Lecture 15: Reaction-diffusion equations

$u = u(t,x)$, $x \in \mathbb{R}^N$, $t>0$, $u \in \mathbb{R}^m$

$$(*) \quad \partial_t u \; \underbrace{-\Delta u}_{\text{(local) diffusion}} = \underbrace{f(u)}_{\text{reaction term}}$$

- excitible medium: more generally $f = f(t,x,u)$
- Δu — comes from particles moving according to Brownian motion (in a rough way, the population tends spread out uniformly, to move towards areas where there are fewer individuals)

"Intuitive" probabilistic justification:

Let the population consist of finite number n of individuals. Consider a discrete space:

$$\{\lambda k : k \in \mathbb{Z}^N\} \subset \mathbb{R}^N, \quad \lambda > 0$$

For a given individual we denote:

$p(t,x)$ – probability that the individual is at point x at time t.

$$X_k(t,x) = \begin{cases} 1, & \text{if } k\text{-th individual is at point } x \text{ at time } t \\ 0, & \text{otherwise} \end{cases}$$

Then $U(t,x) = \frac{1}{n}\sum_{k=1}^{n} X_k(t,x)$ - normalized distrib. of the population

Assuming the movements of individuals are independent of each other, $U(t,x) \to p(t,x)$.

At each instant an individual can:

— move to a neighbouring point with prob. $q < \frac{1}{2n}$

— do not move with probability $1 - q\cdot 2n$

Note that the probability q does not depend on the position in time and space, nor on the previous position $\Rightarrow$ random walk $\Rightarrow$

$$p(t+\tau, \lambda k) = (1-2nq)\, p(t,\lambda k) + q\sum_{j=1}^{n}\left[p(t,\lambda(k+e_j)) + p(t,\lambda(k-e_j))\right]$$

Assume that there exists a regular $p(t,x)$ for which the same relation is true for all x, t. So

$$\partial_t p + O(\tau) = \frac{q\lambda^2}{\tau} \sum_{j=1}^{n} \frac{\partial^2 p}{\partial x_j^2} + O\left(\frac{\lambda^3}{\tau}\right)$$

Now let $\lambda, \tau \to 0$ such that $\frac{q\lambda^2}{\tau} \to D \in (0, +\infty)$

Thus, we get $\partial_t p = D \cdot \Delta p$.

Examples : (1) population dynamics (ecology) : u - concentration density

$$u_t - u_{xx} = f(u)$$

For a moment forget about diffusion and consider an ODE : $u_t = f(u)$, $u(0) = u_0$

Cases : (a) $f(u) = r u$ (Malthus equation '1798)

Solution: $u(t) = u_0 e^{rt}$, $r \in \mathbb{R}$

r - growth rate, the population grows infinitely (which is not natural)

(b) $f(u) = ru\left(1 - \frac{u}{K}\right)$ (logistic equation, ~ 1838)

$r \in \mathbb{R}$, $K \in \mathbb{R}$

Explicit solution: $u(t) = \dfrac{K}{1 + \left(\frac{K}{u_0} - 1\right) e^{-rt}}$

We observe, that :

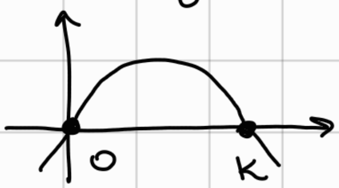
(i) whenever $u_0 > 0$, the solution is well-defined for $\forall t > 0$, $u(t) > 0$ and $u(t) \xrightarrow[t \to +\infty]{} K$

(ii) $u_0 = 0 \Rightarrow u(t) \equiv 0$

This corresponds to a more general fact that we will see later !

→ When u increases, there is a competition for resourses. Here K is called the capacity of environment

More general: monostable equations: $\dot{u} = f(t,u)$



assumptions: $f(0) = f(K) = 0$, f - Lipch in u

$f > 0$ for $u \in (0,K)$

$f < 0$ for $u \notin [0,K]$

Sometimes, there is an extra assumption: $\frac{f(u)}{u} \downarrow$

Lemma: $f: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ - continuous, loc. Lipschitz in u

(i) If $f(t,0) = 0$ $\forall t$, then if $u(0) > 0 \Rightarrow u(t) > 0$ $\forall t$

(ii) If u, v - two solutions and $u(0) > v(0)$, then $u(t) > v(t)$ (in the domain where both sol. exist)

(iii) If $u' \le f(t, u(t))$ and $v' > f(t, v(t))$ and $u(0) \le v(0)$, then $u(t) < v(t)$ $\forall t$.

Rmk 1: when u satisfies the differential inequality $u' \le f(t, u(t))$ we say that u is a sub-solution; otherwise super-solution

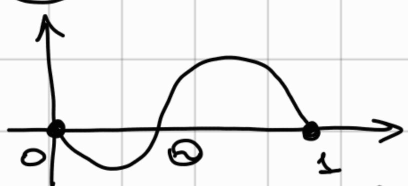
Rmk 2: these statements are true for a single equation, but in general are not true for systems of eqs.

Rmk 3: items (ii) and (iii) are the so-called "comparison theorems" in this very simple setting. We will see more of them for reaction-diffusion eqs.

Here $u=0$ is unstable equilibrium point (asymp)
$u=K$ is stable equilibrium point (asymp)

Thus, the name "monostable" (1 stable point)

c) $f(u) = u(1-u)(u-\theta)$, Bistable equations

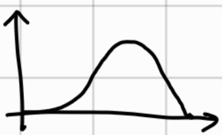


strong Allee effect

or more general assumptions:

- $f(0) = f(\theta) = f(1) = 0$
- $f > 0$ for $u \in (\theta, 1)$
- $f < 0$ for $u \in (0, \theta)$

Weak Allee effect:



monostable equation without condition $\frac{f(u)}{u}$ is decreasing

Theorem: for $u(0) \in [0,1]$ the equation admits global-in-time solution $u(t) \in [0,1]$ $\forall t \in \mathbb{R}$

Moreover, if $u(0) < \theta \Rightarrow u(t) \xrightarrow[t \to +\infty]{} 0$

$u(0) > \theta \Rightarrow u(t) \to 1$

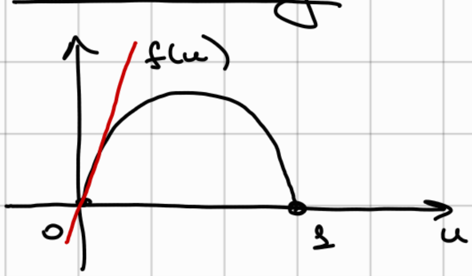
(the small population will turn off – may be not enough sexual partners or can not form big enough groups for fighting against predators)

This theorem explains the term "bistable":

$u=0$ and $u=1$ are stable equilibrium state

$u=\theta$ – unstable equilibrium state

Concluding: we will consider 3 different $f(u)$:



F-KPP

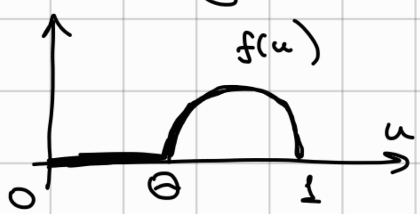
Fisher, Kolmogorov Petrovskii, Piskunov (1937)

Monostable

Bistable

- monostable case with condition that $f(u)$ lies below the tangent line at $u=0$ (think of $f(u) = u(1-u)$)

There is also a case of ignition / combustion non-linearity: $f(u) = 0$, $u \in [0,\theta]$



Rmk: there is one more notion of stability: linear stability

state α is called linearly stable if $f'(\alpha) < 0$

state α — // — linearly unstable if $f'(\alpha) > 0$

Thm: $f \in C^1$ in the vicinity of α ($f(\alpha) = 0$)

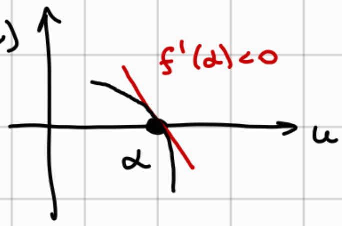
(i) If $f'(\alpha) < 0$ and $u(0)$ is sufficiently close to α, then $u(t) \to \alpha$ as $t \to +\infty$

(ii) If $f'(\alpha)>0$, then no solution (except $u \equiv \alpha$) converges to α as $t \to +\infty$.

On the other hand, if $u(0)$ is close enough to α, then $u(t) \to -\alpha$ as $t \to -\infty$.

<u>Proof</u>:

⌈▶ (i)



$f(u)>0$ for $u \in [\alpha-\varepsilon, \alpha)$

$f(u)<0$ for $u \in (\alpha, \alpha+\varepsilon]$

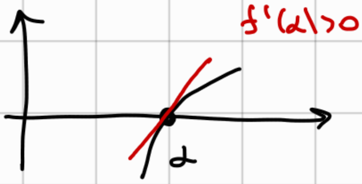
$$\dot{u} = f(u)$$

If $u(0)<\alpha \Rightarrow u(t)<\alpha$ and $\uparrow$

If $u(0)>\alpha \Rightarrow u(t)>\alpha$ and $\downarrow$

$u(t)$ can converge only to α.

(ii)



$f(u)<0$ for $u \in [\alpha-\varepsilon, \alpha)$

$f(u)>0$ for $u \in (\alpha, \alpha+\varepsilon]$

$$\dot{u} = f(u)$$

If $u(0)<\alpha \Rightarrow \dot{u} = f(u)<0 \Rightarrow u \downarrow$ and $u(t)<u(0)<\alpha$

If $u(0)>\alpha \Rightarrow \dot{u} = f(u)>0 \Rightarrow u \uparrow$ and $u(t)>u(0)>\alpha$

⌊ ■

There are many-many ways to generalize these equations:

Δu ⇝ $\int_\Omega K(x-y)\,u(y)\,dy$ – non-local diffusion

⇝ general (uniformly elliptic) term

$$\sum_{i,j=1}^{N} a_{ij}(t,x)\, \partial_i \partial_j u$$

with condition $\exists\ 0<\alpha<\beta<+\infty$:

$$\forall \xi \in \mathbb{R}^N, \forall t>0, \forall x \in \Omega \quad \alpha \|\xi\|^2 \le \sum a_{ij}(t,x)\,\xi_i \xi_j \le \beta \|\xi\|^2$$

$f(u)$ ⇝ $f(t,x,u)$ – depend on space x and time t

$u \in \mathbb{R}$ ⇝ $\vec{u} \in \mathbb{R}^n$ – many species (Lotka-Volterra, predator-prey system, competitive media)

$\Omega \subset \mathbb{R}^N$ ⇝

more complex geometries

line of "fast" diffusion ("roads" in forests)

etc....

Other contexts: → combustion theory (propagation of flame, thermo-diffusive model)

→ probability (BBM - Branching Brownian Motion McKean representation)

→ statistical physics etc....

Reaction-diffusion eqs: problem statement

$$(*) \qquad \partial_t u = D\Delta u + f(t,x,u)$$

- $t \in (0,+\infty)$
- $x \in \Omega$ → $\Omega = \mathbb{R}^N$; → $\Omega \subset \mathbb{R}^N$ - bounded, connected with reg. boundary
- $D > 0$
- $u \in \mathbb{R}$ - scalar
- $f(u)$ is of one of the types above

\+ Initial condition: $u\big|_{t=0} = u_0(x) \in C(\Omega) \cap L^\infty(\Omega)$

\+ Boundary conditions:

(Neumann) $\partial_n u = 0$ for $(t,x) \in (0,+\infty) \times \partial\Omega$

(Dirichlet) $u = 0$ for — // —

(Robin) $\partial_n u + qu = 0$ for — // —

Interpretations:

Neumann: no individuals cross the boundary (in any direction)

Dirichlet: exterior of Ω is extremely unfavorable so population density is zero at boundary

Robin: there is a flow of individuals entering ($q>0$) or leaving the domain ($q<0$)

We consider classical solution u which satisfies

$$(**) \begin{cases} u \in C^0([0,+\infty) \times \bar\Omega) \\ \partial_t u \in C^0((0,+\infty)\times\Omega) \\ \forall i \quad \partial_{x_i} u \in C^0((0,+\infty)\times\bar\Omega) \\ \forall i,j \quad \partial_{x_i x_j} u \in C^0((0,+\infty)\times\Omega) \end{cases}$$

and equation $(*)$, initial and one of the boundary cond.

If $\Omega = \mathbb{R}^N$ we also assume some growth cond. at infinity: $\forall T>0 \ \exists A, B > 0$:

$$|u(t,x)| \le A e^{B|x|}, \quad x \in \mathbb{R}^N, \ t>0$$

What are the important topics ?

① Comparison theorems: roughly speaking

if $u(0,x) \le v(0,x)$ are both solutions of (*)

then $u(t,x) \le v(t,x)$ $\forall t>0$

Closely connected to maximum principle for parabolic PDEs.

This can be very helpful:

example 1:
$$u_t = \Delta u + u(1-u)$$
$$u(0,x) \in [0,1] \quad \forall x \in \mathbb{R}^N$$

- $u \equiv 0$ is solution and $u(0,x) \ge 0$
  $\Rightarrow$ $u(t,x) \ge 0$
- $u \equiv 1$ is solution and $u(0,x) \le 1$
  $\Rightarrow$ $u(t,x) \le 1$

Thus, $u(0,x) \in [0,1] \Rightarrow u(t,x) \in [0,1]$

example 2:
$$u_t = \Delta u - u^3$$
$$u|_{t=0} = u_0 \in [m, M], \quad x \in \Omega \subset \mathbb{R}^N$$

Consider $\begin{cases} \dot{v} = -v^3 \\ v(0) = m \end{cases}$ and $\begin{cases} \dot{w} = -w^3 \\ w(0) = M \end{cases}$

These are sub and supersolutions:

$$v(t) \le u(x,t) \le w(t)$$

$$-\frac{dv}{v^3} = dt \Rightarrow \frac{1}{2v^2} - \frac{1}{2m^2} = t \Rightarrow v = \left(\frac{1}{m^2} + 2t\right)^{-\frac{1}{2}}$$

$v(t) \to 0$ as $t \to \infty$

Analogously, $w(t) \to 0$ as $t \to \infty$

Thus, if u exists, then
$$\underset{\substack{\downarrow \\ 0}}{v(t)} \le u(x,t) \le \underset{\substack{\downarrow \\ 0}}{w(t)} \Rightarrow u \xrightarrow[t \to +\infty]{} 0$$

→ well-posedness of (*) : $\exists!$, cont. dependence

→ special solutions : traveling waves (planar)
take direction $e \in \mathbb{R}^N$ and consider a solution of the form:

$$u(t,x) = \tilde{u}(\underbrace{x \cdot e}_{\text{scalar product}} - vt)$$

$\tilde{u} : \mathbb{R} \to \mathbb{R}$

v - speed of propagation

We will see that for different nonlinearities there exist travelling waves (TW)

$x \in \mathbb{R}^1$: FKPP: $\exists\, c^* : \forall c \geq c^*\ \exists$ TW

Bistable: $\exists!\, c : \exists$ TW

→ $x \in \mathbb{R}^1$: long-time behaviour as $t \to +\infty$
for some initial data (like  Heaviside)
the solution u of (*) "converges" to a TW

§ Maximum principle
for parabolic equations

This is an extension of the results that we have seen for ODEs. First, some definitions:

Def 1: $u(t,x)$ is called sub-solution of (*) if it satisfies (**) and inequalities:

$$\partial_t u \leq \Delta u + f(t,x,u)$$

and on the boundary (if applicable): on $\partial\Omega$

(Neumann) $\partial_n u \leq 0$; (Dirichlet) $u \leq 0$; (Robin) $\partial_n u + qu \leq 0$

If $\Omega = \mathbb{R}^N$, then $|u| \leq A e^{B|x|}$, $A, B > 0$

Analogously, $v(t,x)$ is called a super solution if all inequalities are reversed (except $|v| \le A e^{B|x|}$)

We want to prove the following theorem:

Theorem (comparison principle)

Let u and v be sub- and super-solutions of the reaction-diffusion eq $(*)$.

(i) If $u(0,x) \le v(0,x)$ for $x \in \overline{\Omega}$, then $u(t,x) \le v(t,x)$ for $t>0$, $x \in \overline{\Omega}$

(ii) If moreover, $u(t_0,x_0) = v(t_0,x_0)$ for some $t_0 > 0$, $x_0 \in \Omega$, then $u \equiv v$.

(iii) If Ω is bounded and the boundary condition is of Neumann or Robin type, then (ii) is true even for $x_0 \in \partial\Omega$

Note that the difference $(u-v)$ satisfies

$$\partial_t (u-v) \le \Delta(u-v) + f(t,x,u) - f(t,x,v)$$

Thanks to regularity of u, v, f we can rewrite this equation as follows: $w = u-v$

$$(1) \quad \partial_t w \le \Delta w + g(t,x) w$$

where

$$g(t,x) = \begin{cases} \dfrac{f(t,x,u) - f(t,x,v)}{u-v} & \text{if } u \ne v \\ \partial_u f(t,x,u) & \text{if } u = v. \end{cases}$$

is continuous and uniformly bdd function

So we reduced a problem to studying the linear eq (1) and showing $w \le 0$ $\forall\, t>0$, $x \in \overline{\Omega}$.

Linear problem and maximum principle

Let us consider a more general case:

(2) $\partial_t u = \Delta u + \sum b_i(t,x)\,\partial_i u + c(t,x)\,u$

Let b_i, c be uniformly bdd.

Thm 1 (weak maximum principle)

(i) Let be a sub-solution of linear eq (2).
If $u(0,x) \le 0$, then $u(t,x) \le 0 \quad \forall t>0$

(ii) Let v be super-solution of linear eq (2).
If $v(0,x) \ge 0$, then $v(t,x) \ge 0 \quad \forall t>0$.

because $u(x_0,t_0)=0 \Rightarrow u \equiv 0$

Thm 2 (strong maximum principle)

(i) Let u be a subsolution of (2) and $u(0,x) \le 0$.
If $\exists\, t_0>0, x_0 \in \Omega : u(t_0,x_0)=0 \Rightarrow u \equiv 0$ on $[0,t_0]\times\Omega$

(ii) Let v be a supersolution of (2) and $v(0,x) \ge 0$.
If $\exists\, t_0>0, x_0\in\Omega : v(t_0,x)=0 \Rightarrow v \equiv 0$ on $[0,t_0]\times\Omega$

(iii) If Ω is bdd, then for Neumann and Robin the same statement as in (i), (ii) are true if $x_0 \in \partial\Omega$.

Rmk: it is clear that it is sufficient to consider subsolutions. For the supersolutions just consider $v=-u$.

Proof of maximum principle:

We will prove in 2 cases: (a) Ω - bdd, Dirichlet
(b) $\Omega = \mathbb{R}^N$

First, let's prove the simple case:

Lemma: let u be a subsolution with strict ineq:

$\partial_t u - \Delta u - \sum b_i(t,x)\,\partial_i u - c(t,x)\,u < 0$, $u(0,\cdot)<0$, $\boxed{u|_{\partial\Omega}<0}$

$\Rightarrow u(t,x)<0$

Proof of lemma:

▸ Indeed, take first time $t_0>0$ such that $u(x_0,t_0)=0$ for $x_0\in\Omega$.

At this point: $\partial_t u \geq 0$

$\Delta u \leq 0$ (the local picture )

$\partial_i u = 0$ (as it is local maximum)

$u = 0$

$\Rightarrow \partial_t u - \Delta u - \sum b_i \partial_i u - cu \geq 0$ (!?) ■