Traveling waves in gravitational fingering instability: A dynamical systems approach

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Introduction

A long-term goal is the proof of the linear growth of the length $h(t)$ of mixing zone in 2D viscous incompressible porous media (IPM) eq:

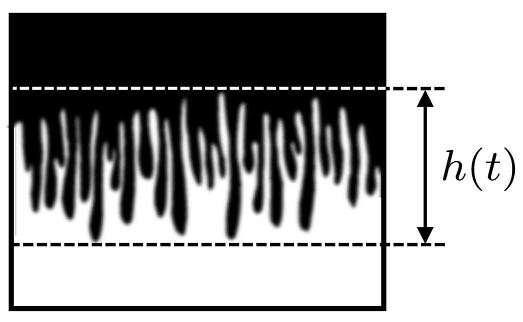
$$\text{(transport equation)} \quad \partial_t c + \operatorname{div}(uc) = \nu \Delta c, \tag{1}$$

$$\text{(incompressibility condition)} \quad \operatorname{div}(u) = 0, \tag{2}$$

$$\text{(Darcy's law)} \quad u = -\nabla p - (0, c). \tag{3}$$

For the unstable stratification (heavy fluid up, light — down), we observe *gravitational (viscous) fingering* instability (see fig. below)

$c = c(t, x, y)$ — transported concentration;
$u = u(t, x, y)$ — velocity of the fluid;
$p = p(t, x, y)$ — pressure;
$\nu \geq 0$ — an inverse of the Péclet number.



$$\text{Estimates [1, 4]:} \quad h(t) \sim \alpha t, \ \text{for some } \alpha \in [1.34, 2]. \tag{4}$$

Objetives

Show in a simplified setting that there are two (interconnected) *mechanisms of possible decrease in the size of the mixing zone:*

1. the convection of fluid in the transverse direction of the flow;
2. the presence of the intermediate concentration, that is the typical concentration inside the finger is $c^* \in (-1, 1)$.

Model formulation: two-tubes model of gravitational fingering

Consider discretization of (1)–(3) in x-direction: fluid flows in two real lines with distance l between them, governed by eq. $(i = 1, 2)$:

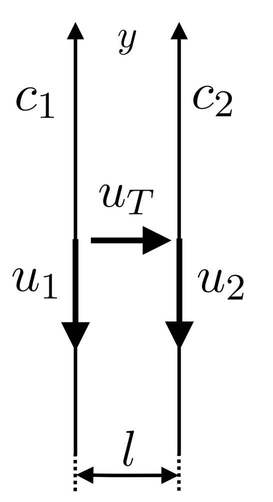
$$\text{(transport equation)} \quad \partial_t c_i + \partial_y(u_i c_i) - \partial_{yy} c_i = (-1)^i f, \tag{5}$$

$$\text{(incompressibility)} \quad \frac{u_T}{l} + \partial_y u_1 = 0, \ -\frac{u_T}{l} + \partial_y u_2 = 0, \tag{6}$$

$$\text{(Darcy's law)} \quad u_i = -\partial_y p_i - c_i, \quad u_T = \frac{p_1 - p_2}{l}. \tag{7}$$

$c_{1,2}(t, y)$ — concentrations in tubes 1,2;
$u_{1,2}(t, y)$ — velocity of the fluid in tubes 1,2;
$p_{1,2}(t, y)$ — pressure in tubes 1,2;

$$f = \begin{cases} \frac{u_T}{l} \cdot c_1, & u_T \geq 0, \\ \frac{u_T}{l} \cdot c_2, & u_T \leq 0. \end{cases} \tag{8}$$



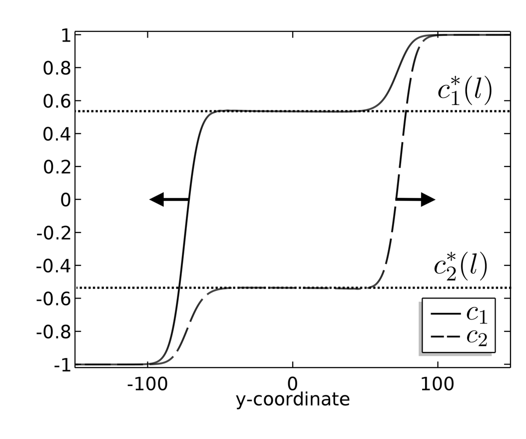
Definition: function $g(t, y)$ is *traveling wave* with speed $v \in \mathbb{R}$ connecting states g_- and g_+, if $g(t, y) = g(y - vt)$ and $g(\pm\infty) = g_\pm$.

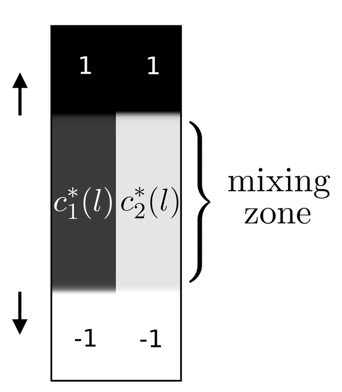
Main theorem [5, Theorem 2.2]

For sufficiently small values of $l > 0$ there exist two intermediate concentrations $c_1^*(l) \in (-1, 1)$, $c_2^*(l) \in (-1, 1)$ and two traveling wave (TW) solutions of the system (5)–(8) that connect the states:

$$(-1, -1) \xrightarrow{TW} (c_1^*(l), c_2^*(l)) \xrightarrow{TW} (1, 1). \tag{9}$$

The speeds of the traveling waves approach $-1/4$ and $1/4$ as $l \to 0$.





Corollary.
For two-tubes model (5)–(8)

$$h(t) \sim \alpha(l) \cdot t,$$

$$\alpha(l) \to \frac{1}{2} \text{ as } l \to 0.$$

Scheme of proof

- Step I: *Study traveling waves (TW),* $g = (c_1, c_2, u_1, u_2, p_1 - p_2)$*:*

$$g = g(\xi), \quad \xi = y - vt, \quad v \text{ — speed of TW} \tag{10}$$

- Step II: *Study combination of 2 TW (so-called propagating terrace)*

Idea of proof (for steps I and II)

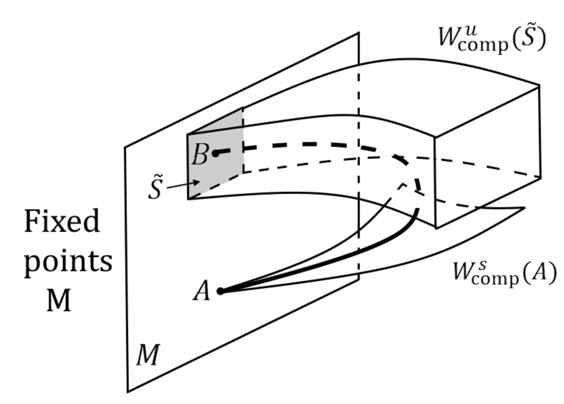
1. Study the singular limit $l = 0$;
2. Prove that statements remain valid for small $l > 0$.

Step I: finding traveling waves = finding heteroclinic orbits in dynamical system

Put a traveling wave ansatz (10) into (5)–(8) for the case $u_T \geq 0$ and get a traveling wave dynamical system (TWDS), here $q = p_1 - p_2$:

$$\begin{aligned} \partial_\xi c_1 &= g_1, \\ \partial_\xi g_1 &= g_1(u_1 - v), \\ \partial_\xi c_2 &= g_2, \\ \partial_\xi g_2 &= g_2(u_2 - v) + (c_1 - c_2)\frac{q}{l^2}, \\ \partial_\xi q &= u_2 - u_1 + c_2 - c_1, \\ \partial_\xi u_1 &= -\frac{q}{l}, \\ \partial_\xi u_2 &= \frac{q}{l}. \end{aligned} \tag{11}$$

After change of variables TWDS becomes slow-fast system as $l \to 0$.



$l = 0$: trajectories can be found explicitly. We represent them as transversal intersection of some stable and unstable manifolds.

$l > 0$: using geometric singular perturbation theory [2, 3], it can be proven that this trajectory exists for TWDS (11).

Step II: propagating terrace

$l = 0$: For every $v < 0$ there exist exactly two traveling wave solutions $c_{1,2}(y - vt)$ satisfying $(c_1, c_2)(-\infty) = (-1, -1)$. Denote

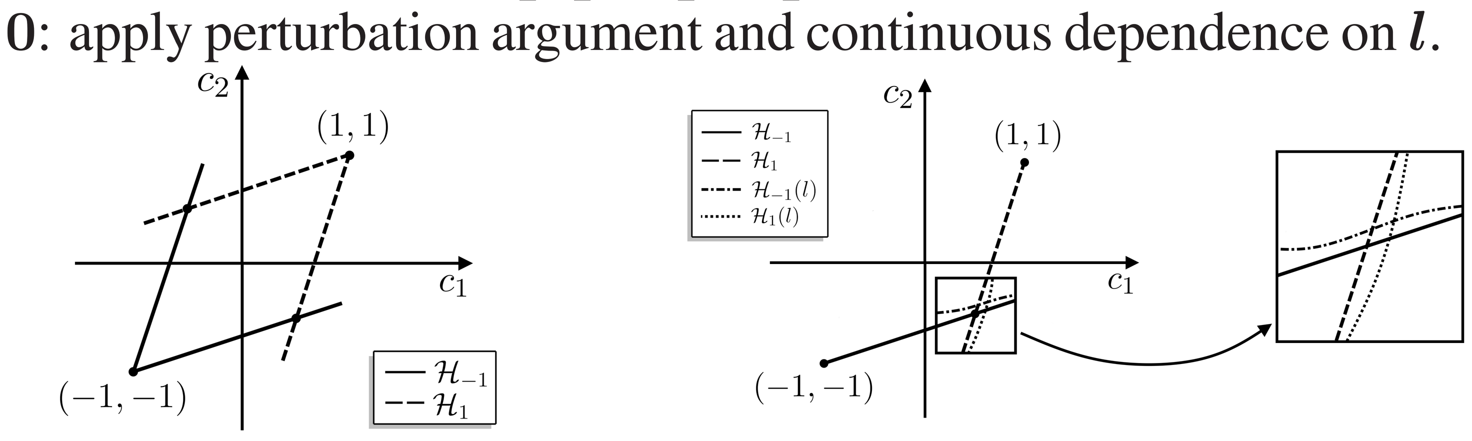
$$\mathcal{H}_{-1} := \{(c_1, c_2)(+\infty)\} = \{(-2v - 1, -6v - 1)\} \cup \{(-6v - 1, -2v - 1)\}, v < 0$$

Analogously, for every $v > 0$ there exist exactly two traveling wave solutions $c_{1,2}(y - vt)$ satisfying $(c_1, c_2)(+\infty) = (1, 1)$. Denote

$$\mathcal{H}_1 := \{(c_1, c_2)(-\infty)\} = \{(-2v + 1, -6v + 1)\} \cup \{(-6v + 1, -2v + 1)\}, v > 0$$

Finally, $\mathcal{H}_{-1} \cap \mathcal{H}_1 = \{(-\frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, -\frac{1}{2})\}$.

$l > 0$: apply perturbation argument and continuous dependence on l.



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