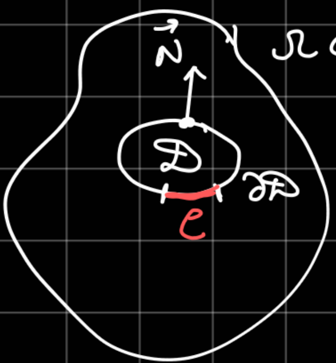
Lecture 5: Conservation & Balance laws

Plan:
1. General definition
2. Example 1: fluid dyn (conservation of mass)
3. Example 2: scalar conservation law

1 Balance law



$\mathcal{D} \subset \Omega$ with Lipshitz boundary (smooth)

$\vec{N}$ - normal vector towards the exterior of the domain $\mathcal{D}$

$$\boxed{\text{production in } \mathcal{D} = \text{flux through the boundary of } \partial\mathcal{D}}$$

- production in $\mathcal{D}$ is some measure (Radon) $\mathcal{P}$
- flux

$$\mathcal{F}_{\mathcal{D}}(e) = \int_{e} q_{\mathcal{D}}(x)\, dS(x)$$

$$\mathcal{P}(\mathcal{D}) = \int_{\partial\mathcal{D}} q_{\mathcal{D}}(x)\, dS(x) \qquad (*)$$

Assume:

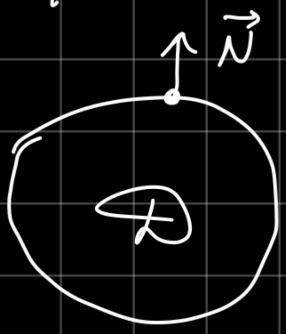
$$q_{\mathcal{D}_1}(x) = q_{\mathcal{D}_2}(x)$$
$$\forall x \in e$$

Take-home (Tuesday)

$$\boxed{\operatorname{div} A = 0}$$

Miracles :

Consequences of (*):



(1) $\exists \; a_{\vec{N}}(x) = q_{\mathcal{D}}(x)$ $\forall x \in \Omega$ for any $\mathcal{D} \subset \Omega$ s.t. $\mathcal{D}$ has $\vec{N}$ as a normal vector at x.

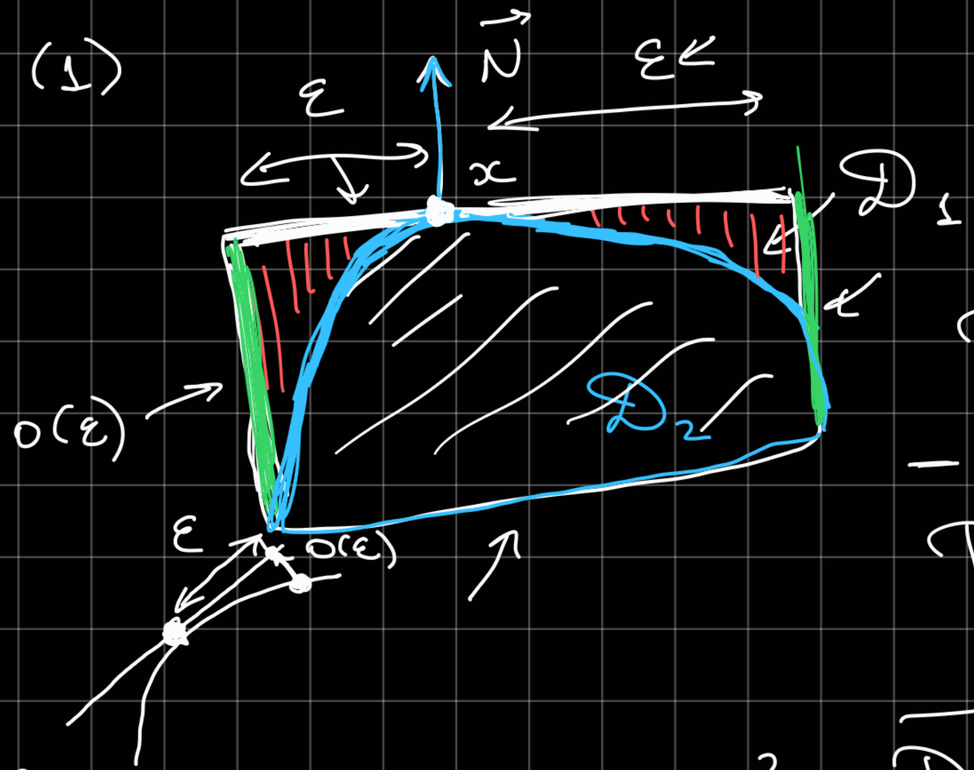
(2) $\exists \; \vec{A}(x) : \Omega \to \mathbb{R}^d$:

$$a_{\vec{N}}(x) = \vec{A}(x) \cdot \vec{N}$$

(3) $\exists$ PDE: $\operatorname{div} \vec{A} = P$

$$P(\mathcal{D}) = \int_{\partial \mathcal{D}} \underbrace{q_{\mathcal{D}}(x)}_{\vec{A}(x)\cdot\vec{N}} dS(x)$$

(1)



$\varepsilon \to 0$

$q_{\mathcal{D}_1}(x) \overset{?}{=} q_{\mathcal{D}_2}(x)$

(*)

$P(\mathcal{D}_1) = \int q_{\mathcal{D}_1}(x) dS$

$-$

$P(\mathcal{D}_2) = \int q_{\mathcal{D}_2}(x) dS$

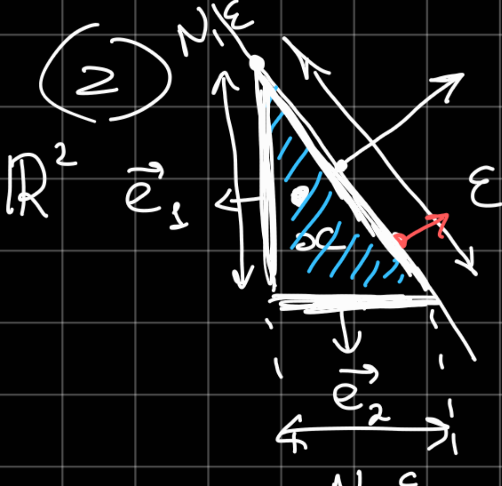
$\varepsilon^2 \sim P(\;) = \int q_{\mathcal{D}_1}(x) dS - \int q_{\mathcal{D}_2}(x) dS$

$\int \sim o(\varepsilon)$

$\Rightarrow \int q_{\mathcal{D}_1}(x) dS(x) = \int q_{\mathcal{D}_2}(x) dS(x)$

$$\Rightarrow \quad \exists \; a_{\vec{N}}(x) = q_{\mathcal{D}}(x)$$ Cauchy

(2)



$\vec{N} = (N_1, N_2)$

$$\varepsilon^2 \sim P(\Delta) = \int_{\Delta} a_{\vec{N}}(x)\,dx$$

$$a_{\vec{N}}(x) \cdot \cancel{\varepsilon} = a_{e_2}(x) \cdot N_2 \cancel{\varepsilon} + a_{e_1}(x) \cdot N_1 \cancel{\varepsilon}$$

$$\Rightarrow \quad a_{\vec{N}}(x) = a_{e_1}(x) N_1 + a_{e_2}(x) N_2$$

$$\Rightarrow \quad a_N(x) = \vec{A}(x) \cdot \vec{N}$$

$$A(x) = (a_{e_1}, a_{e_2})$$

(3)

$$\int_{\partial\mathcal{D}} \vec{A}(x) \cdot \vec{N} \, dS(x) = \int_{\mathcal{D}} \operatorname{div}(\vec{A})\,dx$$

Green-Gauss theorem

$$P = \int_{\mathcal{D}} p(x)\,dx$$

$$\Rightarrow \quad \operatorname{div}(\vec{A}) = p$$ – balance law

$$\boxed{\operatorname{div}(\vec{A}) = 0}$$ – conservation law

Dafermos

Example 1: Fluid flow, continuum mechanics

different scales

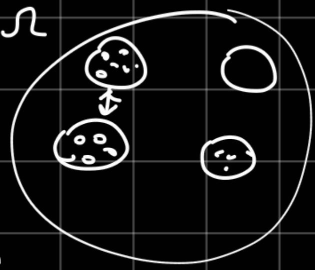
1. atoms / molecules

   o

2. representative small volume

3. domain (macroscale)

Ω



- Eulerian vs. Lagrangian point of view

Eulerian: (x,t) - fix

- velocity: $u(x,t) = (u_1, \dots, u_d) : \mathbb{R}^d \times [0,\infty) \to \mathbb{R}^d$ has units $\left[\frac{L}{T}\right]$
- density: $\rho(x,t) : \mathbb{R}^d \times [0,+\infty) \to \mathbb{R}$ with units $\left[\frac{M}{L^d}\right]$
- pressure: $p(x,t) : \mathbb{R}^d \times [0,+\infty) \to \mathbb{R}$ with units $\left[M L^{-d+2} T^{-2}\right]$

Lagrangian: particles, $a \in \mathbb{R}^d$

trajectories of particles

flow map $X(t,a) = (X_1, \dots, X_d)$ - position of particle a at time t

$$(**) \begin{cases} \partial_t X(t,a) = u(t, X(t,a)) & \leftarrow \text{ODE} \\ X(0,a) = a \end{cases}$$

ODE theory (Cauchy-Lipschitz theorem):

$u \in C_t \mathrm{Lip}_x \Rightarrow \exists!$ solution to $(**)$

$X(t,\cdot)$ – is C^1-diffeo $: \mathbb{R}^d \to \mathbb{R}^d$

Define inverse: $A(t,x) : [0,+\infty) \times \mathbb{R}^d \to \mathbb{R}^d$

$$A(t, X(t,a)) = a \qquad X(t, A(t,x)) = x$$

$$\forall x, a \in \mathbb{R}^d$$

"back-to-labels" map (a – "labels")

Incompressibility condition: "$\operatorname{div} u = 0$"

Take $V \subset \Omega$ – volume of fluid

$$V(t) = X(t, V) = \{ X(t,a) : a \in V \}$$

$$V \xrightarrow{t} V(t)$$

Def: velocity field is called incompressible if

$$\rightarrow \quad |V(t)| \equiv |V|$$

$\uparrow$ Lebesgue measure of V

Lemma: $u \in C_t \operatorname{Lip}_x$

u is incompressible $\Longleftrightarrow$ $\operatorname{div} u = 0$ (u is divergence-free)

Proof:

$$V(t) = \int_{V(t)} 1 \cdot dx \quad ; \quad a \in V \subset \mathbb{R}^d$$

$$\int_{V(t)} f(x,t)\,dx = \int_V f(X(t,a), t) \cdot \boxed{\det(\nabla_a X)}(t,a)\,da$$

$a \mapsto X(t,a)$ $\qquad \det(\nabla_a X)(t,a) = J(t,a)$

$$\rightarrow \quad J(t,a) = \sum_{i_1,\dots,i_d=1}^{d} \varepsilon_{i_1 \dots i_d} \frac{\partial X_{i_1}}{\partial a_1} \cdot \ldots \cdot \frac{\partial X_{i_d}}{\partial a_d}$$

Exercise: $\partial_t J(t,a) = J(t,a) \cdot (\operatorname{div} u)(t, X(t,a))$

Corollary: $J(t,a) \equiv 1 \iff \operatorname{div}(u) = 0$

$$J(t,a) = J(0,a) \cdot e^{\int_0^t (\operatorname{div} u)(s, X(s,a))\,ds}$$

$$J(0,a) = 1$$

$$J(t,a) = J(0,a) \quad \forall t \Rightarrow \operatorname{div}(u) = 0$$

$$V(t) = \int_{V(t)} 1\,dx = \int_V J(t,a)\,da = \int_V da = V$$

iff $\operatorname{div}(u) = 0$

$\blacksquare$

Transport equation

Let's $f(t,x) : [0,+\infty) \times \mathbb{R}^d \to \mathbb{R}$ - scalar

Eulerian: $\partial_t f$ - change of f at (t,x)

Lagrangian: $\partial_t f(t, X(t,a)) =: D_t f$ - convective derivative

$$= \partial_t f + u \cdot \nabla f$$

Thm (transport thm): $V \xrightarrow{t} V(t)$

u - velocity field, $u \in C^1$; f be C^1

$V(t)$ is pushforward of V by the flow map $X(t,a)$

$$\frac{d}{dt}\left(\int_{V(t)} f(x,t)\,dx\right) = \int_{V(t)} (\partial_t f + \operatorname{div}(fu))(t,x)\,dx$$

Proof:

$$\int_{V(t)} f(x,t)\,dx = \int_V f(X(t,a), t)\,\underline{J(t,a)}\,da$$

$$\frac{d}{dt}\left(\int_{V(t)} f(x,t)\,dx\right) = \int_V \underline{(D_t f)}\,(X(t,a),t)\,J(t,a)\,da$$

$$+ \int_V f(X(t,a),t)\cdot \underbrace{\partial_t J(t,a)}_{J(t,a)\cdot \operatorname{div}(u)}\,da =$$

$$= \int_V \Big(\partial_t f + \underbrace{u\cdot\nabla f + f\cdot \operatorname{div}(u)}_{\operatorname{div}(uf)}\Big)(X(t,a),t)\cdot J(t,a)\,da$$

$$= \int_V \Big(\partial_t f + \operatorname{div}(fu)\Big)(X(t,a),t)\cdot J(t,a)\,da =$$

$$= \int_{V(t)} \Big(\partial_t f + \operatorname{div}(fu)\Big)\,dx \qquad \blacksquare$$

Conservation of mass : $\rho(x,t)$

$$m(t,V) = \int_V \rho(x,t)\,dx$$

$$\frac{d}{dt}\, m(t,V(t)) = 0$$

<u>Thm</u> : conservation of mass is equivalent to the following integral eq:

$$\int_{V(t)} (\rho_t + \operatorname{div}(\rho u))\,dx = 0$$

If ρ_t and $\operatorname{div}(\rho u)$ are C, then

$$\boxed{\rho_t + \operatorname{div}(\rho u) = 0}$$

<u>Rmk</u> : scalar transport eq

Proof: $\ulcorner$ $0 = \frac{d}{dt} m(t, X(t,a)) = \frac{d}{dt} \int_{V(t)} \rho(x,t)\,dx$

$$= \int_{V(t)} \underbrace{\left(\rho_t + \operatorname{div}(\rho u)\right)}_{\text{are continuous}} dx.$$

$$f(x) = \lim_{\varepsilon \to 0} \frac{1}{|B_\varepsilon(x)|} \int_{B_\varepsilon(x)} f(y)\,dy$$

$\llcorner$ $\Rightarrow$ $\rho_t + \operatorname{div}(\rho u) = 0.$ ∎

Rmk: $0 = \rho_t + \operatorname{div}(\rho u) = \rho_t + u \cdot \nabla \rho + \operatorname{div}(u)\rho$

incompressibility $\Rightarrow$ $\operatorname{div}(u) = 0$

$$D_t \rho = \rho_t + u \cdot \nabla \rho = 0$$

Next time

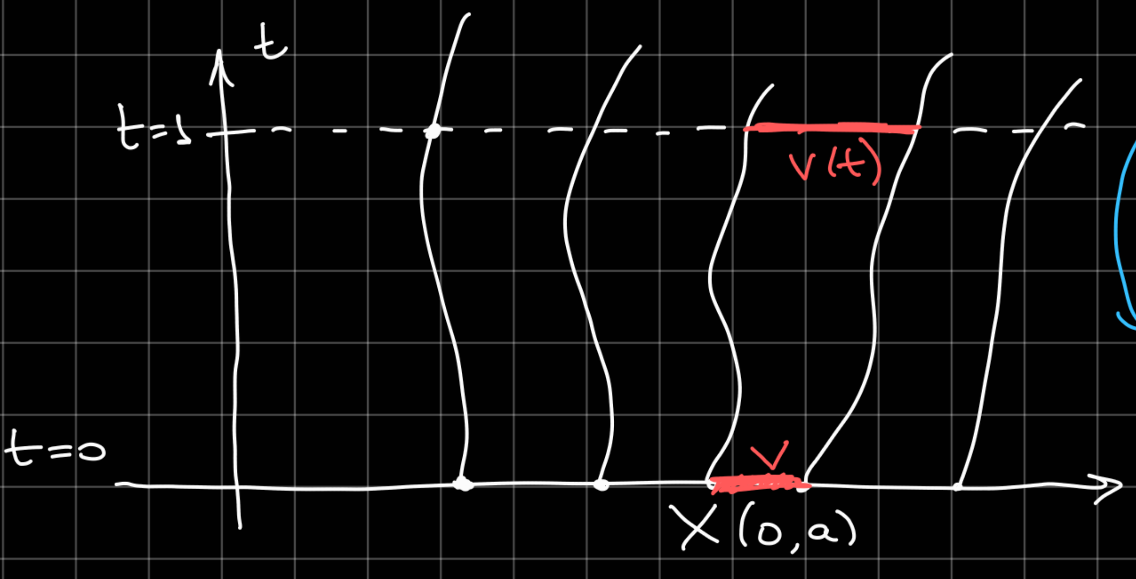
$\rho_t + (\rho \underline{\underline{u}})_x = 0$

$u = u(\rho)$

$u(\rho) = \frac{\rho}{2} \Rightarrow$ Burgers

$$\frac{d}{dt}\left(\rho(t, X(t,a))\right) = 0$$

$\Rightarrow$ $\rho(t, X(t,a)) = \text{const}.$



$u \in C_t \underline{\mathrm{Lip}_x}$
linear in ρ

$\rho_t + \underline{u} \cdot \nabla \rho = 0$

Lecture 6

Last time: Balance laws: $\operatorname{div} A = P$

Conservation laws: $\operatorname{div} A = 0$

Example 1: fluid flow: $a \to X(t,a)$ – flow map under velocity field

$u(x,t)$ $\quad a \in \mathbb{R}^d$

x

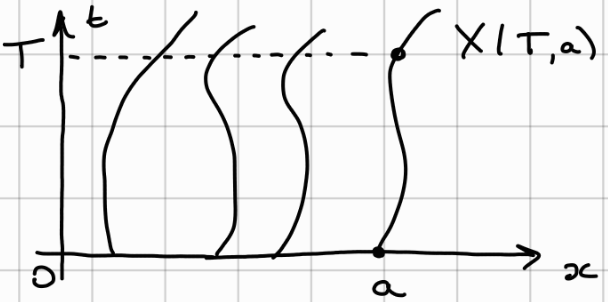
$u(t,x)$ – velocity field

$\rho(t,x)$ – density

$$\begin{cases} \partial_t X = u(X,t) \\ X(0,a) = a \end{cases}$$

• Conservation of mass = scalar transport equation

$$\boxed{\partial_t \rho + \operatorname{div}(\rho u) = 0}$$

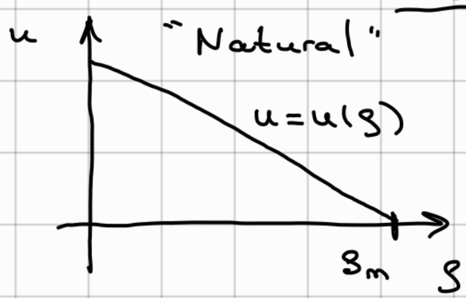


trajectories do not intersect for given $u \in C^0_t \mathrm{Lip}_x$

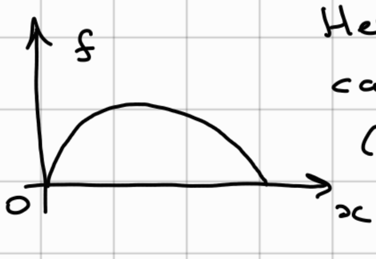
Rmk: $\begin{cases} \operatorname{div} u = 0 \\ \partial_t \rho + \operatorname{div}(\rho u) = 0 \end{cases}$

$\Rightarrow$ $\rho(t, X(t,a)) = \text{const}$ density is conserved along the trajectory for incompressible flow

Example 2: traffic flow: cars choose their velocity depending on "density" of cars nearby

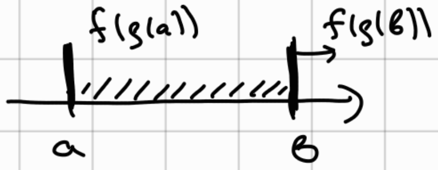


$\Rightarrow$ $\boxed{\rho_t + f(\rho)_x = 0}$ scalar conservation law



Here $f(\rho)$ is called flux (flow function)

ρ_m – density of cars corresp. to "bumper-to-bumber"



$$\frac{d}{dt}\int_a^b \rho(x,t)\,dx = f(\rho(b)) - f(\rho(a))$$

Rmk: 1) taking $u(\rho) = \frac{\rho}{2}$ $\Rightarrow$ Burgers eq: $\rho_t + \left(\frac{\rho^2}{2}\right)_x = 0$

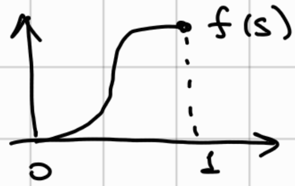
We will analyze it in detail today.

2) for oil recovery the simplest 1-dim model for displacement water–oil is again

$s_t + (f(s))_x = 0$ for

s - water saturation

$f(s)$ - fractional flow function



$f(s)$: $f(0)=0$, $f(1)=1$, $f\uparrow$ and S-shaped

- One can easily create more sophisticated models such as: take drivers anticipation into account

If a driver observe an upstream increase in the density, they show a tendency to brake slightly

$$u - v(\rho) \sim -\rho_x$$

The simplest law: $u = v(\rho) - \varepsilon \rho_x$, $0<\varepsilon \ll 1$

which leads to the "weakly" parabolic eq:

$$\rho_t + f(\rho)_x = \varepsilon(\rho\rho_x)_x$$

Example 3: wave equation ! $u_{tt} - c^2 u_{xx} = 0$

$\operatorname{div}(u_t, -c^2 u_x) = 0$

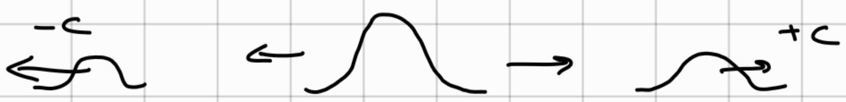
Consider $U = \begin{pmatrix} u_x \\ u_t \end{pmatrix} \Rightarrow U_t + AU_x = 0$

$A = \begin{pmatrix} 0 & -1 \\ -c^2 & 0 \end{pmatrix}$

Indeed, this is just: $\begin{cases} u_{xt} - u_{tx} = 0 \\ u_{tt} - c^2 u_{xx} = 0 \end{cases}$

Eigenvalues of A: $\det\begin{vmatrix} 0-\lambda & -1 \\ -c^2 & -\lambda \end{vmatrix} = \lambda^2 - c^2$, $\lambda_\pm = \pm c$

They correspond to propagation modes:



This is general fact that we will see in the future:

$U \in \mathbb{R}^d$, $F: \mathbb{R}^d \to \mathbb{R}^d$ $\quad U_t + (F(U))_x = 0$ - system of conservation laws

Then for "smooth" solutions we have:

$$U_t + \underbrace{F'(U)}\cdot U_x = 0$$

eigenvalues of this matrix play an important role !

If they are real, they correspond to velocity of propagation of waves.

Example 4: isentropic (= constant entropy) gas dynamics (p-system)

in Lagrangian coordinates:
$$\begin{cases} v_t - u_x = 0 \\ u_t + p(v)_x = 0 \end{cases} \Rightarrow v_{tt} + p(v)_{xx} = 0$$

Rmk: $v_t = u_x \Rightarrow$ (in a simply connected regions) $\exists\ \varphi$: $v = \varphi_x$, $u = \varphi_t$

$$\Rightarrow \quad \varphi_{tt} + (p(\varphi_x))_x = 0$$
$$\varphi_{tt} + p'(\varphi_x)\cdot\varphi_{xx} = 0$$ – nonlinear wave equation

And many other examples:

- conservation of mass, conservation of momentum $\Big\} \Rightarrow \begin{cases} \partial_t u + u\cdot\nabla u = \nabla p + f \\ \operatorname{div}(u) = 0 \end{cases}$

  This is Euler equations for ideal fluid (1755, second PDE ?)
- Navier-Stokes eqs (1845): adds viscosity
$$\partial_t u + (u\cdot\nabla)u - \nu\Delta u = \nabla p$$
- gas dynamics
- electromagnetism (Maxwell eqs)
- magneto-hydrodynamics (M.H.D.) – motion of fluid in the presence of electromagnetic field (think of a Sun)

Etc ......

Burgers equation

$$u_t + \left(\frac{u^2}{2}\right)_x = 0 \qquad\qquad u_t + u\cdot u_x = 0$$

Observation 1: if $u \in C^1$ for all $t>0$, then u is monotonically nondecreasing in x for all $t>0$.

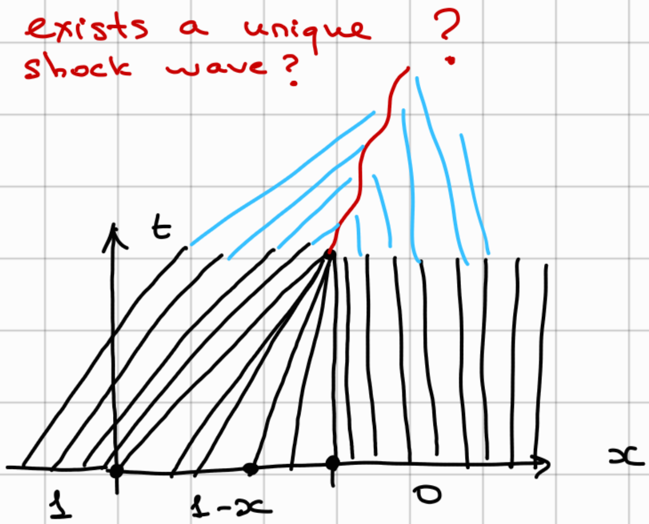
$$u\,(x(s),\ t(s)) = \text{const}$$

$$u_t \cdot t_s + u_x \cdot x_s = 0$$

$$\begin{cases} t_s = 1 \\ x_s = u \end{cases} \Rightarrow \begin{array}{l} u = \text{const on} \\ \text{straigh lines} \\ x = x_0 + u_0(x_0)\,t \end{array}$$

If $u \in C^1$ for $\forall\, t>0$, then characteristics should not intersect $\Rightarrow$ $u_0(x_1) < u_0(x_2)$ if $x_1 < x_2$ $\Rightarrow$ u_0 is non-decreasing ($u(x,0)$) $\Rightarrow$ $u(x,t)$ is non-decreasing in x

Exercise 2 from list 1:



$$x = x_0 + (1-x_0)\,t$$

$t=1$: $\quad x = 1$

At $t=1$ there is a blow-up

Rmk: In general scalar conservation law:

$$u_t + f(u)_x = 0$$

$$u_t + \underline{\underline{f'(u)}} \cdot u_x = 0$$

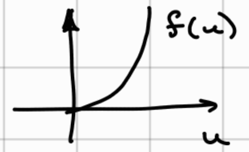
Characteristics are $x = x_0 + f'(u_0(x_0))\,t$

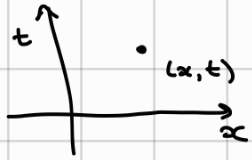
$u \in C^1\ \forall t>0 \Rightarrow f'(u_0(x_1)) < f'(u_0(x_2))$ if $x_1 < x_2$, otherwise characteristics will intersect that leads to a blow-up!

So no matter how smooth f and u_0 are, the solution $u(x,t)$ must become discontinuous

This is a purely non-linear phenomenon!!!

- Assume $f \in C^2$ and $f'' > 0$





$$u_0\big(x - t f'(u(x,t))\big) = u(x,t)$$

$$u_t = u_0' \cdot \big(- f'(u(x,t)) - t f''(u(x,t)) \cdot u_t\big)$$

$$u_t\,(1 + t f'' u_0') = - u_0' f'$$

$$u_t = - \frac{u_0' f'}{1 + t f'' u_0'}$$

Analogously, $\displaystyle u_x = \frac{u_0'}{1 + t f'' u_0'}$

If $u_0' \geq 0$ (and $f'' > 0$) u_t and u_x stay bounded.

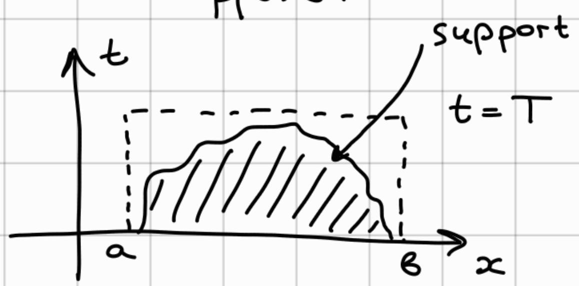
If $u_0' < 0$, then u_t and u_x become unbounded as $1 + t f'' u_0'$ tends to 0.

So we need a notion of <u>weak</u> solution!

Weak solutions to conservation laws

$$\begin{cases} u_t + f(u)_x = 0 & (1) \\ u\big|_{t=0} = u_0(x) \end{cases}$$

Let u be a classical solution and $\varphi \in C^1$ with compact support:



$\mathrm{supp}(\varphi) \subset D = [a,b] \times [0,T]$

that is φ is zero at $x = a$, $x = b$, $t = T$

Multiply (1) by φ and integrate over $\mathbb{R} \times \mathbb{R}_+$

$$\iint_{t>0} (u_t + f(u)_x)\,\varphi \,dx\,dt = \iint_D (u_t + f(u)_x)\,\varphi \,dx\,dt =$$

$$= \int_a^b\!\!\int_0^T (u_t + f(u)_x)\,\varphi\,dx\,dt = \int_a^b u\cdot\varphi\Big|_0^{\cancel{T}\,\to\,0} dx - \int_a^b\!\!\int_0^T u\cdot\varphi_t\,dx$$

$$+ \int_0^T f(u)\cdot\varphi\Big|_a^b{}^{\to\,0}\,dt - \int_0^T\!\!\int_a^b f(u)\cdot\varphi_x\,dx\,dt =$$

$$= -\int_a^b u_0(x)\,\varphi(x)\,dx \;-\; \int_a^b\int_0^T \left(u\varphi_t + f(u)\,\varphi_x\right) dx\,dt$$

$$\Rightarrow \quad \iint_{t\geq 0} \left(u\,\varphi_t + f(u)\,\varphi_x\right) dx\,dt \;+\; \int_{t=0} u_0(x)\,\varphi(x)\,dx = 0 \qquad (2)$$

$u \in C^1$ and satisfies (1) $\Rightarrow$ u satisfies (2)

But in (2) u not necessarily needs to be C^1!

It can be measurable / bounded.

Definition: A bounded measurable function $u(x,t)$ is called a weak solution of IVP:

$$u_t + f(u)_x = 0\,, \quad u\big|_{t=0} = u_0(x)$$

$\uparrow$ bounded / meas.

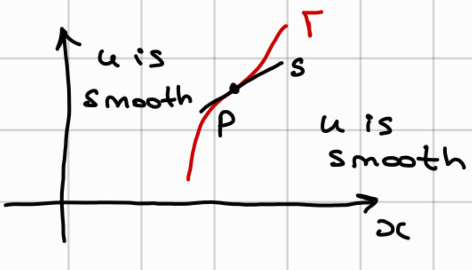
provided that

$$(2) \qquad \iint_{t\geq 0} \left(u\varphi_t + f(u)\varphi_x\right) dx\,dt + \int_{t=0} u_0\varphi\,dx = 0$$

for all $\varphi \in C_0^1$ (φ is C^1 with compact supp)

Rmk: it is clear that if u is in fact C^1, then the original eq. is true: $u_t + f(u)_x = 0$

Lemma (Rankine-Hugoniot condition)



Let Γ be a smooth curve across which u has a jump discontinuity. Take $P \in \Gamma$ and

$u_\ell = \lim_{(x,t)\to P} u$ from "the left"

$u_r = \lim_{(x,t)\to P} u$ from "the right"

Let the tangent line of Γ at P have the slope

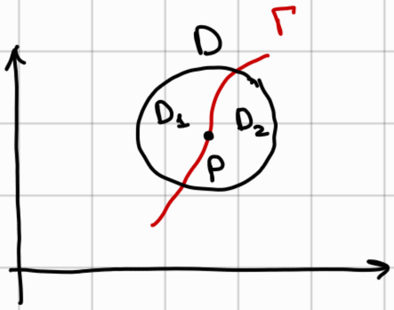
$s = \dfrac{dx}{dt}$. Then: (3) $\boxed{s\cdot(u_\ell - u_r) = f(u_\ell) - f(u_r)}$

Often a jump across the shock is denoted:

$[g(u)] = g(u_\ell) - g(u_r)$, thus we have $s\,[u] = [f]$

This is called the Rankine-Hugoniot condition

Proof :



Let D be a small ball centered at P and let Γ devide D into two regions D_1 and D_2 (see fig)

Let $\varphi \in C_0^1$ on D and consider

$$0 = \iint_D (u\varphi_t + f(u)\varphi_x)\,dx\,dt = \iint_{D_1} + \iint_{D_2}$$

Divergence theorem : (Green-Gauss theorem)

$$\int_{\partial\Omega} P\,dx + Q\,dy = \iint_{\Omega}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dx\,dy$$

$$\iint_{D_1} u\varphi_t + f(u)\varphi_x\,dx\,dt = \iint_{D_1} (u\varphi)_t + (f(u)\varphi)_x\,dx\,dt =$$

as $u \in C^1(D_1)$ and $u_t + f(u)_x = 0$

$$= \int f(u)\varphi\,dt - u\varphi\,dx = \int_{t_1}^{t_2} f(u)\varphi\,dt - u\varphi\,dx =$$

$$= \int_{t_1}^{t_2} \left[f(u_e)\varphi(u_e) - u_e\cdot\varphi(u_e)\cdot s\right] dt$$

Similarly, $\iint_{D_1} u\varphi_t + f(u)\varphi_x\,dx\,dt = -\int_{t_1}^{t_2}\left(f(u_r) - su_r\right)\varphi(u_r)\,dt$

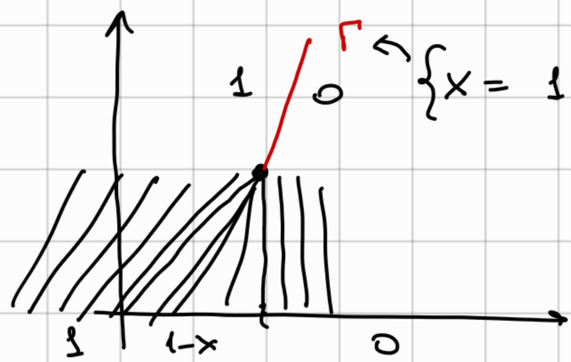
minus because of orientation of ∂D_2 :

Combining together:

$$0 = \int_{t_1}^{t_2}\left([f] - s[u]\right)\varphi(u_e)\,dt$$

Since φ was arbitrary, we get $[f] - s[u] = 0$ relation (3). ■

Example: Burgers eq:



$$s = \frac{\left[\frac{u^2}{2}\right]\Big|_0^1}{[u]\Big|_0^1} = \frac{1}{2}$$

u_e-left state

u_r-right state

in general $s = \frac{u_e + u_r}{2}$

Lecture 7

Scalar conservation law: $\begin{cases} u_t + (f(u))_x = 0 \\ u\big|_{t=0} = u_0(x) \end{cases}$ (*)

$u: \mathbb{R}\times\mathbb{R}_+ \to \mathbb{R}$ - bounded, measurable

$f: \mathbb{R}\to\mathbb{R}$, $f\in C^2$, $f''>0$ on the convex hull of values of u_0

We understand solutions in weak sense:

$$\iint_{t>0} (u\varphi_t + f(u)\varphi_x)\,dx\,dt + \int_{t=0} u_0\varphi\,dx = 0 \qquad (**)$$

for every test function $\varphi\in C_0^1$.

We want to prove theorems on $\exists$, $!$ and asymptotic behavior of solutions to (*). From exercise session 1 we remember that we need some extra conditions for this

$\boxed{\text{Thm 1}}$ ($\exists$):

Let $u_0\in L_\infty(\mathbb{R})$, $f\in C^2(\mathbb{R})$, $f''>0$ on $\{u: |u|\le \|u_0\|_\infty\}$

Then there exists a solution with the following properties:

(a) $|u(x,t)| \le \|u_0\|_\infty = M$, $(x,t)\in\mathbb{R}\times\mathbb{R}_+$

(b) $\exists\, E>0$ (which depends on M, $\mu=\min\{f''(u): |u|\le M\}$ and $A=\max\{|f'(x)| : |u|\le\|u_0\|\}$) ~~such~~

such that $\forall\, a>0, t>0, x\in\mathbb{R}$

$$\frac{u(x+a, t) - u(x,t)}{a} < \frac{E}{t} \qquad (E)$$

(c) u is stable and depends continuously on u_0: if $v_0\in L_\infty(\mathbb{R})$ with $\|v_0\|_\infty \le \|u_0\|_\infty$ and v is the corresponding constructed solution of (*) with initial data v_0, then for $\forall\, x_1, x_2\in\mathbb{R}$ with $x_1<x_2$ and $\forall\, t>0$

$$\int_{x_1}^{x_2} |u(x,t) - v(x,t)|\,dx \le \int_{x_1-At}^{x_2+At} |u_0(x) - v_0(x)|\,dx.$$

~~Remarks: 1)~~

$\boxed{\text{Thm 2}}$ ($!$):

Let $f\in C^2$, $f''>0$ and let u and v be 2 solutions of (*) satisfying condition (E). Then $u=v$ almost everywhere in $t>0$.

Rmk: we call the solution from Thm 1 (that is satisf. (E) may be there exist more solutions which do not satisfy cond. (E) or (c)

2) property (a) is not valid for systems! Sup-norm of solution can increase! It is non-trivial to prove the bounds on the sup-norm.

3) Cond. **(E)** implies some regularity : u is of locally bounded total variation (for $\forall t$ as a function of x)

Indeed, let c_1 be a constant such that $c_1 > \frac{E}{t}$ and let $v = u - c_1 x$. Then

$$v(x+a,t) - v(x,t) = u(x+a,t) - u(x,t) - c_1 a < a\left(\frac{E}{t} - c_1\right) < 0$$

Thus, v is a non-decreasing function, and v is a function of local bounded total variation.

Since $c_1 x$ is also of bounded total variation, then u is of local bounded total variation.

($\Rightarrow$ countable number of jump discontinuities)

4) finite speed of propagation:

$$v = v_0 \equiv 0 \quad \Rightarrow \quad \int_{x_1}^{x_2} |u(x,t)|\, dx \le \int_{x_1 - At}^{x_2 + At} |u_0(x)|\, dx$$

Before proving thm 1 and 2 let us understand better condition (E). Let us give some equivalent formulations and interpetations.

Lemma: ⓐ A smooth solution $u(x,t)$ $(\forall t)$ satisfies condition (E)

ⓑ If u has a discontinuity at point x_0 : (but is smooth to the left and to the right of x_0) and $\lim\limits_{x \to x_0 - 0} u(x,t) = u_L$ and $\lim\limits_{x \to x_0 + 0} u(x,t) = u_R$

satisfies condition (E) $\Rightarrow u_L > u_R$.

(discontinuities can be only down).

Proof:

ⓐ Indeed, let us write:

$$u(x,t) = u_0(\,x - t f'(u(x,t)))$$

$$u_x = u_0' \cdot (1 - t f'' \cdot u_x) \Rightarrow u_x = \frac{u_0'}{1 + t f'' u_0'}.$$

If u is smooth for $\forall t > 0$, then $u_0' > 0$.

Then $u_x \le \frac{u_0'}{t f'' u_0'} = \frac{E}{t}$ for $E = \frac{1}{\inf f''}$.

Using Lagrange theorem : $\dfrac{u(x+a,t)-u(x,t)}{a} = u_x(\xi,t)$

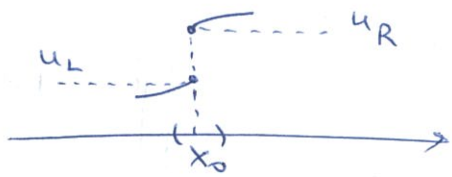
for some $\xi \in [x, x+a)$, and (a) is proved

(b) Either $u_L > u_R$ or $u_L < u_R$ (the case $u_L = u_R$ is not a discontinuity).

- For $u_L < u_R$ the converse of cond. (E) is true:

$\forall E > 0 \;\; \exists x, a > 0, t :$ $\dfrac{u(x+a)-u(x)}{a} > \dfrac{E}{t}$.

Indeed, fix E and take small enough neighbourhood of x_0 such that



- for $x \in (x_0-\delta, x_0)$ $|u-u_L| \leq \varepsilon = \dfrac{u_R-u_L}{4}$
- for $x \in (x_0, x_0+\delta)$ $|u-u_R| \leq \varepsilon = \dfrac{u_R-u_L}{4}$.

This means that for $\forall x_1 \in (x_0-\delta, x_0)$ and $x_2 \in (x_0, x_0+\delta)$

$$u(x_2)-u(x_1) \geq \frac{u_R-u_L}{2}.$$

Fix t and take small a :

$x_2 - x_1 = a$

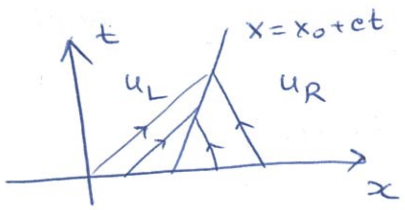
$x_1 \in (x_0-\delta, x_0)$

$x_2 \in (x_0, x_0+\delta)$

$$\frac{u(x_2)-u(x_1)}{a} \geq \frac{u_R-u_L}{2a} = \frac{E}{t}.$$

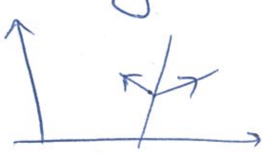
- For $u_L > u_R$ $\dfrac{u(x+a)-u(x)}{a} \leq 0$, thus $\forall E > 0$ is ok ∎

<u>Lemma 2</u> (Remark) : u satisfies condition (E) and is a shock wave solution $u = \begin{cases} u_L, & x < ct \\ u_R, & x > ct \end{cases}$ then $f'(u_L) > c > f'(u_R)$ (<u>Lax condition</u>)



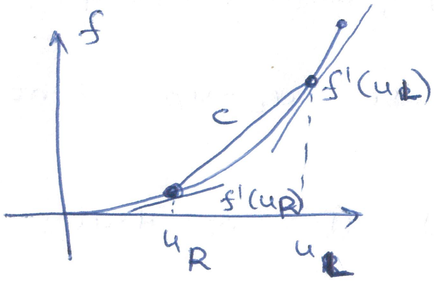
" Characteristics come to the line of discontinuity "

The converse situation corresponds to the case



in some sense where " the information" appears on the discontinuity - which is not

We will generalize the Lax condition to the case of systems.

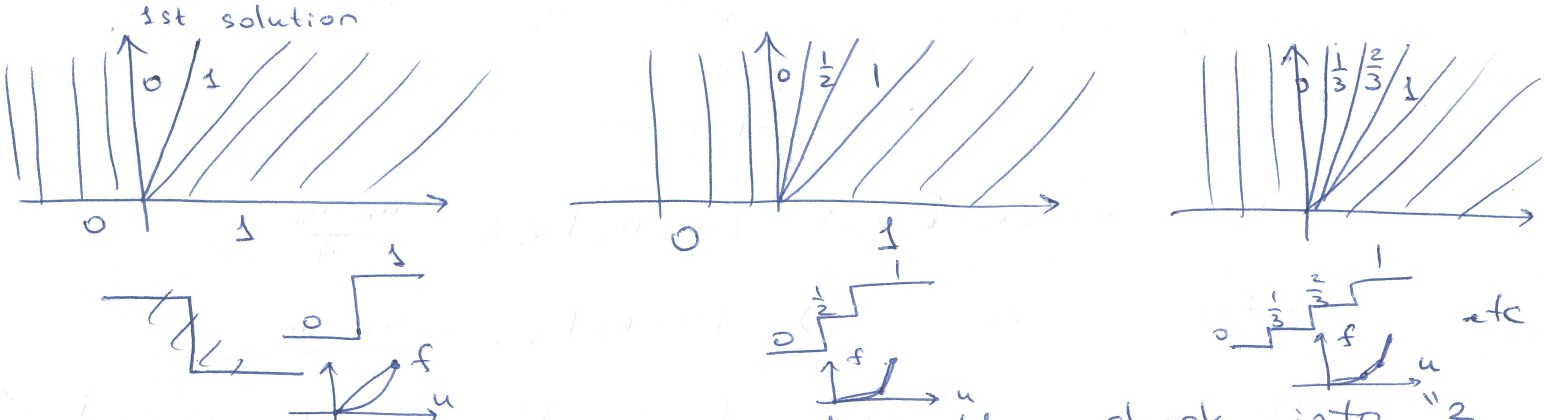


Indeed, $f'' > 0 \Rightarrow$ (see picture)

$$f'(u_L) > c = \frac{f(u_L) - f(u_R)}{u_L - u_R} > f'(u_R)$$

<u>Remark</u> (on Liu criterion) "internal stability of shock"

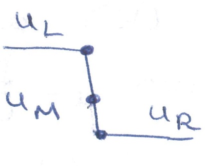
Remember the situation with Burgers equation:



In some sense if we divide the shock into "2 smaller" shocks, they could have a tendency of eigher gluing into 1 shock (some kind of stability) or going further one from another (instability)

Condition (E) $\Rightarrow$ this kind of internal stability of a shock, more precisely the inequalities

$$c(u_L, u_R) = \frac{f(u_R) - f(u_L)}{u_R - u_L} \leq c(u_L, u_M) = \frac{f(u_L) - f(u_M)}{u_L - u_M}$$



$$\geq \quad c(u_M, u_R) = \frac{f(u_R) - f(u_M)}{u_R - u_M}$$

$\forall u_M \in (\min(u_L,$ $\max(u_L, u$

If $\begin{cases} u_M \to u_L \\ u_M \to u_R \end{cases}$ we have Lax condition.

Vanishing viscosity criterion for shock waves.

- We think of equation $u_t + (f(u))_x = 0$ as a first approximation to the following parabolic eq

$$u_t + (f(u))_x = \underbrace{\varepsilon u_{xx}}_{\text{small regularizing term}}, \quad \varepsilon > 0 \qquad (P)$$

Rmk 1: it is well-known (and we see in future when dealing with reaction-diffusion equations) that solutions of (P) are very regular (no shocks)

Rmk 2: equation (P) is a combination of 2 "opposite" effects

$\to$ $u_t + (f(u))_x = 0$ $\rightsquigarrow$ creates shocks:

$\to$ $u_t = \varepsilon u_{xx}$ $\rightsquigarrow$ "smooths":

As a consequence of this confrontation there exist very special solutions, called travelling waves (TW) such that:

$$u(x,t) = v(x-ct)$$

for $c \in \mathbb{R}$ and v - some smooth profile.

They look like "smoothed" shocks !!!

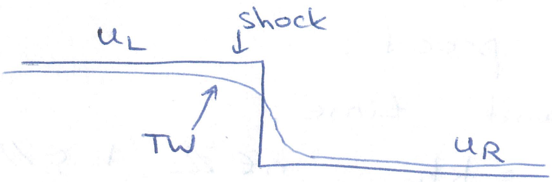
This motivates the following definition:

Def 1 (vanishing viscosity criterion for shock waves):

A shock wave is an entropy solution if it is a limit in L^1_{loc} of a travelling wave solution of (P) as $\varepsilon \to 0$.

$f \in C^2$, $f'' > 0$

Lemma: a shock wave is an entropy solution in sense of def 1, iff $u_L > u_R$.



Proof: Let's look for travelling wave solutions for eq. (P): $v\left(\frac{x-ct}{\varepsilon}\right)$: $v(-\infty) = u_L$, $v(+\infty) = u_R$

$$-c v' + (f(v))' = \varepsilon v''$$

Integrate $\int_{-\infty}^{+\infty}$: $-c(u_R - u_L) + f(u_R) - f(u_L) = 0$

Interesting feature: it is exactly RH condition

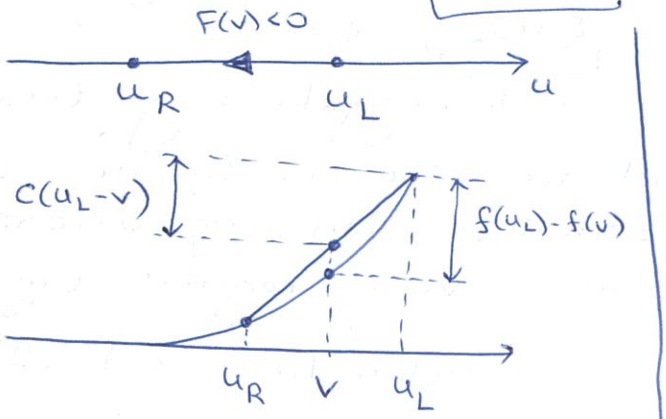
OK, let us integrate $\int_{-\infty}^{\xi}$: $-c(v(\xi) - u_L) + (f(v(\xi)) - f(u_L))$
$= v'(\xi)$

ODE) $v' = f(v) - f(u_L) - c(v - u_L) = F(v)$

Note that RHS $F(u_L) = 0$ and $F(u_R) = 0$ (due to RH!)

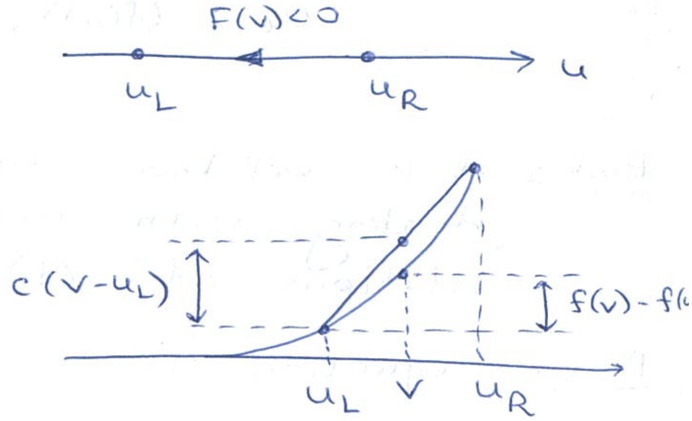
Thus u_L and u_R are two fixed points of this ODE

Consider 2 cases: $\boxed{u_L > u_R}$ and $\boxed{u_L < u_R}$.



In this case: $F(v) < 0$ $\forall v \in (u_R, u_L)$

And there exists a solution v of ODE: $v(-\infty) = u_L$; $v(+\infty) = u_R$



In this case: $F(v) < 0$ $\forall v \in (u_L, u_R)$

And there DOES NOT exist a solution v of ODE $v(-\infty) = u_L$; $v(+\infty) = u_R$