Lecture 15: Reaction-diffusion equations

$u = u(t,x)$, $x \in \mathbb{R}^N$, $t>0$, $u \in \mathbb{R}^m$

$$(*) \quad \partial_t u - \underbrace{\Delta u}_{\text{(local) diffusion}} = \underbrace{f(u)}_{\text{reaction term}}$$

- excitible medium: more generally $f = f(t,x,u)$
- Δu — comes from particles moving according to Brownian motion (in a rough way, the population tends spread out uniformly, to move towards areas where there are fewer individuals)

"Intuitive" probabilistic justification:

Let the population consist of finite number n of individuals. Consider a discrete space:

$$\{\lambda k : k \in \mathbb{Z}^N\} \subset \mathbb{R}^N, \quad \lambda > 0$$

For a given individual we denote:

$p(t,x)$ – probability that the individual is at point x at time t.

$$X_k(t,x) = \begin{cases} 1, & \text{if } k\text{-th individual is at point } x \text{ at time } t \\ 0, & \text{otherwise} \end{cases}$$

Then $$U(t,x) = \frac{1}{n} \sum_{k=1}^{n} X_k(t,x)$$ - normalized distrib. of the population

Assuming the movements of individuals are independent of each other, $U(t,x) \to p(t,x)$.

At each instant an individual can:

- move to a neighbouring point with prob. $q < \frac{1}{2n}$
- do not move with probability $1 - q \cdot 2n$

Note that the probability q does not depend on the position in time and space, nor on the previous position $\Rightarrow$ random walk $\Rightarrow$

$$p(t+\tau, \lambda k) = (1-2nq)\, p(t,\lambda k) + q \sum_{j=1}^{n} \left[p(t, \lambda(k+e_j)) + p(t, \lambda(k-e_j)) \right]$$

Assume that there exists a regular $p(t,x)$ for which the same relation is true for all x, t. So

$$\partial_t p + O(\tau) = \frac{q\lambda^2}{\tau} \sum_{j=1}^{n} \frac{\partial^2 p}{\partial x_j^2} + O\left(\frac{\lambda^3}{\tau}\right)$$

Now let $\lambda, \tau \to 0$ such that $\frac{q\lambda^2}{\tau} \to D \in (0, +\infty)$

Thus, we get $\partial_t p = D \cdot \Delta p$.

Examples : (1) population dynamics : u - concentration
(ecology) density

$$u_t - u_{xx} = f(u)$$

For a moment forget about diffusion and consider an ODE : $u_t = f(u)$, $u(0) = u_0$

Cases : (a) $f(u) = ru$ (Malthus equation '1798)

Solution: $u(t) = u_0 e^{rt}$, $r \in \mathbb{R}$

r - growth rate, the population grows infinitely (which is not natural)

(b) $f(u) = ru\left(1 - \frac{u}{K}\right)$ (logistic equation, ~ 1838)

$r \in \mathbb{R}$, $K \in \mathbb{R}$

Explicit solution: $u(t) = \dfrac{K}{1 + \left(\frac{K}{u_0} - 1\right) e^{-rt}}$

We observe, that :

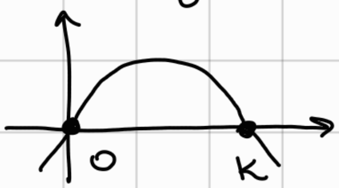
(i) whenever $u_0 > 0$, the solution is well-defined for $\forall t > 0$, $u(t) > 0$ and $u(t) \xrightarrow[t \to +\infty]{} K$

(ii) $u_0 = 0 \Rightarrow u(t) \equiv 0$

This corresponds to a more general fact that we will see later !

→ When u increases, there is a competition for resourses. Here K is called the capacity of environment

More general: monostable equations: $\dot{u} = f(t,u)$



assumptions: $f(0) = f(K) = 0$, f - Lipsch in u

$f > 0$ for $u \in (0, K)$

$f < 0$ for $u \notin [0, K]$

Sometimes, there is an extra assumption: $\frac{f(u)}{u} \downarrow$

<u>Lemma</u>: $f: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ - continuous, loc. Lipschitz in u

(i) If $f(t, 0) = 0$ $\forall t$, then if $u(0) > 0 \Rightarrow u(t) > 0$ $\forall t$

(ii) If u, v - two solutions and $u(0) > v(0)$, then $u(t) > v(t)$ (in the domain where both sol. exist)

(iii) If $u' \le f(t, u(t))$ and $v' > f(t, v(t))$ and $u(0) \le v(0)$, then $u(t) < v(t)$ $\forall t$.

<u>Rmk 1</u>: when u satisfies the differential inequality $u' \le f(t, u(t))$ we say that u is a sub-solution; otherwise super-solution

<u>Rmk 2</u>: these statements are true for a single equation, but in general are not true for systems of eqs.

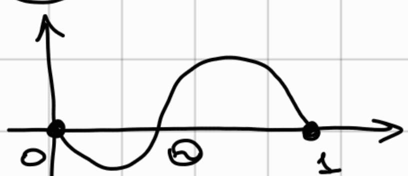
<u>Rmk 3</u>: items (ii) and (iii) are the so-called "comparison theorems" in this very simple setting. We will see more of them for reaction-diffusion eqs.

Here $u = 0$ is unstable equilibrium point (asymp)

$u = K$ is stable equilibrium point (asymp)

Thus, the name "monostable" (1 stable point)

(c) $f(u) = u(1-u)(u-\theta)$, Bistable equations

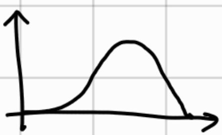


strong Allee effect

or more general assumptions:

- $f(0) = f(\theta) = f(1) = 0$
- $f > 0$ for $u \in (\theta, 1)$
- $f < 0$ for $u \in (0, \theta)$

Weak Allee effect:



monostable equation without condition $\frac{f(u)}{u}$ is decreasing

Theorem: for $u(0) \in [0,1]$ the equation admits global-in-time solution $u(t) \in [0,1]$ $\forall t \in \mathbb{R}$

Moreover, if $u(0) < \theta \Rightarrow u(t) \underset{t\to+\infty}{\longrightarrow} 0$

$u(0) > \theta \Rightarrow u(t) \to 1$

(the small population will turn off - may be not enough sexual partners or can not form big enough groups for fighting against predators)

This theorem explains the term "bistable":

$u=0$ and $u=1$ are stable equilibrium state

$u=\theta$ - unstable equilibrium state

Concluding: we will consider 3 different $f(u)$:

F-KPP

Fisher, Kolmogorov

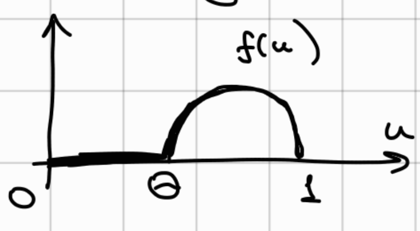
Petrovskii, Piskunov (1937)

Monostable

Bistable

- monostable case with condition that $f(u)$ lies below the tangent line at $u=0$ (think of $f(u) = u(1-u)$)

There is also a case of ignition / combustion non-linearity: $f(u) = 0$, $u \in [0,\theta]$



Rmk: there is one more notion of stability: linear stability

state α is called linearly stable if $f'(\alpha) < 0$

state α —//— linearly unstable if $f'(\alpha) > 0$

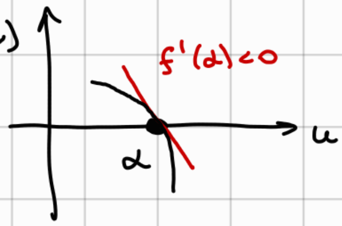
Thm: $f \in C^1$ in the vicinity of α ($f(\alpha) = 0$)

(i) If $f'(\alpha) < 0$ and $u(0)$ is sufficiently close to α, then $u(t) \to \alpha$ as $t \to +\infty$

(ii) If $f'(\alpha)>0$, then no solution (except $u\equiv\alpha$) converges to α as $t\to+\infty$.
On the other hand, if $u(0)$ is close enough to α, then $u(t)\to-\alpha$ as $t\to-\infty$.

<u>Proof</u>:

(i)



$f(u)>0$ for $u\in[\alpha-\varepsilon,\alpha)$
$f(u)<0$ for $u\in(\alpha,\alpha+\varepsilon]$

$\dot u = f(u)$

If $u(0)<\alpha \Rightarrow u(t)<\alpha$ and $\uparrow$
If $u(0)>\alpha \Rightarrow u(t)>\alpha$ and $\downarrow$

$u(t)$ can converge only to α.

(ii)

$f(u)<0$ for $u\in[\alpha-\varepsilon,\alpha)$
$f(u)>0$ for $u\in(\alpha,\alpha+\varepsilon]$

$\dot u = f(u)$

If $u(0)<\alpha \Rightarrow \dot u = f(u)<0 \Rightarrow u\downarrow$ and $u(t)<u(0)<\alpha$
If $u(0)>\alpha \Rightarrow \dot u = f(u)>0 \Rightarrow u\uparrow$ and $u(t)>u(0)>\alpha$ ∎

There are many-many ways to generalize these equations:

$\Delta u \rightsquigarrow \int_\Omega K(x-y)\,u(y)\,dy$ – non-local diffusion

$\rightsquigarrow$ general (uniformly elliptic) term $\sum_{i,j=1}^{N} a_{ij}(t,x)\,\partial_i\partial_j u$

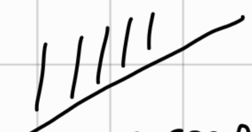
with condition $\exists\ 0<\alpha<\beta<+\infty$:

$\forall \xi\in\mathbb{R}^N, \forall t>0, \forall x\in\Omega \quad \alpha\|\xi\|^2 \le \sum a_{ij}(t,x)\,\xi_i\xi_j \le \beta\|\xi\|^2$

$f(u) \rightsquigarrow f(t,x,u)$ – depend on space x and time t

$u\in\mathbb{R} \rightsquigarrow \vec u\in\mathbb{R}^n$ – many species (Lotka-Volterra, predator-prey system, competitive media)

$\Omega\subset\mathbb{R}^N \rightsquigarrow$



more complex geometries

line of "fast" diffusion ("roads" in forests)

etc....

Other contexts: → combustion theory (propagation of flame, thermo-diffusive model)
→ probability (BBM - Branching Brownian Motion, McKean representation)
→ statistical physics etc....

Reaction-diffusion eqs: problem statement

$$(*) \qquad \partial_t u = D \Delta u + f(t,x,u)$$

- $t \in (0,+\infty)$
- $x \in \Omega$ → $\Omega = \mathbb{R}^N$ or → $\Omega \subset \mathbb{R}^N$ - bounded, connected with reg. boundary
- $D > 0$
- $u \in \mathbb{R}$ - scalar
- $f(u)$ is of one of the types above

\+ Initial condition: $u\big|_{t=0} = u_0(x) \in C(\Omega) \cap L^\infty(\Omega)$

\+ Boundary conditions:

(Neumann) $\partial_n u = 0$ for $(t,x) \in (0,+\infty) \times \partial\Omega$

(Dirichlet) $u = 0$ for — // —

(Robin) $\partial_n u + qu = 0$ for — // —

Interpretations:

Neumann: no individuals cross the boundary (in any direction)

Dirichlet: exterior of Ω is extremely unfavorable so population density is zero at boundary

Robin: there is a flow of individuals entering ($q>0$) or leaving the domain ($q<0$)

We consider classical solution u which satisfies

$$(**) \begin{cases} u \in C^0([0,+\infty) \times \bar\Omega) \\ \partial_t u \in C^0((0,+\infty) \times \Omega) \\ \forall i \quad \partial_{x_i} u \in C^0((0,+\infty) \times \bar\Omega) \\ \forall i,j \quad \partial_{x_i x_j} u \in C^0((0,+\infty) \times \Omega) \end{cases}$$

and equation $(*)$, initial and one of the boundary cond.

If $\Omega = \mathbb{R}^N$ we also assume some growth cond. at infinity: $\forall T > 0 \quad \exists A, B > 0$:

$$|u(t,x)| \le A e^{B|x|}, \quad x \in \mathbb{R}^N, \quad t > 0$$

What are the important topics?

(1) Comparison theorems: roughly speaking
if $u(0,x) \le v(0,x)$ are both solutions of (*)
then $u(t,x) \le v(t,x)$ $\forall t>0$

Closely connected to maximum principle for parabolic PDEs.

This can be very helpful:

example 1: $u_t = \Delta u + u(1-u)$
$u(0,x) \in [0,1]$ $\forall x \in \mathbb{R}^N$

- $u \equiv 0$ is solution and $u(0,x) \ge 0$
 $\Rightarrow$ $u(t,x) \ge 0$
- $u \equiv 1$ is solution and $u(0,x) \le 1$
 $\Rightarrow$ $u(t,x) \le 1$

Thus, $u(0,x) \in [0,1] \Rightarrow u(t,x) \in [0,1]$

example 2: $u_t = \Delta u - u^3$
$u|_{t=0} = u_0 \in [m, M]$, $x \in \Omega \subset \mathbb{R}^N$

Consider $\begin{cases} \dot{v} = -v^3 \\ v(0) = m \end{cases}$ and $\begin{cases} \dot{w} = -w^3 \\ w(0) = M \end{cases}$

These are sub and supersolutions:

$$v(t) \le u(x,t) \le w(t)$$

$$-\frac{dv}{v^3} = dt \Rightarrow \frac{1}{2v^2} - \frac{1}{2m^2} = t \Rightarrow v = \left(\frac{1}{m^2} + 2t\right)^{-\frac{1}{2}}$$

$v(t) \to 0$ as $t \to \infty$

Analogously, $w(t) \to 0$ as $t \to \infty$

Thus, if u exists, then

$$\underset{\substack{\downarrow \\ 0}}{v(t)} \le u(x,t) \le \underset{\substack{\downarrow \\ 0}}{w(t)} \Rightarrow u \xrightarrow[t \to +\infty]{} 0$$

→ well-posedness of $(*)$: $\exists!$ cont. dependence

→ special solutions : traveling waves (planar)

take direction $e \in \mathbb{R}^N$ and consider a solution of the form:

$$u(t,x) = \tilde{u}(\underbrace{x \cdot e}_{\text{scalar product}} - vt)$$

$$\tilde{u} : \mathbb{R} \to \mathbb{R}$$

v - speed of propagation

We will see that for different nonlinearities there exist travelling waves (TW)

$x \in \mathbb{R}^1$: FKPP : $\exists\ c^*$: $\forall c \geq c^*$ $\exists$ TW

Bistable : $\exists!\ c$: $\exists$ TW

→ $x \in \mathbb{R}^1$: long-time behaviour as $t \to +\infty$

for some initial data (like 

Heavyside)

the solution u of $(*)$ "converges" to a TW

§ Maximum principle

for parabolic equations

This is an extension of the results that we have seen for ODEs. First, some definitions:

Def 1: $u(t,x)$ is called sub-solution of $(*)$ if it satisfies $(**)$ and inequalities:

$$\partial_t u \leq \Delta u + f(t,x,u)$$

and on the boundary (if applicable): on $\partial\Omega$

(Neumann) $\partial_n u \leq 0$; (Dirichlet) $u \leq 0$; (Robin) $\partial_n u + gu \leq 0$

If $\Omega = \mathbb{R}^N$, then $|u| \leq A e^{B|x|}$, $A, B > 0$

Analogously, $v(t,x)$ is called a super solution if all inequalities are reversed (except $|v| \le A e^{B|x|}$)

We want to prove the following theorem:

Theorem (comparison principle)

Let u and v be sub- and super-solutions of the reaction-diffusion eq $(*)$.

(i) If $u(0,x) \le v(0,x)$ for $x \in \bar{\Omega}$, then $u(t,x) \le v(t,x)$ for $t>0$, $x \in \bar{\Omega}$

(ii) If moreover, $u(t_0,x_0) = v(t_0,x_0)$ for some $t_0 > 0$, $x_0 \in \Omega$, then $u \equiv v$.

(iii) If Ω is bounded and the boundary condition is of Neumann or Robin type, then (ii) is true even for $x_0 \in \partial\Omega$

Note that the difference $(u-v)$ satisfies

$$\partial_t (u-v) \le \Delta(u-v) + f(t,x,u) - f(t,x,v)$$

Thanks to regularity of u, v, f we can rewrite this equation as follows: $w = u - v$

$$(1) \quad \partial_t w \le \Delta w + g(t,x) w$$

where

$$g(t,x) = \begin{cases} \dfrac{f(t,x,u) - f(t,x,v)}{u-v} & \text{if } u \neq v \\ \partial_u f(t,x,u) & \text{if } u = v. \end{cases}$$

is continuous and uniformly bdd function

So we reduced a problem to studying the linear eq (1) and showing $w \le 0$ $\forall\, t>0$, $x \in \bar{\Omega}$.

Linear problem and maximum principle

Let us consider a more general case:

(2) $\partial_t u = \Delta u + \sum b_i(t,x)\,\partial_i u + c(t,x)\,u$

Let b_i, c be uniformly bdd.

Thm 1 (weak maximum principle)

(i) Let be a sub-solution of linear eq (2).
If $u(0,x) \le 0$, then $u(t,x) \le 0$ $\forall t>0$

(ii) Let v be super-solution of linear eq (2).
If $v(0,x) \ge 0$, then $v(t,x) \ge 0$ $\forall t>0$.

because $u(x_0,t_0)=0 \Rightarrow u\equiv 0$

Thm 2 (strong maximum principle)

(i) Let u be a subsolution of (2) and $u(0,x)\le 0$.
If $\exists\, t_0>0, x_0\in\Omega: u(t_0,x_0)=0 \Rightarrow u\equiv 0$ on $[0,t_0]\times\Omega$

(ii) Let v be a supersolution of (2) and $v(0,x)\ge 0$.
If $\exists\, t_0>0, x_0\in\Omega: v(t_0,x)=0 \Rightarrow v\equiv 0$ on $[0,t_0]\times\Omega$

(iii) If Ω is bdd, then for Neumann and Robin the same statement as in (i), (ii) are true if $x_0\in\partial\Omega$.

Rmk: it is clear that it is sufficient to consider subsolutions. For the supersolutions just consider $v=-u$.

Proof of maximum principle:

We will prove in 2 cases: (a) Ω - bdd, Dirichlet
(b) $\Omega=\mathbb{R}^N$

First, let's prove the simple case:

Lemma: let u be a subsolution with strict ineq:

$\partial_t u - \Delta u - \sum b_i(t,x)\partial_i u - c(t,x)\,u < 0$, $u(0,\cdot)<0$, $\boxed{u|_{\partial\Omega}<0}$

$\Rightarrow u(t,x)<0$

Proof of lemma:

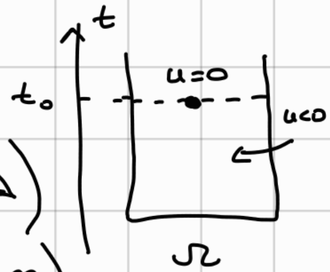
Indeed, take first time $t_0 > 0$ such that $u(x_0, t_0) = 0$ for $x_0 \in \Omega$.

At this point: $\partial_t u \geq 0$

$\Delta u \leq 0$ (the local picture )

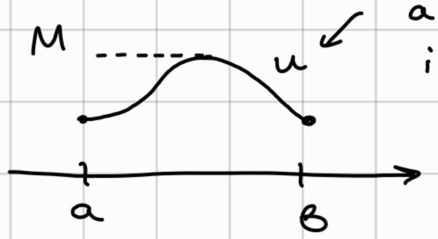
$\partial_i u = 0$ (as it is local maximum)

$u = 0$



$\Rightarrow \partial_t u - \Delta u - \sum b_i \partial_i u - cu \geq 0$ (!?) ∎

Lecture 16: Maximum principles for ODEs.



a non-constant function that achives its maximum over an interval $[a,b]$

Let $[a,b] \subset \mathbb{R}$

$u \in C^2((a,b)) \cap C^0([a,b])$

Consider a differential operator:

$$L = -\frac{d^2}{dx^2} + g\frac{d}{dx} + h$$

- g, h - bounded functions on (a,b)

Let

$$M = \max_{[a,b]} u$$

Question: how inequalities for Lu can lead to conclusions about M?

Lemma 1 (basic lemma for $h \equiv 0$):

Let $h \equiv 0$ and $Lu < 0$. Then u can equal to M only at the endpoints $x = a$ or $x = b$.

Proof:

▶ By contradiction: suppose $\exists x_0 \in (a,b) : u(x_0) = M$

Then $u'(x_0) = 0$

$u''(x_0) \leq 0$

$$\overset{L}{\Rightarrow} \quad Lu\Big|_{x_0} \geq 0 \quad (!?) \quad ■$$

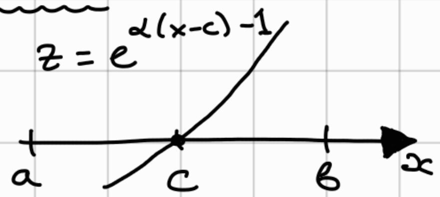
Thm 1 (one-dimensional maximum principle for $h \equiv 0$)

Let $h \equiv 0$ and $Lu \leq 0$.

Then if $\exists\, c \in (a,b) : u(c) = M \Rightarrow u \equiv M$.

Proof:

▶ $z = e^{\alpha(x-c)} - 1$



Suppose $u \not\equiv M \Rightarrow$

$\exists\, d \in (a,b)$ such that $u(d) < M$ (w.l.o.g. $d > c$)

We would like to construct a "barrier" $z(x)$ such that for $w = u + \varepsilon z$:

$$Lw < 0 \quad \text{on } (a, \beta)$$

and we could apply lemma 1.

Take

$$z = e^{\alpha(x-c)} - 1.$$

$z(c) = 0$, $z > 0$ for $x \in (c, \beta)$

$$Lz = (-\alpha^2 + g\alpha)e^{\alpha(x-c)}$$

Since g is bounded we can choose $\alpha > 0$ large enough such that $Lz < 0$

Thus, $Lw = Lu + \varepsilon Lz < 0$.

Moreover, $w(a) = u(a) + \varepsilon \underset{\wedge\atop 0}{z(a)} < u(a) \leq M$

$$w(d) = \underset{\wedge\atop M}{u(d)} + \varepsilon z(d) < M$$

by taking very small ε we can guarantee that $w(d) < M$

Thus, we have a contradiction with Lemma 1. So, $u \equiv M$. ∎

Rmk: this idea of "adding a small barrier" is very useful and we will encounter this many times in future.
The choice of z is not unique!

Thm 2 (one-dimensional Hopf lemma for $h \equiv 0$)
Let $h \equiv 0$ and $Lu \leq 0$.
If $u(a) = M$, then either $u'(a) < 0$ or $u \equiv M$
Similarly, if $u(\beta) = M$, then either $u'(\beta) > 0$ or $u \equiv M$

Rmk: the essense of the Hopf lemma is in strict inequality $u'(a) < 0$. Because the non-strict inequality is straight forward: if $u(a) = M \Rightarrow u'(a) \le 0$. So if the maximum is on the boundary, this point can not be a critical point (unless $u \equiv$ constant)

Proof:

$\ulcorner$ ▶

Let $u(a) = M$ and by contradiction $\exists\, d \in (a,b) : u(d) < M$

We can use the same "barrier"

$$z = e^{\alpha(x-a)} - 1$$

and consider $w = u + \varepsilon z$.

First, $Lw < 0$ for sufficiently large α.

And $w(a) = M > w(d)$ for sufficiently small ε.

So w achieves its maximum at $x = a$.

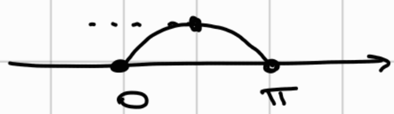
$$w'(a) = u'(a) + \varepsilon\alpha \le 0$$

$\llcorner \Rightarrow u'(a) \le -\varepsilon\alpha < 0$. ■

Interestingly, if we relax condition $h \equiv 0$, the statements are no longer valid. Consider the following counter-example:

- $Lu = -u'' - u$

  Take $Lu = 0$



$\sin(x)$!

- $Lu = -u'' + u$, $x \in [-1,1]$

Look for the solution of the form $u = -x^2 + a,\ a \in \mathbb{R}$

$$Lu = 2 - x^2 + a \le 0, \quad a \le x^2 - 2 \quad \forall x \in [-1,1]$$

Take $a = -2$

In these examples $h \cdot M \le 0$. If $h \cdot M \ge 0$, then everything ok!

Thm 3 (one-dimensional maximum principle for $h \not\equiv 0$)

exercise

Let $h \geq 0$ and $M \geq 0$.

If $Lu \leq 0$ on (a, b)

then u can attain maximum at some point $c \in (a, b)$ only if $u \equiv M$.

Rmk: this theorem should also work for $h \leq 0, M \leq 0$

Thm 4 (one-dimensional Hopf lemma for $h \geq 0$)

exercise

Let $h \geq 0$.

Let $Lu \leq 0$ on (a, b) and $M \geq 0$.

If $u(a) = M$, then either $u'(a) < 0$ or $u \equiv M$.

Similarly, if $u(b) = M$, then either $u'(b) > 0$ or $u \equiv M$.

Thm 5 (comparison principle)

Let $h \equiv 0$, $f \in C^1$

$$Lu \leq f(x) \quad x \in (a, b)$$
$$Lv \geq f(x), \quad x \in (a, b)$$

Then if $\begin{cases} u(a) \leq v(a) \\ u(b) \leq v(b) \end{cases}$, then $u(x) \leq v(x)$ $\forall x\ (a, b)$

Moreover, if $\exists x_0 :\ u(x_0) = v(x_0) \Rightarrow u \equiv v$

Proof:

$w = u - v$; $\left.\begin{array}{l} Lw \leq 0 \\ w(a) \leq 0 \\ w(b) \leq 0 \end{array}\right\} \Rightarrow w(x) \leq 0$ as maximum is obtained on the boundary $\begin{bmatrix} x = a \\ x = b \end{bmatrix}$

And if $w(x_0) = 0$ fo some $x_0 \in (a, b) \Rightarrow w \equiv 0$ ∎

Rmk: if $f = f(x, u)$ the theorem does not easily work without any other assumptions

Rmk: The above strong max. principles say that subsolution u and supersolution v can NOT touch at a point: either $u \equiv v$ or $u < v$

This "untouchability" condition can be very helpful. Consider such an example.

Example: consider a boundary value problem:

$$(1) \quad \begin{cases} -u'' = e^u, & x \in [0,L] \\ u(0) = u(L) = 0 \end{cases}$$

One can interpret the "u" as an equilibrium temperature: conditions $u(0) = u(L) = 0$ say that we have a "cold" boundary, while e^u is the "heating term". They compete with each other and non-negative solution corresponds to an equilibrium between these two effects. We would like to show that if the length of the interval L is suff. large, then no such equilibrium is possible. The physical reason is that the cold boundary is too far from the middle of the interval so that the heating term wins.

Task: show that for large enough $L > 0$ there is no non-negative solution of (1)

Step 1: consider $w = u + \varepsilon \Rightarrow (2) \begin{cases} -w'' = e^{-\varepsilon} e^w \\ w(0) = w(L) = \varepsilon \end{cases}$

Step 2: consider family of functions:

$$V_\lambda(x) = \lambda \sin\left(\frac{\pi x}{L}\right)$$

They are solutions of the following problem:

$$(3)\ \begin{cases} -V_\lambda'' = \frac{\pi^2}{L^2} V_\lambda \\ V_\lambda(0) = V_\lambda(L) = 0 \end{cases}$$

Step 3: Notice that for L large enough

$$e^{-\varepsilon} e^{s} > \frac{\pi^2}{L^2} s, \qquad \forall s > 0.$$

Thus, w as solution of (2) is a supersolution to (3): $w(0) = w(L) = \varepsilon > 0$

$$\begin{cases} -w'' \geq \frac{\pi^2}{L^2} w \\ w(0) = w(L) \geq 0 \end{cases}$$

We assume that $w \geq 0$.

Clearly, for small enough $\lambda > 0$

$$V_\lambda(x) < w(x).$$

Step 4: (Sliding method) Now start increasing λ until some $\lambda_0 > 0$ s.t. the graphs of V_λ and w "touch" at some point:

$$\lambda_0 = \sup\{\lambda > 0 : V_\lambda(x) \leq w(x), \ 0 \leq x \leq L\}$$

Look at the difference: $p = V_\lambda - w$

- $-p'' \leq \frac{\pi^2}{L^2} p \qquad p(x) \leq 0$
- $p(0) = p(L) = -\varepsilon$

In addition, $\exists\, x_0 : \ p(x_0) = 0$. It can not be in (a, b) because of maximum principle and it can not be on the boundary (!?)

Exercise (for interest):

Show that $\exists L_1 > 0$ so that non-negative solution of (1) exists for all $0 < L < L_1$ and does not exist for all $L > L_1$.

Exercise (for now): consider

$$\begin{cases} -u'' - cu' = f(u), & x \in [-L, L] \\ u(-L) = 1, \ u(L) = 0 \end{cases}$$

Prove that if solution exists, then it is unique and decreasing ($u' < 0$)

Hint: use sliding method for 2 solutions u and v, e.g. consider

$$v_h(x) = v(x+h)$$

- Strong maximum principle for any h with assumption $M = 0$.

Thm 6 (one-dimensional maximum principle for $h \not\equiv 0$)

Let $M = 0$.

If $Lu \leq 0$ on (a, b),

exercise

then u can attain maximum at some point $c \in (a, b)$ only if $u \equiv 0$.

Rmk: no assumptions on the sign of h!

Thm 7 (comparison principle):

$f \in C^1$

$$Lu \leq f(x, u) \qquad x \in (a, b)$$

$$Lv \geq f(x, u), \qquad x \in (a, b)$$

Then if $\begin{cases} u(x) \leq v(x) & \forall x \ (a, b) \\ \exists x_0 : & u(x_0) = v(x_0) \end{cases} \Rightarrow u \equiv v$

Lecture 17: Maximum principle for linear parabolic PDEs

Let us consider a linear parabolic PDE:

(1) $\quad \partial_t u = \Delta u + \sum b_i(t,x)\,\partial_i u + c(t,x)\,u =: -Lu$

Here:
- $x \in \Omega$ (either bounded open connected set or $\mathbb{R}^N$, $N \geq 1$)
- $t \geq 0$
- $u : [0,+\infty) \times \Omega \to \mathbb{R}$ – scalar function
- coefficients b_i, c are continuous and uniformly bdd (=bounded)

Initial condition: $u(0,x) = u_0(x)$

Boundary conditions:
- Ω – bdd: (Dirichlet) $u|_{\partial\Omega} = 0$

  (Neumann) $\frac{\partial u}{\partial n}\Big|_{\partial\Omega} = 0$

  (Robin) $\frac{\partial u}{\partial n} + q u\Big|_{\partial\Omega} = 0$
- $\Omega = \mathbb{R}^N$: $\exists A, B > 0$: $|u| \leq A e^{B|x|}$, $x \in \Omega$

Def: u – subsolution of (1) if $\partial_t u + Lu \leq 0$ and either $u|_{\partial\Omega} \leq 0$ or $\frac{\partial u}{\partial n}\Big|_{\partial\Omega} \leq 0$ or $\frac{\partial u}{\partial n} + qu\Big|_{\partial\Omega} \leq 0$

Analogously, v – supersolution if $\partial_t v + Lv \leq 0$ —//—

Thm 1 (weak maximum principle = weak MP)

(i) Let u be a subsolution of (1) s.t. $u(0,x) \leq 0$, Then $\forall t > 0$ $u(t,x) \leq 0$.

(ii) Let v be a supersolution of (1) s.t. $u(0,x) \geq 0$, Then $\forall t > 0$ $v(t,x) \geq 0$.

Rmk: it is clear that it is sufficient to consider subsolutions. For the supersolutions just consider $v = -u$.

Thm 2 (strong maximum principle = strong MP)

(i) Let u be a subsolution of (1) and $u(0,x) \leq 0$.
If $\exists\, t_0 > 0, x_0 \in \Omega : u(t_0, x_0) = 0 \Rightarrow u \equiv 0$ on $[0,t_0]\times\Omega$

(ii) Let v be a supersolution of (2) and $v(0,x) \geq 0$.
If $\exists\, t_0 > 0, x_0 \in \Omega : v(t_0, x) = 0 \Rightarrow v \equiv 0$ on $[0,t_0]\times\Omega$

(iii) If Ω is bdd, then for Neumann and Robin the same statement as in (i), (ii) are true if $x_0 \in \partial\Omega$.

Proof of maximum principle (weak and strong):

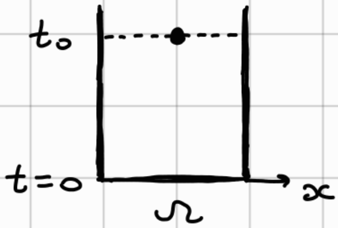
Case 1 : Dirichlet boundary conditions

Lemma 1 : Let $\partial_t u - Lu < 0$, $u(0,x) < 0$, $u|_{\partial\Omega} < 0$
Then $\forall\, t > 0$ $u(t,x) < 0$.

Proof

By contradiction. Let t_0 be the first time when $\exists\, x_0 \in \Omega : u(x_0, t_0) = 0$

At this point : $\partial_t u \geq 0$



$$-Lu \leq 0 \Leftarrow \begin{cases} \Delta u \leq 0 \\ \partial_{x_i} u = 0 \\ u = 0 \end{cases}$$

$$\Rightarrow \partial_t u + Lu \geq 0 \quad (!?)$$

Thus at $\forall\, x \in \Omega, t > 0$ $u(t,x) < 0$ ■

Observation : take $u = e^{kt} w$ for some $K \in \mathbb{R}$

$u < 0 \iff v < 0$ and $u \leq 0 \iff w \leq 0$

But now w satisfies :

$$\partial_t w - \Delta w - \sum b_i \partial_i w - (c-K)w \underset{(\leq)}{<} 0$$

Taking $K > \max |c|$ we can guarantee that $c - K < 0$, or taking $K < -\max|c|$ we have $c - k > 0$.

Let's take $K \geq \max |c| + 1$, and thus $c - K \leq -1$.
In order not to change the notation we stay with letter "u" and consider $c \leq -1 < 0$ in (1).

Now we are ready to prove thm 1 (i).
By contradiction. Take the first moment $t_0 > 0$ s.t. $\exists\, x_0 \in \Omega$: $u(t_0, x_0) = \delta$ for some $\delta > 0$.
At this point (t_0, x_0) :

$$\left.\begin{array}{l} \partial_t u \geq 0 \\ \Delta u \leq 0 \\ \partial_{x_i} u = 0 \end{array}\right\} \Rightarrow -Lu \leq c\delta \leq -\delta$$

$$\Rightarrow \quad \partial_t u + Lu \geq \delta > 0 \quad (!?)$$

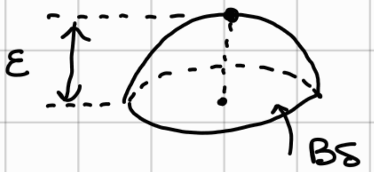
Thus, for all $x \in \Omega$, $t > 0$ $\quad u(t,x) \leq 0$.
We have proven the weak MP for Dirichlet $\overset{bc}{\swarrow}$
|

Let's prove the strong maximum principle for Dirichlet $\overset{bc}{\swarrow}$.

<u>Lemma 2</u>: Let u be subsolution of (1) with Dirichlet and $u(0,x) < 0$ $\forall x \in \Omega \Rightarrow u(t,x) < 0$ $\forall t > 0$

<u>Proof</u>:

$\ulcorner$ ▶ It is enough to consider $\Omega = B_\delta(0)$.
The idea is to construct a "barrier"
Let $w = u + \varepsilon(\delta^2 - |x|^2)^2 e^{-\alpha t}$



Take $\varepsilon > 0$ so small s.t. $w(0,x) < 0$. Moreover, $w\big|_{\partial B_\varepsilon} = u\big|_{\partial B_\varepsilon} \leq 0$

We can choose α such that w is a subsolution
Indeed, $\partial_i (\delta^2 - |x|^2)^2 = 2(\delta^2 - |x|^2)\cdot(-2x_i)$

$$\partial^2_{ii} (\delta^2 - |x|^2)^2 = -4(\delta^2 - |x|^2) + 8x_i^2$$

Then $(-L)(\delta^2 - |x|^2)^2 = (\Delta + \Sigma b_i \partial_i + c)(\delta^2 - |x|^2)^2 =$

$$= 8|x|^2 - 4N(\delta^2 - |x|^2) - 4b\cdot x(\delta^2 - |x|^2) + c(\delta^2 - |x|^2)^2$$

By estimating $|B(t,x)| \le \|B\|_\infty$ and $|c(t,x)| \le \|c\|_\infty$ we obtain:

$$(\partial_t + L)\, z \le \varepsilon e^{-\alpha t} \Big[-\alpha \cdot (\delta^2 - |x|^2)^2 - 8|x|^2 + 4N(\delta^2 - x^2) + $$
$$+ 4|x| \cdot \|B\|_\infty (\delta^2 - |x|^2) + \|c\|_\infty (\delta^2 - |x|^2)^2 \Big]$$

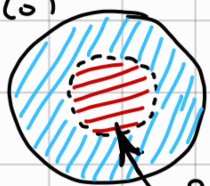
We would like: $(\partial_t + L)\, z \le 0$.

<u>Naive idea</u>: just take $\alpha > 0$ very big and then the first term $-\alpha(\delta^2 - |x|^2)^2$ will be very negative and dominate all other (positive) terms.

<u>Bad news</u>: the term $-\alpha^2(\delta^2 - |x|^2)^2$ is small close to the boundary of the $B_\delta(0)$. So the previous idea works only inside some smaller ball $B_{\delta'}(0) \subset B_\delta(0)$ $(0 < \delta' < \delta)$

<u>What to do?</u> Divide the ball into 2 parts:

$B_\delta(0)$



$B_{\delta'}(0)$

(1) $B_\delta(0) \setminus B_{\delta'}(0)$

(2) $B_{\delta'}(0)$

and estimate $(\partial_t + L)\, z$ in each part separately.

(1) If δ' is close to δ, then all terms that have $(\delta^2 - |x|^2)$ are small and the dominating term is $-8|x|^2$. Take δ' such that $\forall\ x \in B_\delta(0) \setminus B_{\delta'}(0)$ the following ineq is true

$$8|x|^2 > (\delta^2 - x^2) \cdot \big[4N + 4|x| \cdot \|B\|_\infty + \|c\|_\infty \cdot (\delta^2 - |x|^2) \big]$$

Or

$$8(\delta')^2 > (\delta^2 - (\delta')^2) \cdot \big[4N + 4\delta \|B\|_\infty + \delta^2 \cdot \|c\|_\infty \big]$$

Such δ' exists as $8(\delta')^2 \approx 8\delta^2$ when $\delta' \approx \delta$ and right hand side is almost 0.

Thus, for $x \in B_\delta \setminus B_{\delta'}$: $(\partial_t + L)\, z \le -\alpha \varepsilon e^{-\alpha t} (\delta^2 - |x|^2)^2 < 0$

(2) Now take α so big such that for all $x \in B_{\delta'}(0)$ we have:

$$\alpha\,(\delta^2 - |x|^2)^2 > (\delta^2 - |x|^2)\left[4N + 4\cdot|x|\cdot\|b\|_\infty + \|c\|_\infty(\delta^2 - |x|^2)\right]$$

Divide by $\delta^2 - |x|^2$ and it is enough to have

$$\alpha\cdot\left(\delta^2 - (\delta')^2\right)^2 > \delta^2\left[4N + 4\delta'\cdot\|b\|_\infty + \|c\|_\infty\,\delta^2\right]$$

$$\alpha > \frac{\delta^2\left[4N + 4\delta'\,\|b\|_\infty + \|c\|_\infty\,\delta^2\right]}{\left(\delta^2 - (\delta')^2\right)^2}$$

(remember, here δ' is already some fixed value)

Thus, for $x \in B_{\delta'}(0)$: $(\partial_t + L)z < -8\varepsilon e^{-\alpha t}|x|^2 < 0$

$$\Rightarrow (\partial_t + L)\,w = \underbrace{(\partial_t + L)\,u}_{\leq 0} + \underbrace{(\partial_t + L)\left(\varepsilon(\delta^2 - |x|^2)e^{-\alpha t}\right)}_{\leq 0} \leq 0$$

$\Rightarrow$ $w \leq 0$ $\Rightarrow$ $u < w \leq 0$; q.e.d.

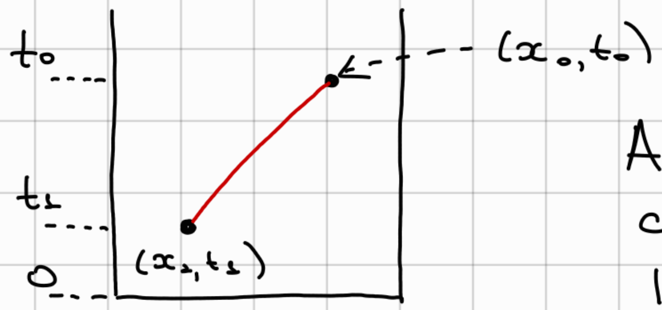
weak MP

Now let's finish proving the strong MP for (D).

Take (t_0, x_0) : $u(t_0, x_0) = 0$.

It is enough to prove that $u \equiv 0$ for $t \in [a, t_0]$, $x \in \Omega$

By contradiction, there exists a point (t_1, x_1), $t_1 < t_0$ such that $u(t_1, x_1) < 0$.



By continuity $u < 0$ in $B_\delta(t_1, x_1)$ – ball in Ω

Assume that the segment connecting x_1 and x_0 in Ω lies in Ω (e.g. Ω - convex)

If necessary, take smaller δ s.t. $B_\delta(x) \subset \Omega$ for all x in this segment $[x_1, x_0]$ (this can be done by compactness of segment)

Now consider $w(t, x) = u\left(t,\, x + \frac{t - t_1}{t_0 - t_1}(x_0 - x_1)\right)$

$$\partial_t w = \partial_t u + \sum_{i=1}^{N} \partial_i u$$

Clearly, w satisfies the equation of type (1)

By previous lemma : $w(t_1, x_1) = u(t_1, x_1)$

$w(t_0, x_1) = u(t_0, x_0)$

$$w(t_1, x_1) < 0 \Rightarrow w(t_0, x_1) < 0 \Rightarrow u(t_0, x_0) < 0 \; (!?)$$

It is easy to generalize this argument for arbitrary connected domains Ω, as there

exists a path between x_1 and x_0 and this path can be approximated by segments. ∎

Both weak and strong MP for Dirichlet bc are proven (case 1)

Case 2 : Neumann and Robin bc.

Lemma 3 (Hopf lemma)

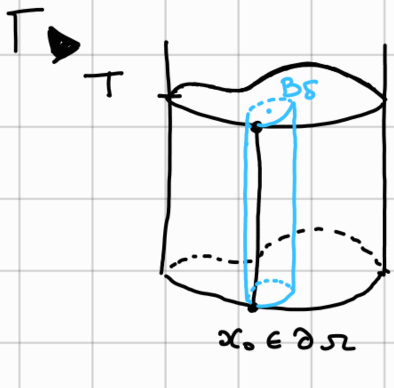
Let u be subsolution of (1) with NO boundary conditions. And let $u(t,x) < 0$ for all $t \in [0,T]$ and $x \in \Omega$.

If $u(T, x_0) = 0$ at $x_0 \in \partial\Omega$, then

$$\frac{\partial u}{\partial n}(T, x_0) > 0 .$$

Rmk : the sign $\frac{\partial u}{\partial n} \geq 0$ is clear, the important statement in lemma is STRICT inequal.

Proof :



By contradiction.

Let $\exists x_0 \in \partial\Omega$ s.t.

$$u(T, x_0) = \frac{\partial u}{\partial n}(T, x_0) = 0$$

Take a ball $B_\delta \subset \Omega$ s.t. $x_0 = \partial B_\delta \cap \partial\Omega$ (this is just some condition on regularity of $\partial\Omega$)

For simplicity we can always assume that the center of the ball B_δ is in the origin and the normal $n = (-1, 0, \dots 0)$

As $u<0$ in $\Omega \times [0,T]$, then $\forall\ 0<r<\delta$

$$\sup_{t\in[0,T]} \ \sup_{x\in B_r} u(t,x) < 0.$$

Consider

$$w = u + \varepsilon_1(t-T) + \varepsilon_2\left[e^{-\alpha|x|^2} - e^{-\alpha\delta^2}\right]$$

$\alpha, \varepsilon_1, \varepsilon_2 > 0$ will be chosen soon.

We want to prove: for domain $A := B_\delta(0) \setminus \overline{B_\rho(0)}$
$0<r<\delta$

$$\begin{cases} \partial_t w + Lw \le 0, & x\in A,\ t\in[0,T] & (a)\\ w(0,x) < 0, & x\in A & (b)\\ w\big|_{\partial A}(t,x) \le 0 & & (c)\end{cases}$$

Thus, by Dirichlet weak MP $\Rightarrow$ $w(T,x) \le 0$ for $x\in A$

This will be a contradiction with

$$w(T,-\delta,0\dots0) = u\Big|_{\substack{x=x_0\\ t=T}} = 0$$

$$\frac{\partial}{\partial n} w(T,\cdot) = -\partial_{x_1} w(T,-\delta,0\dots0) = -\partial_{x_1} u + \varepsilon_2\alpha\cdot 2x_1 e^{-\alpha|x|^2}\Big|_{\dots} =$$

$$= 0 - \varepsilon_2\cdot 2\alpha\delta\cdot e^{-\alpha\delta^2} < 0$$

Let's show (a), (b), (c).

(a) $\partial_t w + Lw = \partial_t w - \Delta w - b\cdot\nabla w - cw \le$

$$\le \varepsilon_1(1+CT) - \varepsilon_2 e^{-\alpha|x|^2}\cdot\left[4\alpha^2|x|^2 - 2N\alpha - 2C\alpha|x| - C\right]$$

where C is $\max(\|b_i\|_\infty, \|c\|_\infty)$.

$$L\left[e^{-\alpha|x|^2} - e^{-\alpha\delta^2}\right] = \sum \frac{d}{dx_i}\left(-2\alpha x_i e^{-\alpha|x|^2}\right) + \sum b_i\left(-2\alpha x_i e^{\dots}\right)$$

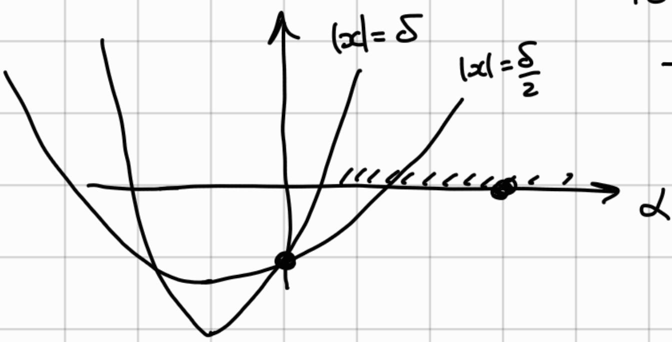
$$+ c\left(e^{-\alpha|x|^2} - e^{-\alpha\delta^2}\right) = -2\alpha N e^{-\alpha|x|^2} + 4\alpha^2 \sum x_i^2 e^{-\alpha|x|^2}$$

$$+ \quad \sum' B_i \left(-2\alpha x_i e^{-\alpha|x|^2}\right) + c\left(e^{-\alpha|x|^2} - e^{-\alpha\delta^2}\right)$$

Fix $\alpha > 0$ s.t. $4\alpha^2|x|^2 - 2N\alpha - C\alpha|x| - C \geq \alpha$

for $x \in B_\delta \setminus B_{\delta/2}$: $\quad d_1\alpha^2 + d_2\alpha + d_3 \geq 0$

This can be done if $|x|$ is not close to 0, e.g. $|x| > \delta/2$ (that's why we take the domain A to be a ring!)



Then w is a subsolution in A if

(cond 1) $\quad \dfrac{\varepsilon_2}{\varepsilon_1} \geq \dfrac{(1+cT)e^{\alpha\delta^2}}{\alpha}$

(b) $\quad w(0,x) = u(0,x) \underbrace{- \varepsilon_1 T + \varepsilon_2\left[e^{-\alpha|x|^2} - e^{-\alpha\delta^2}\right]}_{\leq 0 \text{ for } x \in B_\delta \setminus \overline{B_r}}$ < 0

(cond 2) $\quad \dfrac{\varepsilon_2}{\varepsilon_1} \leq \dfrac{T}{e^{-\alpha r^2} - e^{-\alpha\delta^2}}$

If we choose r very close to δ, then

RHS of (cond 1) < RHS of (cond 2)

(c) Boundary consists of 2 pieces: $\partial B_\delta, \partial B_r$

- Clear that $\quad w(t, \partial B_\delta) = u(t, \partial B_\delta) + \varepsilon_1(t-T)$
  $\forall t \in [0,T] \quad \leq 0 + \varepsilon_1(t-T) \leq 0$

- $w(t, \partial B_r) = \underset{\hat{0}}{u(t, \partial B_r)} + \varepsilon_1(t-T) + \varepsilon_2\left(e^{-\alpha r^2} - e^{-\alpha\delta^2}\right)$

  It is enough to take small $\varepsilon_2 > 0$, e.g.

$$\varepsilon_2 < \frac{-\sup_{t\in[0,\infty)} u(t, \partial B_r) \neq 0}{e^{-\alpha r^2} - e^{-\alpha\delta^2}}$$

  Then $w(t, \partial B_r) < 0$. ∎

Next time we will finish the proof of the weak MP for Neumann / Robin bc.

Lecture 18 : Today we will finish proving the maximum principles (weak and strong) for the Neumann, Robin b.c. and $\Omega = \mathbb{R}^N$ and briefly talk about the existence of the solutions to react.-diff. eqs.

Case 2 (Neumann, Robin b.c.)

W.l.o.g. $c < -1$.

- Want to prove : $u(0,x) \le 0 \Rightarrow u(t,x) \le 0 \quad \forall t>0, x \in \Omega$

By contradiction: $\exists \delta > 0$ and $\exists (t_0, x_0)$: $u(t_0, x_0) = \delta$ and t_0 is the first time when $u(t_0, x_0) = \delta$: for $0 \le t < t_0$ $\forall x \in \overline{\Omega}$ $u(t,x) < \delta$.

(a) If $x_0 \in \Omega$, then the same argument as for Dirichlet case gives a contradiction:

$$(\partial_t + L) u \ge -cu \ge \delta > 0 \quad (!?)$$

(b) If $x_0 \in \partial\Omega$, we are in the context of the Hopf lemma for $w = u - \delta$.

Indeed, $w(t_0, x_0) = 0$ and $w(t,x) < 0$ if $\begin{cases} x \in \Omega \\ 0 \le t \le t_0 \end{cases}$ and w is a subsolution:

$$\partial_t w - \Delta w - b \cdot \nabla w - cw \le -\delta c \le 0$$

Thus, by Hopf lemma

$$\frac{\partial u}{\partial n}(t_0, x_0) = \frac{\partial w}{\partial n}(t_0, x_0) > 0$$

which contradicts the inequality $\frac{\partial u}{\partial n} \le 0$ for the Neumann b.c.

This is a contradiction also for Robin b.c. as

$$\left(\frac{\partial u}{\partial n} + q u\right)\Big|_{(t_0, x_0)} > q u\Big|_{(t_0,x_0)} > q\delta > 0$$

We assume $q > 0$ for the Robin b.c.

So we have proven the weak MP for (N), (R)

Let's prove the strong maximum principle for (N) and (R). As we already know $u(x,t) \leq 0$ $\forall\ x \in \Omega$ and $x \in \partial\Omega$, we can apply the same argument as for the case of the Dirichlet b.c. In particular, if $u \not\equiv 0$, then $u < 0$ $\forall t > 0, x \in \Omega$. Apply the Hopf lemma again to see that $u < 0$ for $x \in \partial\Omega$, $t > 0$. ■

Case 3: $\Omega = \mathbb{R}^N$. Take $w = u\varphi(x)$, where $\varphi \in C^\infty(\Omega)$ is strictly positive and

$$\frac{|\nabla\varphi|}{\varphi},\ \frac{|\Delta\varphi|}{\varphi} \in L^\infty(\mathbb{R}^N)$$

and

$$\varphi(x) = e^{-2B|x|} \quad \text{for } |x| \gg 1.$$

$$w_t = u_t\,\varphi$$

$$\partial_i w = \partial_i u \cdot \varphi + \overset{w/\varphi}{\overset{\downarrow}{u}} \cdot \partial_i \varphi$$

$$\partial^2_{ii} w = \partial_{ii} u \cdot \varphi + \partial_i u \cdot \partial_i \varphi + \overset{\frac{w}{\varphi}}{\overset{\downarrow}{u}}\, \partial_{ii}\varphi$$

$$\Rightarrow \quad (\partial_t + L)u = (\partial_t + L)w - w \cdot \frac{\nabla\varphi \cdot b}{\varphi} - \nabla u \cdot \nabla\varphi - w \cdot \frac{\Delta\varphi}{\varphi}$$

Under the above conditions this operator is of the same type as for u, but for w we have:

$$|w| = |u\, e^{-2B|x|}| \leq A \cdot e^{-B|x|} \underset{\text{as } |x|\to\infty}{\longrightarrow} 0$$

So the proof of the weak MP stays the same. The proof of the strong MP did not involve that Ω is bounded.

Well-posedness of reaction-diffusion eq.

$$(*) \quad \begin{cases} u_t = \Delta u + f(t,x,u) , & \Omega \subseteq \mathbb{R}^N \begin{smallmatrix} \nearrow \text{bdd, open} \\ \searrow \mathbb{R}^N \end{smallmatrix} \\ u(0,x) = u_0(x) \\ + \text{ b.c.} \end{cases}$$

Assume $f \in C^1(\mathbb{R})$, $u_0 \in C^0(\bar{\Omega})$.

1. $\exists$ (existence)
2. $!$ (uniqueness)
3. Continuos dependence on initial data

Thm (uniqueness of solution to react.-diff. eq.)
Let u, v be 2 solutions with the same initial conditions (D) or (N) or (R), then $u \equiv v$.

Proof: just by comparison theorem!

Thm (continuity of initial data)
Let u, v be 2 solutions with the same boundary conditions (D) or (N) or (R), but different initial data u_0, v_0. Then, $\forall t > 0$ $\exists K = \|\partial_u f\|_\infty$:

$$\|u(t,\cdot) - v(t,\cdot)\|_\infty \leq \|u_0 - v_0\|_\infty e^{Kt}$$

Proof

▶ $w = u - v$: $\partial_t w - \Delta w = g(t,x) w$

g is uniformly bounded and $|g(t,x)| \leq K = \|\partial_u f\|_\infty$ and $M e^{Kt}$ is a supersolution.

Taking $M = \|u - v\|_\infty$ we arrive at

$$u - v \leq \|u - v\|_\infty e^{Kt}$$

Analogously, $v - u \leq \|u - v\|_\infty e^{Kt}$ ∎

Thm (continuous dependence on f)

Let $f_n \in C^1(\mathbb{R})$ and $f_n \to f$, $\partial_u f_n \to \partial_u f$ (uniformly)

Let u_n and u be solutions with reaction term f_n and f, respectively, and the same initial and boundary conditions.

Then $u_n \to u$ (locally uniformly in t)

Existence (only formulations and sketches of proof)

(1) linear case:

(1a) $\partial_t u = \Delta u + Ku$

Easy to pass to the heat equation by the change of variables: $u = e^{Kt} w$:

$$w_t = \Delta w$$

We know a lot about the heat eq!

(1b) non-homogeneous heat equation:

$$\begin{cases} \partial_t u = \Delta u + g(t,x) \\ u(0,x) = u_0(x) \\ + \text{b.c. or boundness at } |x| \to +\infty \end{cases}$$

We can write an explicit formula for $\Omega = \mathbb{R}^N$

$$u(t,x) = \int_{\mathbb{R}^N} K(x-y,t)\, u_0(y)\, dy + \int_0^t \int_{\mathbb{R}^N} K(x-y,t-s)\, g(s,y)\, ds\, dy$$

where $K(x,t) = \dfrac{1}{(4\pi t)^{N/2}} e^{-\frac{|x|^2}{4t}}$ – heat kernel

(2) non-linear case: $\partial_t u = \Delta u + f(t,x,u)$

Let f, u_0 satisfy the following assumptions:

(U_0): $\exists M > 0 : |u_0| \leq M$

(F): $f \in C^1$, $f(t,x,0) \in L^\infty$, $\partial_u f \in L^\infty$

In particular, • we can fix $K > 1$, $\forall u \in \mathbb{R}$

$$|f(t,x,u)| \leq K(1+|u|).$$

• Moreover, $\forall u \geq -M$, $t \geq 0$, $x \in \Omega$

$f(t,x,u) \leq K(1+u+M)$,

• and $\forall u \leq M$, $t \geq 0$, $x \in \Omega$,

$f(t,x,u) \geq K(-1+u-M)$

Observation: we can always assume $\partial_u f > 0$ by using the following trick:

$$u(t,x) = e^{-Nt} \tilde{u}(t,x), \text{where } N = \sup |\partial_u f|$$

$$\partial_t u + Lu = f(t,x,u)$$

$$(\partial_t + L)\tilde{u} = \underbrace{N\tilde{u} + e^{Nt} f(t,x,e^{-Nt}\tilde{u})}_{\tilde{f}(t,x,\tilde{u})}$$

$$\partial_{\tilde{u}} \tilde{f} = N + \cancel{e^{Nt}} \partial_u f \cdot \cancel{e^{-Nt}} > 0.$$

So in this section (existence) we will assume $\boxed{\partial_u f > 0.}$

Thm (existence of solution to reaction-diff. eq)

Under the above conditions on Ω, u_0, f there exists a solution of $(*)$ for b.c. (D) or (N) or (R).

Idea: approximate the solution by a sequence of solutions $(u^k)_{k=1}^\infty$ of some linear probl. which solutions we already know.

(monotone iteration method)

Sketch of proof:

⌈

▸ First, consider $\underline{u}$ – the solution of the eq:

$$(\underline{U}) \quad \begin{cases} \partial_t \underline{u} - \Delta \underline{u} = K(-1 + \underline{u} - M) \\ \underline{u}(0,x) = u_0(x) \\ + \text{b.c.} \end{cases}$$

Solution exists (after change of variables we obtain just a heat equation)

Clearly, M is a supersolution of $(\overline{U}) \Rightarrow \underline{u} \le M$

Thus, $K(-1 + \underline{u} - M) \le f(t,x,\underline{u})$ by assumption (F).

Hence, $\underline{u}$ is a sub-solution of (*)

Analogously, consider $\overline{u}$ – the solution of

$$(\overline{U}) \quad \begin{cases} \partial_t \overline{u} - \Delta \overline{u} = K(1 + \overline{u} + M) \\ \overline{u}(0,x) = u_0(x) \\ + \text{b.c.} \end{cases}$$

Solution exists and $\overline{u}$ is a supersolution of (*)

Moreover, $\underline{u} < \overline{u}$ for $t > 0$ (by strong comp. thm)

Second, let's built an approximating seq. Take $u^0 = \underline{u}$, and consider u^1 the solution of the following non-homogeneous heat eq:

$$\begin{cases} \partial_t u^1 - \Delta u^1 = f(t,x,u^0) \\ u^1(0,x) = u^0(x) \\ + \text{b.c.} \end{cases}$$

By comparison principle: $u^0 \le u^1$.

Due to monotonicity of f:

$f(t,x,u^0) = f(t,x,\underline{u}) < f(t,x,\overline{u})$,

and comparison principle, we have

$$u^1 \leq \overline{u}$$

In total, we get: $\underline{u} = u^0 \leq u^1 \leq \overline{u}$.

Proceeding for $k = 1, 2, 3, \ldots$ as follows:

$$\partial_t u^{k+1} - \Delta u^{k+1} = f(t,x,u^k)$$

we obtain

$$\boxed{\underline{u} = u^0 \leq u^k \leq u^{k+1} \leq \overline{u} \qquad \forall k \in \mathbb{N}}$$

Third, at each point (t,x) the sequence converges $u^k(t,x) \to u(t,x)$.

We would like to pass to the limit in the equation and get:

$$\partial_t u - \Delta u = f(t,x,u).$$

Nevertheless, we know only that $u^k \to u$, but don't know the same result about the derivatives!

It would be enough to know that:

- $\|\partial_{x_i} u\|_{C^{0,\alpha}([\tau,T]\times K)} \leq \tilde{C}$
- $\|\partial_t u\|_{C^{0,\alpha}([\tau,T]\times K)} \leq \tilde{C}$ (est.)
- $\|\partial_{x_i x_j} u\|_{C^{0,\alpha}([\tau,T]\times K)} \leq \tilde{C}$

for constant $\tilde{C}$ depending on τ, T, K

Here $C^{0,\alpha}([\tau,T]\times K)$ is a space of bounded and α-Hölder $(0<\alpha<1)$ continuos functions, that is $g\in C^{0,\alpha}$ means there exists a constant $C>0$:

(K: compact set)

$$|g(t_1,x_1)-g(t_2,x_2)| \le C\left(|t_1-t_2|^\alpha + |x_1-x_2|^\alpha\right)$$

supplied with the norm:

$$\|\cdot\|_{C^{0,\alpha}(\dots)} = \|\cdot\|_{L^\infty(\dots)} + C.$$

Why enough to know estimates (est.)?
Because of Arzela-Ascoli theorem:

| a set of functions f_n defined on a compact set, whose $C^{0,\alpha}$-norm is bounded, admits a subsequence which converges in C^0.

So by using (est.) and Arzela-Ascoli theorem, we can (several times) take a convergent subsequence, and pass to the limit in the equation. By uniqueness of the limit this is u that satisfies the reaction-diffusion eq.

Rmk: let us put under the carpet how to obtain estimates like (est.)
Sometimes they are called Schauder estimates and are based on fine properties of the heat kernel and the exact formula for solution: $u_t=\Delta u+g$

Just for the sake of completeness, let me give the formulation of Schauder-type estimates:

Thm (Schauder estimates):

means on every compact $[\tau,T]\times K \subset (0,\infty)\times\Omega$

Let $0<\alpha'<\alpha<1$, $g \in C^{0,\alpha}_{loc}((0,+\infty)\times\Omega)$.

Let u be a solution of

$$\begin{cases} u_t - \Delta u = g(x,t) \\ u(0,x) = u_0(x) \in C^0(\Omega) \end{cases}$$

Then:

- for all $0<\tau<T<+\infty$ and $\forall K \subset \overline{\Omega}$

$$\|u\|_{C^{0,\alpha}([\tau,T]\times K)} + \|\partial_{x_i} u\|_{C^{0,\alpha}([\tau,T]\times K)} \leq$$

$$\leq C\cdot\left[\|g\|_{L^\infty([0,T+1]\times\overline{\Omega})} + \|u\|_{L^\infty([0,T+1]\times\overline{\Omega})}\right]$$

- for all $0<\tau<T<+\infty$ and $\forall K' \subset K \subset \Omega$, $K' \neq K$ – two compact sets:

$$\|\partial_t u\|_{C^{0,\alpha'}([\tau,T]\times K')} + \|\partial_{x_i x_j} u\|_{C^{0,\alpha'}([\tau,T]\times K')} \leq$$

$$\leq C\cdot\left[\|g\|_{C^{0,\alpha}([\frac{\tau}{2},T+1]\times K)} + \|u\|_{L^\infty([0,T+1]\times K)}\right]$$

Here constant C may depend on τ, T, K, K', α.

Last comment on the proof of the thm ∃.
As $\underline{u}$ and $\overline{u}$ satisfy the initial cond. and $\underline{u} \leq u^k \leq \overline{u}$, then u^k will also satisfy the initial condition. ∎

Lecture 19: Existence of travelling wave (TW) solutions to reaction-diffusion eqs

$$(*) \qquad u_t = \Delta u + f(u), \qquad u: \mathbb{R}_+ \times \mathbb{R}^N \to \mathbb{R}$$

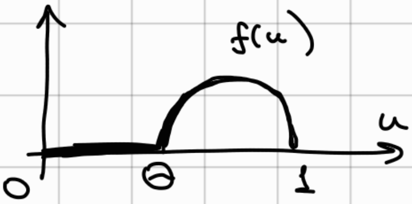
Candidates for the reaction term $f(u)$:

F-KPP — Fisher, Kolmogorov, Petrovskii, Piskunov (1937) | Monostable | Bistable

- monostable case with condition that $f(u)$ lies below the tangent line at $u=0$ (think of $f(u)=u(1-u)$)

$$f'(0) = \sup_{u\in(0,1]} \frac{f(u)}{u}$$

There is also a case of ignition / combustion non-linearity: $f(u)=0,\ u\in[0,\theta]$



Consider $u(0,x) = u_0(x) \in [0,1] \Rightarrow u(t,x) \in [0,1]$ by comparison principle.

We are interested in traveling wave (TW) solutions (sometimes are also called traveling fronts = TF)

Fix direction $\vec{e} \in \mathbb{R}^N$ and consider the solution of the form: $\tilde{u}: \mathbb{R} \to [0,1]$ such that

$$(**) \qquad u(t,x) = \tilde{u}(x\cdot\vec{e} - ct), \qquad c\in\mathbb{R} - \text{speed of TW (apriori unknown)}$$

<u>Rmk 1:</u> $\tilde{u}$ is constant on hyperplanes orthogonal to $\vec{e}$ and for this reason sometimes is called planar TW.

<u>Rmk 2:</u> for simplicity of notation we will omit "$\sim$" and just write u istead of $\tilde{u}$

Putting form $(**)$ into $(*)$, we get an ODE:

$$(TW)_\infty \begin{cases} -u'' - cu' = f(u) \\ u(-\infty) = 1,\ u(+\infty) = 0 \ , \quad u'(-\infty) = u'(+\infty) = 0 \end{cases}$$

Question: for which $c \in \mathbb{R}$ does there exist a solution of $(TW)_\infty$ problem?

Thm (existence of TW solutions)

(i) In the bistable and combustion cases there exists a unique $c \in \mathbb{R}$ for which there exists a solution of $(TW)_\infty$.

Moreover, • u is unique and decreasing;

• sign of c coincides with the sign of $\int_0^1 f(u)\,du$.

(ii) In the monostable case $\exists\, c^* > 0$ such that there exists a solution (TW) iff $c \geq c^*$. When it exists the solution is unique and is decreasing.

(iii) In FKPP case $c^* = 2\sqrt{f'(0)}$.

Rmk: (i) in the bistable case if $c > 0$, this means that the state one invades 0; if $c < 0$ the state 0 invades 1; if $c = 0$ there is a co-existence of two states.

(ii) The sign of speed c is easy to understand: multiply $(TW)_\infty$ by u' and $\int_{-\infty}^{+\infty} \ldots d\xi$:

$$-\int_{-\infty}^{+\infty} u''\cdot u' - c\int_{-\infty}^{+\infty} u'^2 = \int_{-\infty}^{+\infty} f(u)\underbrace{u'd\xi}_{du} \quad \Big| \begin{matrix} -\infty \mapsto 1 \\ +\infty \mapsto 0 \end{matrix}$$

$$\cancel{-\frac{1}{2}(u')^2\Big|_{-\infty}^{+\infty}} \to 0 \qquad - c\int_{\mathbb{R}}(u')^2 = \int_1^0 f(u)\,du \Rightarrow \operatorname{sign}(c)=\operatorname{sign}\int_0^1 f$$

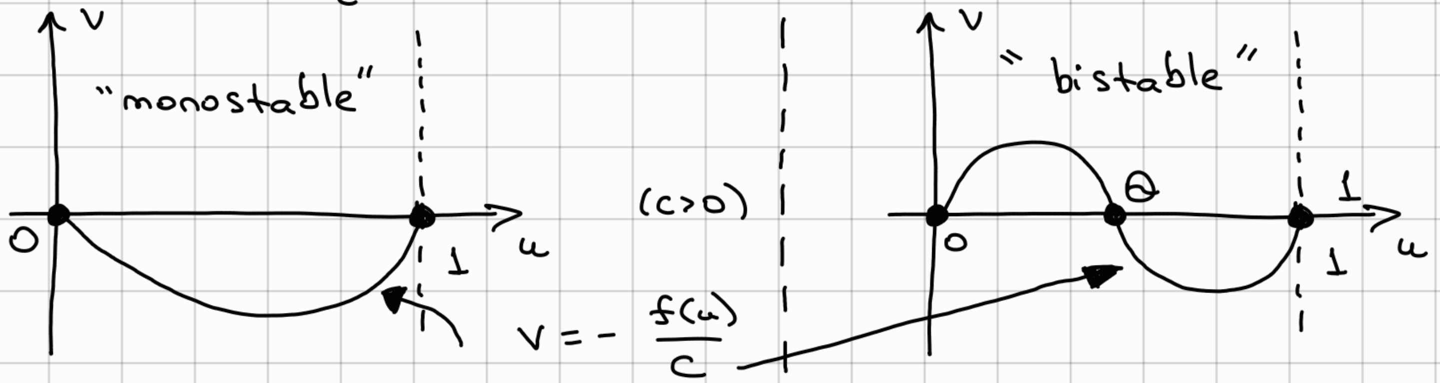
Proof:

There exists 2 proofs: $\nearrow$ "dynamical" (phase plane method) $\searrow$ PDE

Sketch of "dynamical" proof

Write $(TW)_\infty$ as a system of two ODEs of first order: $u'=v$

$$\begin{cases} u' = v \\ v' = -cv - f(u) \end{cases} \qquad \underline{u \in [0,1]}$$



Eq. $(TW)_\infty$ has a solution u $\Leftrightarrow$ $\exists$ heteroclinic orbit $\begin{pmatrix} u(\xi) \\ v(\xi)\end{pmatrix}$ such that $\begin{pmatrix} u(\xi) \\ v(\xi)\end{pmatrix}_{\xi\to-\infty} = \begin{pmatrix} 1 \\ 0\end{pmatrix}$ and $\begin{pmatrix} u(\xi) \\ v(\xi)\end{pmatrix}_{\xi\to+\infty} = \begin{pmatrix} 0 \\ 0\end{pmatrix}$

Step 1: Zoom into vicinicity of fixed point:

$\begin{cases} u=1 \\ v=0\end{cases}$ and $\begin{cases} u=0 \\ v=0\end{cases}$ | $\begin{cases} u=0 \\ v=0\end{cases}$ and $\begin{cases} u=\theta \\ v=0\end{cases}$ and $\begin{cases} u=1 \\ v=0\end{cases}$

Consider a linearized system at equilibrium point $(\alpha,0)$

$$\begin{cases} u' = v \\ v' = -cv - f'(\alpha)\cdot u \end{cases} \qquad \begin{pmatrix} u \\ v\end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -f'(\alpha) & -c\end{pmatrix}\begin{pmatrix} u \\ v\end{pmatrix}$$

Eigenvalues are: $\begin{vmatrix} -\lambda & 1 \\ -f'(\alpha) & -c-\lambda\end{vmatrix} = \lambda^2 + c\lambda + f'(\alpha)$

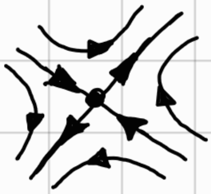
$$\lambda_\pm = \frac{-c \pm \sqrt{c^2-4f'(\alpha)}}{2} \quad ; \qquad r_\pm = \begin{pmatrix} 1 \\ \lambda_\pm\end{pmatrix}$$

$\boxed{\begin{cases} u=1 \\ v=0 \end{cases}}$ $f'(1)>0 \quad \Rightarrow \quad \lambda_{\pm} \in \mathbb{R}$

In particular, $\lambda_+ > 0$ and $\lambda_- < 0$.

This is a saddle point.

Local behavior in the vicinity of $\binom{1}{0}$:



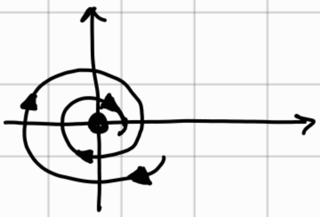
This picture is for the linearized system. By Grobman–Hartman theorem similar picture is true for the original nonlinear system.

Notice that there is exactly one orbit that leaves the point $\binom{1}{0}$, and our goal is to understand for which c it enters $\binom{0}{0}$ without crossing $\{u=0\}$ (we want $u \in [0,1]$)

$\boxed{\begin{cases} u=0 \\ v=0 \end{cases}}$ Local behavior depends on the sign $(c^2 - 4f'(0))$, and is different for monostable and bistable cases

Case I : $\boxed{\text{monostable}}$

- If $0 < c < 2\sqrt{f'(0)}$, then $\lambda_{\pm} \in \mathbb{C} \setminus \mathbb{R}$ and this is a spiral point. This would immediately make $u<0$ at some point along the orbit. This is forbidden as $u \in [0,1]$.

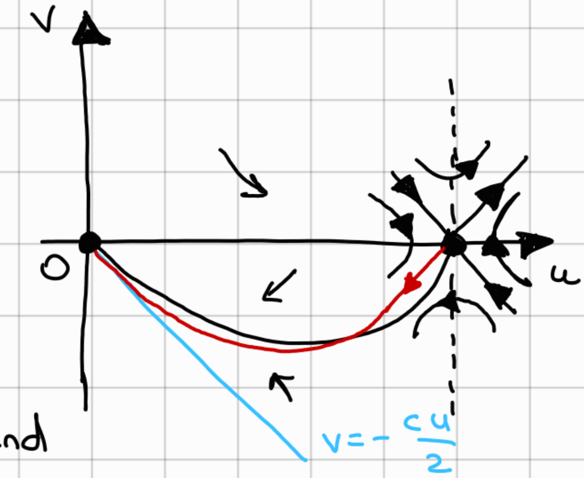


- So $c \geq 2\sqrt{f'(0)}$ (look at the statement for the FKPP case!)
- For $c > 2\sqrt{f'(0)}$ $\lambda_{\pm} < 0$, so $\binom{0}{0}$ is a node

For the FKPP case $f'(0) = \sup\limits_{u \in [0,1]} \frac{f(u)}{u}$.

Lemma: let $c > \sup\limits_{u\in(0,1]} \frac{f(u)}{u}$. Then the orbit $\binom{u}{v}$ s.t. $\binom{u}{v}\Big|_{\xi\to-\infty} = \binom{1}{0}$, does not intersect the line $v = -\frac{c}{2}u$ in the quarter plane $\{v<0\} \cap \{u>0\}$

Rmk: as a consequence we get that this orbit comes to point $\binom{0}{0}$ as $t\to\infty$ without crossing $u=0$.



Proof

1) At $t\to-\infty$ the "red" orbit is above $v=-\frac{cu}{2}$

2) By contradiction:
$\exists\, \xi_0 \in \mathbb{R}$ - the first point of intersection of $v=-\frac{cu}{2}$ and the "red" orbit

At this point we have $\begin{cases} -u''(\xi_0) - cu'(\xi_0) = f(u(\xi_0)) \\ u'(\xi_0) = -\frac{c}{2}u(\xi_0) \end{cases}$

For $c > 2\sqrt{\sup \frac{f(u)}{u}}$, we obtain

$$u''(\xi_0) = -cu'(\xi_0) - f(u(\xi_0)) > -\frac{c\,u'(\xi_0)}{2}$$

This is a contradiction because this means that $\binom{u}{v}$ was already under the line $v = -\frac{cu}{2}$ for $\xi<\xi_0$.

line $\binom{u}{u'} = \begin{pmatrix} \xi \\ -\frac{c\xi}{2} \end{pmatrix} \Rightarrow \binom{u'}{u''} = \begin{pmatrix} 1 \\ -\frac{c}{2} \end{pmatrix}$

$u' = -\frac{cu}{2}$

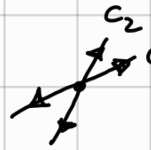
$\frac{d}{d\xi}\binom{u}{u'} = \binom{u'}{u''} = u'\begin{pmatrix} 1 \\ a \\ \vee \\ -\frac{c}{2} \end{pmatrix}$

Thus the angle as $\xi<\xi_0$ is smaller as in the picture.

Thus, in a monostable case there exists at least 1 necessary orbit

In fact, we can say more. There is some monotonicity argument in how trajectories depend on c. Here are 2 observations:

Observation 1 : locally in the vicinity of point $\begin{pmatrix}1\\0\end{pmatrix}$ the trajectory $\begin{pmatrix}u_1\\v_1\end{pmatrix}$ for c_1 is above the trajectory $\begin{pmatrix}u_2\\v_2\end{pmatrix}$ for c_2 if $c_1 > c_2$



Indeed, their tangent vector is $\begin{pmatrix}1\\ \lambda_+\end{pmatrix}$ and $\lambda_+ = \dfrac{-c + \sqrt{c^2 - 4\, f'(1)}}{2}$ is a decreasing function of c.

Observation 2 : in fact, these two trajectories for $c_1 > c_2$ do not intersect in the whole strip $\{u' < 0\} \cap \{0 < u < 1\}$

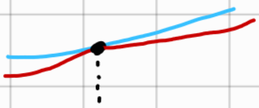
By contradiction, assume they intersect at some point (say, $\xi = 0$)

$$\begin{cases} u_1(0) = u_2(0) \\ u_1'(0) = u_2'(0) \end{cases}$$

This means :

$$u_1''(0) = -c_1 u_1'(0) - f(u_1(0)) > -c_2 u_2'(0) - f(u_2(0)) = u_2''(0)$$

$$\frac{d}{d\xi}\begin{pmatrix}u_1\\u_1'\end{pmatrix}\Bigg|_0 = \begin{pmatrix}u_1'(0)\\u_1''(0)\end{pmatrix}$$



$$\frac{d}{d\xi}\begin{pmatrix}u_2\\u_2'\end{pmatrix}\Bigg|_0 = \begin{pmatrix}u_2'(0)\\u_2''(0)\end{pmatrix}$$

So the intersection can not exist, at most they can "touch".

So this monotonicity argument teaches us, that the set of c such that there exists a front is of the form :

either $[c^*, +\infty)$ or $(c^*, +\infty)$

c^* is necessarily finite by previous lemma

It suffices to prove that for $c = c^*$ there exists a trajectory between $\binom{1}{0} \to \binom{0}{0}$

A continuity argument works:
if for some c the trajectory does not give a front, then it crosses the $\{u=0\}$ - axis. One can show that for $\tilde{c}$ close to $c^{(\tilde{c}>c)}$ the orbit also crosses the $\{u=0\}$ -axis, which will lead to a contradiction. This continuity of an orbit w.r.t. c is non-trivial, but we omit the proof.

This finishes the proof for the general monostable case.

Rmk: Notice that for FKPP case

$$f'(0) = \sup_{u \in (0,1]} \frac{f(u)}{u}, \text{ thus}$$

$$c^* = 2\sqrt{f'(0)}, \text{ and}$$

item (iii) is also proven.

Next time we will prove the theorem for the bistable case, and, may be give a PDE proof of this theorem.

Lecture 20: We want to finish proving theorem:

$$(TW)_\infty \begin{cases} -u'' - cu' = f(u) \\ u(-\infty) = 1,\ u(+\infty) = 0,\quad u'(-\infty) = u'(+\infty) = 0 \end{cases}$$

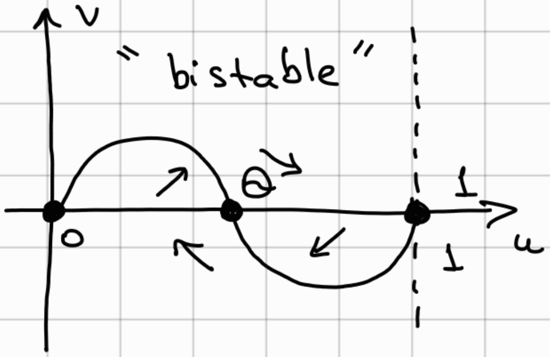
Thm (existence of TW solutions)

(i) In the bistable (and combustion) cases there exists a unique $c \in \mathbb{R}$ for which there exists a solution of $(TW)_\infty$.

Moreover,
- u is unique and decreasing;
- sign of c coincides with the sign of $\int_0^1 f(u)\,du$.

Proof (only sketch)

Let $\int_0^1 f(u)\,du > 0$ (the other case is done similarly)



$u \in [0,1]$

$$\begin{cases} u' = v \\ v' = -cv - f(u) \end{cases}$$

fixed points: $\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}\theta\\0\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}$

$\alpha = \{0, \theta, 1\}$

$$\lambda_\pm = \frac{-c \pm \sqrt{c^2 - 4f'(\alpha)}}{2}; \qquad r_\pm = \begin{pmatrix}1\\ \lambda_\pm\end{pmatrix}$$

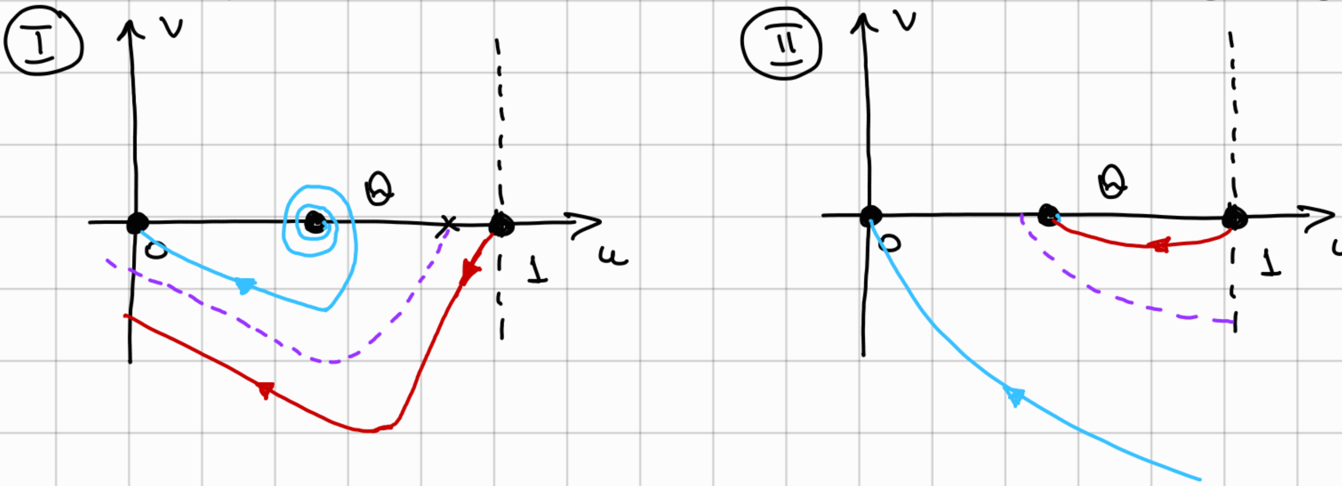
Rmk: $f'(0)$ and $f'(1)$ are negative $\Rightarrow$ $\lambda_\pm \in \mathbb{R}$ and, moreover, $\lambda_+ > 0$, $\lambda_- < 0$ thus both 0 and 1 are saddle points.

So the only way to have an orbit from $\binom{0}{0}$ to $\binom{1}{0}$ is when the unstable manifold (trajectory) from point $\binom{1}{0}$ coincides with the stable manifold of point $\binom{0}{0}$.

It is natural that this is a rare situation (despite the FKPP case where $\binom{0}{0}$ was node and locally all trajectories are attracted by $\binom{0}{0}$).

<u>Idea</u>: find two c s.t. we have:

"blue" is above "red"   "blue" is below "red"



Then by continuity there exists c^* where "blue" and "red" intersect and, thus, coincide.

(I) Take $c \leq 0$. It can be proven that trajectory passing through point $\binom{\gamma}{0}$ will necesserily intersect the axis $u=0$ for $v<0$. This is a natural "barrier" between the "blue" and "red" orbits.
No proof. Moreover, the set of c with such property is open, and by monotonicity is, say $(-\infty, c_1)$.

(II) Take $c \gg 1$. Notice that the restriction of f on $[\theta, 1]$ is of monostable type, for at least for one c_2 the "red" orbit will go $\binom{1}{0} \mapsto \binom{\theta}{0}$ (and as a consequence for all $c > c_2$)

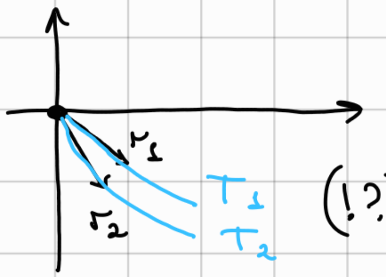
As a result there is a segment $[c_1, c_2]$, for which there exists an orbit from $\binom{1}{0} \mapsto \binom{0}{0}$

It remains to show that $[c_1, c_2]$ consists of 1 point. By contradiction, assume $c_1 < c_2$ such that the unstable "red" trajectory of $\binom{1}{0}$ converges to $\binom{0}{0}$. Let's call them T_1, T_2.

- As before, by monotonicity in c, T_1 is not above T_2.
- On the other hand the tangent vector for $T_{1,2}$ at point $\binom{0}{0}$ is $r_{1,2} = \binom{1}{\lambda_-^{1,2}}$

$$\lambda_-^{1,2} = \frac{-c_{1,2} - \sqrt{c_{1,2}^2 - 4f'(0)}}{2}$$

Notice that $\lambda_-^1 > \lambda_-^2$, which gives a contradiction (see picture below):



(!?) as T_1 is NOT above T_2

$\blacksquare$

"PDE" proof of existence of TW solutions.

Step 1: Let $a \geq 1$ and consider

$$(TW)_a \quad \begin{cases} -u'' - cu' = f(u) & \text{in } (-a, a) \\ u(-a) = 1, \quad u(+a) = 0 \end{cases}$$

Proposition 1: $\forall a, c \quad \exists!\ u = u_{a,c}$, $0 < u < 1$, $u' < 0$.

Proof:

▶ $\exists$

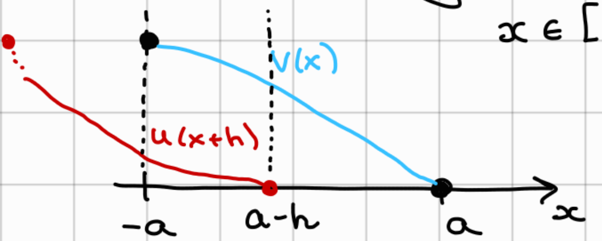
$$\left.\begin{array}{l} u \equiv 0 \ \text{- subsolution} \\ u \equiv 1 \ \text{- supersolution} \end{array}\right\} \Rightarrow \exists \text{ solution } 0 < u < 1$$

(e.g. Perron's method)

(!) Sliding method :
take 2 solutions : u, v of $(TW)_a$

Let's prove that $u \leq v$ and $v \leq u$ (and thus, $u \equiv v$)

$\boxed{u \leq v}$ Notice that $\forall h > 0$ $u(x+h)$ also satisfies the equation $-u'' - cu' = f(u)$, as the eq. is translation invariant.
Consider $u(x+h)$ for $0 < h < 2a$ and h being close to $2a$. Then on the interval $x \in [-a, a-h]$ $u(x+h) \leq v(x)$ as $u(x+h)$ is close to $u(a) = 0$ and $v(x)$ is close to $v(-a) = 1$ (and are continuous in x)



Start decreasing h (that is moving the graph $u(x+h)$ to the right) and consider h_0 s.t:

$$h_0 = \inf \{ h^* \in (0, 2a) : u(x+h) < v(x) \quad \forall x \in [-a, a-h] \quad \forall h \in (h^*, 2a) \}$$

That is the "first moment" when the graphs $u(x+h)$ and $v(x)$ touch, that is

$$\begin{cases} u(x+h_0) \leq v(x) & \forall x \in [-a, a-h] \\ \exists x_0 \in [-a, a-h] : & u(x_0 + h_0) = v(x_0) \end{cases}$$

- If $h_0 = 0$, then $u \leq v$ and this is what we want
- If $h_0 > 0$, then notice that $x_0 \neq -a$ as $u(-a + h_0) < 1 = v(-a)$. Also $x_0 \neq a - h_0$ as $u(a - h_0 + h_0) = u(a) = 0 < v(a - h_0)$.

So $\exists\, x_0 \in (-a, a-h)$: $u(x_0+h_0) = v(x_0)$

But this is a contradiction with the strong maximum principle as $u(x+h)$ is a subsolution and v is a solution, so they can not touch in an interior point of the domain. Thus, h_0 can not be positive.

$\boxed{v \leq u}$ Exchanging the positions of u and v in the previous argument, we get
$v \leq u$.

Thus, we have proven the uniqueness.

$\boxed{u' < 0}$ Let's again use the sliding method, but now for $u(x+h)$ and $u(x)$.
Again for $h \approx 2a$ we have
$$u(x+h) < u(x).$$

Take
$$h_0 = \inf \{ h^* \in (0, 2a) : \quad u(x+h) < u(x) \quad \forall x \in [-a, a-h] \;\; \forall h \in (h^*, 2a) \}$$

- If $h_0 = 0$, then $\forall h \in (0, 2a)$ $u(x+h) < u(x)$, and this gives $u' \leq 0$ (only non-strict inequality)

- If $h_0 > 0$, then by the same argument (strong maximum principle) we get a contradic.

Now let's show that, indeed, $u' < 0$ (strict ineq.)

Differentiate the eq. in $(TW)_a$:
$$-u''' - cu'' = f'(u) \cdot u'$$

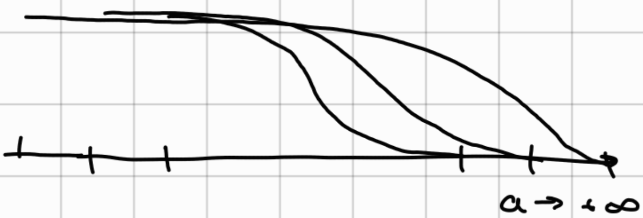
Denote $v = u'$ and consider $f'(u)$ as known function:

Then: $\quad -v'' - cv' = g(\xi) v$

$V_0 \equiv 0$ is a solution of $\longrightarrow$ and $v = u' \leq 0$ is also a solution. By strong maximum principle, either $v \equiv 0$ or $v < 0$.

L As $u \not\equiv 0 \Rightarrow v \not\equiv 0 \Rightarrow v = u' < 0$. ■

We want now to fix c and take limit $a \to +\infty$. But the theorem says that only for some special c there exist a solution. Why this is happening? For many choices of c the solution will "run away" and in "almost all" points converge to 1 or 0:



or

So in the limit you get zero information as the solution converges to 1 or 0 (steady states that we already know)

"Pinning": let's restrict ourselves only to such solutions that have a prescribed value at 0:

Proposition 2: $\exists!\, c$ s.t. the corresponding u satisfies an extra condition $u_{a,c}(0) = \theta$, where θ is:

- bistable case: the unstable equil. $\theta \in (0,1)$
- ignition case: $\sup\{u<1 : f(u) = 0\}$
- monostable case: $\forall\, \theta \in (0,1)$

Proof

▶ For a moment assume no condition $u_{a,c}(0) = \theta$

Consider a mapping : $c \mapsto u_c$

It is decreasing and continuous.

Why decreasing ?

- Take a solution u for some value c_1

Then it is a supersolution for $c_2 > c_1$

(due to sign $u' < 0$)

$$-u'' \underbrace{-u'}_{>0} c_2 - f(u) > -u'' - u' c_1 - f(u) = 0$$

$$\Rightarrow \quad u_{c_2} < u_{c_1}$$

exercise
$$\begin{cases} \text{As } c \to +\infty & u_c(x) \to 0 \quad \text{in } (-a, a] \\ \text{As } c \to -\infty & u_c(x) \to 1 \quad \text{in } [-a, a) \end{cases}$$

All the above gives the unique c : $u_{a,c}(0) = \theta$

Let's prove an apriori bound on c from Prop.2 (to be able to get limit of c when $a \to \infty$)

Lemma : Let $m = \sup\limits_{s \in (0,1]} \frac{f(s)}{s}$.

$$\forall \delta > 0 \quad \exists A > 0 \quad \text{s.t.} \quad \forall a \geq A \quad c \leq 2\sqrt{m} + \delta .$$

Proof:

▶ Consider a problem:

$$(Z) \quad \begin{cases} -z'' - cz' - mz = 0 & \text{in } (-a, a) \\ z(-a) = 1, \quad z(a) = 0 \end{cases}$$

The solution u of $(TW)_a$ is a sub-solution of (Z)

$$mu = \sup_{v \in (0,1]} \frac{f(v)}{v} \cdot u \geq \frac{f(u)}{u} \cdot u = f(u)$$

$$-mu \leq -f(u)$$

Claim: the operator $\mathcal{L} = -\partial^2_{xx} - c\partial_x - m$ satisfies the maximum principle (MP) in $(-a,a)$ for $c > 2\sqrt{m}$ provided a is large enough.

(no proof for a moment)

Assume by contradiction that $c > 2\sqrt{m} + \delta$

Then by claim the operator $\mathcal{L} = -\partial^2_{xx} - c\partial_x - m$ satisfies the maximum principle, thus, for $w = u - z$ we have $Lw \leq 0$ and $w(-a) = w(a) = 0$

$\Rightarrow$ $w \leq 0$ $\Rightarrow$ $u \leq z$ for a large enough.

But we can find z explicitly.

Indeed,

$$z(x) = \frac{e^{r_+(x-a)} - e^{r_-(x-a)}}{e^{-2r_+ a} - e^{-2r_- a}}, \text{ where}$$

r_+, r_- are the 2 real roots of:

$$r^2 + cr + m = 0$$

Notice that $z(0) = \dfrac{1}{e^{-r_+ a} + e^{-r_- a}} \xrightarrow[a \to +\infty]{} 0$

and thus, $u(0) \to 0$, which is a contradiction with "pinning" condition $u(0) = \theta$.

$\blacksquare$

Rmk: one can bound c from below:

consider $v(x) = 1 - u(-x)$

$$\begin{cases} -v'' + cv' = -f(1-v) \\ v(-a) = 1, \; v(a) = 0 \end{cases}$$

$$\Rightarrow \quad -c \leq 2\sqrt{m'} + \delta \quad \text{where } m' = \sup_{s\in(0,1]} \left(-\frac{f(1-s)}{s}\right)$$

$$c \geq -2\sqrt{m'} - \delta$$

So if c is too negative, then we $u(0)$ will go to 1 and can not satisfy the "pinning" condition $u(0) = \theta$.

Step 2:

So we can pass to the limit $a \to +\infty$ and there exists a convergent subsequence $c_a \to c$, $u_a \to u$.

If u_a' and u_a'' are bounded then by Arzela-Ascoli theorem we can take a convergent subseq. and pass to the limit in the eq.

$$(*) \quad \begin{cases} -u'' - cu' = f(u) & \text{in } \mathbb{R}, \ u \in [0,1] \\ u(0) = \theta & u' \leq 0 \end{cases}$$

$\boxed{\text{Monostable case}}$ We have shown that there exists at least 1 solution of $(*)$. Also we know that $u' \leq 0$ and for $\xi \leq 0$ $u \in [\theta, 1]$, so there exists a limit $u(-\infty) = u_0$

Also, $\left.\begin{array}{l} u'(-\infty) = 0 \\ u''(-\infty) = 0 \end{array}\right) \Rightarrow u_0 : \ f(u_0) = 0$

This means that $u_0 = 1$.

Analogously, $u(+\infty) = 0$.

$\boxed{\text{Bistable case}}$ The same reasoning does not work for the bistable case as there could happen: 

that $u \equiv \theta$.

Lecture 21: We finish "PDE" proof for existence of TW solutions for reaction-diffusion eq.

$$u_t = \Delta u + f(u)$$

and formulate the invasion / extinction criteria for monostable / bistable nonlinear.

- But first let's prove a version of (MP) that we left without proof in the previous class.

Lemma: Let $\mathcal{L} = -\frac{d^2}{dx^2} - c\frac{d}{dx} - m$ on $(-a,a)$.
Here $c, m \in \mathbb{R}$. Assume $\boxed{c > 2\sqrt{m}}$.
If
$$\begin{cases} \mathcal{L}z \leq 0 \\ z(a) \leq 0 \\ z(-a) \leq 0 \end{cases} \Rightarrow z(x) \leq 0 \quad \forall x \in (-a,a)$$

Proof:

▶ Trick (Liouville transform): $\mathcal{L}z = 0$.
consider $z = e^{-c/2\, x}\varphi$
(this should kill the first order term in $\mathcal{L}$). Indeed,

$$\partial_x z = -\frac{c}{2} e^{-\frac{c}{2}x}\varphi(x) + e^{-c/2\, x}\varphi'(x)$$
$$\partial^2_{xx} z = \frac{c^2}{4} e^{-c/2\, x}\varphi - c e^{-c/2\, x}\varphi' + e^{-c/2\, x}\varphi''$$

$$\Rightarrow \mathcal{L}z = -\frac{c^2}{4} e^{-c/2\, x}\varphi + \cancel{c e^{-c/2\, x}\varphi'} - e^{-c/2\, x}\varphi''$$
$$+ \frac{c^2}{2} e^{-c/2\, x}\varphi - \cancel{c e^{-c/2\, x}\varphi'} - m e^{-c/2\, x}\varphi =$$
$$= e^{-c/2\, x}\cdot\left[-\varphi'' + \varphi\left(\frac{c^2}{4} - m\right)\right]$$

Notice that sign (z) = sign (φ).

If $\exists\, x_0 \in (-a,a)$: $\varphi(x_0) > 0$ (w.l.o.g. x_0 is argmax of φ), then $\varphi''(x_0) \le 0$ and we have

$$-\varphi'' + \underbrace{\left(\frac{c^2}{4} - m\right)}_{>0} \underset{>0}{\varphi}\Big|_{x_0} > 0 \quad (!?) \quad Lz \le 0.$$

$$\Rightarrow \varphi \le 0 \Rightarrow z \le 0 \quad \blacksquare$$

- Travelling wave solutions satisfy the equation: $(TW)_\infty \begin{cases} -u'' - cu' = f(u) \\ u(-\infty) = 1,\ u(+\infty) = 0 \end{cases}$

$\boxed{\text{Monostable case}}$: we have shown that $\exists \lim\limits_{a\to+\infty} c_a = c$ such that $\exists$ solution of $(TW)_\infty$ with this c. Let's show that the solution of $(TW)_\infty$ $\exists$ for $[c, +\infty)$.

The following lemma is true only for monostable case (we use the fact that $f(u) > 0$, $u \in (0,1)$)

Lemma : If $\exists$ solution of $(TW)_\infty$ for c, then $\forall\, c_1 \ge c$ there also exists a solution of $(TW)_\infty$.

Proof :

Let u_c be a solution with c, then u_c is a supersolution for $c_1 > c$ and $u_c' < 0$. So is $u_c(\cdot + r)$, $r \in \mathbb{R}$

Introduce a finite-domain approximation:

$$\begin{cases} -v'' - c_1 v' = f(v) & \text{in } (-a,a) \\ v(-a) = u_c(-a+r) \\ v(+a) = u_c(a+r) \end{cases}$$

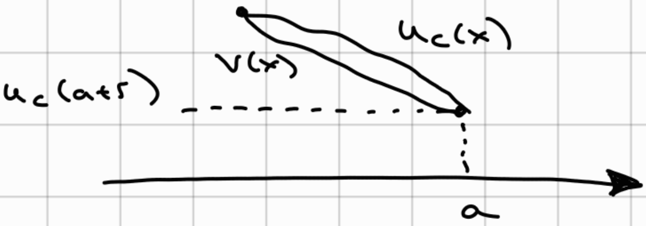
$u_c(\cdot + r)$ is a supersolution

$u(a+r)$ is a subsolution (it is constant)

Here we use $f(a) > 0$ $\forall a \in (0,1)$

$\Rightarrow \exists$ a solution $v(x)$:

$$u_c(a+r) < v(x) < u_c(x+r)$$

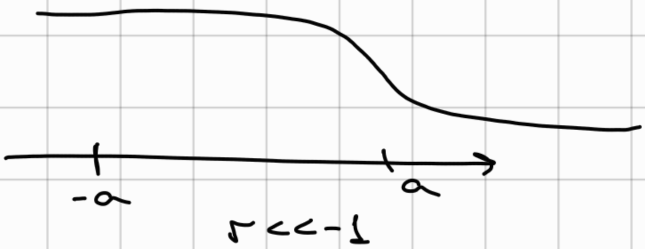


$$\Rightarrow \underbrace{\overset{=}{v(a)}\; u_c(a+r) < v(x) < u_c(-a+r)\; \overset{=}{v(-a)}}$$

Actually, the sliding method works! and only needs

$\Rightarrow$ v is unique and decreasing

By the same argument as before
$\exists!\, r$ s.t. $v(0) = \theta$



By continuity there exists r s.t. $v(0) = \theta$
Again tending $a \to \infty$ we get a limit
$v_a \to v$ and get a solution for c_1. ∎

<u>Rmk</u> : the set of c for which there exists a solution of $(TW)_\infty$ is closed. Indeed, if we have a sequence of solutions (c_n, u_n) with $c_n \to c$ w.l.o.g. $u_n(0) = \theta$ so we can pass to the limit and get a solution of $(TW)_\infty$ with c.

$\boxed{\text{Bistable case}}$ We pass to the limit as
- c_a is bounded
- $u_a(0) = \theta$
- $u_a' \leq 0$

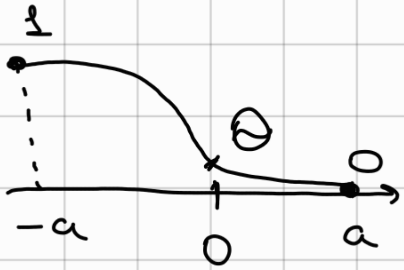
$\Rightarrow \quad c_a \to c \,,\ u_a \to u \quad : \quad \begin{cases} -u'' - cu' = f(u) \\ u(0) = \theta \end{cases}$

So we need to show that $u \not\equiv \theta$.
It suffices to show that: $u_a'(0) \not\to 0$
Then $u'(0) < 0$, then the problem is reduced to "2 monostable" cases and $u(-\infty) = 1$ and $u(+\infty) = 0$.

Let's show that $u_a'(0) \underset{a\to\infty}{\not\to} 0$.

Lemma : $\displaystyle\int_{-a}^{0} f(u_a(x))\,dx \geq \delta > 0 \quad \forall a \geq 1.$

Proof :



Consider $\displaystyle\int_{-a}^{0} f(u_a)\,u_a'\,dx = F(\theta) - F(1)$ where $F(z) = \displaystyle\int_0^z f(u)\,du$

As $\|u_a'\|_{L^\infty} \leq K$ and $f(u) > 0$, $u \in (\theta, 1)$

So $\delta' \leq \left|\displaystyle\int_{-a}^{0} f(u_a)\,u_a'\right| \leq \displaystyle\int_{-a}^{0} f(u)\,du \cdot \|u_a'\|_\infty$

$\Rightarrow \displaystyle\int_{-a}^{0} f(u)\,du \geq \frac{\delta'}{\|u_a'\|_\infty} \geq \frac{\delta'}{K}$ ■

$-u'' - cu' = f(u)$

Integrate this $\displaystyle\int_{-a}^{0}$: $\quad -u'\Big|_{-a}^{0} - cu\Big|_{-a}^{0} = \displaystyle\int_{-a}^{0} f(u)\,dx \geq \delta$

$$-u'(0) + \underbrace{u'(-a)}_{\leq 0} - c\Big[\underset{\theta}{\underset{\shortparallel}{u(0)}} - \underset{1}{\underset{\shortparallel}{u(-a)}}\Big] \geq \delta$$

$$\Rightarrow \quad \boxed{c(1-\theta) - u'(0) \geq \delta}$$

We used the path from $-a$ to 0.

Let's use the other path from 0 to a:

$-u'' - cu = f(u)$

Integrate this $\int_0^a$: $\quad -u'\Big|_0^a - cu\Big|_0^a = \underbrace{\int_0^a f(u(x))\,dx}_{\geq 0}$

$-u'(a) + u'(0) + c\theta \leq 0$

$$u'(0) \leq -c\theta + \underbrace{u'(a)}_{\leq 0} \leq -c\theta$$

Thus, $\boxed{u'(0) \leq -c\theta}$

Combining $\begin{cases} u'(0) \leq -\delta - c(1-\theta) \\ u'(0) \leq -c\theta . \end{cases}$

When $|c| < \dfrac{\delta}{(1-\theta)2}$, then $u'(0) \leq -\dfrac{\delta}{2}$
(c small)

Otherwise $u'(0) \leq -\dfrac{\delta\theta}{(1-\theta)2}$ again strictly negative

- Uniqueness of c^* for bistable case is a consequence of a sliding method (exercise)

Invasion, extinction and asymptotic speed of propagation

$$(*) \begin{cases} u_t = \Delta u + f(u) & \text{in } \mathbb{R}^N \\ u(0,x) = u_0(x), & u_0 \not\equiv 0, \ \underbrace{0 \le u_0 < 1}_{\text{for simplicity}} \end{cases}$$

Thm 1 (invasion for FKPP case)

Assume that $\lim_{s \to 0^+} \frac{f(s)}{s^{1+\frac{2}{N}}} > 0$ (C1)

Then $\forall u_0(x)$ we have $u(t,x) \to 1$ as $t \to \infty$

Rmk 1: sometimes this is called "hair-trigger effect" – even small amount of species will invade everything (under the cond. (C1)). Cond. (C1) is sharp – there are counterexamples when (C1) is not true.

Thm 2 (extinction and invasion for bistable)

(i) (extinction) $\exists \delta > 0$ s.t. if $\int_{\mathbb{R}^N} (u_0 - \theta) < \delta$, then

$$u(t,x) \to 0 \quad \text{as } t \to \infty \quad \forall x \in \mathbb{R}^N$$

(ii) (invasion) $\exists \eta > 0, R > 0$ s.t. if $u_0 \ge \theta + \eta$ on $\overline{B}_R$,

then $u(t,x) \to 1$ as $t \to \infty$ $\forall x \in \mathbb{R}^N$

Rmk 1: if there are not too many species then you have extinction, but if you have enough species on a big enough domain, you will have invasion.

Rmk 2: simpler version of (i): in $u_0 < \theta - \eta$ then $u \to 0$ (straighforward)

Rmk 3: Take $u_0 = \mathbb{1}_{B_R}$ for bistable case:

R small - extinction
R large - invasion

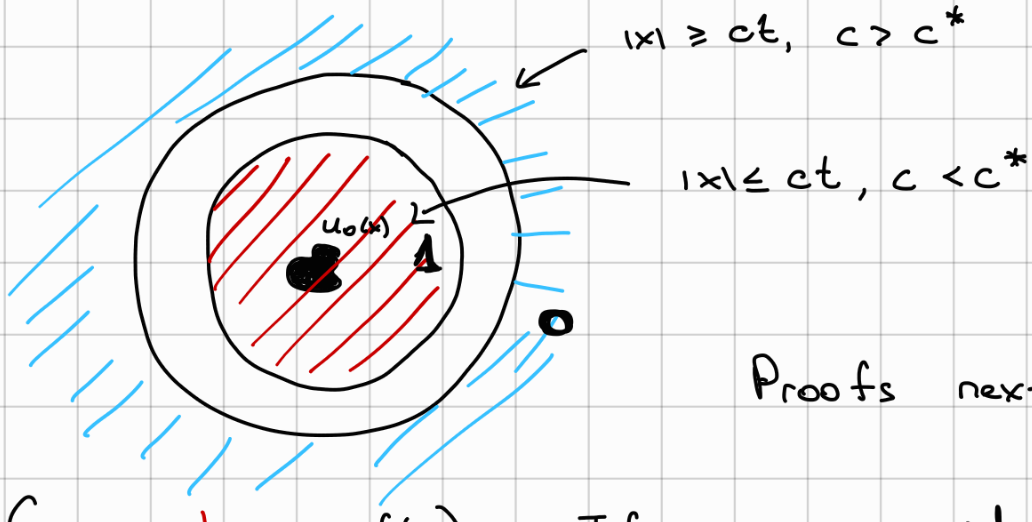
There is a threshold result: $\exists R^*$:
$\forall R < R^*$ extinction and $R > R^*$ invasion.
[Zlatoš' 2006 - 1-dim ; Du & Matano' 2010 - N-dim]

Thm 3 (Principle of asymptotic speed of propagation)
Assume that u_0 has compact support and that there is invasion. Then,

(1) $\forall c > c^*$, $\lim_{t\to\infty} \left\{ \sup_{|x| \geq ct} u(t,x) \right\} = 0$

(2) $\forall c < c^*$ $\lim_{t\to\infty} \left\{ \sup_{|x| \leq ct} |1 - u(t,x)| \right\} = 0$

Rmk: c^* - minimum speed of TW for monostable
c^* - is the unique speed of TW for bistable case



Proofs next time.

Rmk: $\begin{cases} u_t = d\Delta u + f(u), \\ u(0,x) = u_0(x). \end{cases}$ If one considers the eq. with diffusion coef. d then for Fisher-KPP case
$c^* = 2\sqrt{d f'(0)}$. [change of variable $\sqrt{d}\cdot x$]