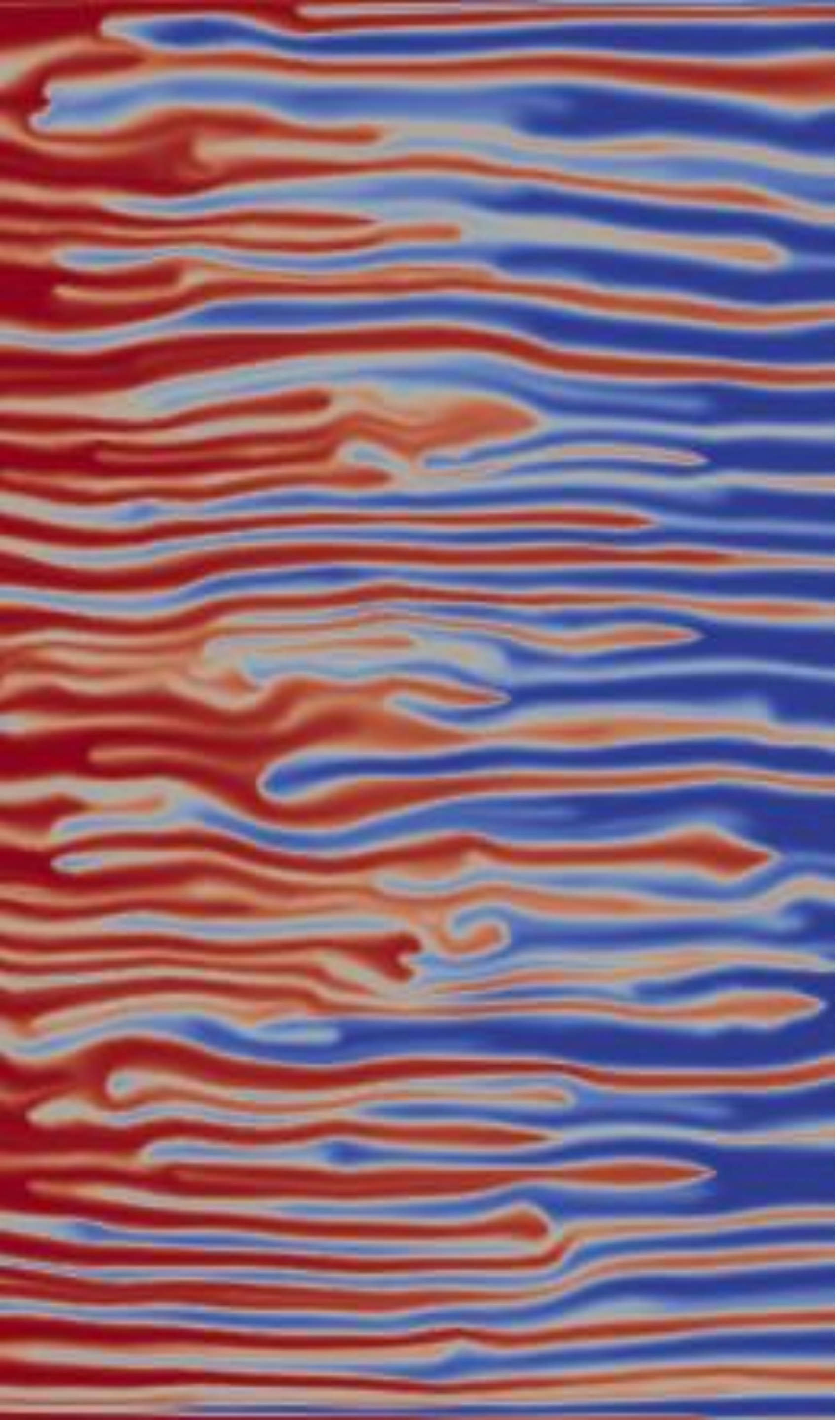
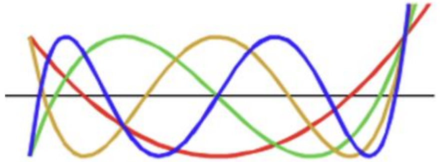


On the linear growth of the mixing zone in a semi-discrete model of Incompressible Porous Medium (IPM) eq



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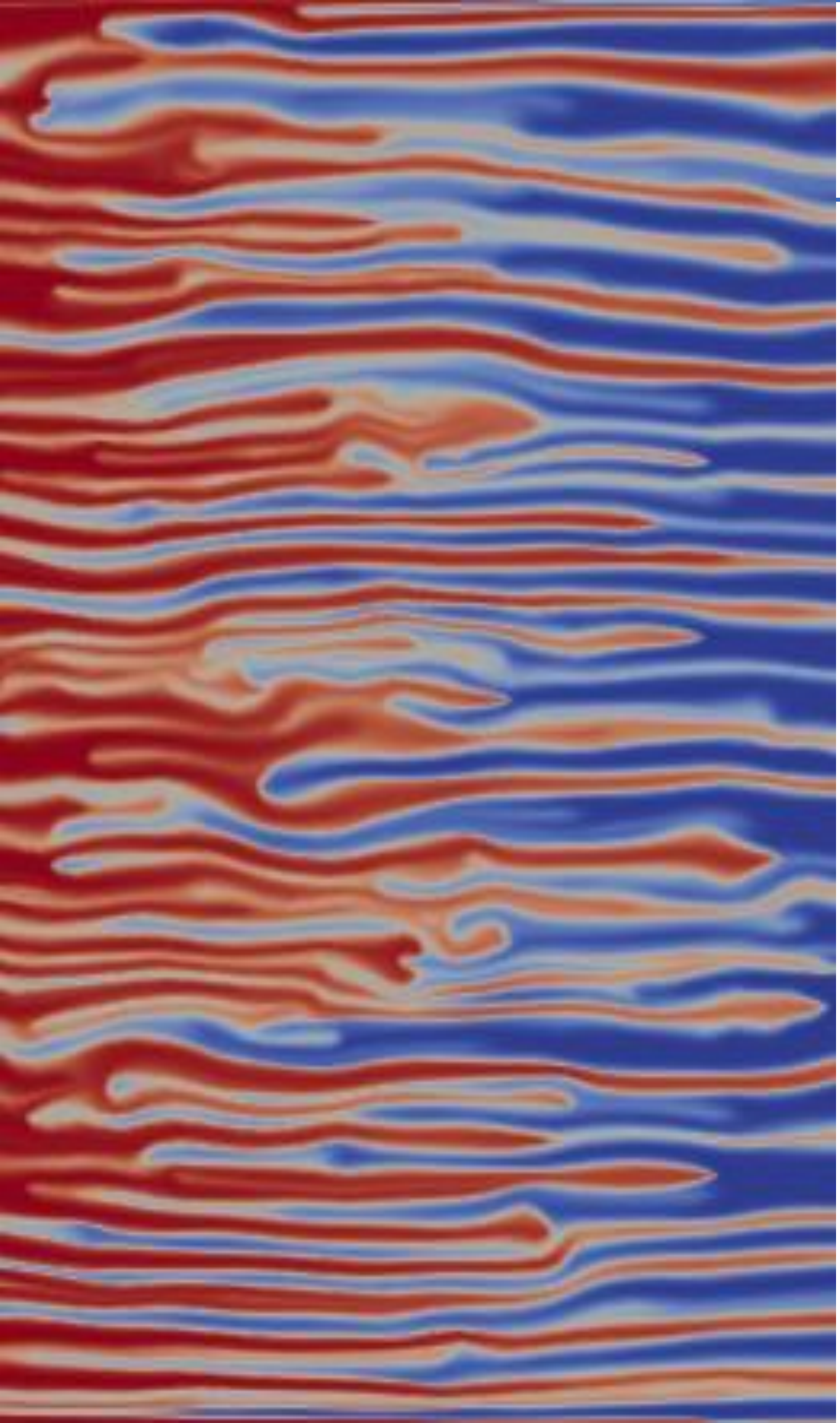
11 June 2024

Talk is based on:

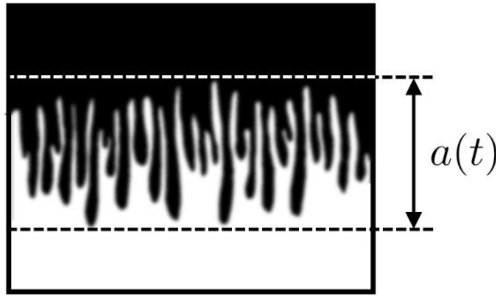
1. Yu. Petrova, S. Tikhomirov, Ya. Efendiev, arXiv: 2401.05981.
   "Propagating terrace in a two-tubes model of gravitational fingering"
2. F. Bakharev, A. Enin, S. Matveenko, D. Pavlov, Yu. Petrova, N. Rastegaev, S. Tikhomirov, arXiv:2310.14260.
   "Velocity of viscous fingers in miscible displacement: Intermediate concentration"



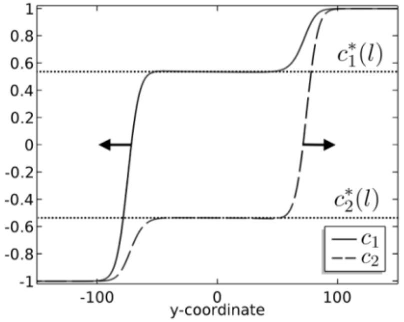
Outline

1. Motivation

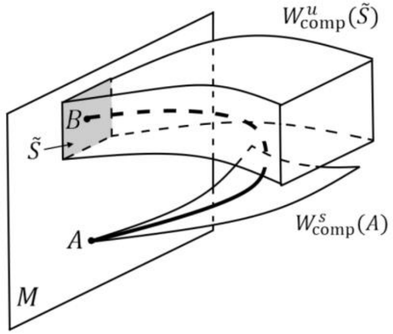
   Miscible displacement in porous media:
   - viscous fingering
   - gravitational fingering

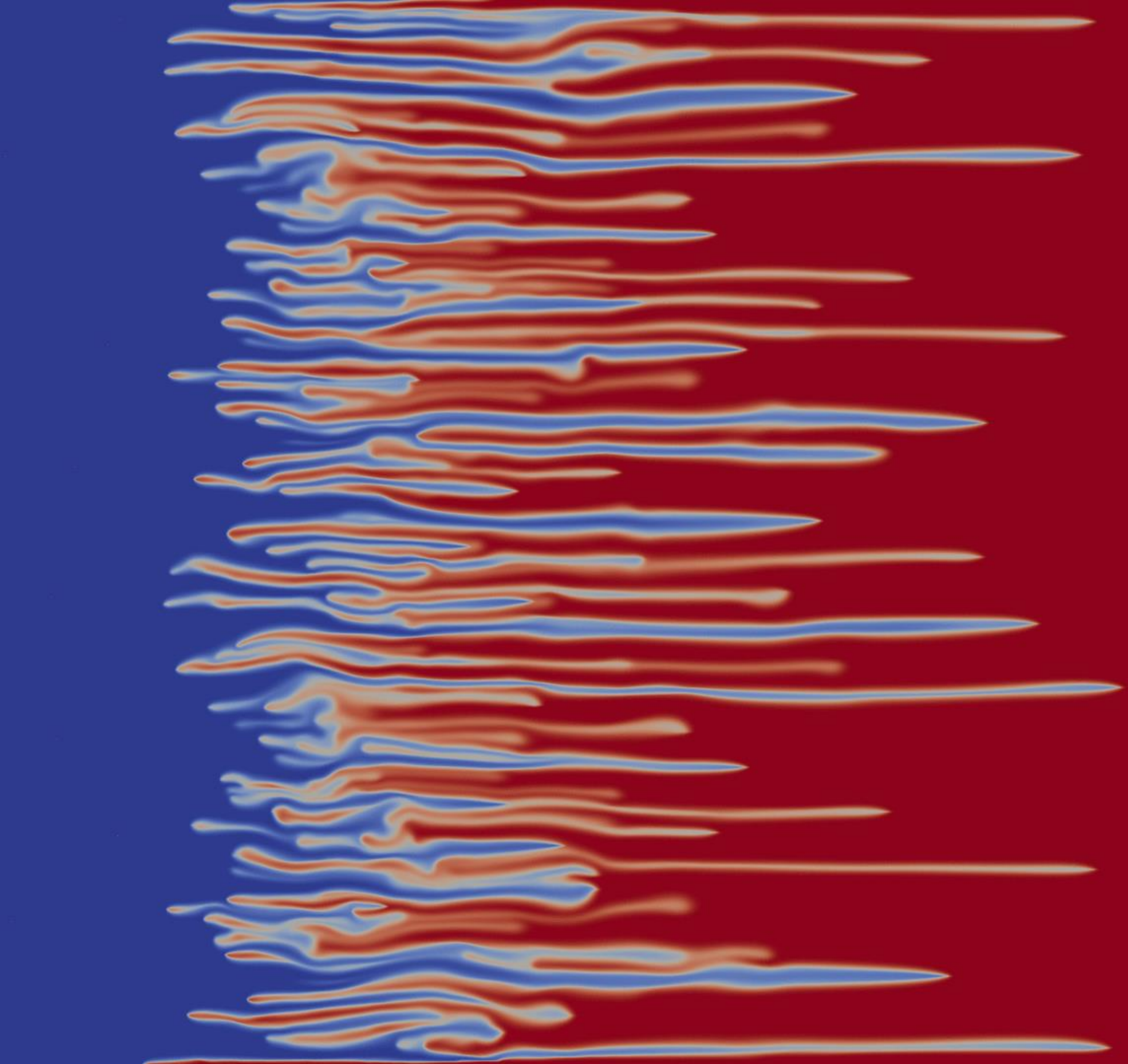


2. Problem statement
   - Two-tubes model
   - Main theorem



3. Sketch of the proof of the main theorem
   - Traveling waves
   - Geometric singular perturbation theory





"Miscible displacement in porous media"
Credit: Pavlov Dmitrii, St. Petersburg State University

Homsy , 1987 "Viscous Fingering in Porous Media"

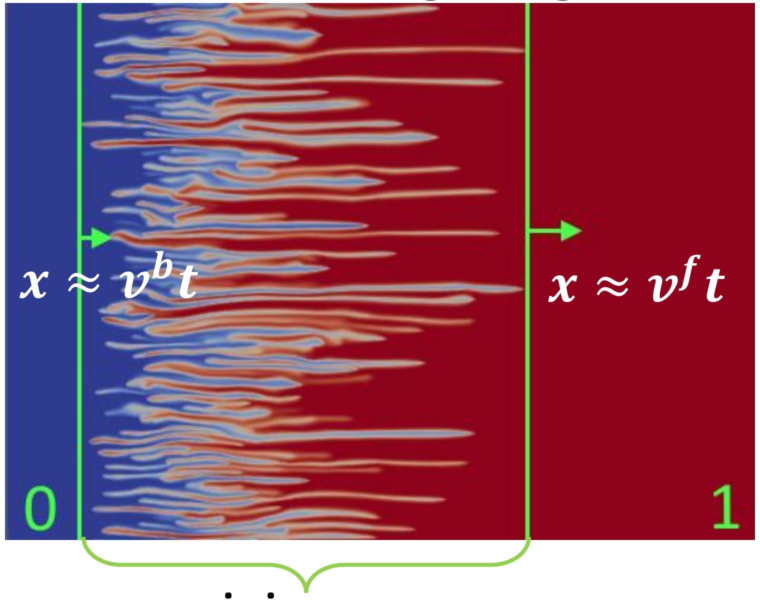
Viscous fingering phenomenon

water (blue color)

polymerized water (red color) 1

Incompressible Porous Medium eq – IPM, 2D (Two formulations)

Viscous fingering



$$c_t + div(uc) = \varepsilon \cdot \Delta c$$

$$div(u) = 0$$

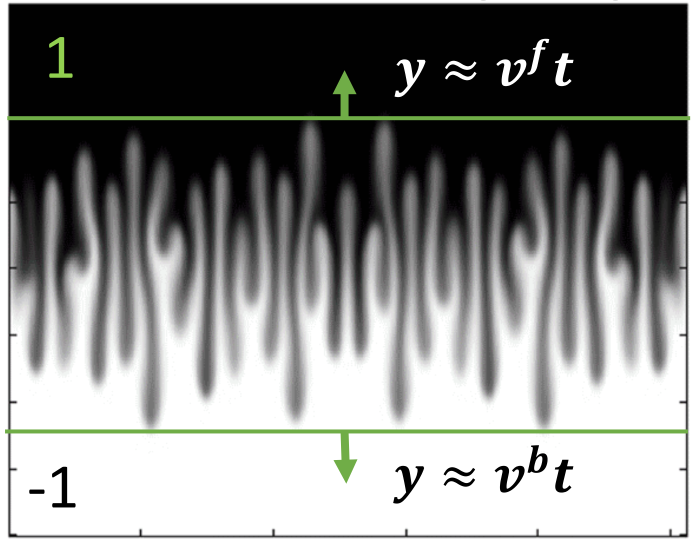
(viscosity) $\quad u = -m(c)\, K\, \nabla p$

(gravity) $\quad u = -\nabla p - (0, c)$

$c = c(t, x, y)$ – concentration
$u = u(t, x, y)$ – velocity
$p = p(t, x, y)$ – pressure

$\varepsilon \geq 0$ – diffusion
$m(c)$ – mobility
K – permeability

Gravitational fingering



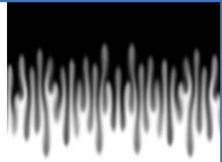
- many laboratory and numerical experiments show *linear growth of the mixing zone* [1], [2]

Question: how to find speeds v^b and v^f of propagation?

[1] Nijjer J., Hewitt D., and Neufeld J. The dynamics of miscible viscous fingering from onset to shutdown. Journal of Fluid Mechanics, 2018.

*[2] Bakharev, F., Enin, A., Groman, A., Kalyuzhnyuk, A., Matveenko, S., **Petrova, Y.**, Starkov, I. and Tikhomirov, S., Velocity of viscous fingers in miscible displacement: Comparison with analytical models. Journal of Computational and Applied Mathematics, 2022.*

IPM: $\varepsilon = 0$ (without diffusion)



Active scalar:

$$\boxed{\begin{aligned} &c_t + u \cdot \nabla c = 0 \\ &u = A(c) \end{aligned}}$$

$$u = \nabla^{\perp}(-\Delta)^{-1}\partial_1 c \qquad \text{(Biot-Savart law)}$$

<u>Discontinuous initial data</u>: free boundary problem (Muskat problem) – ill-posed for unstable stratification

2011 - A. Córdoba, D. Córdoba, F. Gancedo (Annals of Mathematics)
"Interface evolution: the Hele-Shaw and Muskat problems"

<u>Existence</u>: smooth initial data

2007 – D. Cordoba, F. Gancedo, R. Orive (JMP): local well-posedness for initial data H^s

global solution vs finite-time blow-up? open

2017 – Elgindi (ARMA): global solution for small perturbations of $c = -y$

2023 – Kiselev, Yao (ARMA): if solutions stay "smooth" for all times, then there is blow-up at $t = +\infty$

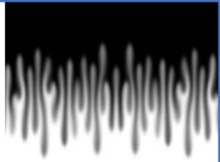
<u>Uniqueness</u>: non-uniqueness of weak solutions – by convex integration

2011 – Córdoba, Faraco, Gancedo (ARMA)

2012 – L. Szekelyhidi Jr.

…and many others…

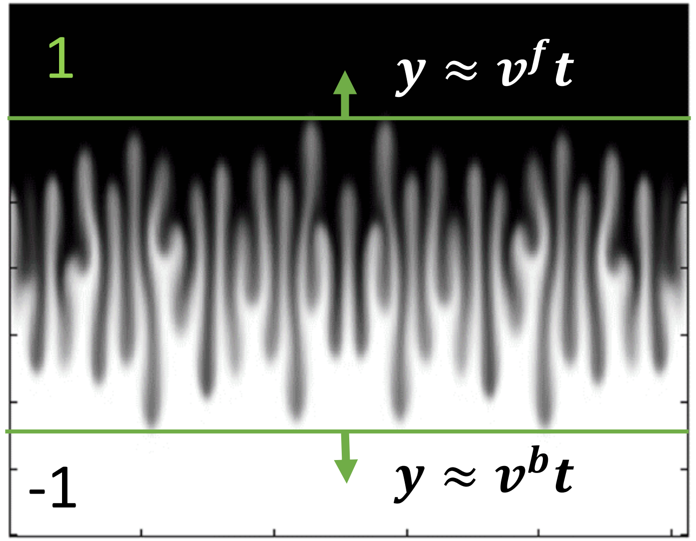
IPM: $\varepsilon > 0$ (with diffusion)



Estimates on the growth:

2005 - Otto, Menon. Proved estimates

- Full model (IPM) $v^f \le 2$
- Simplified model (TFE) $v^f \le 1$



Transverse Flow Equilibrium = TFE

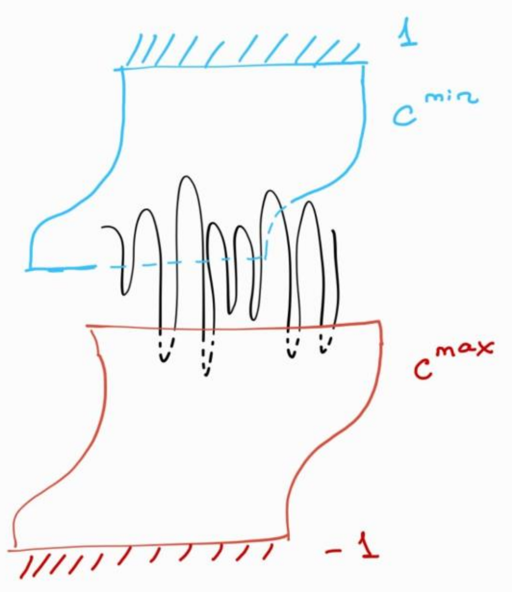
$p(t,x,y) \approx p(t,y)$

$$\boxed{\begin{gathered} c_t + u \cdot \nabla c = \varepsilon \Delta c \\ div(u) = 0 \\ u = (u^1, u^2), \quad u^2 = \bar{c} - c \end{gathered}}$$

Idea of proof (TFE): comparison to 1D Burgers eq ($\bar{c} \le 1$ then $u^2 \le 1 - c$)

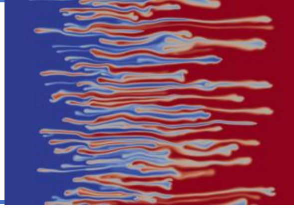
$$\boxed{c_t^{\max} + (1 - c^{\max}) \cdot \partial_y c^{\max} = \varepsilon c_{yy}^{\max}}$$

Theorem (Otto, Menon): If $c(0,x,y) \le c^{\max}(0,y)$,
then $c(t,x,y) \le c^{\max}(t,y)$ for any $t > 0$.



Question: Are those estimates sharp?

Viscosity-driven fingers



$$c_t + div(uc) = \varepsilon \cdot \Delta c$$
$$div(u) = 0$$
$$u = -m(c)\,\nabla p = -1/\mu(c)\nabla p$$

TFE model (viscosity case)

$$u = (u^1, u^2), \quad u^2 = \frac{m(c)}{\int_0^1 m(c)}$$

- Empirical models of velocities:

Koval (1963)	$v^f = M_e \qquad v^b = 1/M_e \qquad M_e = \left(\alpha \cdot M^{0.25} + (1-\alpha)\right)^4$
Todd-Longstaff (1972)	$v^f = M_e \qquad v^b = 1/M_e \qquad M_e = M^\omega$
TFE model (Yortsos, 1995, 2005)	$v^f \le \frac{\overline{m}(0,1)}{m(1)} \qquad v^b \ge \frac{v^f}{M} \qquad \overline{m}(c_1,c_2) = \frac{1}{c_2 - c_1}\int_{c_1}^{c_2} m(c)dc$

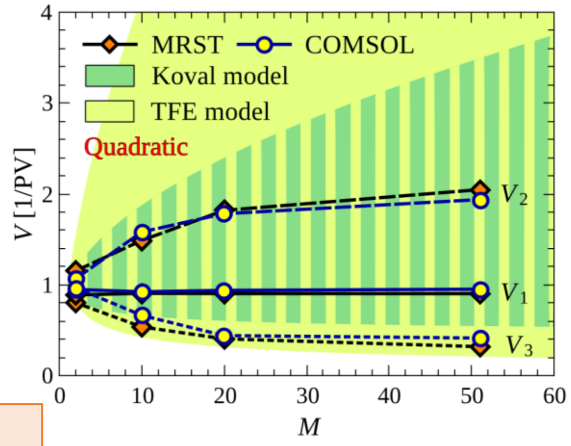
- Ratio of viscosities $M = \frac{\mu(1)}{\mu(0)}$
- "Effective viscosity" M_e

- Yortsos, Salin (2006): comparison to Burgers-type 1D eq:

$$c_t^{\max} + \frac{m(c^{\max})}{m(1)} \cdot \partial_y c^{\max} = \varepsilon c_{yy}^{\max}$$

- If $\quad c(0,x,y) \le c^{\max}(0,y)$,
  then $\quad c(t,x,y) \le c^{\max}(t,y)$ for any $t > 0$.

TFE estimates are too pessimistic!



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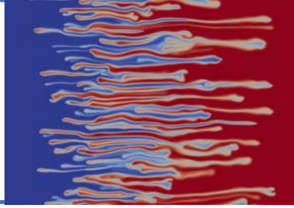

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- Ratio of viscosities

$$M = \frac{\mu(1)}{\mu(0)}$$

- "Effective viscosity"

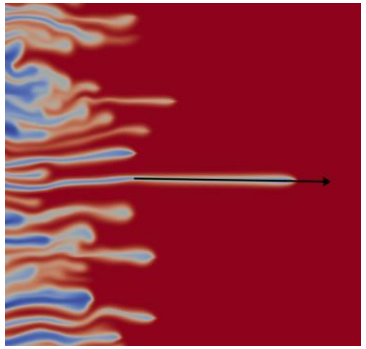
$$M_e$$

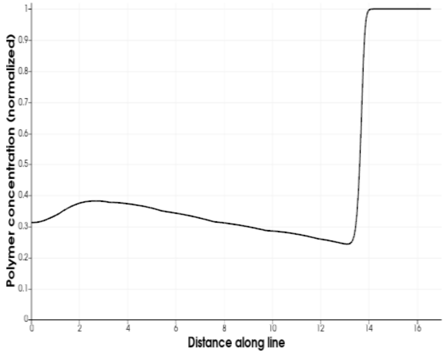
- What could be a mechanism of slow-down?

$$X^f(t, C) = \max_x \{\exists y \colon c(t, x, y) \le C\}$$

$$X^f(t, C) \sim v^f(C) \cdot t$$

$$\boxed{v^{TFE}(C) = \frac{\overline{m}(C, 1)}{m(1)}}$$





Theorem TFE, viscosity (2024, arXiv:2401.05981)

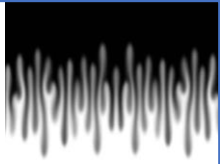
If there exists $C_1 \in [0,1]$ and $l_1 \in \mathbb{R}$:

$$X^f(t, C_1) \le v^{TFE}(C_1) \cdot t + l_1$$

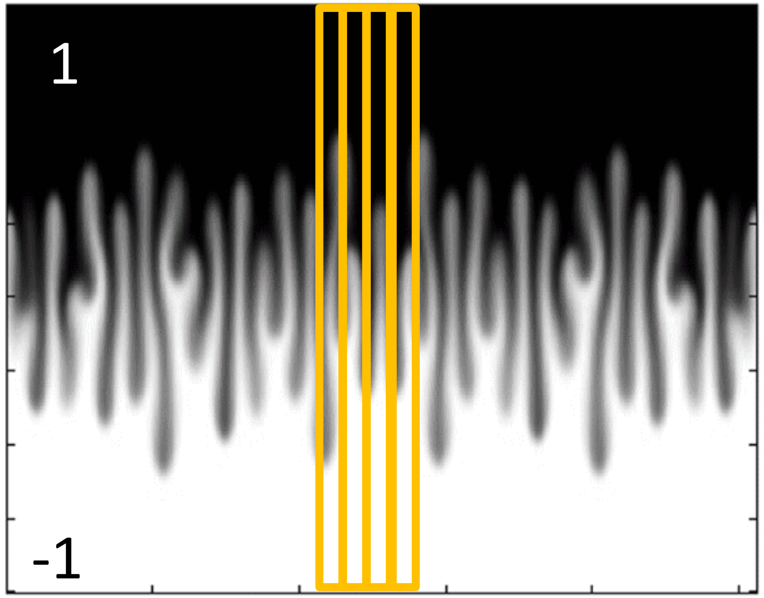
Then for any $C_2 > C_1$ there exists l_2

$$X^f(t, C_2) \le v^{TFE}(C_1) \cdot t + l_2$$

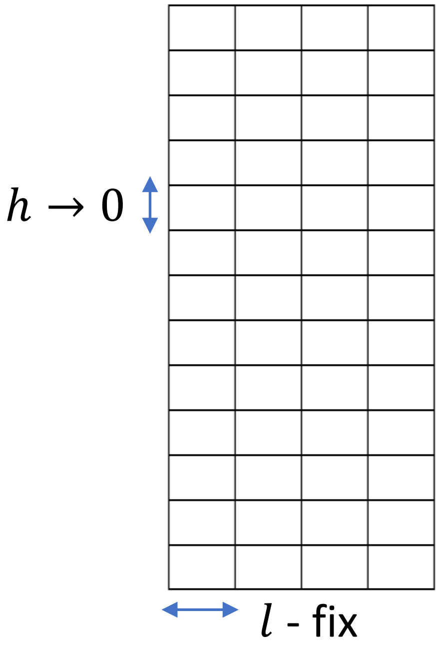
IDEA: semi-discrete model of gravitational fingering



- Discretize in horizontal direction
- Take n tubes, $n = 2,3,4, \ldots$

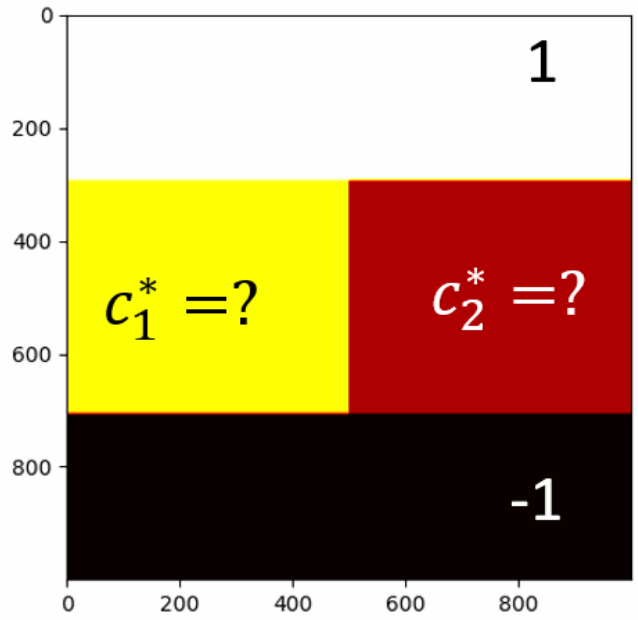


Limit of
numerical scheme



- Finite volume
- Upwind

- For simplicity, $n = 2$



$y \approx v^f \cdot t$

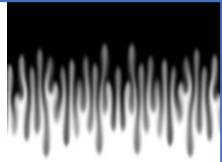
$y \approx v^b \cdot t$

We observe two travelling waves:
$c(y,t) = c(y - vt)$

Tubes (layer, lane,…) models:

2019 — A. Armiti-Juber, C. Rohde “On Darcy- and Brinkman-type models for two-phase flow in asympt. flat domains”
2006 — J.C. Da Mota, S. Schecter “Combustion fronts in a porous medium with two layers”
2019 — H. Holden, N. Risebro “Models for dense multilane vehicular traffic”

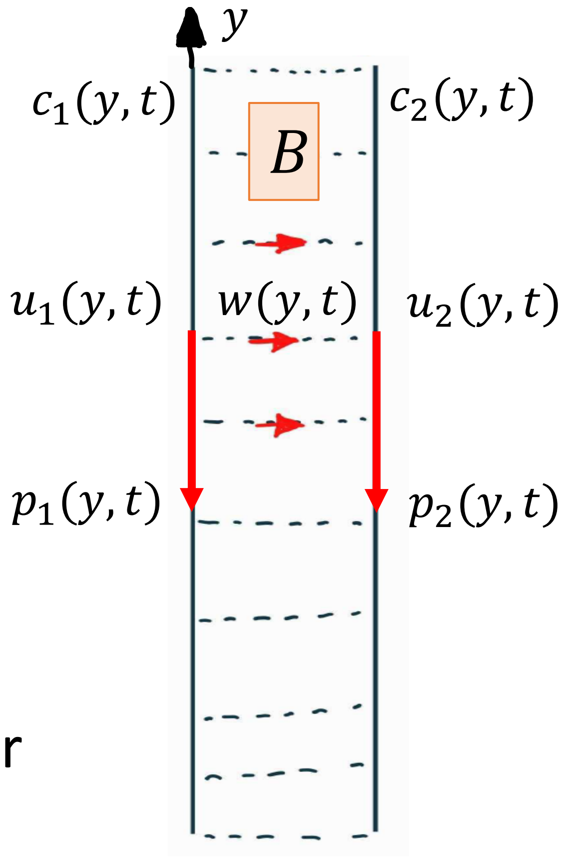
Two-tubes model



1. Original equation on c: $c_t + div(uc) - \Delta c = 0$

   Two-tubes equations on c:

$$\partial_t c_1 + \partial_y(u_1 c_1) - \partial_{yy} c_1 = \boxed{-B}$$
$$\partial_t c_2 + \partial_y(u_2 c_2) - \partial_{yy} c_2 = \boxed{+B}$$



2. Original equation on p: $u = -\nabla p - (0, c)$

   Two-tubes equations on p:

$$u_1 = -\partial_y p_1 - c_1$$
$$u_2 = -\partial_y p_2 - c_2$$
$$w = \frac{p_1 - p_2}{l^2}$$

l - parameter

3. Original equation on u: $div(u) = 0$

   Two-tubes equations on u:

$$\partial_y u_1 + w = 0$$

$$\boxed{B} = \begin{cases} w \cdot c_1, & w > 0, \\ w \cdot c_2, & w < 0 \end{cases}$$

Main result

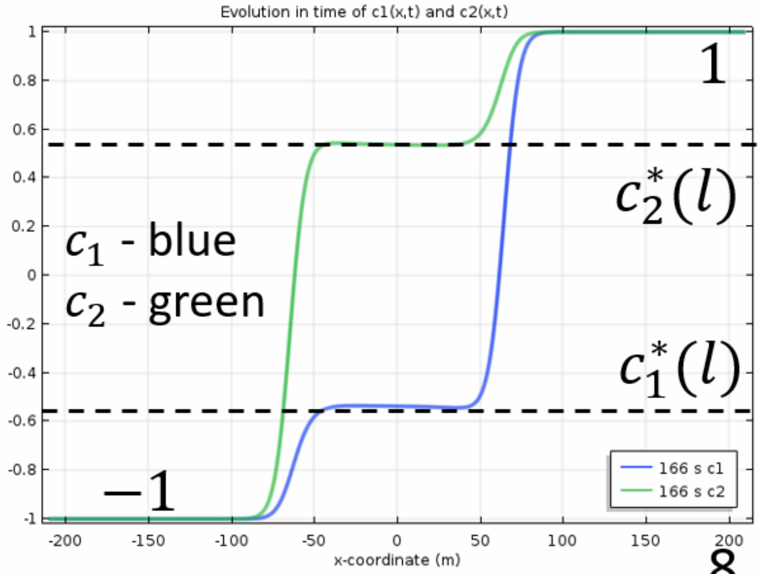
$$
(*)\begin{cases}
\partial_t c_1 + \partial_y(u_1 c_1) - \partial_{yy} c_1 = -B \\
\partial_t c_2 + \partial_y(u_2 c_2) - \partial_{yy} c_2 = B \\
u_1 = -\partial_y p_1 - c_1 \\
u_2 = -\partial_y p_2 - c_2 \\
\partial_y u_1 = -\partial_y u_2 = \dfrac{p_2 - p_1}{l}
\end{cases}
$$

$$
B = \begin{cases}
-\partial_y u_1 \cdot c_1, & \partial_y u_1 < 0, \\
+\partial_y u_2 \cdot c_2, & \partial_y u_1 > 0
\end{cases}
$$

Remark: $\lim_{l\to 0} c_1^*(l) = -0.5$ $\quad \lim_{l\to 0} v^b(l) = -0.25$

$\lim_{l\to 0} c_2^*(l) = +0.5$ $\quad \lim_{l\to 0} v^f(l) = +0.25$

As $t \to \infty$ we observe:



Theorem (Efendiev, P., Tikhomirov, 2024, arXiv: 2401.05981)

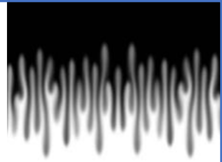
Consider a two-tube model with gravity $(*)$.

Then for all *$l > 0$ sufficiently small* there exists $c_1^*(l), c_2^*(l)$ such that there exist two travelling waves (TW):

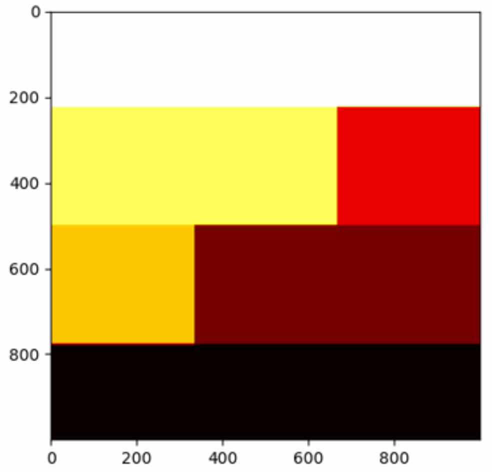
TW1 with speed $v^b(l)$: $\quad (-1,-1) \quad \to \quad (c_1^*,(l)\ c_2^*(l))$

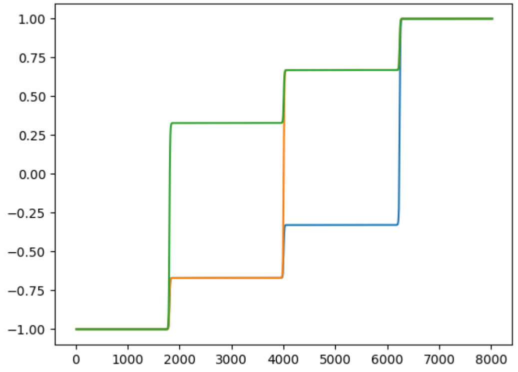
TW2 with speed $v^f(l)$: $\quad (c_1^*,(l)\ c_2^*(l)) \quad \to \quad (1,1).$

Many tubes: numerics

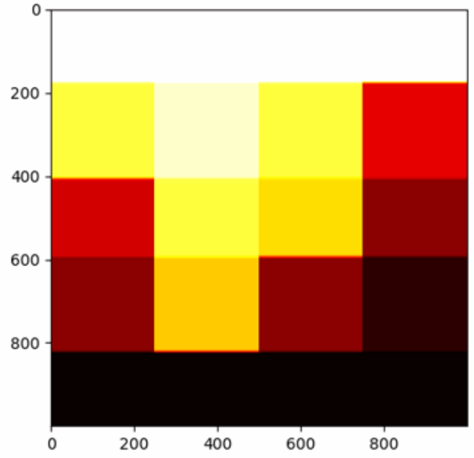


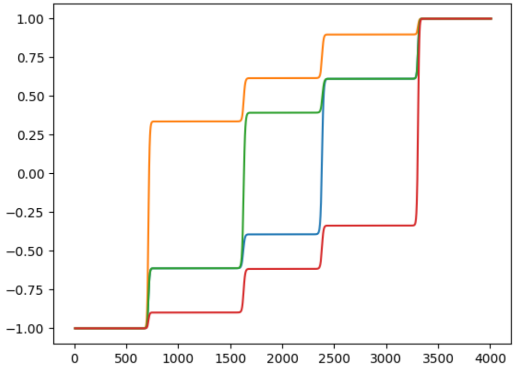
3 tubes



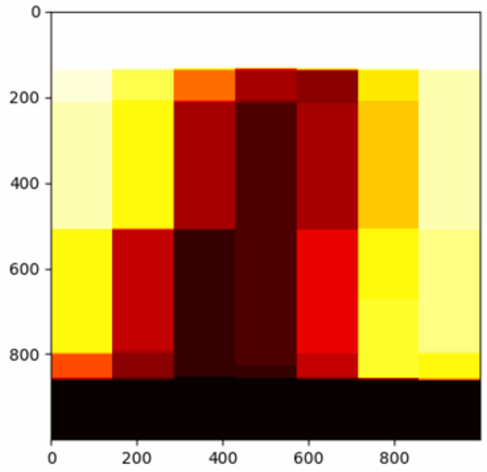


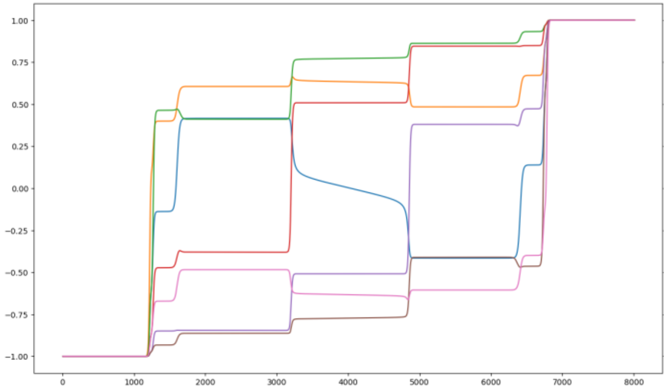
4 tubes





7 tubes





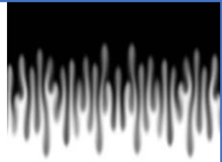
Questions: (1) explain the structure of “asymptotic solutions” for n tubes

(2) find speed of growth of the mixing zone

(3) understand the behaviour as $n \to \infty$. Do we approximate 2-dim IPM?

(4) can we repeat this story for the many tubes viscous fingering model?

Main ingredient in proof: comparison with TFE equations



Step 1: traveling wave ansatz $\xi = y - vt \Rightarrow$ traveling wave dynamical system (TWDS) – it is a slow-fast system!

$l = 0$

heteroclinic orbits can be found explicitly!

Orbit $\in W^s \pitchfork W^u$
+ geometric singular perturbation theory

$l > 0$

heteroclinic orbits persist under small perturbations

This step is broken for viscous fingers!

Two-tubes TFE equations

$$(**)\begin{cases} \partial_t c_1 + \partial_y(u_1 c_1) - \partial_{yy} c_1 = -B \\ \partial_t c_2 + \partial_y(u_2 c_2) - \partial_{yy} c_2 = B \\ u_1 = -u_2 = \dfrac{c_2 + c_1}{2} - c_1 \\ B = \begin{cases} -\partial_y u_1 \cdot c_1, & \partial_y u_1 < 0, \\ +\partial_y u_2 \cdot c_2, & \partial_y u_1 > 0 \end{cases} \end{cases}$$

This system can be seen a hyperbolic system in non-conservative form (for fixed choice of B) with parabolic regularization:

$$C_t + A(C)C_y = C_{yy}$$

We solve the Riemann problem:

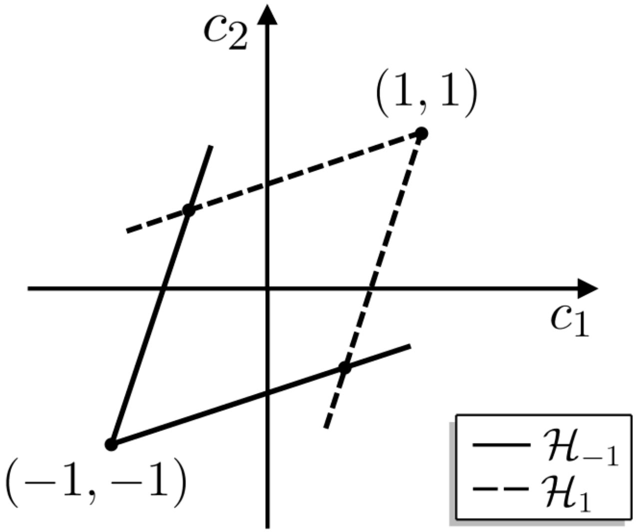
$$C = (c_1, c_2)\Big|_{t=0} = \begin{cases} (+1,+1), & y \geq 0 \\ (-1,-1), & y \leq 0 \end{cases}$$

Selection criteria for discontinuous solutions: vanishing viscosity

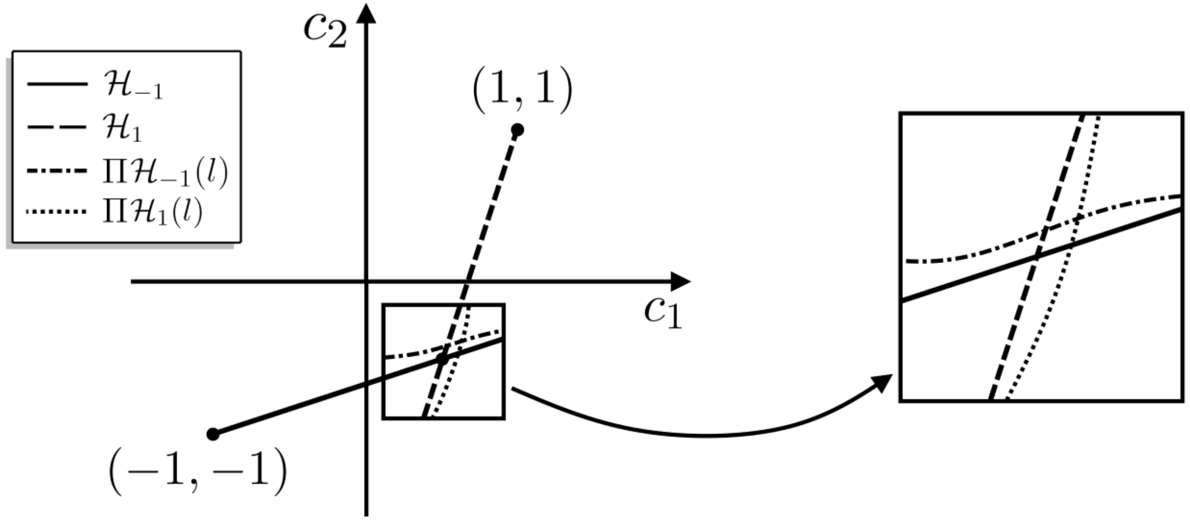
Proof: step 2

Step 2: combining 2 traveling waves

TFE



IPM for small $l > 0$



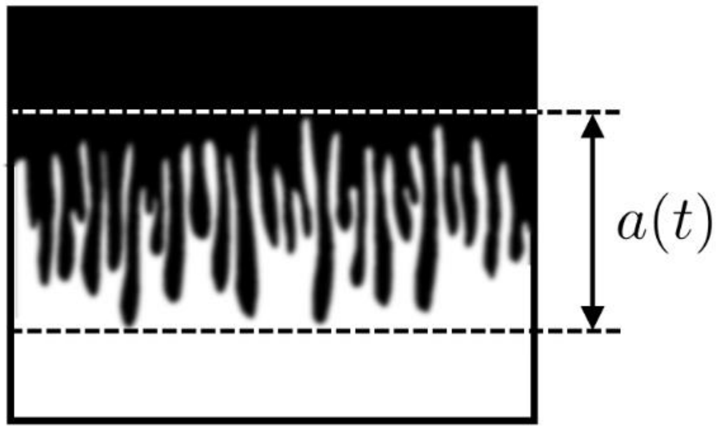
- “Temple-like” system (rarefaction and shock curves coincide and are linear)
- Similar explicit linear structure for $n = 3$ tubes (in progress)
- Starting from $n \geq 4$ appear also non-linear families and complex eigenvalues in some subdomains of $(c_1, \dots, c_n)$ (numerical evidence)...

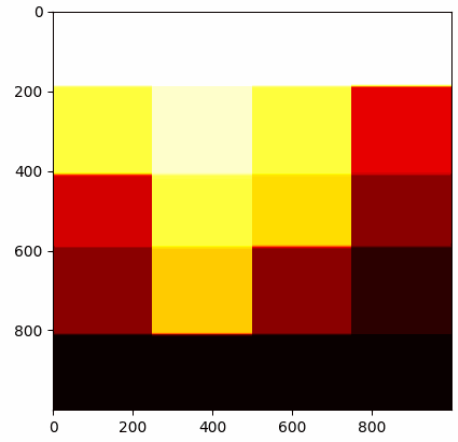
…Many interesting open questions….

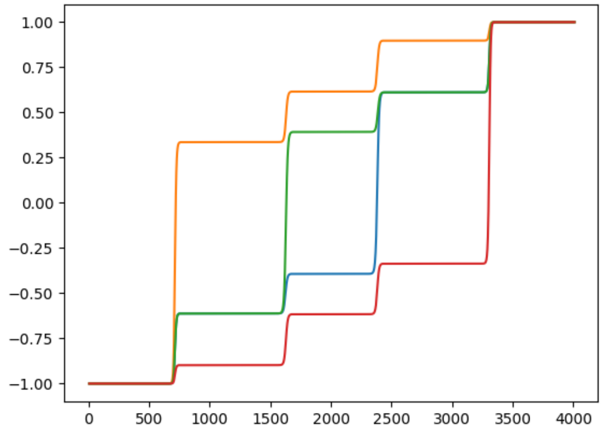
Thank you for your attention!

yu.pe.petrova@gmail.com

https://yulia-petrova.github.io/







For more details see arXiv: 2401.05981

Any questions, comments and ideas are very welcome!