On the linear growth of the mixing zone in a semi-discrete model of Incompressible Porous Medium (IPM) equation

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We present a semi-discrete model (5)–(8) of the two-dimensional viscous incompressible porous medium (IPM) equation describing gravitational fingering instability. The IPM equation describes evolution of concentration carried by the flow of incompressible fluid determined via Darcy's law in the field of gravity:

$$\partial_t c + \operatorname{div}(uc) = \nu \Delta c, \tag{1}$$

$$\operatorname{div}(u) = 0, \tag{2}$$

$$u = -\nabla p - (0, c). \tag{3}$$

Here $c = c(t, x, y)$ is the transported concentration, $u = u(t, x, y)$ is the vector field describing the fluid motion, $p = p(t, x, y)$ is the pressure, and $\nu \geq 0$ is a dimensionless parameter equal to an inverse of the Peclet number. Usually the spatial domain (x, y) is either the whole space $\mathbb{R}^2$ or cylinder $[0, 1] \times \mathbb{R}$ with periodic or no-flux boundary conditions, but here we consider a discretization in x.

We are interested in studying the *exact rate of the linear growth of mixing zone* (see Fig. 1) formed when the initial condition is close to the unstable stratification:

$$c(0, x, y) = \begin{cases} +1, & y \geq 0, \qquad \text{(heavy fluid)} \\ -1, & y < 0. \qquad \text{(light fluid)} \end{cases} \tag{4}$$

The theoretical bounds on the speed of the linear growth for (1)–(3) are obtained in [2], see also numerical results both for gravitational and viscous fingering [3, 4, 6]. *Our goal* is to answer the question: can the bounds in [2] be improved?

The semi-discrete model consists of a system of advection-reaction-diffusion equations on concentrations $c_k = c_k(t, y)$, velocities $u_k = u_k(t, y)$, pressures $p_k = p_k(t, y)$, describing motion of miscible liquids in several vertical tubes (n real lines, $y \in \mathbb{R}$, $k = 1..n$) and interflow between them (governed by velocities $w_{k+1/2}$).

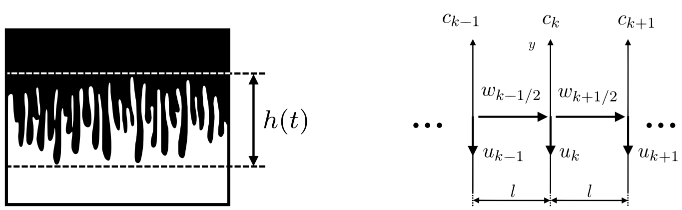

Figure 1. Left: gravitational fingering instability, $h(t) \sim \alpha t$ — scaling of the size of the mixing zone. Right: the n-tubes model

The advantage of the semi-discrete model is that it explicitly shows the possible (interconnected) mechanisms of slowing down the fingers' growth: (1) the convection in the transverse direction of the flow; (2) intermediate concentration, that is the typical concentration inside the finger is $c^* \in (-1, 1)$, see also discussion in [5].

Let $n \in \mathbb{N}, n \geq 2$, be the number of tubes. The *n-tubes IPM model* is obtained as a formal limit of the upwind finite-volume scheme and reads as follows, $k = 1..n$:

(5) (transport eq. in k-th tube) $\quad \partial_t c_k + \partial_y(u_k c_k) - \nu \partial_{yy} c_k = f_{k-1/2} - f_{k+1/2}$,

(6) (incompressibility condition) $\quad l \cdot \partial_y u_k - w_{k-1/2} + w_{k+1/2} = 0$.

Function $f_{k+1/2}$ is responsible for the interflow between k-th and $(k+1)$ tubes:

$$f_{k+1/2} = \begin{cases} c_k \cdot \frac{w_{k+1/2}}{l}, & w_{k+1/2} \geq 0, \text{ (fluid flows from tube } k \text{ to } (k+1)) \\ c_{k+1} \cdot \frac{w_{k+1/2}}{l}, & w_{k+1/2} \leq 0. \text{ (fluid flows from tube } (k+1) \text{ to } k) \end{cases} \tag{7}$$

The velocities u_k and $w_{k+1/2}$ are given by the Darcy's law:

$$u_k = -\partial_y p_k - c_k, \qquad w_{k+1/2} = \frac{p_{k+1} - p_k}{l}. \tag{8}$$

Here $l > 0$ is a parameter equal to the distance between the tubes. We assume that the last, n-th tube, is connected with the 1-st tube, thus all the indexes in the equations should be understood modulo n.

Numerical modelling shows that the typical asymptotic solution as $t \to \infty$ for initial data close to (4) for a small number of tubes looks like a stacked combination of traveling waves which we call a propagating terrace (see Fig. 2).

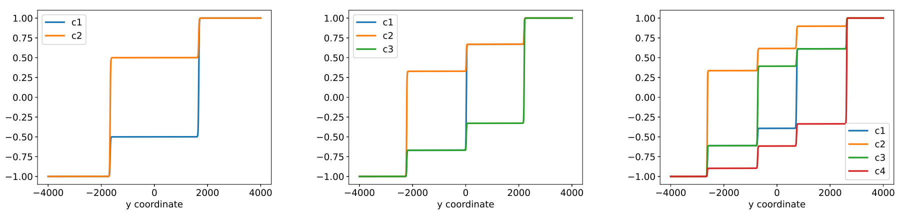

FIGURE 2. Typical asymptotic solution c_k, $k = 1..n$ for $n = 2, 3, 4$ tubes

In the talk we present a rigorous justification of the existence of a propagating terrace in the simplest setting of two tubes (see preprint [1]). The *main result* of [1] claims that for fixed $\nu > 0$ and sufficiently small values of $l > 0$ there exist two intermediate concentrations $c_1^*(l) \in (-1, 1)$, $c_2^*(l) \in (-1, 1)$ and two traveling wave (TW) solutions that connect the states:

$$(-1, -1) \xrightarrow{TW} (c_1^*(l), c_2^*(l)) \xrightarrow{TW} (1, 1). \tag{9}$$

Moreover, the speeds of the traveling waves approach $-1/4$ and $1/4$ as $l \to 0$.

The main tool in the proof in [1] is geometric singular perturbation theory. We represent the travelling wave dynamical system for the n-tubes IPM model as a singular perturbation of the pressure-free transverse flow equilibrium (TFE) model.[1] The only difference between the n-tubes IPM and n-tubes TFE models

[1]In the literature describing viscous fingering TFE model has an analogue called vertical (flow) equilibrium, see e.g. [7].

is the Darcy's law — instead of (8), TFE model states

$$u_k = \left(\frac{1}{n}\cdot\sum_{i=1}^{n} c_i\right) - c_k =: \bar{c} - c_k, \qquad \sum_{i=1}^{n} w_{k-1/2} = 0. \tag{10}$$

The TFE model turns out to be easier to analyze, in particular, for the two-tubes TFE model (5)–(7) and (10) we find explicit solutions in terms of traveling waves.[2]

There are several interesting *open questions* on the n-tubes IPM/TFE models:

(1) describe all possible asymptotic solutions of (5)–(8) as $t \to \infty$ for $n \geq 3$. For $n \geq 4$, we observe that the asymptotic solution is non-unique. In many situations a propagating terrace as in Fig. 2 appear.
   - how to determine the constant states between the traveling waves?
   - the speeds of the fastest and the slowest traveling waves play an important role as they determine the rate of growth of the mixing zone — how to find them explicitly?

(2) study the limit as the number of tubes $n \to \infty$. Do the solutions of the system (5)–(8) approximate the solutions of the original IPM model (1)–(3) as $n \to \infty$? In which sense?

(3) study the well-posedness of the "hyperbolic" problem (5)–(8) with $\nu = 0$. What can we say about the vanishing viscosity limit as $\nu \to 0$?

(4) study the existence of the propagating terrace (9) for the two-tubes model for all values of $l > 0$ (now the result of [1] is valid only for l small enough). Study the stability properties of the propagating terrace.

(5) study the above questions for the n-tubes model for viscous fingering:

$$u = -Km(c)\nabla p, \quad K \text{ — permeability tensor, } m(c) \text{ — mobility function.}$$

References

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[2] In this case the Hugoniot loci form straight lines and coincide with the rarefaction curves, making the 2-tubes TFE system to be of Temple type [8].