

# On the impact of diffusion ratio on vanishing viscosity solutions of Riemann problems for chemical flooding models

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Joint work with Fedor Bakharev, Aleksandr Enin and Nikita Rastegaev: arXiv:2111.15001

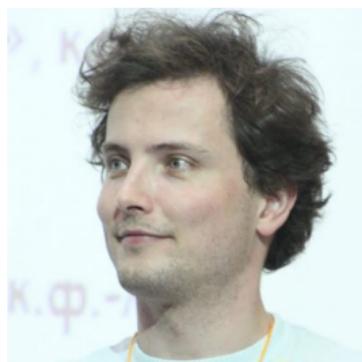


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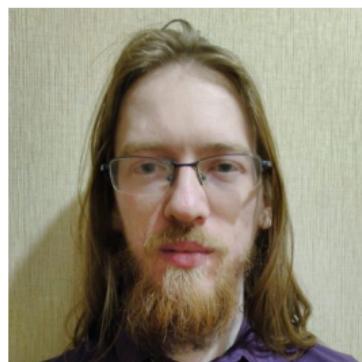


## Joint work with

Chebyshev Laboratory, St Petersburg State University, Russia



Fedor Bakharev



Aleksandr Enin



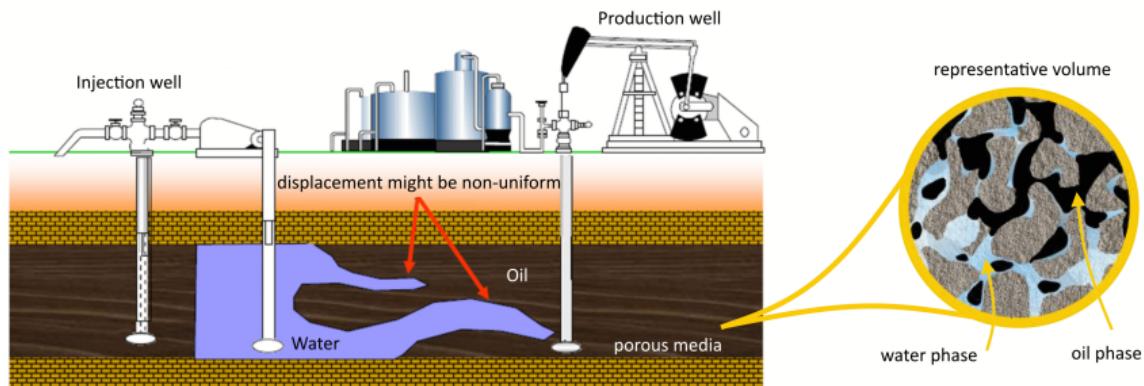
Nikita Rastegaev

- The talk is based on arXiv:2111.15001:  
F. Bakharev, A. Enin, Yu. Petrova, N. Rastegaev “Impact of dissipation ratio on vanishing viscosity solutions of the Riemann problem for chemical flooding model”
- Collaboration with Russian petroleum company GazpromNeft (2018–2021)

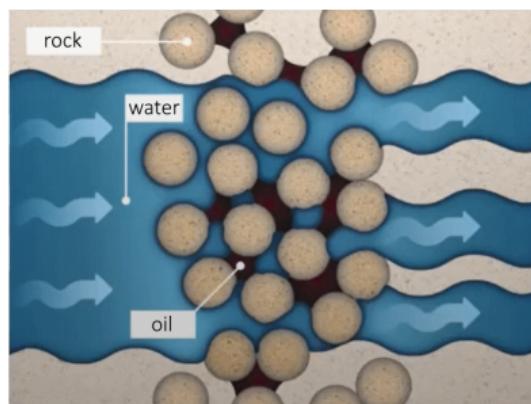
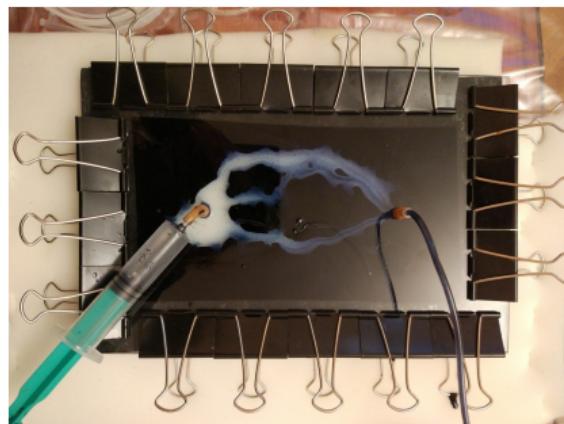
# Motivation

We are interested in the mathematical model of oil recovery. Some features:

- Porous media (averaged models of flow)
- Unknown variables:  $s(t, x)$  — the averaged water saturation in small volume  
 $1 - s$  — the average oil saturation in representative volume
- Relatively small speeds ( $\approx 1$  meter per day): Navier-Stokes  $\rightarrow$  Darcy's law
- Multiphase flow: oil, water, gas.
- Applications to EOR (enhanced oil recovery) methods: chemical, thermal, gas etc



# Problems: macroscopic and microscopic sweep efficiency



- happens due to very viscous oil or inhomogeneous media

- local entrapment of oil in pores due to high capillary pressure

## Possible solution

- Inject gas ( $\text{CO}_2$ , natural) to decrease the oil viscosity
- Add **chemicals (polymer)** to increase the water viscosity
- Add **chemicals (surfactant)** that reduce the surface tension etc

# Fundamental research: two main directions

## 1-dim in spatial variable

- Stable displacement



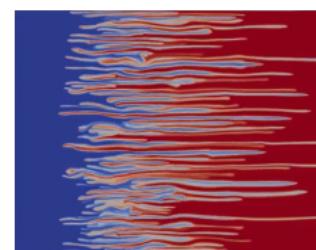
- main question: find an exact solution to a Riemann problem
- **hyperbolic conservation laws**

$$\begin{aligned} s_t + f(s, c)_x &= 0, \\ (cs + a(c))_t + (cf(s, c))_x &= 0. \end{aligned}$$

Example: chemical flooding model

## 2-dim (or 3-dim) in spatial variable

- Unstable displacement



- source of instability: water and oil/polymer have different viscosities
- **viscous fingering phenomenon**

$$\begin{aligned} c_t + u \cdot \nabla c &= \varepsilon \Delta c, \\ \operatorname{div}(u) &= 0, \\ u &= -\nabla p / \mu(c). \end{aligned}$$

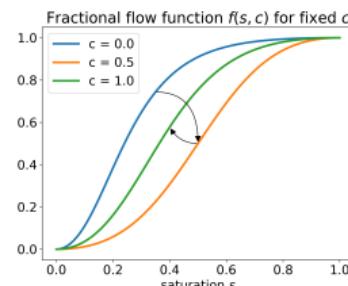
Example: Peaceman model

# Problem statement

Chemical flooding can be described as the system of conservation laws ( $x \in \mathbb{R}, t > 0$ ):

$$\begin{aligned} s_t + f(s, c)_x &= 0, && \text{(conservation of water)} \\ (cs + a(c))_t + (cf(s, c))_x &= 0. && \text{(conservation of chemical)} \end{aligned} \tag{1}$$

- $s = s(x, t)$  — water phase saturation;
- $f(s, c)$  — fractional flow function (usually *S*-shaped);
- $c = c(x, t)$  — concentration of a chemical agent in water;
- $a(c)$  — adsorption of a chemical agent on a rock (usually increasing, concave).



Initial data:

$$(s, c)|_{t=0} = \begin{cases} (1, 1), & \text{if } x \leq 0, \\ (0, 0), & \text{if } x > 0, \end{cases} \tag{2}$$

Aim:

Find a solution to initial-value problem (1)–(2) when  $f$  depends non-monotonically on  $c$ .

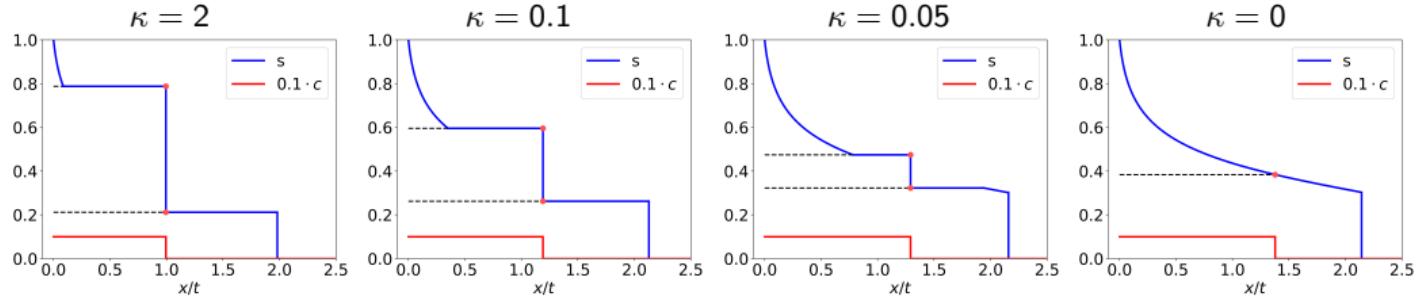
NB: Non-monotone dependence appears in surfactant flooding, low salinity water flooding etc

# Preview of results

- weak solutions  $\Rightarrow$  non-uniqueness of solutions to a Riemann problem
- use vanishing viscosity criterion — add small diffusion/capillary terms

$$\begin{aligned} s_t + f(s, c)_x &= \varepsilon_c (A(s, c)s_x)_x, \\ (cs + a(c))_t + (cf(s, c))_x &= \varepsilon_c (cA(s, c)s_x)_x + \varepsilon_d c_{xx}. \end{aligned}$$

- $f(s, c)$  monotone in  $c \Rightarrow$  uniqueness of vanishing viscosity solution (1988)
- Main idea of the talk:**  $f(s, c)$  non-monotone in  $c \Rightarrow$  exist multiple vanishing viscosity solutions, depending on ratio  $\kappa = \varepsilon_d / \varepsilon_c$



NB: the appeared shock is known as undercompressive (transitional).

# Hyperbolic systems of conservation laws\*

$$G(u)_t + F(u)_x = 0 \quad (3)$$

Here

- $G(u)$  — accumulation function (conserved quantities)
- $F(u)$  — flux function (flux of conserved quantities)

Simplest example: wave equation

$$y_{tt} - c^2 y_{xx} = 0 \quad (\text{J. d'Alambert, 1750})$$

can be rewritten as a system of two first-order equations on the state-vector  $u = \begin{pmatrix} y_x \\ y_t \end{pmatrix}$

$$u_t + Du_x = 0, \quad \text{with} \quad D = \begin{pmatrix} 0 & -1 \\ -c^2 & 0 \end{pmatrix}$$

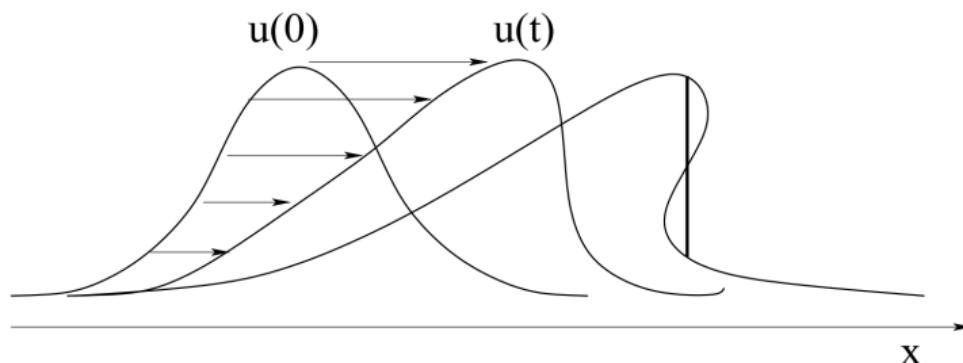
- eigenvalues  $\lambda_1 = c$  and  $\lambda_2 = -c$  are real, the system is hyperbolic. Solution contain two wave modes that propagate at the velocities  $\lambda_1$  and  $\lambda_2$ .

\* For more details see e.g. "Hyperbolic conservation laws: an illustrated tutorial" by Alberto Bressan.

# Hyperbolic systems of conservation laws

$$u_t + \left( \frac{u^2}{2} \right)_x = 0 \quad (\text{Burger's equation, 1948})$$

- non-linearity implies **wave speed**  $\lambda(u) = u$  depends on state  $u$
- So the wave can spread (**rarefaction wave**) or concentrate (**shock wave**)



$$u_t + (f(u))_x = 0 \quad (\text{Buckley-Leverett equation})$$

- existence, uniqueness was established by Olga Oleinik (1957)

# Riemann problem (1858)

- Riemann solved the initial-value problem with data having a **single jump**

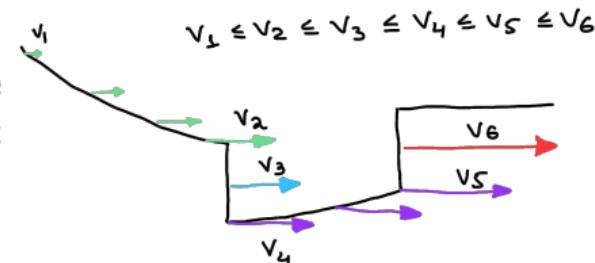
$$u|_{t=0} = \begin{cases} u^L, & x \leq 0; \\ u^R, & x > 0. \end{cases}$$

- took advantage of the **scale invariance** of the equations and the data:

$$u(\alpha x, \alpha t) = u(x, t) \quad \text{for all } \alpha > 0$$

- solution to a Riemann problem is important because:
  - it appears in a long-term behavior of Cauchy problem
  - helps to prove the existence of solutions to Cauchy problem (Glimm's method)
  - helps to construct numerical solution (Godunov method)

Any solution to a Riemann problem consists of a sequence of rarefaction or shock waves (and constant states) that are **compatible by speeds**



# Shock waves: RH condition and admissibility criteria

- discontinuous solutions are defined in the sense of distributions (**weak form**)
- for a shock wave from  $u^-$  to  $u^+$  moving with velocity  $v$ , the weak condition amounts to the following **Rankine-Hugoniot** condition (RH)

$$-v G(u^-) + F(u^-) = -v G(u^+) + F(u^+) \quad (\text{RH})$$

- RH means conservation: what flows into left side flows out of the right side
- Problems from the perspectives of both mathematics and physics:
  - if all RH solutions are allowed, a Riemann problem has multiple solutions
  - some RH solutions violate physical principles
- **Vanishing viscosity criteria:** consider a diffusive system of conservation laws

$$G(u)_t + F(u)_x = \varepsilon [B(u) u_x]_x, \quad \varepsilon \rightarrow 0$$

# Travelling wave solutions of diffusive system (Hopf, 1948)

- $u(x, t) = \hat{u}(\xi)$  with  $\xi := x - v t$  for a fixed shock velocity  $v$
- reduction to first-order system of ordinary differential equations:

$$\varepsilon B(\hat{u}) \hat{u}_\xi = -v [G(\hat{u}) - G(u^-)] + F(\hat{u}) - F(u^-)$$

- $u^-$  and  $u^+$  are fixed points and we look for an orbit connecting them

$$\hat{u}(-\infty) = u^-, \quad \hat{u}(+\infty) = u^+$$

- diffusive terms cause a shock wave to have a thin, smooth internal structure as a result of balancing nonlinear focusing and diffusive spreading
- travelling wave solution approaches the jump discontinuity in  $L^1$  as  $\varepsilon \rightarrow 0^+$

Questions?

## Historical review: zero adsorption

$$\begin{aligned} s_t + f(s, c)_x &= 0, \\ (sc)_t + (cf(s, c))_x &= 0. \end{aligned} \quad (s, c)|_{t=0} = \begin{cases} (s_L, c_L), & \text{if } x \leq 0, \\ (s_R, c_R), & \text{if } x > 0, \end{cases} \quad (4)$$

Can be rewritten in a more symmetric form:

$$\begin{aligned} s_t + (sg(s, b))_x &= 0, & b = sc &\text{ — total amount of chemicals} \\ b_t + (bg(s, b))_x &= 0. & g = f/s &\text{ — new flux function} \end{aligned}$$

- 1980 — elasticity theory (Keyfitz, Kranzer):  $g(s, b) = \tilde{g}(s^2 + b^2)$ .
- 1980 — polymer flooding (Isaacson, Glimm, Temple):  $f(s, c)$  monotone in  $c$
- many-many generalisations (delta shocks etc)

Interesting properties:

- has coordinate system of Riemann invariants
- rarefaction and shock curves coincide

Problems:

- **non-strictly** hyperbolic system
- has **contact discontinuities**

Work in progress with D. Marchesin, B. Plohr: justification of Isaacson-Glimm admissibility criterion & construction of a solution for the case when  $f(s, c)$  is non-monotone in  $c$

## Historical review: non-zero adsorption

$$\begin{aligned} s_t + f(s, c)_x &= 0, \\ (sc + a(c))_t + (cf(s, c))_x &= 0. \end{aligned} \quad (s, c)|_{t=0} = \begin{cases} (s_L, c_L), & \text{if } x \leq 0, \\ (s_R, c_R), & \text{if } x > 0, \end{cases} \quad (5)$$

### Proposition (Johansen-Winther, 1988 (JW))

The solution  $c(x, t)$  is monotone in  $x$  for any  $t > 0$ . If  $a(c)$  is concave, then

- is constant if  $c_L = c_R$ ;
- contains several rarefaction waves if  $c_L < c_R$ ;
- contains exactly one shock wave if  $c_L > c_R$ .

Historical review:

- 1988 — JW,  $f(s, c)$  monotone in  $c$ . Found a unique vanishing viscosity solution.

When  $f(s, c)$  non-monotone in  $c$ , multiple vanishing viscosity solutions are possible.

- 2017 — W. Shen, gave some examples in Lagrangian coordinates. See also
- 1986 — Entov, Kerimov, non-rigorous consideration of the non-monotone case.

## Reduced problem

$$\begin{aligned} s_t + f(s, c)_x &= 0, \\ (sc + a(c))_t + (cf(s, c))_x &= 0. \end{aligned} \quad (s, c)|_{t=0} = \begin{cases} (1, 1), & \text{if } x \leq 0, \\ (0, 0), & \text{if } x > 0, \end{cases} \quad (6)$$

### Proposition (Johansen-Winther, 1988 (JW))

*There exists  $u^- = (s^-, 1)$  and  $u^+ = (s^+, 0)$  such that the solution to a Riemann problem has the following structure:*

$$(1, 1) \xrightarrow{c=1} u^- \xrightarrow{c \text{ jumps from 1 to 0}} u^+ \xrightarrow{c=0} (0, 0). \quad (7)$$

### Historical review:

- 1988 — JW,  $f(s, c)$  monotone in  $c$ . Found a unique vanishing viscosity solution.

When  $f(s, c)$  non-monotone in  $c$ , multiple vanishing viscosity solutions are possible.

- 2017 — W. Shen, gave some examples in Lagrangian coordinates. See also
- 1986 — Entov, Kerimov, non-rigorous consideration of the non-monotone case.

# Dissipative system

To define a shock wave between  $u^-$  and  $u^+$  we consider dissipative system:

$$\begin{aligned}s_t + f(s, c)_x &= \varepsilon_c (A(s, c)s_x)_x, \\(cs + \alpha)_t + (cf(s, c))_x &= \varepsilon_c (cA(s, c)s_x)_x + \varepsilon_d c_{xx}, \\ \alpha_t &= \varepsilon_r^{-1} (a(c) - \alpha).\end{aligned}$$

- $\varepsilon_c$  — dimensionless capillary pressure
- $\varepsilon_d$  — dimensionless diffusion term
- $\varepsilon_r$  — dimensionless relaxation time
- $A(s, c)$  — capillary pressure term
- $\alpha = \alpha(x, t)$  — dynamic adsorption

We consider two particular cases:

## Capillarity and Diffusion

$$\begin{aligned}s_t + f(s, c)_x &= \varepsilon_c (A(s, c)s_x)_x, \\(cs + a(c))_t + (cf(s, c))_x &= \varepsilon_c (cA(s, c)s_x)_x + \varepsilon_d c_{xx},\end{aligned}$$

## Capillarity and Dynamic Adsorption

$$\begin{aligned}s_t + f(s, c)_x &= \varepsilon_c (A(s, c)s_x)_x, \\(cs + \alpha)_t + (cf(s, c))_x &= \varepsilon_c (cA(s, c)s_x)_x, \\ \alpha_t &= \varepsilon_r^{-1} (a(c) - \alpha).\end{aligned}$$

# Restrictions on $f$ and $a$

(F1)  $f \in C^2([0, 1]^2); f(0, c) = 0; f(1, c) = 1;$

(F2)  $f_s(s, c) > 0$  for  $s \in (0, 1)$ ,  $c \in [0, 1]$ ;  
 $f_s(0, c) = f_s(1, c) = 0$ ;

(F3)  $f$  is S-shaped in  $s$ ;

(F4)  $f$  is non-monotone in  $c$ :

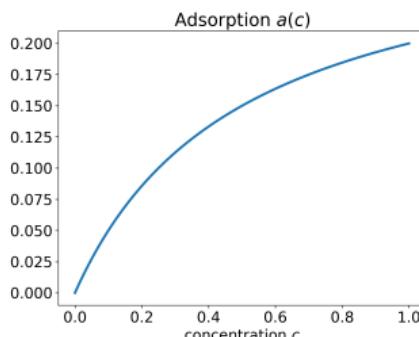
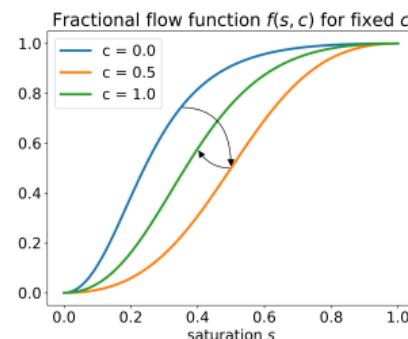
$\forall s \in (0, 1) \exists c^*(s) \in (0, 1)$ :

- $f_c(s, c) < 0$  for  $0 < s < 1$ ,  $0 < c < c^*(s)$ ;
- $f_c(s, c) > 0$  for  $0 < s < 1$ ,  $c^*(s) < c < 1$ ;

(A)  $A$  is bounded from zero and infinity;

(a)  $a \in C^2$ ,  $a(0) = 0$ ,  $a$  is strictly increasing and concave.

NB: assumptions are just the simplest possible ones. Could be generalised.



# Travelling wave dynamical system

$$\begin{aligned} s_t + f(s, c)_x &= \varepsilon_c (A(s, c)s_x)_x, \\ (cs + a(c))_t + (cf(s, c))_x &= \varepsilon_c (cA(s, c)s_x)_x + \varepsilon_d c_{xx}. \end{aligned} \tag{8}$$

Searching for travelling wave solutions  $s = s(\xi)$ ,  $c = c(\xi)$ ,  $\xi := \varepsilon_c^{-1}(x - vt)$  with boundary conditions

$$s(\pm\infty) = s^\pm, \quad c(-\infty) = 1, \quad c(+\infty) = 0,$$

we arrive at

$$\begin{aligned} A(s, c)s_\xi &= f(s, c) - v(s + d_1), \\ \kappa c_\xi &= v(d_1 c - d_2 - a(c)). \end{aligned} \tag{9}$$

- Parameters:  $\kappa = \varepsilon_d/\varepsilon_c$  and  $v$ .  $d_1 = a(1)$  and  $d_2 = 0$  — some constants.
- Note that  $u^\pm$  are fixed points of dynamical system (9);
- We are only interested in the trajectories connecting two saddle points (or saddle-nodes) due to compatibility of speeds in

$$(1, 1) \rightarrow u^- \xrightarrow{c\text{-shock}} u^+ \rightarrow (0, 0).$$

# Main result

Consider a dynamical system under assumptions (F1)–(F4), (A), (a):

$$\begin{aligned} A(s, c)s_\xi &= f(s, c) - v(s + d_1), \\ \kappa c_\xi &= v(d_1 c - d_2 - a(c)). \end{aligned}$$

Theorem (Bakharev, Enin, P., Rastegaev, 2021, arxiv:2111.15001)

There exist  $0 < v_{\min} < v_{\max} < \infty$ , such that for every  $\kappa = \varepsilon_d/\varepsilon_c \in (0, +\infty)$ , there exist unique

- points  $s^-(\kappa) \in [0, 1]$  and  $s^+(\kappa) \in [0, 1]$ ;
- velocity  $v(\kappa) \in [v_{\min}, v_{\max}]$ ,

such that there exists a travelling wave, connecting two saddle points  $u^-(\kappa) = (s^-(\kappa), 1)$  and  $u^+(\kappa) = (s^+(\kappa), 0)$  with velocity  $v(\kappa)$ . Moreover,  $v(\kappa)$  is monotone and continuous;  $v(\kappa) \rightarrow v_{\min}$  as  $\kappa \rightarrow \infty$ ;  $v(\kappa) \rightarrow v_{\max}$  as  $\kappa \rightarrow 0$ .

# Solution construction algorithm

1. From  $\kappa$  we calculate  $v(\kappa)$  (binary search).
2. From  $v$  we determine  $s^-(v)$  and  $s^+(v)$  via Rankine-Hugoniot condition.
3. Construct waves  $(1, 1) \rightarrow (s^-(v), 1)$  and  $(s(v), 0) \rightarrow (0, 0)$ .

Example: “boomerang” model:

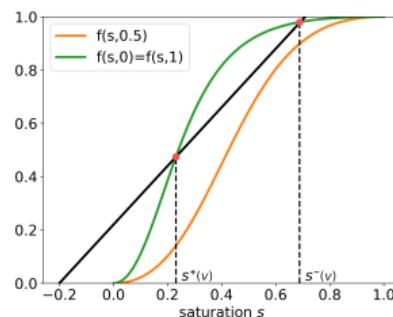


Figure 1: Flux functions

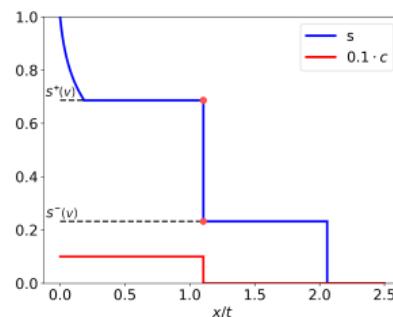


Figure 2: Solution  $s$  and  $c$

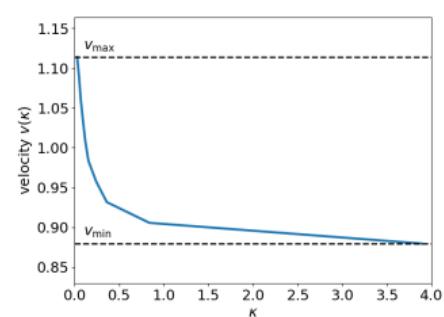


Figure 3: Function  $v(\kappa)$

# We would like to investigate...

## Convergence

Does the Riemann problem solution of the dissipative system (8) converge to the non-dissipative solution as  $\varepsilon_c \rightarrow 0$  with fixed  $\kappa$ ?

## Asymptotic stability as $t \rightarrow \infty$

Does the solution of a Cauchy problem for the dissipative system (8) with the correct values at  $\pm\infty$  tend to undercompressible travelling wave as  $t \rightarrow +\infty$ ?

I believe that “yes”, but...

- Difficulty: comparison theorems do not work for systems
- Hope 1: without diffusion terms the system decouples (in Lagrangian coordinates)
- Hope 2: “steepness” argument works? (like for Fisher-KPP reaction-diffusion eq.)
- any ideas?

# Scheme of proof

The Theorem can be divided into simpler statements:

- $\forall v \in [v_{\min}, v_{\max}] \quad \exists! \kappa(v)$ : there is a saddle-to-saddle travelling wave with  $\kappa(v)$ .
- $\kappa(v)$  is continuous.
- $\nexists v_1 \neq v_2 : \kappa(v_1) = \kappa(v_2)$ , thus  $\kappa(v)$  is monotone.
- $\kappa(v) \rightarrow \infty$  as  $v \rightarrow v_{\min}$ .
- $\kappa(v) \rightarrow \kappa_{\text{crit}} \geq 0$  as  $v \rightarrow v_{\max}$ .
- When  $\kappa < \kappa_{\text{crit}}$  and  $v = v_{\max}$  there is a saddle to saddle-node travelling wave

$\kappa(v)$  is monotone and continuous thus there exists an inverse function satisfying the Theorem.

# Phase portrait, fixed points, isoclines

We consider travelling wave dynamical system:

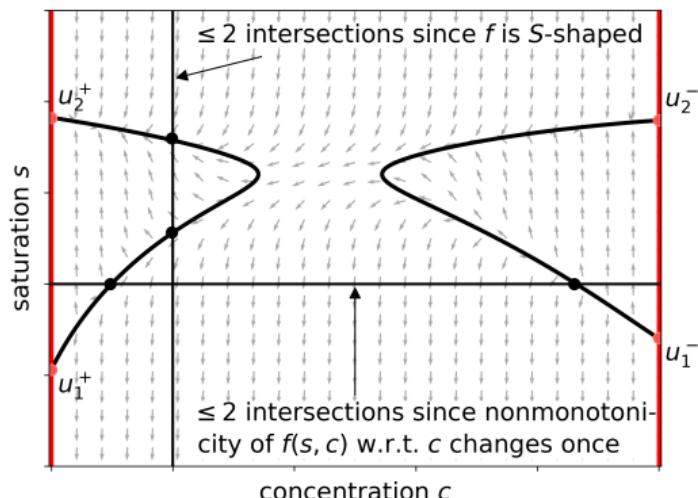
$$\begin{aligned} A(s, c)s_\xi &= f(s, c) - v(s + d_1), \\ \kappa c_\xi &= v(d_1c - d_2 - a(c)), \end{aligned}$$

Isoclines:

**red lines** are  $d_1c - d_2 - a(c) = 0$ ,  
**black lines** are  $f(s, c) - v(s + d_1) = 0$ .

Fixed points:

$u_1^+$  and  $u_2^-$  — saddle points;  
 $u_2^+$  — attractor;  $u_1^-$  — repeller



Aim:

find all pairs  $(\kappa, v)$  for which there exists a trajectory  
from saddle point  $u_2^-$  to saddle point  $u_1^+$

# Nullcline configurations: main and intermediate classes

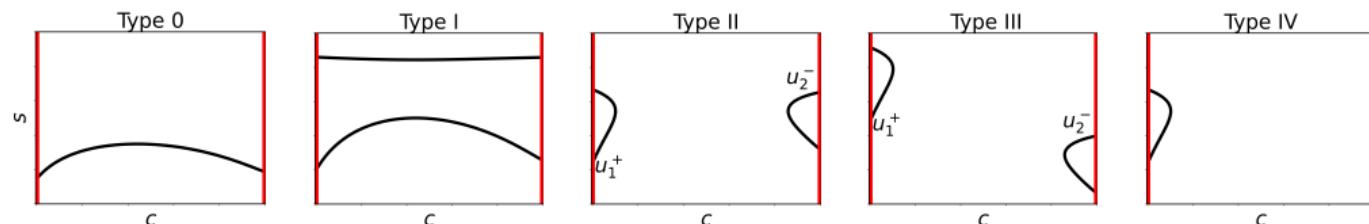


Figure 4: Five wide classes of nullcline configurations

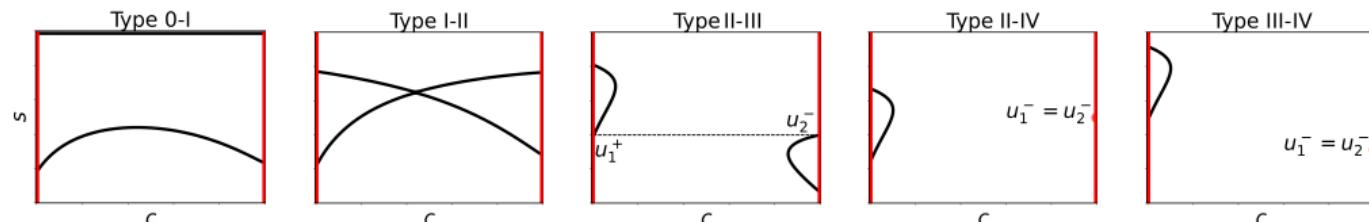


Figure 5: Intermediate types of nullcline configurations, appearing under assump. (F1)–(F4)

- Only Type II nullcline configuration has saddle-to-saddle connections.
- Type I-II corresponds to  $v_{\min}$ .
- Type II-III or Type II-IV correspond to  $v_{\max}$ .

# Nullcline configurations: monotone dependence on $v$

black lines  $f(s, c) = v(s + d_1)$

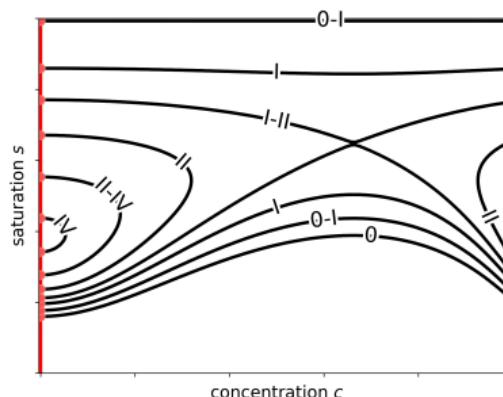
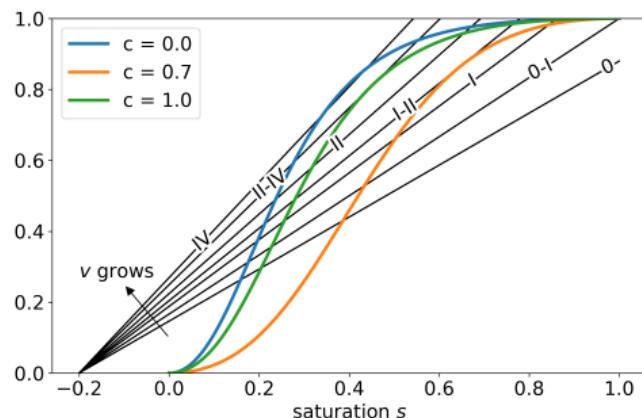


Figure 6: nullcline configuration evolution as  $v$  grows: Type 0 → Type I → Type II → Type IV

# Nullcline configurations: bad cases

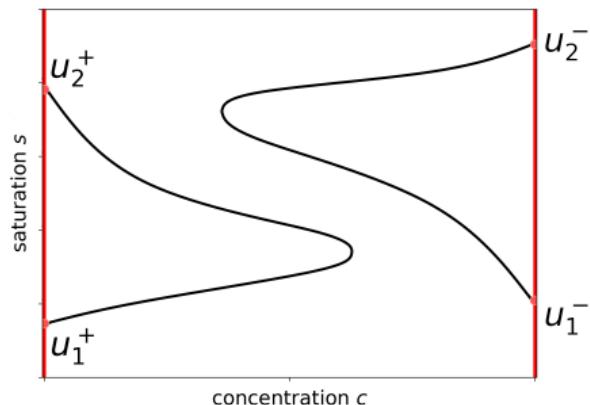


Figure 7: If  $f$  is not S-shaped.

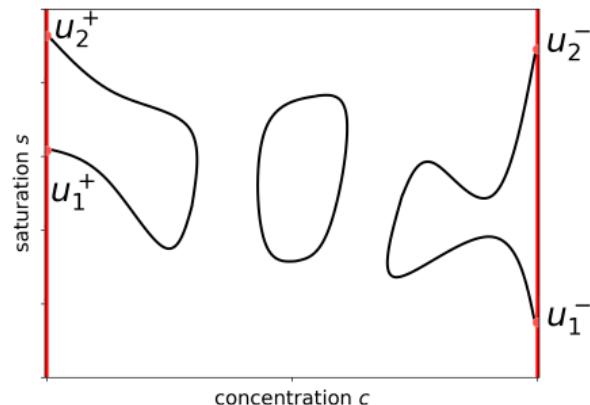


Figure 8: If non-monotonicity is more complex.

We believe that the similar result is true without conditions (F3)–(F4).

# Type II configuration: for every $v$ there exist $\kappa$

$$\frac{s_\xi}{c_\xi} = \kappa \cdot \frac{v^{-1}f(s, c) - (s + d_1)}{A(s, c)(d_1c - d_2 - a(c))}$$

Used property: continuous and monotonous dependence of trajectories on  $\kappa$ .

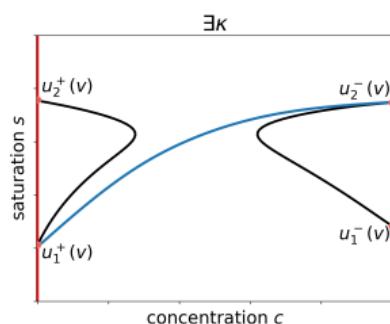
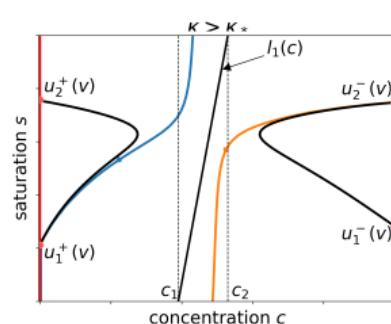
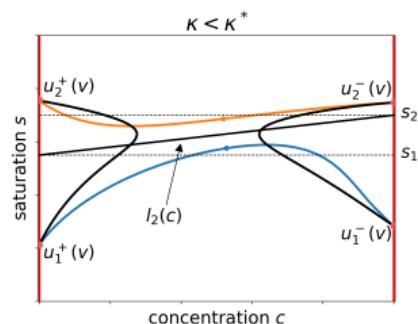


Figure 9:  $\kappa \ll 1$ .

Figure 10:  $\kappa \gg 1$ .

Figure 11:  $\exists \kappa$ .

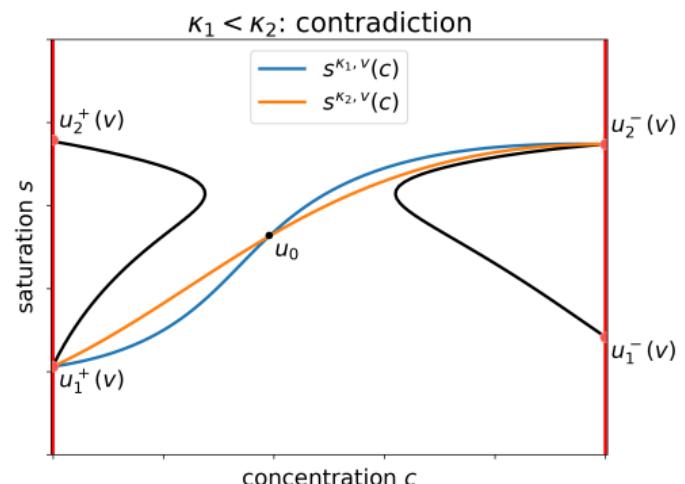
# Type II config.: $\kappa(v)$ is unique for every $v \in (v_{\min}, v_{\max})$ .

If there are  $\kappa_1 < \kappa_2$  for one  $v$ , then the corresponding trajectories must intersect, which leads to a contradiction.

The slope

$$\frac{s_\xi}{c_\xi} = \kappa \cdot \frac{v^{-1}f(s, c) - (s + d_1)}{A(s, c)(d_1c - d_2 - a(c))}$$

is positive at the intersection point  $(s, c)$ , so it strictly increases when  $\kappa$  increases.



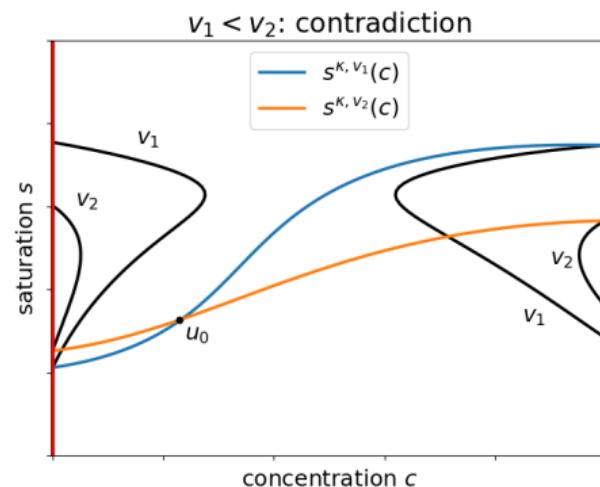
NB: this property might be lost for more complex nullcline configurations.

## Type II configuration: monotonicity of $\kappa(v)$

If  $\kappa(v_1) = \kappa(v_2)$  for  $v_1 < v_2$ , then the corresponding trajectories must intersect, which leads to a contradiction. The slope

$$\frac{s_\xi}{c_\xi} = \kappa \cdot \frac{v^{-1}f(s, c) - (s + d_1)}{A(s, c)(d_1c - d_2 - a(c))}$$

is positive at the intersection point  $(s, c)$ , so it strictly increases when  $v$  increases.



# Possible directions for future research

- General classes of  $f$  and  $a$ : when the dependence  $v(\kappa)$  is nontrivial?
- Construct solution to any Riemann problem  $(s_L, c_L) \rightarrow (s_R, c_R)$
- Convergence and asymptotic stability as  $t \rightarrow \infty$  (as mentioned above)
- Consider a three-phase flow with chemicals (water, oil and gas): travelling wave dynamical system will become three-dimensional, thus the analysis will be more complex.

Vielen Dank für Ihre Aufmerksamkeit!

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# Literature

## Own works:

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