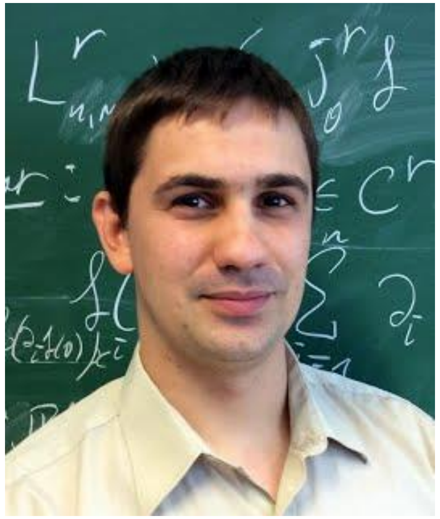
Two tubes model of miscible displacement: traveling waves and normal hyperbolicity

Yulia Petrova

Pontifícia Universidade Católica do Rio de Janeiro (PUC-Rio)
Rio de Janeiro, Brazil

yulia-petrova.github.io



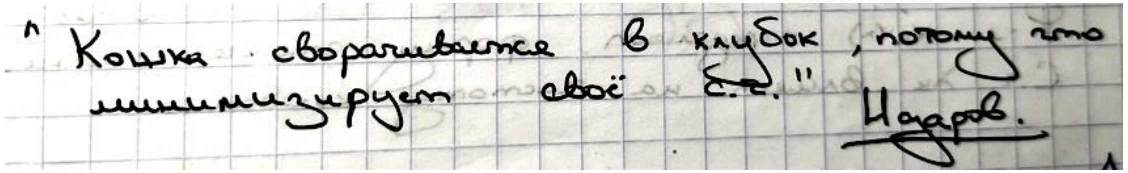
Sergey Tikhomirov
PUC-Rio
Brazil

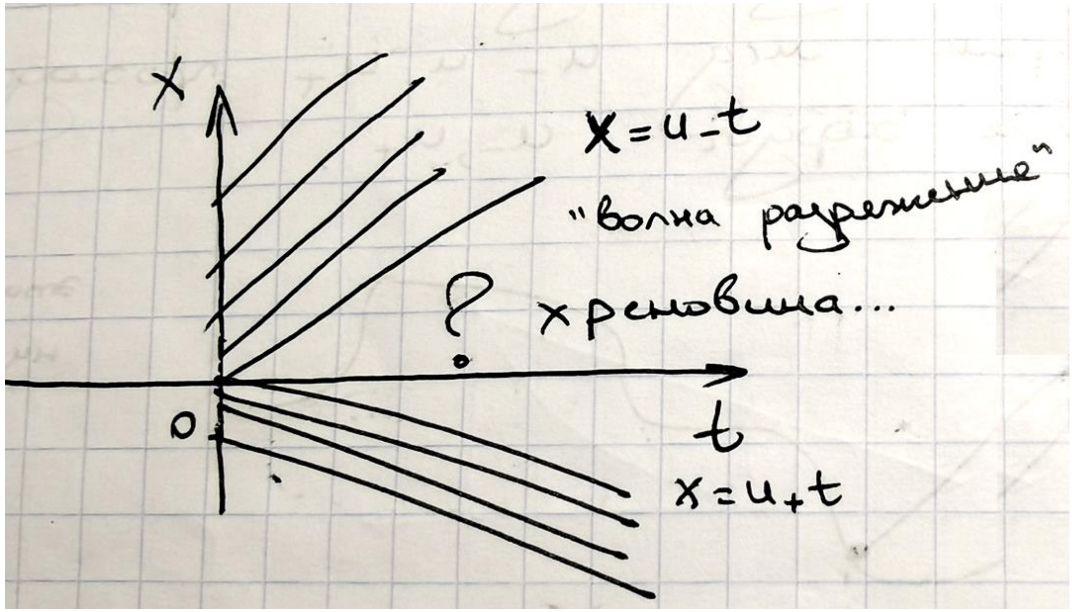
Yalchin Efendiev
Texas A&M
USA

8 мая 2023
Мини-конференция молодых ученых,
посвященная 60-летию А.И. Назарова

Первое научное знакомство с АИ было в 2010…

Вырезки из конспекта по практикам по матфизике:





Спасибо за терпение быть научным руководителем!

Выпуск кафедры матфизики, 2013 год

Отмечу две характеристики АИ как математика:

1. Умение разговаривать на разных языках

   В частности, переводить с одного языка на другой

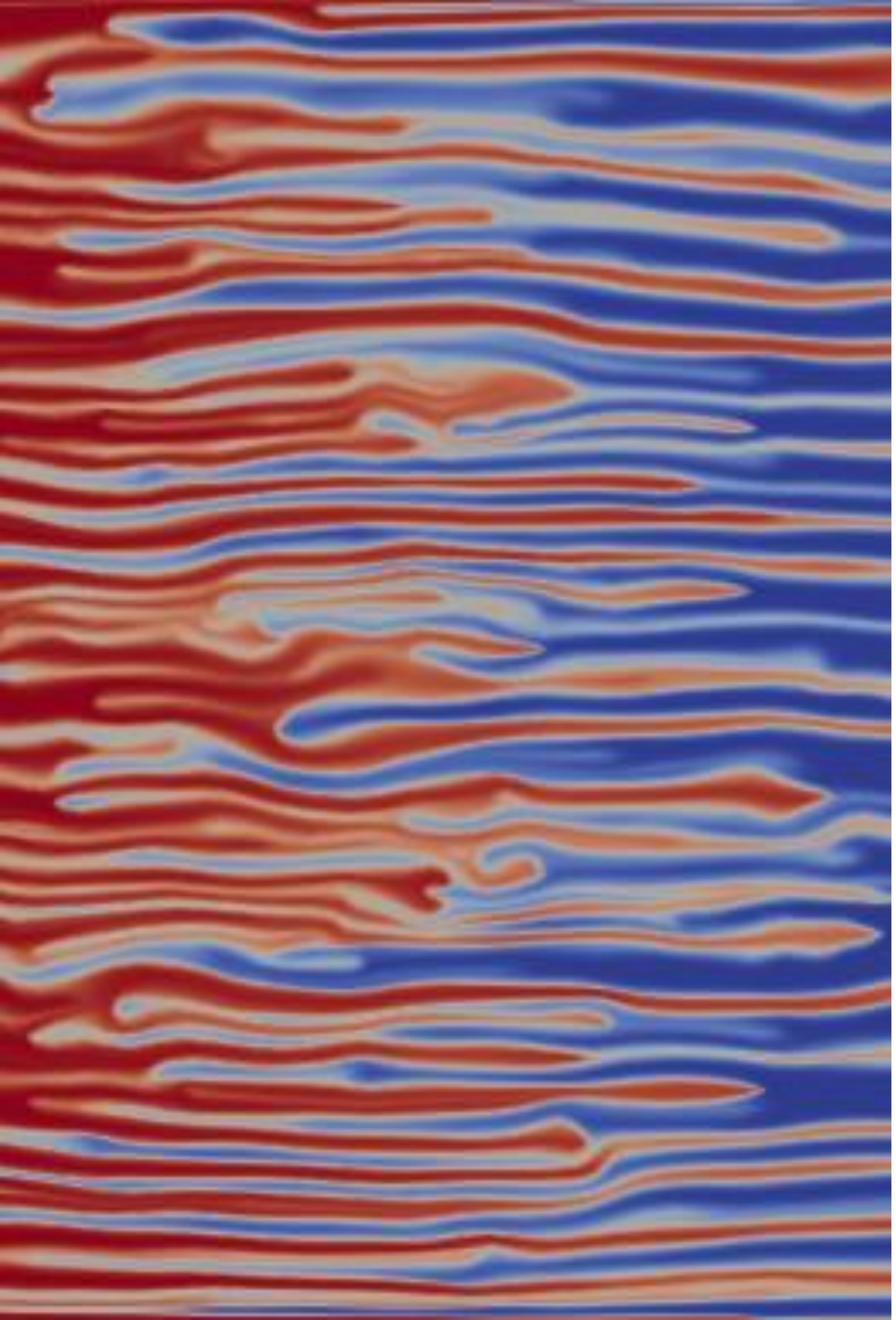
   - яркий пример: спектральная теория в приложении к задачам теории вероятностей и мат статистики)

2. «Лягушка»

Как говорил физик Фримен Дайсон:

«Бывают учёные-птицы, а бывают и учёные-лягушки. Птицы парят в вышине и обозревают обширные пространства математики, сколько видит глаз. Лягушки же копошатся далеко внизу и видят только растущие поблизости цветы. Для них наслаждение — внимательно разглядывать конкретные объекты; задачи они решают последовательно, одну за другой»

Поэтому сегодня я расскажу о *конкретной* задаче, про которую полезно говорить на *разных языках*

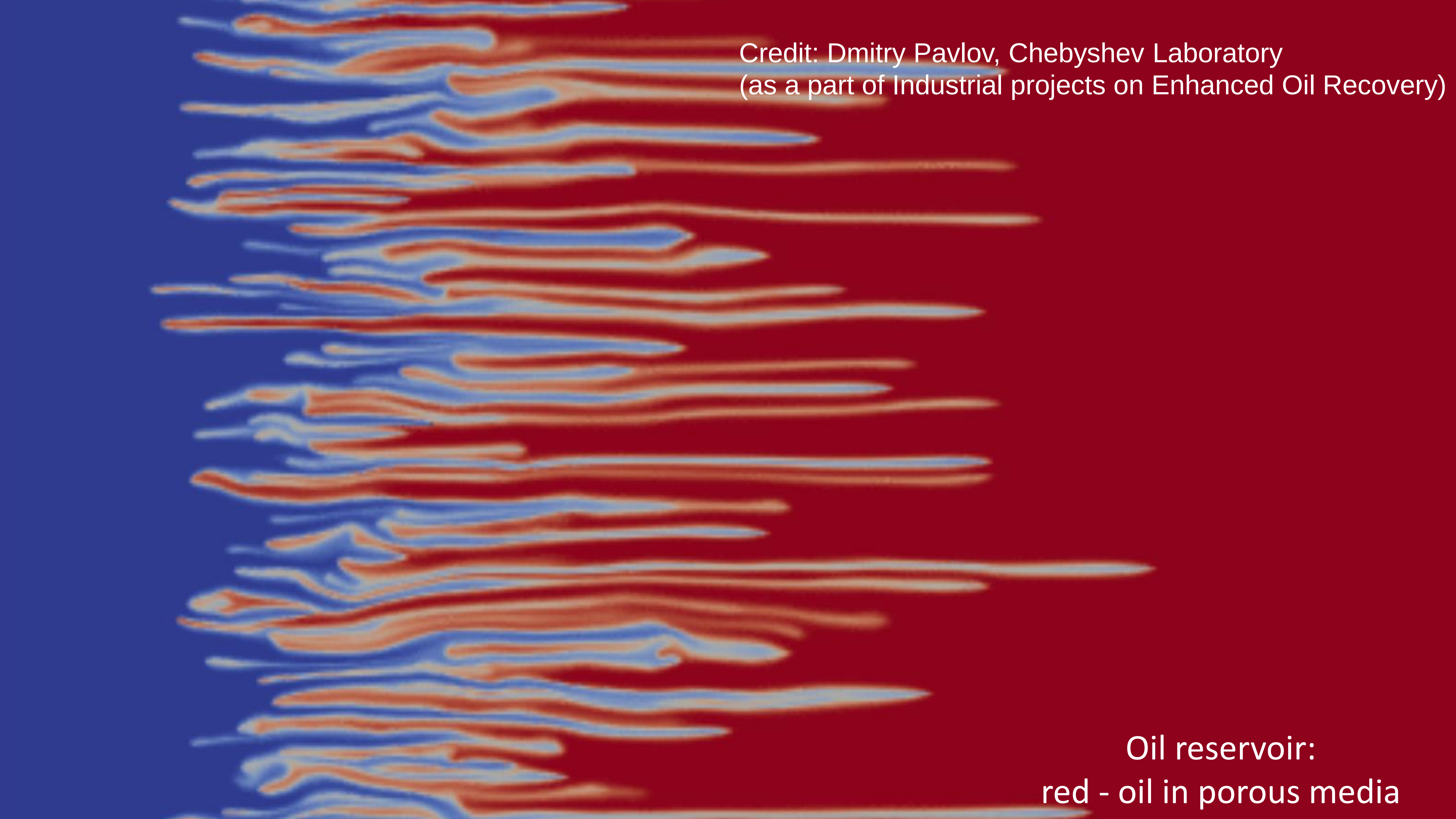


Outline

1. General phenomenon:
   - Viscous fingers
   - Gravitational fingers

2. Mathematical model (2-dim):
   Incompressible Porous Medium (IPM) equation
   - Well-posedness & Dynamics

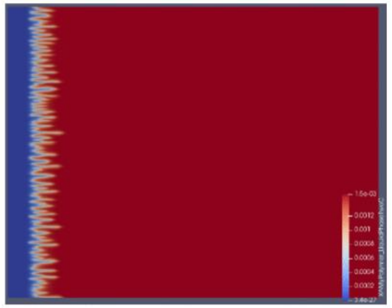
3. “Toy” model (2-tubes):
   - Theorem on gravitational fingering
   - Conjectures

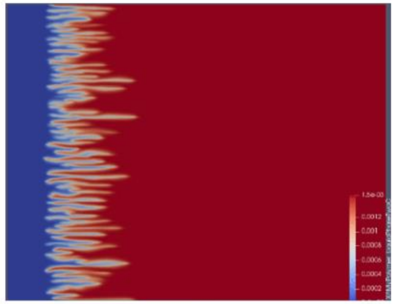

Credit: Dmitry Pavlov, Chebyshev Laboratory
(as a part of Industrial projects on Enhanced Oil Recovery)
Oil reservoir:
red - oil in porous media

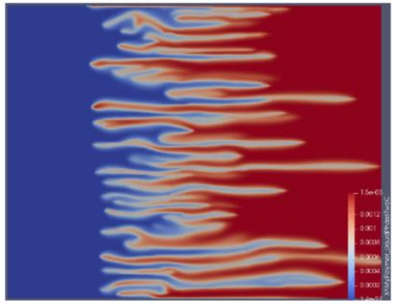
Two settings (Incompressible porous medium eqs - IPM)

1. Viscosity-driven fingers: 2-dim

$$\begin{aligned} c_t + u \cdot \nabla c &= \ \varepsilon\, \Delta c \\ div\, u &= 0 \\ u &= -k \cdot m(c) \nabla p \end{aligned}$$





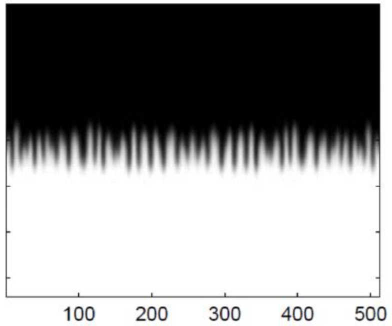


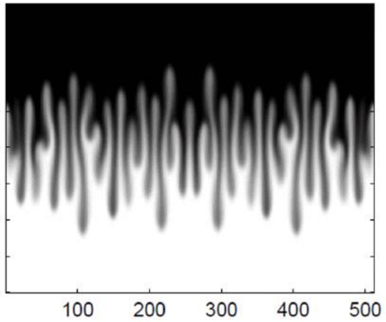
- c – concentrations of viscous spices (transport equation) $\quad c \in [0, 1]$
- u – velocity of fluid (incompressibility condition)
- p – pressure
  velocity is defined by Darcy law and mobility of liquid $m(c)$;
  $m(c)$ – decreasing function, e.g.
  $$m(c) = \mathrm{e}^{-\mathrm{ac}}$$

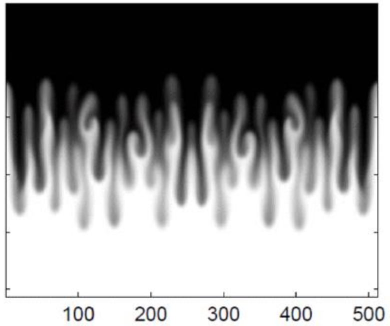
We did a lot of numerical simulations.
Motivation of statement of the problem.

2. Gravity-driven fingers: 2-dim

$$\begin{aligned} c_t + u \cdot \nabla c &= \ \varepsilon\, \Delta c \\ div\, u &= 0 \\ u &= -\nabla p - (0, c) \end{aligned}$$







- c – concentrations of heavy spices (transport equation) $\quad c \in [-1, 1]$
- u – velocity of fluid (incompressibility condition)
- p – pressure.

  velocity is defined by Darcy law and gravitation

We have some theorems for “toy model”

Questions of interest: $\varepsilon = 0$ (no diffusion)

1. Well-posedness:
   - active scalar: $u = A(c)$ – singular integral operator (like in SQG)

$$u = \nabla^{\perp}(-\Delta)^{-1}\partial_1 c$$

(Biot-Savart law)

$$\boxed{\begin{aligned} &c_t + u \cdot \nabla c = 0 \\ &u = A(c) \end{aligned}}$$

   - existence of a global solution vs finite-time blow-up, e.g.: T. Elgindi (2017), A. Castro, D. Cordoba, D. Lear (2018), A. Kiselev, Y. Yao (2023)

The best result (up to Jan 2023):

Kiselev, A. and Yao, Y., 2023. Small scale formations in the incompressible porous media equation. Archive for Rational Mechanics and Analysis, 247(1), p.1.

"Informally" (only conditional result):

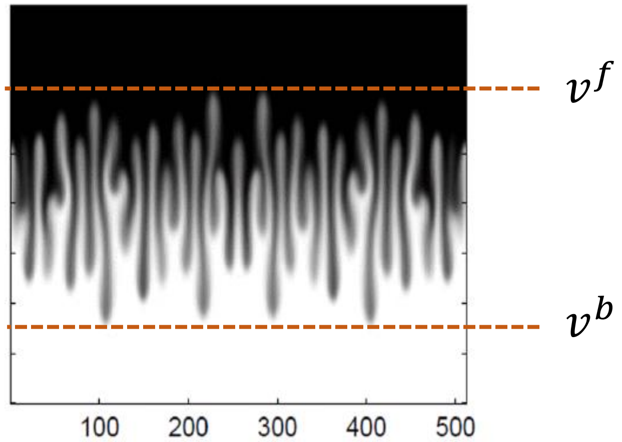
Let solution stay smooth for all $t > 0$ (in an "appropriate" Sobolev space). Then at least as $t \to \infty$ the Sobolev norm blows-up.

   - non-uniqueness of solutions (convex integration technique): D. Córdoba, D. Faraco, F. Gancedo (2011), R. Shvydkoy (2011), L. Szekelyhidi Jr. (2012)

- Related models: generalized Buckley-Leverett equation – N. Chemetov, W. Neves (2014)
  Muskat pr. & Hele-Shaw (free boundary) – A. Cordoba, D. Cordoba, F. Gancedo (2011) etc.

Questions of interest: $\varepsilon > 0$ Goal: EXACT speed of growth

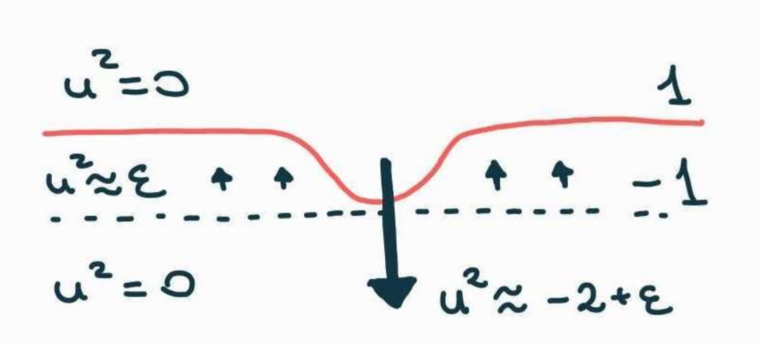
2. Dynamics of the mixing zone:
   - experiments show linear growth of the mixing zone
   - mathematically rigorous results: F. Otto, G. Menon (2005)
     - Simplified model: transverse flow equilibrium (TFE)



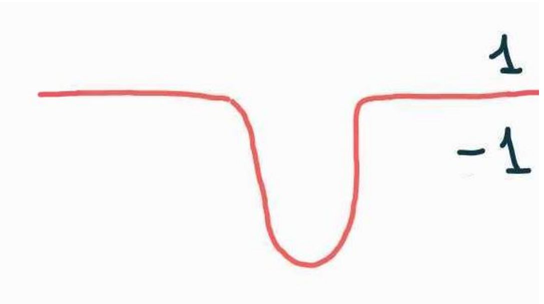
$$
\boxed{\begin{gathered} c_t + u \cdot \nabla c = \varepsilon\, \Delta c \\ div\, u = 0 \\ u = (u^1, u^2), \qquad u^2 = \bar{c} - c \end{gathered}}
$$

Why fingers appear? It is a hair-trigger effect!

No flow



Velocity u changes due to concentration c



Concentration c changes due to velocity u

Three methods to obtain estimates on linear growth

Energy estimates

F. Otto, G. Menon (2005)

Work both for IPM and TFE models

Gravitational potential energy $E(t)$

$$\limsup_{t\to\infty} \frac{E(t)}{t^2} \leq \frac{1}{6}$$

Mean perimeter $P(t)$

$$\limsup_{t\to\infty} \frac{1}{t^2}\int_0^t P^2(s) \leq \frac{\pi}{9}$$

Are NOT sharp enough!

Comparison theorems

F. Otto, G. Menon (2005)

Known results only for TFE model

Consider 1d equation (viscous Burgers!)

$$c_t^{max} + (1 - c^{max})c_y^{max} = \varepsilon\,(c^{max})_{yy}$$

$$c_t^{min} + (-1 - c^{min}) \cdot c_y^{min} = \varepsilon\,\left(c^{min}\right)_{yy}$$

Comparison theorem (Otto, Menon,2005)

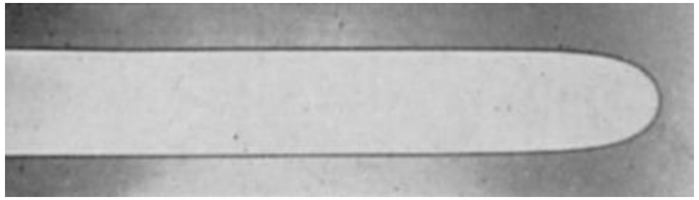
- If $c(0,x,y) < c^{max}(0,y)$, then
  $$c(t,x,y) \leq c^{max}(t,y)$$
- If $c(0,x,y) > c^{min}(0,y)$, then
  $$c(t,x,y) \geq c^{min}(t,y)$$

Neglect advection in *transverse* direction!

Traveling waves

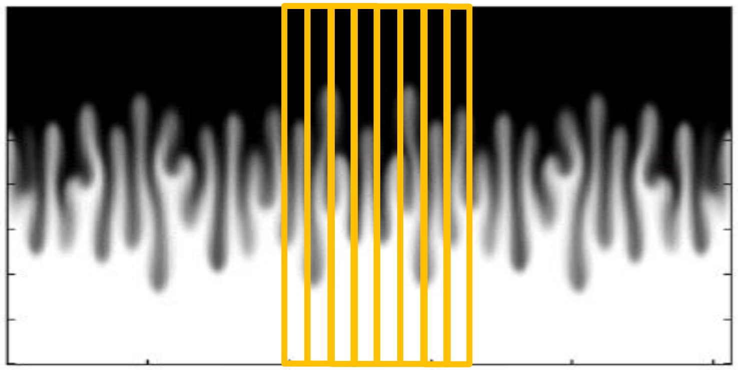
Solutions of the special form: “$c(x - vt)$”

Saffman-Taylor “fingers” (1958)



$\varepsilon = 0$

Many-tubes model (Tikhomirov et al)



Is the flow in *transverse* direction important?

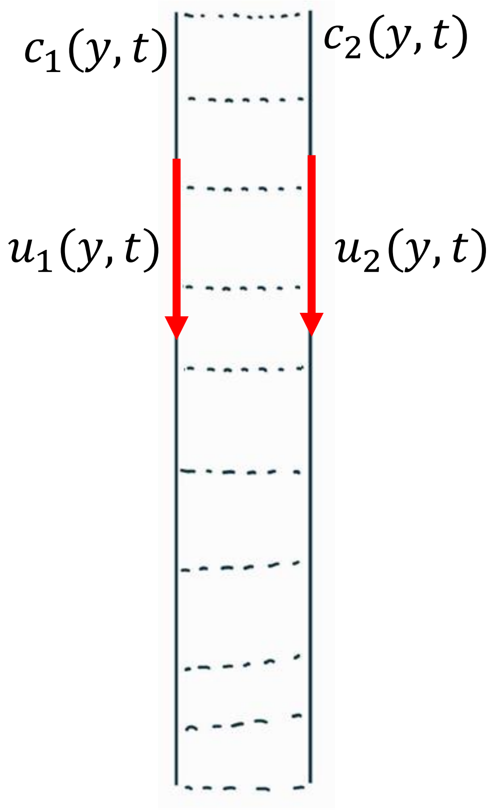
Two-tubes model (with gravity)

Original equations

$$c_t + div(uc) = \varepsilon \, \Delta c$$
$$div\, u = 0$$

Two-tube equations

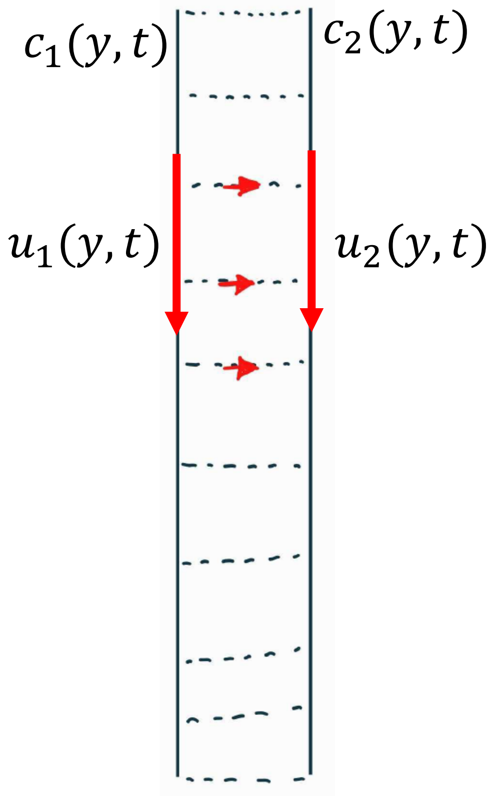
$$\partial_t c_1 + \partial_y (u_1 c_1) - \varepsilon \, \partial_{yy} c_1 = 0$$
$$\partial_t c_2 + \partial_y (u_2 c_2) - \varepsilon \, \partial_{yy} c_2 = 0$$



Two-tubes model (with gravity)

Original equations

$$c_t + div(uc) = \varepsilon\, \Delta c$$
$$div\, u = 0$$



Two-tube equations: inclusion of transverse flow

$$\partial_t c_1 + \partial_y(u_1 c_1) - \varepsilon\, \partial_{yy} c_1 = \boxed{-(-1)^{1,2} \partial_y u_{1,2} \cdot c_{1,2}}$$
$$\partial_t c_2 + \partial_y(u_2 c_2) - \varepsilon\, \partial_{yy} c_2 = \boxed{(-1)^{1,2} \partial_y u_{1,2} \cdot c_{1,2}}$$

$$(-1)^{1,2} \partial_y u_{1,2} \cdot c_{1,2} = \begin{cases} -\partial_y u_1 \cdot c_1, & \partial_y u_1 < 0, \\ +\partial_y u_2 \cdot c_2, & \partial_y u_1 > 0 \end{cases}$$

Model for velocities is different for IPM and TFE:

- TFE: $\mathrm{u} = \bar{c} - c, \quad \mathrm{u}_1 = \frac{\mathrm{c}_1 + \mathrm{c}_2}{2} - \mathrm{c}_1, \qquad \mathrm{u}_2 = \frac{\mathrm{c}_1 + \mathrm{c}_2}{2} - \mathrm{c}_2$
- IPM: we need to introduce pressure (...not today...)

Initial condition:

$$c_{1,2}(y,0) = -1, y < 0$$
$$c_{1,2}(y,0) = +1, y > 0$$

Main result (TFE model with gravity)

Theorem (Efendiev, P., Tikhomirov, 2023+)

Consider a two-tube model with gravity.
Then there exists unique (up to swap) c_1^*, c_2^* such that TFE two-tubes system has travelling waves

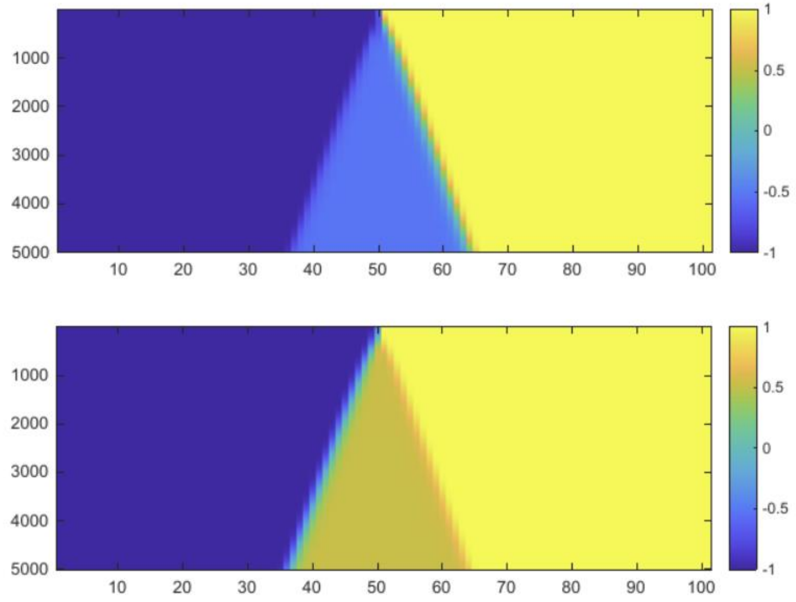
$$(-1,-1) \to (c_1^*, c_2^*) \to (1,1)$$

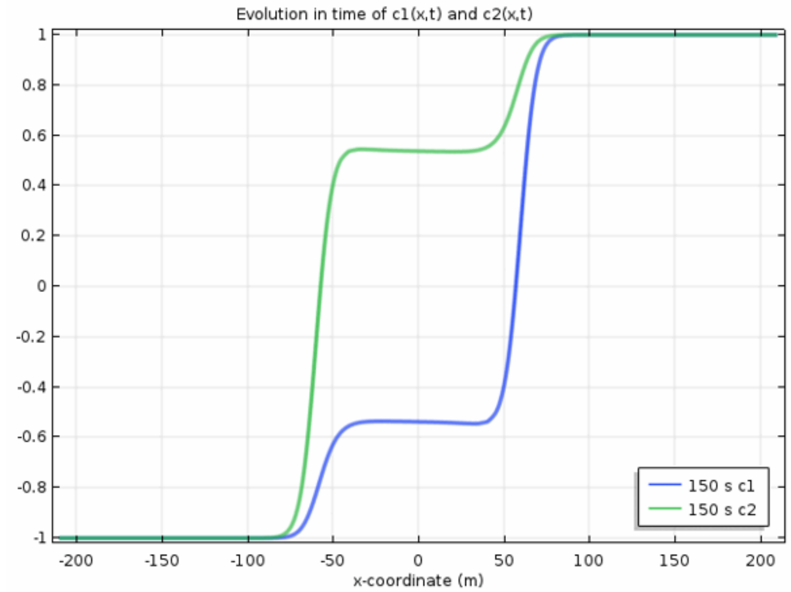
Moreover,

$$c_1^* = -\frac{1}{2}, \quad c_2^* = \frac{1}{2},$$
$$v^b = -\frac{1}{4}, \quad v^f = \frac{1}{4}.$$

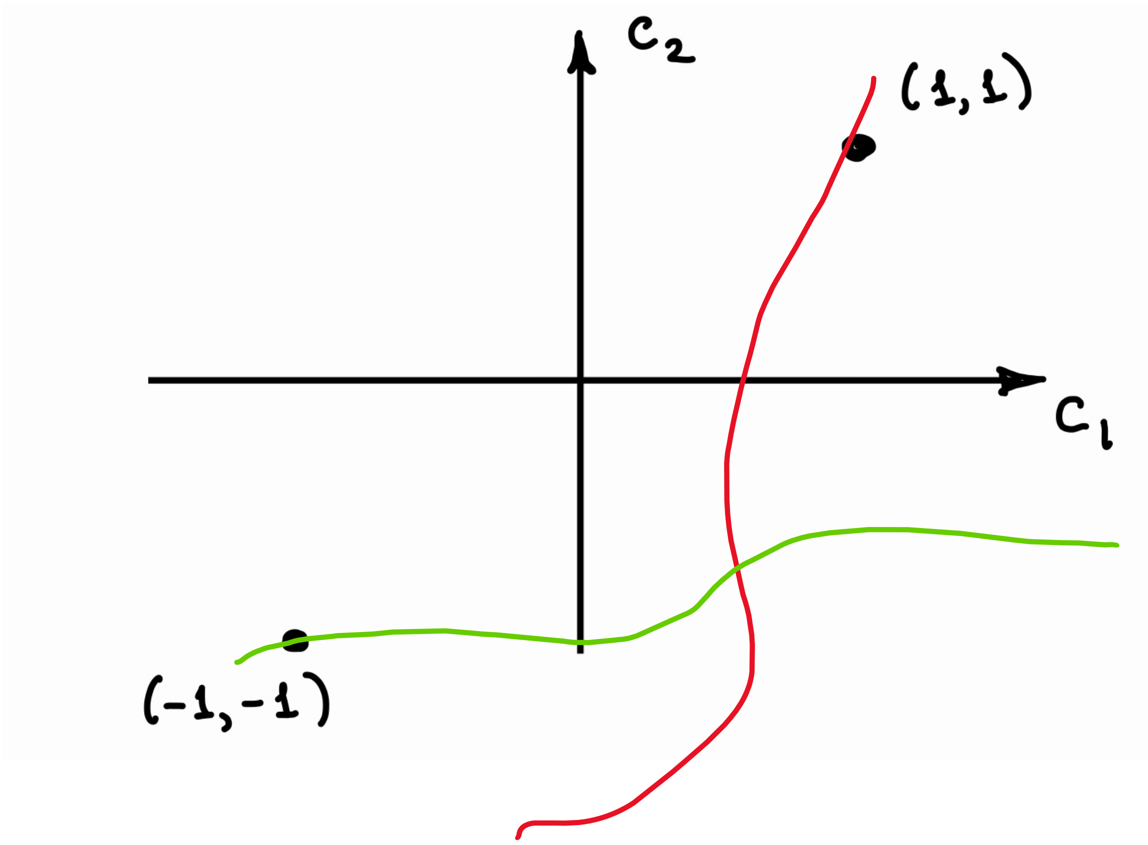
Including in the system cross-flow automatically “slows down” fingers (through creating an intermediate concentration)

time





Scheme of proof



1. What are the states (c_1^*, c_2^*) such that there exists a travelling wave with velocity v^f
$$(c_1^*, c_2^*) \to (1{,}1)?$$

For any $v^f \in \mathbb{R}$ there exists a unique such (c_1^*, c_2^*). We get a curve, parametrized by v^f.

2. What are the states (c_1^*, c_2^*) such that there exists a travelling wave with velocity v^b
$$(-1,-1) \to (c_1^*, c_2^*)?$$

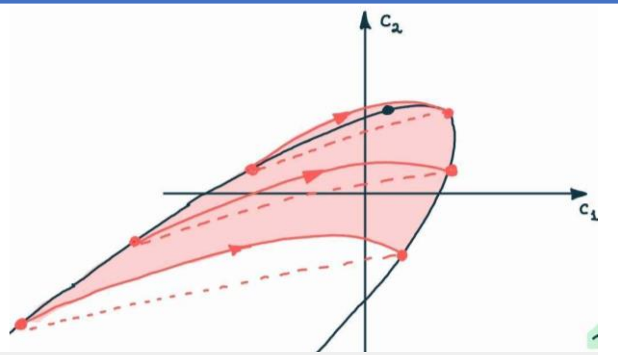
For any $v^b \in \mathbb{R}$ there exists a unique such (c_1^*, c_2^*). We get a curve, parametrized by v^b.

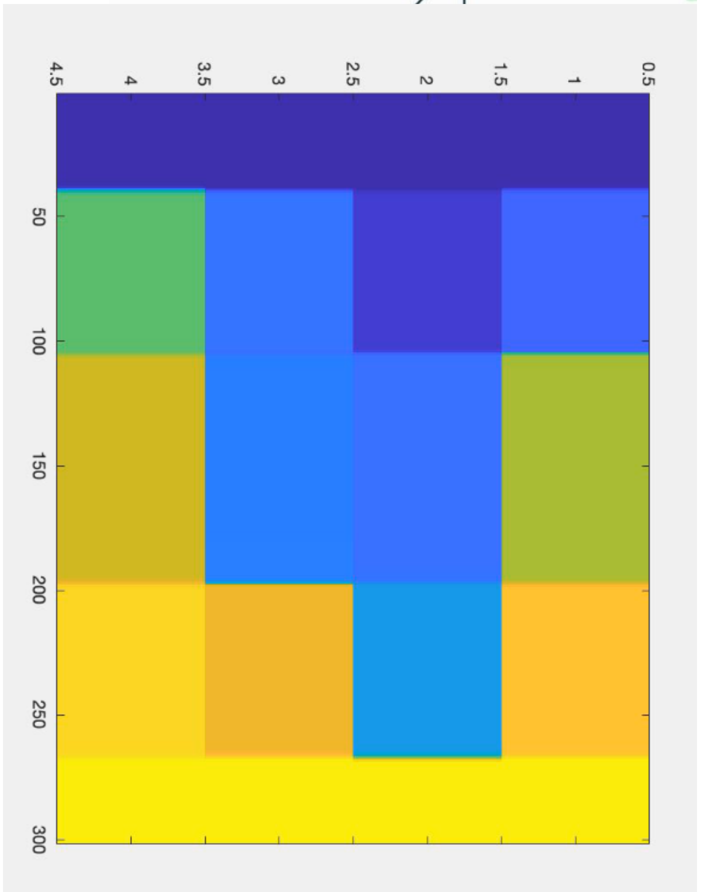
3. The intersection of these curves gives the desired intermediate concentration!

Discussion

1. TFE – the proof is based on analytic expressions for heteroclinic orbits and invariant manifolds for the corresponding 3-dim traveling wave dynamical system

2. IPM – similar result is true: IPM considered as a singular perturbation of TFE model (normal hyperbolicity)





Questions for future:

1. Does the n-tube model posses a system of n travelling waves?
   How to determine their constant states?
   Can we go to the limit as the number of tubes $n \to \infty$?

2. Can we prove similar results for viscosity-driven fingers?

Александр Ильич, с днем рождения!

СПАСИБО за внимание!
Приезжайте в гости в Рио-де-Жанейро!

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4. Homsy, G.M., 1987. Viscous fingering in porous media. Annual review of fluid mechanics, 19(1), pp.271-311.

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Thank you very much!

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