Dynamical systems.

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Place: COMSATS University, Lahore Campus, Pakistan

Dates: 25–29 November 2019

Contents

1 Lectures

Sketches of lecture 1. 25 November 2019

1.1 The notion of dynamical system

Let's start with defining the notion of ¡¡abstract¿¿ dynamical system.
Let I be one of $\mathbb{Z}, \mathbb{R}, \mathbb{Z}^+, \mathbb{R}^+$ — time.
Let M be a metric space with ¡¡dist¿¿ — space, for us M is one of $\mathbb{R}^n$, S^1, $\mathbb{T}^2$.

Definition 1. *A map $\Phi : I \times M \to M$ is called a dynamical system if the following assumptions hold:*

(DS1) $\Phi(0, x) = x$ *(initial moment)*

(DS2) $\Phi(t_1, \Phi(t_2, x)) = \Phi(t_1 + t_2, x)$ *for any* $t_1, t_2 \in I$ *(composition property)*

(DS3) $\Phi(t, x)$ *is continious as a function of two variables x and t.*

If $\Phi : \mathbb{Z} \times M \to M$ then Φ is called a dynamical system with discrete time.
If $\Phi : \mathbb{R} \times M \to M$ then Φ is called a dynamical system with continious time (flow).

During this course we will more concentrate on $\mathbb{Z}, \mathbb{Z}^+$.
Examples:

1. Let $f : M \to M$ be a homeomorphism that is

   (a) f is continious

   (b) f is injective

   (c) f is subjective

   (d) from (b) and (c) we see that there exists f^{-1}. Also f^{-1} needs to be continious.

   Let $\Phi : \mathbb{Z} \times M \to M$ and define

$$\Phi(n, x) := \begin{cases} \underbrace{f \circ \ldots \circ f}_{n \text{ times}}(x), & n > 0 \\ x, & n = 0 \\ \underbrace{f^{-1} \circ \ldots \circ f^{-1}}_{n \text{ times}}(x), & n < 0 \end{cases} \tag{1}$$

Statement 1. $\Phi(n, x)$ *defined in 1 is a dynamical system with discrete time.*

Proof: (DS1) by definition
(DS3) composition of homeomorhisms is a homeomorphism
(DS2) If $t_1, t_2 > 0$ it is obvious. If $t_1 > 0$, $t_2 < 0$ and $t_1 + t_2 > 0$ then $f^{-1} \circ f^{-1} \circ \ldots \circ (f^{-1} \circ f) \circ f \circ \ldots \circ f = f \circ \ldots \circ f$ — correct due to cancellations. Other cases are similar. $\square$

Theorem 1. *If $\Phi : \mathbb{Z} \times M \to M$ a dynamical system. Define $f : M \to M$ such that $f(x) = \Phi(1, x)$. Then*

(a) f is a homeomorphism;

(b) $\Phi(n, x)$ can be obtained by (1).

Proof: (a) f is continious by (DS3), f^{-1} exists and is continious as $f^{-1}(x) = \Phi(-1, x)$. (b) follows from (DS2) $\square$

2. Continious time dynamical systems are related to ODEs.

   Remark 1. *Not all continious time dynamical systems appear from ODEs*

   Let $x \in \mathbb{R}^n$, $V : \mathbb{R}^n \to \mathbb{R}^n$. Consider autonomous differential equation:

$$\dot{x} = V(x), \text{ where } x : \mathbb{R} \text{ (time)} \to \mathbb{R}^n \text{ (space)}. \tag{2}$$

   Definition 2. *We say that equation (2) has a local solution with initial condition x_0 if there exists an $\varepsilon > 0$ and a differentible map $x : (-\varepsilon, \varepsilon) \to \mathbb{R}^n$ such that $x(0) = x_0$ and*

$$\frac{d}{dt} x(t) = V(x(t)), \textit{ for } t \in (-\varepsilon, \varepsilon). \tag{3}$$

   If there exists function $x : \mathbb{R} \to \mathbb{R}^n$ satisfying (3) we say that x is a global solution.

   We call this Cauchy problem.

   Why I make such complications in the definition?

   Example: $\dot{x} = x^2$, for $x(0) = 0$. Prove that the solution is $x = \tan(t)$ for $t \in (-\pi/2, \pi/2)$. It blow-ups in a finite time.

   However we will consider another type of equations and in fact will not consider anything where there is no ¡¡global solution¿¿.

Definition 3. *$V : M \to M$ is called a Lipshitz function if there exists $L > 0$ such that $|V(x) - V(y)| \leqslant L|x - y|$ for any $x, y \in M$.*

Example: $V(x) = x$, $V(x) = \sin(x)$.
Sufficient condition: if $|D(V)(x)| \leqslant L$ then V is a Lipshitz function.

Theorem 2. *(without proof) Let $\dot{x} = V(x)$.*

- *Assume $V(x)$ — continious function. Then exists at least one local solution.*
- *Assume $V(x)$ — Lipshitz function, then for any $x_0 \in \mathbb{R}^n$ there exists and unique global solution $x(t, x_0)$ of the equation (3). Moreover it is a continious function in x_0.*

Remark 2. *There are milder condidtions which guarantee uniqueness. In particular for all examples in Mark worksheets it is unique. But still we need some assumption.*

Example: $\dot{x} = 3x^{2/3}$, $x(0) = 0$. There are at least 2 solutions.

Important: if you know that there exists solution and it is unique and continious then the corresponding ODEs give rise to dynamical system.

So assume that V is Lipshitz and define map $\Phi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ as $\Phi(t, x_0) = x(t, x_0)$. Let us show that it is a dynamical system. Let's check 3 properties (DS1)–(DS3).

(DS1) $x(0, x_0) = x_0$ by initial data

(DS3) $x(t, x_0)$ — continious (in fact I omitted from you the proof)

(DS2) We need to prove that $\Phi(t_2, \Phi(t_1, x_0)) = \Phi(t_1 + t_2, x_0)$.
Let's denote $x_1 = \Phi(t_1, x_0)$, $g_1(t) = \Phi(t, x_1)$ and $g_2(t) = \Phi(t + t_1, x_0)$. Let's prove that $g_1(t)$ and $g_2(t)$ are solutions of the same equation, passing through the same point x_1. For $g_1(t)$ is clear. Let's prove that $g_2(t)$ is a solution, that is 2 properties are satisfied:

- (initial condition) $g_2(0) = \Phi(t_1, x_0) = x_1$
- (differential equation) $\dot{g}_2(t) = \frac{d}{dt}\Phi(t + t_1, x_0) = V(\Phi(t + t_1, x_0)) = V(g_2(t))$

So $g_2(t)$ is also a solution and by uniqueness we have that they coincide, that is $g_1 = g_2$.

Remark 3. *it is important that $V(x)$ **does not depend on time!** Assume $\dot{x} = V(x, t)$ — analogue of theorem 1. We can repeat $\Phi(t, x) = \ldots$*

(DS1) we have automatic

(DS2) continious os OK

(DS3) will be a problem here!

Sometimes I will consider only positive times. In that case I will consider $I = \mathbb{Z}^+$ or $I = \mathbb{R}^+$.

1.2 Fixed points. Periodic points.

Def: Let $\Phi : I \times M \to M$.

- a point $x \in M$ is called a *fixed point* for $\Phi(t, x)$ if $\Phi(t, x) = x$ for all $t \in I$.
- a point $x \in M$ is called a *peroidic point* for $\Phi(t, x)$ if $\exists \tau \in I$, $\tau > 0$, such that $\Phi(\tau, x) = x$. Additionally for continious time ($I = \mathbb{R}$, $I = \mathbb{R}^+$) we ask x not to be a fixed point.

Definition 4. *Let x be a periodic point for $\Phi(t, x)$. Let's consider it's period*

$$\tau_0 := \inf\{\tau > 0 : x = \phi(\tau, x)\}$$

Theorem 3. $\tau_0 > 0$

Proof: if $I = \mathbb{Z}$ then it is obvious. If $I = \mathbb{R}$ then we need to prove.

Assume $\tau_0 = 0$, then $\Phi(\tau_0, x) = x$, and nothing :(

x is periodic point means that

1. $\exists \tau > 0$: $\Phi(\tau, x) = x$.
2. $\exists \tau_1 > 0$: $\Phi(\tau_1, x) \neq x$

We want to prove by contradiction. What does it mean that $\tau_0 = 0$? It means that exists a sequence $\varepsilon_k \to 0$ such that $\Phi(\varepsilon_k, x) = x$. Let's prove that in this case the second property doesn't hold that is: $\forall \tau_1 > 0$ $\Phi(\tau_1, x) = x$.

Let's note that if $\Phi(\tau, x) = x$ then $\Phi(\tau + \tau, x) = \Phi(\tau, \Phi(\tau, x)) = \Phi(\tau, x) = x$. So $\Phi(n\tau, x) = x$. Let's prove that $\Phi(\tau, x)$ is dense in M. Then by continuity of Φ we will come to a contradiction. ToDo: finish proof. □

Definition 5. *Let $\Phi(t, x)$ be a dynamical system, then orbit (=trajectory) $O(x)$ is defined as*

$$O(x) := \{\Phi(t, x), \,|\, t \in I\}$$

Remark:

- to find fixed points you need to solve the equation $f(t, x) = x$ (usually you can do it).
- to find periodic points you need to solve the equation $f^m(t, x) = x$ (usually more difficult)

Theorem 4. *$x \in M$ is a periodic point iff it's orbit $O(x)$ is finite.*

Proof: If $x \in M$ is a periodic point then it's orbit $O(x)$ is finite — clear.
Vice versa needs to be proved. □

Let us consider $\Phi : \mathbb{Z}_0^+ \times M \to M$. We have $f(x) = \Phi(1, x)$ — continious. No need for $f^{-1} = \Phi(-1, x)$ to exist. This corresponds to the systems which can go to the future but can not go to the past.

Definition 6. *The point x is called preperiodic for $\Phi : \mathbb{Z}_0^+ \times M \to M$ iff $\exists m$ such that $\Phi(m, x)$ is periodic.*

Statement 2. *If $\Phi : \mathbb{Z}_0^+ \times M \to M$ then $O(x)$ is finite iff x is preperiodic.*

Proof: the same reasoning as in theorem 4. □

More examples of dynamical systems (that we will consider in worksheets):

1. **Rotation of a circle**

   Let's denote S^1 — a standard circle, that is $S^1 = \mathbb{R}/\mathbb{Z}$.

   We introduce an equivalence relation: $x \sim y$ if $x - y \in \mathbb{Z}$. And $f : S^1 \to S^1$ is defined as $f(x) = x + \alpha$.

   Need to prove that it is correct: if $x \sim y$ then $f(x) \sim f(y)$.

2. Let's $f : S^1 \to S^1$ such that $f(x) = 2x$.

3. **Automorphism of a torus**

   A torus $\mathbb{T}^2 = S^1 \times S^1$, $(x, y) \in \mathbb{T}^2$ then $x \in S^1$, $y \in S^1$. So $(x_1, y_1) \sim (x_2, y_2)$ iff $x_1 - x_2 \in \mathbb{Z}$ and $y_1 - y_2 \in \mathbb{Z}$.

   Let's define $f : \mathbb{T}^2 \to \mathbb{T}^2$ as $f(x) = Ax$ for

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

   It is invertible.

Sketches of lecture 2. 26 November 2019

1.3 Why exists the solution of ODE?

Let's discuss the sketch of the proof of the existence and uniqueness of the solution of ODE.

Theorem 5. *Consider the following Cauchy problem $\dot{x} = V(x)$, $x(0) = x_0$.*

- *Assume $V(x)$ — continious function. Then exists at least one local solution.*
- *Assume $V(x)$ — Lipshitz function, then for any $x_0 \in \mathbb{R}^n$ there exists and unique global solution $x(t, x_0)$ of the equation (3). Moreover it is a continious function in x_0.*

Proof:
Idea 1: Consider the following integral equation:

$$x(t) = x_0 + \int\limits_0^t V(x(\tau))\, d\tau$$

Proof of uniqueness: consider 2 solutions: □

1.4 Lyapunov stability theory

Let $\Phi : I^+ \times M \to M$, here $I^+ = \mathbb{Z}_0^+$, $I^+ = \mathbb{R}_0^+$.

Definition 7. *We say that trajectory $x(t) = \Phi(t, x_0)$ is Lyapunov stable if for any $\varepsilon > 0$ $\exists \delta > 0$ such that for $\forall x_1 \in B_\delta(x_2)$ then for $y(t) = \Phi(t, x_1)$ holds $|y(t) - x(t)| \leqslant \varepsilon$*

Definition 8. *We say that $x(t)$ is asymptotically stable if:*

1. *$x(t)$ is Lyapunov stable*
2. *$\exists \delta > 0$ such that if $x_1 \in B_\delta(x_1)$ and $y(t) = \Phi(t, x_1)$ then $|y(t) - x(t)| \to 0$ as $t \to \infty$.*

Examples:

1. maps on a plane
2. **Rotation of a circle**
3.

Let $f : M \to M$ correspond to a dynamical system with discrete time. Let p be a fixed point of f. We are interested in such a question: if a trajectory of p is stable or not?

Definition 9. *Function $V : M \to \mathbb{R}$ is called Lyapunov-type function if*

1. *$\exists p \in M$ such that $V(p) = 0$;*
2. *$V(x) \geqslant 0$ for $\forall x \in M$;*
3. *If $V(x) = 0$ then $x = p$.*
4. *$V(x)$ is continious.*

Examples: Let $M = \mathbb{R}^2$, $p = (0, 0)$. Test yourself: if the following function is Lyapunov function?

1. $V(x, y) = \sqrt{x^2 + y^2}$ — yes
2. $V(x, y) = x$ — no
3. $V(x, y) = (x - y)^2$ — no
4. $V(x, y) = (x - y)^2 + (x + y)^2$ — yes
5. $V(x, y) = x^2 + 2y^2$ — yes
6. $V(x, y) = x^2 - xy + y^2$ — yes

Why do we need Lyapunov functions?

Sketches of lecture 3. 27 November 2019

Let M be a metric space.

Goal: we want to have a theorem (see theorem 6):
if [some conditions which are easy to check] then [Lyapunov/asymptotic stability].

Lemma 1. *Let $V : \mathbb{R}^n \to \mathbb{R}^n$ — Lyapunov-type function and $V(p) = 0$. Consider $x_k \in \mathbb{R}^n$ — bounded. Then $V(x, y) \to 0$ iff $x_k \to p$.*

Proof: If $x_k \to 0$ then $V(x_k) \to 0$ — clear.
Let $V(x_k) \to 0$. Assume $x_k \not\to p$. There exists a subsequence x_{n_k} such that $\operatorname{dist}(x_{n_k}, p) > \delta$. $\square$

Theorem 6. *Assume that $V : B(r, p) \to \mathbb{R}$ — Lyapunov-type function. Let M be a metric space, $f : M \to M$ and $p \in M$ is a fixed point of f, that is $f(p) = p$. If*

- *$\exists \rho < r : f(B(\rho, p)) \subset B(r, p)$.*

- *$V(f(x)) \leqslant V(x)$ for $x \in B(\rho, p)$.*

Then p is Lyapunov stable.

Proof: Lyapunov stability means that
Step 1:
Step 2: $\square$

Theorem 7. *Assume that $V : B(r, p) \to \mathbb{R}$ — Lyapunov-type function. Let M be a metric space, $f : M \to M$ and $p \in M$ is a fixed point of f, that is $f(p) = p$. If*

- *$\exists \rho < r \ f(B(\rho, p)) \subset B(r, p)$.*
- *$V(f(x)) < V(x)$ for $x \in B(\rho, p) \setminus \{p\}$.*

Then p is asymptotically stable.

Proof: $\square$

Examples:

1. Consider the rotation of a circle:
$$\begin{pmatrix} f(x) \\ f(y) \end{pmatrix} = \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$
Let's consider $V(x, y) = x^2 + y^2$ — Lyapunov-type function. We see that
$$V(f(x), f(y)) = (f(x))^2 + (f(y))^2 = x^2 + y^2 = V(x, y).$$
So theorem 7 can be applied and $x = y = 0$ is Lyapunov stable point.

2. Consider the following linear map:
$$\begin{pmatrix} f(x) \\ f(y) \end{pmatrix} = \begin{pmatrix} 0.9 & 0 \\ 0 & 0.8 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$
Let's consider $V(x, y) = x^2 + y^2$.

3. Consider the following linear map:
$$\begin{pmatrix} f(x) \\ f(y) \end{pmatrix} = \begin{pmatrix} 0.9 & 0.1 \\ 0 & 0.9 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$
Let's consider $V(x, y) = x^2 + y^2$. We need the following inequality:

   No!

   Let's consider $V(x, y) = a^2x^2 + b^2y^2$ and find a and b such that $V(x, y)$ is nesessary Lyapunov function.

Theorem 8. *Let $S \subset M$ — closed subset (you can think of it as a sector). Assume that $V : S \to \mathbb{R}$ — continious function (no need for Lyapunov-type function). Let $f : M \to M$ and $p \in M$ is a fixed point of f, that is $f(p) = p$. If*

- *$\exists \rho$ such that if $x \in B(\rho, p) \cap S \setminus \{p\}$ then*
  1. *$x \in B(r, p) \cap S$.*
  2. *$V(f(x)) > V(x)$.*
- *For any $\delta > 0$ there $\exists x \in B(\delta, p)$ such that $V(x) > 0$.*

Then p is not Lyapunov stable.

Proof: $\square$

Lemma 2. *$f : M \to M$ and $p \in M$ is a fixed point of f. Let $m > 0$. The following statements are equivalent:*

1. *f is Lyapunov/asymptotically stable*
2. *f^m is Lyapunov/asymptotically stable*

Proof: (1) to (2) — obvious
(2) to (1). What is Lyapunov stability:

$$\forall \varepsilon > 0$$

$\square$

Sketches of lecture 4. 28 November 2019

Consider $f : \mathbb{R}^n \to \mathbb{R}^n$, $f(0) = 0$

$$f(x) = Ax + g(x)$$

a C^1-function, $A = Df(0)$, $g(x) = o(x)$.

Theorem 9. *If all eigenvalues $|\lambda_i| < 1$ of A, then f is asymptotically stable.*

Proof:
Case 1: Let $\|A\| < 1$, then exists $\mu < 1$ such that for any x holds $|Ax| \leqslant \mu x$.
Choose $\mu_1 \in (\mu, 1)$. There exists small ρ such that $\forall x \in B(\rho, 0)$ holds $|g(x)| < (\mu_1 - \mu)|x|$. Then

$$|f(x)| = |Ax + g(x)| \leqslant |Ax| + |g(x)| \leqslant \mu|x| + (\mu_1 - \mu)|x| = \mu_1|x| < |x|$$

So by the Theorem f is asymptotically stable.
Case 2: Let $\|A\| \geqslant 1$. Then $\exists m$ such that $\|A^m\| < 1$ (see Worksheet for the diagonalizible matrices, general case we omit).
So we have:

$$f^m(x) = A^m x + g_m(x), \qquad g_m(x) = o(x).$$

Using case 1 we get that f^m is asymptotically stable. By the Theorem we get that f is asymptotically stable. $\square$

Theorem 10. *$f : \mathbb{R}^n \to \mathbb{R}^n$, $f(0) = 0$, $f(x) = Ax + g(x)$, $g(x) = o(x)$ as $x \to 0$. If exists eigenvalue of A, $|\lambda_i| > 1$, then f is not stable.*

Proof: Let's prove in a simplest case: $n = 2$ and

$$A = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$$

Take $x_0 = (\varepsilon, 0)$ then $x_n = (\lambda^n \varepsilon, 0)$.

Let

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \qquad V(x_1, x_2) = |x_1| - |x_2|$$

and $S = \{(x_1, x_2) : V(x) \geqslant 0\}$.

Fix $\varepsilon > 0$. Let's have $\rho > 0$, if $x \in B(\rho, 0)$ then $|g(x)| < \varepsilon |x|$ and $\lambda - 2\varepsilon > 1$, $\mu + 2\varepsilon < 1$.

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \lambda x_1 \\ \mu x_2 \end{pmatrix} + \begin{pmatrix} g_1(x_1, x_2) \\ g_2(x_1, x_2) \end{pmatrix}$$

Then

$$V(f(x)) = |\lambda x_1 + g_1(x_1, x_2)| - |\mu x_1 + g_2(x_1, x_2)| \geqslant$$

□

Remark 4. *The previous theorem is true for any n.*

1.5 Derivates along the system

Let $\dot{x} = F(x)$, $x \in \mathbb{R}^n$, $F : \mathbb{R}^n \to \mathbb{R}^n$. We have existence and uniqueness. We have flow (continious dynamical system) $\Phi(t, x)$. Take one solution $x(t)$ and arbitrary function $W : \mathbb{R}^n \to \mathbb{R}$.

We want to understand how $W(x(t))$ is changing with a time. Calculate derivative:

$$\frac{d}{dt} W(x(t)) = DW(x(t)) \cdot \dot{x} = DW(x(t)) \cdot F(x(t)).$$

Examples:

1. Consider

$$\dot{x} = -y$$
$$\dot{y} = x$$

Let's consider $V(x, y) = x^2 + y^2$ and take derivative along the trajectory:

$$\frac{d}{dt} V(x, y) = \begin{pmatrix} 2x & 2y \end{pmatrix} \cdot \begin{pmatrix} -y \\ x \end{pmatrix} = 0$$

So the function V is constant along the trajectories.

2. System corresponding to penduluum with friction:

$$\dot{x} = y$$
$$\dot{y} = -\sin(x) - by$$

Consider $W = -\cos(x) + \frac{y^2}{2}$. Take derivative along the trajectory:

$$\frac{d}{dt}W = \begin{pmatrix}\sin(x) & y\end{pmatrix} \cdot$$

So the energy decreases along the trajectories.

3. Machine learning example: consider a function which you want to minimize, say $E(x, y) = x^2 + xy + 3y^2 + y^4$. To find it's minimum you can run such a system:

$$\dot{x} = -2x - y = -\frac{\partial E}{\partial x}$$
$$\dot{y} = -x - 6y - 4y^3 = -\frac{\partial E}{\partial y}$$

Take derivative of $W(x, y) = E(x, y)$ along the trajectory:

$$\frac{d}{dt}W = \begin{pmatrix}2x + y & x + 6y + 4y^3\end{pmatrix} \cdot \begin{pmatrix}-2x - y \\ -x - 6y - 4y^3\end{pmatrix} < 0$$

1.6 Stability for continious time dynamical systems

Plan:

1. Create connection between stability for continious time and discrete time
2. Give criteria similar to Theorems in discrete case

Statement 3. *Let $\Phi(t, x)$ — dynamical system with continious time. Let p be fixed point ($\Phi(t, x) = x$ for any t). Then for any $\varepsilon_1 > 0$ there exists $\varepsilon_2 > 0$ such that if $x \in B(\varepsilon_2, p)$ for any $t \in [0, 1]$ $\Phi(t, x) \in B(\varepsilon_1, p)$.*

Theorem 11. *$\Phi(t, x)$ — continious time, p — fixed point of Φ. Let $f(x) = \Phi(1, x)$ — continious function. Then the following statents are equivalent:*

1. *p is Lyapunov stable for Φ*
2. *p is Lyapunov stable for f*

Proof: From (1) to (2) is obvious.
From (2) to (1): □

Theorem 12. *$\dot{x} = F(x)$ and $F(p) = 0$. Let $V : B(r, p) \to \mathbb{R}$ be a Lyapunov-type function.*

1. *If $\frac{d}{dt}V(x) \leqslant 0$, $x \in B(r, p)$ then p is Lyapunov stable for Φ*

2. $\frac{d}{dt}V(x) < 0$, $x \in B(r,p)$, $x \neq p$ *then* p *is asymptotically stable.*

Proof: Let $f(x) = \Phi(1,x)$.

$$V(f(x)) = V(\Phi(1,x)) = V(\Phi(0,x)) + \int\limits_0^1 \frac{d}{dt} V(\Phi(t,x))\,dt = V(x) + \int\limits_0^1 \frac{d}{dt} V(\Phi(t,x))\,dt \leqslant V(x)$$

$\square$

Remark 5. *If we have a system* $\dot{x} = F(x)$ *and a function* W*. We denote the derivative along the trajectory (with circle above* W*)* $W(x) = \frac{d}{dt}W(x(t)) = DW(x) \cdot F(x)$*.*

1.7 Topological conjugacy

Definition 10. *Let* $f : X \to X$ *and* $g : Y \to Y$*, where* X *and* Y *are metric spaces. We say that* f *and* g *are topologically conjugated (¡¡similar¿¿), and denote it* $f \sim g$*, if there exists* $h : X \to Y$ *— homeomorphism such that*

$$h \circ f = g \circ h$$

Remark 6. *You have already seen this — it is just the change of variables. For matrices it corresponds for diagonalisation (or to make Jordan blocks)*

Examples:

1. $f(x) = x + 1$, $g(y) = y + 2$, $x \in \mathbb{R}, y \in \mathbb{R}$. Then $h(0) = 0$, $h(1) = 2$, $h(2) = 4$, $h(3) = 6$, so we guess that $h(x) = 2x$. Let's check: $2(x+1) = 2x + 2$.

2. $f(x) = \frac{x}{2}$, $g(y) = \frac{y}{4}$, $x \in \mathbb{R}, y \in \mathbb{R}$. Check that we can take h as follows

$$h(x) = \begin{cases} x^2, & x > 0 \\ 0, & x = 0 \\ -x^2, & x < 0 \end{cases}$$

Properies of conjugacy:

1. $f \sim f$

2. $f \sim g$ then $g \sim f$. Just take $\tilde{h} = h^{-1}$

3. If $f_1 \sim f_2$ and $f_2 \sim f_3$ then $f_1 \sim f_3$. Proof.

   Properties 1–3 mean that $f \sim g$ is an equivalence relation.

4. If $f \sim g$ then $f^n \sim g^n$. Indeed,

$$\begin{aligned} f^n &= (h \circ g \circ h^{-1})^n = (h \circ g \circ h^{-1}) \circ (h \circ g \circ h^{-1}) \circ \ldots \circ (h \circ g \circ h^{-1}) = \\ &= h \circ g \circ (h^{-1} \circ h) \circ g \circ (h^{-1} \circ h) \ldots (h^{-1} \circ h) \circ g \circ h^{-1}) = h \circ g^n \circ h^{-1}, q.e.d. \end{aligned}$$

   This property helps us to understand example 2: we have $h(0) = 0$. Let's $h(1) = 1$, $f^n(1) = \frac{1}{2^n}$, $g^n(1) = \frac{1}{4^n}$. So we can guess now that $h(x) = x^2$.

Remark 7. *Conjugacy map is not necessarily unique!*

5. If $f \sim g$ and f generates a dynamical system $\Phi(n, x)$, g generates a dynamical system $\Psi(n, x)$. Then $\forall n \in \mathbb{Z}$ holds $\Phi(n, x) \sim \Psi(n, x)$.

Lemma 3. *Let $f : X \to X$, $g : Y \to Y$, $h : X \to Y$ and $h \circ f = g \circ h$. Then*

1. *$h(O(x, f)) = O(h(x), g)$. So orbit is mapped into the orbit.*

2. *$x \in Per(f)$ iff $h(x) \in Per(g)$*

3. *p is fixed point for f and is Lyapunov stable (asymptotically stable) iff $h(p)$ is fixed and is Lyapunov stable (asymptotically stable).*

   Proof: *Let's prove that if p is Lyapunov stable for f then $h(p)$ is Lyapunov stable for g. We want*

$$\forall \varepsilon > 0\, \exists \delta > 0 \text{ such that if } y_0 \in B(\delta, h(p)) \text{ then } y_n = g^n(y_0) \in B(\varepsilon, h(p))$$

   There $\exists$

   Remark 8. *Here we strongly used that h is a homeomorphism! That is that h is continious and h^{-1} is also continious. So h^{-1} is as important as h.*

   Remark 9. *It is clear why h should be minimum homeomorphism. Why not diffeomorphism? Let's remember example 2:*

$$h(x) = \begin{cases} x^2, & x > 0 \\ 0, & x = 0 \\ -x^2, & x < 0 \end{cases} \qquad h'(x) = \begin{cases} 2x, & x > 0 \\ 0, & x = 0 \\ -2x, & x < 0 \end{cases}$$

   Here $h'(x)$ is continios and it is OK. But if we look for h^{-1} then

$$h^{-1}(x) = \begin{cases} \sqrt{x}, & x > 0 \\ 0, & x = 0 \\ -\sqrt{|x|}, & x < 0 \end{cases}$$

   And h^{-1} is not even differentiable.

1.8 Grobman-Hartman theorem

Let $f : \mathbb{R}^n \to \mathbb{R}^n$ and $f \in C^1$ — diffeo, p is a fixed point, $A = Df(p)$ — differential.

Definition 11. *Matrix A is called hyperbolic if for all eigenvalues λ_i holds $|\lambda_i| \neq 1$.*

So not allowed eigenvalues like $1, -1, a + bi$ (with $a^2 + b^2 = 1$).

Definition 12. *Point p is called hyperbolic if it's differential $A = Df(p)$ is hyperbolic.*

Theorem 13. *(Grobman-Hartman theorem)*
If A is a hyperbolic matrix then there exists U — neighbourhood of p and homeo $h : \mathbb{R}^n \to \mathbb{R}^n$ such that $h \circ f = A \circ h$ (for $x \in U$).

Remark 10. *Why it is important that A is hyperbolic? Let's take $f(x) = x + x^3$, $f(0) = 0$, $Df(x) = x$, $A = \mathrm{Id}$*

Remark 11. *Even for linear maps with eigenvalues modulo 1 we cannot say anything about Lyapunov stability. For example:*

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ — \textit{Lyapunov stable}}$$

and

$$B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ — \textit{not Lyapunov stable}}$$

Possible conclusions:

Let's take $n = 2$, $\lambda > 1$, $\mu < 1$ and

$$A = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$$

Consider $f(x) = Ax + r(x)$, $r(x) = o(x)$. And take U — some neighbourhood of 0.

Definition 13. *Set of stable points of f (locally)*

$$W^s_{loc}(0) := \{x_0 : x_n \to 0, x_n \in U\}$$

Definition 14. *Set of unstable points of f (locally)*

$$W^u_{loc}(0) := \{x_0 : x_{-n} \to 0, x_{-n} \in U\}$$

Definition 15. *Set of stable points of f*

$$W^s(0) := \{x_0 : x_n \to 0\} = \cup_{n \geqslant 0} f^{-n}(W^s_{loc}(0))$$

Definition 16. *Set of unstable points of f*

$$W^u(0) := \{x_0 : x_{-n} \to 0\}$$

Write text about the trajectories of nonlinear system if we know the trajectories of linear system.

Examples:

1. Bouncing ball (table tennis rocket)

$$t_{n+1} = t_n + \frac{2}{g} v_n$$
$$v_{n+1} = v_n + 2b\cos(t_{n+1})$$

Here $t_n \mathrm{mod} 2\pi$.

What are fixed points?

$$\frac{2}{\pi} v = 2\pi k, \; \cos(t) = 0 \text{ then } t = \frac{\pi}{2} + \pi n$$

Let's find differential of f

$$\begin{pmatrix} 1 & \frac{2}{g} \\ -2b\sin(t + \frac{2}{g}v) & 1 - 2b\sin(t + \frac{2}{g}v) \cdot \frac{2}{g} \end{pmatrix}$$

This map is hyperbolic.

2 Problems

26 November, 2019

2.1 Problems-1: examples of non-existence or non-uniqueness of solutions of ODEs

7. Consider equation $\dot{x} = \sqrt{|x|}$. Let $x(0) = 0$.

   (a) Find at least two solutions $x : \mathbb{R} \to \mathbb{R}$.

   (b*) Find all solutions of the problem.

   NB: If right hand side is not a Lipshitz function there might be not unique solution.

8. Consider a function $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined as follows:

$$f(x,y) = \begin{cases} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, & y \geqslant 2x \\ \begin{pmatrix} 0 \\ 1 \end{pmatrix}, & y < 2x \end{cases}$$

   Note that f is not continious. Prove that the system of differential equations

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = f(x,y)$$

   has no solution for $t > 0$ given an initial condition $x(0) = y(0) = 0$.
   NB: if $f(x,y)$ is not continuous there might be no solution of the ODE.

9. Consider a metric space M. Let $\Phi(t,x)$ ($t \in \mathbb{R}$, $x \in M$) be a continuous-time dynamical system (flow). If map $f : M \to M$ satisfies $f(x) = \Phi(1,x)$ then we say that f is embedded into a flow.

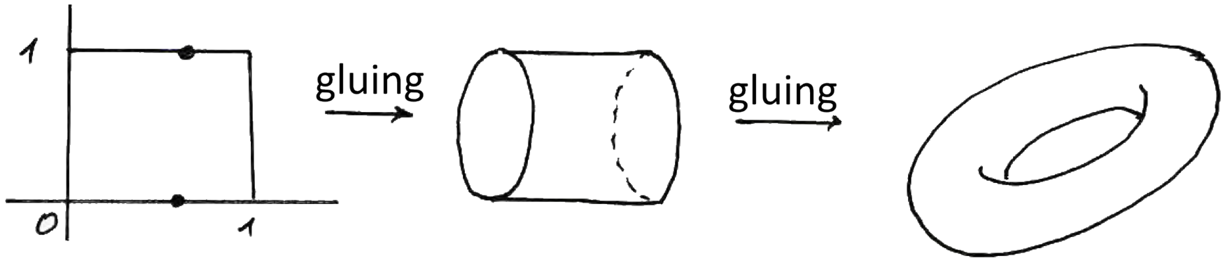
   (a) Let $M = \mathbb{R}$ and $f(x) = -x$. Prove that it cannot be embeed into the flow.

   (b) Let $M = \mathbb{R}^2$ and $f(x) = -x$. Prove that it can be embeed into the flow.

   (c) Let $M = \mathbb{R}^2$. Construct a map $f : M \to M$, which cannot be embedded into a flow.

2.2 Problems-1: periodic points for maps.

We will use the following notation: $S^1 = \mathbb{R}/\mathbb{Z}$ — a standard circle, $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ — a torus.



1. Consider a map $f : S^1 \to S^1$ defined as $f(x) = 2x$.
   (a) Prove that it is well-defined, so if $x \sim y$ then $f(x) \sim f(y)$. Prove that f is a continious map but not a homeomorphism.
   (b) Find all periodic points of period 3, and all periodic points of period 4.
   (c) Find 3 different preperiodic points, which are not periodic.
   (d) Find all periodic points and all preperiodic points.
2. Consider a map $f : S^1 \to S^1$ defined as $f(x) = x + \alpha$ for $\alpha \in \mathbb{R}$.
   (a) Imagine α is rational. Find all periodic points.
   (b) Imagine α is irrational. Find all periodic points.
3. Consider a map $f : \mathbb{T}^2 \to \mathbb{T}^2$ defined as

$$f(x, y) = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{4}$$

   (a) Prove that there exists an inverse map f^{-1}.
   (b) Find all fixed points and all periodic points of the map f.

NB: this map is often called Arnold's cat map due to russian mathematician Vladimir Arnold (1937 — 2010), who demonstrated the effects of this map in the 1960s using an image of a cat (google it!).

2.3 Problems-1: extra problems: transitivity

Def: Let M be a metric space. A map $f : M \to M$ is called *transitive* if there exists a point $x \in M$ such that its positive semitrajectory

$$O^+(x, f) = \{f^n(x) \,|\, n \in \mathbb{N} \cup \{0\}\}$$

is dense in M. That is $\overline{O^+(x, f)} = M$.

4. (a) For which $\alpha \in \mathbb{R}$ the map $f : S^1 \to S^1$ defined as $f(x) = x + \alpha$ is transitive?

   (b*) Prove that the map $f : S^1 \to S^1$ defined as $f(x) = 2x$ is transitive.
   Hint: prove that for any interval I on a circle there exists n such that $f^n(I) = S^1$.

   (c**) Prove that Arnold's cat map defined by formula (4) is transitive.

5. Prove that the decimal expansion of 2^n for $n \in \mathbb{N}$ can start with arbitrary natural number.
   Hint: use problem 4a.

6**. Let M be a metric space and $f : M \to M$. Prove that if there exists a point $x \in M$ which semitrajectory is dense in M then there exists a point $y \in M$ which trajectory is dense in M.

2.4 Problems-1: extra problems: ω, α-limit set

Def: Let M be a metric space, a map $f : M \to M$. A set $A \subset M$ is called *ω-limit set* of a trajectory $O(x, f)$ (and is denoted as $\omega(x, f)$) if it is a set of all limit points of all sequences $f^{n_k}(x)$ as $n_k \to +\infty$. That is

$$\omega(x, f) := \{\lim f^{n_k}(x) \,|\, n_k \in \mathbb{N}, n_k \to +\infty\}$$

In a similar way an α-limit set can be defined:

$$\alpha(x, f) := \{\lim f^{n_k}(x) \,|\, n_k \in \mathbb{N}, n_k \to -\infty\}$$

An important notion in the theory of dynamical systems is the notion of *nonwandering point.*
Def: a point $x_0 \in M$ is called *wandering* for the map f if

$$\exists U \text{ — a neighborhood of } x_0, \text{ and } \exists N \in \mathbb{N} \text{ such that } \forall n \in \mathbb{N} : |n| > N \qquad f^n(U) \cap U = \emptyset$$

A point x_0 is called *nonwandering* if it is not wandering. Realize that it is equivalent to the following:

$$\forall U \text{ — a neighborhood of } x_0, \text{ and } \forall N \in \mathbb{N} \text{ such that } \exists n \in \mathbb{N} : |n| > N \qquad f^n(U) \cap U \neq \emptyset$$

Def: a collection of all nonwandering points for the map f is called a *nonwandering set* and is denoted by $\Omega(f)$.

7. Prove the following statements:

   (a) If $x \in \mathrm{Per}(f)$ then $x \in \omega(x, f)$.

   (b) $\omega(x, f) \cup \alpha(x, f) \subset \Omega(f)$ for any $x \in M$.

   (c) If M is compact then $\Omega(f) \neq \emptyset$

8. Find the nonwandering set $\Omega(f)$:

   (a) for the map $f : S^1 \to S^1$ defined as $f(x) = 2x$

   (b) for the map $f : S^1 \to S^1$ defined as $f(x) = x + \alpha$

   (c) for the the Arnold's cat map.

12*. Find an example of a map $f : M \to M$ for which does not hold

$$\bigcup_{x \in M} (\omega(x, f) \cup \alpha(x, f)) \neq \Omega(f)$$

.

13. Let $x \in \mathbb{R}^n$. Consider differential equation

$$\dot{x} = f(x),$$

where $f \in C^0$. Assume that for any $x, y \in \mathbb{R}^n$ the following inequality holds

$$< x - y, f(x) - f(y) > \leq 0,$$

where $< \cdot, \cdot >$ denotes the scalar product. Prove that for any $x_0 \in \mathbb{R}^n$ there exists not more than one solution for $t \geq 0$, satisfying $x(0) = x_0$.

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2.5 Problems-2: Lyapunov functions

1. Test yourself. Which of the following functions are Lyapunov-type functions for $M = \mathbb{R}^2$ and $p = 0$? Recall that function $V(x, y)$ is of Lyapunov type iff $V(x, y) \geq 0$ and $V(x, y) = 0$ only for $(x, y) = 0$.

   (a) $V(x, y) = |x| + 2|y|$

   (b) $V(x, y) = x^2 + 2y^2$

   (c) $V(x, y) = x^2 - 3xy + y^2$

   (d) $V(x, y) = |x - y| + x^2 - 2xy + y^2$

2. For which $b \in \mathbb{R}$ the function

$$V(x, y) = x^2 + bxy + 2y^2$$

   is Lyapunov-type function for $p = (0, 0)$?

3. Consider a map $f : \mathbb{R} \to \mathbb{R}$ defined by the formula:

   (a) $f(x) = x - x^3$

   (b) $f(x) = x - x^2$

   and a semi-dynamical system generated by it. Note that 0 is a fixed point. Is it

   i. Lyapunov stable?

   ii. Asymptotically stable?

4. Find Lyapunov function V, which allows to prove stability of a map $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined as

$$f(x) = \begin{pmatrix} 0.9 & 1 \\ 0 & 0.9 \end{pmatrix} x$$

   Hint: Try to find $V(x, y)$ in the form $ax^2 + by^2$ for some $a, b \in \mathbb{R}^+$.

2.6 Problems-2: stability of linear maps

5. Consider $x \in \mathbb{R}^n$, $f : \mathbb{R}^n \to \mathbb{R}^n$ such that $f(x) = Ax$, and corresponding dynamical system with discrete time. Here A is a matrix of size $n \times n$ (not necessarily invertible).

   (a) Prove that the following statements are equivalent:

      i. Every semitrajectory x_t is asymptotically stable

      ii. Fixed point $x = 0$ is asymptotically stable

   (b) Prove that the following statements are equivalent:

i. Fixed point $x = 0$ is asymptotically stable

ii. $\|A^k\| \to 0$ as $k \to \infty$

6. Let $A \in M_{n\times n}$ and diagonalizable, that is $\exists S \in M_{n\times n}$ — invertible such that

$$A = S^{-1}JS$$

where $J = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$ is a diagonal matrix

$$J = \begin{pmatrix} \lambda_1 & 0 & 0 & \dots & 0 \\ 0 & \lambda_2 & 0 & \dots & 0 \\ 0 & 0 & \lambda_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \lambda_n \end{pmatrix}$$

Prove that:

(a) If all eigenvalues $|\lambda_i| < 1$, $i = 1, \dots, n$, then $\|A^k\| \to 0$ as $k \to \infty$.

(b) If $\exists|\lambda_j| > 1$ then $\|A^k\| \to \infty$ as $k \to \infty$.

(c) If for any $|\lambda_i| \leqslant 1$ and $\exists|\lambda_l| = 1$ then

i. $\|A^k\| \not\to 0$

ii. $\exists C > 0$ such that $\|A^k\| < C$.

Extra problems

7. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a C^1-diffeomorphisms and p is its fixed point ($f(p) = p$). Assume that all eigenvalues λ_i of $A = Df(p)$ satisfies $\lambda_i \neq 1$. Prove the following **Theorem.** For any $\delta > 0$ there exists $\varepsilon > 0$ such that for any map $g : \mathbb{R}^n \to \mathbb{R}^n$ satisfying $|f(x) - g(x)| < \varepsilon$, $\|Df(x) - Dg(x)\| < \varepsilon$ there exist $q \in B_\delta(p)$ – fixed point for g (that is $g(q) = q$).

8. Let $\Phi : I^+ \times \mathbb{R} \to \mathbb{R}$ be a semi-dynamical system. Assume that x_0 is a fixed point of Φ and there exists a $\delta > 0$ such that

$$\operatorname{dist}(\Phi(t, y_0), x_0) \to 0, \qquad t \to \infty$$

for any $y_0 \in \mathbb{R}$ with $\operatorname{dist}(y_0, x_0) < \delta$. Show that the fixed point x_0 is Lyapunov stable for the cases

(a) $I^+ = \mathbb{R}^+$;

(b) $I^+ = \mathbb{Z}^+$.

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2.7 Problems-3: derivatives along the system

1. Consider the system of equations:

$$\dot{x} = y$$
$$\dot{y} = -x - 2y$$

   Find derivative of the function $V(x, y) = x^2 + y^2$ along the flow generated by those equations.

2. Consider function $V : \mathbb{R}^n \to \mathbb{R}$, $V \in C^1$ and consider dynamical system generated by a differential equation

$$\dot{x} = -\mathrm{grad} V(x).$$

   Such systems are called *gradient flows*. (Note that we do not assume that V is a Lyapunov-type function).

   (a) Find derivative of V along the system.

   (b) Prove that the system does not have periodic trajectories (fixed point are allowed)

3. Let $H : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be a C^2-function. Consider the system of differential equations:

$$\dot{q} = \frac{\partial H}{\partial p},$$
$$\dot{p} = -\frac{\partial H}{\partial q},$$

   Find the derivative of the function H along the system.

4. (a) Consider second order differential equation

$$m\ddot{x} + k(x + x^3) = 0,$$

   where $m, k > 0$. Prove that along any solution of this differential equation the value

$$\frac{m\dot{x}^2}{2} + k\Big(\frac{x^2}{2} + \frac{x^4}{4}\Big)$$

   is constant.

   (b) Consider the dynamical system generated by differential equations

$$\dot{x} = y,$$
$$\dot{y} = -\alpha(x + x^3) - \beta y,$$

   where $\alpha, \beta > 0$. Consider function $V(x, y) = y^2 + \alpha(x^2 + x^4/2)$. Find the derivative of V along the system.

(c) Note that this function V allows to prove Lyapunov stability of the point 0, but not asymptotical stability. Find Lyapunov function W which allows to prove asymptotic stability.

5. Let $K(t,s) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $f(t) : \mathbb{R} \to \mathbb{R}$ be two continuous bounded functions. Prove that there exists $T > 0$ such that for any $x_0 \in R$ there exists a unique solution to the following equation

$$x(t) = f(t) + \int_0^t K(t,s)x(s)ds$$

for $|t| \leq T$.

Hint: Use same methods as for existence and uniqueness of differential equations.
Remark: Such equations are called linear Volterra integral equation of the second kind.

6. Consider the Lotka-Volterra model (also known as the predator–prey equations)

$$\dot{x} = (\alpha - \beta y)x$$
$$\dot{y} = (\delta x - \gamma)y,$$

with $\alpha, \beta, \gamma, \delta > 0$. Here x is the number of prey (for example, rabbits); y is the number of some predator (for example, foxes).

Assume that solutions are unique.

(a) Show that if solutions begin in the first quadrant, they stay in the first quadrant. This shows that predator and prey population can't become negative. Determine what happens if we begin with only predators or only preys.

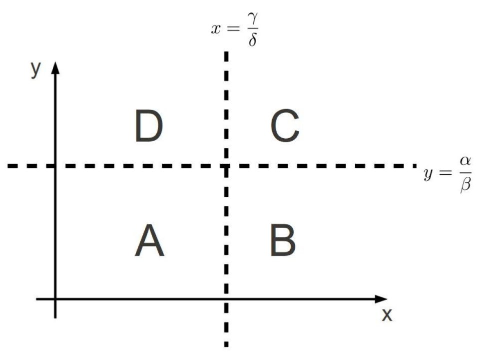
(b) Divide to four regions (see figure below. Regions don't contain the separating line). Show that solution that begins in A, does the circle $A \to B \to C \to D \to A$.

(c) Define

$$H(x,y) := \alpha \ln(y) - \beta y - \delta x + \gamma \ln(x).$$

Show that $H(x,y) = C$ along solutions.

(d) Use this fact and uniqueness of the solution to show that every nonfixed solution in the first quadrant is periodic.
Hint: look at values of $H(x,y)$ along the line $y = \alpha/\beta$.



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2.8 Problems-4: topological conjugacy

1. (a) Let $a, b \in \mathbb{R}$, $a \neq 0, 1$. Consider maps $f, g : \mathbb{R} \to \mathbb{R}$, defined as $f(x) = ax$, $g(x) = ax + b$. Prove that $f \sim g$ and find a map S such that $f \circ S = S \circ g$.

   (b) Find exact formula for $g^n(x)$.

2. (a) Consider maps $A, B : \mathbb{R} \to \mathbb{R}$ defined as follows $Ax := 2x$, $Bx := 3x$. Find a homeomorphism $h : \mathbb{R} \to \mathbb{R}$, such that $A \circ h = h \circ B$

   (b) Consider $a, b \in \mathbb{R}$, $a, b \neq 0$. When maps $f(x) = ax$ and $g(y) = by$ are topologically conjugated? Find a map S such that $f \circ S = S \circ g$.

3. Consider a diffeomorphism $F : \mathbb{R}^2 \to \mathbb{R}^2$ defined as follows

$$F \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x/2 \\ 2y + x^2 \end{pmatrix}.$$

   Let $A = DF(0)$.

   (a) Find a map $h : \mathbb{R}^2 \cap B(0, 1/2) \to \mathbb{R}^2$, such that

$$F \circ h(x) = h \circ A(x), \quad x \in B(0, 1/4).$$

   (b) Find stable manifold of point 0.

Extra problem

4. Let $a \in (0, 1)$. Consider maps $f(x, y) = (a^2 x + y^2, ay)$, $g(x, y) = (a^2 x, ay)$.

   (a) Prove that there exists a C^1-diffeomorphism $h : \mathbb{R}^2 \to \mathbb{R}^2$ such that $h \circ f = g \circ h$.

   (b) Prove that it is not possible to construct a C^2-diffeomorphism $h : \mathbb{R}^2 \to \mathbb{R}^2$ such that $h \circ f = g \circ h$.