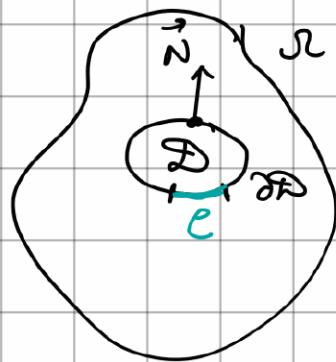


## Lecture 5: Conservation & Balance laws

Plan:

1. General definition
2. Example 1: fluid dyn (conservation of mass)
3. Example 2: scalar conservation law

### (1) Balance law



$D \subset \mathbb{R}$  with Lipschitz boundary (smooth)

$N$  - normal vector towards the exterior of the domain  $D$

$$\text{[production in } D] = \text{flux through the boundary of } \partial D$$

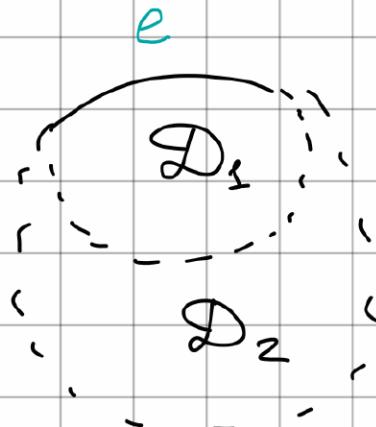
- production in  $D$  is some measure (Radon)  $P$

- flux

$$F_D(e) = \int_E q_D(x) dS(x)$$

$$P(D) = \int_{\partial D} q_D(x) dS(x) \quad (*)$$

Assume:



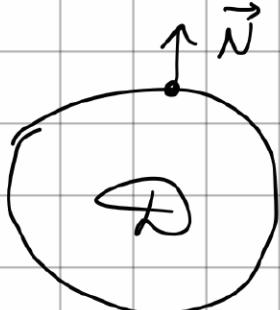
$$q_{D_1}(x) = q_{D_2}(x)$$

$$\forall x \in E$$

Take-home  
(Tuesday)

$$\boxed{\operatorname{div} A = 0}$$

Miracles : (1)  
Consequences of (\*) :



$$\exists a_{\mathcal{D}}(x) = q_{\mathcal{D}}(x)$$

$\forall x \in \mathcal{R}$

for any  $\mathcal{D} \subset \mathcal{R}$  s.t.

$\mathcal{D}$  has  $\vec{N}$  as a normal vector at  $x$ .

$$(2) \quad \exists \vec{A}(x) : \mathcal{R} \rightarrow \mathbb{R}^d :$$

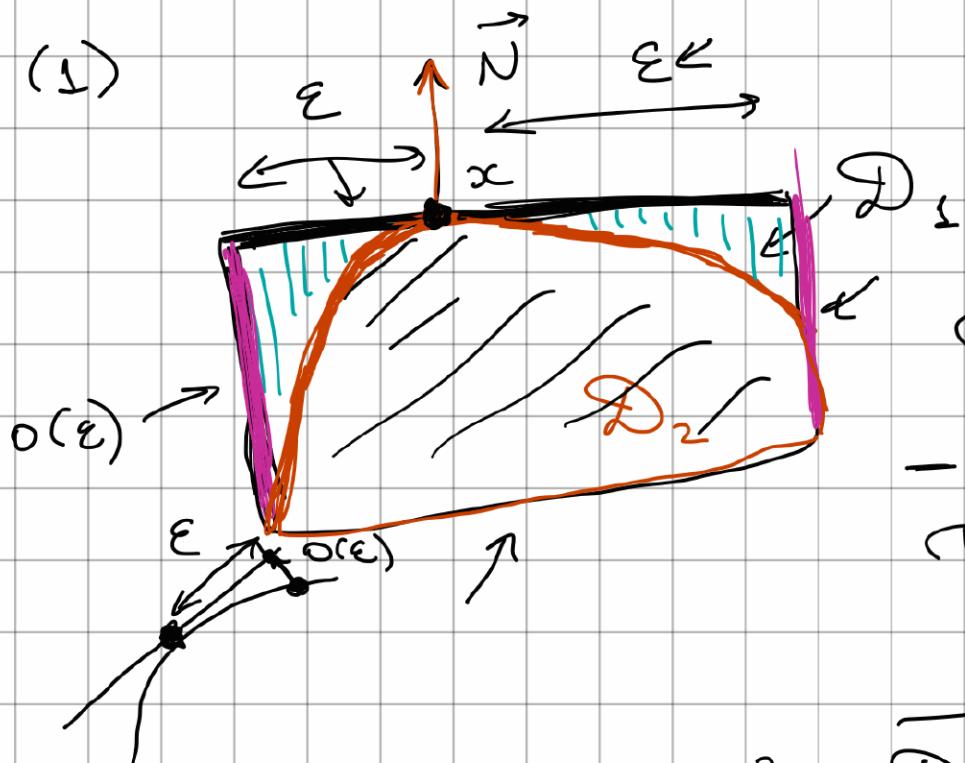
$$a_{\vec{N}}(x) = \vec{A}(x) \cdot \vec{N}$$

$$(3) \quad \exists \text{PDE} : \operatorname{div} \vec{A} = P$$

$$P(\mathcal{D}) = \int_{\partial \mathcal{D}} q_{\mathcal{D}}(x) dS(x)$$

$$\frac{\text{"}}{\vec{A}(x) \cdot \vec{N}}$$

(1)



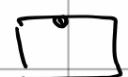
$$\epsilon \rightarrow 0$$

?

$$q_{\mathcal{D}_1}(x) = q_{\mathcal{D}_2}(x)$$

(\*)

$$P(\mathcal{D}_1) = \int q_{\mathcal{D}_1}(x) dS$$



$$P(\mathcal{D}_2) = \int q_{\mathcal{D}_2}(x) dS$$



$$\int \sim O(\epsilon)$$

||

$$\epsilon^2 \sim \overline{P(\mathcal{D}_1)} = \int q_{\mathcal{D}_1}(x) dS$$

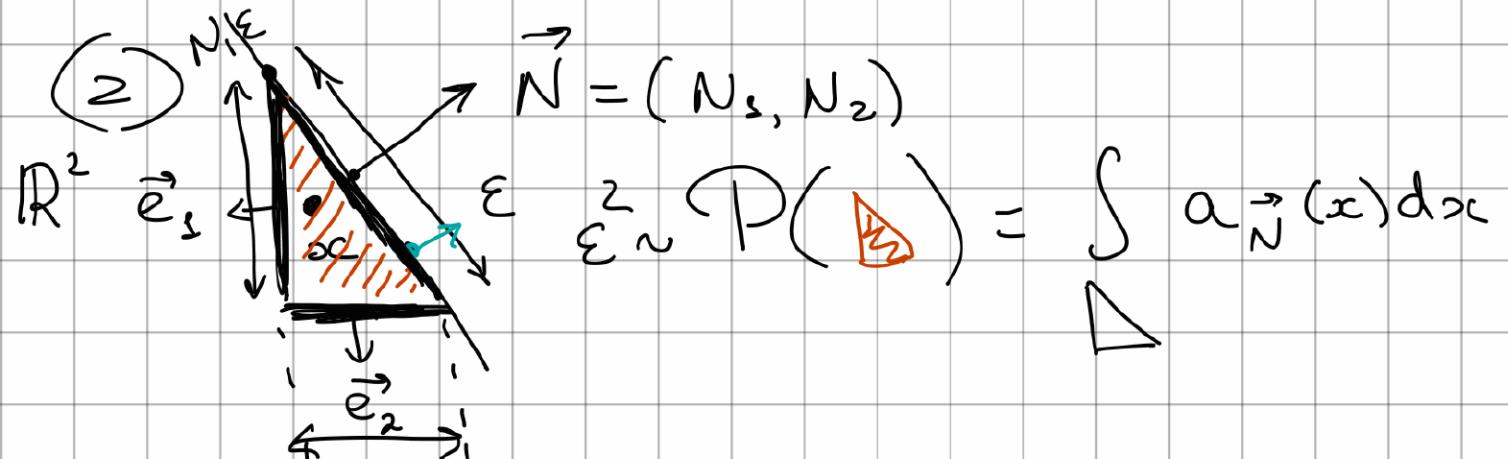


$$\Rightarrow \underline{\int q_{\mathcal{D}_1}(x) dS(x)} = \int q_{\mathcal{D}_2}(x) dS(x)$$

$$- \int q_{\mathcal{D}_2}(x) dS$$



$$\Rightarrow \exists a_{\vec{N}}(x) = q_{\vec{N}}(x) \quad \underline{\text{Cauchy}}$$



$$a_{\vec{N}}(x) \cdot \underline{\epsilon} = a_{e_1}(x) \cdot \underline{N_1 \epsilon} +$$

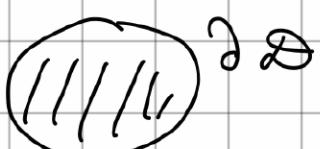
$$+ a_{e_2}(x) \cdot \underline{N_2 \epsilon}$$

$$\Rightarrow a_{\vec{N}}(x) = a_{e_1}(x) N_1 + a_{e_2}(x) N_2$$

$$\Rightarrow a_{\vec{N}}(x) = \vec{A}(x) \cdot \vec{N}$$

$$\vec{A}(x) = (a_{e_1}, a_{e_2})$$

$$(3) \int_{\partial D} \vec{A}(x) \cdot \vec{N} dS(x) = \int_D \text{div}(\vec{A}) dx$$



||| |||  $\int_D$   
Green-Gauss  
theorem

$$P = \int_D p(x) dx$$

$$\Rightarrow \text{div}(\vec{A}) = P \quad - \text{balance law}$$

$$\boxed{\text{div}(\vec{A}) = 0} \quad - \text{conservation law}$$

# Dafermos

Example 1 : Fluid flow, continuum mechanics

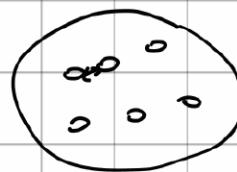
different scales

1. atoms / molecules

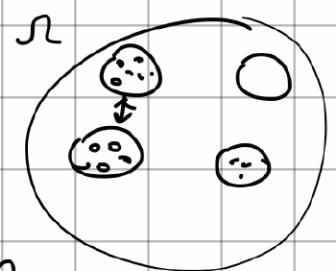
o

2. representative

small volume



3. domain  
(macroscale)



- Eulerian vs. Lagrangian point of view

Eulerian :  $(x, t) \mapsto x$

- velocity :  $u(x, t) = (u_1, \dots, u_d) : \mathbb{R}^d \times [0, \infty) \rightarrow \mathbb{R}^d$   
has units  $\left[ \frac{\text{L}}{\text{T}} \right]_{\mathbb{R}^d}$
- density :  $\rho(x, t) : \mathbb{R}^d \times [0, \infty) \rightarrow \mathbb{R}$   
with units  $\left[ \frac{\text{M}}{\text{L}^d} \right]$
- pressure :  $p(x, t) : \mathbb{R}^d \times [0, \infty) \rightarrow \mathbb{R}$   
with units  $\left[ \text{ML}^{-d+2} \text{T}^{-2} \right]$

Lagrangian : particles,  $a \in \mathbb{R}^d$   
trajectories of particles

flow map

$X(t, a) = (X_1, \dots, X_d)$  - position  
of particle  $a$  at time  $t$

$$(**) \left\{ \begin{array}{l} \partial_t X(t, a) = u(t, X(t, a)) \\ X(0, a) = a \end{array} \right. \leftarrow \text{ODE}$$

ODE theory (Cauchy-Lipschitz theorem) :

$u \in C_t \text{Lip}_x \Rightarrow \exists! \text{ solution to } (**)$

$X(t, \cdot)$  - is  $C^1$ -diffeo :  $\mathbb{R}^d \rightarrow \mathbb{R}^d$

Define inverse:  $A(t, x) : [0, +\infty) \times \mathbb{R}^d \rightarrow \mathbb{R}^d$

$$A(t, X(t, a)) = a \quad X(t, A(t, a)) = a$$

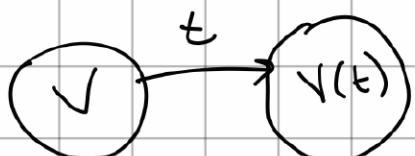
$\forall x, a \in \mathbb{R}^d$

"back-to-labels" map ( $a$  - "labels")

Incompressibility condition : " $\operatorname{div} u = 0$ "

Take  $V \subset \mathbb{R}$  - volume of fluid

$$V(t) = X(t, V) = \{X(t, a) : a \in V\}$$



Def.: velocity field is called incompressible if

$$\rightarrow |V(t)| = |V| \quad \uparrow \text{Lebesgue measure of } V$$

Lemma:  $u \in C_t \operatorname{Lip}_x$

$u$  is incompressible  $\Leftrightarrow \operatorname{div} u = 0$  ( $u$  is divergence-free)

Proof:



$$V(t) = \int_{V(t)} \mathbf{f} \cdot d\mathbf{x} ; \quad a \in V \subset \mathbb{R}^d$$

$$\int_{V(t)} \mathbf{f}(x, t) d\mathbf{x} = \int_V \mathbf{f}(X(t, a), t) \cdot \det(\nabla_a X) da$$

$\downarrow$   
 $a \mapsto X(t, a)$

$$\rightarrow J(t, a) = \sum_{i_1, \dots, i_d=1}^d \varepsilon_{i_1 \dots i_d} \frac{\partial X_{i_1}}{\partial a_1} \dots \frac{\partial X_{i_d}}{\partial a_d}$$

Exercise:  $\partial_t J(t, a) = J(t, a) \cdot (\operatorname{div} a)(t, X(t, a))$

Corollary :  $J(t, \alpha) = \underline{1} \Leftrightarrow \frac{\operatorname{div}(u)}{\int_0^t (J(s, \alpha)) ds} = 0$

 $\rightarrow J(t, \alpha) = J(0, \alpha) \cdot e^{\int_0^t \operatorname{div}(u) ds}$

$$J(0, \alpha) = \underline{1}$$

$$J(t, \alpha) = J(0, \alpha) \quad \forall t \Rightarrow \operatorname{div}(u) = 0$$

$$V(t) = \int_{V(t)} \underline{d}x = \int_V J(t, \alpha) d\alpha = \int_V d\alpha = V$$

L

$$\text{if } \operatorname{div}(u) = 0$$

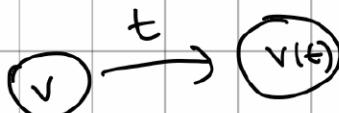
## Transport equation

Let's  $f(t, x) : [0, \infty) \times \mathbb{R}^d \rightarrow \mathbb{R}$  - scalar

Eulerian:  $\partial_t f$  - change of  $f$  at  $(t, x)$

Lagrangian:  $\partial_t f(t, X(t, \alpha)) =: D_t f$  - convection derivative

$$\partial_t f + u \cdot \nabla f$$



Thm (transport thm):

$u$  - velocity field,  $u \in C^1$ ;  $f$  be  $C^1$

$V(t)$  is pushforward of  $V$  by the flow map  $X(t, \alpha)$

$$\frac{d}{dt} \left( \int_{V(t)} f(x, t) dx \right) = \int_V (\partial_t f + \operatorname{div}(fu))(t, x) dx$$

Proof:

$$\int_{V(t)} f(x, t) dx = \int_V f(X(t, \alpha), t) \underline{J(t, \alpha) da}$$

$$\begin{aligned}
 \frac{d}{dt} \left( \underset{V(t)}{\int} f(x, t) dx \right) &= \underset{V}{\int} (\partial_t f)(x(t, \alpha), t) J(t, \alpha) \\
 &\quad + \underset{V}{\int} f(x(t, \alpha), t) \cdot \partial_t J(t, \alpha) d\alpha = \\
 &= \underset{V}{\int} \left( \partial_t f + \underbrace{u \cdot \nabla f + f \cdot \operatorname{div}(u)}_{\operatorname{div}(uf)} \right) (x(t, \alpha), t) \cdot J(t, \alpha) d\alpha \\
 &= \underset{V(t)}{\int} \left( \partial_t f + \operatorname{div}(fu) \right) dx \quad \blacksquare
 \end{aligned}$$

Conservation of mass:  $g(x, t)$

$$m(t, V) = \underset{V}{\int} g(x, t) dx$$

$$\frac{d}{dt} m(t, V(t)) = 0$$

Thm: conservation of mass is equivalent to the following integral eq:

$$\underset{V(t)}{\int} (g_t + \operatorname{div}(gu)) dx = 0$$

If  $g_t$  and  $\operatorname{div}(gu)$  are C, then

$$g_t + \operatorname{div}(gu) = 0$$

Remark: scalar transport eq

Proof:  $\int_0^t = \frac{d}{dt} m(t, X(t, a)) = \int_{\mathbb{R}^n} g(x, t) dx$

$$= \int_{V(t)} \underbrace{(g_t + \operatorname{div}(gu))}_{\text{are continuous}} dx.$$

$$f(x) = \lim_{\varepsilon \rightarrow 0} \frac{1}{|B_\varepsilon(x)|} \int_{B_\varepsilon(x)} f(y) dy$$

$\Rightarrow g_t + \operatorname{div}(gu) = 0.$  ■

Remark:  $0 = g_t + \operatorname{div}(gu) = g_t + u \cdot \nabla g + \operatorname{div}(u)g$

incompressibility  $\Rightarrow \operatorname{div}(u) = 0$  □

Next time

$$g_t + (gu)_x = 0$$

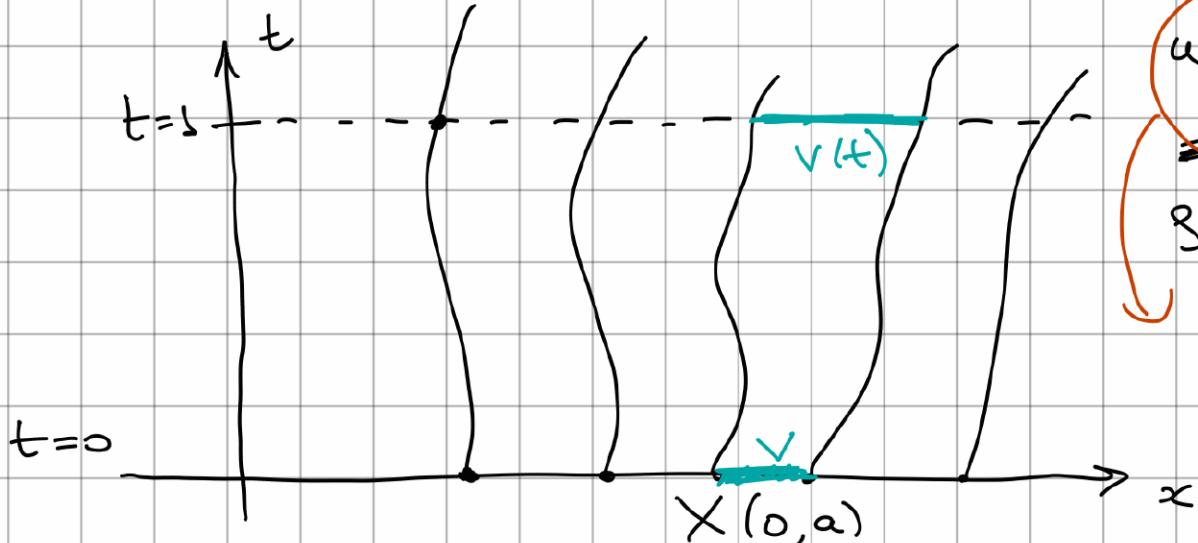
$$D_t g = g_t + u \cdot \nabla g = 0$$

$$u = u(g)$$

$$u(g) = \frac{g}{2} \Rightarrow \text{Burgers}$$

$$\frac{d}{dt} (g(t, X(t, a))) = 0$$

$$\Rightarrow g(t, X(t, a)) = \text{const.} \quad \leftarrow$$



## Lecture 6

Last time: Balance laws:  $\operatorname{div} A = P$

Conservation laws:  $\operatorname{div} A = 0$

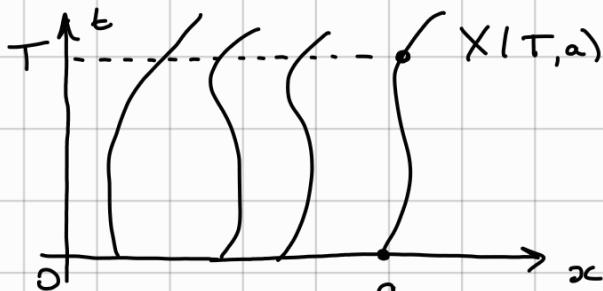
Example 1: fluid flow:  $a \rightarrow X(t, a)$  - flow map under velocity field

$u(x, t)$   $u(t, x)$  - velocity field  $\in \mathbb{R}^d$   
 $s(t, x)$  - density

$$\begin{cases} \partial_t X = u(X, t) \\ X(0, a) = a \end{cases}$$

• Conservation of mass = scalar transport equation

$$\partial_t s + \operatorname{div}(su) = 0$$

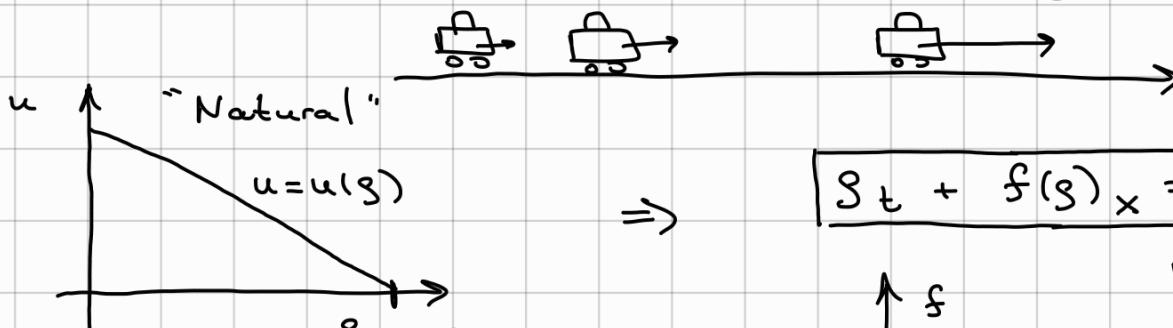


trajectories do not intersect for given  $u \in C^1_t \operatorname{Lip}_x$

Rmk:  $\begin{cases} \operatorname{div} u = 0 \\ \partial_t s + \operatorname{div}(su) = 0 \end{cases}$

$\Rightarrow s(t, X(t, a)) = \text{const}$   
density is conserved along the trajectory for incompressible flow

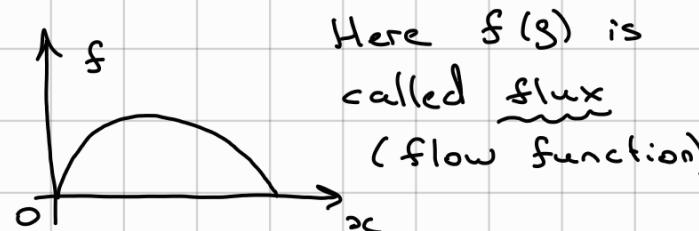
Example 2: traffic flow: cars choose their velocity depending on "density" of cars nearby



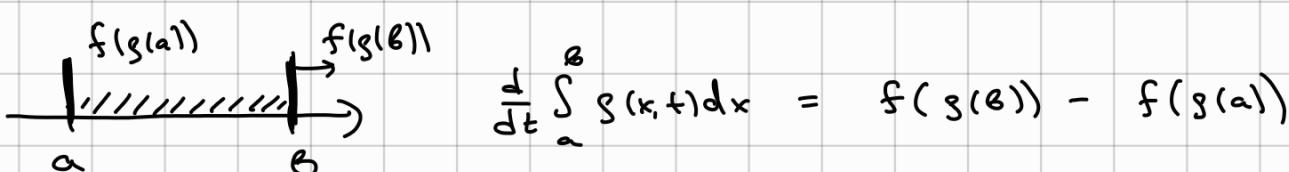
$s_m$  - density of cars corresp. to "bumper-to-bumper"

$$s_t + f(s)_x = 0$$

scalar conservation law



Here  $f(s)$  is called flux (flow function)



$$\frac{d}{dt} \int_a^b s(x, t) dx = f(g(b)) - f(g(a))$$

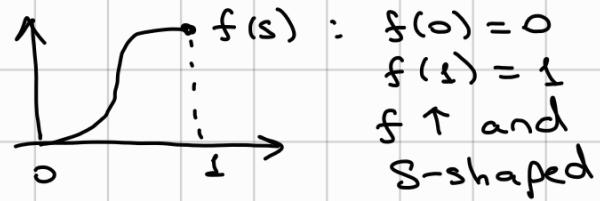
Rmk: i) taking  $u(s) = \frac{s}{2} \Rightarrow$  Burgers eq:  $s_t + \left(\frac{s^2}{2}\right)_x = 0$   
We will analyze it in detail today.

2) for oil recovery the simplest 1-dim model for displacement water-oil is again

$$s_t + (f(s))_x = 0 \quad \text{for}$$

$s$ - water saturation

$f(s)$ - fractional flow function



- One can easily create more sophisticated models such as: take drivers anticipation into account

If a driver observe an upstream increase in the density, they show a tendency to brake slightly

$$u - v(s) \sim -g_x$$

The simplest law:  $u = v(s) - \varepsilon g_x$ ,  $0 < \varepsilon \ll 1$   
which leads to the "weakly" parabolic eq:

$$s_t + f(s)_x = \varepsilon (gg_x)_x$$

Example 3: wave equation!  $u_{tt} - c^2 u_{xx} = 0$

$$\operatorname{div}(u_t, -c^2 u_x) = 0$$

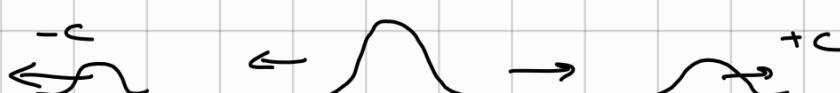
Consider  $\mathbf{U} = \begin{pmatrix} u_x \\ u_t \end{pmatrix} \Rightarrow \mathbf{U}_t + A \mathbf{U}_x = 0$

$$A = \begin{pmatrix} 0 & -1 \\ -c^2 & 0 \end{pmatrix}$$

Indeed, this is just:  $\begin{cases} u_{xt} - u_{tx} = 0 \\ u_{tt} - c^2 u_{xx} = 0 \end{cases}$

Eigenvalues of  $A$ :  $\det \begin{vmatrix} 0 - \lambda & -1 \\ -c^2 & 0 \end{vmatrix} = \lambda^2 - c^2$ ,  $\lambda_{\pm} = \pm c$

They correspond to propagation modes:



This is general fact that we will see in the future:

$$\mathbf{U} \in \mathbb{R}^d, F: \mathbb{R}^d \rightarrow \mathbb{R}^d \quad \mathbf{U}_t + (F(\mathbf{U}))_x = 0 \quad \text{- system of conservation laws}$$

Then for "smooth" solutions we have:

$$\mathbf{U}_t + \underbrace{F'(\mathbf{U}) \cdot \mathbf{U}_x}_{\text{eigenvalues of this matrix}} = 0$$

eigenvalues of this matrix play an important role!

If they are real, they correspond to velocity of propagation of waves.

Example 4: isentropic (=constant entropy) gas dynamics  
( $p$ -system)

in Lagrangian coordinates :

$$\begin{cases} v_t - u_x = 0 \\ u_t + p(u)_x = 0 \end{cases} \Rightarrow v_{tt} + p(v)_{xx} = 0$$

Rmk:  $v_t = u_x \Rightarrow$  (in a simply connected regions)  
 $\exists \Phi : v = \Phi_x$   
 $u = \Phi_t$

$$\Rightarrow \Phi_{tt} + (p(\Phi_x))_x = 0$$

$$\Phi_{tt} + p'(\Phi_x) \cdot \Phi_{xx} = 0 \quad -\text{nonlinear wave equation}$$

And many other examples :

- conservation of mass
- conservation of momentum

$$\Rightarrow \begin{cases} \partial_t u + u \cdot \nabla u = \nabla p + f \\ \operatorname{div}(u) = 0 \end{cases}$$

This is Euler equations for ideal fluid  
(1755, second PDE !)

- Navier - Stokes eqs (1845): adds viscosity

$$\partial_t u + (u \cdot \nabla) u \rightarrow \Delta u = \nabla p$$

- gas dynamics

- electromagnetism (Maxwell eqs)

- magneto-hydrodynamics (M.H.D.) - motion of fluid in the presence of electromagnetic field  
(think of a Sun)

Etc .....

Burgers equation

$$u_t + \left( \frac{u^2}{2} \right)_x = 0$$

$$u_t + u \cdot u_x = 0$$

Observation 1: if  $u \in C^1$  for all  $t > 0$ , then  
in  $\infty$  for all  $t > 0$ .



$$u(x(s), t(s)) = \text{const}$$

$$u_t \cdot t_s + u_x \cdot x_s = 0$$

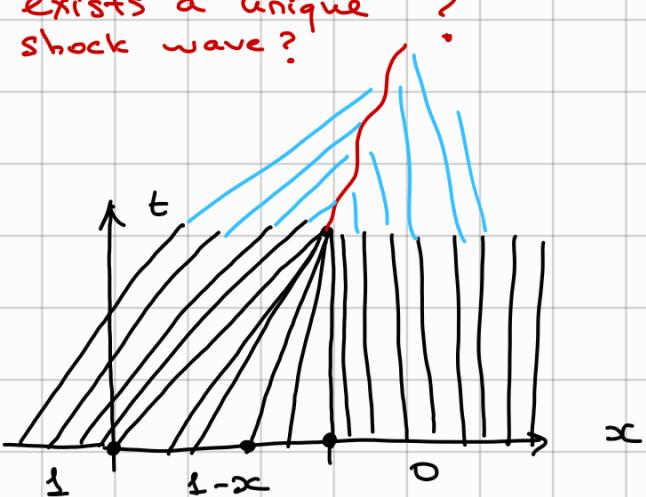
$$\begin{cases} t_s = 1 \\ x_s = u \end{cases} \Rightarrow u = \text{const on straight lines}$$

$$x = x_0 + u_0(x_0)t$$

If  $u \in C^1$  for  $\forall t > 0$ , then characteristics should not intersect  $\Rightarrow u_0(x_1) < u_0(x_2)$  if  $x_1 < x_2 \Rightarrow u_0$  is non-decreasing ( $u(x, 0)$ )  $\Rightarrow u(x, t)$  is non-decreasing in  $x$

Exercise 2 from list 1:

exists a unique  
shock wave?



$$x = x_0 + (1-x_0)t$$

$$t = 1 : x = 1$$

At  $t = 1$  there is a blow-up

Rmk: In general scalar conservation law:

$$u_t + f(u)_x = 0$$

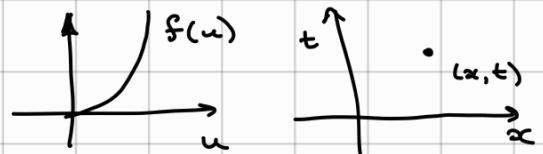
$$u_t + \underline{f'(u) \cdot u_x} = 0$$

Characteristics are  $x = x_0 + f'(u_0(x_0))t$

$u \in C^1 \wedge t > 0 \Rightarrow f'(u_0(x_1)) < f'(u_0(x_2))$  if  $x_1 < x_2$ , otherwise characteristics will intersect that leads to a blow-up!

So no matter how smooth  $f$  and  $u_0$  are, the solution  $u(x, t)$  must become discontinuous This is a purely non-linear phenomenon!!!

- Assume  $f \in C^2$  and  $f'' > 0$



$$u_0(x - t f'(u(x,t))) = u(x,t)$$

$$u_t = u'_0 \cdot (-f'(u(x,t)) - t f''(u(x,t)) \cdot u_x)$$

$$u_t (1 + t f'' u'_0) = -u'_0 f'$$

$$u_t = -\frac{u'_0 f'}{1 + t f'' u'_0}$$

Analogously,  $u_x = \frac{u'_0}{1 + t f'' u'_0}$

If  $u'_0 \geq 0$  (and  $f'' > 0$ )  $u_t$  and  $u_x$  stay bounded.

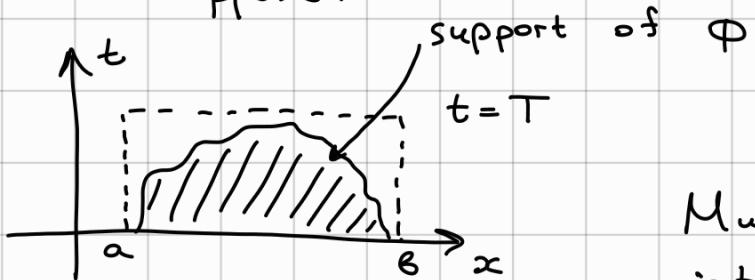
If  $u'_0 < 0$ , then  $u_t$  and  $u_x$  become unbounded as  $1 + t f'' u'_0$  tends to 0.

So we need a notion of weak solution!

Weak solutions to conservation laws

$$\begin{cases} u_t + f(u)_x = 0 \\ u|_{t=0} = u_0(x) \end{cases} \quad (\star)$$

Let  $u$  be a classical solution and  $\varphi \in C^1$  with compact support:



$\text{supp } (\varphi) \subset D = [a, b] \times [0, T]$   
that is  $\varphi$  is zero at  $x=a, x=b, t=T$

Multiply  $(\star)$  by  $\varphi$  and integrate over  $\mathbb{R} \times \mathbb{R}_+$

$$\begin{aligned} \iint_{t>0} (u_t + f(u)_x) \varphi \, dx dt &= \iint_D (u_t + f(u)_x) \varphi \, dx dt = \\ &= \iint_{a \leq x \leq b} (u_t + f(u)_x) \varphi \, dx dt = \int_a^b u \cdot \varphi \Big|_0^T \, dx - \iint_{a \leq x \leq b} u \cdot \varphi_t \, dx \\ &\quad + \int_0^T \int_a^b f(u) \cdot \varphi \Big|_a^b \, dt - \int_0^T \int_a^b f(u) \cdot \varphi_x \, dx dt = \end{aligned}$$

$$= - \int_0^T u_0(x) \varphi(x) dx - \iint_{\Omega \times (0,T)} (u \varphi_t + f(u) \varphi_x) dx dt$$

$$\Rightarrow \iint_{\Omega \times [0,T]} (u \varphi_t + f(u) \varphi_x) dx dt + \int_{t=0} u_0(x) \varphi(x) dx = 0 \quad (2)$$

$u \in C^1$  and satisfies (1)  $\Rightarrow u$  satisfies (2)

But in (2)  $u$  not necessarily needs to be  $C^1$ .  
It can be measurable / bounded.

Definition: A bounded measurable function  $u(x,t)$  is called a weak solution of IVP:

$$u_t + f(u)_x = 0, \quad u|_{t=0} = u_0(x) \quad \uparrow \text{bounded/meas.}$$

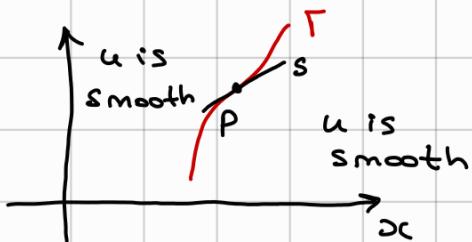
provided that

$$(2) \quad \iint_{\Omega \times [0,T]} (u \varphi_t + f(u) \varphi_x) dx dt + \int_{t=0} u_0 \varphi dx = 0$$

for all  $\varphi \in C_0^\infty$  ( $\varphi$  is  $C^\infty$  with compact supp)

Rmk: it is clear that if  $u$  is in fact  $C^1$ , then the original eq. is true:  $u_t + f(u)_x = 0$

Lemma (Rankine-Hugoniot condition)



Let  $\Gamma$  be a smooth curve across which  $u$  has a jump discontinuity. Take  $P \in \Gamma$  and

$$u_e = \lim_{(x,t) \rightarrow P} u \quad \text{from "the left"}$$

$$u_r = \lim_{(x,t) \rightarrow P} u \quad \text{from "the right"}$$

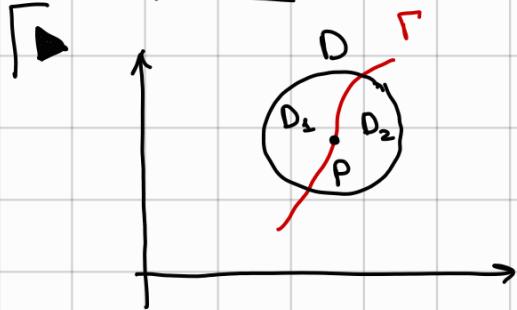
Let the tangent line of  $\Gamma$  at  $P$  have the slope

$$s = \frac{dx}{dt}. \quad \text{Then: (3)} \quad \boxed{s \cdot (u_e - u_r) = f(u_e) - f(u_r)}$$

Often a jump across the shock is denoted:

$[g(u)] = g(u_e) - g(u_r)$ , thus we have  $S[u] = [f]$   
This is called the Rankine-Hugoniot condition

Proof :



Let  $D$  be a small ball centered at  $P$  and let  $\Gamma$  devide  $D$  into two regions  $D_1$  and  $D_2$  (see fig)

Let  $\varphi \in C_0^1$  on  $D$  and consider

$$0 = \iint_D (u\varphi_t + f(u)\varphi_x) dx dt = \iint_{D_1} + \iint_{D_2}$$

Divergence theorem :  
(Green-Gauss theorem)

$$\int_Q P dx + Q dy = \iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy$$

$$\iint_{D_1} u\varphi_t + f(u)\varphi_x dx dt = \iint_{D_2} (u\varphi)_t + (f(u)\varphi)_x dx dt =$$

↑  
 $D_2$

as  $u \in C^1(D_2)$  and  $u_t + f(u)_x = 0$

$$= \int_{t_1}^{t_2} \int f(u) \varphi dt - u \varphi dx = \int_{t_1}^{t_2} f(u) \varphi dt - u \varphi dx =$$

~~F~~  $t_2$   $t_1$

$$= \int_{t_1}^{t_2} [f(u_e) \varphi(u_e) - u_e \cdot \varphi(u_e) \cdot s] dt$$

Similarly,

$$\iint_{D_1} u\varphi_t + f(u)\varphi_x dx dt = - \int_{t_1}^{t_2} (f(u_r) - s u_r) \varphi(u_r) dt$$

minus because of orientation of  $\partial D_2$ : ~~F~~

Combining together:

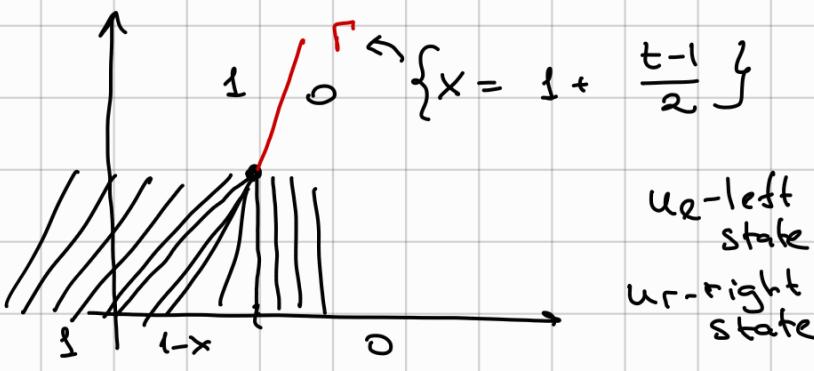
$$0 = \int_{t_1}^{t_2} ([f] - s[u]) \varphi(u_e) dt$$

Since  $\varphi$  was arbitrary, we get  $[f] - s[u] = 0$  relation (3). ■

Example: Burgers eq :

$$s = \frac{[u^2] /_0^L}{[u] /_0^L} = \frac{1}{2}$$

in general  $s = \frac{u_e + u_r}{2}$



$u_e$ -left state

$u_r$ -right state

Lecture 7 | Scalar conservation law:  $\begin{cases} u_t + (f(u))_x = 0 \\ u: \mathbb{R}^+ \rightarrow \mathbb{R} - \text{bounded, measurable} \\ f: \mathbb{R} \rightarrow \mathbb{R}, f \in C^2, f'' > 0 \text{ on the convex hull of values of } u_0 \end{cases} \quad (*)$

We understand solutions in weak sense:

$$\underset{t>0}{\int} \int (u \varphi_t + f(u) \varphi_x) dx dt + \int_{t=0}^{t=0} u_0 \varphi dx = 0 \quad (**)$$

for every test function  $\varphi \in C_0^1$ .

We want to prove theorems on  $\exists$ ,  $!$  and asymptotic behavior of solutions to  $(*)$ . From exercise session 1 we remember that we need some extra conditions for this

Thm1 ( $\exists$ ):

Let  $u_0 \in L^\infty(\mathbb{R})$ ,  $f \in C^2(\mathbb{R})$ ,  $f'' > 0$  on  $\{u: |u| \leq \|u_0\|_\infty\}$ .

Then there exists a solution with the following properties:

a)  $|u(x,t)| \leq \|u_0\|_\infty = M$ ,  $(x,t) \in \mathbb{R} \times \mathbb{R}^+$

b)  $\exists E > 0$  (which depends on  $M$ ,  $\mu = \min \{f''(u) : |u| \leq M\}$  and  $A = \max \{|f'(x)| : |u| \leq \|u_0\|_\infty\}$ ) such

such that  $\forall a > 0, t > 0, x \in \mathbb{R}$

$$\frac{u(x+a, t) - u(x, t)}{a} < \frac{E}{t} \quad (E)$$

c)  $u$  is stable and depends continuously on  $u_0$ : if  $v_0 \in L^\infty(\mathbb{R})$  with  $\|v_0\|_\infty \leq \|u_0\|_\infty$  and  $v$  is the corresponding constructed solution of  $(*)$  with initial data  $v_0$ , then for  $\forall x_1, x_2 \in \mathbb{R}$  with  $x_1 < x_2$  and  $\forall t > 0$

$$\int_{x_1}^{x_2} |u(x, t) - v(x, t)| dx \leq \int_{x_1-At}^{x_2+At} |u_0(x) - v_0(x)| dx.$$

Remarks //  $\exists$ )

Thm2 (!): Let  $f \in C^2$ ,  $f'' > 0$  and let  $u$  and  $v$  be 2 solutions of  $(*)$  satisfying condition (E). Then  $u = v$  almost everywhere in  $t > 0$ .

Rmk: we call the solution from Thm1 (that is satisfied)  $\Gamma_1$

may be there exist more solutions which do not satisfy cond. (E) or (a)

- 2) property (a) is not valid for systems! Sup-norm of solution can increase! It is non-trivial to prove the bounds on the sup-norm. locally
- 3) Cond. (E) implies some regularity:  $u$  is of bounded total variation (for  $\forall t$  as a function of  $x$ )  
 Indeed, let  $c_1$  be a constant such that  $c_1 > \frac{E}{t}$  and let  $v = u - c_1 x$ . Then  
 $v(x+a, t) - v(x, t) = u(x+a, t) - u(x, t) - c_1 a < a \left( \frac{E}{t} - c_1 \right) < 0$   
 Thus,  $v$  is a non-decreasing function, and  $v$  is a function of local bounded total variation.  
 Since  $c_1 x$  is also of bounded total variation, then  $u$  is of local bounded total variation.  
 $(\Rightarrow$  countable number of jump discontinuities)

- 4) finite speed of propagation:

$$v = v_0 \equiv 0 \Rightarrow \int_{x_1}^{x_2} |u(x, t)| dx \leq \int_{x_1 - At}^{x_2 + At} |u_0(x)| dx$$

Before proving thm 1 and 2 let us understand better condition (E). Let us give some equivalent formulations and interpretations.

Lemmat: a) A smooth solution  $u(x, t)$  satisfies condition (E)

b) If  $u$  has a discontinuity at point  $x_0$ : (but is smooth to the left and to the right of  $x_0$ )  
 $\lim_{x \rightarrow x_0^-} u(x, t) = u_L$  and  $\lim_{x \rightarrow x_0^+} u(x, t) = u_R$  and  
 $satisfies condition (E) \Rightarrow u_L > u_R.$

(discontinuities can be only down).

Proof:

► a) Indeed, let us write:

$$u(x, t) = u_0(x - t f'(u(x, t)))$$

$$u_x = u'_0 \cdot (1 - t f''(u_x)) \Rightarrow u_x = \frac{u'_0}{1 + t f''(u'_0)}$$

If  $u$  is smooth for  $\forall t > 0$ , then  $u'_0 > 0$ .

$$\text{Then } u_x \leq \frac{u'_0}{t f''(u'_0)} = \frac{E}{t} \text{ for } E = \frac{1}{\inf f''}.$$

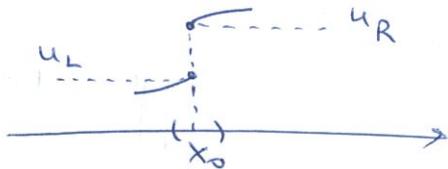
Using Lagrange theorem:  $\frac{u(x+a, t) - u(x, t)}{a} = u_x(\xi, t)$ ,  
 for some  $\xi \in (x, x+a)$ , and (a) is proved

(b) Either  $u_L > u_R$  or  $u_L < u_R$  (the case  $u_L = u_R$  is not a discontinuity).

• For  $u_L < u_R$  the converse of cond. (E) is true:

$$\forall E > 0 \exists x, a > 0, t : \frac{u(x+a, t) - u(x, t)}{a} > \frac{E}{t}.$$

Indeed, fix  $E$  and take small enough neighbourhood of  $x_0$  such that



• for  $x \in (x_0 - \delta, x_0)$   $|u - u_L| \leq \varepsilon = \frac{u_R - u_L}{4}$

• for  $x \in (x_0, x_0 + \delta)$   $|u - u_R| \leq \varepsilon = \frac{u_R - u_L}{4}$ .

This means that for  $\forall x_1 \in (x_0 - \delta, x_0)$  and  $x_2 \in (x_0, x_0 + \delta)$   $u(x_2) - u(x_1) \geq \frac{u_R - u_L}{2}$ .

Fix  $t$  and take <sup>small</sup>  $a$ :

$$x_2 - x_1 = a$$

$$x_1 \in (x_0 - \delta, x_0)$$

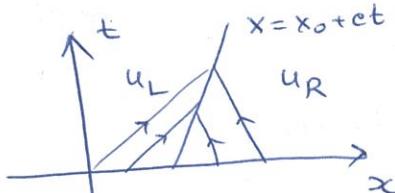
$$x_2 \in (x_0, x_0 + \delta)$$

$$\frac{u(x_2) - u(x_1)}{a} \cancel{\leq \frac{E}{t}}$$
  

$$\frac{u_R - u_L}{2a} = \frac{E}{t}$$

• For  $u_L > u_R$   $\frac{u(x+a, t) - u(x, t)}{a} \leq 0$ , thus  $\forall E > 0$  is ok

Lemma 2 (Remark):  $u$  satisfies condition (E) and is a shock wave solution  $u = \begin{cases} u_L, & x < ct \\ u_R, & x > ct \end{cases}$  then  $f'(u_L) > c > f'(u_R)$  (Lax condition)



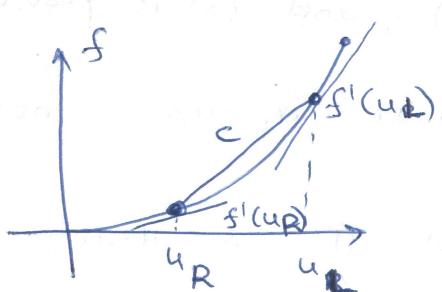
"Characteristics come to the line of discontinuity"

The converse situation corresponds to the case "information" appears on the discontinuity,



in some sense where "the information" appears on the discontinuity, which is not

We will generalize the Lax condition to the case of systems.



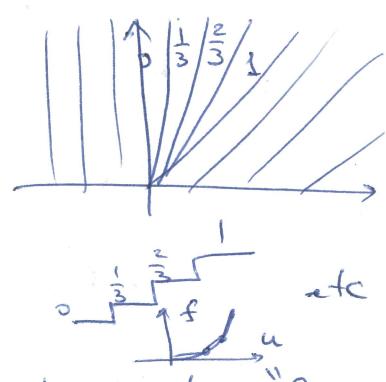
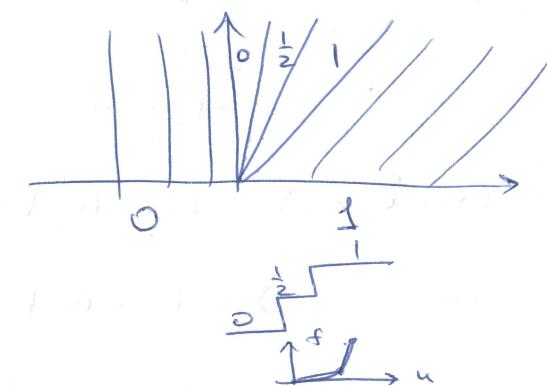
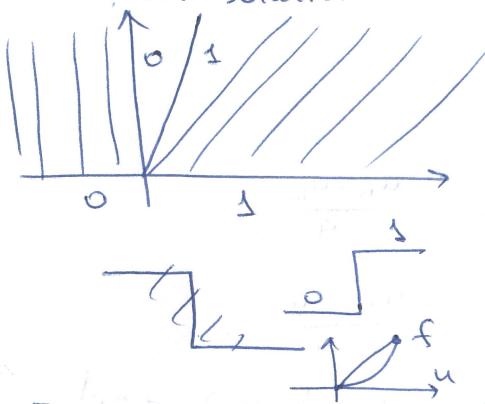
Indeed,  $f'' > 0 \Rightarrow$  (see picture)

$$f'(u_L) > c = \frac{f(u_L) - f(u_R)}{u_L - u_R} > f'(u_R)$$

Remark (on Liu criterion) "internal stability of shock"

Remember the situation with Burgers equation:

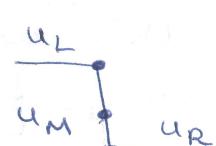
1st solution



In some sense if we divide the shock into "smaller" shocks, they could have a tendency of either glueing into 1 shock (some kind of stability) or going further one from another (instability)

Condition (E)  $\Rightarrow$  this kind of internal stability of a shock, more precisely the inequalities

$$c(u_L, u_R) = \frac{f(u_R) - f(u_L)}{u_R - u_L} \leq c(u_L, u_M) = \frac{f(u_L) - f(u_M)}{u_L - u_M}$$



$$c(u_M, u_R) = \frac{f(u_R) - f(u_M)}{u_R - u_M} \quad \text{if } u_M \in (\min(u_L, u_R), \max(u_L, u_R))$$

If  $u_M \rightarrow u_L$  we have Lax condition.

Vanishing viscosity criterion for shock waves.

We think of equation  $u_t + (f(u))_x = 0$  as a first approximation to the following parabolic eq

$$u_t + (f(u))_x = \varepsilon u_{xx}, \quad \varepsilon > 0 \quad (\text{P})$$

small regularizing term

Rmk 1: it is well-known (and we see in future when dealing with reaction-diffusion equations) that solutions of (P) are very regular (no shocks) "opposite"

Rmk 2: equation (P) is a combination of 2 effects

$\rightarrow u_t + (f(u))_x = 0 \rightsquigarrow$  creates shocks:  $\overbrace{\quad}^{\text{shock}} \rightarrow \overbrace{\quad}^{\text{shock}}$

$\rightarrow u_t = \varepsilon u_{xx} \rightsquigarrow$  "smooths":  $\overbrace{\quad}^{\text{"smooths"}} \rightarrow \overbrace{\quad}^{\text{"smooths"}}$

As a consequence of this confrontation there exist very special solutions, called travelling waves (TW) such that:

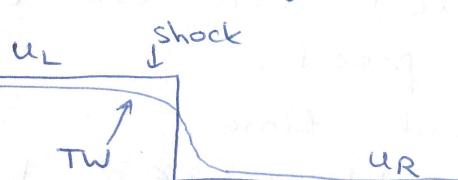
$$u(x, t) = v(x - ct) \quad \text{TW}$$

for  $c \in \mathbb{R}$  and  $v$  - some smooth profile.

They look like "smoothed" shocks!!!

This motivates the following definition:

Defn(vanishing viscosity criterion for shock waves):  
A shock wave is an entropy solution if is a limit<sup>in  $L^\infty$</sup>  of a travelling wave solution of (P)  
 $f \in C^2, f'' > 0$   
as  $\varepsilon \rightarrow 0$ .



Lemma: a shock wave is an entropy solution in sense of defn, iff  $u_L > u_R$ .

Proof: Let's look for travelling wave solutions for eq. (P):  $v\left(\frac{x-ct}{\varepsilon}\right)$ :  $v(-\infty) = u_L, v(+\infty) = u_R$

$$-cv' + (f(v))' = \varepsilon v''$$

Integrate  $\int_{-\infty}^{+\infty}$ :  $-c(v_R - v_L) + f(u_R) - f(u_L) = 0$

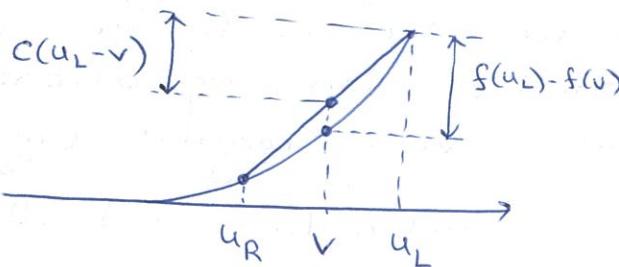
Interesting feature: it is exactly RH condition  
OK, let us integrate  $\int_{-\infty}^{+\infty}$ :  $-c(v(\xi) - u_L) + (f(v(\xi))) - f(u_L) = v'(\xi) \int_{-\infty}^{+\infty}$

ODE)  $v' = f(v) - f(u_L) - c(v - u_L) = F(v)$   
 Note that RHS  $F(u_L) = 0$  and  $F(u_R) = 0$  (due to RH!)

Thus  $u_L$  and  $u_R$  are two fixed points of this ODE

Consider 2 cases:  $u_L > u_R$  and  $u_L < u_R$ .

$$F(v) < 0$$

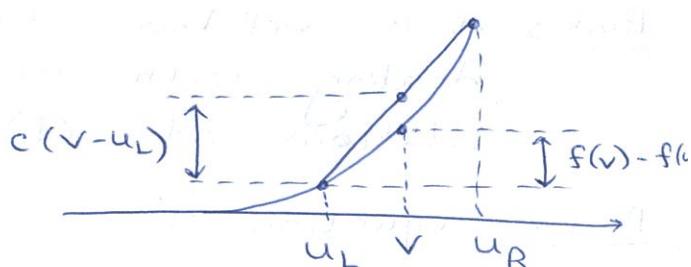


In this case:  $F(v) < 0 \quad \forall v \in (u_R, u_L)$

And there exists a solution  $v$  of ODE:  
 $v(-\infty) = u_L; v(+\infty) = u_R$

L

$$F(v) < 0$$



In this case:  $F(v) < 0 \quad \forall v \in (u_L, u_R)$

And there DOES NOT exist a solution  $v$  of ODE  
 $v(-\infty) = u_L; v(+\infty) = u_R$

Lecture 8: Scalar conservation law :  $\begin{cases} u_t + (f(u))_x = 0 \\ u|_{t=0} = u_0(x) \end{cases} \quad (*)$

$u: \mathbb{R} \times \mathbb{R}_+ \rightarrow \mathbb{R}$  - bounded, measurable

$f: \mathbb{R} \rightarrow \mathbb{R}$ ,  $f \in C^2$ ,  $f'' > 0$ . As we will see it is enough to define  $f$  on the convex hull of values  $u_0$

We understand solutions in weak sense :

$$\int_0^\infty \left[ u \varphi_t + f(u) \varphi_x \right] dx dt + \int_{t=0} u_0 \varphi dx = 0 \quad (**)$$

for every test function  $\varphi \in C_0^\infty$ .

Define  $M := \|u_0\|_\infty$ ,  $A := \max_{|u| \leq M} |f'(u)|$ ,  $\mu := \min_{|u| \leq M} f''(u)$

Today we will start proving theorem on existence.

Thm 1 (E): Let  $u_0 \in L_\infty(\mathbb{R})$ ;  $f \in C^2(\mathbb{R})$ ,  $f'' > 0$  on  $\{u: |u| \leq M\}$

There exists a solution with the following properties

(a)  $|u(x,t)| \leq M$ ,  $(x,t) \in \mathbb{R} \times \mathbb{R}_+$

(b)  $\exists E = E(M, \mu, A) > 0$  such that  $\forall x_0, \forall t > 0$

$$\frac{|u(x+a, t) - u(x, t)|}{a} < \frac{E}{t} \quad (E)$$

"entropy" cond.

(c)  $u$  is stable and depends continuously on  $u_0$ :  
 if  $v_0 \in L_\infty(\mathbb{R})$  with  $\|v_0\|_\infty \leq \|u_0\|_\infty$  and  $v$  is the corresponding constructed solution of (f)  
 with initial data  $v_0$ , then for  $\forall x_1, x_2 \in \mathbb{R}$   
 with  $x_1 < x_2$  and  $t > 0$

$$\int_{x_1}^{x_2} |u(x, t) - v(x, t)| dx \leq \int_{x_1-At}^{x_2+At} |u_0(x) - v_0(x)| dx \quad (S)$$

"stability"

How to prove this theorem?

There exist (at least) 5 approaches:

- (a) Calculus of variations and Hamilton-Jacobi theory
- (b) Vanishing viscosity method
- (c) Non-linear semigroup theory

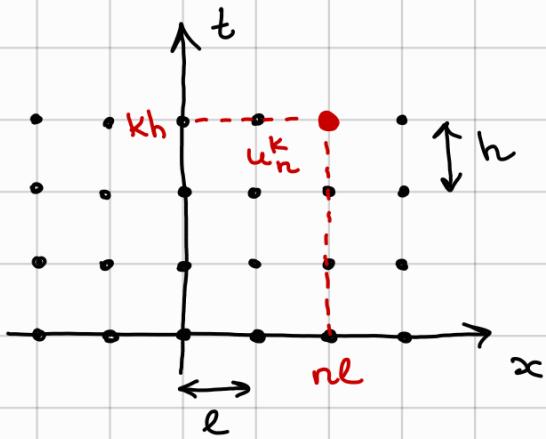
(d) Method of characteristics

(e) Finite-difference method

We will follow Smoller (Chapter 16) and use (e).

Here is the scheme of the proof:

Step 2: discretization in space and time



$$x_n = nl, \quad n \in \mathbb{Z} \quad l = \Delta x > 0$$

$$t_k = kh, \quad k \in \mathbb{N} \cup \{0\} \quad h = \Delta t > 0$$

$$u_n^k = u(nl, kh)$$

Consider a finite-difference (explicit) scheme:

(D) 
$$u_n^{k+1} = \frac{u_{n+1}^k + u_{n-1}^k}{2} - \frac{h}{2e} \cdot (f(u_{n+1}^k) - f(u_{n-1}^k))$$

$$u_n^0 = u_0(nl), \quad n \in \mathbb{Z}$$

In what follows we will always assume:

$$\frac{Ah}{e} \leq 1$$

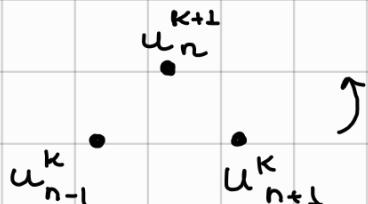
(CFL condition)

Courant-Friedrichs-Lowy

It is important for the stability of the numerical scheme and tells that the time step  $h$  should be small enough.

First, we will formulate and prove some properties of solutions  $u_n^k$  of a discrete eq. (D):

(1a) solution exists (evident!)



(1b) if  $|u_n^0| \leq M$ , then  $|u_n^k| \leq M \quad \forall k \in \mathbb{N}$   
(boundedness)

$$(1c) \exists E = E(M, A, \mu) > 0 : \frac{u_n^k - u_{n-2}^k}{2\ell} \leq \frac{E}{kh} \quad (\text{E-disc})$$

discrete entropy condition

NB: the discrete entropy condition is a natural consequence of a finite difference approximation (D).

(1d) local bounded variation:  $\forall X > 0$  and  $Kh > 0$   
 $\exists c(X, \alpha)$  (but independent of  $h$  and  $\ell$ ):

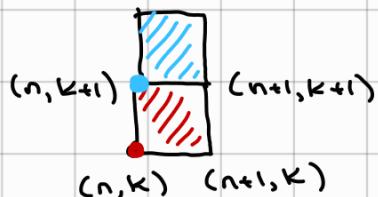
$$\sum_{|nl| \leq X/\ell} |u_{n+2}^k - u_n^k| \leq C \quad \text{and some other...}$$

Step 2: We will prove convergence as  $h, \ell \rightarrow 0$ .

Consider  $U_{h,e}(x, t) = u_n^k$  if

$$nl \leq x \leq (n+s)\ell$$

$$kh \leq t \leq (k+s)h$$



We will prove that there exists

subsequence  $U_{h_i, e_i}$  of  $U_{h,e}$  such that  $U_{h_i, e_i} \rightarrow u(x,t)$   
- some measurable function

Step 3: We will prove that this limiting function satisfies integral equality (\*\*)  
and all properties of theorem on  $\mathbb{Z}$ .

Proof of the theorem 1.

Lemma 1 (boundedness of  $u_n^k$ ):  $|u_n^k| \leq M$ ,  $\forall n \in \mathbb{Z}, k \in \mathbb{N}$

This is an exercise 2 from list 3.

Lemma 2 (discrete entropy condition)

If  $C = \min \left( \frac{\mu}{2}, \frac{A}{4M} \right)$ , then

$$\frac{u_n^k - u_{n-2}^k}{2\ell} \leq \frac{E}{kh} \quad \text{where } E = \frac{1}{C}.$$

Proof:

Let  $z_n^k = \frac{u_n^k - u_{n-2}^k}{2\ell}$  and first let us prove  
some recurrent relation for  $z_n^{k+1}$  of the form  
"  $z_n^{k+1} = A z_{n+1}^k + B z_{n-1}^k + C$ "

$$z_n^{k+1} = \frac{1}{2} [z_{n+1}^k + z_{n-1}^k] - \frac{h}{(2e)^2} (f(u_{n+1}^k) - f(u_{n-1}^k)) + \frac{h}{(2e)^2} (f(u_{n-1}^k) - f(u_{n-3}^k))$$

Notice that due to  $f \in C^2$  we can write

$$f(u_{n+1}^k) = f(u_{n-1}^k) + f'(u_{n-1}^k)(u_{n+1}^k - u_{n-1}^k) + f''(\theta_1) \frac{(u_{n+1}^k - u_{n-1}^k)^2}{2}$$

for some  $\theta_1$  between  $u_{n+1}^k$  and  $u_{n-1}^k$

$$= f(u_{n-1}^k) + f'(u_{n-1}^k) \cdot 2l \cdot z_{n+1}^k + f''(\theta_1) \cdot \frac{(2l)^2}{2} (z_{n+1}^k)^2$$

Analogously,

$$f(u_{n-3}^k) = f(u_{n-1}^k) - f'(u_{n-1}^k) \cdot 2l \cdot z_{n-1}^k + f''(0_2) \frac{(2e)^2}{2} (z_{n-1}^k)^2$$

Thus,

$$\text{Thus, } z_n^{k+1} = z_{n+1}^k \cdot \left[ \frac{1}{2} - \frac{h}{2e} f'(u_{n-1}^k) \right] + z_{n-1}^k \left[ \frac{1}{2} + \frac{h}{2e} f'(u_{n-1}^k) \right] - \frac{h}{2} \cdot \left[ f''(Q_1) \cdot (z_{n+1}^k)^2 + f''(Q_2) \cdot (z_{n-1}^k)^2 \right]$$

Note that  $A + B = 1$  and  $A, B \geq 0$

Define  $\tilde{z}_n^k = \max \{ z_{n-1}^k, z_{n+1}^k, 0 \}$

If  $\tilde{z}_n^k = 0 \quad \forall n$ , then (E-disc) is true since

$$z_{jk} = \hat{z}_{jk} = 0 \leq \frac{1}{\pi} \int_{\Gamma} f(z) dz \quad \forall \epsilon > 0.$$

Thus, w.l.o.g. we can assume  $\tilde{z}_n^k \neq 0$ . Suppose

$\tilde{z}_n^k = z_{n+1}^k$  (the other case is similar)

$$\begin{aligned} z_{n+1}^{k+1} &\leq z_{n+1}^k \cdot [A + B] - h \frac{M}{2} \left( z_{n+1}^k \right)^2 \leq \\ &\leq \tilde{z}_n^k + h.c. \left( \tilde{z}_n^k \right)^2. \end{aligned}$$

Notice that

$$|z_n^k| \leq \frac{M}{\ell} \leq \frac{M}{A h} \leq \frac{M}{n} \cdot \frac{1}{4MC} = \frac{1}{4ch}$$

**CFL**

$$c = \frac{A}{4M}$$

$$\text{Let } M^k = \max_{n \in \mathbb{Z}} \left\{ \tilde{x}_n^k \right\} \geq 0.$$

Let  $\varphi(y) = y - c \cdot h \cdot y^2$ . Since  $\varphi' = 1 - 2chy$ ,  $\varphi$  is

increasing if  $y \leq \frac{1}{2ch}$ . But we have  
 $\tilde{z}_n^k \leq M^k \leq \frac{1}{4ch} < \frac{1}{2ch}$ .

So that  $\varphi(\tilde{z}_n^k) \leq \varphi(M^k)$  and we have

$$\tilde{z}_n^k - ch(\tilde{z}_n^k)^2 \leq M^k - ch(M^k)^2$$

Thus,  $\tilde{z}_n^{k+1} \leq M^k - ch(M^k)^2 \quad \forall n \in \mathbb{Z}$ .

It follows that

$$M^{k+1} \leq M^k - ch(M^k)^2 \quad (M)$$

Claim:  $M^k = \frac{1}{ch k + 1/\mu_0}$ .

Suppose we have proven claim. Let us see how it helps to prove lemma 2. Indeed,

$$z_n^k \leq M^k \leq \frac{1}{ch k + 1/\mu_0} \leq \frac{1}{ch k} = \frac{E}{h k}, E = \frac{1}{c}.$$

Proof of claim: first - intuition why such estimate could be true

Inequality (M) for  $M^k$  is a discrete analog of ODE inequality:  $\varphi' \leq -ch \varphi^2$

if it was an equality  $\varphi' = -ch \varphi^2$ , then the solution is:

$$\frac{d\varphi}{\varphi^2} = -ch dt$$

$$-\frac{1}{\varphi} = -cht + C_1$$

$$\varphi(t) = \frac{1}{cht - C_1} \quad \text{and with } \varphi(0) = \varphi_0$$

we will have

$$\varphi(t) = \frac{1}{cht + 1/\varphi_0}$$

So one can try to prove

$$\varphi(t) \leq \frac{1}{cht + 1/\varphi_0}$$

Second, let us make the formal proof.

We will do it by induction.

Base :  $k=0$  : - clear :  $M^0 = \frac{1}{1/\mu^0} = M^0$ .

$k > 0$  : suppose that

$$M^k = \frac{1}{ch k + 1/\mu^0}$$

and we want to prove that

$$M^{k+1} = \frac{1}{ch(k+1) + 1/\mu^0}$$

We have:  $\frac{1}{M^k} \geq ch k + \frac{1}{\mu^0}$ , so

$$1 - ch M^k \geq 1 - ch k M^k \geq \frac{M^k}{\mu^0} \geq 0.$$

Thus  $1 - (ch M^k)^2 \geq 0$ .

We have  $M^{k+1} \leq M^k (1 - ch M^k)$ , so that

$$\frac{M^{k+1}}{1 - ch M^k} \leq M^k \leq \frac{M^k}{1 - (ch M^k)^2}$$

and thus  $M^{k+1} \leq \frac{M^k}{1 + ch M^k} = \frac{1}{ch + 1/\mu^k} \leq \frac{1}{ch(k+1) + 1/\mu^0}$  q.e.d. ■

L

Lemma (space estimate) : For any  $X > 0$  and  $kh \geq \alpha > 0$ , there is a constant  $C = C(X, \alpha, N)$  (but independent on  $h, e$ ) such that:

$$\sum_{|nl| \leq X/e} |u_{n+2}^k - u_n^k| \leq C$$

Proof :

► Set  $v_n^k = u_n^k - c_1 nl$ , where  $c_1$  is chosen so large that  $E/\alpha < c_1$ . Then

$$\begin{aligned} v_{n+2}^k - v_n^k &= u_{n+2}^k - u_n^k - 2c_1 l \leq \frac{2E}{kh} - 2c_1 l \leq \\ &\leq 2E \left( \frac{E}{\alpha} - c_1 \right) < 0, \text{ so } v_n^k \text{ is decreas. inn} \end{aligned}$$

Thus

$$\begin{aligned} \sum_{|nl| \leq X/e} |u_{n+2}^k - u_n^k| &\leq \sum |v_{n+2}^k - v_n^k| + \sum 2c_1 l = \\ &= - \sum (v_{n+2}^k - v_n^k) + 2c_1 l \left( \frac{2X}{e} + 1 \right) \leq 4M + 2c_1 X + c_2 X \end{aligned}$$

L telescopic sum  $\leq 4(M + c_1 X)$

q.e.d. ■

Lecture 9 : We continue proving theorem on existence of entropy solution for scalar conslaw.

Lemma 4 (time estimate -  $u_n^k$  are  $L^2$  locally Lipschitz in  $k$ )

If  $h/e \geq \delta > 0$  and  $h, e \leq 1$ , then exists  $L > 0$  (independent of  $h, e$ ) such that

if  $k > p$ , where  $(k-p)$  is even and  $p\hbar \geq 2 > 0$ , then

$$\sum_{|n| \leq X/e} |u_n^k - u_n^p| e \leq L(k-p) h$$

A similar estimate holds if  $(k-p)$  is odd.

Proof:

Let us express  $u_n^k$  in terms of  $u_n^p$  where  $(k-p)$  is even.

$$u_n^k = \frac{1}{2} (u_{n+1}^{k-1} + u_{n-1}^{k-1}) - \frac{h}{2e} f'(Q) (u_{n+1}^{k-1} - u_{n-1}^{k-1}) = \\ = u_{n+1}^{k-1} \left( \frac{1}{2} - \frac{h}{2e} f'(Q) \right) + u_{n-1}^{k-1} \left( \frac{1}{2} + \frac{h}{2e} f'(Q) \right)$$

$$\text{or } u_n^k = a_{n+1}^{k-1} u_{n+1}^{k-1} + a_{n-1}^{k-1} u_{n-1}^{k-1}, \text{ where} \\ a_{n+1}^{k-1} + a_{n-1}^{k-1} = 1 \text{ and } a_{n+1}^{k-1}, a_{n-1}^{k-1} \geq 0.$$

Applying this to  $u_{n-1}^k$  and  $u_{n+1}^k$  gives a formula:

$$u_n^{k+1} = A u_{n+2}^{k-1} + B u_n^{k-1} + C u_{n-2}^{k-1}$$

where  $A, B, C \geq 0$ ,  $A+B+C=1$ .

$$\text{Hence, } |u_n^{k+1} - u_n^{k-1}| \leq A |u_{n+2}^{k-1} - u_n^{k-1}| + C |u_{n-2}^{k-1} - u_n^{k-1}|$$

Multiplying this by  $\Delta x = e$  and summing, we

get :

$$\sum_{|n| \leq X/e} |u_n^{k+1} - u_n^{k-1}| \Delta x \leq C \Delta x$$

↑  
Lemma 3

Now if  $(k-p)$  is even, we can do this operation several times and using the triangle inequality, we get :

$$\sum_{|n| \leq X/e} |u_n^k - u_n^p| \Delta x \leq \sum_{i=p}^{k-2} \sum_{|n| \leq X/e} |u_n^{i+2} - u_n^i| \Delta x \leq (k-p) C \Delta x \leq$$

$$\leq \frac{\Delta t}{\delta} (k-p) c = L(k-p) h \quad \text{for } L = \frac{c}{\delta}, h = \Delta t$$

Lemma 5 (stability): Let  $u_n^k$  and  $v_n^k$  be solutions to the finite-difference scheme (D) corresponding to the initial conditions  $u_n^0$  and  $v_n^0$ , respectively, where

$$\sup_{n \in \mathbb{Z}} |u_n^0| \leq M \quad \text{and} \quad \sup_{n \in \mathbb{Z}} |v_n^0| \leq M$$

Then, if  $k > 0$ ,

$$\sum_{|n| \leq N} |u_n^k - v_n^k| \cdot \ell \leq \sum_{|n| \leq N+k} |u_n^0 - v_n^0| \cdot \ell$$

Proof:

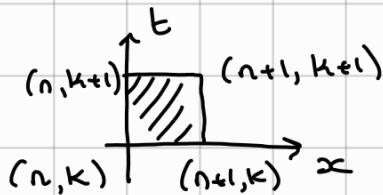
►  $w_n^k = u_n^k - v_n^k$ . From (D) we have

$$\begin{aligned} w_n^{k+1} &= u_n^{k+1} - v_n^{k+1} = \frac{u_{n+1}^k + u_{n-1}^k}{2} - \frac{h}{2e} (f(u_{n+1}^k) - f(u_{n-1}^k)) \\ &\quad - \frac{v_{n+1}^k + v_{n-1}^k}{2} + \frac{h}{2e} (f(v_{n+1}^k) - f(v_{n-1}^k)) = \\ &= \frac{w_{n+1}^k + w_{n-1}^k}{2} - \frac{h}{2e} (f(u_{n+1}^k) - f(v_{n+1}^k)) \\ &\quad + \frac{h}{2e} (f(u_{n-1}^k) - f(v_{n-1}^k)) = \\ &= w_{n+1}^k \underbrace{\left[ \frac{1}{2} - \frac{h}{2e} f'(Q_1) \right]}_{\geq 0 \text{ due to CFL}} + w_{n-1}^k \underbrace{\left[ \frac{1}{2} + \frac{h}{2e} f'(Q_2) \right]}_{\geq 0} \end{aligned}$$

Now proceed by induction.

$$\begin{aligned} \sum_{|n| \leq N} |w_n^{k+1}| &\leq \sum_{|n| \leq N} |w_{n+1}^k| \cdot A_{n+1}^k + \sum_{|n| \leq N} |w_{n-1}^k| \cdot B_{n-1}^k = \\ &= \sum_{m=N}^{N+1} |w_m^k| A_m^k + \sum_{m=-N}^{N-1} |w_m^k| \cdot B_m^k \leq \\ &\leq \sum_{|m| \leq N+1} |w_m^k| A_m^k + \sum_{|m| \leq N+1} |w_m^k| \cdot B_m^k \leq \sum_{|m| \leq N+1} |w_m^k| \end{aligned}$$

Step 2: Rather than define  $u_n^k$  in mesh points let us continue  $u_n^k$  as a piecewise constant function in the upper half plane.



$$U_{h,e}(x,t) = u_n^k \text{ if } nl \leq x \leq (n+1)l \\ kh \leq t \leq (k+1)h$$

So we have a family of functions  $\{U_{h,e}\}$  and would like to choose a convergent subsequence  $U_{h_i, e_i}$  as  $h_i, l_i \rightarrow 0$   $i \rightarrow \infty$ .

Lemma 6 (convergence : the set of functions  $\{U_{h,e}\}$  is compact in the topology of  $L_1$ -convergence on compacta)

There exists a subsequence  $\{U_{h_i, e_i}\}_{i \in \mathbb{N}}$  which converges to a measurable function  $u(x,t)$  in the sense that for  $\forall X > 0, t > 0, T > 0$  both

$$\int_{|x| \leq X} |U_{h_i, e_i}(x,t) - u(x,t)| dx \rightarrow 0 \text{ as } h_i, l_i \rightarrow 0$$

and

$$\int_0^T \int_{|x| \leq X} |U_{h_i, e_i}(x,t) - u(x,t)| dx dt \rightarrow 0.$$

Furthermore, the function  $u(x,t)$  satisfies :

- (a)  $\sup_{\substack{x \in \mathbb{R} \\ t > 0}} |u(x,t)| \leq M$ ; (b) inequality (S)  
(stability)

Proof :

► First, take  $t = \text{const}$  and consider  $U_{h,e}(x,t)$  as functions of  $x$ . By Lemma 1 and Lemma 3 the set of functions  $\{U_{h,e}\}$  is bounded and have uniformly bounded total variation on each bounded interval in  $x$ .

Helly's theorem (simple version):

A uniform bounded sequence of monotone, real functions admits a convergent subsequence.

Helly's theorem (generalized version):

A uniform bounded sequence of BV<sub>loc</sub> (locally of bounded variation) real functions admits a convergent subsequence on every compact set.

Rmk: a function of BV<sub>loc</sub> can be written as a sum of increasing and decreasing functions (on each compact interval). This is why the generalized version of the Helly's theorem is true.

So by Helly's theorem on each interval

we have a convergent subsequence  $\{U'_{h,e}\}$ .

By a standard diagonal process we can construct a subsequence  $\{U''_{h,e}\}$  from  $\{U'_{h,e}\}$  which converges at every  $x \in \mathbb{R}$  for this particular  $t = \text{const} > 0$ .

Second, take  $\{t_m\}_{m=1}^{\infty}$  - a countable and dense subset of  $(0, T)$ , e.g.  $\mathbb{Q} \cap (0, T)$ .

For  $t=t_1$  we have  $\{U'_{h_1,e_1}\}$  a convergent subsequence.

For  $t=t_2$  take a convergent sub.  $\{U'_{h_2,e_2}\}$  from  $\{U'_{h_1,e_1}\}$  etc. So we have:

$$t=t_1: U'_{h_1,e_1}$$

$$t=t_2: U'_{h_2,e_2}$$

$$t=t_3: U'_{h_3,e_3}$$

$$t=t_4: U'_{h_4,e_4}$$

$$U'_{h_1,e_1}$$

$$U'_{h_2,e_2}$$

$$U'_{h_3,e_3}$$

$$U'_{h_4,e_4}$$

$$U'_{h_3,e_3}$$

$$U'_{h_4,e_4}$$

$$U'_{h_1,e_1}$$

$$U'_{h_2,e_2}$$

$$U'_{h_4,e_4}$$

$$U'_{h_1,e_1}$$

$$U'_{h_3,e_3}$$

$$U'_{h_2,e_2}$$

...

By a standard diagonal process, we can choose a subsequence  $\{U'_{h_i,e_i}\}$  which converges for all  $\{t_m\}_{m=1}^{\infty}$  and all  $x \in \mathbb{R}$ .

Third, we want to show that there is a convergence for all  $t \in (0, T)$ . So that in the limit we indeed obtain a function defined in the strip  $0 < t < T$ .

Let  $U_i = U_{e_i, h_i}$  and we want to show that

$$I_{ij} = \int_{-X}^X |U_i(x, t) - U_j(x, t)| dx \rightarrow 0 \quad \forall i, j \rightarrow \infty$$

i.e. that  $\{U_i\}$  is a Cauchy sequence in  $L_1(|x| \leq X)$

For  $t \in (0, T)$  we find a subsequence  $\{t_{m_s}\} \subset \{t_m\}$  such that  $t_{m_s} \rightarrow t$  as  $s \rightarrow \infty$ . Let  $\tau_s = t_{m_s}$ . Then

$$\begin{aligned} I_{ij}(t) &\leq \int_{-X}^X |U_i(x, t) - U_i(x, \tau_s)| dx + \int_{-X}^X |U_i(x, \tau_s) - U_j(x, \tau_s)| dx \\ &\quad + \int_{-X}^X |U_j(x, t) - U_j(x, \tau_s)| dx =: I_1 + I_2 + I_3 \end{aligned}$$

For  $t = \tau_s$  we have a convergence of  $U_i$ , thus for  $s$  large enough we have  $I_2 < \varepsilon/3$

Let's estimate  $I_1$ :

$$\begin{aligned} I_1 &= \int_{-X}^X |U_i(x, [\frac{t}{h_i}] h_i) - U_i(x, [\frac{\tau_s}{h_i}] h_i)| dx = \\ &= \sum_{\substack{(n+1)l_i \\ |n| < \frac{X}{e_i} + 1}} \int_{nl_i}^{(n+1)l_i} |U_i(x, [\frac{t}{h_i}] h_i) - U_i(x, [\frac{\tau_s}{h_i}] h_i)| dx = \\ &= \sum_{\substack{|n| < \frac{X}{e_i} + 1}} |u_n^{[t/h_i]} - u_n^{[\tau_s/h_i]}| l_i \leq [h_i] \left[ \frac{t}{h_i} \right] - \left[ \frac{\tau_s}{h_i} \right] \uparrow \\ &\quad \text{Lemma 4} \end{aligned}$$

$$\leq [t - \tau_s] < \frac{\varepsilon}{3} \quad \text{for } s \text{ large enough.}$$

Analogously,  $I_3 < \frac{\varepsilon}{3}$ . Thus  $I_{ij} \leq 3 \cdot \frac{\varepsilon}{3} = \varepsilon$ .

We have proved pointwise limit for every  $t \in (0, T)$ , that is  $\exists u(x, t) \in L_1(|x| \leq X)$  (in part, measurable)

Fourth, let us show that  $I_{ij} \rightarrow 0$  uniformly in  $t$ ,  $0 \leq t \leq T$ . Indeed, fix  $\varepsilon > 0$ . Choose finite subset  $\mathcal{F} \subset \{t_m\}$  such that if  $0 \leq t \leq T$  there is a  $t_m \in \mathcal{F}$  such that  $L(t-t_m) < \frac{\varepsilon}{3}$ . Then we choose  $i, j$  so large that  $I_{ij} < \frac{\varepsilon}{3}$  for all  $t_m \in \mathcal{F}$  (it is possible because  $\mathcal{F}$  is finite). This reasoning gives us the desired uniformity int.

Fifth, using uniform convergence, we have

$$\forall \tau \in (0, T] \quad \int_{\tau}^T I_{ij} dt \rightarrow 0.$$

Now we write

$$\int_0^T = \int_0^\tau + \int_\tau^T : \quad \underbrace{\int_\tau^T}_{< \frac{\varepsilon}{2} \text{ for } i, j \text{ suffic. large}} + \underbrace{\int_0^\tau}_{< \frac{\varepsilon}{2} \text{ if } 8N\chi_\tau < \varepsilon} |U_i - U_j| dx dt = \iint_{\Omega \times (0, T)} |U_i - U_j| dx dt + \int_{\Omega \times [\tau, T]} |U_i - U_j| dx dt < \varepsilon$$

That means

$$\int_0^T I_{ij} dt \rightarrow 0 \quad \text{as } i, j \rightarrow +\infty.$$

Sixth, since local convergence in  $L_1$  implies pointwise convergence a.e. of a subsequence, we see

$$|U_i| \leq M \Rightarrow |u| \leq M$$

and Lemma 5  $\Rightarrow$  (S)

Step 3: Let us show that the limiting function  $u(x, t)$ , indeed, satisfies the properties from thm.

Lemma 7 (entropy inequality) :  $u$  satisfies (E).

Proof:

It is sufficient to show that if  $(x_1 - x_2) > 2l_i$  and  $t > h_i$  then

$$\frac{U_i(x_1, t) - U_i(x_2, t)}{x_1 - x_2} < \frac{2E}{t - h_i}.$$

Let  $x_1 > x_2$  and note that

$$U_i(x_j, t) = U_i(x_j - \gamma_j, [\frac{t}{h_i}] h_i) \quad j=1,2$$

for some  $0 \leq \gamma_j < l_j$ . Thus,

$$\frac{U_i(x_1, t) - U_i(x_2, t)}{x_1 - x_2} = \frac{1}{x_1 - x_2} \sum_{\substack{\text{over all integers} \\ \text{in the interval } [x_2 - \gamma_2, x_1 - \gamma_1]}} (u_n^k - u_{n-2}^k) \quad \text{for } k = [\frac{t}{h_i}]$$

Using Lemma 2, we have

$$\begin{aligned} \frac{U_i(x_1, t) - U_i(x_2, t)}{x_1 - x_2} &\leq \frac{E(x_1 - \gamma_1 - x_2 + \gamma_2)}{[\frac{t}{h_i}] h_i (x_1 - x_2)} \leq \frac{E(x_1 - \gamma_1 - x_2 + \gamma_2)}{(t - h_i)(x_1 - x_2)} \\ &= \frac{E}{t - h_i} + \frac{E(\gamma_2 - \gamma_1) < l_i}{(t - h_i)(x_1 - x_2)} < \frac{2E}{t - h_i} \blacksquare \end{aligned}$$