ON SOLUTIONS OF A RIEMANN PROBLEM FOR A CHEMICAL FLOODING MODEL

Yulia P. Petrova

St Petersburg, Russia

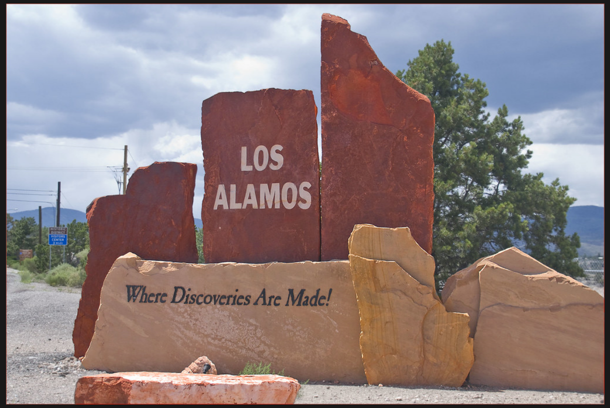
Bradley J. Plohr

Los Alamos, USA

Dan Marchesin

Rio de Janeiro, Brasil

St Petersburg University



75 anos impa
Instituto de Matemática Pura e Aplicada

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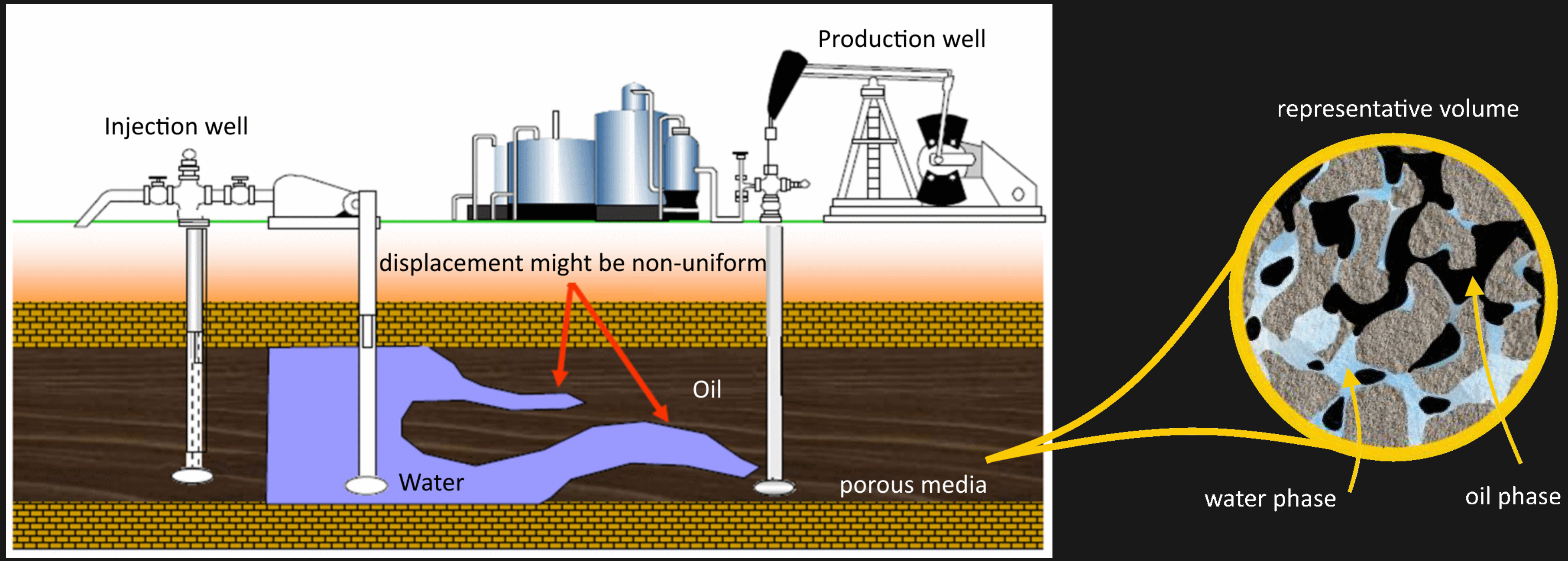
MOTIVATION: ENHANCED OIL RECOVERY (EOR)

We are interested in the mathematical model of oil recovery. Some features:

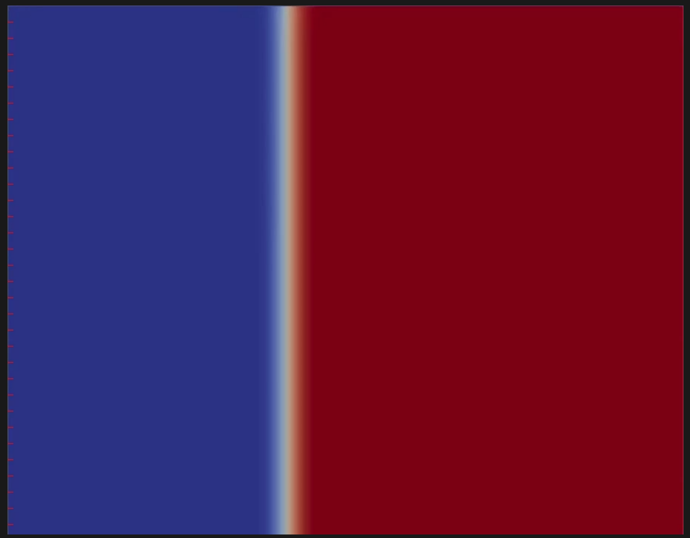
- *porous media* (averaged models of flow)
- unknown variables: $s \in [0, 1]$ - water saturation, $1 - s$ - oil concentration
- relatively *small speeds* (≈ 1 meter per day): Navier-Stokes $\longrightarrow$ Darcy's law
- multiphase flow: oil, water, gas.
- applications to EOR methods: thermal, gas, *chemical flooding*



TWO MAIN DIRECTIONS OF INVESTIGATION

Stable dislacement

- 1-dim in spatial variable



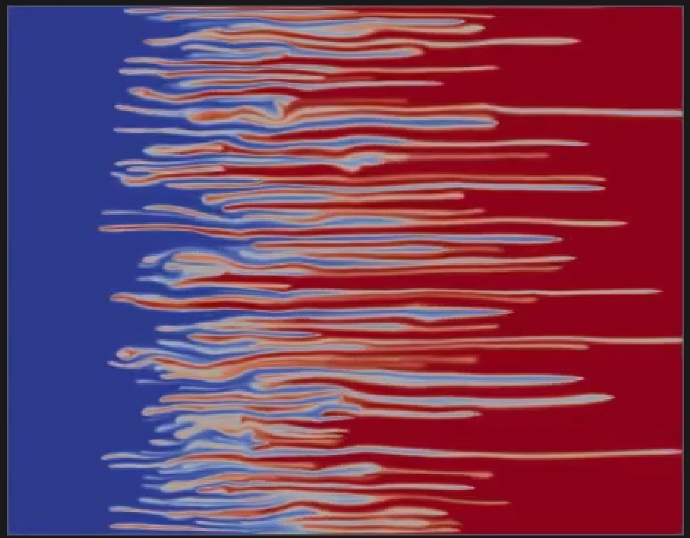
- main question: find an exact solution for a Riemann problem
- hyperbolic conservation laws

$$s_t + f(s,c)_x = 0,$$
$$(cs)_t + (cf(s,c))_x = 0.$$

Example: chemical flooding model

Unstable displacement

- 2-dim (or 3-dim)



- source of instability: water and oil/polymer have different viscosities
- viscous fingering phenomenon

$$c_t + u \cdot \nabla c = \varepsilon \Delta c,$$
$$\mathrm{div}(u) = 0, \quad u = -\frac{1}{\mu(c)} \nabla p.$$

Example: Peaceman model

PREVIEW OF RESULTS

We will consider a chemical flooding model (1980' E. Isaacson)

$$\begin{aligned} s_t + f(s,c)_x &= 0, \\ (cs)_t + (cf(s,c))_x &= 0. \end{aligned} \qquad (M_0)$$

- contact discontinuities $\Rightarrow$ non-uniqueness of solutions to a Riemann problem
- vanishing viscosity criterion doesn't help
- Isaacson-Glimm admissibility criterion needs to be justified

Main idea: add small physical effect (chemical's adsorption on the rock, α small)

$$\begin{aligned} s_t + f(s,c)_x &= 0, \\ (cs + \alpha a(c))_t + (cf(s,c))_x &= 0. \end{aligned} \qquad (M_\alpha)$$

Vanishing adsorption criterion: the admissible contacts for M_0 are the L^1_{loc} limits of a family of admissible solutions for M_α as $\alpha \to 0$.

HYPERBOLIC SYSTEMS OF CONSERVATION LAWS

$$G(U)_t + F(U)_x = 0$$

- $G(U)$ is the *accumulation function* (conserved quantities)
- $F(U)$ is the *flux function* (the flow of conserved quantities)

Simplest example: the *wave equation*

$$y_{tt} - c^2\, y_{xx} = 0 \qquad \text{(J. d'Alembert, 1747)}$$

can be written as a system of two first-order equations on the state vector $U = (y_x \quad y_t)^T$

$$U_t + A\,U_x = 0 \quad \text{with} \quad A := \begin{pmatrix} 0 & -1 \\ -c^2 & 0 \end{pmatrix}$$

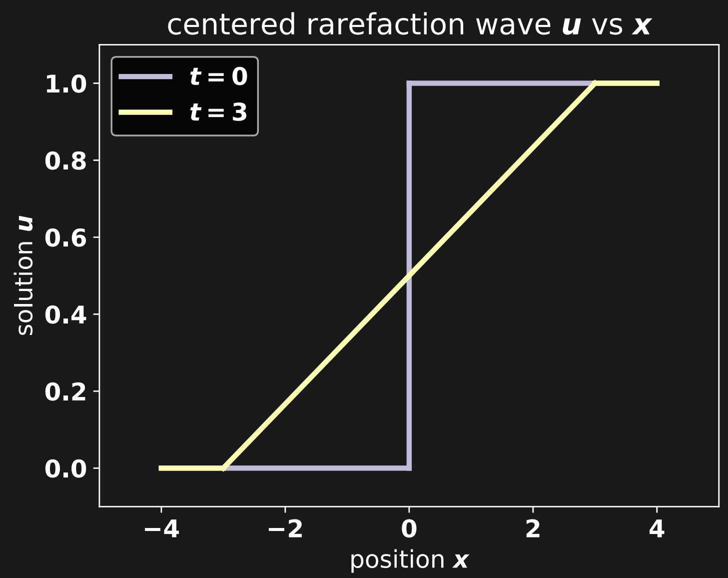
- the eigenvalues $\lambda_1 = -c, \lambda_2 = +c$ of A are *real*, the system is *hyperbolic*. Solutions contain 2 *wave modes* propagating at the *velocities* λ_1 and λ_2

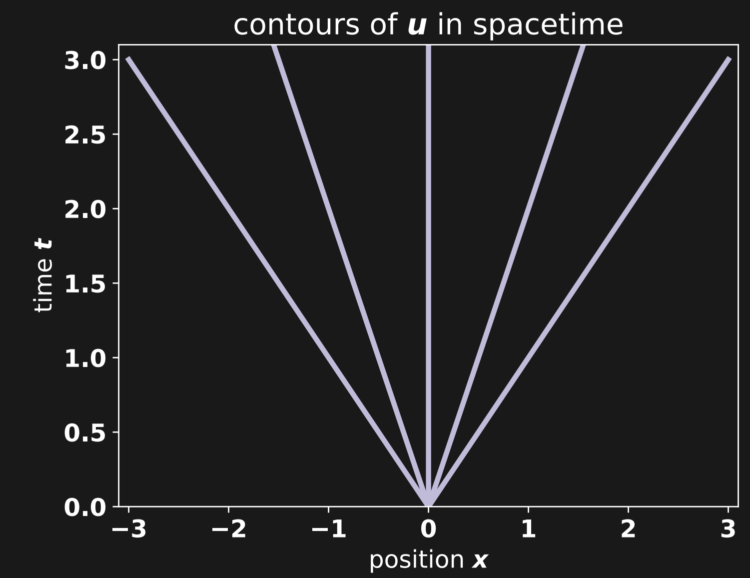
HYPERBOLIC EQUATIONS: NON-LINEAR SCALAR EQUATION

- *inviscid Burgers' equation* (1948)

$$u_t + \left(\tfrac{1}{2}u^2\right)_x = 0$$

- non-linearity implies *wave speed* $\lambda(u) = u$ *depends on the state* u
- therefore waves can spread (*rarefaction wave*) or focus (*shock wave*)
- *Buckley-Leverett model* for two-phase (water/oil) flow in a porous medium





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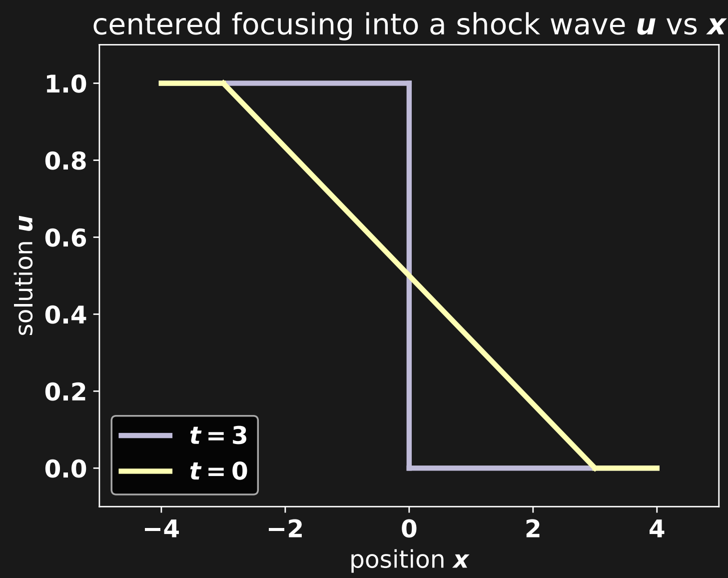
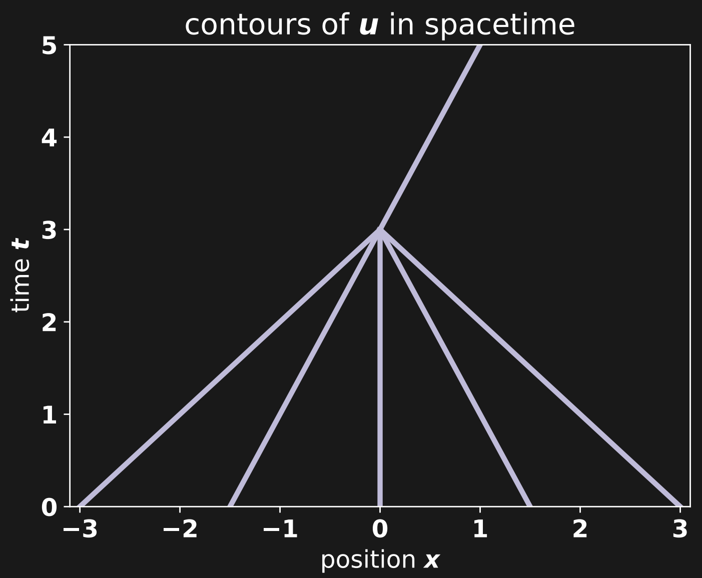




GAS DYNAMICS: PROTOTYPE SYSTEM OF CONSERVATION LAWS

- gas state U comprises the *density*, *velocity*, and *specific energy*
- physical content of equations: *conservation of mass, momentum, and energy*
- 3 wave modes:
  - left and right-facing nonlinear sonic modes (*shock* and *rarefaction waves*)
  - *contact mode* (e.g., a cold front on a weather map)
    - pressure and velocity do not jump, but temperature can
    - wave neither focuses nor spreads

WORK OF RIEMANN (1858)

- Riemann solved the initial-value problem with data having a *single jump*

$$U(x,0) = U_L, \quad x < 0; \qquad U(x,0) = U_R, \quad x > 0$$

- took advantage of the *scale invariance* of the equations and the data:

$$U(\alpha\, x, \alpha\, t) = U(x,t) \quad \text{for all } \alpha > 0$$

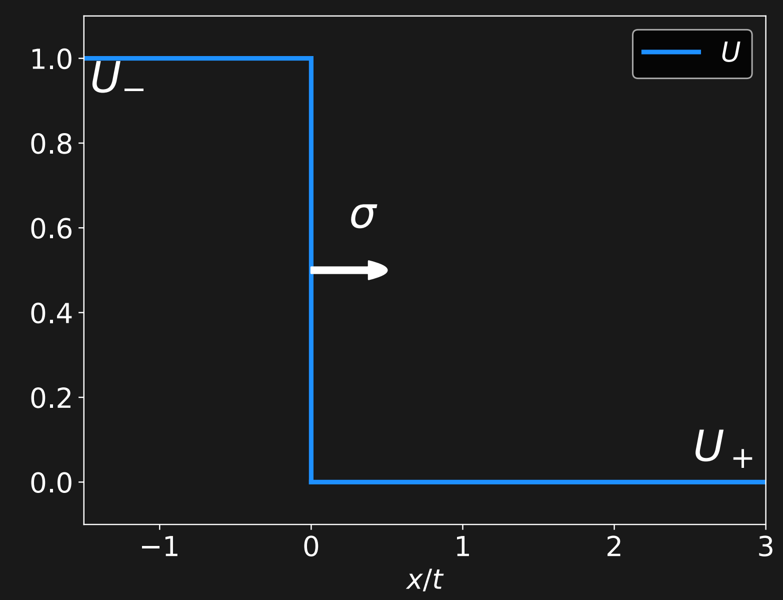
- solution to a Riemann problem is important because:
  - it appears in a long-term behavoir of Cauchy problem
  - helps to prove the existence of solutions to Cauchy problem (Glimm's method, random choice method)
  - helps to construct numerical solution (Godunov method)

SHOCK WAVES: RH CONDITION

- discontinuous solutions are defined in the sense of distributions (*weak form*)
- for a shock wave from U_- to U_+ moving with velocity σ, the weak condition amounts to the following *Rankine-Hugoniot condition*

$$-\sigma G(U_-) + F(U_-) = -\sigma G(U_+) + F(U_+)$$

- RH means *conservation*: what flows into left side flows out of the right side



SHOCK WAVES: ADMISSIBILITY CRITERIA

- *Problems* from the perspectives of both mathematics and physics:
  - if all RH solutions are allowed, a Riemann problem has multiple solutions
  - some RH solutions violate physical principles
- physical *selection criteria* for gas dynamics relate to
  - second law of thermodynamics: *entropy cannot decrease*
  - neglected physical effects (*viscosity and heat conduction*) with small coefficients nonetheless having a large influence (as they are multiplied by large solution gradients)
- *vanishing viscosity criterion*: consider a diffusive system of conservation laws

$$G(U)_t + F(U)_x = \epsilon[B(U)\,U_x]_x$$

TRAVELING WAVE SOLUTIONS OF THE DIFFUSIVE SYSTEM (HOPF, 1948)

- $U(x,t) = \hat{U}(\xi)$ with $\xi := x - \sigma\, t$ for fixed *shock velocity* σ
- reduction to first-order system of ordinary differential equations:

$$\epsilon\, B(\hat{U})\, \hat{U}_\xi = -\sigma \left[G(\hat{U}) - G(U_-)\right] + F(\hat{U}) - F(U_-)$$

- U_- and U_+ are fixed points and we look for an orbit connecting them

$$\hat{U}(-\infty) = U_-\,, \qquad \hat{U}(+\infty) = U_+$$

- diffusive terms cause a shock wave to have a *thin, smooth internal structure* in which nonlinear focusing balances diffusive spreading
- traveling wave solution *approaches* the jump discontinuity in L^1 as $\epsilon \to 0^+$

DIFFUSIVE TERMS ARE NOT A PANACEA

- Contact discontinuities do not have diffusive profiles (there is lack of nonlinear focusing to balance diffusive spreading)
- equilibria are not isolated:

$$-\sigma \left[G(\hat{U}) - G(U_-)\right] + F(\hat{U}) - F(U_-)$$

  vanishes all along the contact curve

- Ad hoc admissibility criteria need to be justified

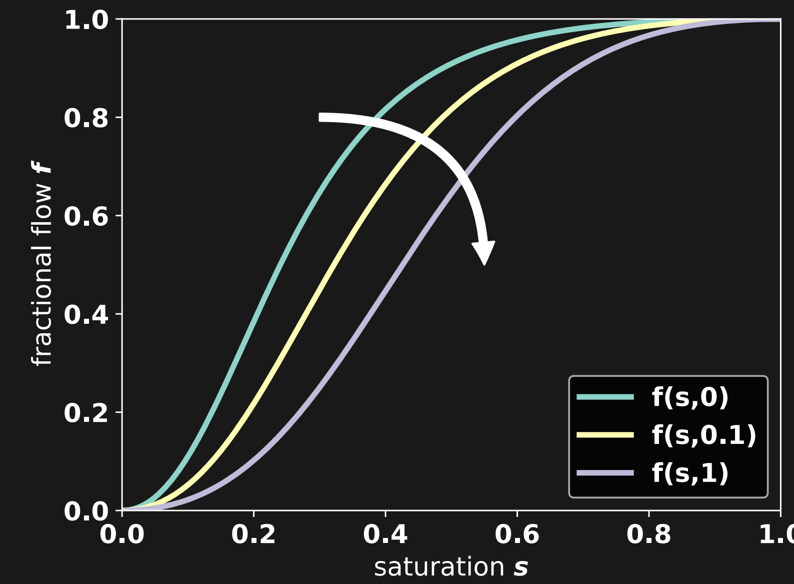
QUESTIONS? COMMENTS?

GLIMM-ISAACSON MODEL (KKIT MODEL)

Two-phase oil-water flow with *polymer* in the water to increase its viscosity

$$s_t + f(s,c)_x = 0$$
$$(cs)_t + (cf(s,c))_x = 0$$

- $s \in [0,1]$ - water saturation
- $c \in [0,1]$ - polymer concentration in water
- f - fractional flow function: affected by polymer
  - S-shaped in s
  - f is monotone in c



Initial data:

$$(s,c)\big|_{t=0} = \begin{cases} (s_L, c_L), & \text{if } x \leqslant 0, \\ (s_R, c_R), & \text{if } x \geqslant 0. \end{cases}$$

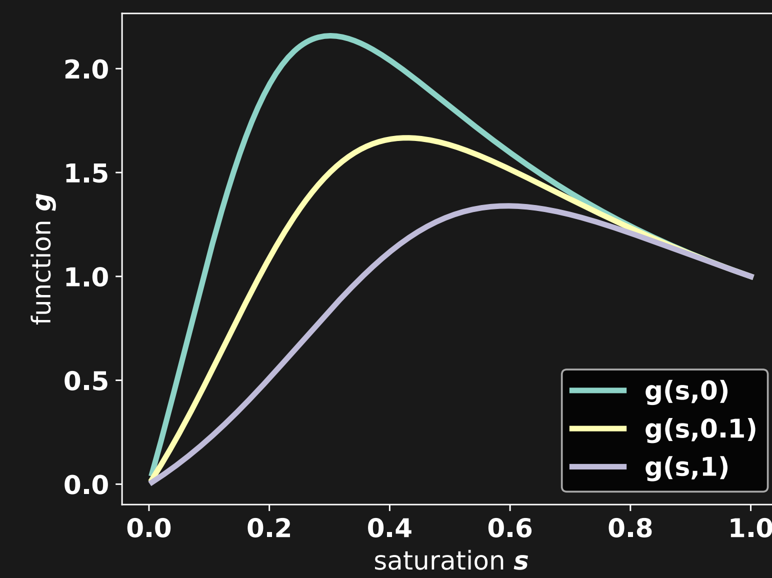
Question: find an exact solution $s(x,t)$ and $c(x,t)$ to this initial value problem

GLIMM-ISAACSON MODEL (KKIT MODEL)

Two-phase oil-water flow with *polymer* in the water to increase its viscosity (a symmetric form)

$$
\begin{aligned}
s_t + (sg(s,b))_x &= 0 \\
b_t + (bg(s,b))_x &= 0
\end{aligned}
$$

- $s \in [0,1]$ - water saturation
- $b = sc \in [0,1]$ - total amount of polymer
- $g = f/s$



Initial data:

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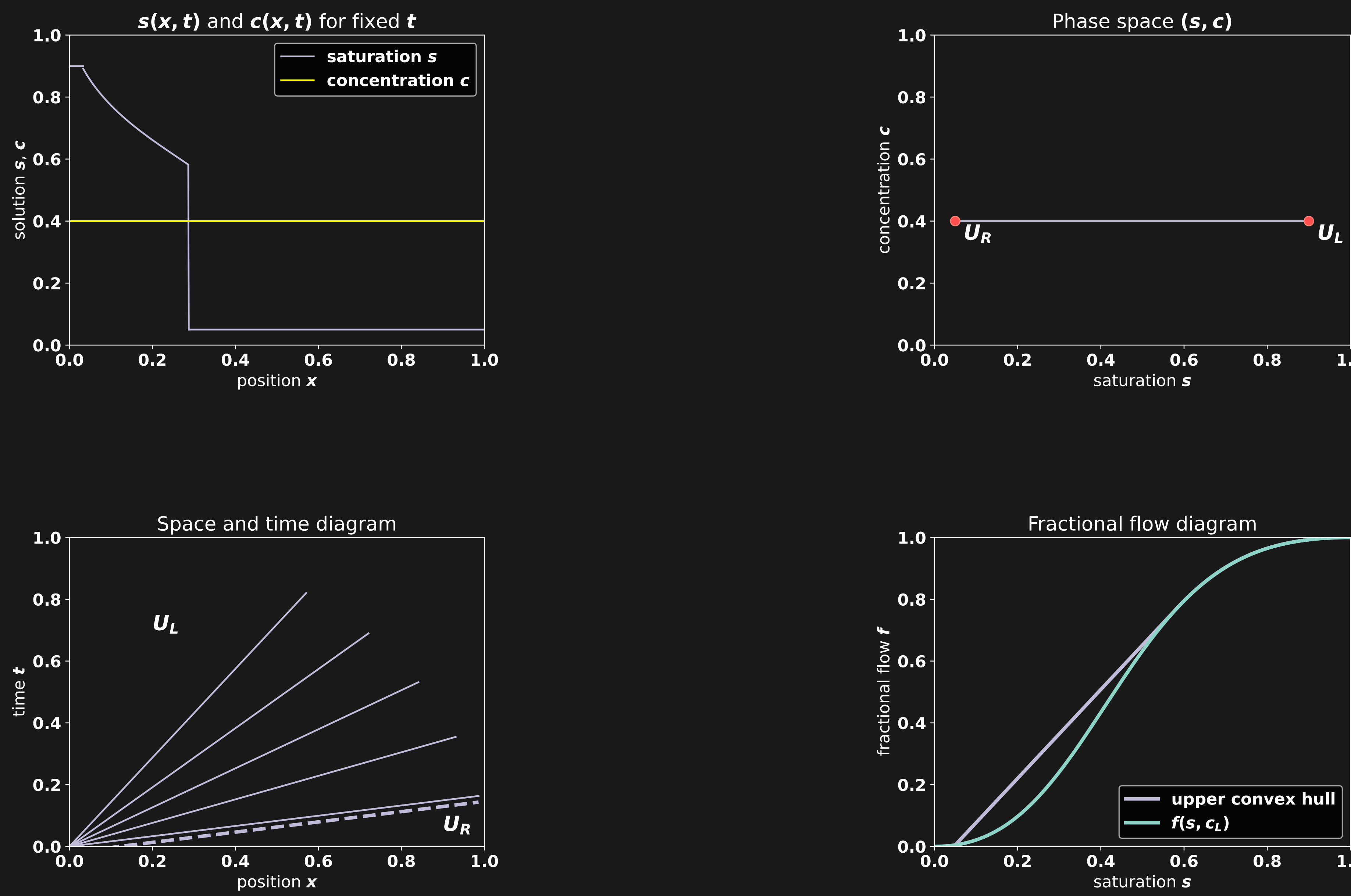
$$

Question: find an exact solution $s(x,t)$ and $b(x,t)$ to this initial value problem

FOUR WAYS TO REPRESENT A SOLUTION: BL SOLUTION



CHARACTERISTIC FAMILIES: S AND C-WAVES

$$\begin{aligned} s_t + f(s,c)_x &= 0, \\ (cs)_t + (cf(s,c))_x &= 0. \end{aligned}$$

Differentiate the equations: $\quad DG(U)U_t + DF(U)U_x = 0$:

$$\begin{pmatrix} 1 & 0 \\ c & s \end{pmatrix} \begin{pmatrix} s \\ c \end{pmatrix}_t + \begin{pmatrix} f_s & f_c \\ f_s & f + cf_c \end{pmatrix} \begin{pmatrix} s \\ c \end{pmatrix}_x = 0.$$

Hence

$$\begin{pmatrix} s \\ c \end{pmatrix}_t + \begin{pmatrix} f_s & f_c \\ 0 & \frac{f}{s} \end{pmatrix} \begin{pmatrix} s \\ c \end{pmatrix}_x = 0.$$

Characteristic speeds: $\quad \lambda^{(s)} = f_s \quad$ and $\quad \lambda^{(c)} = f/s.$

Eigenvectors: $\quad r^{(s)} := \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad$ and $\quad r^{(c)} := \begin{pmatrix} -f_c \\ \lambda^{(s)} - \lambda^{(c)} \end{pmatrix}.$

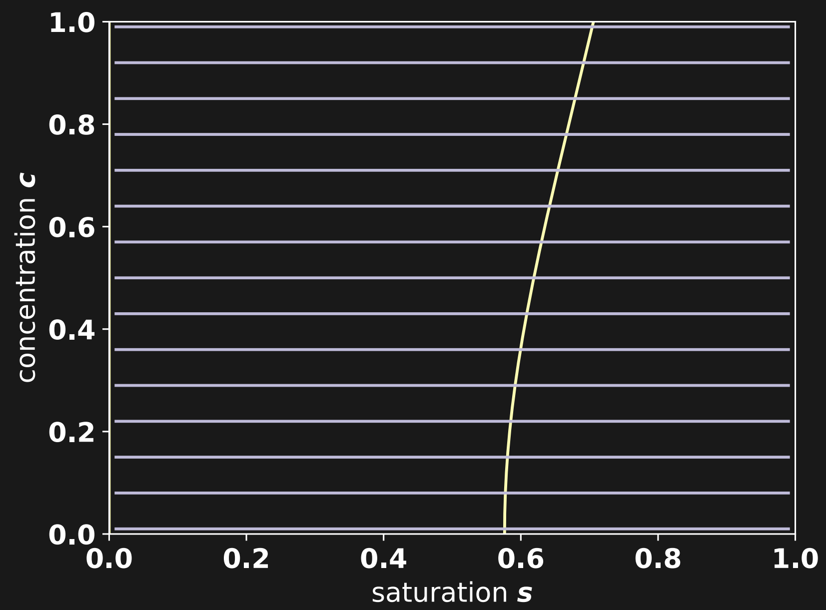
CHARACTERISTIC FAMILIES: S AND C-WAVES

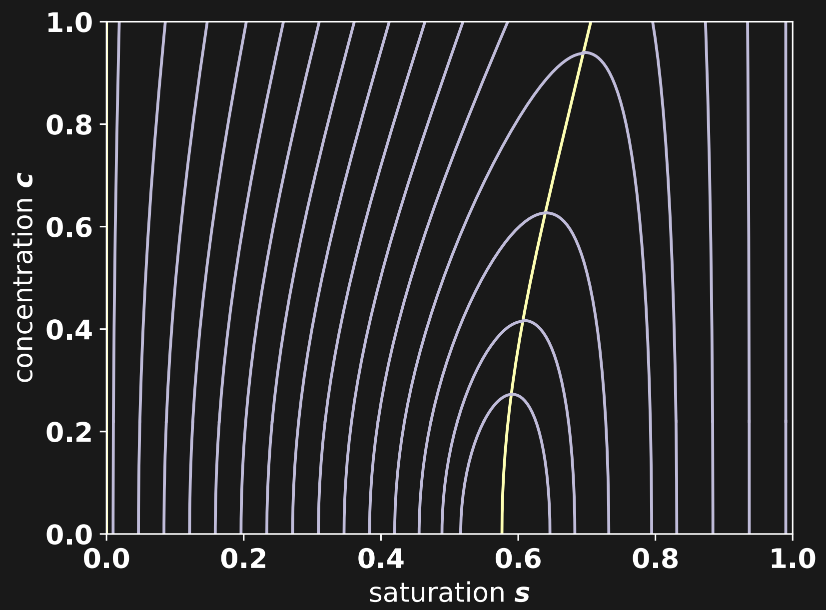
s-waves

- $\lambda^{(s)} = f_s$
- solve the Buckley-Leverett equation $c = const$
- Riemann invariant $c = const$

c-waves

- $\lambda^{(c)} = f/s$
- are contact discontinuities (linearly degenerate field)
- Riemann invariant $f/s = const$



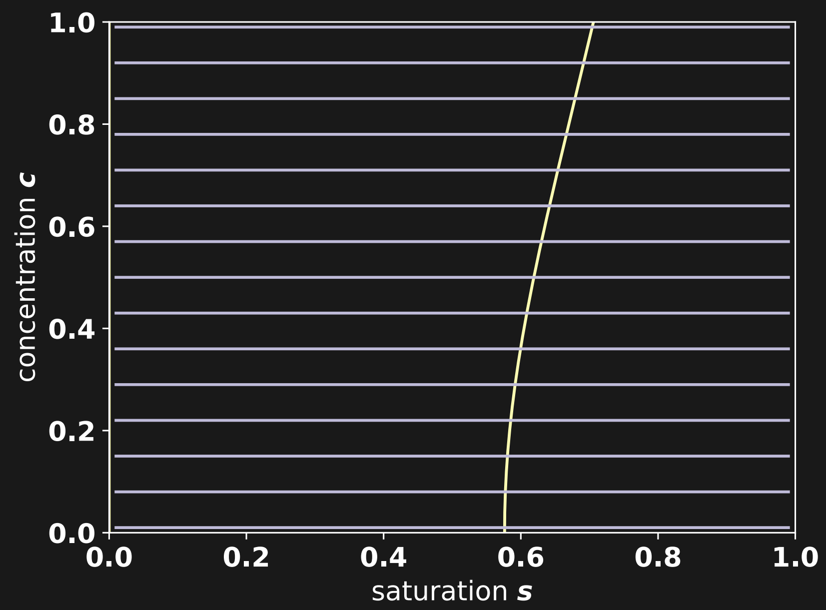


For both families the *rarefaction and shock curves coincide*! But in a different way…

INTERESTING FACTS (TEMPLE, AROUND 1983)

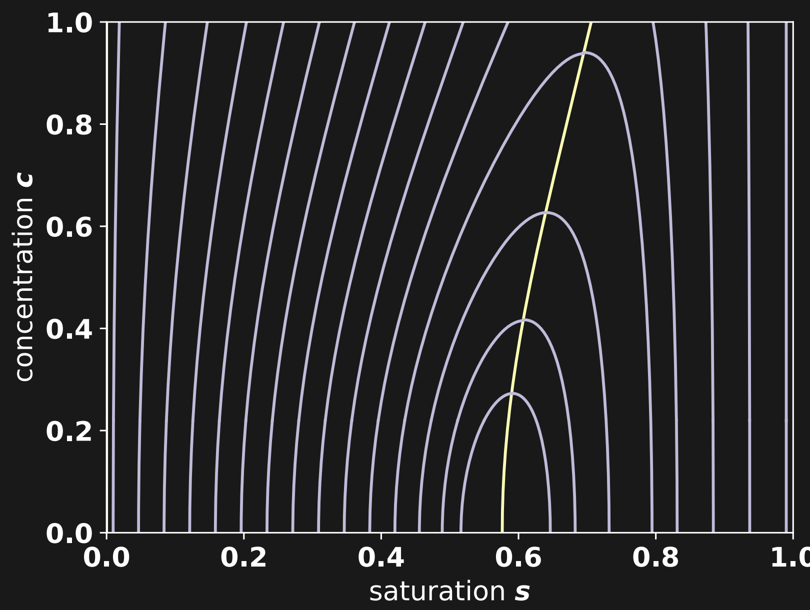
Theorem. (B. Temple) The following statements are equivalent:

- λ-shock and λ-rarefaction coincide on S
- either *S is a straight line in U-space* or *λ is constant on S*
- system reduces to a scalar conservation law on S



line family

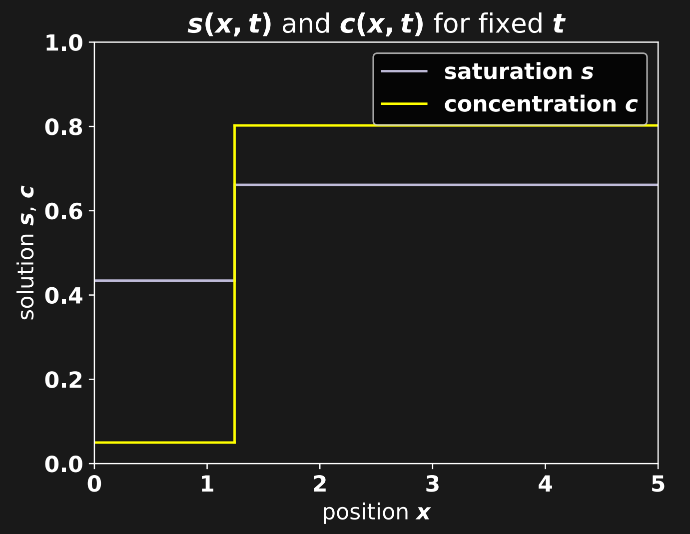
- lines in U-space
- represented by s-waves
- non-linear scalar equation

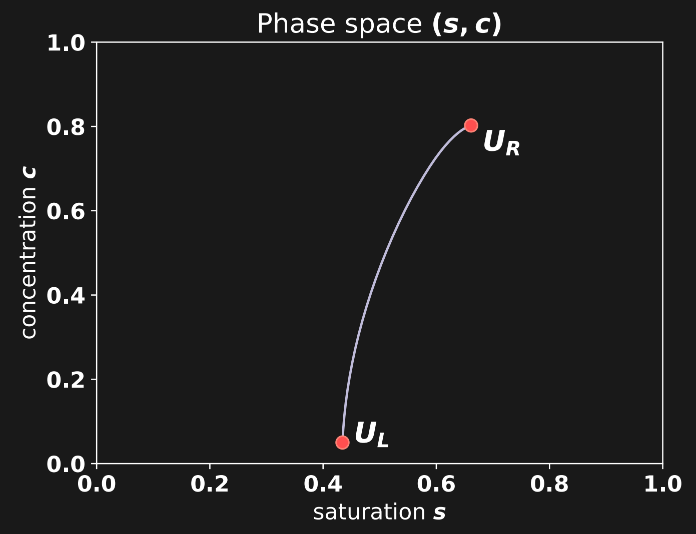


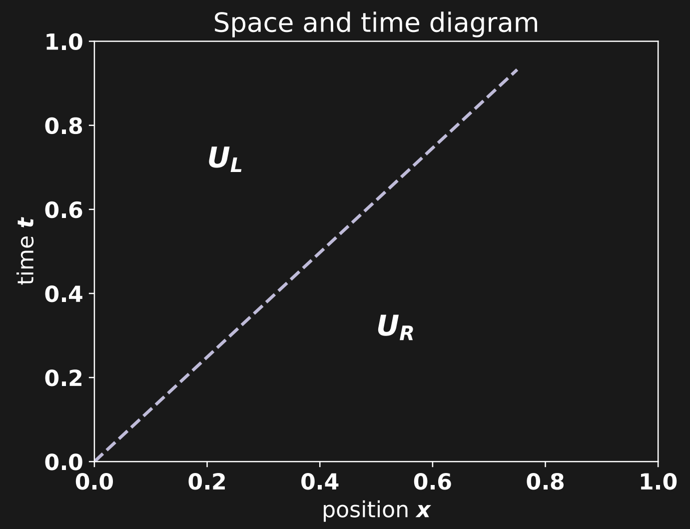
contact family

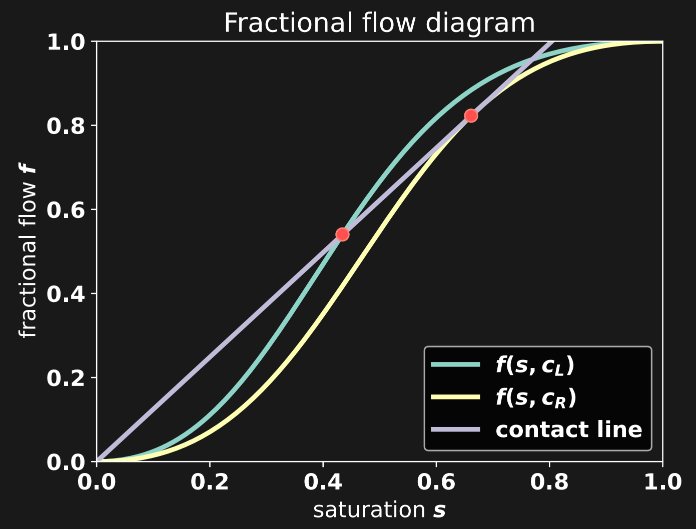
- λ is constant
- represented by c-waves
- linear scalar equation

EXAMPLE: C-WAVE

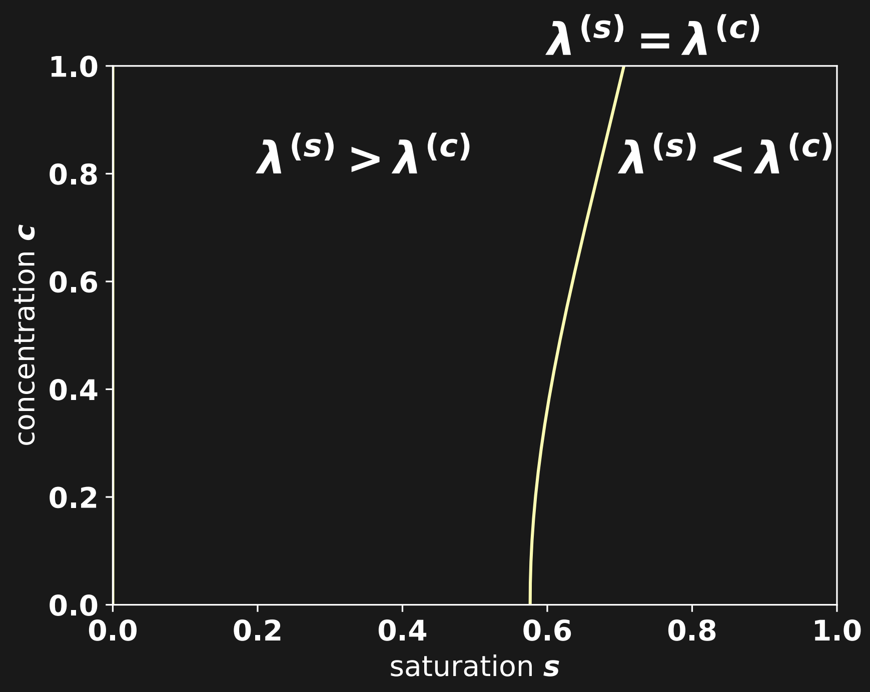


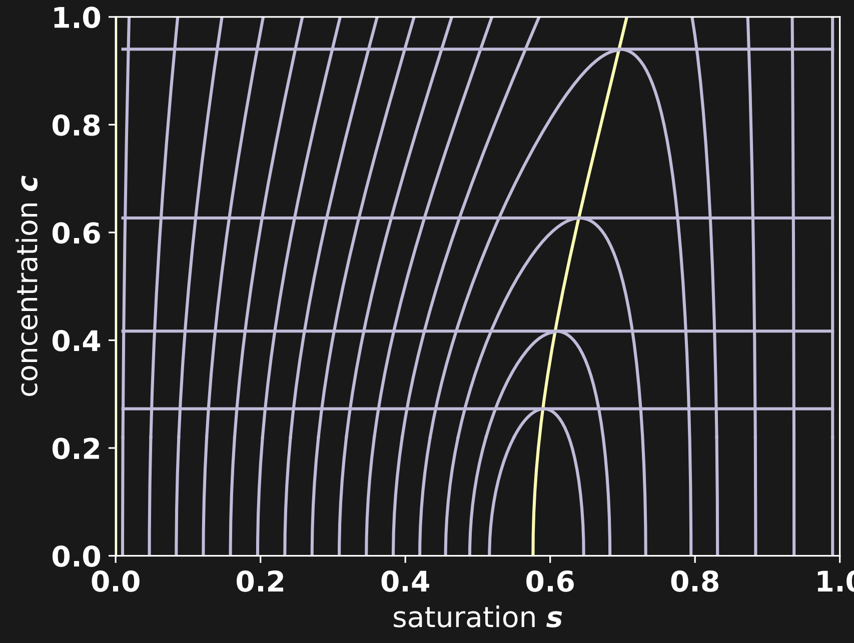






NON-STRICTLY HYPERBOLIC SYSTEM



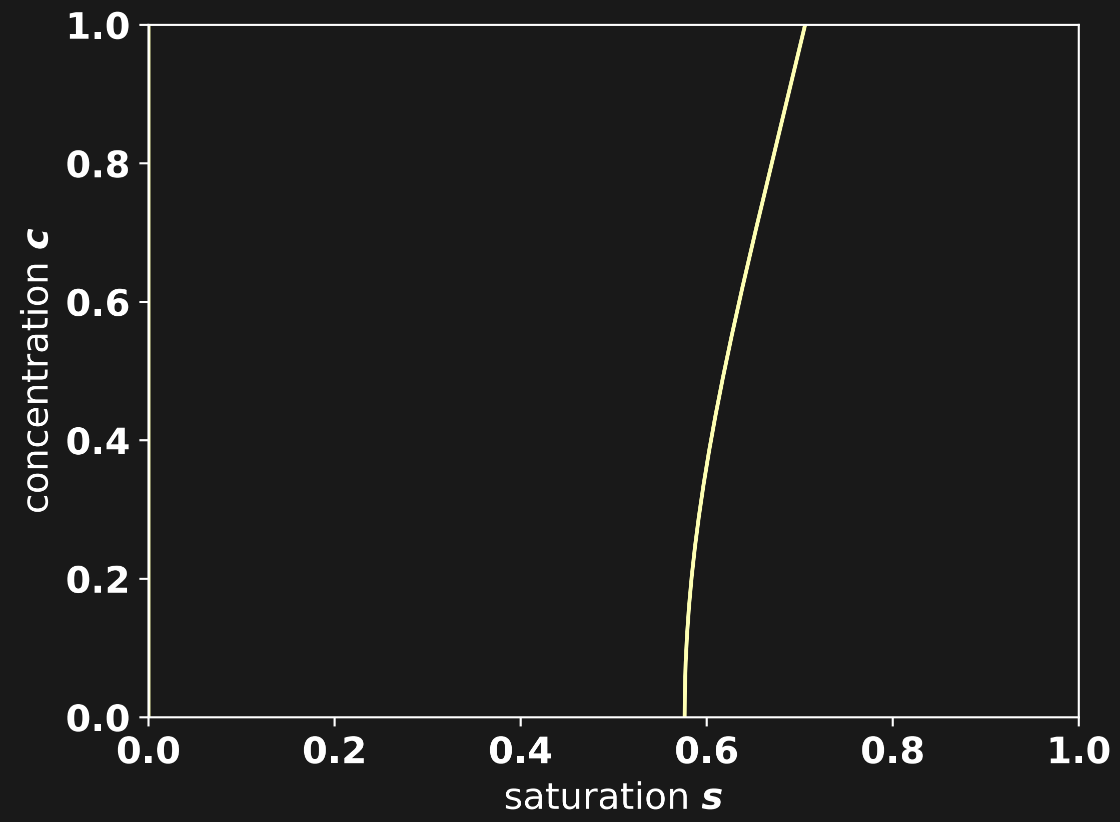


The coordinate system of wave curves is singular and wave speeds coincide on a co-dimension one curve (*coincidence locus*)

$$\lambda^{(s)} = f_s = \frac{f}{s} = \lambda^{(c)}$$

s and c-waves are tangent on coincidence curve.

NON-UNIQUENESS OF SOLUTIONS



- *admissibility criterion* of E. Isaacson and J. Glimm: a contact discontinuity is admissible if and only if its left and right states lie on the same side of the coincidence locus
- *consequence*: existence and uniqueness of solutions for all Riemann problems

What is the (physical) motivation of this criterion?

GENERAL ADMISSIBILITY CRITERION

- a model M_0
- a parameterized family of models M_α with its own admissibility criterion

Definiton: a solution for M_0 is admissible provided it is the L^1_{loc} limit of a family of admissible solutions of M_α as $\alpha \to 0$.

For instance, a solution of M_α could be any wave group: a shock, rarefaction, or composite, or more general wave group that is admissible for M_α.

Example: chemical flooding model M_α, *vanishing adsorption criterion*.

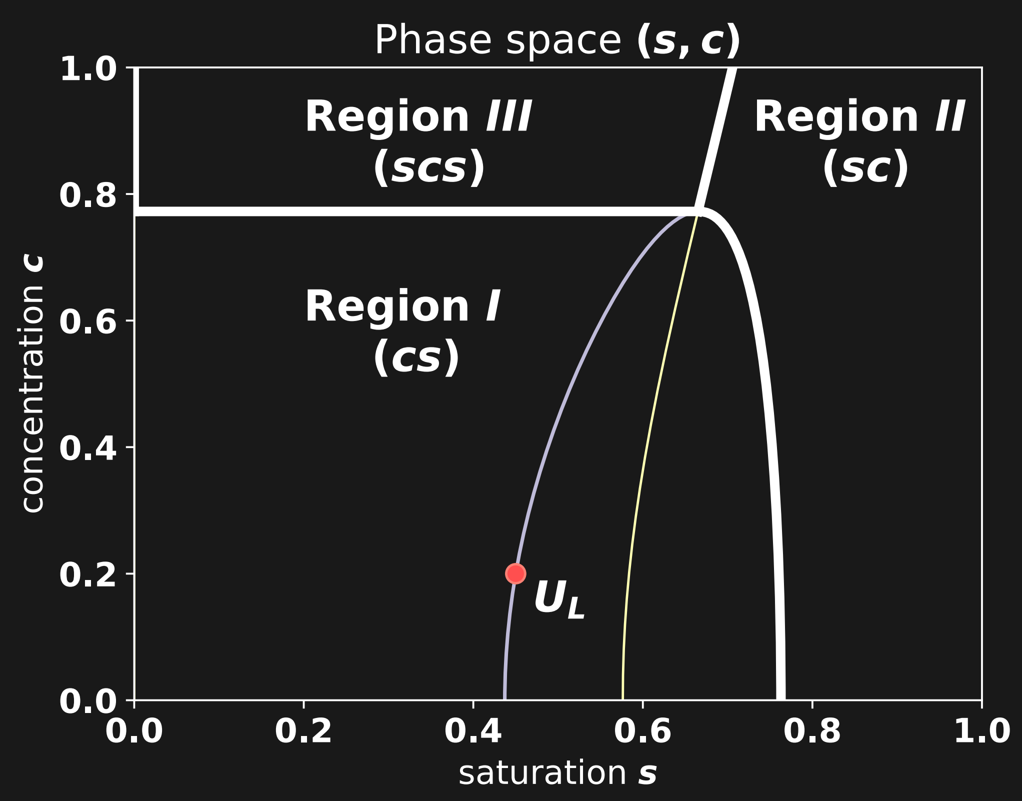
$$\begin{aligned} s_t + f(s,c)_x &= 0, \\ (cs + \alpha a(c))_t + (cf(s,c))_x &= 0. \end{aligned}$$

- if adsorption depends nonlinearly on the concentration, then *contact discontinuities become rarefactions and shock waves*
- For moderate values of c we have $a''(c) \leq 0$. When $a'' \equiv 0$, c-waves are contacts.

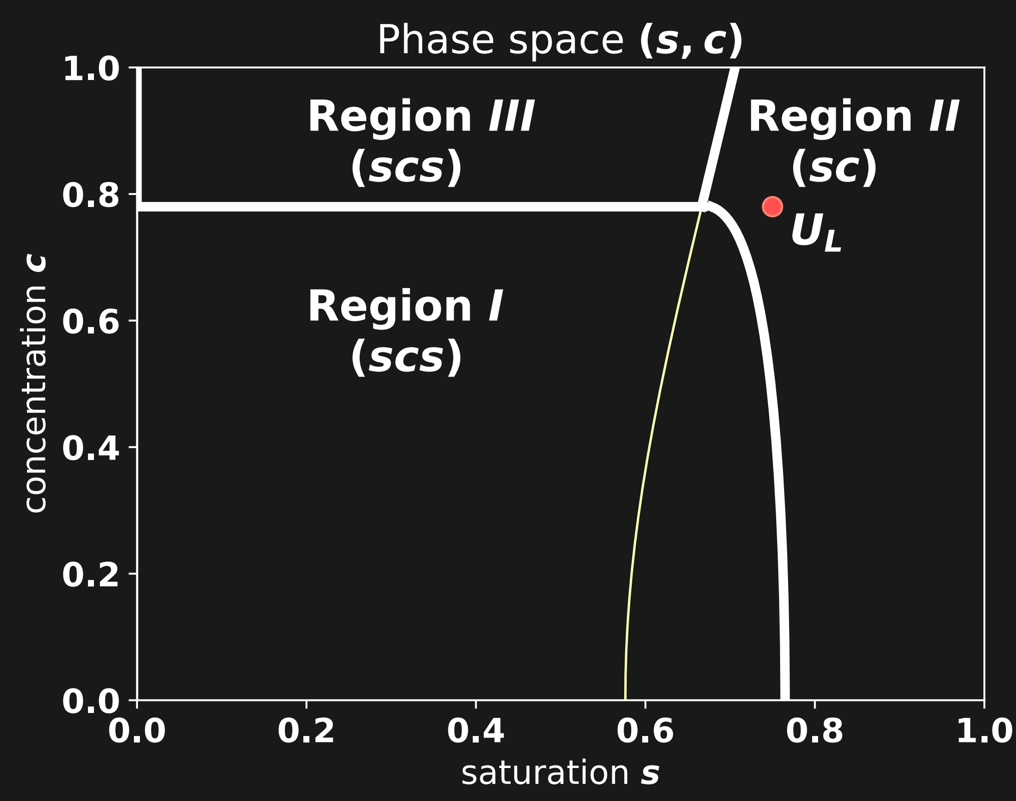
MAIN RESULT

Theorem (P., Marchesin, Plohr ‘2022). The set of admissible Riemann solutions is the same for the Isaacson-Glimm and vanishing adsorption admissibility criteria.

The solutions may be presented by two diagrams (E. Isaacson ‘1980):



U_L to the left of coincidence



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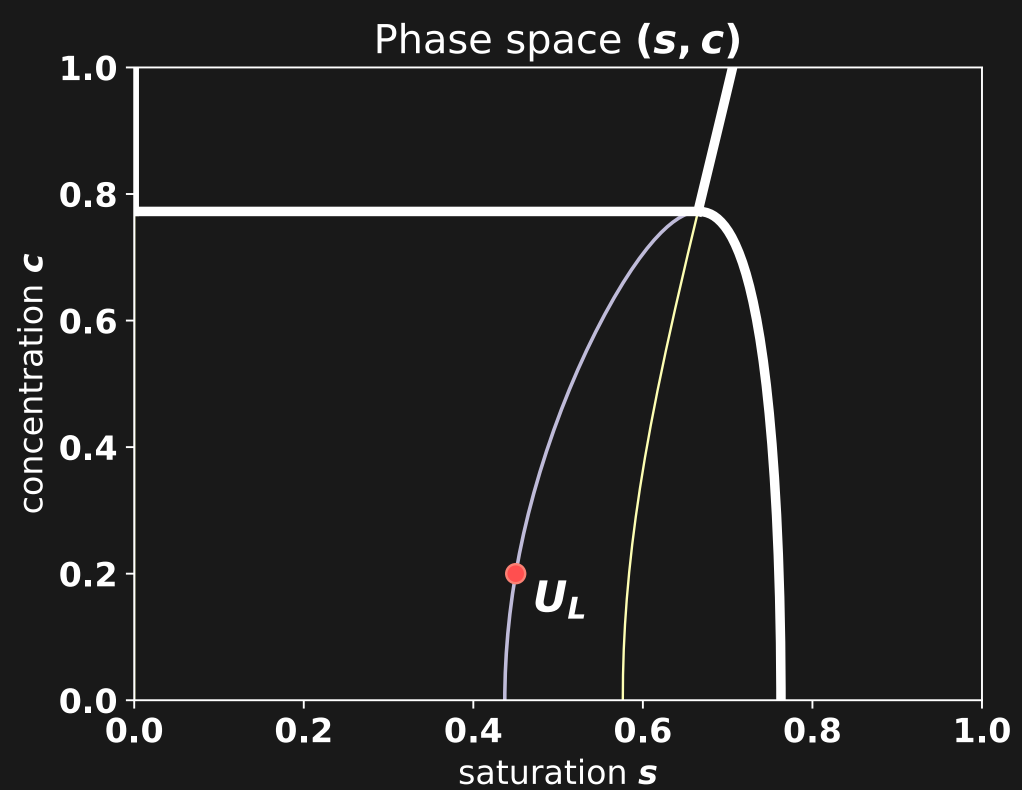

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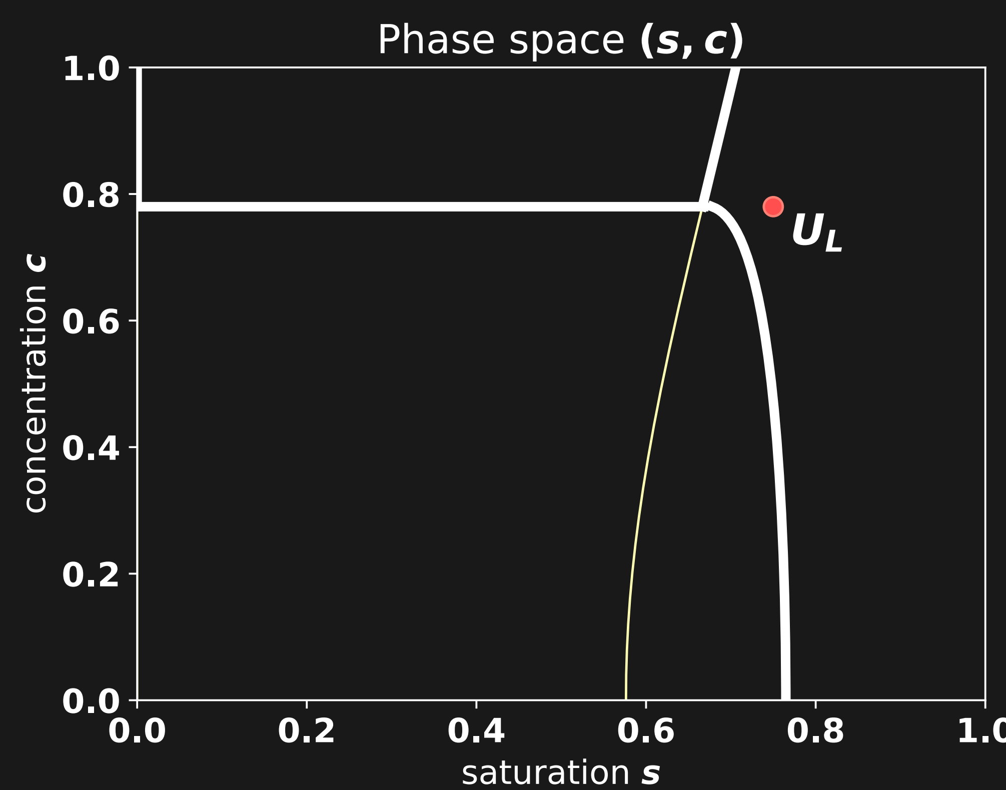
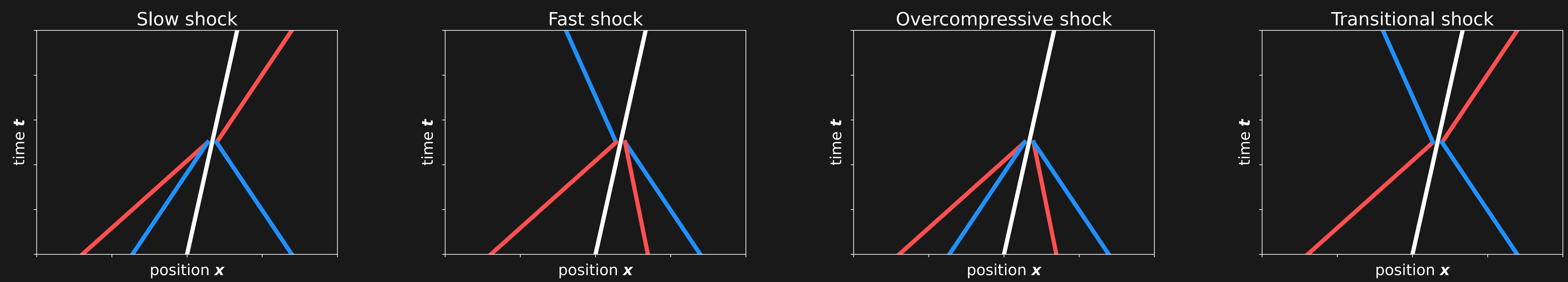


U_L to the right of coincidence

SHOCKS TYPES



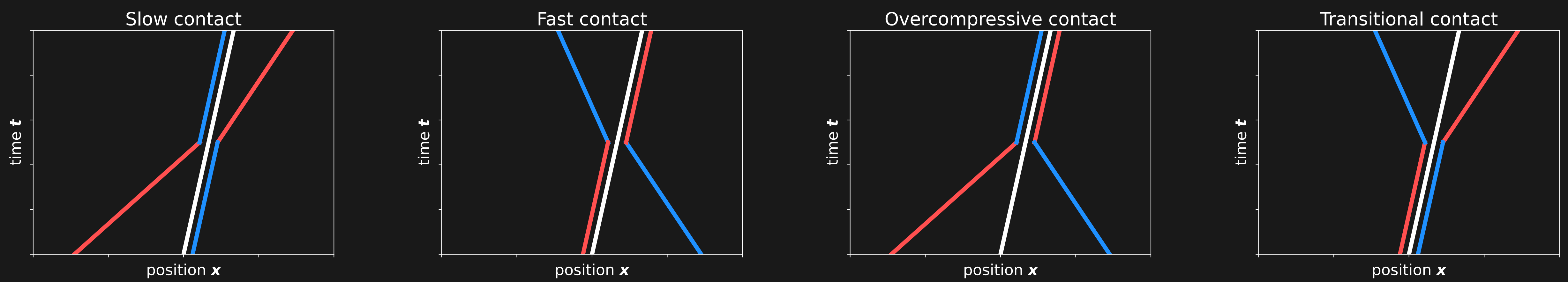
A shock from U_- to U_+ with speed σ is called

- slow if $\lambda_1(U_-) > \sigma > \lambda_1(U_+)$ and $\sigma < \lambda_2(U_-), \lambda_2(U_+)$;
- fast if $\lambda_2(U_-) > \sigma > \lambda_2(U_+)$ and $\sigma > \lambda_1(U_-), \lambda_1(U_+)$;
- overcompressive if $\sigma > \lambda_2(U_+), \lambda_1(U_+)$ and $\sigma < \lambda_1(U_-), \lambda_2(U_-)$;
- undercompressive if $\sigma < \lambda_1(U_-), \lambda_1(U_+)$ and $\sigma > \lambda_2(U_-), \lambda_2(U_+)$.

Undercompressive $\equiv$ transitional.

Fast and slow are also called Lax shocks (classical).

TYPES OF CONTACT DISCONTINUITIES

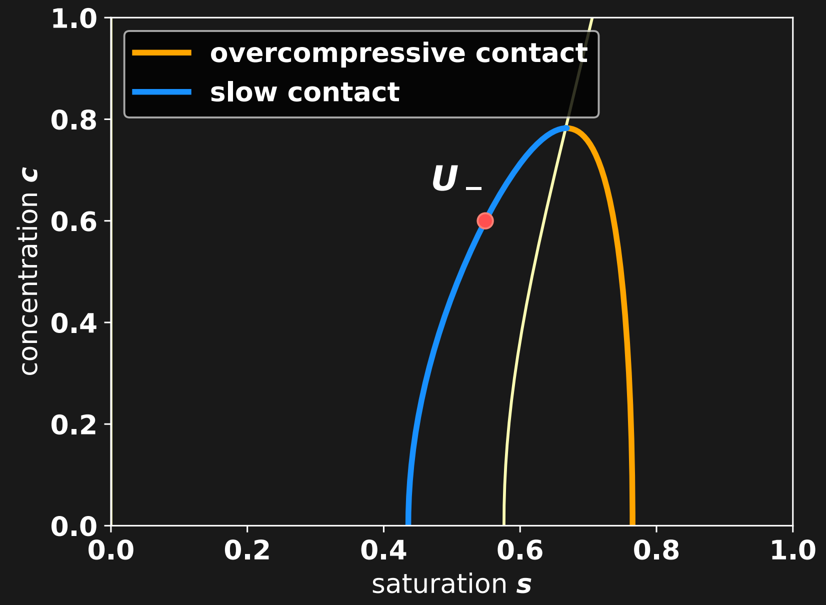


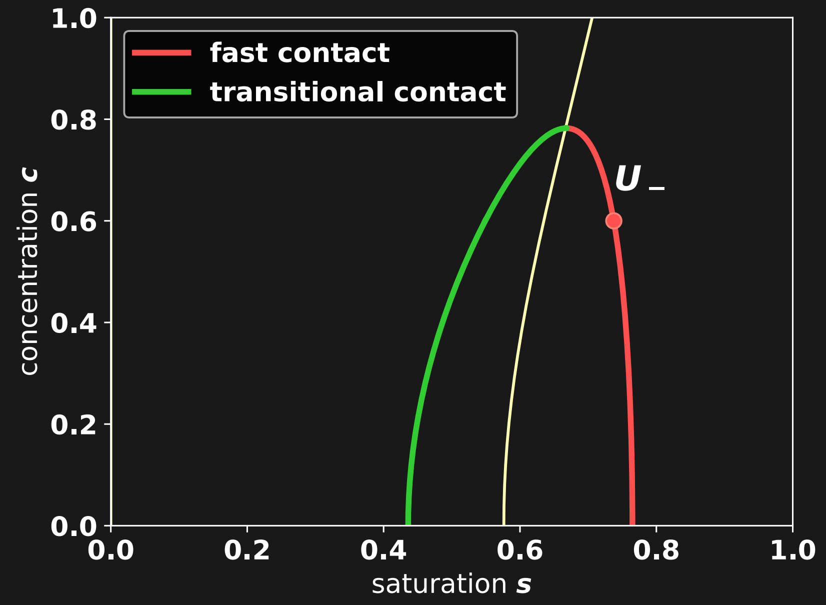
We will call a contact discontinuity from U_- to U_+ with speed σ

- slow if $\lambda_2(U_-) > \lambda_1(U_-) = \sigma = \lambda_1(U_+) < \lambda_2(U_+)$;
- fast if $\lambda_1(U_-) < \lambda_2(U_-) = \sigma = \lambda_2(U_+) > \lambda_1(U_+)$;
- overcompressive if $\lambda_2(U_-) > \lambda_1(U_-) = \sigma = \lambda_2(U_+) > \lambda_1(U_+)$;
- undercompressive if $\lambda_1(U_-) < \lambda_2(U_-) = \sigma = \lambda_1(U_+) < \lambda_2(U_+)$.

Undercompressive $\equiv$ transitional.

MAIN STEP: ADMISSIBLE CONTACTS



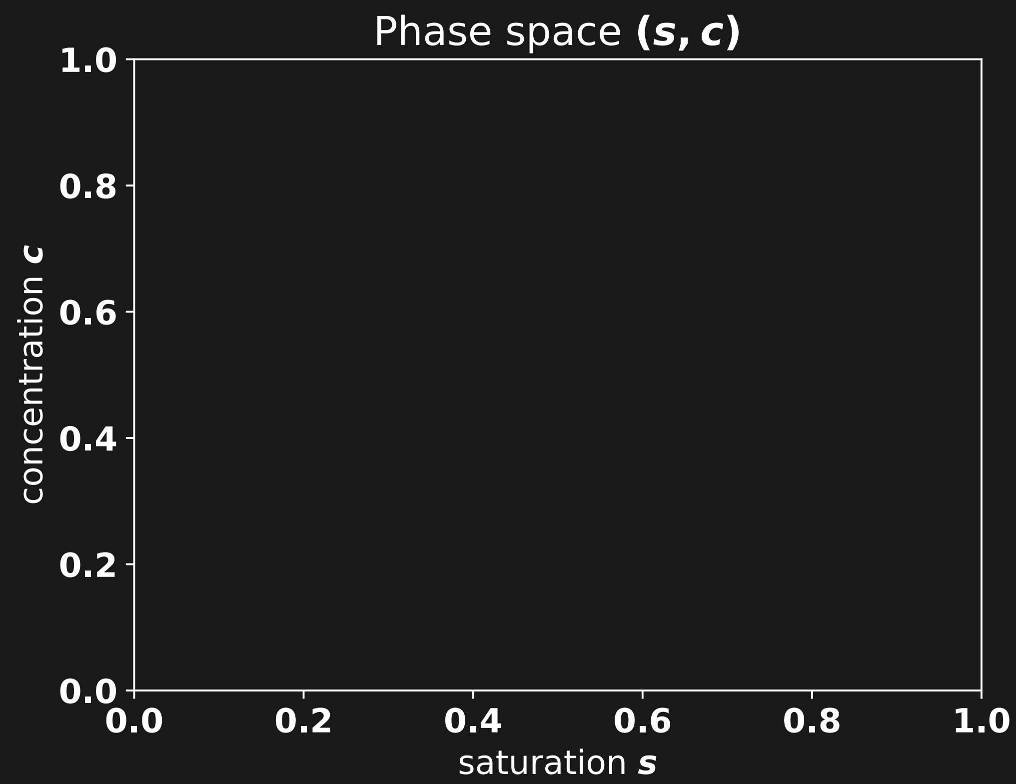


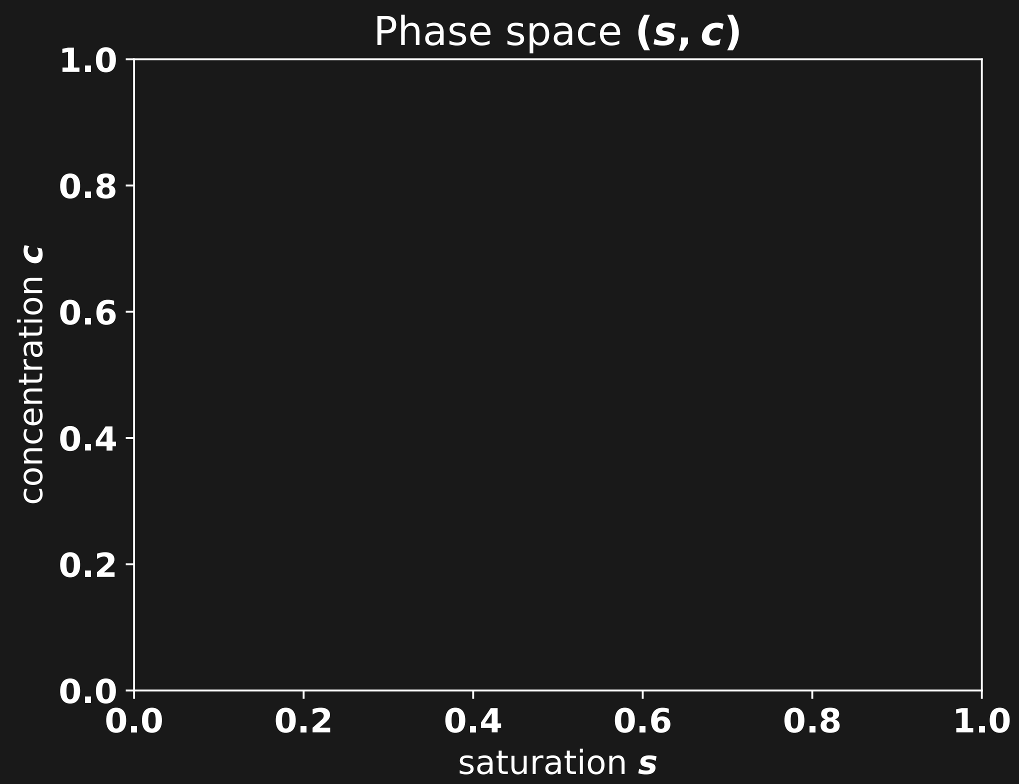
- *Isaacson-Glimm criterion*: slow and fast contact discontinuities are admissible, overcompressive and undercompressive are not.
- *Vanishing adsorption criterion*: slow, fast and overcompressive contact discontinuities are admissible, undercompressive are not.

P.S. Overcompressive contacts can be represented as a sequence of two waves (c and s). Whether or not they are regarded as admissible does not affect the Riemann problem solution.

THEOREM PROOF SKETCH

Slow, fast and overcompressive contact discontinuities are admissible, undercompressive are not.





DOES THERE EXIST A MODEL WITH TRANSITIONAL CONTACT?

DOES THERE EXIST A MODEL WITH TRANSITIONAL CONTACT?

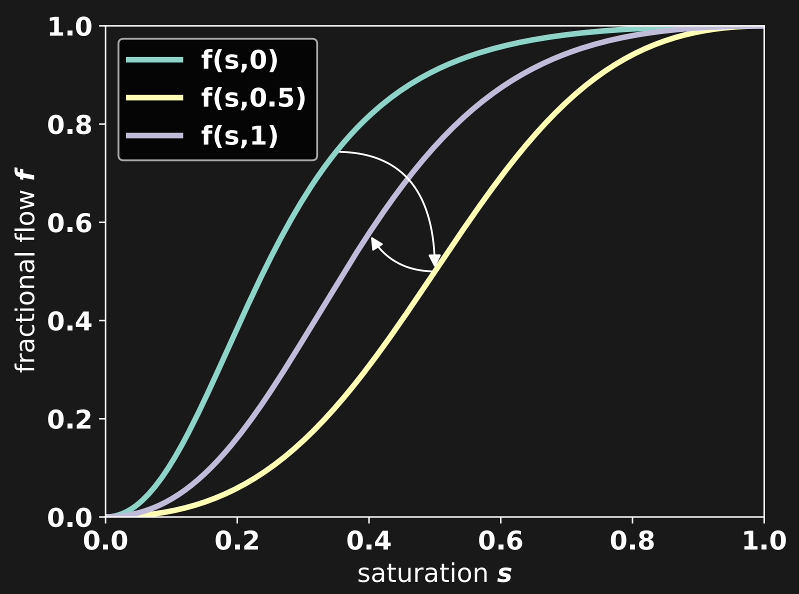
YES!

Two-phase oil-water flow with *surfactant* in the water

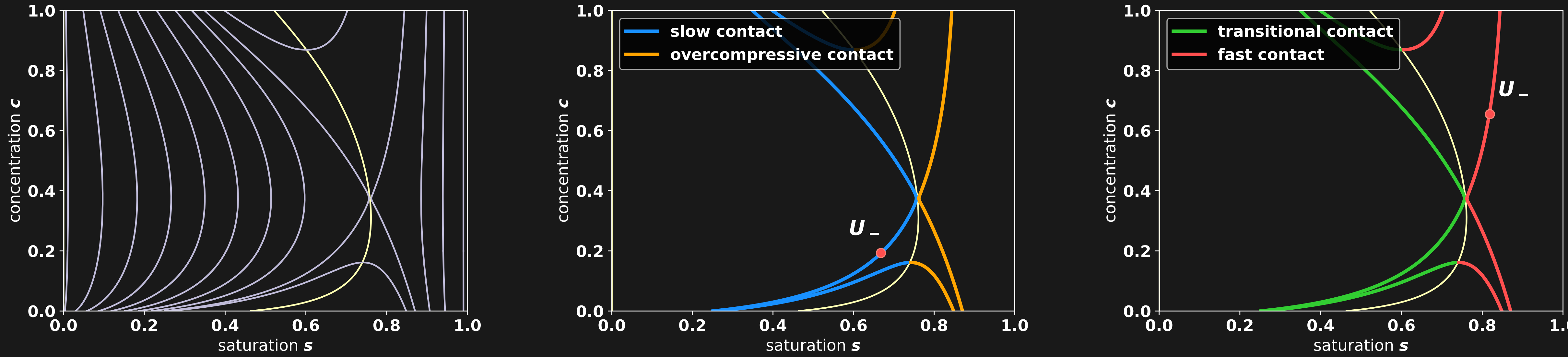
$$s_t + f(s,c)_x = 0$$
$$(cs)_t + (cf(s,c))_x = 0$$

- $s \in [0,1]$ - water saturation
- $c \in [0,1]$ - surfactant concentration
- f - fractional flow function: affected by surfactant
    - S-shaped in s
    - *f is non-monotone in c*

  i.e. f changes its monotonicity in c once
- *Work in progress*



THERE EXIST ADMISSIBLE TRANSITIONAL CONTACTS!



Theorem (Bakharev, Enin, P., Rastegaev '2021).

For a *non-monotone* chemical flooding model with adsorption M_α there *exist transitional shocks*. They correspond to a saddle-to-saddle conection for a travelling wave dynamical system and depend on the ratio of diffusion coefficients.

See also Entov, Kerimov (1986) and Shen (2017).

Consequence: there exist transitional contacts as a limit of transitional shocks

TRANSITIONAL SHOCK

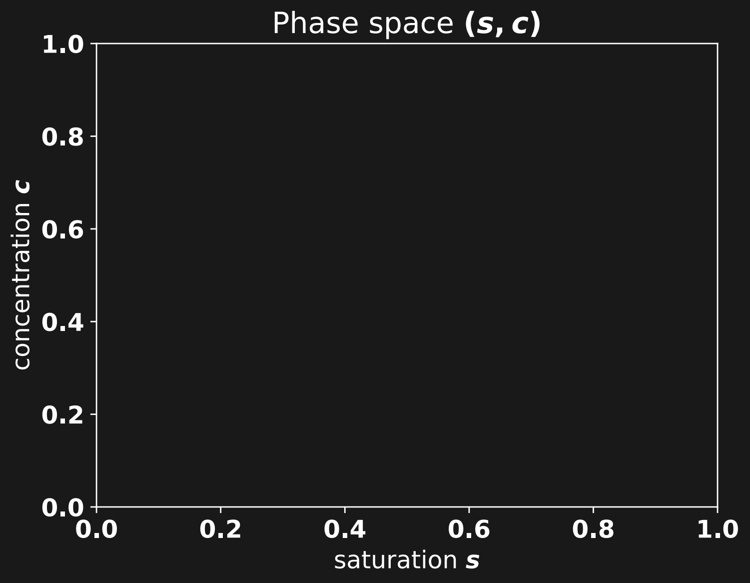
Adsorption, Capillarity and Diffusion:

$$s_t + f(s,c)_x = \varepsilon_c (s_x)_x,$$
$$(cs + a(c))_t + (cf(s,c))_x = \varepsilon_c (cs_x)_x + \varepsilon_d (c_x)_x$$

Parameter: $\kappa = \varepsilon_d / \varepsilon_c$

Theorem (Bakharev, Enin, P., Rastegaev '2021).

There exist $0 < \sigma_{\min} < \sigma_{\max} < \infty$, such that for every $\kappa = \varepsilon_d/\varepsilon_c \in (0, +\infty)$, there exist a unique travelling wave, connecting two saddle points $U_-(\kappa)$ and $U_+(\kappa)$ with velocity $\sigma(\kappa)$. Moreover, $\sigma(\kappa)$ is monotone and continuous; $\sigma(\kappa) \to \sigma_{\min}$ as $\kappa \to \infty$; $\sigma(\kappa) \to \sigma_{\max}$ as $\kappa \to 0$.



Remark: The transitional contact does not depend on κ.

Thank you!

yulia.petrova@impa.br

https://yulia-petrova.github.io

75 anos impa Instituto de Matemática Pura e Aplicada

St Petersburg University

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Own works:

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THANK YOU ONCE AGAIN FOR YOUR ATTENTION!

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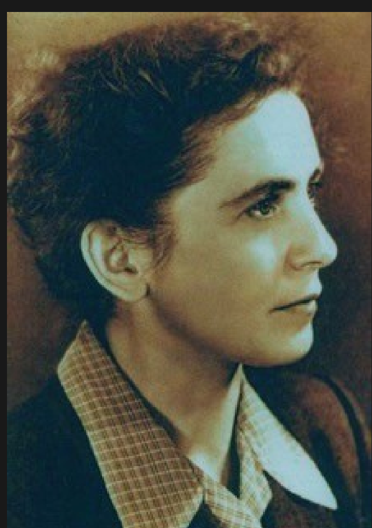

Satellite conferences:

- *23 - 27 May 2022* - Mathematical hydrodynamics: the legacy of Olga Ladyzhenskaya and modern perspectives
- *16 - 23 July 2022* - O.A. Ladyzhenskaya centennial conference on PDE's

Olga Ladyzhenskaya (7 March 1922 – 12 January 2004)