Spectral asymptotics in some problems with integral constraints

Problem statement

We look for the asymptotics of eigenvalues of the problem

$$(-1)^p u^{(2p)}(t) = \lambda u(t) + \mathcal{P}_{n-2p}(t), \qquad t \in [0,1] \tag{1}$$

$$\int_0^1 t^i u(t)\, dt = 0, \qquad i = 0 \dots n-1, \tag{2}$$

where $n, p \in \mathbf{N}$, $n > 2p$, and $\mathcal{P}_{n-2p}(t)$ is a polynomial of degree less than $(n-2p)$ with unknown coefficients.

Theorem 1 (asymptotics of eigenvalues)

$$\lambda_k = \left(\pi k + \frac{2n-p-1}{2} + O\left(\frac{1}{k}\right)\right)^{2p}, \qquad \text{as } k \to \infty. \tag{3}$$

The equivalent problem

$$(-1)^p y^{(2n)}(t) = \lambda y^{(2n-2p)}(t), \quad t \in [0,1] \tag{4}$$

$$y^{(j)}(0) = y^{(j)}(1) = 0, \quad j = 0 \dots n-1. \tag{5}$$

NB: the principle eigenvalue of (4)-(5) gives the sharp constant in the embedding theorem $\overset{\circ}{W}{}_2^n(0,1) \hookrightarrow \overset{\circ}{W}{}_2^{n-p}(0,1)$.
The equivalence can be seen by putting $u(t) = y^{(n)}(t)$.

History (M. Janet, 1931, [1])

Problem (4)-(5) was solved for $n \in \mathbb{N}$ and $p = 1$. For arbitrary p the answer was only formulated without proof and in implicit terms.

Application to small ball asymptotics

Formula (3) can be applied to calculate the asymptotics of

$$\mathbb{P}\{\|X_n(t)\|_{L_2[0,1]} < \varepsilon\} \text{ as } \varepsilon \to 0.$$

Here $X_n(t)$ is so called n-th order detrended Gaussian process

$$X_n(t) := X(t) - \sum_{i=0}^{n-1} a_i t^i,$$

where a_i are determined by relations

$$\int_0^1 t^i X_n(t)\, dt = 0, \;\; i = 0 \dots n-1.$$

$X(t)$, $t \in [0,1]$, is a Green Gaussian process. Namely, $\mathbb{E}X = 0$ and the covariance function $G(s,t) = \mathbb{E}X(s)X(t)$ is the Green function for a

BVP: $\quad Lu := (-1)^p u^{(2p)} = \lambda u \;+\;$ boundary conditions (BC).

NB: for $n > 2p$ the process X_n does not depend on the above boundary conditions.

Theorem 2 (exact small ball asymptotics)

$$\mathbb{P}\left\{\|X_n\|_{L_2[0,1]} < \varepsilon\right\} \sim C\varepsilon^{\gamma} \exp\left(-D\, \varepsilon^{-\frac{2}{2p-1}}\right).$$

Here C is an explicit constant, $\gamma = \frac{1-2np+p^2}{2p-1}$, $D = \frac{2p-1}{2}(2p \sin(\frac{\pi}{2p}))^{-\frac{2p}{2p-1}}$.

History

$n = 1$ $p = 1$	centered Brownian bridge and Brownian motion	2005 — L. Beghin et al [2] 2006 — P. Deheuvels [3]
$n = 2$ $p = 1$	detrended Brownian motion	2012 — X. Ai, W. Li [4]
$n \geqslant 3$ $p = 1$	n-th order detrended Brownian motion	2014 — X. Ai, W. Li [5]
$\forall n, p$ $n > 2p$	n-th order detrended Gaussian process $X_n(t)$	2016 — Yu. Petrova [6]

Proof of the Theorem 1

Step 1: Odd solutions

We can assume that the eigenfunction is odd or even (wrt $t = 1/2$).
If $y(t)$ is an even solution of the eq. (4):

$$(-1)^p y^{(2n)}(t) - \lambda y^{(2n-2p)}(t) = 0,$$

then

$$(-1)^p (y')^{(2n-2)}(t) - \lambda (y')^{(2n-2-2p)}(t) = C$$

and the constant $C = 0$, as the left hand side is odd.

E.v. λ related to even e.f. of (4)-(5) with parameters (n,p)	$\Leftrightarrow$	E.v. λ related to odd e.f. of (4)-(5) with parameters $(n-1,p)$

So we can restrict ourselves only to odd solutions.

Step 2: Determinant

Every odd solution of the equation (4) is of the form:

$$y(t) = a_0 \sin\left(\xi_0(2t-1)\right) + \dots + a_{p-1}\sin\left(\xi_{p-1}(2t-1)\right) + \\ + a_p(2t-1) + \dots + a_{n-1}(2t-1)^{2n-2p-1},$$

here $\xi_k = \frac{1}{2}|\lambda|^{\frac{1}{2p}} e^{\frac{ik\pi}{p}}$, $k = 0 \dots p-1$.
Substituting $y(t)$ into the boundary conditions (5), we get the equation $\Delta_{n,p}(\lambda) = 0$, where $\Delta_{n,p}(\lambda)$ is some determinant.

Step 3: Equation on determinant

$\Delta_{n,p}$ as a function of $\xi_0, \dots, \xi_{p-1}$ satisfies the following relation:

$$\frac{\partial^p}{\partial \xi_0 \dots \partial \xi_{p-1}} \Delta_{n,p} = C \cdot \xi_0 \cdot \dots \cdot \xi_{p-1} \cdot \Delta_{n-1,p}. \tag{6}$$

Step 4: Asymptotics of eigenvalues

$$\Delta_{p,p} = C \begin{vmatrix} \xi_0^{1/2} \mathcal{J}_{1/2}(\xi_0) & \dots & \xi_{p-1}^{1/2} \mathcal{J}_{1/2}(\xi_{p-1}) \\ \xi_0^{3/2} \mathcal{J}_{3/2}(\xi_0) & \dots & \xi_{p-1}^{3/2} \mathcal{J}_{3/2}(\xi_{p-1}) \\ \dots & \dots & \dots \\ \xi_0^{(2p-1)/2} \mathcal{J}_{(2p-1)/2}(\xi_0) & \dots & \xi_{p-1}^{(2p-1)/2} \mathcal{J}_{(2p-1)/2}(\xi_{p-1}) \end{vmatrix}$$

Here $\mathcal{J}_k(x)$ are Bessel functions of the first kind. Using relation (6) we get the following representation for $\Delta_{n,p}$

$$\begin{vmatrix} \xi_0^{(2n-2p+1)/2} \mathcal{J}_{(2n-2p+1)/2}(\xi_0) & \dots & \xi_{p-1}^{(2n-2p+1)/2} \mathcal{J}_{(2n-2p+1)/2}(\xi_{p-1}) \\ \xi_0^{(2n-2p+3)/2} \mathcal{J}_{(2n-2p+3)/2}(\xi_0) & \dots & \xi_{p-1}^{(2n-2p+3)/2} \mathcal{J}_{(2n-2p+3)/2}(\xi_{p-1}) \\ \dots & \dots & \dots \\ \xi_0^{(2n-1)/2} \mathcal{J}_{(2n-1)/2}(\xi_0) & \dots & \xi_{p-1}^{(2n-1)/2} \mathcal{J}_{(2n-1)/2}(\xi_{p-1}) \end{vmatrix}$$

The final equation will be of the form

$$\Delta_{n,p}(\lambda) \cdot \Delta_{n-1,p}(\lambda) = 0.$$

Using asymptotics of Bessel functions we get (3).

References

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