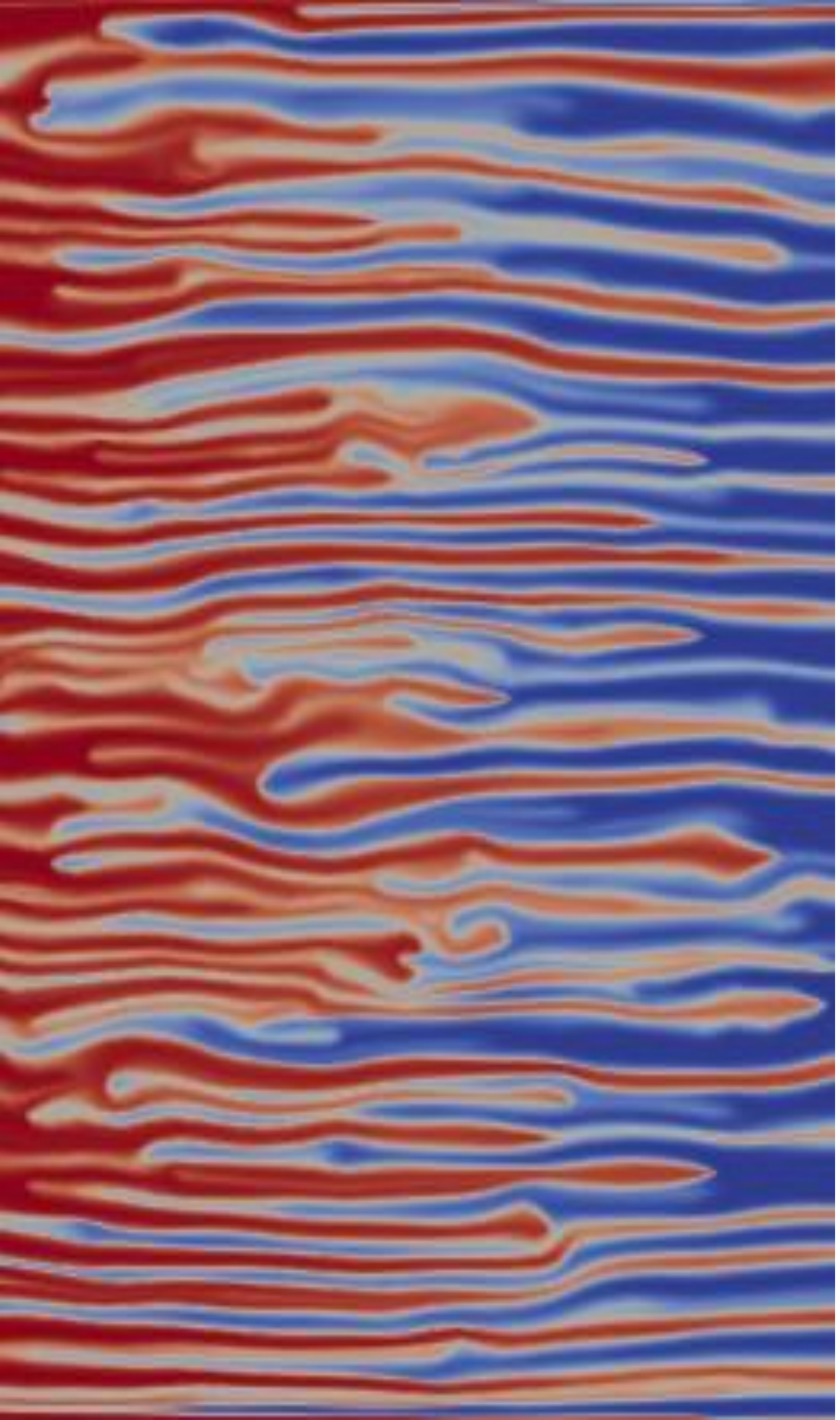

A propagating terrace of two traveling waves in a toy model of Incompressible Porous Medium (IPM) eq

PUC-Rio

CRANED #7

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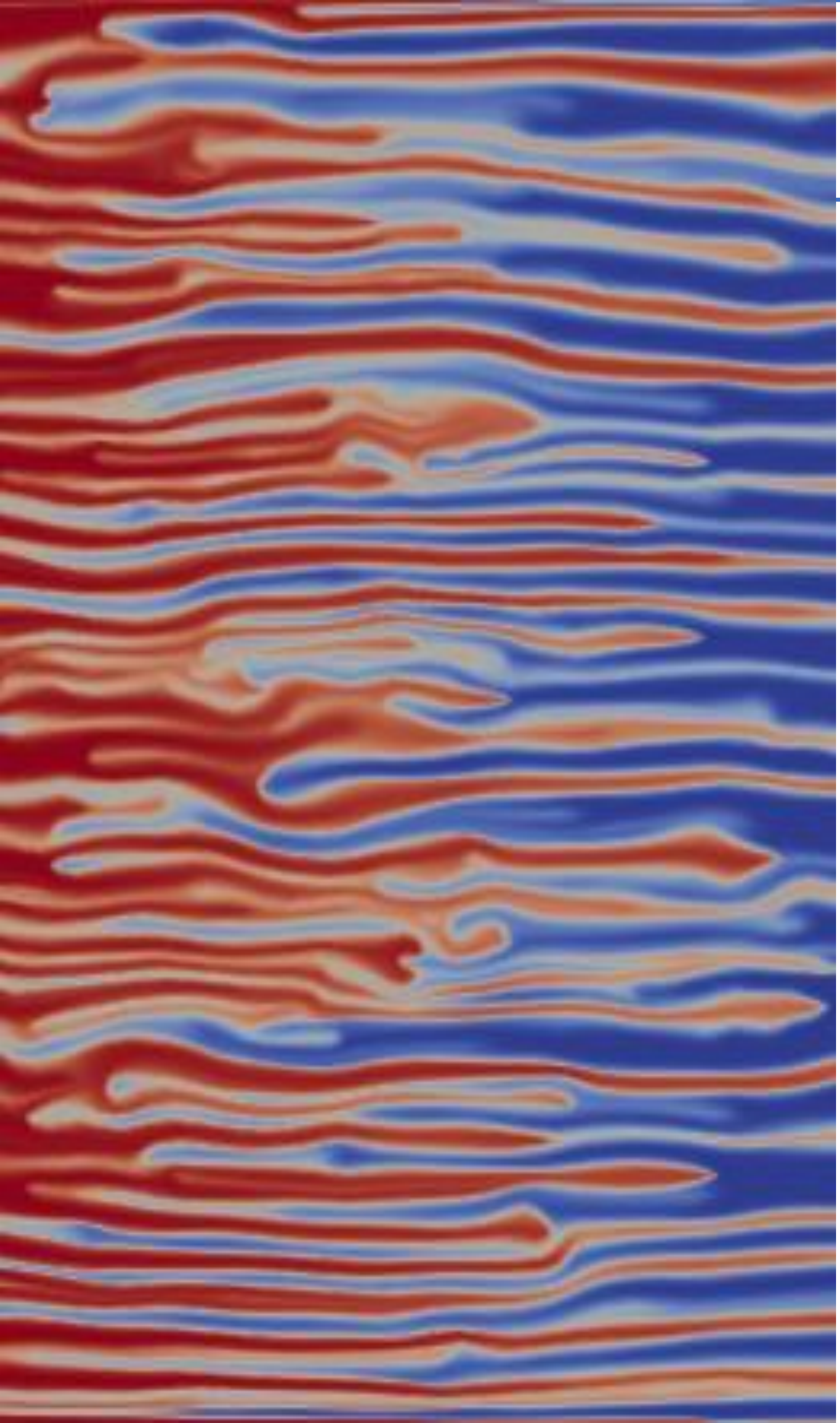
21 June 2024

Based on joint work:

1. *“Propagating terrace in a two-tubes model of gravitational fingering” 2024*
   (with S. Tikhomirov, Ya. Efendiev) Submitted. ArXiv: 2401.05981
2. *“Velocity of viscous fingers in miscible displacement: Intermediate concentration” 2024*
3. *“Velocity of viscous fingers in miscible displacement: Comparison with analytical models”2022*
   (with F. Bakharev, A. Enin, S.Matveenko, D.Pavlov, N.Rastegaev, I. Starkov, S.Tikhomirov) JCAM

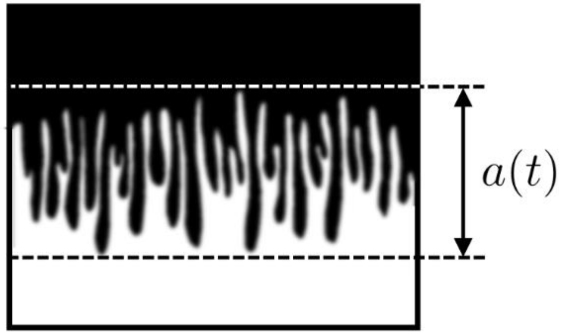


Outline

1. Motivation

   Miscible displacement in porous media

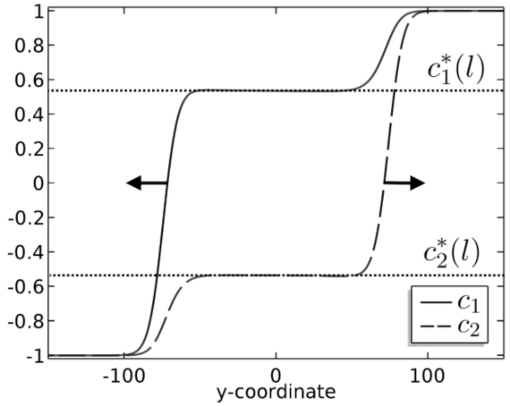
   - viscous fingering
   - gravitational fingering

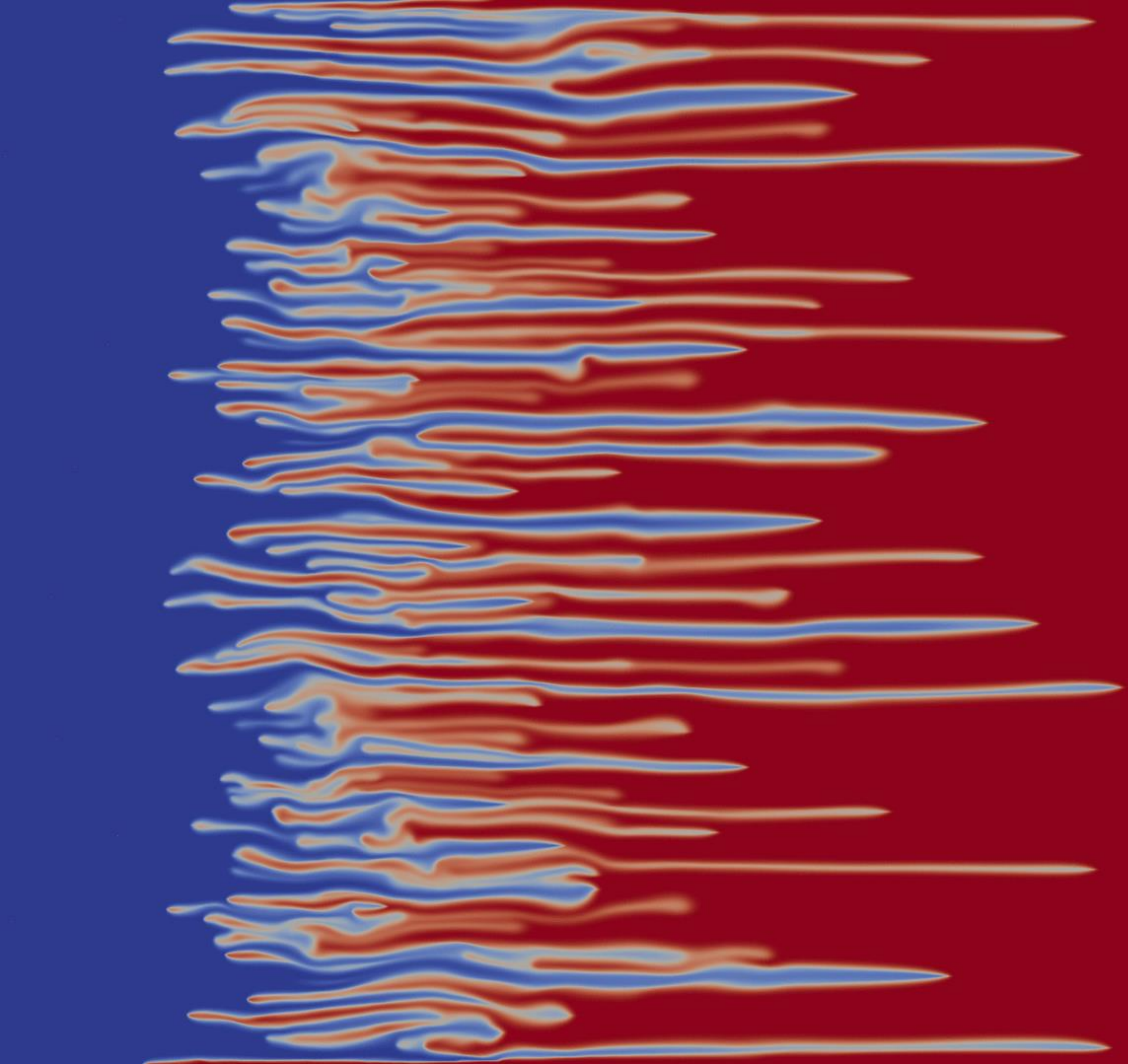


2. Problem statement
   - Two-tubes model
   - Main theorem
   - Sketch of proof:

     - traveling waves

     - slow-fast systems





“Miscible displacement in porous media”
Credit: Pavlov Dmitrii, St. Petersburg State University

Homsy , 1987 “Viscous Fingering in Porous Media”

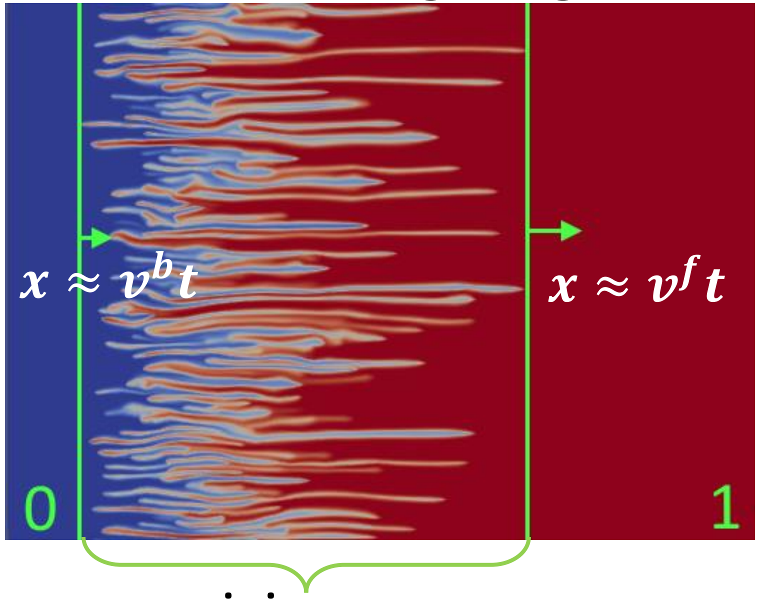
Viscous fingering phenomenon

water (blue color)

polymerized water (red color)

Incompressible Porous Medium eq – IPM, 2D (Two formulations)

Viscous fingering



mixing zone

$$c_t + div(uc) = \varepsilon \cdot \Delta c$$

$$div(u) = 0$$

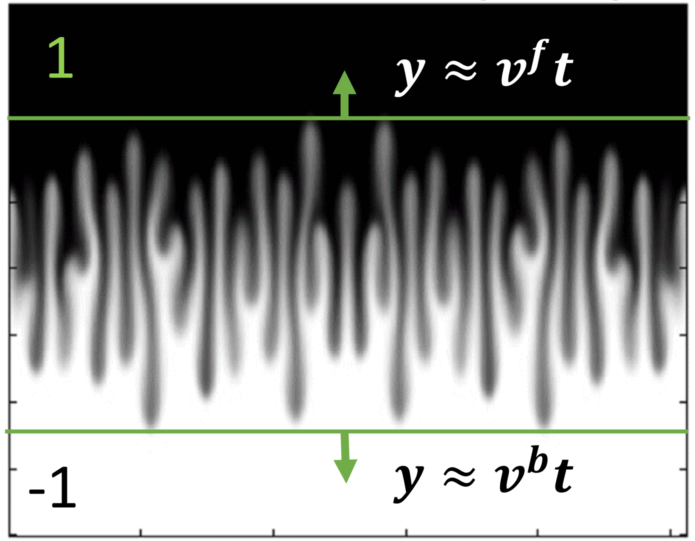
(viscosity) $\quad u = -m(c)\, K\, \nabla p$

(gravity) $\quad u = -\nabla p - (0, c)$

$c = c(t, x, y)$ – concentration
$u = u(t, x, y)$ – velocity
$p = p(t, x, y)$ – pressure

$\varepsilon \geq 0$ – diffusion
$m(c)$ – mobility
K – permeability

Gravitational fingering



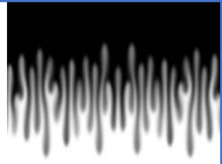
- many laboratory and numerical experiments show *linear growth of the mixing zone* [1], [2]

Question: how to find speeds v^b and v^f of propagation?

[1] Nijjer J., Hewitt D., and Neufeld J. The dynamics of miscible viscous fingering from onset to shutdown. Journal of Fluid Mechanics, 2018.

*[2] Bakharev, F., Enin, A., Groman, A., Kalyuzhnyuk, A., Matveenko, S., **Petrova, Y.**, Starkov, I. and Tikhomirov, S., Velocity of viscous fingers in miscible displacement: Comparison with analytical models. Journal of Computational and Applied Mathematics, 2022.*

IPM: $\varepsilon = 0$ (without diffusion)



Active scalar:

$$\boxed{\begin{aligned} &c_t + u \cdot \nabla c = 0 \\ &u = A(c) \end{aligned}}$$

$$u = \nabla^{\perp}(-\Delta)^{-1}\partial_1 c \qquad \text{(Biot-Savart law)}$$

Discontinuous initial data: free boundary problem (Muskat problem) – ill-posed for unstable stratification

2011 - A. Córdoba, D. Córdoba, F. Gancedo (Annals of Mathematics)
"Interface evolution: the Hele-Shaw and Muskat problems"

Existence: smooth initial data

2007 – D. Cordoba, F. Gancedo, R. Orive (JMP): local well-posedness for initial data H^s

global solution vs finite-time blow-up? open

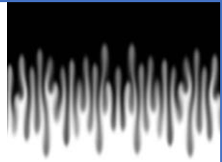
2017 – T. Elgindi (ARMA): global solution for small perturbations of $c = -y$
2023 – S. Kiselev, Y. Yao (ARMA): if solutions stay "smooth" for all times, then there is blow-up at $t = +\infty$

Uniqueness: non-uniqueness of weak solutions – by convex integration

2011 – D. Córdoba, D. Faraco, F. Gancedo (ARMA)
2012 – L. Szekelyhidi Jr.

…and many others…

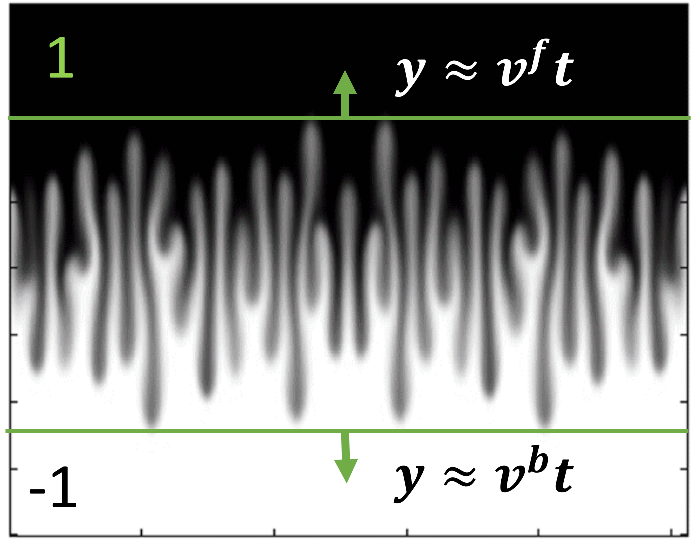
IPM: $\varepsilon > 0$ (with diffusion)



Estimates on the growth:

2005 – F. Otto, G. Menon. Proved estimates

- Full model (IPM) $v^f \leq 2$
- Simplified model (TFE) $v^f \leq 1$



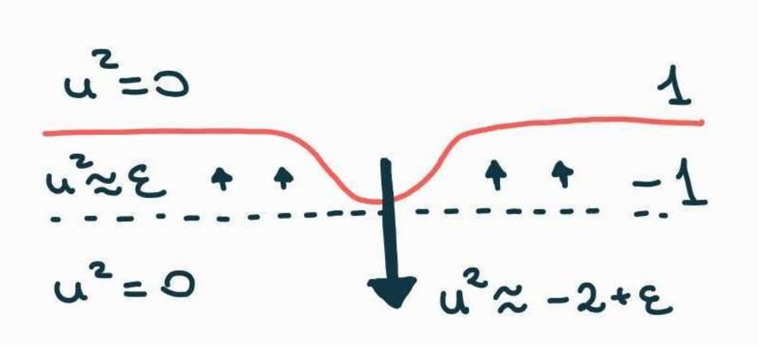
Transverse Flow Equilibrium = TFE

$$p(t,x,y) \approx p(t,y)$$

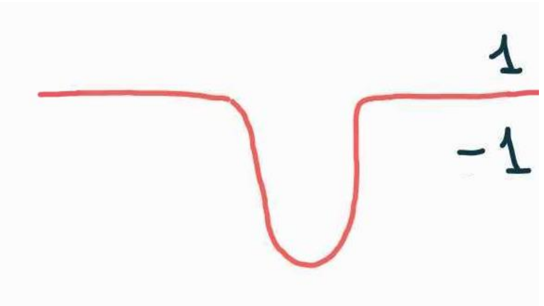
$$\begin{gathered} c_t + u \cdot \nabla c = \varepsilon \Delta c \\ div(u) = 0 \\ u = (u^1, u^2), \quad u^2 = \bar{c} - c \end{gathered}$$

Why fingers appear? It is a hair-trigger effect!

No flow



Velocity u changes due to concentration c



Concentration c changes due to velocity u

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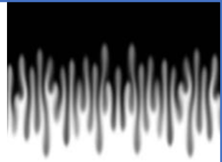

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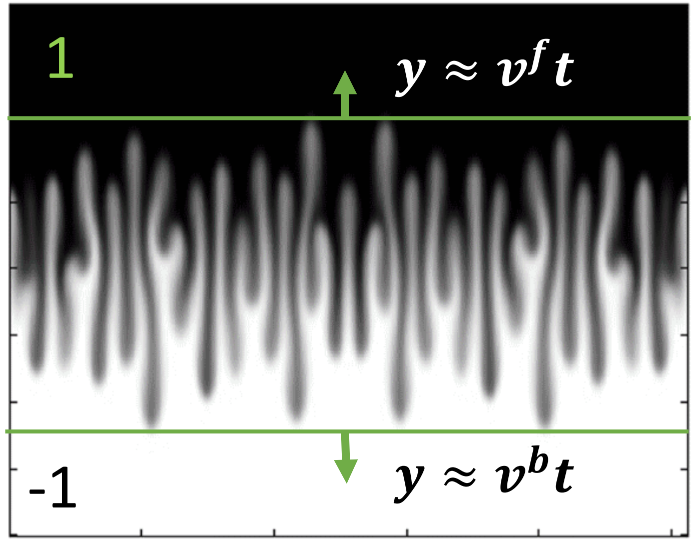

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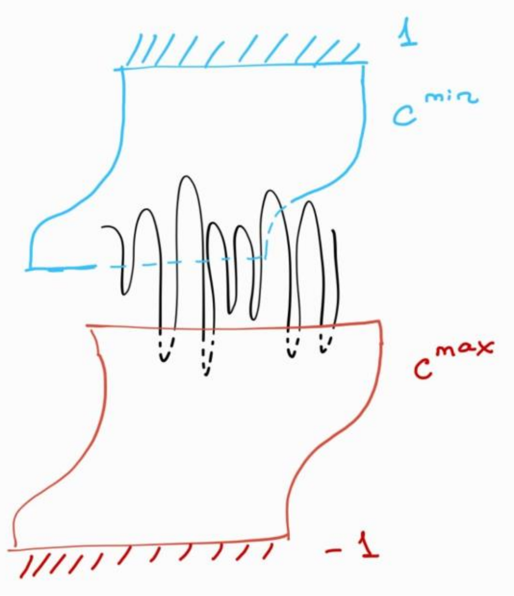
$$\boxed{\begin{gathered} c_t + u \cdot \nabla c = \varepsilon \Delta c \\ div(u) = 0 \\ u = (u^1, u^2), \quad u^2 = \bar{c} - c \end{gathered}}$$

Idea of proof (TFE): comparison to 1D Burgers eq ($\bar{c} \leq 1$ then $u^2 \leq 1 - c$)

$$\boxed{c_t^{\max} + (1 - c^{\max}) \cdot \partial_y c^{\max} = \varepsilon c_{yy}^{\max}}$$

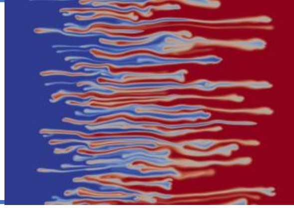


Theorem (Otto, Menon): If $c(0,x,y) \leq c^{\max}(0,y)$,
then $c(t,x,y) \leq c^{\max}(t,y)$ for any $t > 0$.

Question: Are those estimates sharp?

Viscosity-driven fingers

TFE estimates are too pessimistic!



$$c_t + div(uc) = \varepsilon \cdot \Delta c$$
$$div(u) = 0$$
$$u = -m(c)\,\nabla p = -1/\mu(c)\nabla p$$

Ratio of viscosities

$$M = \frac{\mu(1)}{\mu(0)}$$

Empirical models of velocities:

- Koval (1963): $v^f = (0.22 \cdot M^{0.25} + 0.78)^4$
- Todd-Longstaff (1972): $v^f = M^{2/3}$

Transverse Flow Equilibrium = TFE

$$p(t,x,y) \approx p(t,y)$$

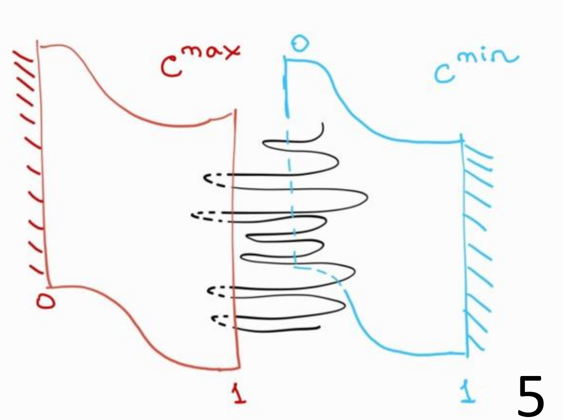
$$u = (u^1, u^2), \quad u^1 = \frac{m(c)}{avg(m(c))}$$

TFE estimate:

$$v^f \le \frac{\int_0^1 m(c)}{m(1)}$$

Idea of proof (TFE estimates): comparison to 1D Burgers-type eq (c $\le$ 1 then $u^1 \le \frac{m(c)}{m(1)}$)

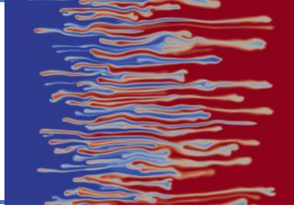
$$c_t^{\max} + \frac{m(c^{\max})}{m(1)} \cdot \partial_x c^{\max} = \varepsilon c_{xx}^{\max}$$



Theorem (Yortsos, Salin, 2006): If $c(0,x,y) \le c^{\max}(0,x)$,
then $c(t,x,y) \le c^{\max}(t,x)$ for any $t > 0$.

Question: Are those estimates sharp?

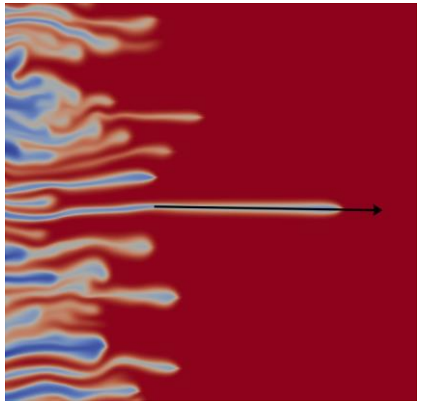
SLOW-DOWN of fingers… Why? Questions?

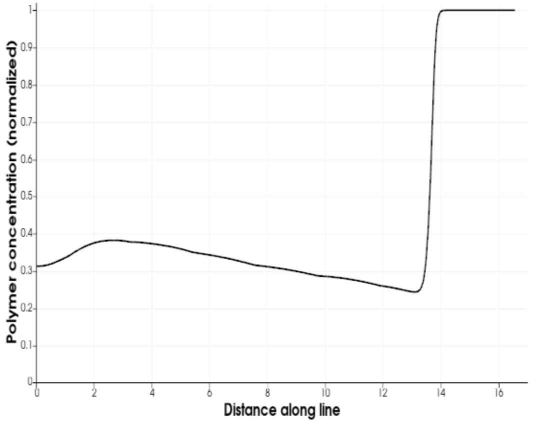


Naïve idea:
(1) flow in the transverse direction?
(2) Intermediate concentration?

$$c_t + div(uc) = \varepsilon \cdot \Delta c$$
$$div(u) = 0$$
$$u = -m(c)\,\nabla p = -1/\mu(c)\nabla p$$

A more careful look at simulations…. Intermediate concentration appears!





TFE estimate: $v^f \leq \frac{\int_0^1 m(c)}{m(1)}$

Improved TFE estimate

$$v^{TFE}(C) = \frac{\frac{1}{1-C}\int_C^1 m(c)}{m(1)}$$

$$X^f(t, C) = \max_x\{\exists y: c(t, x, y) \leq C\}$$

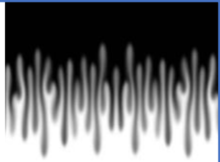
$$X^f(t, C) \sim v^f(C) \cdot t$$

Theorem TFE, viscosity (with many colleagues from St. Petersburg, JCAM, 2024, arXiv:2401.05981)

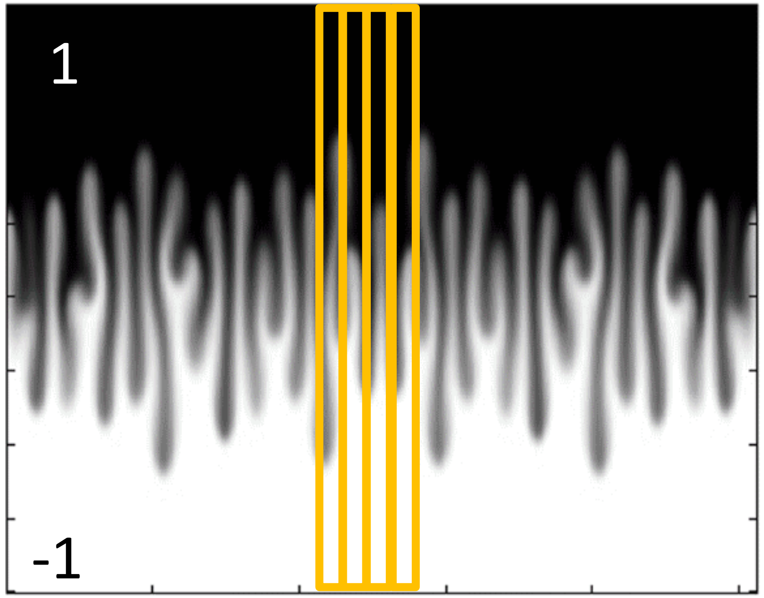
If there exists $C_1 \in [0,1]$ and $l_1 \in \mathbb{R}$: $X^f(t, C_1) \leq v^{TFE}(C_1) \cdot t + l_1$

Then for any $C_2 > C_1$ there exists l_2: $X^f(t, C_2) \leq v^{TFE}(C_1) \cdot t + l_2$

IDEA: semi-discrete model of gravitational fingering

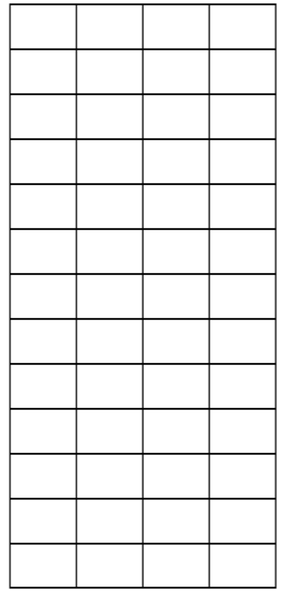


- Discretize in horizontal direction
- Take n tubes, $n = 2,3,4, \dots$



Limit of numerical scheme

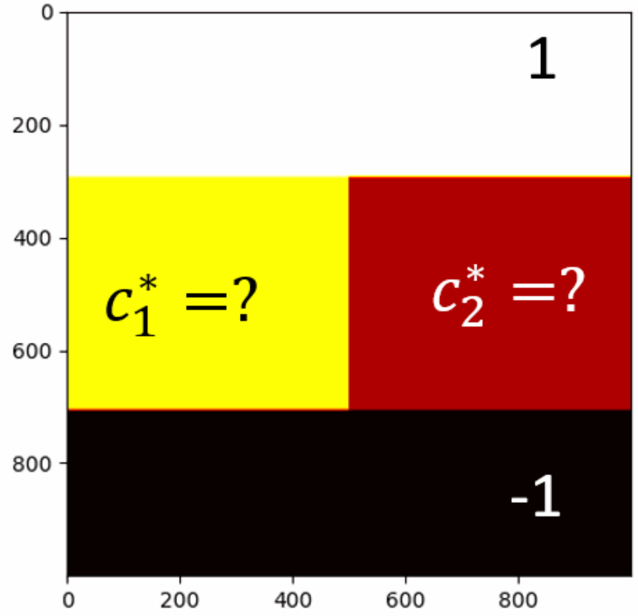
$h \to 0$



l - fix

- Finite volume
- Upwind

- For simplicity, $n = 2$



$y \approx v^f \cdot t$

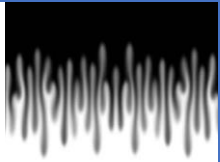
$y \approx v^b \cdot t$

We observe two traveling waves:
$c(y,t) = c(y - vt)$

Tubes (layer, lane,…) models:

2019 — A. Armiti-Juber, C. Rohde “On Darcy- and Brinkman-type models for two-phase flow in asympt. flat domains”
2006 — J.C. Da Mota, S. Schecter “Combustion fronts in a porous medium with two layers”
2019 — H. Holden, N. Risebro “Models for dense multilane vehicular traffic”

Two-tubes model



1. Original equation on c: $c_t + div(uc) - \Delta c = 0$

   Two-tubes equations on c:

$$\partial_t c_1 + \partial_y(u_1 c_1) - \partial_{yy} c_1 = \boxed{-B}$$
$$\partial_t c_2 + \partial_y(u_2 c_2) - \partial_{yy} c_2 = \boxed{+B}$$

2. Original equation on p: $u = -\nabla p - (0, c)$

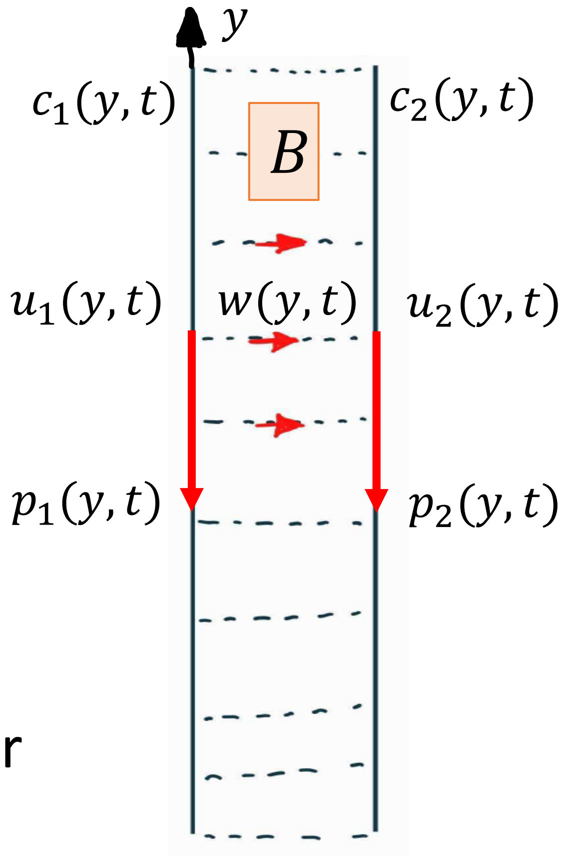
   Two-tubes equations on p:

$$u_1 = -\partial_y p_1 - c_1$$
$$u_2 = -\partial_y p_2 - c_2$$
$$w = -\frac{p_2 - p_1}{l}$$

3. Original equation on u: $div(u) = 0$

   Two-tubes equations on u:

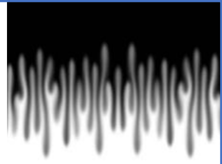
$$\partial_y u_1 + \frac{w}{l} = 0$$



l - parameter

$$\boxed{B} = \begin{cases} \frac{w}{l} \cdot c_1, & w > 0, \\ \frac{w}{l} \cdot c_2, & w < 0 \end{cases}$$

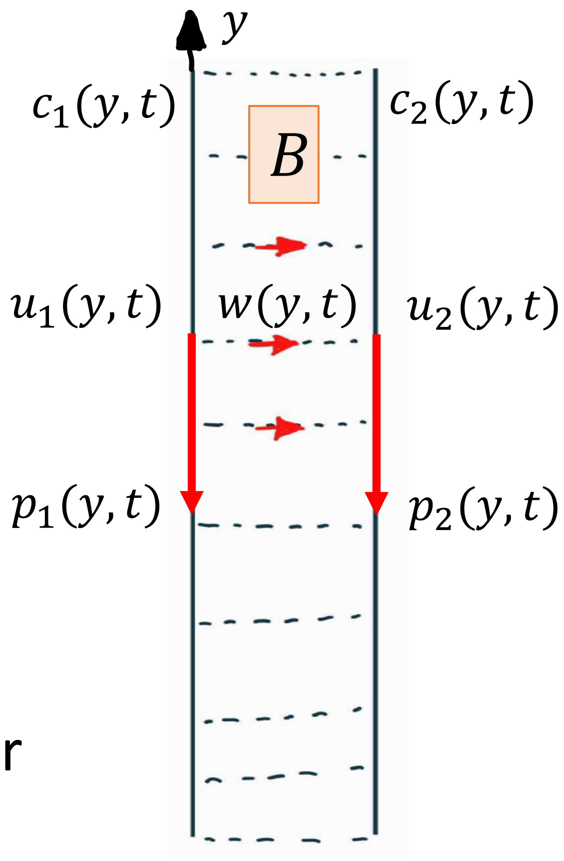
Two-tubes model



1. Original equation on c: $$c_t + div(uc) - \Delta c = 0$$
   Two-tubes equations on c:
   $$\partial_t c_1 + \partial_y(u_1 c_1) - \partial_{yy} c_1 = \boxed{-B}$$
   $$\partial_t c_2 + \partial_y(u_2 c_2) - \partial_{yy} c_2 = \boxed{+B}$$

2. Original equation on p: $$u = -\nabla p - (0, c)$$
   Two-tubes equations on p:
   $$u_1 = -\partial_y p_1 - c_1$$
   $$u_2 = -\partial_y p_2 - c_2$$
   $$\frac{w}{l} = -\frac{p_2 - p_1}{l^2}$$

3. Original equation on u: $$div(u) = 0$$
   Two-tubes equations on u:
   $$\partial_y u_1 + \frac{w}{l} = 0$$

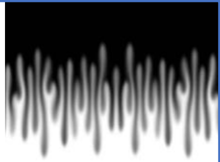


l - parameter

$$\boxed{B} = \begin{cases} \frac{w}{l} \cdot c_1, & w > 0, \\ \frac{w}{l} \cdot c_2, & w < 0 \end{cases}$$

Main result

Questions?



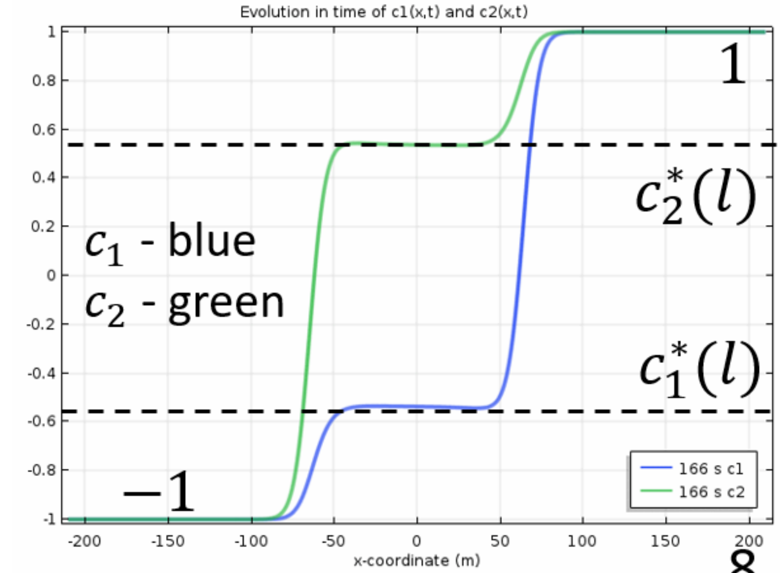
$$
(*)\begin{cases}
\partial_t c_1 + \partial_y (u_1 c_1) - \partial_{yy} c_1 = -B \\
\partial_t c_2 + \partial_y (u_2 c_2) - \partial_{yy} c_2 = B \\
u_1 = -\partial_y p_1 - c_1 \\
u_2 = -\partial_y p_2 - c_2 \\
\partial_y u_1 = -\partial_y u_2 = \dfrac{p_2 - p_1}{l^2}
\end{cases}
$$

$$
B = \begin{cases}
-\partial_y u_1 \cdot c_1, & \partial_y u_1 < 0, \\
+\partial_y u_2 \cdot c_2, & \partial_y u_1 > 0
\end{cases}
$$

Remark: $\lim_{l \to 0} c_1^*(l) = -0.5 \quad \lim_{l \to 0} v^b(l) = -0.25$

$\lim_{l \to 0} c_2^*(l) = +0.5 \quad \lim_{l \to 0} v^f(l) = +0.25$

As $t \to \infty$ we observe:



Theorem (Efendiev, P., Tikhomirov, 2024, arXiv: 2401.05981)

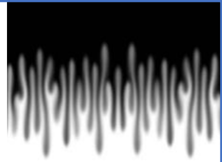
Consider a two-tube model with gravity $(*)$.

Then for all *$l > 0$ sufficiently small* there exists $c_1^*(l), c_2^*(l)$ such that there exist two traveling waves (TW):

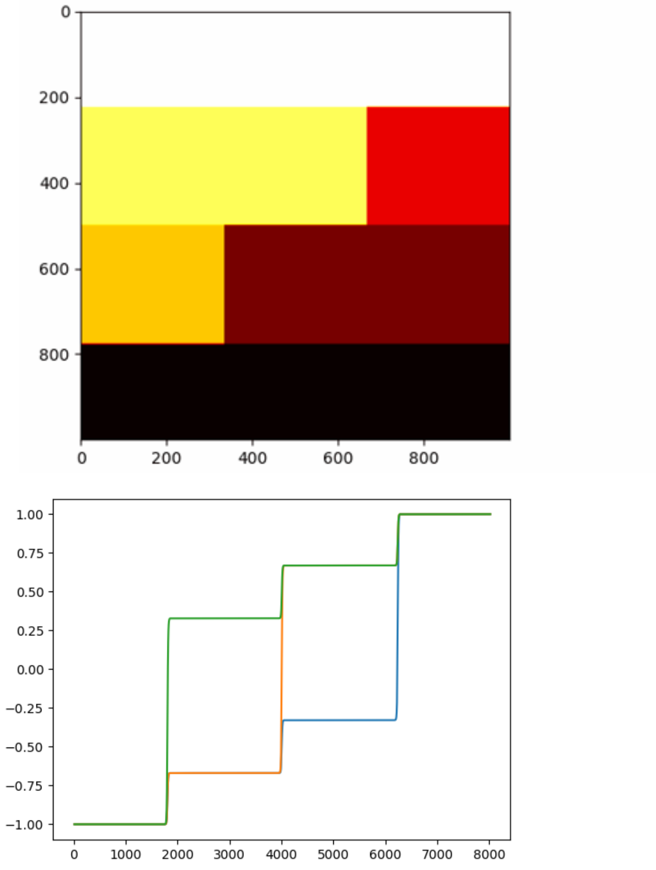
TW1 with speed $v^b(l)$: $(-1,-1) \to (c_1^*,(l)\ c_2^*(l))$

TW2 with speed $v^f(l)$: $(c_1^*,(l)\ c_2^*(l)) \to (1,1)$.

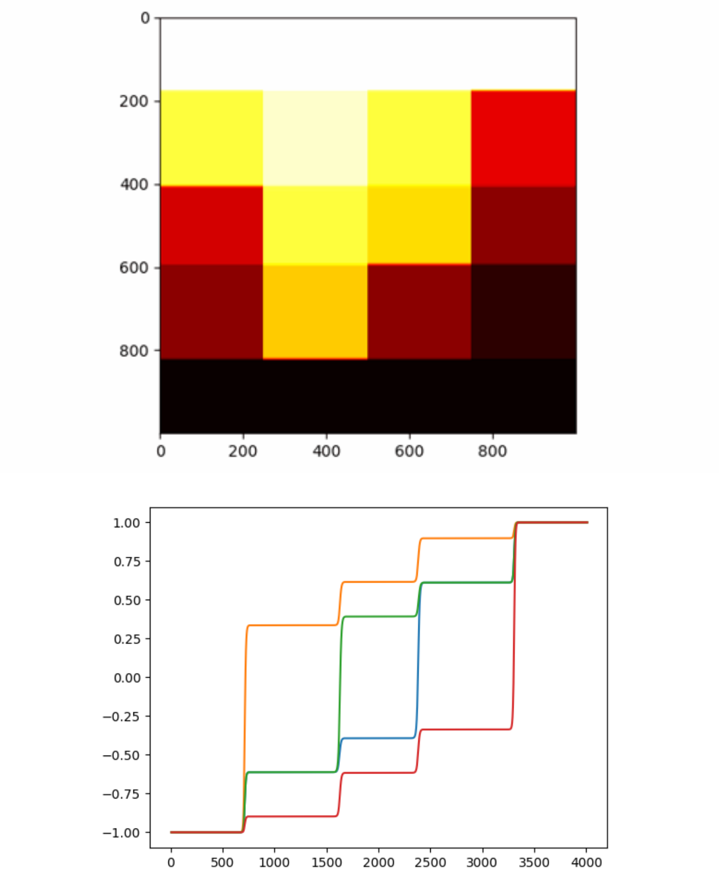
Many tubes: numerics



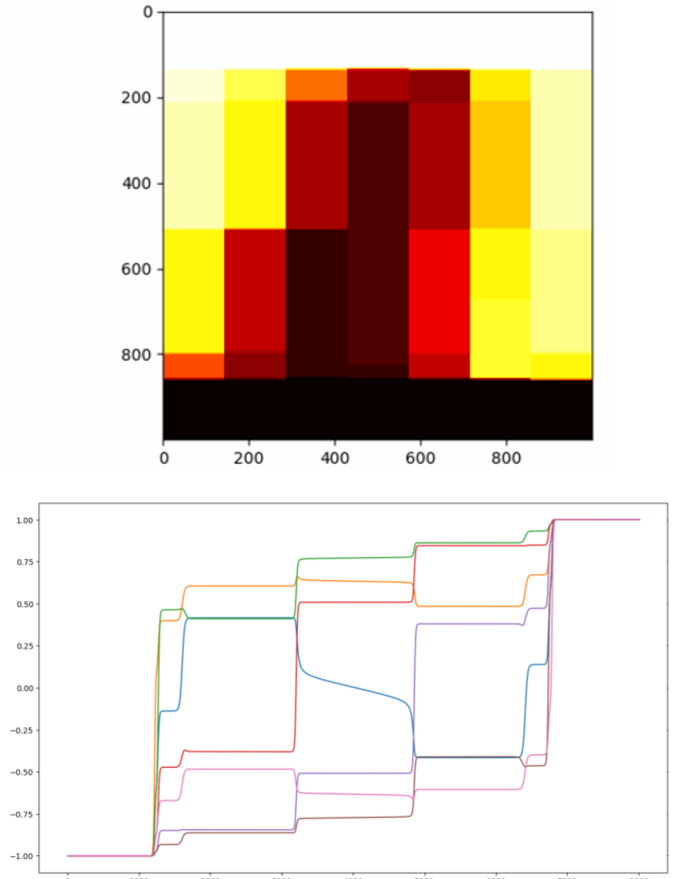
3 tubes



4 tubes



7 tubes



Questions: (open)

(1) explain the structure of “asymptotic solutions” for n tubes and study their stability

(2) find speed of growth of the mixing zone

(3) understand the behaviour as $n \to \infty$. Do we approximate 2-dim IPM?

(4) can we repeat this story for the many tubes viscous fingering model?

Scheme of proof: step 1

Travelling wave (TW) ansatz with fixed v:

$$c_1(t,y) = c_1(y - vt)$$
$$c_2(t,y) = c_2(y - vt)$$

$$u_1(t,y) = u_1(y - vt)$$
$$u_2(t,y) = u_2\,(y - vt)$$

$$p_1(t,y) = p_1(y - vt)$$
$$p_2(t,y) = p_2(y - ct)$$

With condition at $+\infty$:

$$c_1(+\infty) = 1$$
$$c_2(+\infty) = 1$$
$$u_1(+\infty) = 0$$
$$u_2(+\infty) = 0$$
$$(p_1 - p_2)(+\infty) = 0$$

System of ODEs in $\mathbb{R}^6$:

$$\boxed{\begin{cases} \dot{X} = F_v(X,Y) \\ l \cdot \dot{Y} = AY - BX \end{cases}}$$

Here:

- $X = \begin{pmatrix} c_1 \\ c_2 \\ \partial_\xi c_1 \\ \partial_\xi c_2 \end{pmatrix} \in \mathbb{R}^4, \quad Y = \begin{pmatrix} u_1 \\ p_1 - p_2 \end{pmatrix} \in \mathbb{R}^2$
- $A = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}, \qquad B \in M^{2\times 4}, \qquad l \ll 1$

Aim: $\forall v \quad \exists$ TW

find a heteroclinic orbit $\big(X(\xi), Y(\xi)\big), \xi \in \mathbb{R}$ such that $\big(X(+\infty), Y(+\infty)\big) =$ given point.

Scheme of proof: step 1

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$$p_1(t,y) = p_1(y - vt)$$
$$p_2(t,y) = p_2(y - ct)$$

With condition at $+\infty$:

$$c_1(+\infty) = 1$$
$$c_2(+\infty) = 1$$
$$u_1(+\infty) = 0$$
$$u_2(+\infty) = 0$$
$$(p_1 - p_2)(+\infty) = 0$$

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$$\boxed{\begin{cases} \dot{X} = F_v(X,Y) \\ l \cdot \dot{Y} = AY - BX \end{cases}}$$

Here:

- $X = \begin{pmatrix} c_1 \\ c_2 \\ \partial_\xi c_1 \\ \partial_\xi c_2 \end{pmatrix} \in \mathbb{R}^4, \quad Y = \begin{pmatrix} u_1 \\ p_1 - p_2 \end{pmatrix} \in \mathbb{R}^2$
- $A = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}, \qquad B \in M^{2\times 4}, \qquad l \ll 1$

Observation:
for $l \to 0$ this system has a special ``slow-fast'' structure.
Key tool: geometric singular perturbation theory (GSPT) by Fenichel (JDE, 1979)

Scheme of proof: step 2

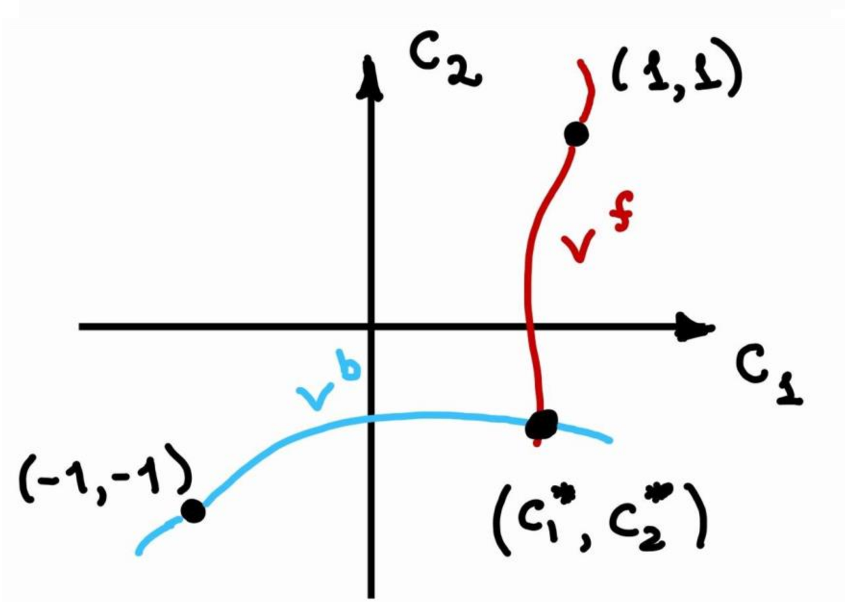
1) For each $v^f \in \mathbb{R}$ we find all points s.t. there exists a TW:

$$(c_1, c_2) \;\to\; (1,1)$$

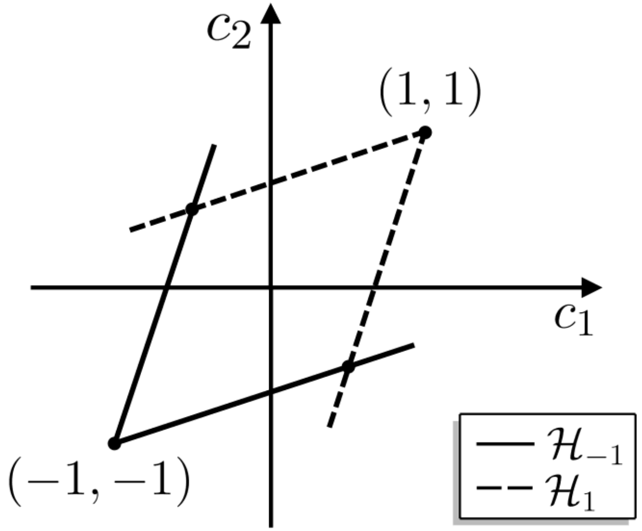
2) For each $v^b \in \mathbb{R}$ we find all points s.t. there exists a TW:

$$(-1,-1) \to (c_1, c_2)$$

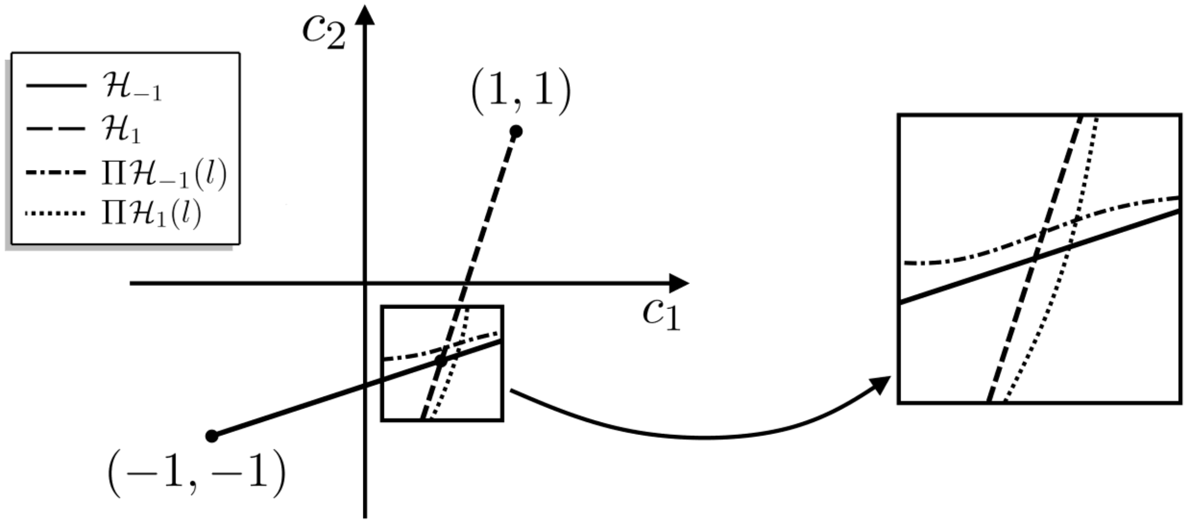
3) Cross fingers 🤞 that these two curves intersect



$l = 0$ – these curves are just straight lines



$0 < l \ll 1$– perturbation argument



But what is a singular limit $l = 0$?..

$$(**)\begin{cases} \partial_t c_1 + \partial_y(u_1 c_1) - \partial_{yy} c_1 = -B \\ \partial_t c_2 + \partial_y(u_2 c_2) - \partial_{yy} c_2 = B \\ \boxed{u_1 = \dfrac{c_2 + c_1}{2} - c_1 = \bar{c} - c_1} \\ B = \begin{cases} -\partial_y u_1 \cdot c_1, & \partial_y u_1 < 0, \\ +\partial_y u_2 \cdot c_2, & \partial_y u_1 > 0 \end{cases} \end{cases}$$

$l = 0$ corresponds to the two-tubes TFE equations (**) !!!

(at closer inspection, no surprise at all)

Question: (open)

TFE as a limit of IPM when $\frac{k_y}{k_x} \to \infty$?

Can we use the connection to prove the linear growth in 2D IPM?

2D IPM model

$$c_t + u \cdot \nabla c = \varepsilon \, \Delta c$$
$$div\, u = 0$$
$$u = -\begin{pmatrix} k_x & 0 \\ 0 & k_y \end{pmatrix} \nabla p - (0, c)$$

$$\xrightarrow{\frac{k_y}{k_x} \to \infty}$$

2D TFE model

$$c_t + u \cdot \nabla c = \varepsilon \, \Delta c$$
$$div\, u = 0$$
$$u = (u^x, u^y)$$
$$u^y = \bar{c} - c$$

Thanks to my collaborators!


Sergey Tikhomirov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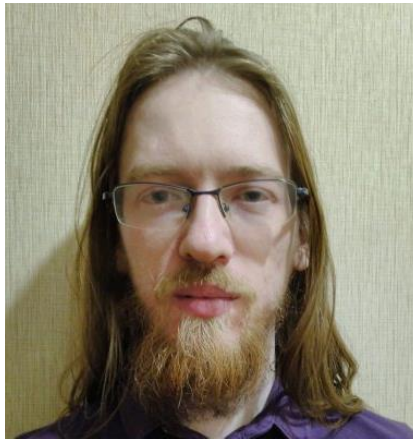
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Thank you for your attention!

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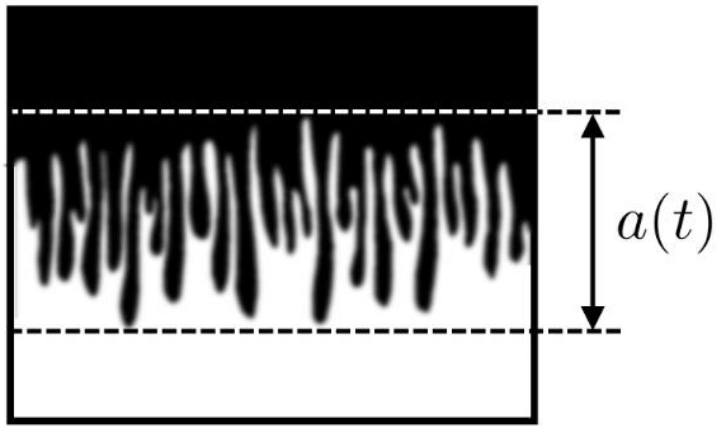

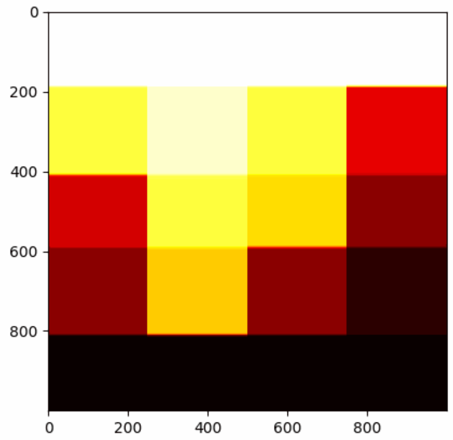

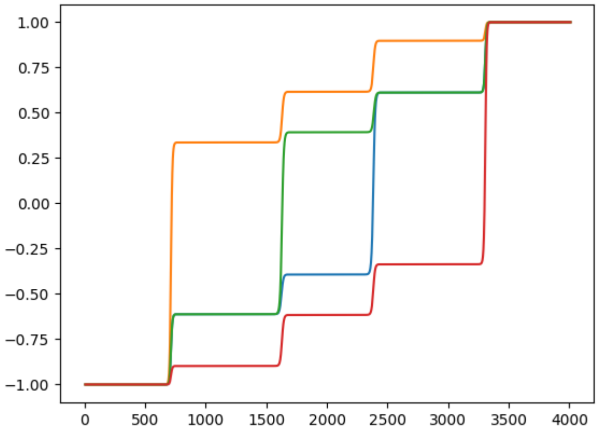

For more details see arXiv:2401.05981 (two-tubes model)
arXiv:2310.14260 (numerics of viscous fingering)
arXiv:2012.02849 (numerics of viscous fingering)

Any questions, comments and ideas are very welcome!

References Muito obrigada pela atenção!

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