

On the impact of diffusion ratio on vanishing viscosity solutions of Riemann problems for chemical flooding models

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Joint work with Fedor Bakharev, Aleksandr Enin and Nikita Rastegaev: arXiv:2111.15001



10 May 2022
Oberseminar "Nonlinear Dynamics"

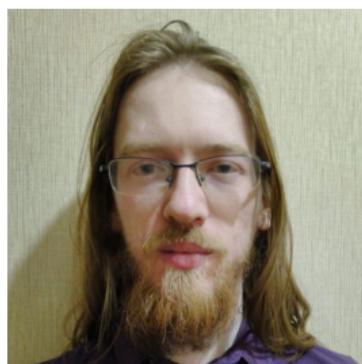


Joint work with

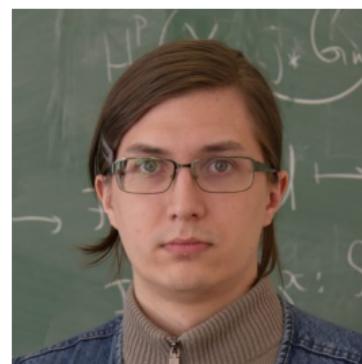
Chebyshev Laboratory, St Petersburg State University, Russia



Fedor Bakharev



Aleksandr Enin



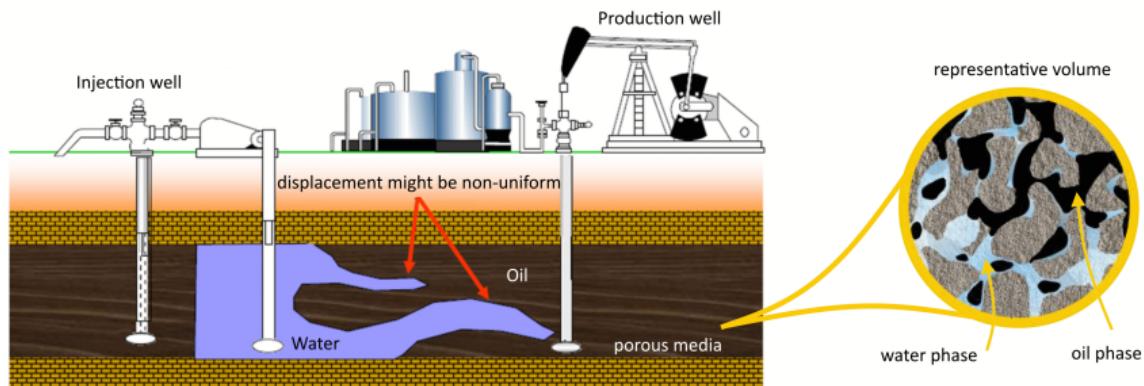
Nikita Rastegaev

- The talk is based on arXiv:2111.15001:
F. Bakharev, A. Enin, Yu. Petrova, N. Rastegaev “Impact of dissipation ratio on vanishing viscosity solutions of the Riemann problem for chemical flooding model”
- Collaboration with Russian petroleum company GazpromNeft (2018–2021)

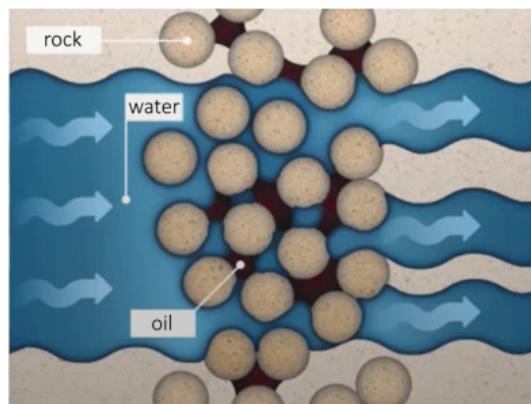
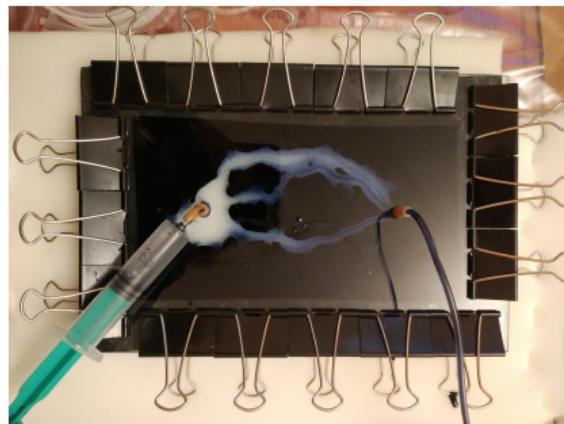
Motivation

We are interested in the mathematical model of oil recovery. Some features:

- Porous media (averaged models of flow)
- Unknown variables: $s(t, x)$ — the averaged water saturation in small volume
 $1 - s$ — the average oil saturation in representative volume
- Relatively small speeds (≈ 1 meter per day): Navier-Stokes \rightarrow Darcy's law
- Multiphase flow: oil, water, gas.
- Applications to EOR (enhanced oil recovery) methods: chemical, thermal, gas etc



Problems: macroscopic and microscopic sweep efficiency



- happens due to very viscous oil or inhomogeneous media

- local entrapment of oil in pores due to high capillary pressure

Possible solution

- Inject gas (CO_2 , natural) to decrease the oil viscosity
- Add **chemicals (polymer)** to increase the water viscosity
- Add **chemicals (surfactant)** that reduce the surface tension etc

Fundamental research: two main directions

1-dim in spatial variable

- Stable displacement



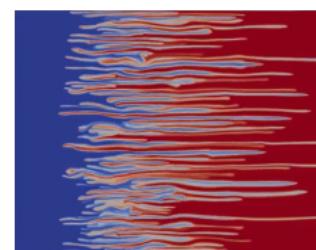
- main question: find an exact solution to a Riemann problem
- **hyperbolic conservation laws**

$$\begin{aligned} s_t + f(s, c)_x &= 0, \\ (cs + a(c))_t + (cf(s, c))_x &= 0. \end{aligned}$$

Example: chemical flooding model

2-dim (or 3-dim) in spatial variable

- Unstable displacement



- source of instability: water and oil/polymer have different viscosities
- **viscous fingering phenomenon**

$$\begin{aligned} c_t + u \cdot \nabla c &= \varepsilon \Delta c, \\ \operatorname{div}(u) &= 0, \\ u &= -\nabla p / \mu(c). \end{aligned}$$

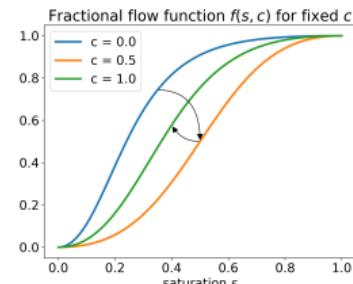
Example: Peaceman model

Problem statement

Chemical flooding can be described as the system of conservation laws ($x \in \mathbb{R}, t > 0$):

$$\begin{aligned} s_t + f(s, c)_x &= 0, && \text{(conservation of water)} \\ (cs + a(c))_t + (cf(s, c))_x &= 0. && \text{(conservation of chemical)} \end{aligned} \quad (1)$$

- $s = s(x, t)$ — water phase saturation;
- $f(s, c)$ — fractional flow function (usually *S*-shaped);
- $c = c(x, t)$ — concentration of a chemical agent in water;
- $a(c)$ — adsorption of a chemical agent on a rock (usually increasing, concave).



Initial data:

$$(s, c)|_{t=0} = \begin{cases} (1, 1), & \text{if } x \leq 0, \\ (0, 0), & \text{if } x > 0, \end{cases} \quad (2)$$

Aim:

Find a solution to initial-value problem (1)–(2) when f depends non-monotonically on c .

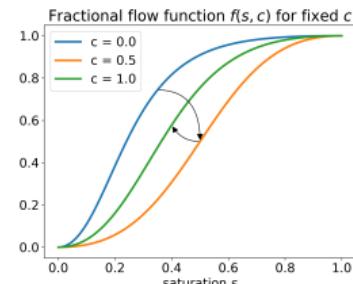
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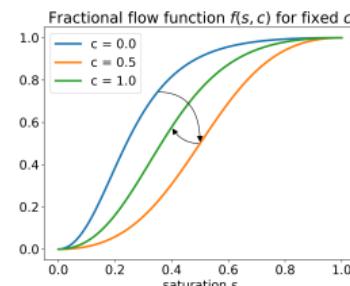
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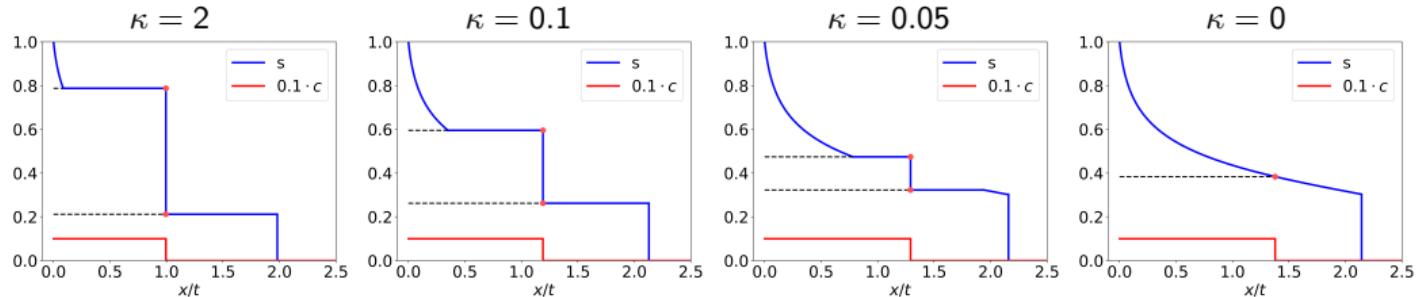
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Preview of results

- weak solutions \Rightarrow non-uniqueness of solutions to a Riemann problem
- use vanishing viscosity criterion — add small diffusion/capillary terms

$$s_t + f(s, c)_x = \varepsilon_c (A(s, c)s_x)_x,$$
$$(cs + a(c))_t + (cf(s, c))_x = \varepsilon_c (cA(s, c)s_x)_x + \varepsilon_d c_{xx}.$$

- $f(s, c)$ monotone in $c \Rightarrow$ uniqueness of vanishing viscosity solution (1988)
- Main idea of the talk:** $f(s, c)$ non-monotone in $c \Rightarrow$ exist multiple vanishing viscosity solutions, depending on ratio $\kappa = \varepsilon_d / \varepsilon_c$



NB: the appeared shock is known as undercompressive (transitional).

Hyperbolic systems of conservation laws*

$$G(u)_t + F(u)_x = 0 \quad (3)$$

Here

- $G(u)$ — accumulation function (conserved quantities)
- $F(u)$ — flux function (flux of conserved quantities)

Simplest example: wave equation

$$y_{tt} - c^2 y_{xx} = 0 \quad (\text{J. d'Alambert, 1750})$$

can be rewritten as a system of two first-order equations on the state-vector $u = \begin{pmatrix} y_x \\ y_t \end{pmatrix}$

$$u_t + Du_x = 0, \quad \text{with} \quad D = \begin{pmatrix} 0 & -1 \\ -c^2 & 0 \end{pmatrix}$$

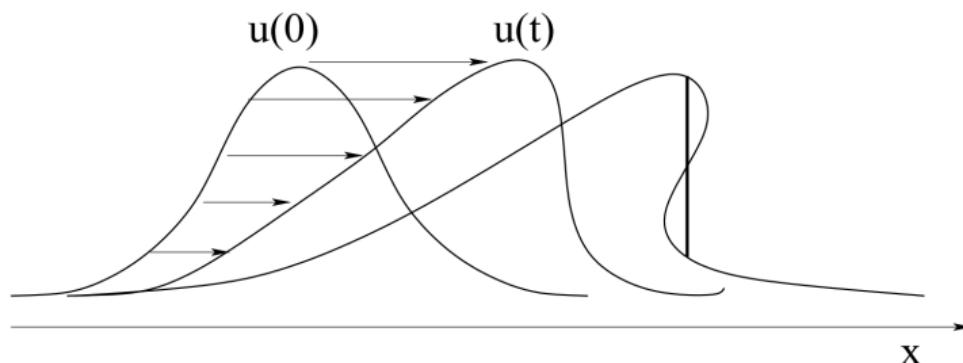
- eigenvalues $\lambda_1 = c$ and $\lambda_2 = -c$ are real, the system is hyperbolic. Solution contain two wave modes that propagate at the velocities λ_1 and λ_2 .

* For more details see e.g. "Hyperbolic conservation laws: an illustrated tutorial" by Alberto Bressan.

Hyperbolic systems of conservation laws

$$u_t + \left(\frac{u^2}{2} \right)_x = 0 \quad (\text{Burger's equation, 1948})$$

- non-linearity implies **wave speed** $\lambda(u) = u$ depends on state u
- So the wave can spread (**rarefaction wave**) or concentrate (**shock wave**)



$$u_t + (f(u))_x = 0 \quad (\text{Buckley-Leverett equation})$$

- existence, uniqueness was established by Olga Oleinik (1957)

Riemann problem (1858)

- Riemann solved the initial-value problem with data having a **single jump**

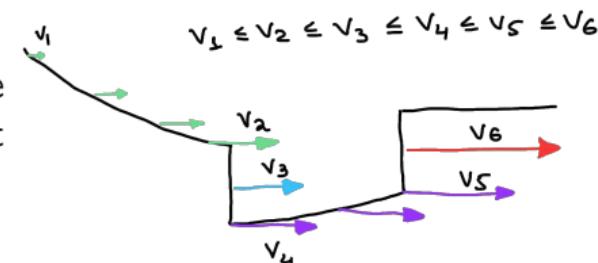
$$u|_{t=0} = \begin{cases} u^L, & x \leq 0; \\ u^R, & x > 0. \end{cases}$$

- took advantage of the **scale invariance** of the equations and the data:

$$u(\alpha x, \alpha t) = u(x, t) \quad \text{for all } \alpha > 0$$

- solution to a Riemann problem is important because:
 - it appears in a long-term behavior of Cauchy problem
 - helps to prove the existence of solutions to Cauchy problem (Glimm's method)
 - helps to construct numerical solution (Godunov method)

Any solution to a Riemann problem consists of a sequence of rarefaction or shock waves (and constant states) that are **compatible by speeds**

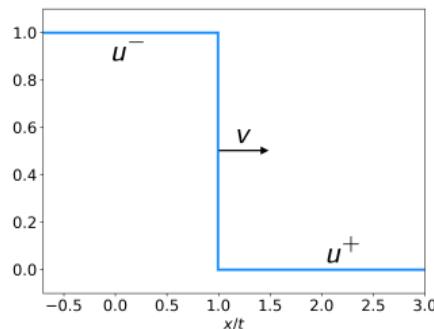


Shock waves: RH condition and admissibility criteria

- discontinuous solutions are defined in the sense of distributions (**weak form**)
- for a shock wave from u^- to u^+ moving with velocity v , the weak condition amounts to the following **Rankine-Hugoniot** condition (RH)

$$-v G(u^-) + F(u^-) = -v G(u^+) + F(u^+) \quad (\text{RH})$$

- RH means conservation: what flows into left side flows out of the right side



- Vanishing viscosity criteria:** consider a diffusive system of conservation laws

$$G(u)_t + F(u)_x = \varepsilon [B(u) u_x]_x, \quad \varepsilon \rightarrow 0$$

Travelling wave solutions of diffusive system (Hopf, 1948)

- $u(x, t) = \hat{u}(\xi)$ with $\xi := x - v t$ for a fixed shock velocity v
- reduction to first-order system of ordinary differential equations:

$$\varepsilon B(\hat{u}) \hat{u}_\xi = -v [G(\hat{u}) - G(u^-)] + F(\hat{u}) - F(u^-)$$

- u^- and u^+ are fixed points and we look for an orbit connecting them

$$\hat{u}(-\infty) = u^-, \quad \hat{u}(+\infty) = u^+$$

- travelling wave solution approaches the jump discontinuity in L^1 as $\varepsilon \rightarrow 0^+$

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Historical review: zero adsorption

$$\begin{aligned} s_t + f(s, c)_x &= 0, \\ (sc)_t + (cf(s, c))_x &= 0. \end{aligned} \quad (s, c)|_{t=0} = \begin{cases} (s_L, c_L), & \text{if } x \leq 0, \\ (s_R, c_R), & \text{if } x > 0, \end{cases} \quad (4)$$

Can be rewritten in a more symmetric form:

$$\begin{aligned} s_t + (sg(s, b))_x &= 0, & b = sc &\text{ — total amount of chemicals} \\ b_t + (bg(s, b))_x &= 0. & g = f/s &\text{ — new flux function} \end{aligned}$$

- 1980 — elasticity theory (Keyfitz, Kranzer): $g(s, b) = \tilde{g}(s^2 + b^2)$.
- 1980 — polymer flooding (Isaacson, Glimm, Temple): $f(s, c)$ monotone in c

Interesting properties:

- has coordinate system of Riemann invariants
- rarefaction and shock curves coincide

Problems:

- non-strictly hyperbolic system
- has contact discontinuities

Work in progress with D. Marchesin, B. Plohr: justification of Isaacson-Glimm admissibility criterion & construction of a solution for the case when $f(s, c)$ is non-monotone in c

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Historical review: non-zero adsorption

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Proposition (Johansen-Winther, 1988 (JW))

The solution $c(x, t)$ is monotone in x for any $t > 0$. If $a(c)$ is concave, then

- is constant if $c_L = c_R$;
- contains several rarefaction waves if $c_L < c_R$;
- contains exactly one shock wave if $c_L > c_R$: $(1, 1) \xrightarrow{c=1} u^- \xrightarrow{\text{c-shock}} u^+ \xrightarrow{c=0} (0, 0)$

Historical review:

- 1988 — JW, $f(s, c)$ monotone in c . Found a unique vanishing viscosity solution.

When $f(s, c)$ non-monotone in c , multiple vanishing viscosity solutions are possible.

- 2017 — W. Shen, gave some examples in Lagrangian coordinates. See also
- 1986 — Entov, Kerimov, non-rigorous consideration of the non-monotone case.

Dissipative system

To define a shock wave between u^- and u^+ we consider dissipative system:

$$\begin{aligned}s_t + f(s, c)_x &= \varepsilon_c (A(s, c)s_x)_x, \\(cs + \alpha)_t + (cf(s, c))_x &= \varepsilon_c (cA(s, c)s_x)_x + \varepsilon_d c_{xx}, \\ \alpha_t &= \varepsilon_r^{-1} (a(c) - \alpha).\end{aligned}$$

- ε_c — dimensionless capillary pressure
- ε_d — dimensionless diffusion term
- ε_r — dimensionless relaxation time
- $A(s, c)$ — capillary pressure term
- $\alpha = \alpha(x, t)$ — dynamic adsorption

We consider two particular cases:

Capillarity and Diffusion

$$\begin{aligned}s_t + f(s, c)_x &= \varepsilon_c (A(s, c)s_x)_x, \\(cs + a(c))_t + (cf(s, c))_x &= \varepsilon_c (cA(s, c)s_x)_x + \varepsilon_d c_{xx},\end{aligned}$$

Capillarity and Dynamic Adsorption

$$\begin{aligned}s_t + f(s, c)_x &= \varepsilon_c (A(s, c)s_x)_x, \\(cs + \alpha)_t + (cf(s, c))_x &= \varepsilon_c (cA(s, c)s_x)_x, \\ \alpha_t &= \varepsilon_r^{-1} (a(c) - \alpha).\end{aligned}$$

Restrictions on f and a

(F1) $f \in C^2([0, 1]^2); f(0, c) = 0; f(1, c) = 1;$

(F2) $f_s(s, c) > 0$ for $s \in (0, 1)$, $c \in [0, 1]$;
 $f_s(0, c) = f_s(1, c) = 0$;

(F3) f is S-shaped in s ;

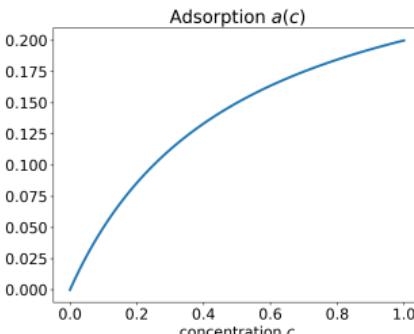
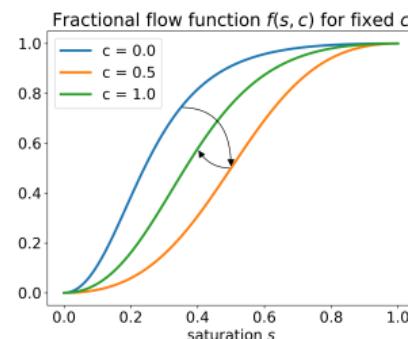
(F4) f is non-monotone in c :
 $\forall s \in (0, 1) \exists c^*(s) \in (0, 1)$:

- $f_c(s, c) < 0$ for $0 < s < 1$, $0 < c < c^*(s)$;
- $f_c(s, c) > 0$ for $0 < s < 1$, $c^*(s) < c < 1$;

(A) A is bounded from zero and infinity;

(a) $a \in C^2$, $a(0) = 0$, a is strictly increasing and concave.

NB: assumptions are just the simplest possible ones. Could be generalised.



Travelling wave dynamical system

$$\begin{aligned} s_t + f(s, c)_x &= \varepsilon_c (A(s, c)s_x)_x, \\ (cs + a(c))_t + (cf(s, c))_x &= \varepsilon_c (cA(s, c)s_x)_x + \varepsilon_d c_{xx}. \end{aligned} \tag{6}$$

Searching for travelling wave solutions $s = s(\xi)$, $c = c(\xi)$, $\xi := \varepsilon_c^{-1}(x - vt)$ with boundary conditions

$$s(\pm\infty) = s^\pm, \quad c(-\infty) = 1, \quad c(+\infty) = 0,$$

we arrive at

$$\begin{aligned} A(s, c)s_\xi &= f(s, c) - v(s + d_1), \\ \kappa c_\xi &= v(d_1 c - d_2 - a(c)). \end{aligned} \tag{7}$$

- Parameters: $\kappa = \varepsilon_d/\varepsilon_c$ and v . $d_1 = a(1)$ and $d_2 = 0$ — some constants.
- Note that u^\pm are fixed points of dynamical system (7);
- We are only interested in the trajectories connecting two saddle points (or saddle-nodes) due to compatibility of speeds in

$$(1, 1) \rightarrow u^- \xrightarrow{c\text{-shock}} u^+ \rightarrow (0, 0).$$

Main result

Consider a dynamical system under assumptions (F1)–(F4), (A), (a):

$$\begin{aligned} A(s, c)s_\xi &= f(s, c) - v(s + d_1), \\ \kappa c_\xi &= v(d_1 c - d_2 - a(c)). \end{aligned}$$

Theorem (Bakharev, Enin, P., Rastegaev, 2021, arxiv:2111.15001)

There exist $0 < v_{\min} < v_{\max} < \infty$, such that for every $\kappa = \varepsilon_d/\varepsilon_c \in (0, +\infty)$, there exist unique

- points $s^-(\kappa) \in [0, 1]$ and $s^+(\kappa) \in [0, 1]$;
- velocity $v(\kappa) \in [v_{\min}, v_{\max}]$,

such that there exists a travelling wave, connecting two saddle points

$u^-(\kappa) = (s^-(\kappa), 1)$ and $u^+(\kappa) = (s^+(\kappa), 0)$ with velocity $v(\kappa)$. Moreover, $v(\kappa)$ is monotone and continuous; $v(\kappa) \rightarrow v_{\min}$ as $\kappa \rightarrow \infty$; $v(\kappa) \rightarrow v_{\max}$ as $\kappa \rightarrow 0$.

Solution construction algorithm

1. From κ we calculate $v(\kappa)$ (binary search).
2. From v we determine $s^-(v)$ and $s^+(v)$ via Rankine-Hugoniot condition.
3. Construct waves $(1, 1) \rightarrow (s^-(v), 1)$ and $(s(v), 0) \rightarrow (0, 0)$.

Example: “boomerang” model:

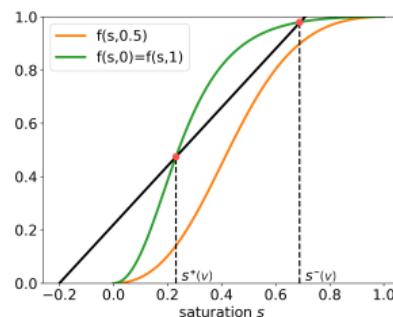


Figure 1: Flux functions

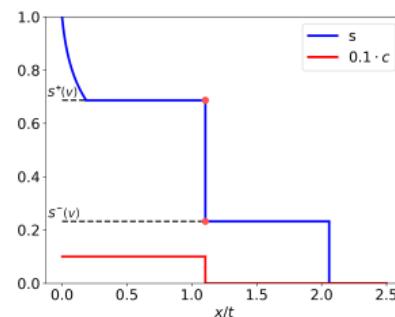


Figure 2: Solution s and c

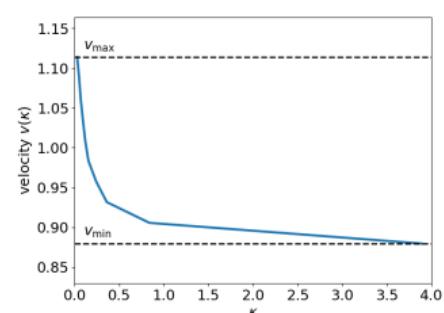


Figure 3: Function $v(\kappa)$

We would like to investigate...

Fix $k = \varepsilon_d / \varepsilon_c$.

$$s_t + f(s, c)_x = \varepsilon_c (A(s, c)s_x)_x,$$

$$(cs + \alpha)_t + (cf(s, c))_x = \varepsilon_c (cA(s, c)s_x)_x + \varepsilon_d c_{xx}.$$

Convergence

Does the Riemann problem solution of the dissipative system converge to the non-dissipative solution as $\varepsilon_c \rightarrow 0$?

Asymptotic stability as $t \rightarrow \infty$

Does the Cauchy problem solution of the dissipative system with the correct values at $\pm\infty$ tend to travelling wave as $t \rightarrow +\infty$?

I believe that “yes”, but...

- Difficulty: comparison theorems do not work for systems
- Hope 1: without diffusion terms the system decouples (in Lagrangian coordinates)
- Hope 2: “steepness” argument works? (like for Fisher-KPP reaction-diffusion eq.)

Scheme of proof

The Theorem can be divided into simpler statements:

- $\forall v \in [v_{\min}, v_{\max}] \quad \exists! \kappa(v)$: there is a saddle-to-saddle travelling wave with $\kappa(v)$.
- $\kappa(v)$ is continuous.
- $\nexists v_1 \neq v_2 : \kappa(v_1) = \kappa(v_2)$, thus $\kappa(v)$ is monotone.
- $\kappa(v) \rightarrow \infty$ as $v \rightarrow v_{\min}$.
- $\kappa(v) \rightarrow \kappa_{\text{crit}} \geq 0$ as $v \rightarrow v_{\max}$.
- When $\kappa < \kappa_{\text{crit}}$ and $v = v_{\max}$ there is a saddle to saddle-node travelling wave

$\kappa(v)$ is monotone and continuous thus there exists an inverse function satisfying the Theorem.

Phase portrait, fixed points, isoclines

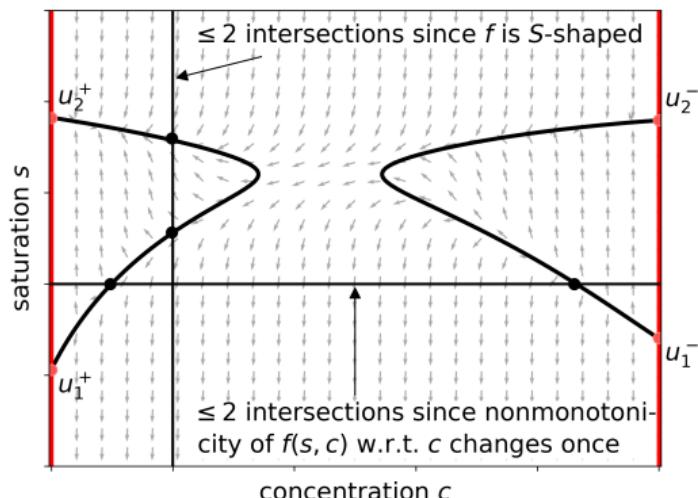
We consider travelling wave dynamical system:

$$\begin{aligned} A(s, c)s_\xi &= f(s, c) - v(s + d_1), \\ \kappa c_\xi &= v(d_1c - d_2 - a(c)), \end{aligned}$$

Isoclines:

red lines are $d_1c - d_2 - a(c) = 0$,
black lines are $f(s, c) - v(s + d_1) = 0$.

Fixed points:
 u_1^+ and u_2^- — saddle points;
 u_2^+ — attractor; u_1^- — repeller



Aim:

find all pairs (κ, v) for which there exists a trajectory
from saddle point u_2^- to saddle point u_1^+

Nullcline configurations: main and intermediate classes

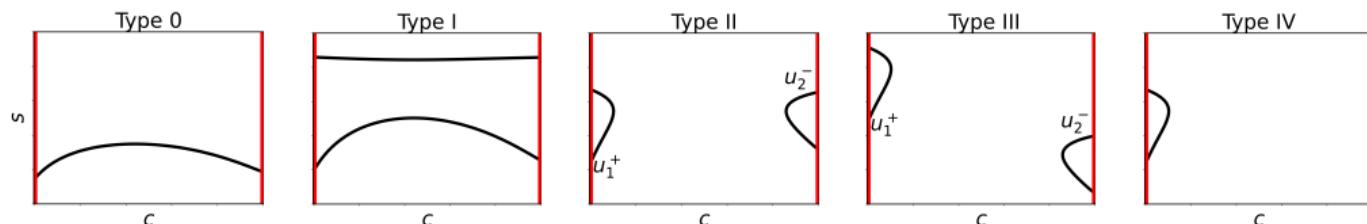


Figure 4: Five wide classes of nullcline configurations

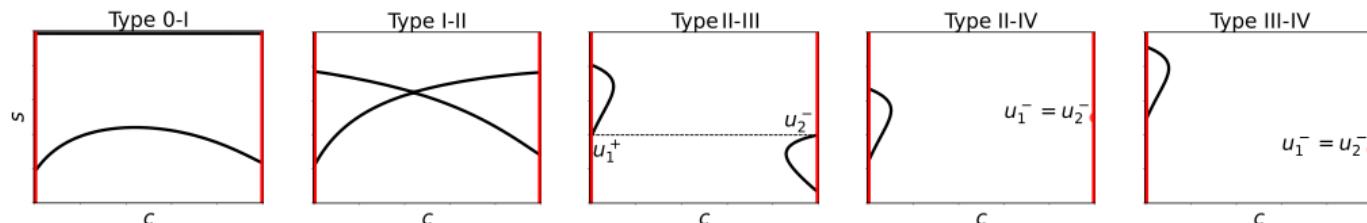


Figure 5: Intermediate types of nullcline configurations, appearing under assump. (F1)–(F4)

- Only Type II nullcline configuration has saddle-to-saddle connections.
- Type I-II corresponds to v_{\min} .
- Type II-III or Type II-IV correspond to v_{\max} .

Nullcline configurations: monotone dependence on v

black lines $f(s, c) = v(s + d_1)$

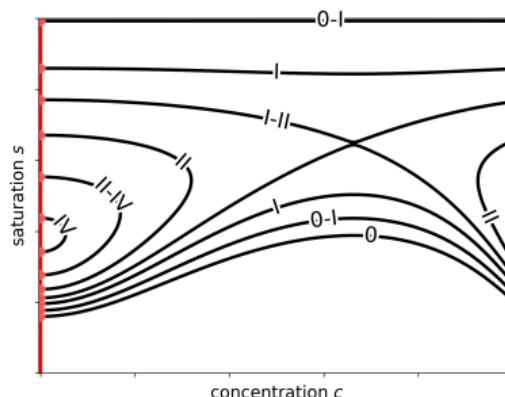
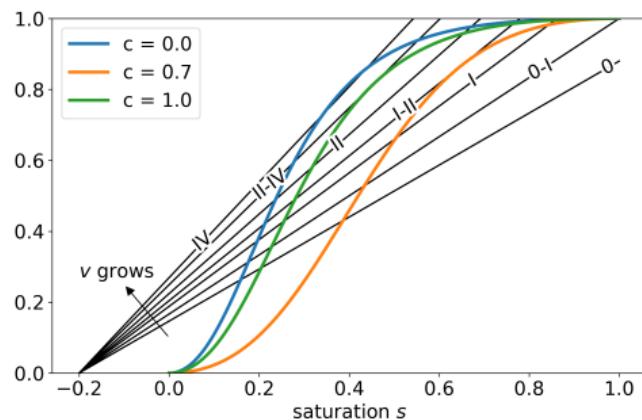


Figure 6: nullcline configuration evolution as v grows: Type 0 → Type I → Type II → Type IV

Nullcline configurations: bad cases

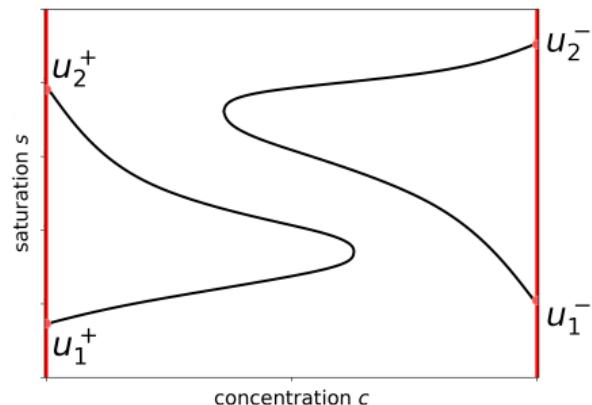


Figure 7: If f is not S-shaped.

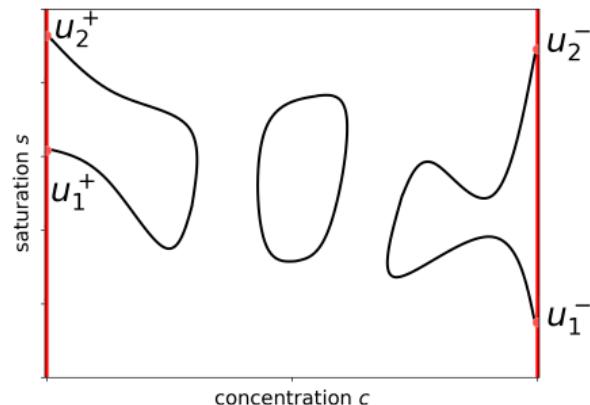


Figure 8: If non-monotonicity is more complex.

We believe that the similar result is true without conditions (F3)–(F4).

Type II configuration: for every v there exist κ

$$\frac{s_\xi}{c_\xi} = \kappa \cdot \frac{v^{-1}f(s, c) - (s + d_1)}{A(s, c)(d_1c - d_2 - a(c))}$$

Used property: continuous and monotonous dependence of trajectories on κ .

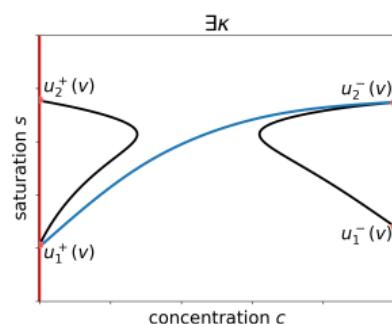
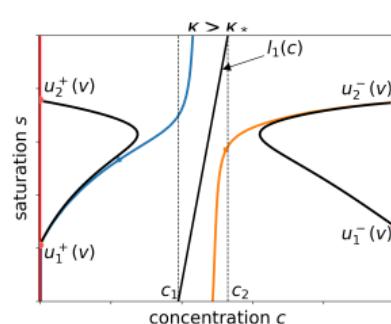
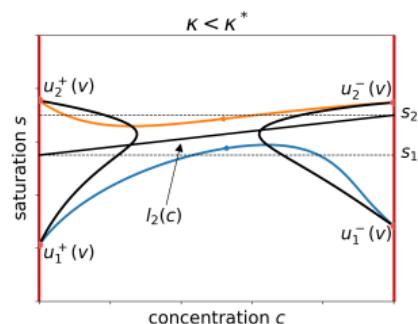


Figure 9: $\kappa \ll 1$.

Figure 10: $\kappa \gg 1$.

Figure 11: $\exists \kappa$.

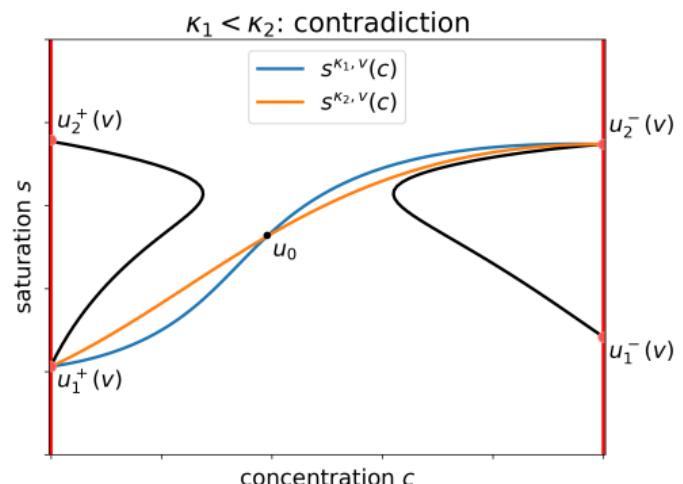
Type II config.: $\kappa(v)$ is unique for every $v \in (v_{\min}, v_{\max})$.

If there are $\kappa_1 < \kappa_2$ for one v , then the corresponding trajectories must intersect, which leads to a contradiction.

The slope

$$\frac{s_\xi}{c_\xi} = \kappa \cdot \frac{v^{-1}f(s, c) - (s + d_1)}{A(s, c)(d_1c - d_2 - a(c))}$$

is positive at the intersection point (s, c) , so it strictly increases when κ increases.



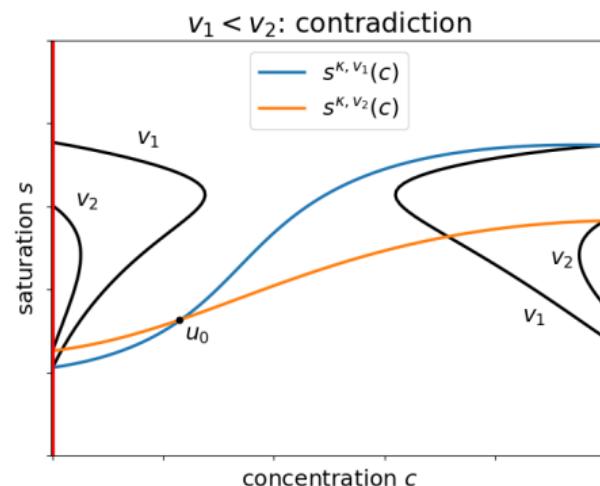
NB: this property might be lost for more complex nullcline configurations.

Type II configuration: monotonicity of $\kappa(v)$

If $\kappa(v_1) = \kappa(v_2)$ for $v_1 < v_2$, then the corresponding trajectories must intersect, which leads to a contradiction. The slope

$$\frac{s_\xi}{c_\xi} = \kappa \cdot \frac{v^{-1}f(s, c) - (s + d_1)}{A(s, c)(d_1c - d_2 - a(c))}$$

is positive at the intersection point (s, c) , so it strictly increases when v increases.



Possible directions for future research

- General classes of f and a : when the dependence $v(\kappa)$ is nontrivial?
- Construct solution to any Riemann problem $(s_L, c_L) \rightarrow (s_R, c_R)$
- Convergence and asymptotic stability as $t \rightarrow \infty$ (as mentioned above)
- Consider a three-phase flow with chemicals (water, oil and gas): travelling wave dynamical system will become three-dimensional, thus the analysis will be more complex.

Vielen Dank für Ihre Aufmerksamkeit!

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Literature

Own works:

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- Johansen, T. and Winther, R., 1988. The solution of the Riemann problem for a hyperbolic system of conservation laws modeling polymer flooding. SIAM journal on mathematical analysis, 19(3), pp.541-566.
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- Entov, V.M. and Kerimov, Z.A., 1986. Displacement of oil by an active solution with a nonmonotonic effect on the flow distribution function. Fluid Dynamics, 21(1), pp.64-70.
- Keyfitz, B.L. and Kranzer, H.C., 1980. A system of non-strictly hyperbolic conservation laws arising in elasticity theory. Archive for Rational Mechanics and Analysis, 72(3), pp.219-241.
- Isaacson E., 1980. Global solution of a Riemann problem for a non-strictly hyperbolic system of conservation laws arising in enhanced oil recovery // Rockefeller University, NY preprint.
- Bressan, A., 2013. Hyperbolic conservation laws: an illustrated tutorial. In Modelling and optimisation of flows on networks (pp. 157-245). Springer, Berlin, Heidelberg.

Introduction

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Review on conservation laws

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Main result

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Proof

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Summarize

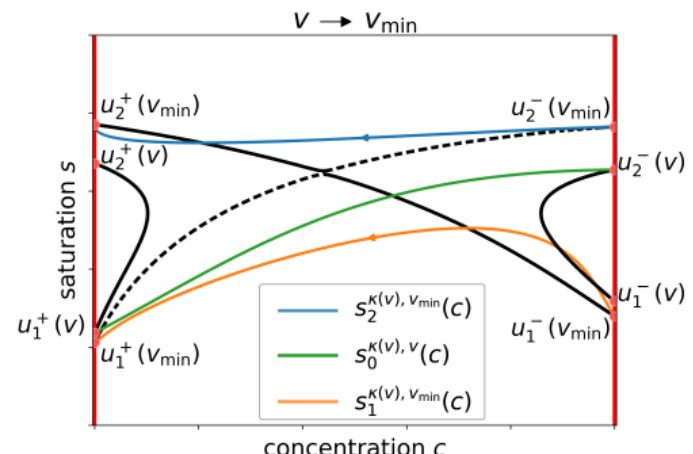
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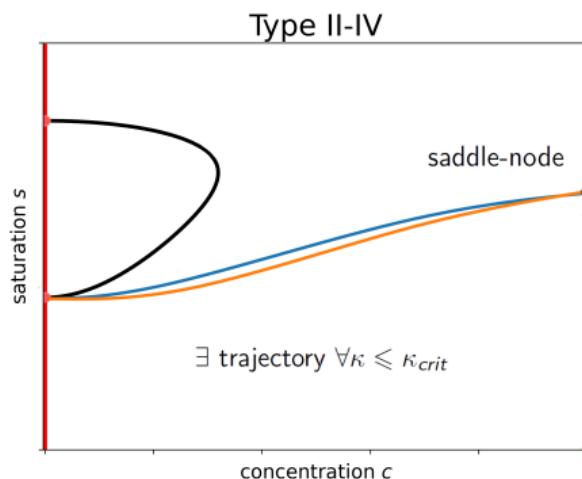
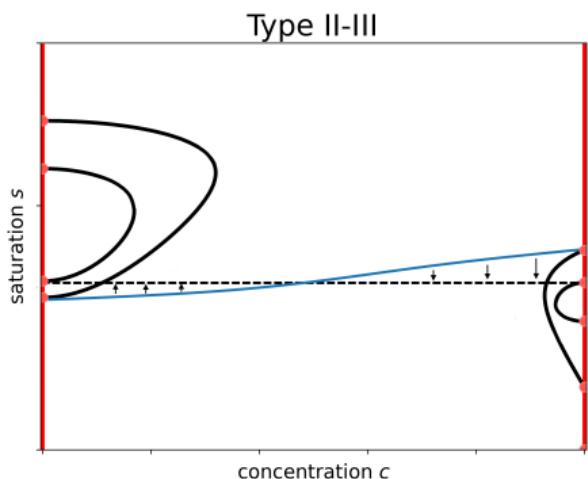
$\kappa \rightarrow +\infty$ as $v \rightarrow v_{\min}$

For any finite κ :

- green orbit is between blue and orange;
- blue orbit is higher than -----;
- orange orbit is lower than -----.

When $v \rightarrow v_{\min}$ the limits of green, blue and orange orbits coincide, which can not happen for any finite κ .



$\kappa \rightarrow 0$ as $v \rightarrow v_{\max}$ $\kappa \rightarrow \kappa_{\text{crit}}$ as $v \rightarrow v_{\max}$ 

Non-strictly hyperbolic system and Lax condition

$$\begin{pmatrix} s_t \\ c_t \end{pmatrix} + \begin{pmatrix} f_s(s, c) & f_c(s, c) \\ 0 & \frac{f(s, c)}{s + a'(c)} \end{pmatrix} \begin{pmatrix} s_x \\ c_x \end{pmatrix} = 0. \quad \lambda_s = f_s(s, c), \quad \lambda_c = \frac{f(s, c)}{s + a'(c)}.$$

Monotone (in c) case is well-studied (Johansen-Winther, 1988). Lax condition gives unique solution, no other c -shocks have structure (no other vanishing viscosity solutions).

Lax condition in non-monotone case sometimes gives physically meaningless solutions.

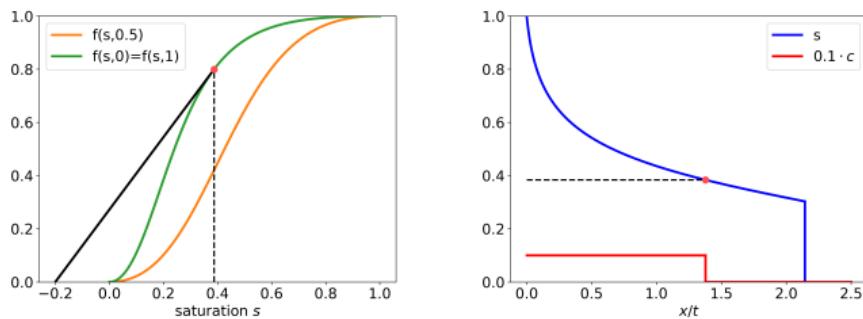


Figure 12: “boomerang” model: solution to a Riemann problem for Lax admissibility criterion
Thus, we study c -shocks that have structure (vanishing viscosity travelling wave solutions).

Decoupling in Lagrangian coordinates

$$\begin{aligned} s_t + f(s, c)_x &= 0, \\ (sc + a(c))_t + (cf(s, c))_x &= 0. \end{aligned} \tag{8}$$

Introduce Lagrangian coordinates (ψ, ϕ) :

$$\begin{aligned} \frac{\partial}{\partial \phi} \left(\frac{s}{f(s, c)} \right) + \frac{\partial}{\partial \psi} \left(\frac{1}{f(s, c)} \right) &= 0, \\ \frac{\partial}{\partial \phi} a(c) + \frac{\partial}{\partial \psi} c &= 0. \end{aligned} \tag{9}$$

- equation on concentration c is decoupled from equation on saturation s
- decoupling is lost if we add diffusion/capillary terms