Propagating terrace in a semi-discrete model of Incompressible Porous Medium (IPM) equation

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Sergey Tikhomirov
(PUC-Rio)

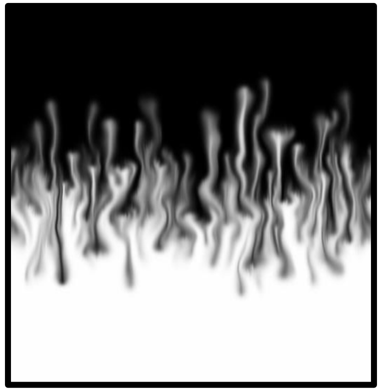
Based on:
Propagating terrace in a two-tubes model of gravitational fingering

ArXiv: 2401.05981
To appear in SIMA

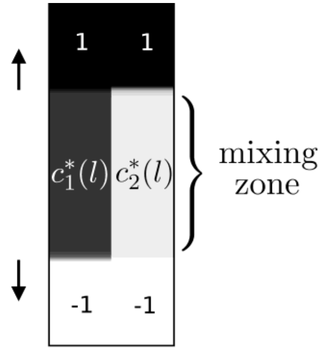
Yalchin Efendiev
(Texas A&M)

Outline

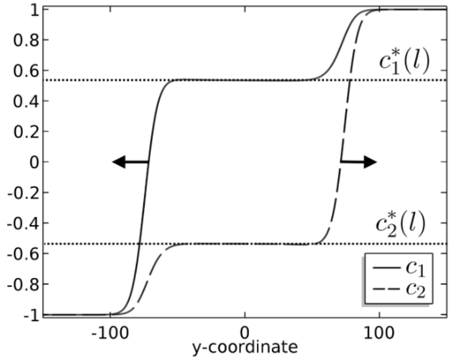
1. Introduction
   Miscible displacement in porous media
   - viscous fingering
   - gravitational fingering



2. Problem statement
   - Two-tubes model
   - Main theorem



3. Proof:
   - traveling waves
   - slow-fast systems
   - geometric singular perturb. theory



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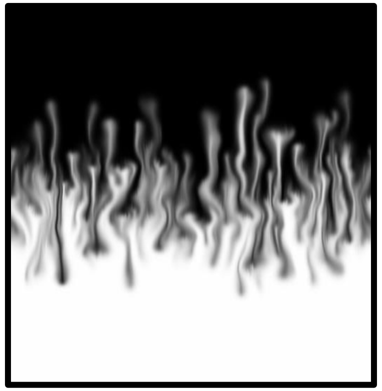

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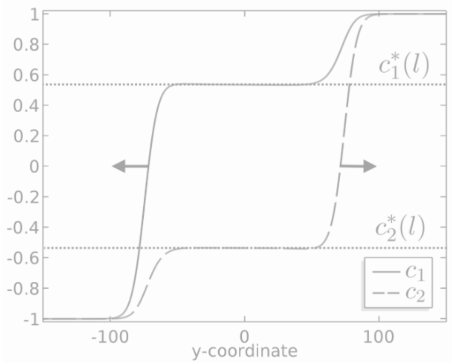
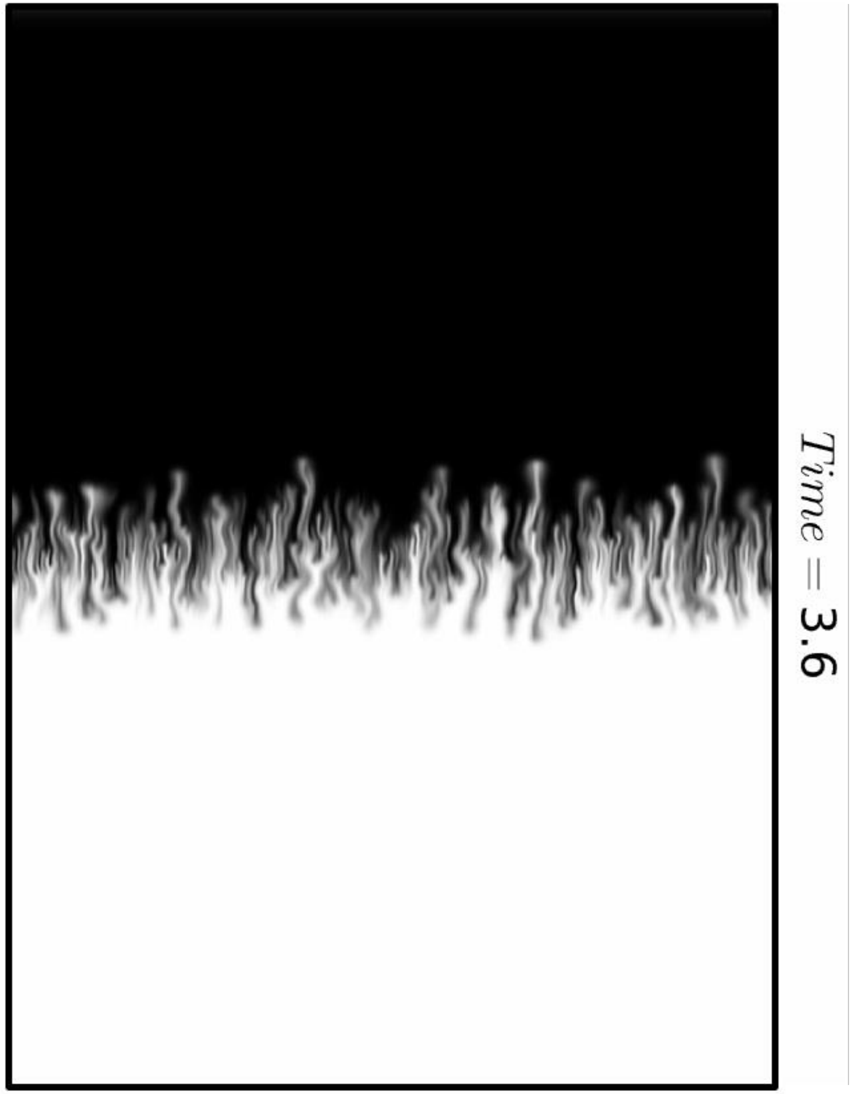


Gravitational fingering instability

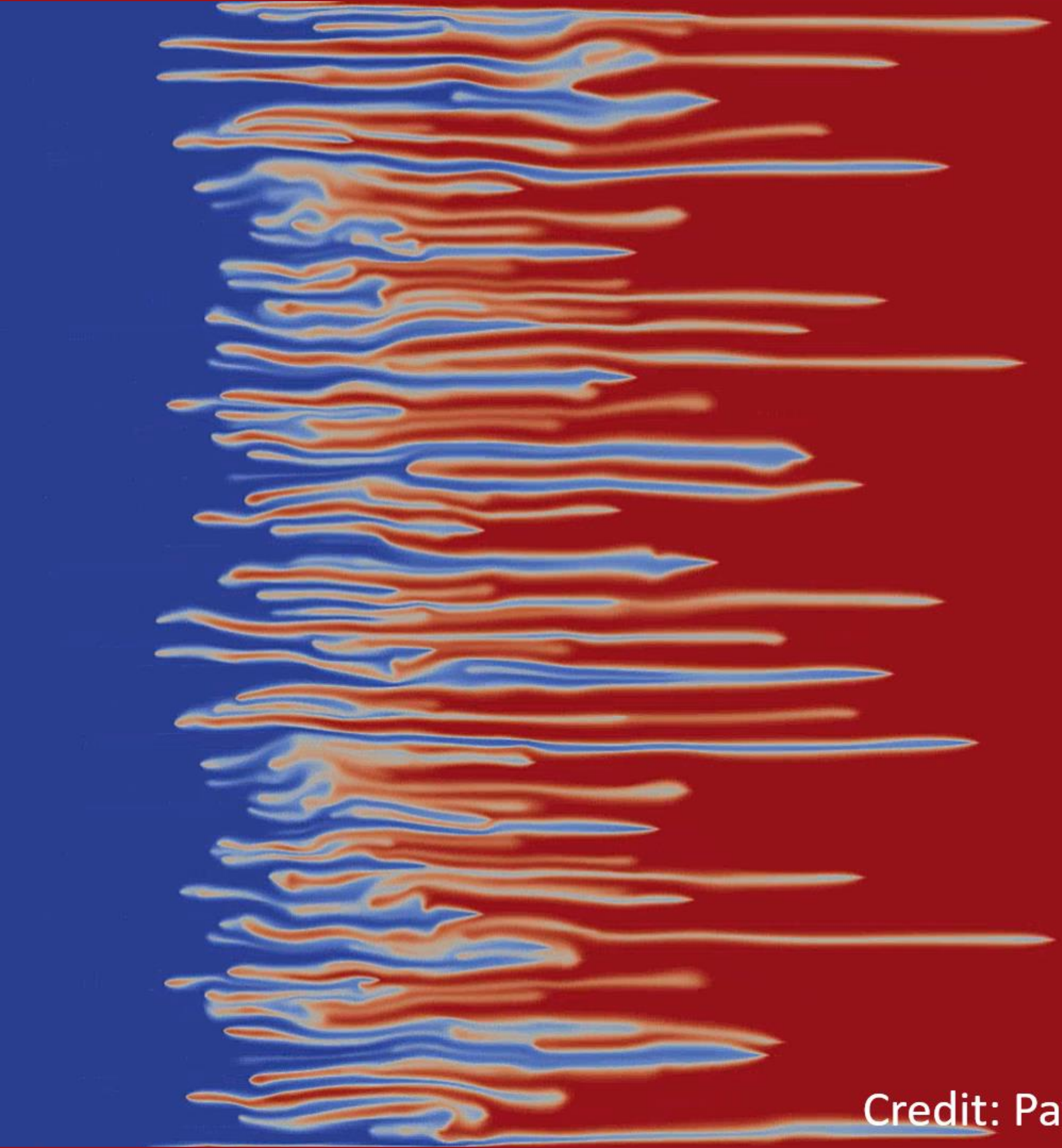
- Miscible displacement
- porous media
  (averaged models of flow)
- Relatively small speeds
  Navier Stokes $\longrightarrow$ Darcy's law
- Applications?



Heavy fluid

Light fluid

Credit: Nicolas Valade, INRIA



Viscous fingering phenomenon

water (blue color)

polymerized water (red color)

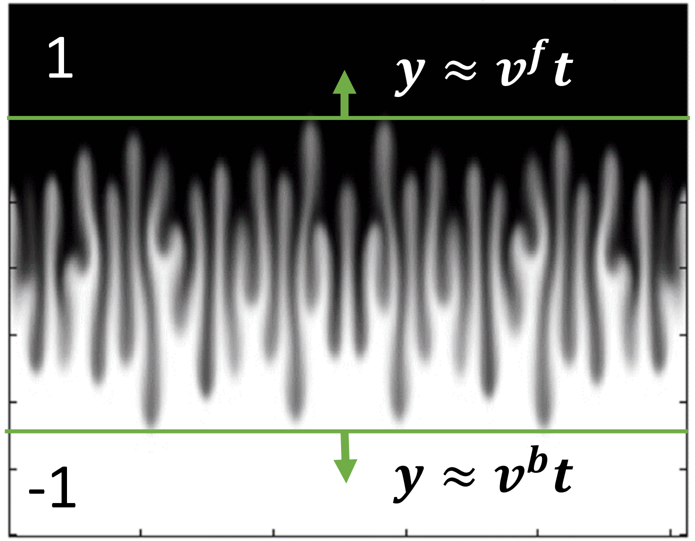
Appears in applications:

Enhanced Oil Recovery

Credit: Pavlov Dmitrii, St. Petersburg State University

Incompressible Porous Medium eq – IPM, 2D (Two formulations)

Gravitational fingering



$$c_t + div(uc) = \varepsilon \cdot \Delta c$$

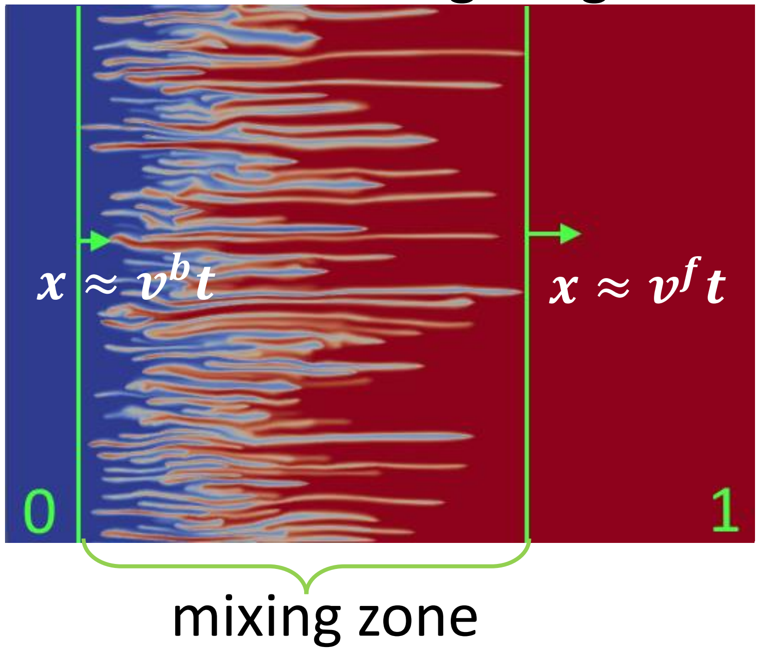
$$div(u) = 0$$

(gravity) $\quad u = -\nabla p - (0, c)$

(viscosity) $\quad u = -m(c)\, K\, \nabla p$

$c = c(t,x,y)$ – concentration $\varepsilon \geq 0$ – diffusion
$u = u(t,x,y)$ – velocity $m(c)$ – mobility
$p = p(t,x,y)$ – pressure K – permeability

Viscous fingering

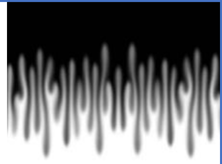


- many laboratory and numerical experiments show *linear growth of the mixing zone*

Question: how to find speeds v^b and v^f of propagation?

1969 – R. Wooding (JFM) *Growth of fingers at an unstable diffusing interface in a porous medium or Hele-Shaw cell*
…
2018 – J. Nijjer, D. Hewitt, J. Neufeld (JFM) *The dynamics of miscible viscous fingering from onset to shutdown.*
2022 – F. Bakharev, A. Enin, A. Groman, A. Kalyuzhnyuk, S. Matveenko, **Y. Petrova**, I. Starkov, S. Tikhomirov (JCAM)

IPM: $\varepsilon = 0$ (without diffusion)



Active scalar:

$$\boxed{\begin{aligned} & c_t + u \cdot \nabla c = 0 \\ & u = A(c) \end{aligned}}$$

$$u = \nabla^{\perp}(-\Delta)^{-1}\partial_1 c \qquad \text{(Biot-Savart law)}$$

Discontinuous initial data: free boundary problem (Muskat problem) – ill-posed for unstable stratification

2011 – A. Córdoba, D. Córdoba, F. Gancedo (Annals of Mathematics)
"Interface evolution: the Hele-Shaw and Muskat problems"

Existence: smooth initial data

2007 – D. Cordoba, F. Gancedo, R. Orive (JMP): local well-posedness for initial data H^s

global solution vs finite-time blow-up? open

2017 – T. Elgindi (ARMA): global solution for small perturbations of $c = -y$

2023 – S. Kiselev, Y. Yao (ARMA): if solutions stay "smooth" for all times, then there is blow-up at $t = +\infty$

Uniqueness: non-uniqueness of weak solutions by convex integration

2011 – D. Córdoba, D. Faraco, F. Gancedo (ARMA)

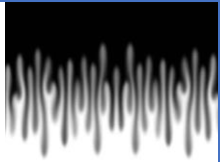
2012 – L. Szekelyhidi Jr. (Annales de l'ENS)

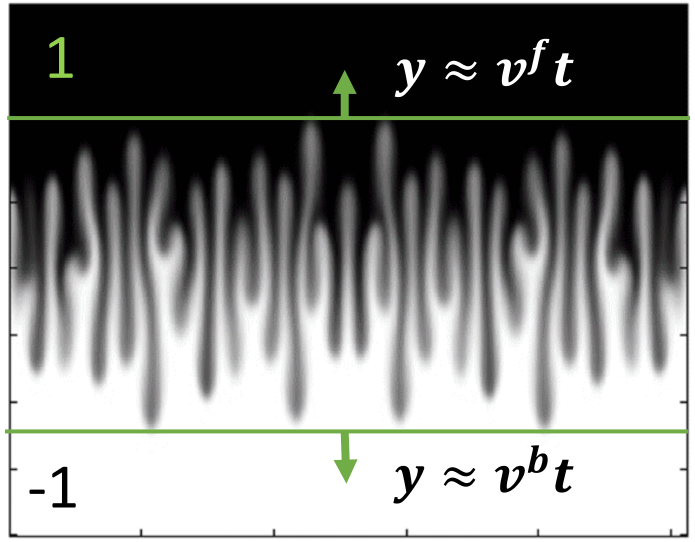
Asymptotic stability of stable stratification:

2024 – R. Bianchini, T. Crin-Brat, M. Paicu (ARMA)

…and many others…

IPM: $\varepsilon > 0$ (with diffusion)





Estimates on the growth:

2005 – F. Otto, G. Menon. Proved estimates

- Full model (IPM) $v^f \leq 2$
- Simplified model (TFE) $v^f \leq 1$

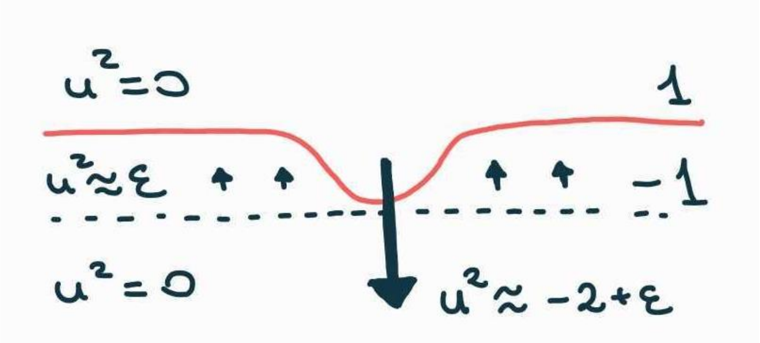
Transverse Flow Equilibrium = TFE

$$p(t,x,y) \approx p(t,y)$$

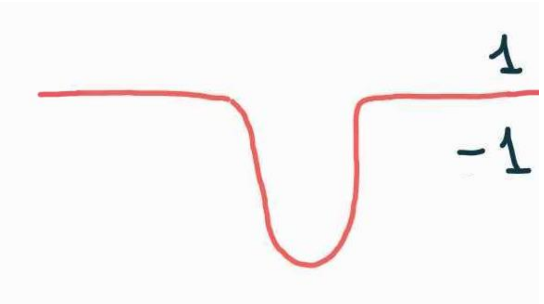
$$c_t + u \cdot \nabla c = \varepsilon \Delta c$$
$$div(u) = 0$$
$$u = (u^1, u^2), \quad u^2 = \bar{c} - c$$

Why fingers appear? It is a hair-trigger effect!

No flow



Velocity u changes due to concentration c



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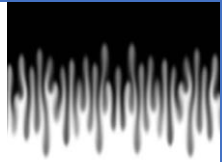

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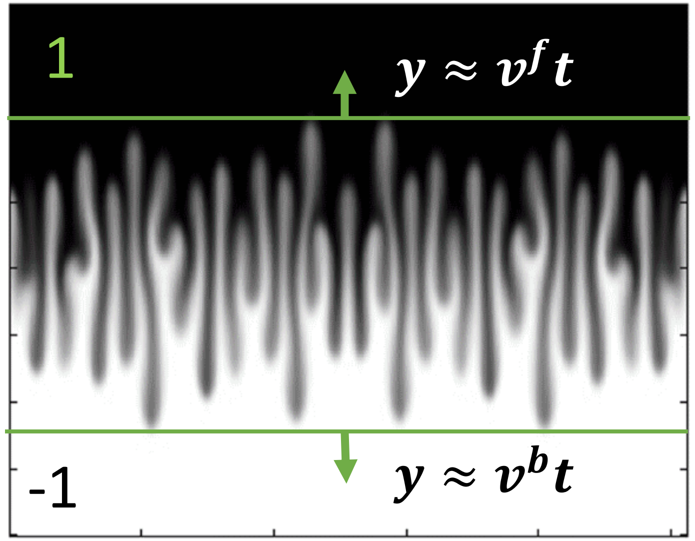


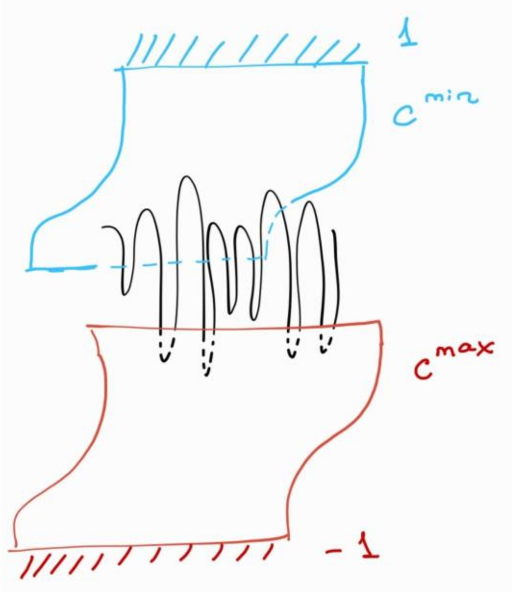
Transverse Flow Equilibrium = TFE

$p(t,x,y) \approx p(t,y)$

$$c_t + u \cdot \nabla c = \varepsilon \Delta c$$
$$div(u) = 0$$
$$u = (u^1, u^2), \quad u^2 = \bar{c} - c$$

Idea of proof (TFE): comparison to 1D Burgers eq ($\bar{c} \leq 1$ then $u^2 \leq 1 - c$)

$$c_t^{\max} + (1 - c^{\max}) \cdot \partial_y c^{\max} = \varepsilon c_{yy}^{\max}$$



Theorem (Otto, Menon): If $c(0,x,y) \leq c^{\max}(0,y)$,
then $c(t,x,y) \leq c^{\max}(t,y)$ for any $t > 0$.

Question: Are those estimates sharp?

Are the estimates sharp?

Estimates on the growth (theory):

2005 – F. Otto, G. Menon

- Full model (IPM) $v^f \leq 2$
- Simplified model (TFE) $v^f \leq 1$

Estimates on the growth (numerics):

2022 – G. Bofetta, S. Musacchio

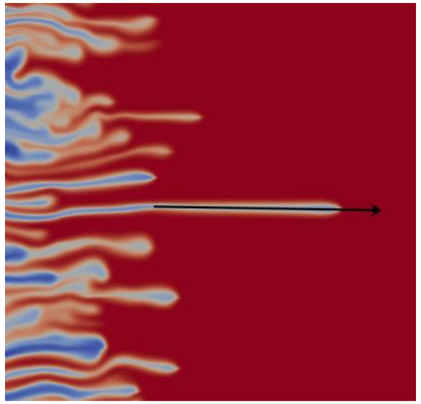
- Full model (IPM, 2D) $v^f \approx 0.67$
- Full model (IPM, 3D) $v^f \approx 0.43$

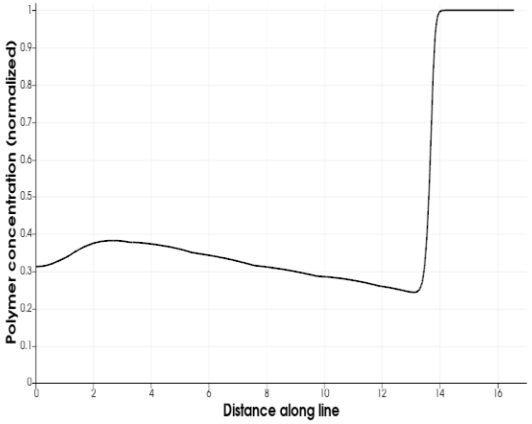
Viscous fingering: this gap in empirical and numerical estimates is even bigger

SLOW-DOWN of fingers… Why?

Two (possible) mechanisms:

1. Transport in transverse direction
2. Intermediate concentration on tip of finger





2024 – F. Bakharev, D. Pavlov, A. Enin, S. Matveenko, **Yu. Petrova**, N. Rastegaev, S. Tikhomirov (JCAM)

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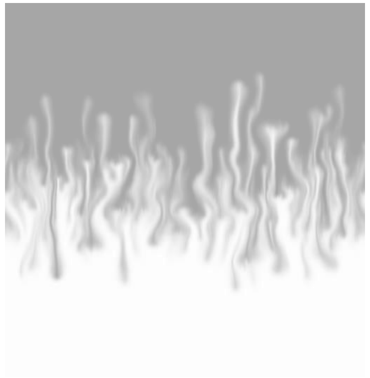

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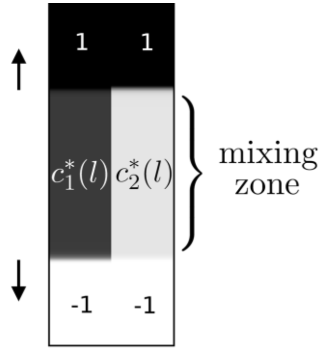

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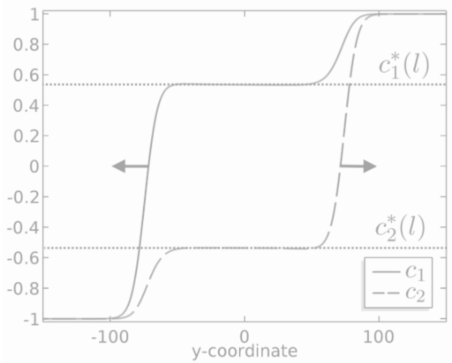
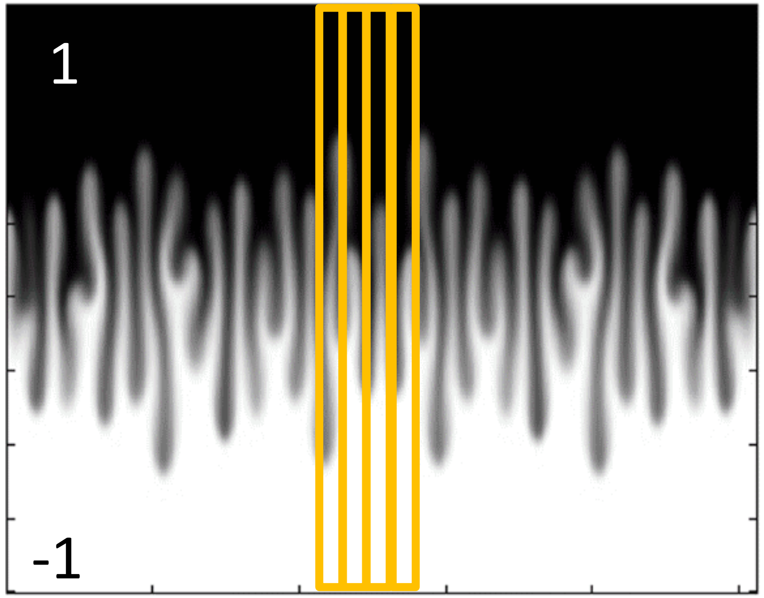


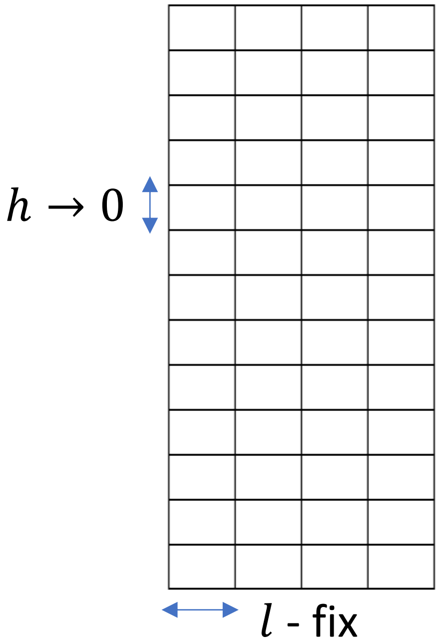
IDEA: semi-discrete model of gravitational fingering

- Discretize in horizontal direction
- Take n tubes, $n = 2,3,4,\ldots$



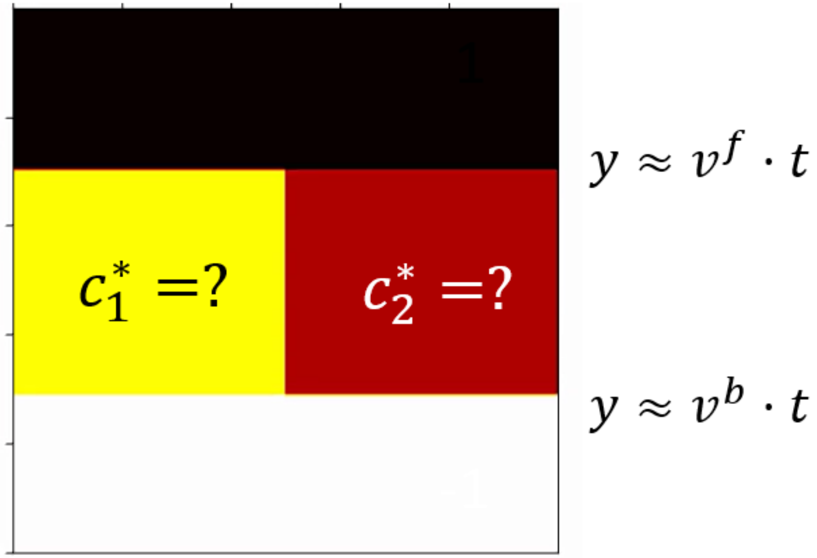
Tubes (layer, lane,…) models:

Limit of
numerical scheme



- Finite volume
- Upwind

- For simplicity, $n = 2$



We observe two traveling waves:
$c(y,t) = c(y - vt)$

1995 — Y. Yortsos "A theoretical analysis of vertical flow equilibrium"
2006 — J.C. Da Mota, S. Schecter "Combustion fronts in a porous medium with two layers"
2019 — A. Armiti-Juber, C. Rohde "On Darcy- and Brinkman-type models for two-phase flow in asympt. flat domains"
2019 — H. Holden, N. Risebro "Models for dense multilane vehicular traffic"

Two-tubes model

1. Original equation on c: $c_t + div(uc) - \Delta c = 0$

   Two-tubes equations on c:

$$\partial_t c_1 + \partial_y(u_1 c_1) - \partial_{yy} c_1 = \boxed{-B}$$
$$\partial_t c_2 + \partial_y(u_2 c_2) - \partial_{yy} c_2 = \boxed{+B}$$

2. Original equation on p: $u = -\nabla p - (0, c)$

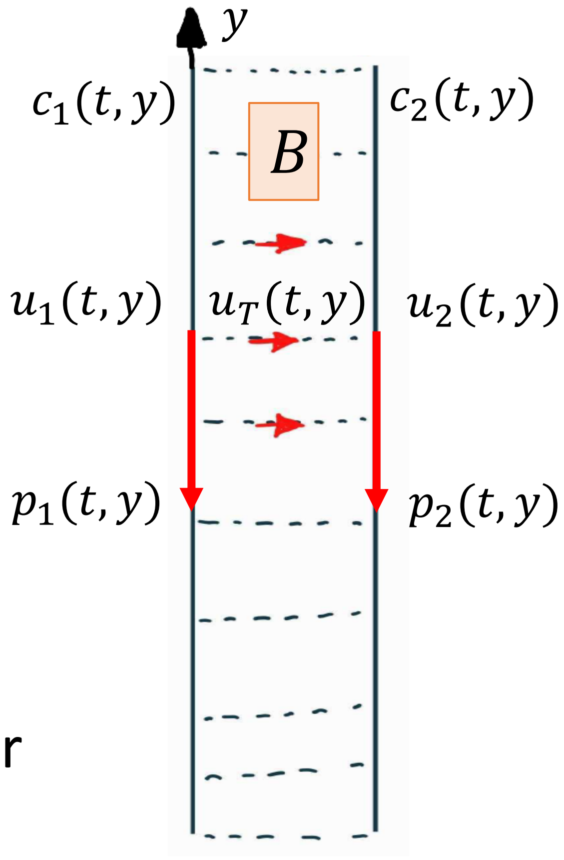
   Two-tubes equations on p:

$$u_1 = -\partial_y p_1 - c_1$$
$$u_2 = -\partial_y p_2 - c_2$$
$$u_T = -\frac{p_2 - p_1}{l}$$

3. Original equation on u: $div(u) = 0$

   Two-tubes equations on u:

$$\partial_y u_1 + \frac{u_T}{l} = 0$$



l - parameter

$$\boxed{B} = \begin{cases} \frac{u_T}{l} \cdot c_1, & u_T > 0, \\ \frac{u_T}{l} \cdot c_2, & u_T < 0 \end{cases}$$

Two-tubes model

1. Original equation on c: $c_t + div(uc) - \Delta c = 0$

   Two-tubes equations on c:

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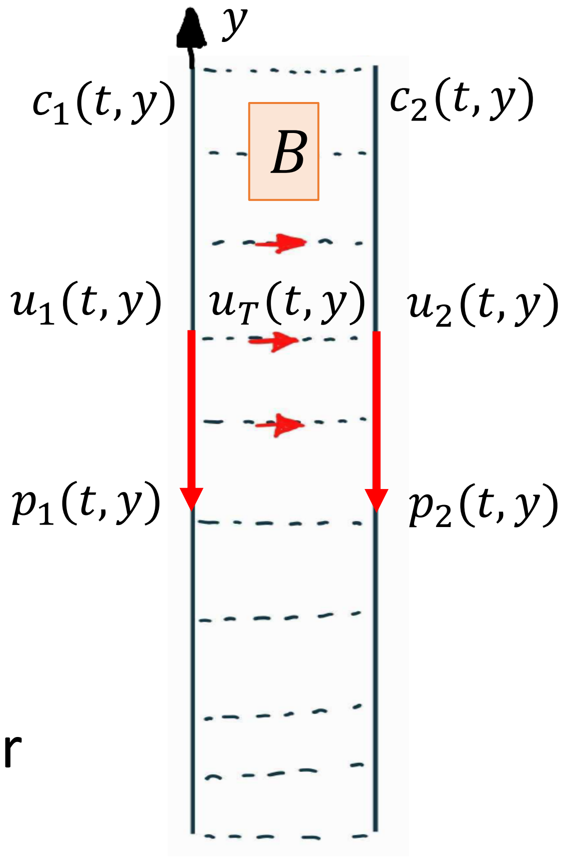


2. Original equation on p: $u = -\nabla p - (0, c)$

   Two-tubes equations on p:

$$u_1 = -\partial_y p_1 - c_1$$
$$u_2 = -\partial_y p_2 - c_2$$
$$\boxed{\frac{u_T}{l}} = -\frac{p_2 - p_1}{l^2}$$

l - parameter

3. Original equation on u: $div(u) = 0$

   Two-tubes equations on u:

$$\partial_y u_1 + \boxed{\frac{u_T}{l}} = 0$$

$$\boxed{B} = \begin{cases} \boxed{\frac{u_T}{l}} \cdot c_1, & u_T > 0, \\ \boxed{\frac{u_T}{l}} \cdot c_2, & u_T < 0 \end{cases}$$

Main result

Questions?

$$(*)\begin{cases} \partial_t c_1 + \partial_y(u_1 c_1) - \partial_{yy} c_1 = -B \\ \partial_t c_2 + \partial_y(u_2 c_2) - \partial_{yy} c_2 = B \\ u_1 = -\partial_y p_1 - c_1 \\ u_2 = -\partial_y p_2 - c_2 \\ \partial_y u_1 = -\partial_y u_2 = \dfrac{p_2 - p_1}{l^2} \end{cases}$$

$$B = \begin{cases} -\partial_y u_1 \cdot c_1, & \partial_y u_1 < 0, \\ +\partial_y u_2 \cdot c_2, & \partial_y u_1 > 0 \end{cases}$$

$$\boxed{\begin{array}{lll} y \to +\infty: & c_{1,2} \to +1; & u_{1,2,T} \to 0 \\ y \to -\infty: & c_{1,2} \to -1; & u_{1,2,T} \to 0 \end{array}}$$

Theorem (P., Tikhomirov, Efendiev, arXiv: 2401.05981, accept. SIMA)

Consider a two-tube model with gravity $(*)$.

Then for all $l > 0$ *sufficiently small* there exists $c_1^*(l), c_2^*(l)$

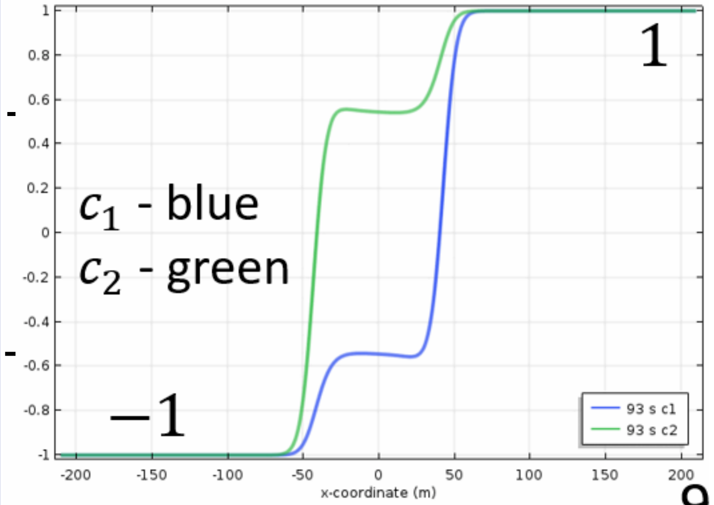
such that there exist two traveling waves (TW):

TW1 with speed $v^b(l)$: $(-1,-1) \to (c_1^*, (l)\ c_2^*(l))$

TW2 with speed $v^f(l)$: $(c_1^*, (l)\ c_2^*(l)) \to (1,1)$.

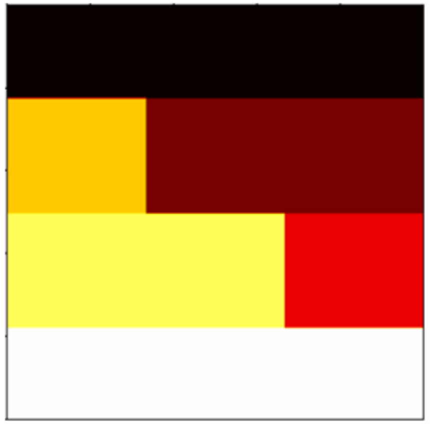
Moreover, $\lim_{l\to 0} c_1^*(l) = -\lim_{l\to 0} c_2^*(l) = -\frac{1}{2}$; $\lim_{l\to 0} v^f(l) = -\lim_{l\to 0} v^b(l) = \frac{1}{4}$.

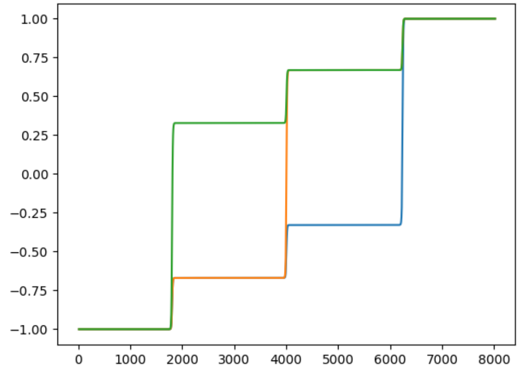
As $t \to \infty$ we observe:



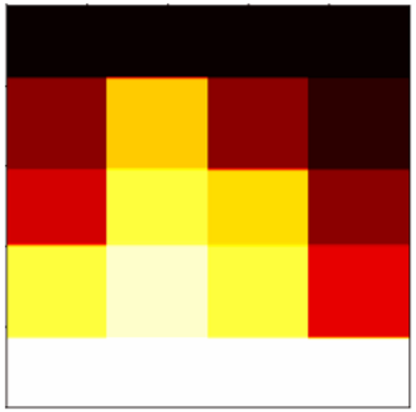
Many tubes: numerics

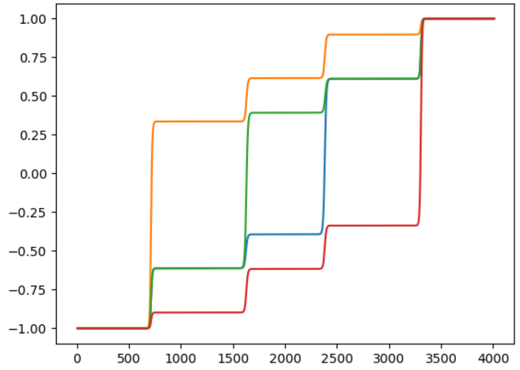
3 tubes



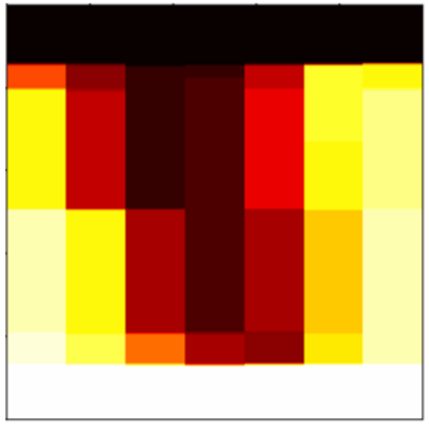


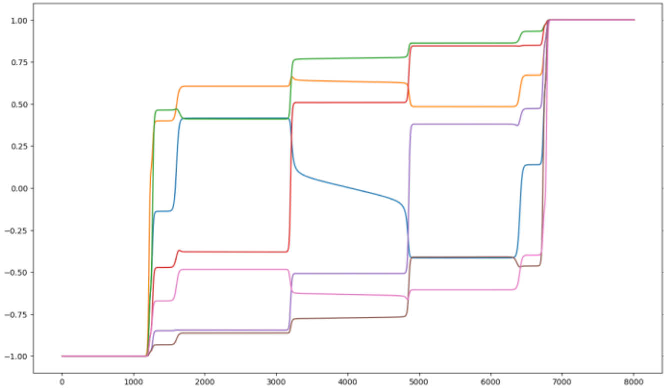
4 tubes





7 tubes





Questions: (open)

(1) explain the structure of "asymptotic solutions" for n tubes and study their stability
(2) find speed of growth of the mixing zone
(3) understand the behaviour as $n \to \infty$. Do we approximate 2-dim IPM?
(4) can we repeat this story for the many tubes viscous fingering model?

Well-posedness for two-tubes model? $t \geq 0$ $y \in \mathbb{R}$

Two-tubes IPM

$$\begin{cases} \partial_t c_1 + \partial_y(u_1 c_1) - \partial_{yy} c_1 = -B \\ \partial_t c_2 + \partial_y(u_2 c_2) - \partial_{yy} c_2 = B \\ u_1 = -\partial_y p_1 - c_1 \\ u_2 = -\partial_y p_2 - c_2 \\ B = \begin{cases} -\partial_y u_1 \cdot c_1, & \partial_y u_1 < 0, \\ +\partial_y u_2 \cdot c_2, & \partial_y u_1 > 0 \end{cases} \\ \partial_y u_1 = -\partial_y u_2 = \dfrac{p_2 - p_1}{l^2} \end{cases}$$

$l \to 0?$

Two-tubes TFE

$$\begin{cases} \partial_t c_1 + \partial_y(u_1 c_1) - \partial_{yy} c_1 = -B \\ \partial_t c_2 + \partial_y(u_2 c_2) - \partial_{yy} c_2 = B \\ u_1 = \dfrac{c_2 + c_1}{2} - c_1 = \bar{c} - c_1 \\ B = \begin{cases} -\partial_y u_1 \cdot c_1, & \partial_y u_1 < 0, \\ +\partial_y u_2 \cdot c_2, & \partial_y u_1 > 0 \end{cases} \end{cases}$$

$l = 0$: singular limit

Initial Condition:

$$c_1(0, y) = c_1^0(y)$$
$$c_2(0, y) = c_2^0(y)$$

Conditions at $y = \pm\infty$:

$$c_{1,2}(t, +\infty) = +1$$
$$c_{1,2}(t, -\infty) = -1$$

$$u_{1,2}(t, \pm\infty) = 0$$
$$(p_2 - p_1)(t, \pm\infty) = 0$$

Questions (open):

1) Does there exist global solution $(c_1, c_2, u_1, u_2, p_1, p_2) \in C([0, \infty]; X)$ for suitable Banach space X?
2) As $t \to \infty$ does solution converge to a propagating terrace (combination of two traveling waves)?
3) Can we rigorously justify the singular limit as $l \to 0$?

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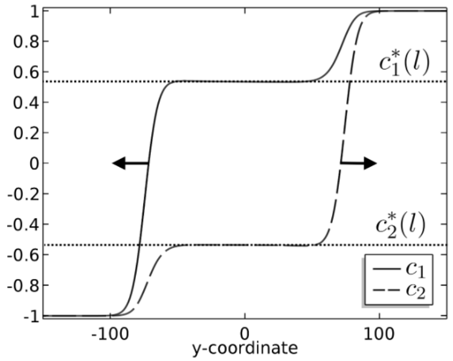

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$$(*)\quad \begin{cases} \partial_t c_1 + \partial_y (u_1 c_1) - \partial_{yy} c_1 = -B \\ \partial_t c_2 + \partial_y (u_2 c_2) - \partial_{yy} c_2 = B \\ u_1 = -\partial_y p_1 - c_1 \\ u_2 = -\partial_y p_2 - c_2 \\ \partial_y u_1 = -\partial_y u_2 = \dfrac{p_2 - p_1}{l^2} \end{cases}$$

$$B = \begin{cases} -\partial_y u_1 \cdot c_1, & \partial_y u_1 < 0, \\ +\partial_y u_2 \cdot c_2, & \partial_y u_1 > 0 \end{cases}$$

$$\boxed{\begin{array}{lll} y \to +\infty: & c_{1,2} \to +1; & u_{1,2,T} \to 0 \\ y \to -\infty: & c_{1,2} \to -1; & u_{1,2,T} \to 0 \end{array}}$$

Theorem (P., Tikhomirov, Efendiev, arXiv: 2401.05981, accept. SIMA)

Consider a two-tube model with gravity $(*)$.

Then for all *$l > 0$ sufficiently small* there exists $c_1^*(l), c_2^*(l)$

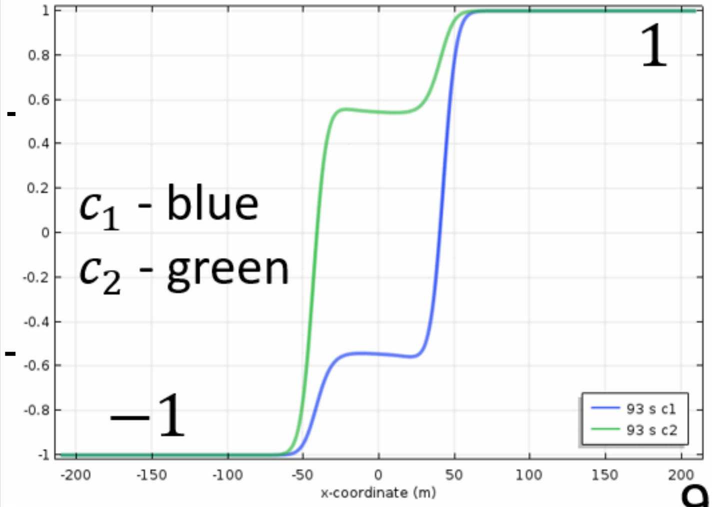
such that there exist two traveling waves (TW):

TW1 with speed $v^b(l)$: $(-1,-1) \to (c_1^*, (l)\; c_2^*(l))$

TW2 with speed $v^f(l)$: $(c_1^*, (l)\; c_2^*(l)) \to (1, 1)$.

Moreover, $\lim\limits_{l\to 0} c_1^*(l) = -\lim\limits_{l\to 0} c_2^*(l) = -\frac{1}{2}$; $\lim\limits_{l\to 0} v^f(l) = -\lim\limits_{l\to 0} v^b(l) = \frac{1}{4}$.

As $t \to \infty$ we observe:



Scheme of proof

Step 1: structure of the set of traveling wave (TW) solutions

$$c_1(t,y) = c_1(y - vt)$$
$$c_2(t,y) = c_2(y - vt)$$

$$u_1(t,y) = u_1(y - vt)$$
$$u_2(t,y) = u_2(y - vt)$$

$$p_1(t,y) = p_1(y - vt)$$
$$p_2(t,y) = p_2(y - ct)$$

Theorem

For sufficiently small $l > 0$ and for each v close to $\frac{1}{4}$ there exists a TW:

$$(c_1^*, c_2^*, u_1^*, u_2^*, p_1^* - p_2^*) \to (1,1,0,0,0)$$

Similarly,

$$(-1,-1,0,0,0) \to (c_1^{**}, c_2^{**}, u_1^{**}, u_2^{**}, p_1^{**} - p_2^{**})$$

Step 2: existence of a propagating terrace of two traveling waves

- Find a common intermediate state $(c_1, c_2, u_1, u_2, p_1 - p_2)$ for traveling waves above

Scheme of proof: step 1

Travelling wave (TW) ansatz with fixed v:

$$c_1(t,y) = c_1(y - vt)$$
$$c_2(t,y) = c_2(y - vt)$$

$$u_1(t,y) = u_1(y - vt)$$
$$u_2(t,y) = u_2\,(y - vt)$$

$$p_1(t,y) = p_1(y - vt)$$
$$p_2(t,y) = p_2(y - ct)$$

With condition at $+\infty$:

$$c_1(+\infty) = 1$$
$$c_2(+\infty) = 1$$
$$u_1(+\infty) = 0$$
$$u_2(+\infty) = 0$$
$$(p_1 - p_2)(+\infty) = 0$$

System of ODEs in $\mathbb{R}^6$:

$$\boxed{\begin{cases} \dot{X} = F_v(X,Y) \\ l \cdot \dot{Y} = AY - BX \end{cases}}$$

Here:

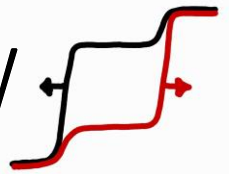
- $X = \begin{pmatrix} c_1 \\ c_2 \\ \partial_\xi c_1 \\ \partial_\xi c_2 \end{pmatrix} \in \mathbb{R}^4, \quad Y = \begin{pmatrix} u_1 \\ p_1 - p_2 \end{pmatrix} \in \mathbb{R}^2$
- $A = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}, \qquad B \in M^{2\times 4}, \qquad l \ll 1$

<u>Obs:</u> for $l \to 0$ this system has a ``slow-fast'' structure

<u>Key tool:</u> geometric singular perturbation theory (GSPT)

1979 – N. Fenichel (JDE); 2020 – M. Wechselberger

Scheme of proof: step 2 – propagating terrace of 2 TW



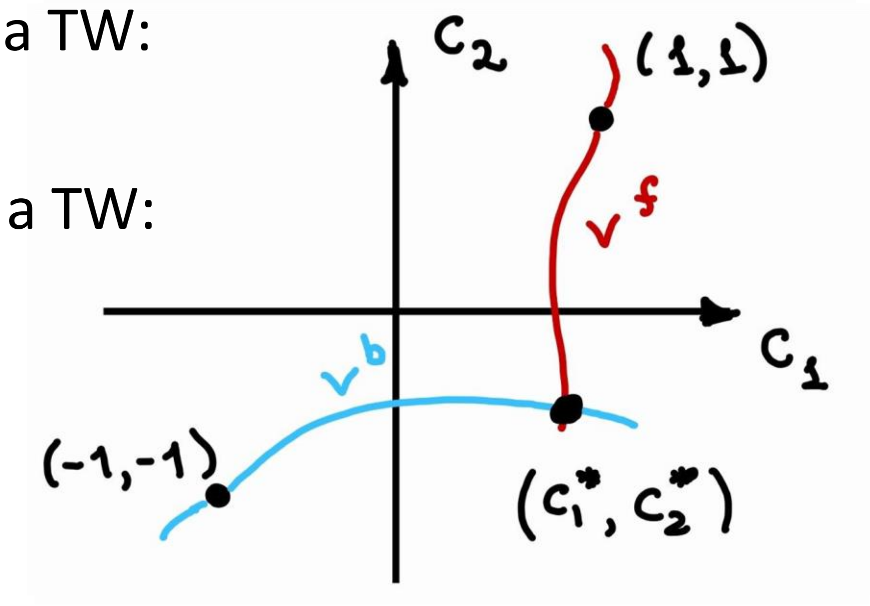
1) For each $v^f \in I_f \subset \mathbb{R}$ we find all points s.t. there exists a TW:

$$(c_1\,, c_2) \;\rightarrow\; (1,1)$$

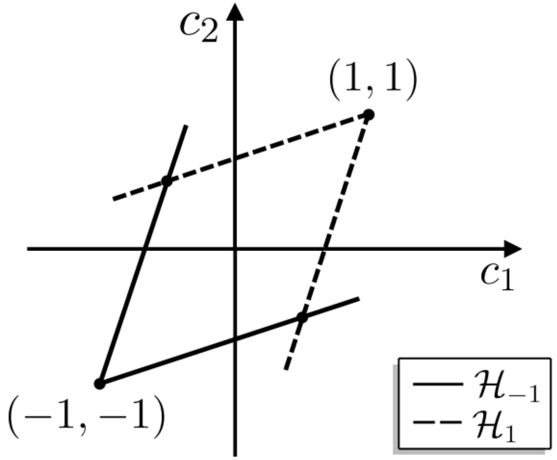
2) For each $v^b \in I_b \subset \mathbb{R}$ we find all points s.t. there exists a TW:

$$(-1,-1) \rightarrow (c_1\,, c_2)$$

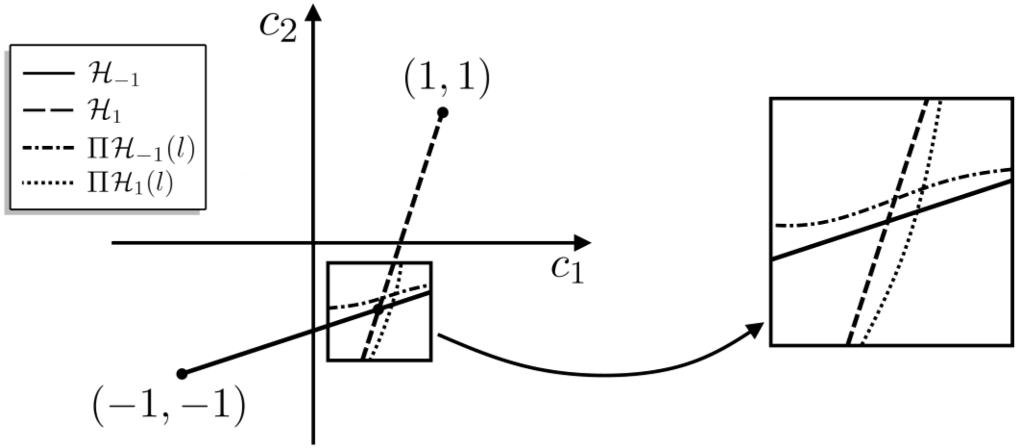
3) Find the intersection points of these two curves



$l = 0$ – these curves are just straight lines



$0 < l \ll 1$– perturbation argument



Slow-fast systems: simple example

Slow system

$$\begin{cases} \dot{x} &= -x \\ \varepsilon \cdot \dot{y} &= x^2 - y \end{cases}$$

Fast system

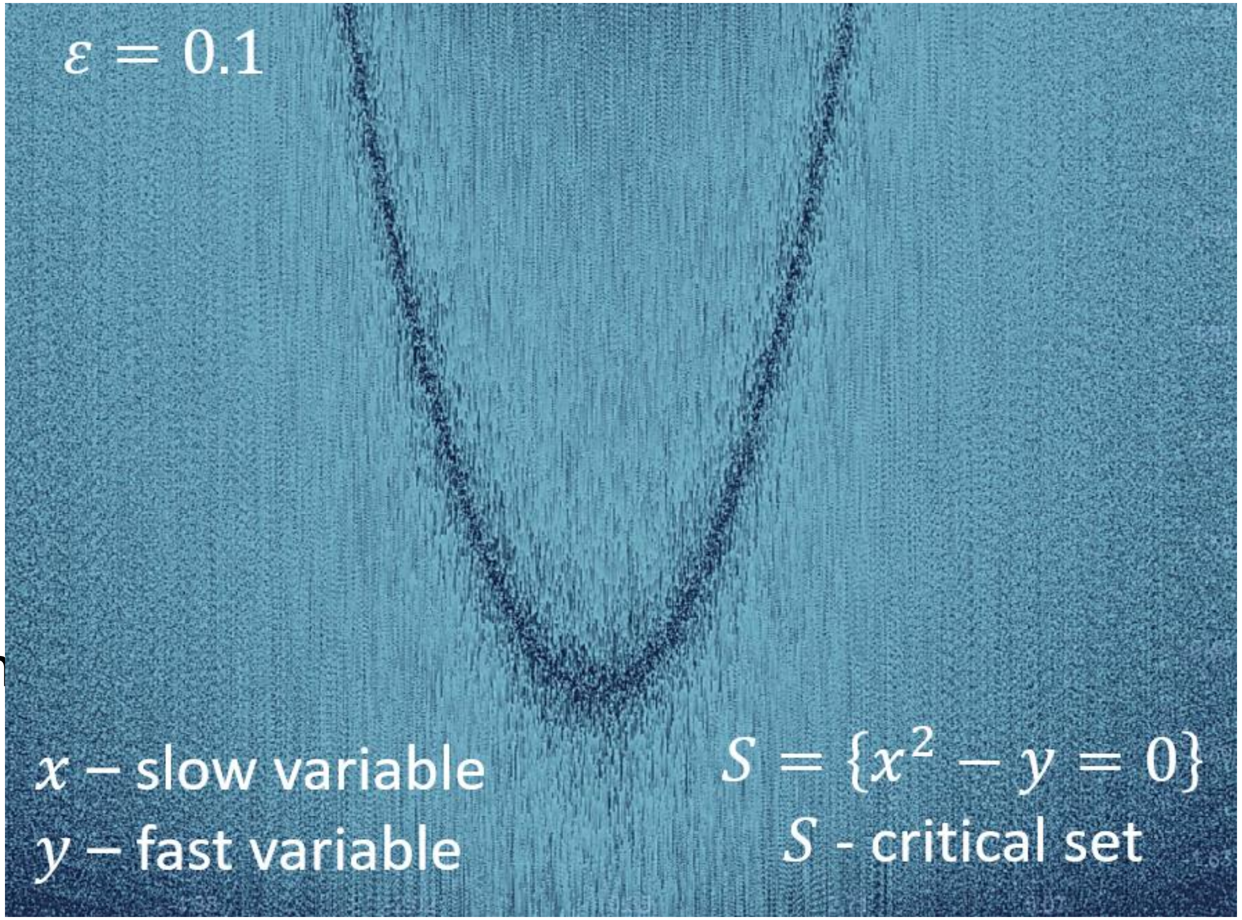
$$\begin{cases} x' &= \varepsilon \cdot (-x) \\ y' &= x^2 - y \end{cases}$$

$t = \varepsilon \cdot s$

$0 < \varepsilon \ll 1$

Formally $\varepsilon \to 0$

Formally $\varepsilon \to 0$



Reduced slow system

$$\begin{cases} \dot{x} &= -x \\ 0 &= x^2 - y \end{cases}$$

Reduced fast system

$$\begin{cases} x' &= 0 \\ y' &= x^2 - y \end{cases}$$

Geometric singular perturbation theory (GSPT)

Slow system (t – slow time)

$$\begin{cases} \dot{X} &= F(X,Y,\varepsilon) \\ \varepsilon \cdot \dot{Y} &= G(X,Y,\varepsilon) \end{cases}$$

$$t = \varepsilon \cdot s$$

$$0 < \varepsilon \ll 1$$

Fast system (s – fast time)

$$\begin{cases} X' &= \varepsilon \cdot F(X,Y,\varepsilon) \\ Y' &= G(X,Y,\varepsilon) \end{cases}$$

Formally $\varepsilon \to 0$

Reduced slow system

$$\begin{cases} \dot{X} &= F(X,Y,0) \\ 0 &= G(X,Y,0) \end{cases}$$

Formally $\varepsilon \to 0$

Reduced fast system

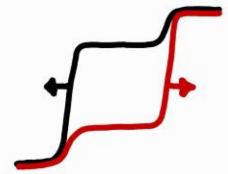
$$\begin{cases} X' &= 0 \\ Y' &= G(X,Y,0) \end{cases}$$

$S = \{G(X,Y,0) = 0\}$ – critical set

empty or consists of isolated points
(regular perturbation problem)

contains a differentiable manifold
(singular perturbation problem)

Normally hyperbolic manifolds ("fast-slow" version)



$$\begin{cases} X' = \varepsilon \cdot F(X,Y,\varepsilon) \\ Y' = G(X,Y,\varepsilon) \end{cases}$$

$(X,Y) \in \mathbb{R}^m \times \mathbb{R}^n$, $\quad F(X,Y,\varepsilon), G(X,Y,\varepsilon)$ – smooth

$S = \{(X,Y) \in \mathbb{R}^{m+n}: \ G(X,Y,0) = 0\}$ – critical manifold

Definition: A smooth compact manifold $S_0 \subset S$ is called *normally hyperbolic* if the $n \times n$ matrix $DG_Y(X,Y,0)$ is hyperbolic for all $(X,Y) \in S_0$.

In particular, S_0 is called:

- attracting, if all eigenvalues of $DG_y(X,Y,0)$ have negative real part
- repelling, if all eigenvalues of $DG_y(X,Y,0)$ have positive real part
- of saddle-type, if it is neither attracting nor repelling

Normal hyperbolicity of critical manifold $\Rightarrow$ ``nice'' perturbation

1979 – N. Fenichel (JDE) Geometric singular perturbation theory for ordinary differential equations
2015 – C. Kuehn, Multiple Time Scale Dynamics (Chapters 1-3) – intro to slow-fast systems & Fenichel's work

Fenichel's theorem (``fast-slow'' version)

Let S_0 be a compact normally hyperbolic submanifold (possibly with boundary) of the critical manifold S of the system

$$\begin{cases} X' = \varepsilon \cdot F\,(X,Y,\varepsilon) \\ Y' = \quad\; G(X,Y,\varepsilon) \end{cases}$$

and that $F, G \in C^r$ $(r \geq 2)$.

Then for $\varepsilon > 0$ sufficiently small, the following hold:

(F1) There exists a locally invariant manifold S_ε diffeomorphic to S_0.

(F2) S_ε has Hausdorff distance $O(\varepsilon)$ from S_0 (as $\varepsilon \to 0$).

(F3) The flow on S_ε converges to the flow of the reduced slow system (as $\varepsilon \to 0$).

(F4) S_ε is C^r-smooth and normally hyperbolic

Remark: S_ε may be not unique

Local invariance means that trajectories can enter or leave S_ε only through its boundaries.

Scheme of proof: step 1 (more detailed)

$$\begin{cases} \dot{X} = F_v(X,Y) \\ l \cdot \dot{Y} = AY - BX \end{cases}$$

- $X \in \mathbb{R}^4$ - slow
- $Y \in \mathbb{R}^2$ - fast
- $l \ll 1$

- Critical manifold:

$$S = \{(X,Y): \quad Y = A^{-1}BX\}, \qquad \dim S = 4$$

- $K \subset S$ (compact) is normally hyperbolic as the matrix

$$A = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix} \quad \text{has eigenvalues} \quad \lambda_{\pm} = \pm\sqrt{2}$$

Thus, by Fenichel's theorem for $l \ll 1$

- For any compact submanifold $K \subset S$ there exists a locally invariant manifold $K_l \subset \mathbb{R}^6$

$$K_l = \{(X,Y): \quad Y = A^{-1}BX + l \cdot h(X,l)\} \qquad \text{for some smooth function } h$$

Result: 6-dim system on (X,Y) $\Rightarrow$ 4-dim system on X on K_l:

$$\boxed{\dot{X} = F_v(X, A^{-1}BX + l \cdot h(X,l))}$$

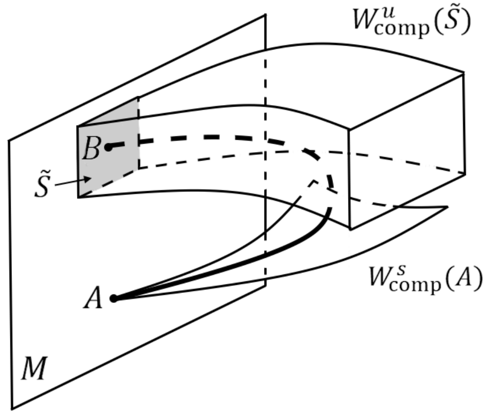
Scheme of proof: step 1 (more detailed)

THE END

We have a perturbation problem ($X \in \mathbb{R}^4$):

$l > 0$: $$\dot{X} = F_v(X, A^{-1}BX + l \cdot h(X,l))$$

$l = 0$: $$\dot{X} = F_v(X, A^{-1}BX)$$



$$\begin{cases} \dot{a} = r \\ \dot{b} = s \\ \dot{r} = -vr - \dfrac{a}{2}(s - r) \\ \dot{s} = -vs - ra \end{cases}$$

Fixed points:
$\{r = s = 0\}$

4-dim …we can find all heteroclinic orbits explicitly when $l = 0$!…

3-dim Obs 1: there is no b in the right hand side!

2-dim Obs 2: there are 2-dim invariant manifolds: $\{s = 2r\}$

1-dim Obs 3: inside these invariant manifolds holds:

$$r = -\left(v + \frac{a}{2}\right)^2 + r_0, \qquad r_0 \in \mathbb{R}$$

Heteroclinic orbit can be represented as a transverse intersection of stable and unstable manifolds
1970 – M. Hirsh, C. Pugh, M. Shub, Invariant manifolds

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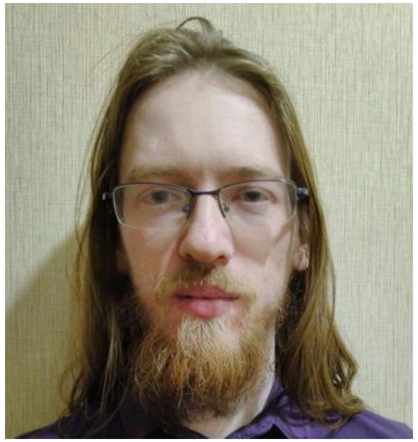

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Thank you for your attention!

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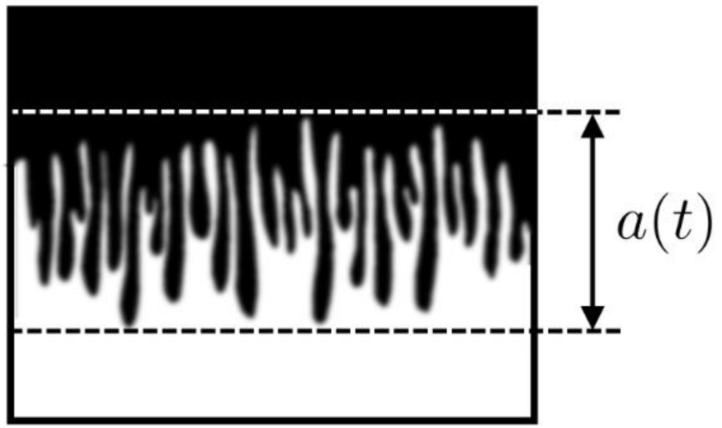

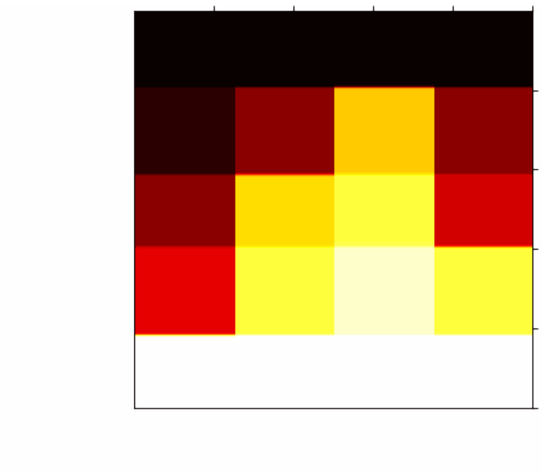

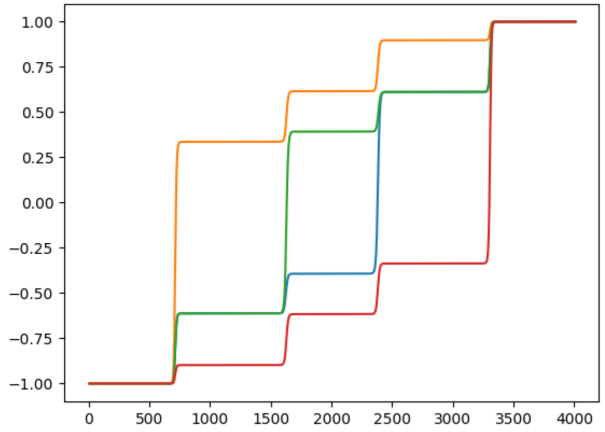

For more details see arXiv:2401.05981 (two-tubes model)
See also: arXiv:2310.14260 (numerics of viscous fingering)
arXiv:2012.02849 (numerics of viscous fingering)

Any questions, comments and ideas are very welcome!

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   ArXiv: 2401.05981*;* 2024. *To appear in SIMA.*
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