Questions for exam on the course:
"Shock waves in conservation laws and reaction-diffusion equations"

Last update: 12/05/2023.

Part 1: Around wave equation.

1. Wave equation: "physical" derivation (balls and springs).
2. Wave equation: derivation from general principles.
3. D'Alambert's formula for 1D wave equation, and well-posedness of Cauchy problem on real line.
4. Inhomogeneous wave equation. Duhamel principle.
5. Mixed initial-boundary value problem for wave equation: existence and uniqueness of solution.
6. Mixed initial-boundary value problem for wave equation: solution by a Fourier series.

Part 2: Conservation and balance laws.

7. Fluid flow: Eulerian vs. Lagrangian point of view; flow map; incompressibility condition.
8. Fluid flow: scalar transport equation, conservation of mass.
9. Scalar conservation law. Weak form of solution. Rankine-Hugoniot condition.
10. Burgers equation: blow-up in finite time, explicit solutions to different Riemann problems, multiplicity of solutions, definition of entropy solution, irreversibility.
11. Scalar conservation law with convex flux function: various interpretations of entropy condition (Lax, Liu, vanishing viscosity).
12. Scalar conservation law with convex flux function: theorem on existence of entropy solution. Lemmas 1 and 2 describing properties for discrete approximation (boundedness, entropy condition).
13. Scalar conservation law with convex flux function: theorem on existence of entropy solution. Lemmas 3, 4 and 5 describing properties for discrete approximation (space and time estimates, stability).
14. Scalar conservation law with convex flux function: theorem on existence of entropy solution. Lemma 6 on convergence and properties of the limiting solution.
15. Scalar conservation law with convex flux function: theorem on existence of entropy solution. Lemmas 7 and 8 on properties of the limiting solution.
16. Scalar conservation law with convex flux function: uniqueness of entropy solution. General plan of proof without technical details.
17. Scalar conservation law with convex flux function: uniqueness of entropy solution. Proof that $|\psi_x^m|$ is bounded using the entropy condition.
18. Scalar conservation law with convex flux function: solution to a Riemann problem for two cases ($u_l < u_r$ and $u_l > u_r$).
19. Systems of conservation laws: weak solution, Rankine–Hugoniot condition, notion of hyperbolic and strictly hyperbolic systems, examples.
20. Systems of conservation laws: notion of genuinely nonlinear and linearly degenerate characteristic family; simple waves. Theorem on existence of k-rarefaction wave.
21. Systems of conservation laws: notion of shock curves (Hugoniot locus). Theorem on structure of shock waves (property (iii) without proof). Notion of Lax admissibility criteria for shocks.
22. Systems of conservation laws: notion of k-contact discontinuity. Theorem on linear degeneracy (shock and rarefaction curves coincide). Example (linear wave equation).
23. Systems of conservation laws: theorem on local solvability of a Riemann problem for strictly hyperbolic systems (each characteristic family is genuinely nonlinear or linearly degenerate).
24. Systems of conservation laws: entropy criteria (Lax, Liu, vanishing viscosity, entropy/entropy-flux).
25. Buckley-Leverett equation (with S-shaped flux function): solution to a Riemann problem for two cases ($u_l < u_r$ and $u_l > u_r$).

Part 3: Intro to reaction-diffusion equations.

26. Reaction-diffusion equations: probabilistic justification of laplacian, examples for nonlinearities (FKPP, monostable, bistable, ignition) and their interpretation in population dynamics. Formulation of the initial-value problem.

27. Maximum principles for linear ODEs of the second order with $h \equiv 0$ (with proofs).

28. Various versions of the maximum principles for linear ODEs of the second order without the assumption that $h \equiv 0$ (with proofs). Counterexamples.

29. The idea of the "sliding method" on two examples.

30. Weak and strong maximum principle for linear parabolic PDEs for bounded domains with Dirichlet boundary conditions (with proof).

31. Weak and strong maximum principle for linear parabolic PDEs for bounded domains with Neumann/Robin boundary conditions (with proof). Hopf lemma.

32. Notions of sub- and supersolution. Comparison theorems for parabolic PDEs (with proof). Application on concrete examples.