On the impact of diffusion ratio on vanishing viscosity solutions of Riemann problems for chemical flooding models

Yulia Petrova [1,2]

[1] IMPA, Instituto de Matematica Pura e Aplicada, Rio de Janeiro, Brazil

[2] St Petersburg State University, Chebyshev Lab, Russia

https://yulia-petrova.github.io/

Joint work with Fedor Bakharev, Aleksandr Enin and Nikita Rastegaev: arXiv:2111.15001

10 May 2022

Oberseminar "Nonlinear Dynamics"

Joint work with

Chebyshev Laboratory, St Petersburg State University, Russia

Fedor Bakharev

Aleksandr Enin

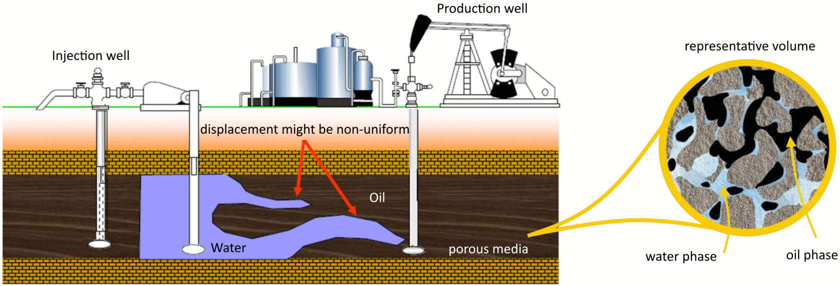
Nikita Rastegaev

- The talk is based on arXiv:2111.15001:
  F. Bakharev, A. Enin, Yu. Petrova, N. Rastegaev "Impact of dissipation ratio on vanishing viscosity solutions of the Riemann problem for chemical flooding model"
- Collaboration with Russian petroleum company GazpromNeft (2018–2021)

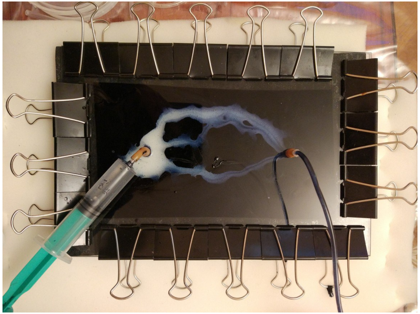
Motivation

We are interested in the mathematical model of oil recovery. Some features:

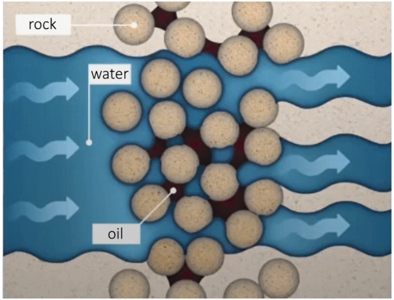
- Porous media (averaged models of flow)
- Unknown variables: $s(t,x)$ — the averaged water saturation in small volume
  $1-s$ — the average oil saturation in representative volume
- Relatively small speeds (≈ 1 meter per day): Navier-Stokes $\to$ Darcy's law
- Multiphase flow: oil, water, gas.
- Applications to EOR (enhanced oil recovery) methods: chemical, thermal, gas etc



Problems: macroscopic and microscopic sweep efficiency



- happens due to very viscous oil or inhomogeneous media



- local entrapment of oil in pores due to high capillary pressure

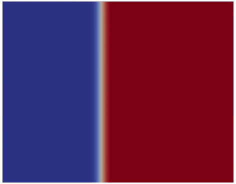
Possible solution

- Inject gas (CO_2, natural) to decrease the oil viscosity
- Add chemicals (polymer) to increase the water viscosity
- Add chemicals (surfactant) that reduce the surface tension etc

Fundamental research: two main directions

1-dim in spatial variable

- Stable displacement



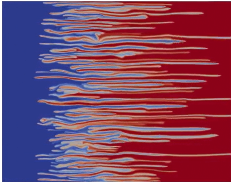
- main question: find an exact solution to a Riemann problem
- hyperbolic conservation laws

$$
\begin{aligned}
s_t + f(s,c)_x &= 0,\\
(cs + a(c))_t + (cf(s,c))_x &= 0.
\end{aligned}
$$

Example: chemical flooding model

2-dim (or 3-dim) in spatial variable

- Unstable displacement



- source of instability: water and oil/polymer have different viscosities
- viscous fingering phenomenon

$$
\begin{aligned}
c_t + u \cdot \nabla c &= \varepsilon \Delta c,\\
\operatorname{div}(u) &= 0,\\
u &= -\nabla p / \mu(c).
\end{aligned}
$$

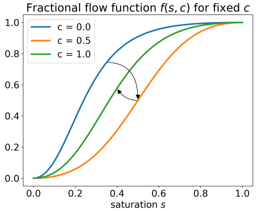
Example: Peaceman model

Problem statement

Chemical flooding can be described as the system of conservation laws ($x \in \mathbb{R}$, $t > 0$):

$$\begin{aligned} s_t + f(s,c)_x &= 0, && \text{(conservation of water)} \\ (cs + a(c))_t + (cf(s,c))_x &= 0. && \text{(conservation of chemical)} \end{aligned} \tag{1}$$

- $s = s(x,t)$ — water phase saturation;
- $f(s,c)$ — fractional flow function (usually S-shaped);
- $c = c(x,t)$ — concentration of a chemical agent in water;
- $a(c)$ — adsorption of a chemical agent on a rock (usually increasing, concave).



Initial data:

$$(s,c)\big|_{t=0} = \begin{cases} (1,1), & \text{if } x \leqslant 0, \\ (0,0), & \text{if } x > 0, \end{cases} \tag{2}$$

Aim:

Find a solution to initial-value problem (1)–(2) when f depends non-monotonically on c.

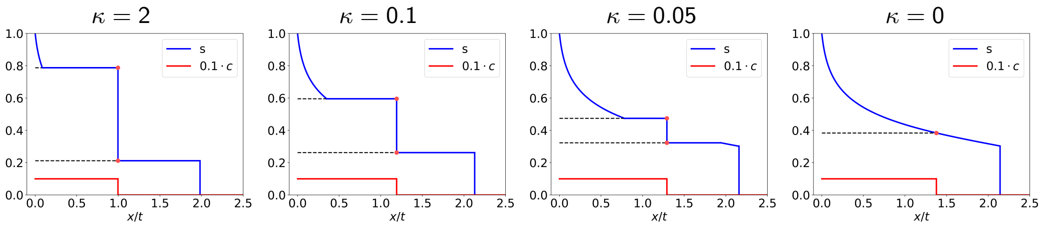
NB: Non-monotone dependence appears in surfactant flooding, low salinity water flooding etc

Preview of results

- weak solutions $\Rightarrow$ non-uniqueness of solutions to a Riemann problem
- use vanishing viscosity criterion — add small diffusion/capillary terms

$$\boxed{\begin{aligned} s_t + f(s,c)_x &= \varepsilon_c (A(s,c)s_x)_x, \\ (cs + a(c))_t + (cf(s,c))_x &= \varepsilon_c (cA(s,c)s_x)_x + \varepsilon_d c_{xx}. \end{aligned}}$$

- $f(s,c)$ monotone in $c \Rightarrow$ uniqueness of vanishing viscosity solution (1988)
- Main idea of the talk: $f(s,c)$ non-monotone in $c \Rightarrow$ exist multiple vanishing viscosity solutions, depending on ratio $\kappa = \varepsilon_d/\varepsilon_c$



NB: the appeared shock is known as undercompressive (transitional).

Hyperbolic systems of conservation laws*

$$G(u)_t + F(u)_x = 0 \tag{3}$$

Here

- $G(u)$ — accumulation function (conserved quantities)
- $F(u)$ — flux function (flux of conserved quantities)

Simplest example: wave equation

$$y_{tt} - c^2 y_{xx} = 0 \qquad \text{(J. d'Alambert, 1750)}$$

can be rewritten as a system of two first-order equations on the state-vector $u = \begin{pmatrix} y_x \\ y_t \end{pmatrix}$

$$u_t + D u_x = 0, \qquad \text{with} \qquad D = \begin{pmatrix} 0 & -1 \\ -c^2 & 0 \end{pmatrix}$$

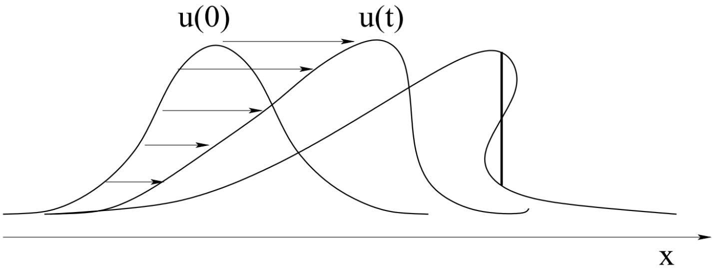
- eigenvalues $\lambda_1 = c$ and $\lambda_2 = -c$ are real, the system is hyperbolic. Solution contain two wave modes that propagate at the velocities λ_1 and λ_2.

* For more detailes see e.g. "Hyperbolic conservation laws: an illustrated tutorial" by Alberto Bressan.

Hyperbolic systems of conservation laws

$$u_t + \left(\frac{u^2}{2}\right)_x = 0 \qquad \text{(Burger's equation, 1948)}$$

- non-linearity implies wave speed $\lambda(u) = u$ depends on state u
- So the wave can spread (rarefaction wave) or concentrate (shock wave)



$$u_t + (f(u))_x = 0 \qquad \text{(Buckley-Leverett equation)}$$

- existence, uniqueness was estabibliblished by Olga Oleinik (1957)

Riemann problem (1858)

- Riemann solved the initial-value problem with data having a single jump

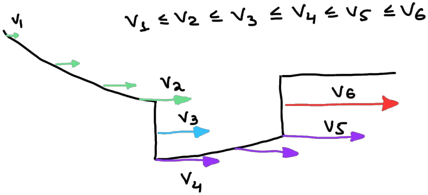
$$u\Big|_{t=0} = \begin{cases} u^L, & x \leqslant 0; \\ u^R, & x > 0. \end{cases}$$

- took advantage of the scale invariance of the equations and the data:

$$u(\alpha\, x, \alpha\, t) = u(x,t) \quad \text{for all } \alpha > 0$$

- solution to a Riemann problem is important because:
  - it appears in a long-term behavior of Cauchy problem
  - helps to prove the existence of solutions to Cauchy problem (Glimm's method)
  - helps to construct numerical solution (Godunov method)

Any solution to a Riemann problem consists of a sequence of rarefaction or shock waves (and constant states) that are compatible by speeds



Shock waves: RH condition and admissibility criteria

- discontinuous solutions are defined in the sense of distributions (weak form)
- for a shock wave from u^- to u^+ moving with velocity v, the weak condition amounts to the following Rankine-Hugoniot condition (RH)

$$-v\,G(u^-)+F(u^-)=-v\,G(u^+)+F(u^+) \qquad \text{(RH)}$$

- RH means conservation: what flows into left side flows out of the right side

- Problems from the perspectives of both mathematics and physics:
  - if all RH solutions are allowed, a Riemann problem has multiple solutions
  - some RH solutions violate physical principles
- Vanishing viscosity criteria: consider a diffusive system of conservation laws

$$G(u)_t+F(u)_x=\varepsilon\,[B(u)\,u_x]_x\,, \qquad \varepsilon\to 0$$

Travelling wave solutions of diffusive system (Hopf, 1948)

- $u(x,t) = \hat{u}(\xi)$ with $\xi := x - v\,t$ for a fixed shock velocity v
- reduction to first-order system of ordinary differential equations:

$$\varepsilon\, B(\hat{u})\, \hat{u}_\xi = -v\left[G(\hat{u}) - G(u^-)\right] + F(\hat{u}) - F(u^-)$$

- u^- and u^+ are fixed points and we look for an orbit connecting them

$$\hat{u}(-\infty) = u^-, \qquad \hat{u}(+\infty) = u^+$$

- diffusive terms cause a shock wave to have a thin, smooth internal structure as a result of balancing nonlinear focusing and diffusive spreading
- travelling wave solution approaches the jump discontinuity in L^1 as $\varepsilon \to 0^+$

Questions?

Historical review: zero adsorption

$$\begin{aligned} s_t + f(s,c)_x &= 0, \\ (sc)_t + (cf(s,c))_x &= 0. \end{aligned} \qquad (s,c)\big|_{t=0} = \begin{cases} (s_L, c_L), & \text{if } x \leqslant 0, \\ (s_R, c_R), & \text{if } x > 0, \end{cases} \tag{4}$$

Can be rewritten in a more symmetric form:

$$s_t + (sg(s,b))_x = 0, \qquad b = sc \text{ — total amount of chemicals}$$

$$b_t + (bg(s,b))_x = 0. \qquad g = f/s \text{ — new flux function}$$

- 1980 — elasticity theory (Keyfitz, Kranzer): $g(s,b) = \tilde{g}(s^2 + b^2)$.
- 1980 — polymer flooding (Isaacson, Glimm, Temple): $f(s,c)$ monotone in c
- many-many generalisations (delta shocks etc)

Interesting properties:

- has coordinate system of Riemann invariants
- rarefaction and shock curves coincide

Problems:

- non-strictly hyperbolic system
- has contact discontinuities

Work in progress with D. Marchesin, B. Plohr: justification of Isaacson-Glimm admissibility criterion & construction of a solution for the case when $f(s,c)$ is non-monotone in c

Historical review: non-zero adsorption

$$
\begin{aligned}
s_t + f(s,c)_x &= 0, \\
(sc + a(c))_t + (cf(s,c))_x &= 0.
\end{aligned}
\qquad
(s,c)\big|_{t=0} = \begin{cases} (s_L, c_L), & \text{if } x \leqslant 0, \\ (s_R, c_R), & \text{if } x > 0, \end{cases}
\tag{5}
$$

Proposition (Johansen-Winther, 1988 (JW))

The solution $c(x,t)$ is monotone in x for any $t > 0$. If $a(c)$ is concave, then

- *is constant if $c_L = c_R$;*
- *contains several rarefaction waves if $c_L < c_R$;*
- *contains exactly one shock wave if $c_L > c_R$.*

Historical review:

- 1988 — JW, $f(s,c)$ monotone in c. Found a unique vanishing viscosity solution.

When $f(s,c)$ non-monotone in c, multiple vanishing viscosity solutions are possible.

- 2017 — W. Shen, gave some examples in Lagrangian coordinates. See also
- 1986 — Entov, Kerimov, non-rigorous consideration of the non-monotone case.

Reduced problem

$$\begin{aligned} s_t + f(s,c)_x &= 0, \\ (sc + a(c))_t + (cf(s,c))_x &= 0. \end{aligned} \qquad (s,c)\big|_{t=0} = \begin{cases} (1,1), & \text{if } x \leqslant 0, \\ (0,0), & \text{if } x > 0, \end{cases} \tag{6}$$

Proposition (Johansen-Winther, 1988 (JW))

There exists $u^- = (s^-, 1)$ *and* $u^+ = (s^+, 0)$ *such that the solution to a Riemann problem has the following structure:*

$$(1,1) \xrightarrow{c=1} u^- \xrightarrow{c \text{ jumps from } 1 \text{ to } 0} u^+ \xrightarrow{c=0} (0,0). \tag{7}$$

Historical review:

- 1988 — JW, $f(s,c)$ monotone in c. Found a unique vanishing viscosity solution.

When $f(s,c)$ non-monotone in c, multiple vanishing viscosity solutions are possible.

- 2017 — W. Shen, gave some examples in Lagrangian coordinates. See also
- 1986 — Entov, Kerimov, non-rigorous consideration of the non-monotone case.

Dissipative system

To define a shock wave between u^- and u^+ we consider dissipative system:

$$
\begin{aligned}
s_t + f(s,c)_x &= \varepsilon_c (A(s,c)s_x)_x, \\
(cs+\alpha)_t + (cf(s,c))_x &= \varepsilon_c (cA(s,c)s_x)_x + \varepsilon_d c_{xx}, \\
\alpha_t &= \varepsilon_r^{-1}(a(c)-\alpha).
\end{aligned}
$$

- ε_c — dimensionless capillary pressure
- ε_d — dimensionless diffusion term
- ε_r — dimensionless relaxation time

- $A(s,c)$ — capillary pressure term
- $\alpha = \alpha(x,t)$ — dynamic adsorption

We consider two particular cases:

Capillarity and Diffusion

$$
\begin{aligned}
&s_t + f(s,c)_x = \varepsilon_c (A(s,c)s_x)_x, \\
&(cs + a(c))_t + (cf(s,c))_x = \varepsilon_c (cA(s,c)s_x)_x + \varepsilon_d c_{xx},
\end{aligned}
$$

Capillarity and Dynamic Adsorpsion

$$
\begin{aligned}
&s_t + f(s,c)_x = \varepsilon_c (A(s,c)s_x)_x. \\
&(cs+\alpha)_t + (cf(s,c))_x = \varepsilon_c (cA(s,c)s_x)_x, \\
&\alpha_t = \varepsilon_r^{-1}(a(c)-\alpha).
\end{aligned}
$$

Restrictions on f and a

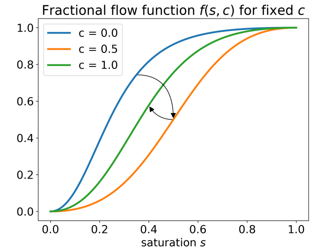
(F1) $f \in C^2([0,1]^2)$; $f(0,c) = 0$; $f(1,c) = 1$;

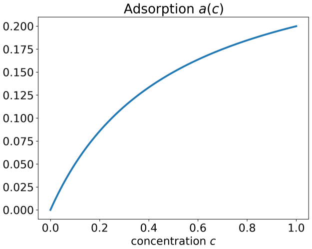
(F2) $f_s(s,c) > 0$ for $s \in (0,1)$, $c \in [0,1]$;
$f_s(0,c) = f_s(1,c) = 0$;

(F3) f is S-shaped in s;

(F4) f is non-monotone in c:
$\forall s \in (0,1)\, \exists c^*(s) \in (0,1)$:

- $f_c(s,c) < 0$ for $0 < s < 1$, $0 < c < c^*(s)$;
- $f_c(s,c) > 0$ for $0 < s < 1$, $c^*(s) < c < 1$;





(A) A is bounded from zero and infinity;

(a) $a \in C^2$, $a(0) = 0$, a is strictly increasing and concave.

NB: assumptions are just the simplest possible ones. Could be generalised.

Travelling wave dynamical system

$$\begin{aligned} s_t + f(s,c)_x &= \varepsilon_c (A(s,c)s_x)_x, \\ (cs + a(c))_t + (cf(s,c))_x &= \varepsilon_c (cA(s,c)s_x)_x + \varepsilon_d c_{xx}. \end{aligned} \tag{8}$$

Searching for travelling wave solutions $s = s(\xi)$, $c = c(\xi)$, $\xi := \varepsilon_c^{-1}(x - vt)$ with boundary conditions

$$s(\pm\infty) = s^{\pm}, \qquad c(-\infty) = 1, \qquad c(+\infty) = 0,$$

we arrive at

$$\boxed{\begin{aligned} A(s,c)s_\xi &= f(s,c) - v(s + d_1), \\ \kappa c_\xi &= v(d_1 c - d_2 - a(c)). \end{aligned}} \tag{9}$$

- Parameters: $\kappa = \varepsilon_d/\varepsilon_c$ and v. $d_1 = a(1)$ and $d_2 = 0$ — some constants.
- Note that $u^{\pm}$ are fixed points of dynamical system (9);
- We are only interested in the trajectories connecting two saddle points (or saddle-nodes) due to compatibility of speeds in

$$(1,1) \to u^- \xrightarrow{\text{c-shock}} u^+ \to (0,0).$$

Main result

Consider a dynamical system under assumptions (F1)–(F4), (A), (a):

$$\boxed{\begin{aligned} A(s,c)s_\xi &= f(s,c) - v(s+d_1), \\ \kappa c_\xi &= v(d_1 c - d_2 - a(c)). \end{aligned}}$$

Theorem (Bakharev, Enin, P.,Rastegaev, 2021, arxiv:2111.15001)

There exist $0 < v_{\min} < v_{\max} < \infty$, *such that for every* $\kappa = \varepsilon_d/\varepsilon_c \in (0, +\infty)$, *there exist unique*

- *points* $s^-(\kappa) \in [0,1]$ *and* $s^+(\kappa) \in [0,1]$;
- *velocity* $v(\kappa) \in [v_{\min}, v_{\max}]$,

such that there exists a travelling wave, connecting two saddle points $u^-(\kappa) = (s^-(\kappa), 1)$ *and* $u^+(\kappa) = (s^+(\kappa), 0)$ *with velocity* $v(\kappa)$. *Moreover,* $v(\kappa)$ *is monotone and continuous;* $v(\kappa) \to v_{\min}$ *as* $\kappa \to \infty$; $v(\kappa) \to v_{\max}$ *as* $\kappa \to 0$.

Solution construction algorithm

1. From κ we calculate $v(\kappa)$ (binary search).
2. From v we determine $s^-(v)$ and $s^+(v)$ via Rankine-Hugoniot condition.
3. Construct waves $(1,1) \to (s^-(v), 1)$ and $(s(v), 0) \to (0,0)$.

Example: "boomerang" model:

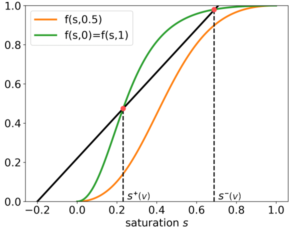


Figure 1: Flux functions

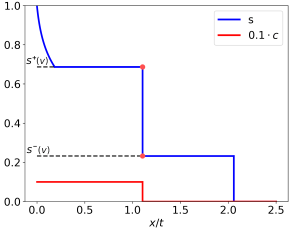


Figure 2: Solution s and c

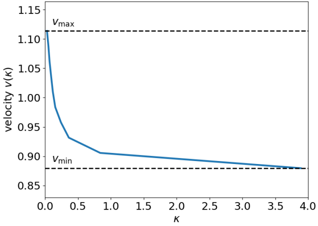


Figure 3: Function $v(\kappa)$

We would like to investigate...

Convergence

Does the Riemann problem solution of the dissipative system (8) converge to the non-dissipative solution as $\varepsilon_c \to 0$ with fixed κ?

Asymptotic stability as $t \to \infty$

Does the solution of a Cauchy problem for the dissipative system (8) with the correct values at $\pm\infty$ tend to undercompressible travelling wave as $t \to +\infty$?

I believe that "yes", but...

- Difficulty: comparison theorems do not work for systems
- Hope 1: without diffusion terms the system decouples (in Lagrangian coordinates)
- Hope 2: "steepness" argument works? (like for Fisher-KPP reaction-diffusion eq.)
- any ideas?

Scheme of proof

The Theorem can be divided into simpler statements:

- $\forall v \in [v_{\min}, v_{\max}] \quad \exists! \kappa(v)$: there is a saddle-to-saddle travelling wave with $\kappa(v)$.
- $\kappa(v)$ is continuous.
- $\nexists v_1 \neq v_2 : \kappa(v_1) = \kappa(v_2)$, thus $\kappa(v)$ is monotone.
- $\kappa(v) \to \infty$ as $v \to v_{min}$.
- $\kappa(v) \to \kappa_{crit} \geqslant 0$ as $v \to v_{max}$.
- When $\kappa < \kappa_{crit}$ and $v = v_{\max}$ there is a saddle to saddle-node travelling wave

$\kappa(v)$ is monotone and continuous thus there exists an inverse function satisfying the Theorem.

Phase portrait, fixed points, isoclines

We consider travelling wave dynamical system:

$$\begin{aligned} A(s,c)s_\xi &= f(s,c) - v(s+d_1), \\ \kappa c_\xi &= v(d_1 c - d_2 - a(c)), \end{aligned}$$

Isoclines:

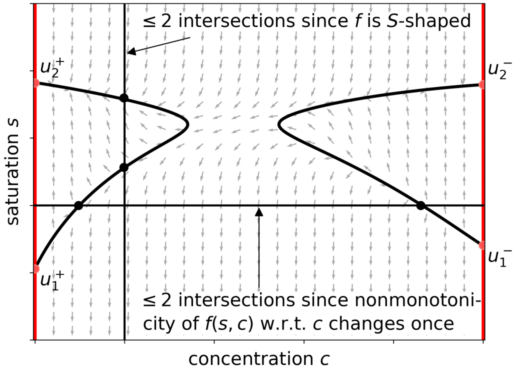
red lines are $d_1 c - d_2 - a(c) = 0$,

black lines are $f(s,c) - v(s+d_1) = 0$.

Fixed points:

u_1^+ and u_2^- — saddle points;

u_2^+ — attractor; u_1^- — repeller



Aim:

find all pairs (κ, v) for which there exists a trajectory from saddle point u_2^- to saddle point u_1^+

Nullcline configurations: main and intermediate classes

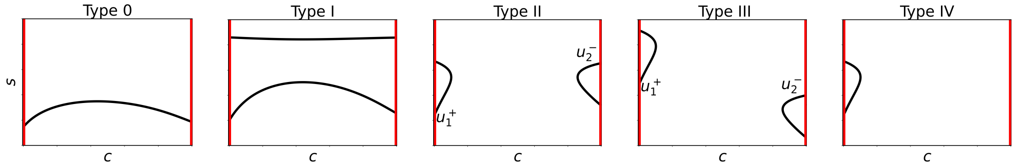


Figure 4: Five wide classes of nullcline configurations

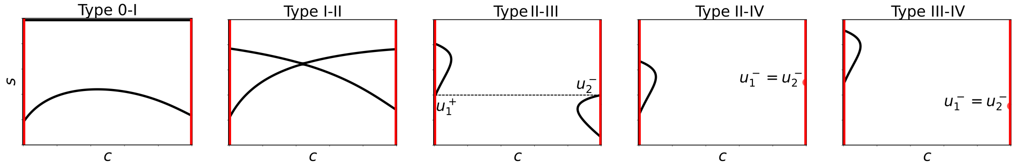
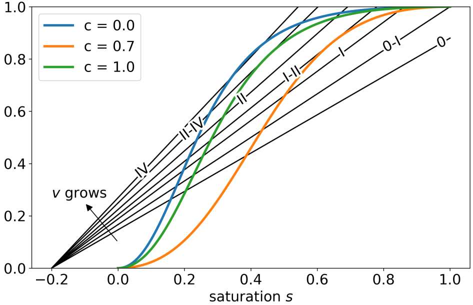


Figure 5: Intermediate types of nullcline configurations, appearing under assump. (F1)–(F4)

- Only Type II nullcline configuration has saddle-to-saddle connections.
- Type I-II corresponds to $v_{\min}$.
- Type II-III or Type II-IV correspond to $v_{\max}$.

Nullcline configurations: monotone dependence on v

black lines $f(s,c) = v(s + d_1)$



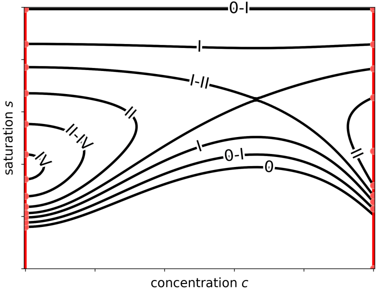


Figure 6: nullcline configuration evolution as v grows: Type 0 $\to$ Type I $\to$ Type II $\to$ Type IV

Nullcline configurations: bad cases

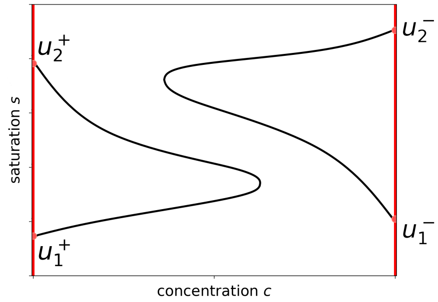


Figure 7: If f is not S-shaped.

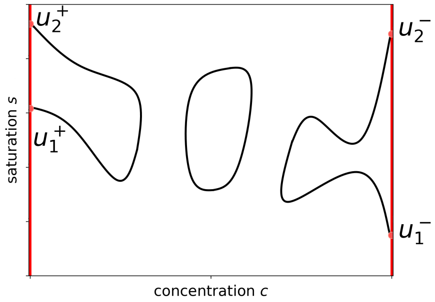


Figure 8: If non-monotonicity is more complex.

We believe that the similar result is true without conditions (F3)–(F4).

Type II configuration: for every v there exist κ

$$\boxed{\frac{s_\xi}{c_\xi} = \kappa \cdot \frac{v^{-1} f(s,c) - (s+d_1)}{A(s,c)(d_1 c - d_2 - a(c))}}$$

Used property: continuous and monotonous dependence of trajectories on κ.

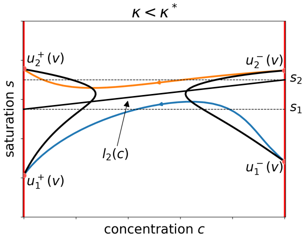


Figure 9: $\kappa << 1$.

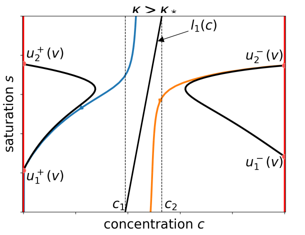


Figure 10: $\kappa >> 1$.

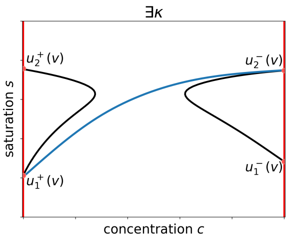


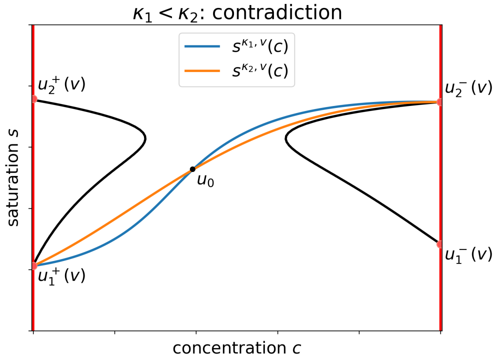
Figure 11: $\exists\kappa$.

Type II config.: $\kappa(v)$ is unique for every $v \in (v_{\min}, v_{\max})$.

If there are $\kappa_1 < \kappa_2$ for one v, then the corresponding trajectories must intersect, which leads to a contradiction.
The slope

$$\boxed{\frac{s_\xi}{c_\xi} = \kappa \cdot \frac{v^{-1} f(s,c) - (s + d_1)}{A(s,c)(d_1 c - d_2 - a(c))}}$$

is positive at the intersection point (s, c), so it strictly increases when κ increases.



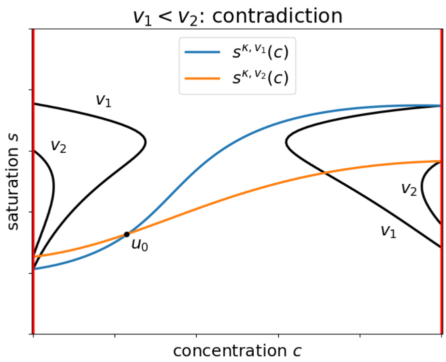
NB: this property might be lost for more complex nullcline configurations.

Type II configuration: monotonicity of $\kappa(v)$

If $\kappa(v_1) = \kappa(v_2)$ for $v_1 < v_2$, then the corresponding trajectories must intersect, which leads to a contradiction. The slope

$$\frac{s_\xi}{c_\xi} = \kappa \cdot \frac{v^{-1} f(s,c) - (s + d_1)}{A(s,c)(d_1 c - d_2 - a(c))}$$

is positive at the intersection point (s, c), so it strictly increases when v increases.



Possible directions for future research

- General classes of f and a: when the dependence $v(\kappa)$ is nontrivial?
- Construct solution to any Riemann problem $(s_L, c_L) \to (s_R, c_R)$
- Convergence and asymptotic stability as $t \to \infty$ (as mentioned above)
- Consider a three-phase flow with chemicals (water, oil and gas):
  travelling wave dynamical system will become three-dimensional, thus the analysis will be more complex.

Vielen Dank für Ihre Aufmerksamkeit!

yulia.petrova@impa.br https://yulia-petrova.github.io/

Literature

Own works:

- F. Bakharev, A. Enin, Yu. Petrova, N. Rastegaev, Impact of dissipation ratio on vanishing viscosity solutions of the Riemann problem for chemical flooding model. arXiv:2111.15001.
- Yu. Petrova, D. Marchesin, B. Plohr. Vanishing adsorption admissibility criteria for contact discontinuities for the Glimm-Isaacson model. Work in progress. See slides.

Other works:

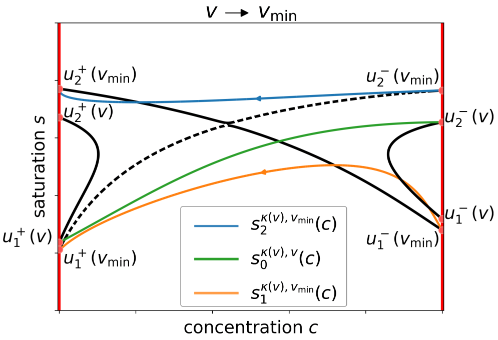
- Johansen, T. and Winther, R., 1988. The solution of the Riemann problem for a hyperbolic system of conservation laws modeling polymer flooding. SIAM journal on mathematical analysis, 19(3), pp.541-566.
- Shen, W., 2017. On the uniqueness of vanishing viscosity solutions for Riemann problems for polymer flooding. Nonlinear Differential Equations and Applications NoDEA, 24(4), pp.1-25.
- Entov, V.M. and Kerimov, Z.A., 1986. Displacement of oil by an active solution with a nonmonotonic effect on the flow distribution function. Fluid Dynamics, 21(1), pp.64-70.
- Keyfitz, B.L. and Kranzer, H.C., 1980. A system of non-strictly hyperbolic conservation laws arising in elasticity theory. Archive for Rational Mechanics and Analysis, 72(3), pp.219-241.
- Isaacson E., 1980. Global solution of a Riemann problem for a non-strictly hyperbolic system of conservation laws arising in enhanced oil recovery // Rockefeller University, NY preprint.
- Bressan, A., 2013. Hyperbolic conservation laws: an illustrated tutorial. In Modelling and optimisation of flows on networks (pp. 157-245). Springer, Berlin, Heidelberg.

$\kappa \to +\infty$ as $v \to v_{\min}$

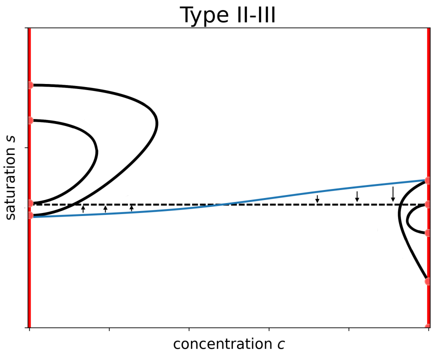
For any finite κ:

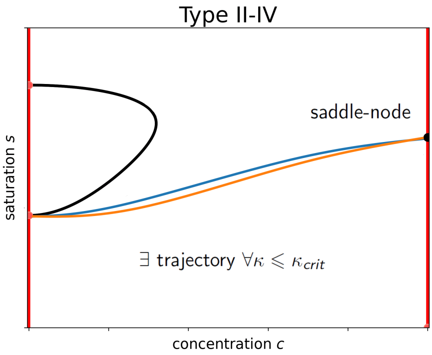
- green orbit is between blue and orange;
- blue orbit is higher than - - - -;
- orange orbit is lower than - - - -.

When $v \to v_{\min}$ the limits of green, blue and orange orbits coincide, which can not happen for any finite κ.



$\kappa \to 0$ as $v \to v_{\max}$ $\kappa \to \kappa_{crit}$ as $v \to v_{\max}$





Non-strictly hyperbolic system and Lax condition

$$\begin{pmatrix} s_t \\ c_t \end{pmatrix} + \begin{pmatrix} f_s(s,c) & f_c(s,c) \\ 0 & \frac{f(s,c)}{s+a'(c)} \end{pmatrix} \begin{pmatrix} s_x \\ c_x \end{pmatrix} = 0. \qquad \lambda_s = f_s(s,c), \quad \lambda_c = \frac{f(s,c)}{s+a'(c)}.$$

Monotone (in c) case is well-studied (Johansen-Winther, 1988). Lax condition gives unique solution, no other c-shocks have structure (no other vanishing viscocity solutions).

Lax condition in non-monotone case sometimes gives physically meaningless solutions.

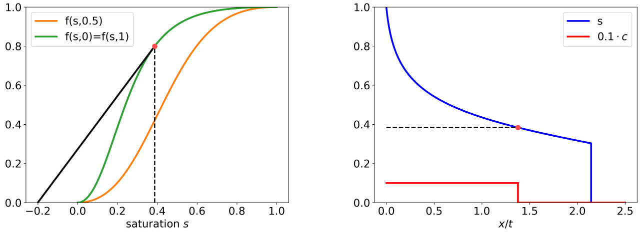


Figure 12: “boomerang” model: solution to a Riemann problem for Lax admissibility criterion

Thus, we study c-shocks that have structure (vanishing viscosity travelling wave solutions).

Decoupling in Lagrangian coordinates

$$\boxed{\begin{aligned} s_t + f(s,c)_x &= 0, \\ (sc + a(c))_t + (cf(s,c))_x &= 0. \end{aligned}} \tag{10}$$

Introduce Lagrangian coordinates (ψ, ϕ):

$$\boxed{\begin{aligned} \frac{\partial}{\partial \phi}\left(\frac{s}{f(s,c)}\right) + \frac{\partial}{\partial \psi}\left(\frac{1}{f(s,c)}\right) &= 0, \\ \frac{\partial}{\partial \phi} a(c) + \frac{\partial}{\partial \psi} c &= 0. \end{aligned}} \tag{11}$$

- equation on concentration c is decoupled from equation on saturation s
- decoupling is lost if we add diffusion/capillary terms