On the linear growth of the mixing zone in a semi-discrete model of Incompressible Porous Medium eq.

Yulia Petrova

PUC-Rio, Pontifical Catholic University of Rio de Janeiro
Department of Mathematics

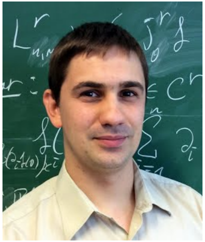
https://yulia-petrova.github.io/

Alma mater:
St Petersburg State University, Russia

Oberwolfach
"Hyperbolic Balance Laws:
Interplay between Scales and Randomness"

26 February – 1 March 2024

Joint work with



Sergey Tikhomirov

(PUC-Rio, Brazil)

Yalchin Efendiev

(Texas A&M, USA)

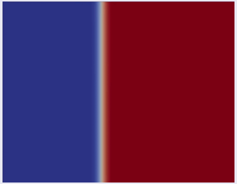
The talk is based on:

- Y. Petrova, S. Tikhomirov, Ya. Efendiev "Propagating terrace in a two-tubes model of gravitational fingering", 2024, arXiv: 2401.05981.
- Y. Efendiev, Y. Petrova, S. Tikhomirov "Transversally Reduced Fingering Model". Work in progress.

Multiphase flow in porous media

1-dim in spatial variable

- Stable displacement



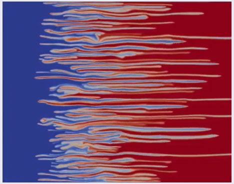
- main question: find an exact solution to a Riemann problem
- system of non-strictly hyperbolic CL

$$
\begin{aligned}
s_t + \quad f(s,c)_x &= 0,\\
(cs)_t + (cf(s,c))_x &= 0.
\end{aligned}
$$

Polymer model

2-dim (or 3-dim) in spatial variable

- Unstable displacement



- source of instability: water and oil/polymer have different viscosities
- viscous fingering phenomenon

$$
c_t + u \cdot \nabla c = \varepsilon \Delta c,
$$

$$
\mathrm{div}(u) = 0, \quad u = -k \cdot m(c) \nabla p.
$$

Incompressible porous media eq

2-dim miscible flow in porous media

1. Transport of species ($\varepsilon = \frac{1}{\mathrm{Pe}} \geq 0$)

$$c_t + u \cdot \nabla c = \varepsilon \Delta c$$

2. Incompressibility condition

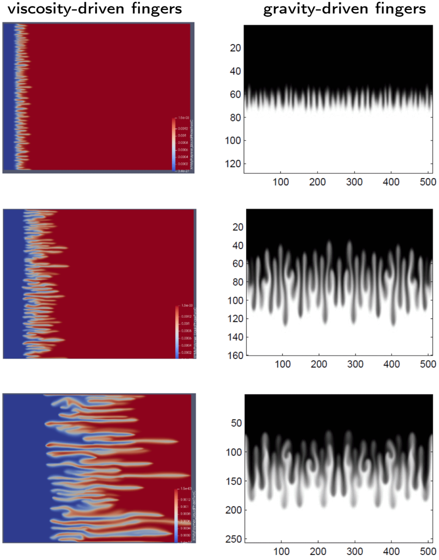
$$\mathrm{div}(u) = 0$$

3a. Darcy's law (viscosity-driven)

$$u = -k \cdot m(c) \cdot \nabla p$$

3b. Darcy's law (gravity-driven)

$$u = -\nabla p - (0, c)$$



Initial data:
unstable stratification (gravity)

$$c\Big|_{t=0} = c_0(t, y) = \begin{cases} +1, & y \geq 0, \\ -1, & y \leq 0. \end{cases}$$

NB: 1 + 2+ 3b for $\varepsilon = 0$ is known as IPM (incompressible porous media equation)

Well-posedness ($\varepsilon = 0$)

Active scalar, e.g. for gravity-driven:

$$c_t + u \cdot \nabla c = 0, \qquad \text{(transport eq)}$$

$$u = \nabla^{\perp}(-\Delta)^{-1}\partial_1 c \qquad \text{(Biot-Savart law)}$$

Well-posedness ($\varepsilon = 0$)

Active scalar, e.g. for gravity-driven:

$$c_t + u \cdot \nabla c = 0, \qquad \text{(transport eq)}$$
$$u = \nabla^{\perp}(-\Delta)^{-1}\partial_1 c \qquad \text{(Biot-Savart law)}$$

- local well-posedness for smooth initial data (H^s): yes
  2007 — D. Cordoba, F. Gancedo, R. Orive (JMP): in $\mathbb{R}^2$

Well-posedness ($\varepsilon = 0$)

Active scalar, e.g. for gravity-driven:

$$c_t + u \cdot \nabla c = 0, \qquad \text{(transport eq)}$$

$$u = \nabla^{\perp}(-\Delta)^{-1}\partial_1 c \qquad \text{(Biot-Savart law)}$$

- local well-posedness for smooth initial data (H^s): yes
  2007 — D. Cordoba, F. Gancedo, R. Orive (JMP): in $\mathbb{R}^2$
- global regularity vs finite-time blow up: open question
  2023 — A. Kiselev, Y. Yao (ARMA): for some initial data it is shown that
  if there exists a smooth global solution exists, then Sobolev norms blow up at ∞

Well-posedness ($\varepsilon = 0$)

Active scalar, e.g. for gravity-driven:

$$c_t + u \cdot \nabla c = 0, \qquad \text{(transport eq)}$$
$$u = \nabla^{\perp}(-\Delta)^{-1}\partial_1 c \qquad \text{(Biot-Savart law)}$$

- local well-posedness for smooth initial data (H^s): yes
  2007 — D. Cordoba, F. Gancedo, R. Orive (JMP): in $\mathbb{R}^2$
- global regularity vs finite-time blow up: open question
  2023 — A. Kiselev, Y. Yao (ARMA): for some initial data it is shown that
  if there exists a smooth global solution exists, then Sobolev norms blow up at ∞
- non-uniqueness of weak solutions:
  2011 — D. Córdoba, D. Faraco, F. Gancedo (ARMA): convex integration
  2012 — L. Szekelyhidi Jr.

Well-posedness ($\varepsilon = 0$)

Active scalar, e.g. for gravity-driven:

$$c_t + u \cdot \nabla c = 0, \qquad \text{(transport eq)}$$
$$u = \nabla^{\perp}(-\Delta)^{-1}\partial_1 c \qquad \text{(Biot-Savart law)}$$

- local well-posedness for smooth initial data (H^s): yes
  2007 — D. Cordoba, F. Gancedo, R. Orive (JMP): in $\mathbb{R}^2$
- global regularity vs finite-time blow up: open question
  2023 — A. Kiselev, Y. Yao (ARMA): for some initial data it is shown that
  if there exists a smooth global solution exists, then Sobolev norms blow up at ∞
- non-uniqueness of weak solutions:
  2011 — D. Córdoba, D. Faraco, F. Gancedo (ARMA): convex integration
  2012 — L. Szekelyhidi Jr.
- stability of the stratified steady states:
  2017 — T. Elgindi (ARMA): small perturbations (in H^s, $s > 20$) are stable
  2019 — A. Castro, D. Cordoba, D. Lear (ARMA)
  2024 — R. Bianchini, T. Crin-Barat, M. Paicu (ARMA)

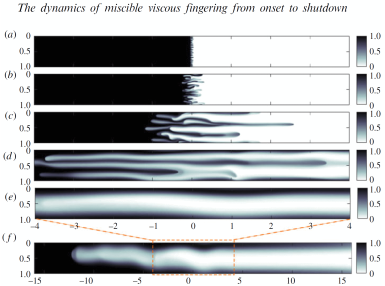
Growth of the mixing zone ($\varepsilon > 0$)

Length of the mixing zone:

$$a(t) = x\big|_{c=0.99} - x\big|_{c=-0.99}$$

Three regimes:

1. an early-time, linearly unstable regime: mixing zone grows diffusively
2. an intermediate-time nonlinear regime: mixing zone grows linearly (independent of $\varepsilon = \frac{1}{Pe}$)
3. a late time, single-finger exchange-flow regime



Nijjer J., Hewitt D., Neufeld J.
The dynamics of miscible viscous fingering from onset to shutdown. JFM.

Growth of the mixing zone ($\varepsilon > 0$)

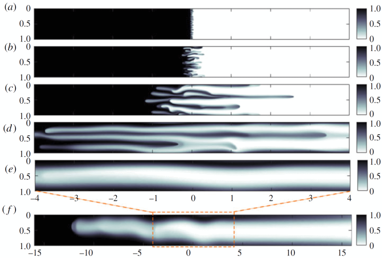
Length of the mixing zone:

$$a(t) = x\big|_{c=0.99} - x\big|_{c=-0.99}$$

Three regimes:

1. an early-time, linearly unstable regime: mixing zone grows diffusively
2. an intermediate-time nonlinear regime: mixing zone grows linearly (independent of $\varepsilon = \frac{1}{Pe}$)
3. a late time, single-finger exchange-flow regime

The dynamics of miscible viscous fingering from onset to shutdown



Nijjer J., Hewitt D., Neufeld J. The dynamics of miscible viscous fingering from onset to shutdown. JFM.

Aim

Find exact speed of propagation: $a(t) \sim \text{const} \cdot t$

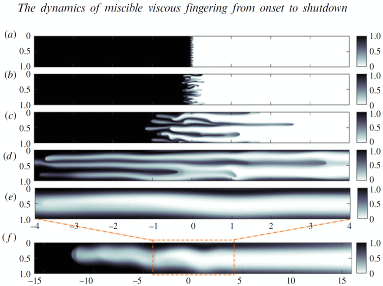
Growth of the mixing zone ($\varepsilon > 0$)

Length of the mixing zone:

$$a(t) = x\big|_{c=0.99} - x\big|_{c=-0.99}$$

Three regimes:

1. an early-time, linearly unstable regime: mixing zone grows diffusively
2. an intermediate-time nonlinear regime: mixing zone grows linearly (independent of $\varepsilon = \frac{1}{Pe}$)
3. a late time, single-finger exchange-flow regime



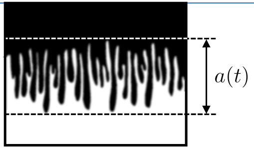
Nijjer J., Hewitt D., Neufeld J.
The dynamics of miscible viscous fingering from onset to shutdown. JFM.

Aim

Find exact speed of propagation: $a(t) \sim \text{const} \cdot t$

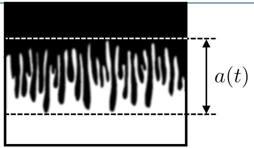
Applications: F. Bakharev, A. Enin, K. Kalinin, Y. Petrova, N. Rastegaev, S. Tikhomirov, "Optimal polymer slugs injection profiles", 2023, JCAM & Patent "Method for chemical flooding of enhanced oil recovery" (optimization of graded viscosity banks technology), 2022.

Theoretical justification of linear growth of mixing zone



2006 — F. Otto, G. Menon
"Dynamic scaling in miscible viscous fingering", CMP

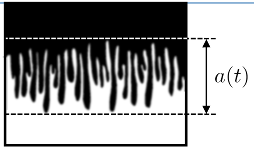
Theoretical justification of linear growth of mixing zone



2006 — F. Otto, G. Menon
"Dynamic scaling in miscible viscous fingering", CMP

- Energy estimates for IPM:

Theoretical justification of linear growth of mixing zone

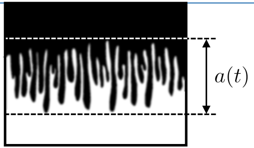


2006 — F. Otto, G. Menon
"Dynamic scaling in miscible viscous fingering", CMP

- Energy estimates for IPM:
  average in transverse direction

$$\bar{c}(t,y) = \fint c(t,x,y)\,dx$$

Theoretical justification of linear growth of mixing zone



2006 — F. Otto, G. Menon
"Dynamic scaling in miscible viscous fingering", CMP

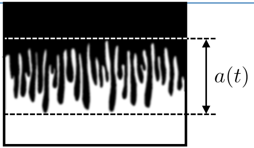
- Energy estimates for IPM:
  average in transverse direction

$$\bar{c}(t,y) = \fint c(t,x,y)\,dx$$

gravitational potential energy

$$E(t) = \int_{\mathbb{R}} y \cdot (c_0 - \bar{c})(t,y)\,dy \qquad \text{satisfies} \qquad \limsup_{t\to\infty} \frac{E(t)}{t^2} \le \frac{1}{6}.$$

Theoretical justification of linear growth of mixing zone



2006 — F. Otto, G. Menon
"Dynamic scaling in miscible viscous fingering", CMP

- Energy estimates for IPM:
  average in transverse direction

$$\bar{c}(t,y) = ⨍ c(t,x,y)\,dx$$

gravitational potential energy

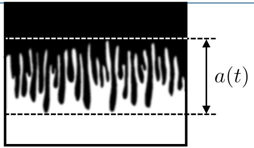
$$E(t) = \int_{\mathbb{R}} y \cdot (c_0 - \bar{c})(t,y)\,dy \qquad \text{satisfies} \qquad \limsup_{t\to\infty} \frac{E(t)}{t^2} \le \frac{1}{6}.$$

- Pointwise estimates for simplified model of Darcy's law (TFE: Wooding, 1969)

Transverse flow equilibrium (TFE)

$$\partial_t c + u \cdot \nabla c = \Delta c$$
$$\operatorname{div}(u) = 0$$
$$u = \bar{c} - c$$

Theoretical justification of linear growth of mixing zone



2006 — F. Otto, G. Menon
"Dynamic scaling in miscible viscous fingering", CMP

- Energy estimates for IPM:
  average in transverse direction

$$\bar{c}(t,y) = ⨍ c(t,x,y)\,dx$$

  gravitational potential energy

$$E(t) = \int_{\mathbb{R}} y \cdot (c_0 - \bar{c})(t,y)\,dy \qquad \text{satisfies} \qquad \limsup_{t\to\infty} \frac{E(t)}{t^2} \le \frac{1}{6}.$$

- Pointwise estimates for simplified model of Darcy's law (TFE: Wooding, 1969)

Transverse flow equilibrium (TFE)

$$\partial_t c + u \cdot \nabla c = \Delta c$$
$$\mathrm{div}(u) = 0$$
$$u = \bar{c} - c$$

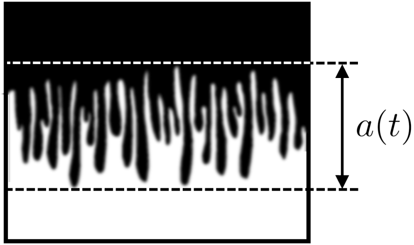
Comparison with 1-dim Burgers: $c^{\max}(t,y)$

$$\partial_t c^{\max} + (1 - c^{\max})\partial_y c^{\max} = \partial_{yy} c^{\max}$$

If $\quad c(0,x,y) \le c^{\max}(0,y),$

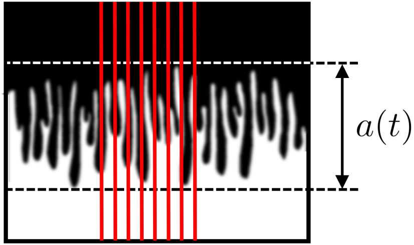
then $\quad c(t,x,y) \le c^{\max}(t,y), \qquad t > 0.$

Idea (arXiv:2401.05981, P., Tikhomirov, Efendiev)



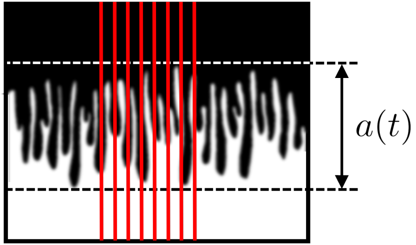
Idea (arXiv:2401.05981, P., Tikhomirov, Efendiev)

- Discretize in transversal direction (horizontal) — “multitubes” model



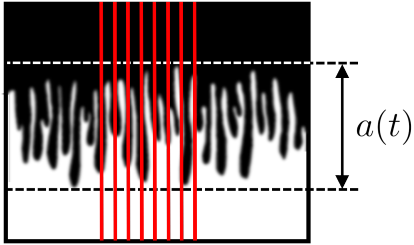
Idea (arXiv:2401.05981, P., Tikhomirov, Efendiev)

- Discretize in transversal direction (horizontal) — "multitubes" model
- Include explicitly the interflow between the tubes.
  Does it affect the speed of fingers?



Idea (arXiv:2401.05981, P., Tikhomirov, Efendiev)

- Discretize in transversal direction (horizontal) — "multitubes" model
- Include explicitly the interflow between the tubes.
  Does it affect the speed of fingers?



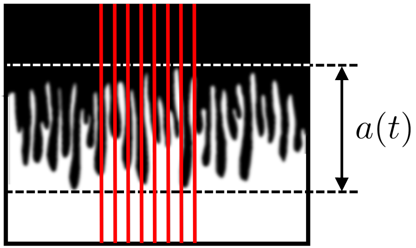
Multilayer / multilane models:

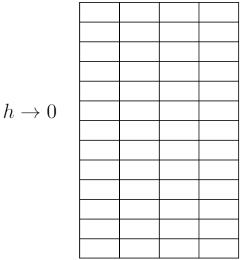
2006 — J.C. Da Mota, S. Schecter "Combustion fronts in a porous medium with two layers", JDDE

2019 — H. Holden, N. Risebro "Models for dense multilane vehicular traffic", SIMA

Idea (arXiv:2401.05981, P., Tikhomirov, Efendiev)

- Discretize in transversal direction (horizontal) — "multitubes" model
- Include explicitly the interflow between the tubes.
  Does it affect the speed of fingers?





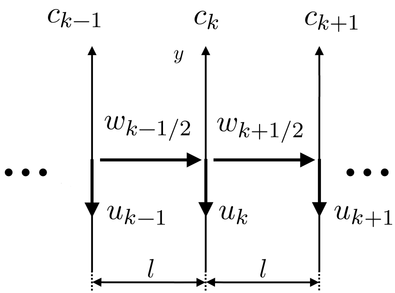
Multilayer / multilane models:

2006 — J.C. Da Mota, S. Schecter "Combustion fronts in a porous medium with two layers", JDDE

2019 — H. Holden, N. Risebro "Models for dense multilane vehicular traffic", SIMA

Multi-tubes model of IPM

Take $n \geq 2$ tubes ($\mathbb{R}$), connected cyclically. As a limit of finite-volume scheme, we get:

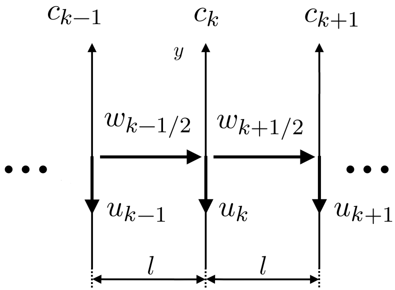


Multi-tubes model of IPM

Take $n \geq 2$ tubes ($\mathbb{R}$), connected cyclically. As a limit of finite-volume scheme, we get:

1. Transport equation in k-th tube, $k = 1, \dots, n$:

$$\partial_t c_k + \partial_y (u_k c_k) = f_{k-1/2} - f_{k+1/2}$$



Multi-tubes model of IPM

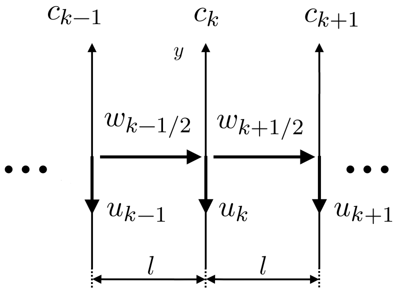
Take $n \geq 2$ tubes ($\mathbb{R}$), connected cyclically. As a limit of finite-volume scheme, we get:

1. Transport equation in k-th tube, $k = 1, \ldots, n$:

$$\partial_t c_k + \partial_y (u_k c_k) = f_{k-1/2} - f_{k+1/2}$$

2. Incompressibility condition:

$$l \cdot \partial_y u_k - w_{k-1/2} + w_{k+1/2} = 0$$



Multi-tubes model of IPM

Take $n \geq 2$ tubes ($\mathbb{R}$), connected cyclically. As a limit of finite-volume scheme, we get:

1. Transport equation in k-th tube, $k = 1, \dots, n$:

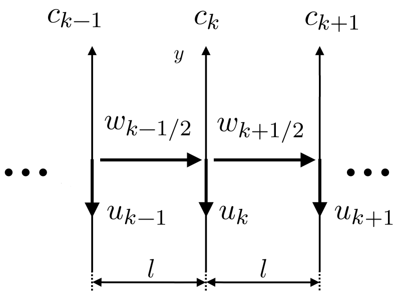
$$\partial_t c_k + \partial_y (u_k c_k) = f_{k-1/2} - f_{k+1/2}$$

2. Incompressibility condition:

$$l \cdot \partial_y u_k - w_{k-1/2} + w_{k+1/2} = 0$$

3. Darcy's law in k-th tube and between tubes:

$$u_k = -\partial_y p_k - c_k,$$
$$w_{k+1/2} = (p_{k+1} - p_k)/l.$$



Multi-tubes model of IPM

Take $n \geq 2$ tubes ($\mathbb{R}$), connected cyclically. As a limit of finite-volume scheme, we get:

1. Transport equation in k-th tube, $k = 1, \ldots, n$:

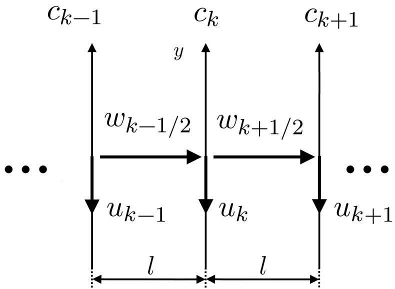
$$\partial_t c_k + \partial_y (u_k c_k) = f_{k-1/2} - f_{k+1/2}$$

2. Incompressibility condition:

$$l \cdot \partial_y u_k - w_{k-1/2} + w_{k+1/2} = 0$$

3. Darcy's law in k-th tube and between tubes:

$$u_k = -\partial_y p_k - c_k,$$
$$w_{k+1/2} = (p_{k+1} - p_k)/l.$$

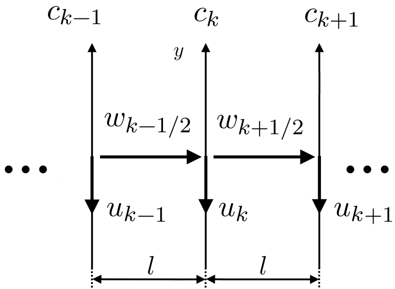


Interflow between tubes

$$f_{k+1/2} = \begin{cases} c_k \cdot \frac{w_{k+1/2}}{l}, & w_{k+1/2} \geq 0, \\ c_{k+1} \cdot \frac{w_{k+1/2}}{l}, & w_{k+1/2} \leq 0. \end{cases}$$

Multi-tubes model of IPM

Take $n \geq 2$ tubes ($\mathbb{R}$), connected cyclically. As a limit of finite-volume scheme, we get:



1. Transport equation in k-th tube, $k = 1, \ldots, n$:

$$\partial_t c_k + \partial_y (u_k c_k) = f_{k-1/2} - f_{k+1/2}$$

2. Incompressibility condition:

$$l \cdot \partial_y u_k - w_{k-1/2} + w_{k+1/2} = 0$$

3. Darcy's law in k-th tube and between tubes:

$$u_k = -\partial_y p_k - c_k,$$
$$w_{k+1/2} = (p_{k+1} - p_k)/l.$$

Interflow between tubes

$$f_{k+1/2} = \begin{cases} c_k \cdot \frac{w_{k+1/2}}{l}, & w_{k+1/2} \geq 0, \\ c_{k+1} \cdot \frac{w_{k+1/2}}{l}, & w_{k+1/2} \leq 0. \end{cases}$$

Initial condition: $c_k\big|_{t=0} = \begin{cases} +1, & y \geq 0, \\ -1, & y \leq 0. \end{cases}$

$$w_{k+1/2}\big|_{y=\pm\infty} = u_k\big|_{y=\pm\infty} = 0,$$
$$c_k\big|_{y=\pm\infty} = \pm 1$$

Multi-tubes model of IPM

Consider $n \geq 2$, tubes, connected cyclically. As a limit of finite-volume scheme, we get:

1. Transport equation in k-th tube, $k = 1, \ldots, n$:

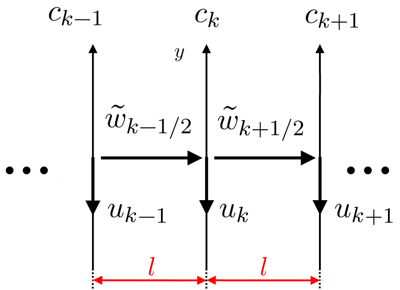
$$\partial_t c_k + \partial_y (u_k c_k) = f_{k-1/2} - f_{k+1/2}$$

2. Incompressibility condition:

$$\partial_y u_k - \tilde{w}_{k-1/2} + \tilde{w}_{k+1/2} = 0$$

3. Darcy's law in k-th tube and between tubes:

$$u_k = -\partial_y p_k - c_k,$$
$$\tilde{w}_{k+1/2} = (p_{k+1} - p_k)/l^2.$$



Interflow between tubes

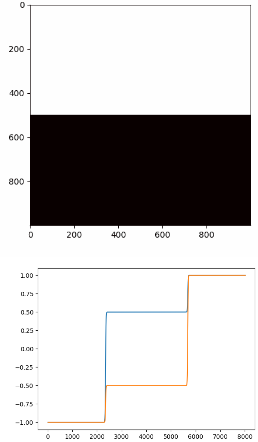
$$f_{k+1/2} = \begin{cases} c_k \cdot \tilde{w}_{k+1/2}, & \tilde{w}_{k+1/2} \geq 0, \\ c_{k+1} \cdot \tilde{w}_{k+1/2}, & \tilde{w}_{k+1/2} \leq 0. \end{cases}$$

Initial condition: $$c_k\big|_{t=0} = \begin{cases} +1, & y \geq 0, \\ -1, & y \leq 0. \end{cases}$$

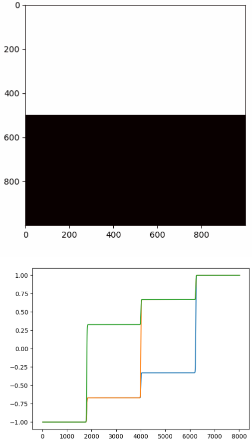
$$\tilde{w}_{k+1/2}\big|_{y=\pm\infty} = u_k\big|_{y=\pm\infty} = 0,$$
$$c_k\big|_{y=\pm\infty} = \pm 1$$

Semi-discrete model of IPM: numerical experiments

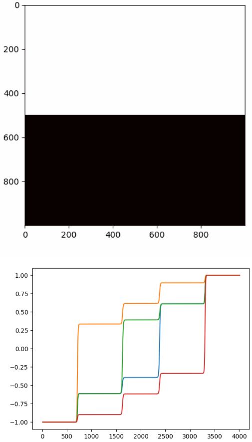
$n = 2$ tubes



$n = 3$ tubes



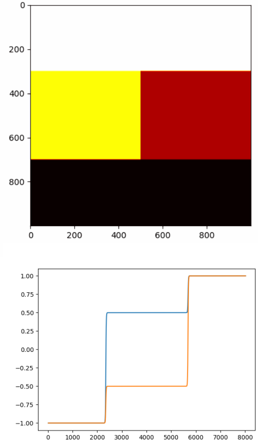
$n = 4$ tubes



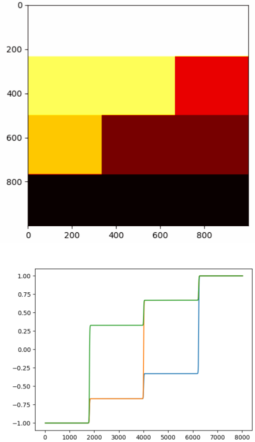
Aims: (1) explain the structure of "asymptotic solutions" for n tubes
(2) find speed of growth of the mixing zone
(3) understand the behavior as $n \to \infty$. Do we approximate 2-dim IPM?

Semi-discrete model of IPM: numerical experiments

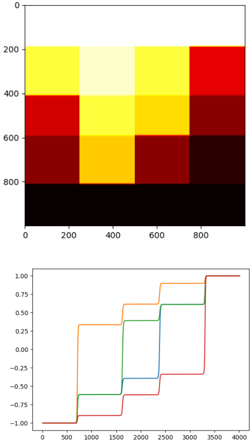
$n = 2$ tubes



$n = 3$ tubes



$n = 4$ tubes

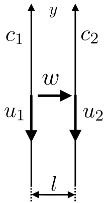


Aims: (1) explain the structure of “asymptotic solutions” for n tubes
(2) find speed of growth of the mixing zone
(3) understand the behavior as $n \to \infty$. Do we approximate 2-dim IPM?

Two-tubes IPM

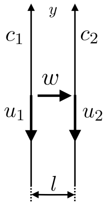
$$(*)\begin{cases} \partial_t c_1 + \partial_y(u_1 c_1) - \partial_{yy} c_1 = -f \\ \partial_t c_2 + \partial_y(u_2 c_2) - \partial_{yy} c_2 = f \\ u_1 = \partial_y p_1 - c_1 \\ u_2 = \partial_y p_2 - c_2 \\ \partial_y u_1 = -\partial_y u_2 = \frac{p_2 - p_1}{l^2} =: \frac{q}{l^2} \end{cases}$$

$$f = \begin{cases} -\partial_y u_1 \cdot c_1, & \partial_y u_1 \leq 0, \\ +\partial_y u_1 \cdot c_2, & \partial_y u_1 \geq 0. \end{cases}$$



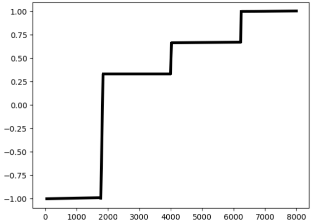
Two-tubes IPM

$$
(*)\begin{cases}
\partial_t c_1 + \partial_y(u_1 c_1) - \partial_{yy} c_1 = -f \\
\partial_t c_2 + \partial_y(u_2 c_2) - \partial_{yy} c_2 = f \\
u_1 = \partial_y p_1 - c_1 \\
u_2 = \partial_y p_2 - c_2 \\
\partial_y u_1 = -\partial_y u_2 = \frac{p_2 - p_1}{l^2} =: \frac{q}{l^2}
\end{cases}
\qquad
f = \begin{cases}
-\partial_y u_1 \cdot c_1, & \partial_y u_1 \leq 0, \\
+\partial_y u_1 \cdot c_2, & \partial_y u_1 \geq 0.
\end{cases}
$$



Definition

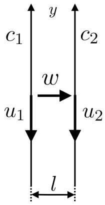
A **propagating terrace** connecting $\alpha \in \mathbb{R}^5$ to $\beta \in \mathbb{R}^5$ is a pair of finite sequences $(\sigma_j)_{0\leq j\leq N}$ and $(g_j)_{1\leq j\leq N}$ such that:



Two-tubes IPM

$$(*)\begin{cases} \partial_t c_1 + \partial_y(u_1 c_1) - \partial_{yy} c_1 = -f \\ \partial_t c_2 + \partial_y(u_2 c_2) - \partial_{yy} c_2 = f \\ u_1 = \partial_y p_1 - c_1 \\ u_2 = \partial_y p_2 - c_2 \\ \partial_y u_1 = -\partial_y u_2 = \frac{p_2 - p_1}{l^2} =: \frac{q}{l^2} \end{cases}$$

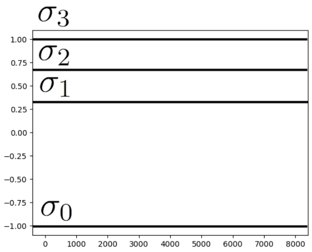
$$f = \begin{cases} -\partial_y u_1 \cdot c_1, & \partial_y u_1 \leq 0, \\ +\partial_y u_1 \cdot c_2, & \partial_y u_1 \geq 0. \end{cases}$$



Definition

A **propagating terrace** connecting $\alpha \in \mathbb{R}^5$ to $\beta \in \mathbb{R}^5$ is a pair of finite sequences $(\sigma_j)_{0 \leq j \leq N}$ and $(g_j)_{1 \leq j \leq N}$ such that:

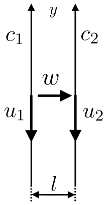
- Each $\sigma_j = (c_{1j}, c_{2j}, u_{1j}, u_{2j}, q_j)$ is a stationary solution of $(*)$ and $\sigma_0 = \alpha$, $\sigma_N = \beta$.



Two-tubes IPM

$$
(*)\begin{cases}
\partial_t c_1 + \partial_y(u_1 c_1) - \partial_{yy} c_1 = -f \\
\partial_t c_2 + \partial_y(u_2 c_2) - \partial_{yy} c_2 = f \\
u_1 = \partial_y p_1 - c_1 \\
u_2 = \partial_y p_2 - c_2 \\
\partial_y u_1 = -\partial_y u_2 = \frac{p_2 - p_1}{l^2} =: \frac{q}{l^2}
\end{cases}
$$

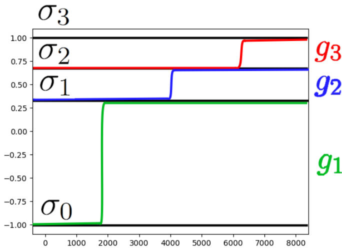
$$
f = \begin{cases}
-\partial_y u_1 \cdot c_1, & \partial_y u_1 \leq 0, \\
+\partial_y u_1 \cdot c_2, & \partial_y u_1 \geq 0.
\end{cases}
$$



Definition

A **propagating terrace** connecting $\alpha \in \mathbb{R}^5$ to $\beta \in \mathbb{R}^5$ is a pair of finite sequences $(\sigma_j)_{0 \leq j \leq N}$ and $(g_j)_{1 \leq j \leq N}$ such that:

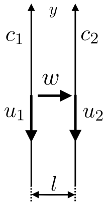
- Each $\sigma_j = (c_{1j}, c_{2j}, u_{1j}, u_{2j}, q_j)$ is a stationary solution of $(*)$ and $\sigma_0 = \alpha$, $\sigma_N = \beta$.
- g_j is a traveling wave solution of connecting σ_{j-1} to σ_j, $1 \leq j \leq N$, that is $g_j = g_j(y - v_j t)$.



Two-tubes IPM

$$(*)\begin{cases} \partial_t c_1 + \partial_y(u_1 c_1) - \partial_{yy} c_1 = -f \\ \partial_t c_2 + \partial_y(u_2 c_2) - \partial_{yy} c_2 = f \\ u_1 = \partial_y p_1 - c_1 \\ u_2 = \partial_y p_2 - c_2 \\ \partial_y u_1 = -\partial_y u_2 = \frac{p_2 - p_1}{l^2} =: \frac{q}{l^2} \end{cases}$$

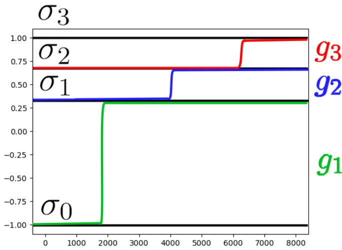
$$f = \begin{cases} -\partial_y u_1 \cdot c_1, & \partial_y u_1 \leq 0, \\ +\partial_y u_1 \cdot c_2, & \partial_y u_1 \geq 0. \end{cases}$$



Definition

A **propagating terrace** connecting $\alpha \in \mathbb{R}^5$ to $\beta \in \mathbb{R}^5$ is a pair of finite sequences $(\sigma_j)_{0\leq j\leq N}$ and $(g_j)_{1\leq j\leq N}$ such that:

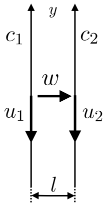
- Each $\sigma_j = (c_{1j}, c_{2j}, u_{1j}, u_{2j}, q_j)$ is a stationary solution of $(*)$ and $\sigma_0 = \alpha$, $\sigma_N = \beta$.
- g_j is a traveling wave solution of connecting σ_{j-1} to σ_j, $1 \leq j \leq N$, that is $g_j = g_j(y - v_j t)$.
- The speed $v_j \in \mathbb{R}$ of each traveling wave g_j satisfies $v_1 \leq v_2 \leq \ldots \leq v_N$.



Two-tubes IPM: theorem

$$(*)\begin{cases} \partial_t c_1 + \partial_y(u_1 c_1) - \partial_{yy} c_1 = -f \\ \partial_t c_2 + \partial_y(u_2 c_2) - \partial_{yy} c_2 = f \\ u_1 = \partial_y p_1 - c_1 \\ u_2 = \partial_y p_2 - c_2 \\ \partial_y u_1 = -\partial_y u_2 = \frac{p_2 - p_1}{l^2} =: \frac{q}{l^2} \end{cases}$$

$$f = \begin{cases} -\partial_y u_1 \cdot c_1, & \partial_y u_1 \leq 0, \\ +\partial_y u_1 \cdot c_2, & \partial_y u_1 \geq 0. \end{cases}$$



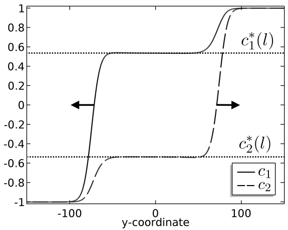
Theorem (arXiv:2401.05981, P., Tikhomirov, Efendiev)

There exists sufficiently small $l_0 > 0$ such that for all $l \in (0, l_0)$ there exist a propagating terrace of two traveling waves with speeds $v_1^*(l)$, $v_2^*(l)$ connecting the states

$$\sigma_0 = (-1, -1, 0, 0, 0),$$
$$\sigma_1 = (c_1^*(l), c_2^*(l), u_1^*(l), u_2^*(l), 0),$$
$$\sigma_2 = (1, 1, 0, 0, 0).$$

Moreover, as $l \to 0$ we obtain:

$$\lim c_1^*(l) = -\lim c_2^*(l) = 1/2, \quad \lim v_2^*(l) = -\lim v_1^*(l) = 1/4.$$



Main ingredient in proof: comparison with TFE equations

- traveling wave ansatz: $\xi = y - vt \Rightarrow$ traveling wave dynamical system (TWDS)

$l = 0$

heteroclinic orbits can be found explicitly!

orbit $\subset \{W^s \pitchfork W^u\}$
+ geometric singular perturbation theory

$l > 0$

heteroclinic orbits persist under small perturbations

Main ingredient in proof: comparison with TFE equations

- traveling wave ansatz: $\xi = y - vt \Rightarrow$ traveling wave dynamical system (TWDS)

$l = 0$

heteroclinic orbits can be found explicitly!

orbit $\subset \{W^s \pitchfork W^u\}$
+ geometric singular perturbation theory

$l > 0$

heteroclinic orbits persist under small perturbations

Two-tubes TFE equations:

$$\partial_t c_1 + \partial_y(u_1 c_1) = -f$$
$$\partial_t c_2 + \partial_y(u_2 c_2) = f$$
$$u_1 = \bar{c} - c_1 = (c_2 - c_1)/2$$
$$u_2 = \bar{c} - c_2 = (c_1 - c_2)/2$$
$$f = \begin{cases} -\partial_y u_1 \cdot c_1, & \partial_y u_1 \leq 0, \\ +\partial_y u_1 \cdot c_2, & \partial_y u_1 \geq 0. \end{cases}$$

Main ingredient in proof: comparison with TFE equations

- traveling wave ansatz: $\xi = y - vt \Rightarrow$ traveling wave dynamical system (TWDS)

$l = 0$

heteroclinic orbits can be found explicitly!

orbit $\subset \{W^s \pitchfork W^u\}$
+ geometric singular perturbation theory

$l > 0$

heteroclinic orbits persist under small perturbations

Two-tubes TFE equations:

$$\partial_t c_1 + \partial_y (u_1 c_1) = -f$$
$$\partial_t c_2 + \partial_y (u_2 c_2) = f$$
$$u_1 = \bar{c} - c_1 = (c_2 - c_1)/2$$
$$u_2 = \bar{c} - c_2 = (c_1 - c_2)/2$$
$$f = \begin{cases} -\partial_y u_1 \cdot c_1, & \partial_y u_1 \leq 0, \\ +\partial_y u_1 \cdot c_2, & \partial_y u_1 \geq 0. \end{cases}$$

This system can be seen a hyperbolic system in non-conservative form (for fixed choice of f):

$$\boxed{S_t + A(S) S_y = 0}$$

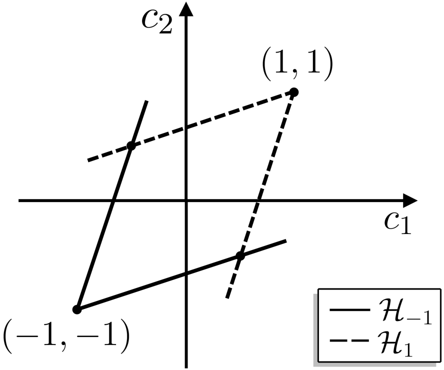
We solve the Riemann problem:

$$S = (c_1, c_2)\big|_{t=0} = \begin{cases} (+1, +1), & y \geq 0 \\ (-1, -1), & y \leq 0 \end{cases}$$

Selection criteria for discontinuous solutions: vanishing viscosity

Two-tubes TFE

Shock curves for $(c_1, c_2) = (1, 1)$ and $(-1, -1)$:



- "Temple-like" system (rarefaction and shock curves coincide and are linear)
- Similar explicit linear structure for $n = 3$ tubes (in progress)
- Starting from $n \geq 4$ appear also non-linear families and complex eigenvalues in some subdomains of $(c_1, \ldots, c_n)$ (numerical evidence)...

Open questions

1. Two-tubes IPM (with diffusion): prove stability of the propagation terrace

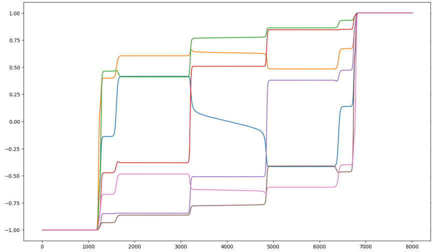
Open questions

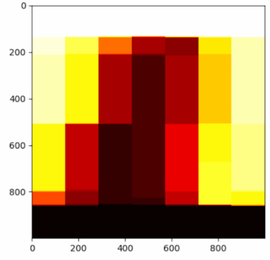
1. Two-tubes IPM (with diffusion): prove stability of the propagation terrace
2. Multi-tubes IPM & TFE (hyperbolic) for $n \geq 2$:
    - well-posedness
    - structure of solution of a Riemann problem $(-1, \dots, -1)$ to $(1, \dots, 1)$

Open questions

1. Two-tubes IPM (with diffusion): prove stability of the propagation terrace
2. Multi-tubes IPM & TFE (hyperbolic) for $n \geq 2$:
   - well-posedness
   - structure of solution of a Riemann problem $(-1, \ldots, -1)$ to $(1, \ldots, 1)$

Observation (for $n \geq 4$): starting from random initial data, we come to different asymptotic states, and their number grows with n.

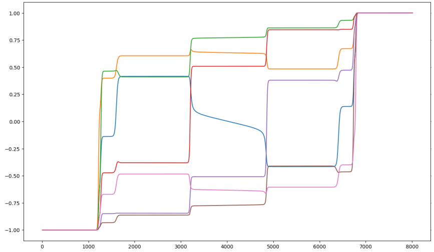


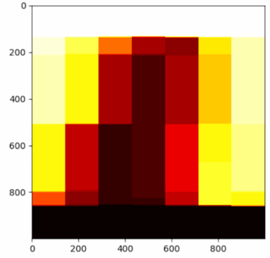


Open questions

1. Two-tubes IPM (with diffusion): prove stability of the propagation terrace
2. Multi-tubes IPM & TFE (hyperbolic) for $n \geq 2$:
   - well-posedness
   - structure of solution of a Riemann problem $(-1, \ldots, -1)$ to $(1, \ldots, 1)$

   Observation (for $n \geq 4$): starting from random initial data, we come to different asymptotic states, and their number grows with n.

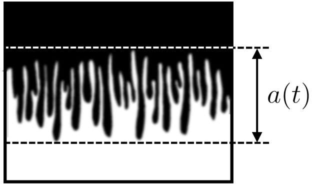


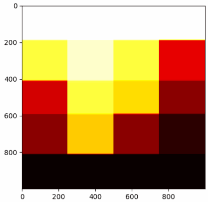


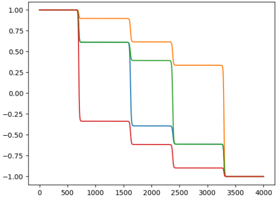
3. Limit $n \to \infty$. Convergence of n-tubes IPM to 2-dim IPM?

Thank you for your attention!

yu.pe.petrova@gmail.com https://yulia-petrova.github.io/







For more details see arXiv: 2401.05981

Any questions, comments and ideas are very welcome!