L_2-small ball asymptotics for Gaussian random functions

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Abstract

This article is a survey of the results on asymptotic behavior of small ball probabilities in L_2-norm. Article ***in progress***.

1 Introduction

The theory of small ball probabilities (also called small deviation probabilities) is extensively studied in recent decades (see the surveys [1, 2, 3]; for the extensive up-to-date bibliography see [4]). Given a random vector X in a Banach space $\mathfrak{B}$, the relation

$$\mathbb{P}\left\{\|X\|_{\mathfrak{B}} < \varepsilon\right\} \sim f(\varepsilon) \qquad \text{as} \quad \varepsilon \to 0 \tag{1}$$

is called an ***exact*** asymptotics of small deviations. Typically, this probability is exponentially small, and often a ***logarithmic*** asymptotics is studied, that is the relation

$$\log\big(\mathbb{P}\{\|X\|_{\mathfrak{B}} < \varepsilon\}\big) \sim f(\varepsilon) \qquad \text{as} \quad \varepsilon \to 0.$$

Theory of small deviations has numerous applications including the accuracy of discrete approximation of random processes and the quantization problem, the calculation of the metric entropy for functional sets, the law of the iterated logarithm in the Chung form, the rate of escape of infinite dimensional Brownian motion (more details in [2, Chapter 7]). It was also observed that the small deviation theory is related to the functional data analysis [5] and nonparametric Bayesian estimation [6, 7].

The discussed topic is almost boundless, so in this paper we focus on the most elaborated (and may be the simplest) case, where X is ***Gaussian*** and the norm is ***Hilbertian***. Let X be a Gaussian random vector in the Hilbert space $\mathfrak{H}$ with $\mathbb{E}X = 0$. Denote by $\mathcal{G}_X$ its covariance operator (it is a compact non-negative operator in $\mathfrak{H}$).

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Let $\mu_k \geq 0$, $k \in \mathbb{N}$, be a non-increasing sequence of eigenvalues of $\mathcal{G}_X$, counted with their multiplicities, and denote by $\{\varphi_k\}$ a complete orthonormal system of the corresponding eigenvectors. It is well known (see, e.g., [8, Chapter 2] or [9])that if ξ_k are i.i.d. standard normal random variables, then we have the following distributional equality [1]

$$X \stackrel{d}{=} \sum_{k=1}^{\infty} \sqrt{\mu_k}\, \varphi_k\, \xi_k,$$

which is usually called the ***Karhunen–Loève expansion***.[2] By the orthonormality of the system φ_k this implies

$$\|X\|_{\mathfrak{H}}^2 \stackrel{d}{=} \sum_{k=1}^{\infty} \mu_k \xi_k^2. \tag{2}$$

Therefore the small ball asymptotics for X in the Hilbertian case is completely determined by the eigenvalues μ_k.

Notice that if $\sum\limits_{k=1}^{\infty} \mu_k < \infty$ then the series in (2) converges almost surely (a.s.), otherwise it diverges a.s. The latter is impossible for $X \in \mathfrak{H}$, so, in what follows we always assume that $\mathcal{G}_X$ is the trace class operator.

The paper is organised as follows. In Section 2 we describe the first attempts towards small ball probabilities in L_2-norm. In Section 3 we formulate two important results in the field: Wenbo Li comparison theorem and D-L-L theorem. Combining these results one may reduce the problem of L_2-small deviations to finding a "good" spectral asymptotics for the corresponding covariance operator. In Section 4 we mention a variety of results concerning the so-called Green Gaussian processes, when the covariance function is the Green function of some ordinary differential operator (ODE). This allows to study the asymptotics of eigenvalues using powerful methods of spectral theory of ODEs.

Let us introduce some notation.

For $\mathcal{O} \subset \mathbb{R}^n$, $\|\cdot\|_{2,\mathcal{O}}$ stands for the norm in $L_2(\mathcal{O})$:

$$\|X\|_{2,\mathcal{O}}^2 = \int\limits_{\mathcal{O}} |X(x)|^2\, dx.$$

Here and later on $f(\varepsilon) \sim g(\varepsilon)$ as $\varepsilon \to 0$ means $\lim\limits_{\varepsilon\to 0} f(\varepsilon)/g(\varepsilon) = 1$.

The space $W_p^m(0,1)$ is the Banach space of functions u having continuous derivatives up to $(m-1)$-th order when $u^{(m-1)}$ is absolutely continuous on $[0,1]$ and $u^{(m)} \in L_p(0,1)$. If $p = 2$ it is a Hilbert space.

2 First works

The oldest results about L_2-small ball probabilities concern classical Gaussian processes on the interval $[0,1]$. R. Cameron and W. Martin [15] proved the relation for the Wiener

[1]For the special choice $\xi_k = \frac{1}{\sqrt{\mu_k}}\,(X,\varphi_k)_{\mathfrak{H}}$ this equality holds almost surely.

[2]Up to our knowledge stochastic functions given by similar series were first introduced by D.D. Kosambi [10]. However, K. Karhunen [11, 12] and M. Loève [13] were the first who proved the optimality in terms of the total mean square error resulting from the truncation of the series. An analogous formula for stationary Gaussian processes was introduced by M. Kac and A.J.F. Siegert in [14].

process $W(t)$

$$\mathbb{P}\{\|W\|_{2,[0,1]} \le \varepsilon\} \sim \frac{4\varepsilon}{\sqrt{\pi}} \exp\bigl(-\frac{1}{8}\varepsilon^{-2}\bigr),$$

while T. Anderson and D. Darling [16] established the corresponding asymptotics for the Brownian bridge $B(t)$

$$\mathbb{P}\{\|B\|_{2,[0,1]} \le \varepsilon\} \sim \frac{\sqrt{8}}{\sqrt{\pi}} \exp\bigl(-\frac{1}{8}\varepsilon^{-2}\bigr). \tag{3}$$

The latter formula describes the lower tails of famous Cramer–von Mises–Smirnov ω^2-statistic.

One should notice that a minor difference (rank one process) between Wiener process and Brownian bridge influences the power term in the asymptotics but not the exponential one.

In general case the problem of L_2-small ball asymptotics was solved by G. Sytaya [17], but in an implicit way. The main ingredient was the Laplace transform and the saddle point technique. The general formulation of result by Sytaya in terms of operator theory states as follows.

Theorem 1 ([17, Theorem 1]). *As $\varepsilon \to 0$ we have*

$$\mathbb{P}\Bigl\{\|X\|_{\mathfrak{H}} \leqslant \varepsilon\Bigr\} \sim \bigl(4\gamma^2\pi\,\mathrm{Tr}(\mathcal{G}_X R_\gamma(\mathcal{G}_X))^2\bigr)^{-1/2}\ \exp\Biggl(-\int\limits_0^\gamma \mathrm{Tr}(\mathcal{G}_X[R_u(\mathcal{G}_X) - R_\gamma(\mathcal{G}_X)])\,du\Biggr). \tag{4}$$

Here $Tr(\mathcal{G})$ is the trace of operator $\mathcal{G}$; $R_u(\mathcal{G})$ is the resolvent operator defined by the formula

$$R_u(\mathcal{G}) = (I + 2u\mathcal{G})^{-1},$$

and $\gamma = \gamma(\varepsilon)$ satisfies the equation $\varepsilon^2 = \mathrm{Tr}(\mathcal{G}_X R_\gamma(\mathcal{G}_X))$.

In terms of eigenvalues μ_k of the covariance operator $\mathcal{G}_X$, this result reads as follows:

Theorem 2 ([17, formula (20)]). *If $\mu_k > 0$ and $\sum\limits_{k=1}^{\infty} \mu_k < \infty$, then as $\varepsilon \to 0$*

$$\mathbb{P}\Bigl\{\sum_{k=1}^{\infty} \mu_k \xi_k^2 \leqslant \varepsilon^2\Bigr\} \sim \bigl(-2\pi\gamma^2 h''(\gamma)\bigr)^{-1/2} \exp\bigl(\gamma h'(\gamma) - h(\gamma)\bigr), \tag{5}$$

where $h(\gamma) = \frac{1}{2}\sum\limits_{k=1}^{\infty} \log(1 + 2\mu_k\gamma)$ and γ is uniquely determined by equation $\varepsilon^2 = h'(\gamma)$ for $\varepsilon > 0$ small enough.

As a particular case the formula (3) for the Brownian bridge was also obtained. However, these theorems are not convenient to use for concrete computations and applications. For example, finding expressions for the function $h(\gamma)$ and determining the implicit relation $\gamma = \gamma(\varepsilon)$ are the two difficulties that arise. The significant simplification was done only in 1997 by T. Dunker, W. Linde and M. Lifshits [18], which we describe in detail in Sec. 3.

The results of [17] were not widely known. In 1976, J. Hoffmann-Jorgensen [19] obtained two-sided estimates for L_2-small ball probabilities. Later the result of Sytaya was rediscovered by I. Ibragimov [20]. See also the work [21] in the context of the problem of natural rate of escape of infinite-dimensional Brownian motion. A. Dembo, E. Mayer-Wolf and O. Zeituni in a series of works [22, 23] gave a probabilistic proof based on large deviation principle.

In 1984 V. Zolotarev [24] suggested an explicit description of the small deviations in the case $\mu_k = \phi(k)$ with a decreasing and logarithmically convex function ϕ on $[1, \infty)$. His method was based on application of the Euler-MacLaurin formula to sums in the right hand side of (5). Unfortunately, no proofs were presented in [24], and, as was shown in [18], this result is not valid without additional assumptions about the function ϕ (in particular, the final formula in [24, Example 2] is not correct, see the end of Sec. 3 below).

One of the natural extensions of the problem under consideration is to find asymptotics (1) for shifted small balls. It was shown in [17, Theorem 1] that if the shift a belongs to the reproducing kernel Hilbert space (RKHS) of the process X, then

$$\mathbb{P}\{\|X - a\|_{\mathfrak{H}} < \varepsilon\} \sim \mathbb{P}\{\|X\|_{\mathfrak{H}} < \varepsilon\} \cdot \exp\bigl(-\|\mathcal{G}_X^{-1/2} a\|_{\mathfrak{H}}^2/2\bigr), \tag{6}$$

where $\|\mathcal{G}_X^{-1/2} a\|_{\mathfrak{H}}$ is in fact the norm of a in RKHS of X. See also some further inequalities between the probabilities of centered and shifted balls in [25] by J. Hoffmann-Jorgensen, L.A. Shepp and R.M. Dudley. A generalised version of equivalence (6) for Gaussian Radon measures on locally convex vector spaces was proved by Ch. Borell [26].

Formula (6) was later generalized by J. Kuelbs, W. Li and W. Linde [27] and W. Li, W. Linde [28], who handled the asymptotics of probability

$$\mathbb{P}\{\|X - f(t)a\|_{\mathfrak{H}} < R(t)\} \qquad \text{as } t \to \infty,$$

for various combinations of f and R, including the case $R(t) \to \infty$ as $t \to \infty$.

3 Second wave

The main difficulty in using Theorem 2 is that the explicit formulas for the eigenvalues of the covariance operators are rarely known. It was partially overcome by the celebrated ***Wenbo Li comparison principle*** [29] (see also [30][3]).

Theorem 3 ([29, 30]). *Let ξ_k be i.i.d. standard normal r.v.'s; μ_k and $\tilde\mu_k$ be two positive nonincreasing summable sequences such that $\prod \mu_k/\tilde\mu_k < \infty$. Then*

$$\mathbb{P}\Bigl\{\sum_{k=1}^{\infty} \tilde\mu_k \xi_k^2 < \varepsilon^2\Bigr\} \sim \mathbb{P}\Bigl\{\sum_{k=1}^{\infty} \mu_k \xi_k^2 < \varepsilon^2\Bigr\} \cdot \biggl(\prod_{k=1}^{\infty} \frac{\mu_k}{\tilde\mu_k}\biggr)^{\frac12}, \qquad \text{as } \varepsilon \to 0. \tag{7}$$

So, if we know sufficiently sharp (and sufficiently simple!) asymptotic approximations $\tilde\mu_k$ for the eigenvalues μ_k of $\mathcal{G}_X$ then formula (7) provides the L_2-small ball asymptotics for X up to a constant. However, to use this idea efficiently, we need explicit expressions of L_2-small ball asymptotics for "model" sequences $\tilde\mu_k$.

[3] In paper [29] this statement was proved under assumption $\sum |1 - \mu_k/\tilde\mu_k| < \infty$. Later F. Gao, J. Hannig, T.-Y. Lee and F. Torcaso [30] relaxed this assumption to the natural one $\prod \mu_k/\tilde\mu_k < \infty$.

In [31] Theorem 3 was extended for the sums $\sum a_k |\xi_k|^p$, $p > 0$, and even more general ones.

The important step in the latter problem was made by M.A. Lifshits in [32] who considered the small ball problem for more general series:

$$\mathcal{S} := \sum_{k=1}^{\infty} \phi(k) Z_k, \tag{8}$$

where $\{\phi(k)\}$ is a non-increasing sequence of positive numbers, $\sum \phi(k) < \infty$, and Z_k are independent copies of a positive random variable Z with finite variance and absolutely continuous distribution. The main restriction imposed in [32] on the distribution function $F(\cdot)$ of Z is

$$c_1 F(\tau) \le F(b\tau) \le c_2 F(\tau) \tag{9}$$

for some $b, c_1, c_2 \in (0,1)$ and for all sufficiently small τ. This assumption implies a polynomial (but not necessarily regular) lower tail behavior of the distribution. For the special case $Z_k = \xi_k^2$ the sum in (8) corresponds to L_2-small ball probabilities (2).

Using the idea of R.A. Davis and S.I. Resnick [33], Lifshits expressed the exact small ball behavior of (8) in terms of the Laplace transform of $\mathcal{S}$. If Z has finite third moment then a quantitative estimate of the remainder term was also given.

The next important step was done by T. Dunker, M.A. Lifshits and W. Linde in [18]. The result is based on the following assumption on the sequence $\phi(k)$:

the sequence $\phi(k)$ admits a positive, logarithmically convex, twice differentiable and integrable extension on the interval $[1, +\infty)$.

Under some additional assumptions[4] on the distribution function F authors significantly simplified the expressions from [32] for the small ball behavior of $\mathcal{S}$. By using Euler–MacLaurin's summation formula they succedeed to express the small ball probabilities in terms of three integrals:

$$\begin{aligned}
I_0(u) :&= \int_1^{\infty} (\log f)(u\phi(t))\, dt, \\
I_1(u) :&= \int_1^{\infty} u\phi(t)(\log f)'(u\phi(t))\, dt, \\
I_2(u) :&= \int_1^{\infty} (u\phi(t))^2 (\log f)''(u\phi(t))\, dt.
\end{aligned}$$

Here $f(u)$ is the Laplace transform of Z,

$$f(u) := \int_0^{\infty} \exp(-u\tau)\, dF(\tau).$$

[4] The results in [18] heavily depend on the "condition $\mathbf{I}$": total variation $\mathbb{V}_{[0,\infty)}(uf'(u)/f(u))$ is finite. This condition together with restriction (9) implies that the distribution function F is regularly varying at zero with some index $\alpha < 0$. Also this condition holds for the case $Z_k = |\xi_k|^p$, $p > 0$. Some relaxations are discussed in [18, Section 5] and later papers of F. Aurzada [34], A.A. Borovkov, P.S. Ruzankin [35, 36], L.V. Rozovsky (see [37] and references therein)

The main result (Theorem 3.1 in [18]) reads as follows:

$$\mathbb{P}\Big\{\mathcal{S} < r\Big\} \sim \sqrt{\frac{f(u\phi(1))}{2\pi I_2(u)}}\exp(I_0(u) + \rho(u) + ur)), \quad \text{as } r \to 0, \tag{10}$$

where $u = u(r)$ is any function satisfying

$$\lim_{r\to 0} \frac{I_1(u) + ur}{\sqrt{I_2(u)}} = 0, \tag{11}$$

and $\rho(u)$ is a bounded function that represents the remainder terms in Euler–MacLauren formula and can be written explicitly via infinite sums.

At a first glance, formula (10) and condition (11) do not seem to be more explicit than Theorem 2. Hovewer, for $Z_k = \xi_k^2$ (in this case $f(u) = (1+2u)^{-\frac{1}{2}}$) the result of [18] turned out to be much more computationally tractable and became the base for derivation of the exact asymptopics for many "model" sequences of eigenvalues $\phi(k)$.

In particular, the following asymptotics were obtained in [18, Section 4].

1. Let $p > 1$. Then, as $\varepsilon \to 0$,

$$\mathbb{P}\Big\{\sum_{k=1}^{\infty} k^{-p}\xi_k^2 \le \varepsilon^2\Big\} \sim \mathcal{C}\cdot\varepsilon^{\gamma}\exp\Big(-D\varepsilon^{-\frac{2}{p-1}}\Big), \tag{12}$$

   where the constants D and γ depend on p as follows:

$$D = \frac{p-1}{2}\left(\frac{\pi}{p\sin\frac{\pi}{p}}\right)^{\frac{p}{p-1}}, \qquad \gamma = \frac{2-p}{2(p-1)},$$

   while the constant $\mathcal{C}$ is given by the following expression:

$$\mathcal{C} = \frac{(2\pi)^{\frac{p}{4}}\big(\sin\frac{\pi}{p}\big)^{\frac{1+\gamma}{2}}}{(p-1)^{\frac{1}{2}}\big(\frac{\pi}{p}\big)^{1+\frac{\gamma}{2}}}.$$

   Notice that this result was first obtained in [24, Example 1].

2. The second result shows that the general asymptotic formula in [24] is not true. Namely, we have as $\varepsilon \to 0$

$$\mathbb{P}\Big\{\sum_{k=0}^{\infty}\exp(-k)\,\xi_k^2 < \varepsilon^2\Big\} \sim \frac{\exp\Big(-\frac{\pi^2}{12} - \frac{1}{4}(\log(\frac{1}{\varepsilon^2}\log\frac{1}{\varepsilon^2}))^2 + \psi_0(\log(\frac{1}{\varepsilon^2}\log\frac{1}{\varepsilon^2}))\Big)}{\pi^{\frac{1}{2}}\varepsilon^{\frac{1}{2}}(\log\frac{1}{\varepsilon^2})^{\frac{3}{4}}},$$

   where ψ_0 is an explicitly given (though complicated) 1-periodic and bounded function. It is shown in [18] that ψ_0 is ***non-constant*** while such term is absent in [24, Example 2].

We also mention the papers [38] and [39], where the logarithmic L_2-small ball asymptotics was obtained for the integrated Brownian motion

$$W_m(t) = \int_0^t W_{m-1}(s)\,ds, \quad m \ge 1; \qquad W_0(t) = W(t). \tag{13}$$

Namely, D. Khoshnevisan and Z. Shi proved the relation

$$\ln \mathbb{P}\left(\|W_1\|_{2,[0,1]} \le \varepsilon\right) \sim -\frac{3}{8}\,\varepsilon^{-\frac{2}{3}}, \quad \text{as } \varepsilon \to 0.$$

X. Chen and W. Li considered a general case and obtained formula

$$\ln \mathbb{P}\left(\|W_m\|_{2,[0,1]} \le \varepsilon\right) \sim D_m\,\varepsilon^{-\frac{2}{2m+1}}, \quad \text{as } \varepsilon \to 0,$$

where

$$D_m = -\frac{2m+1}{2}\left((2m+2)\sin\frac{\pi}{2m+2}\right)^{-\frac{2m+2}{2m+1}}.$$

4 Exact asymptotics: Green Gaussian processes and beyond

4.1 Green Gaussian processes and their properties

As was mentioned above, the results of [29, 30] and [18] allow to obtain the exact (at least up to a constant) small ball asymptotics for Gaussian random vectors from sufficiently sharp eigenvalues asymptotics of the corresponding covariance operators. The first significant progress in this direction was made for the special important class of Gaussian functions on the interval.

The ***Green Gaussian process*** is a Gaussian process X on the interval (say, $[0,1]$) such that its covariance function G_X is the (generalized) Green function of an ordinary differential operator (ODO) on $[0,1]$ with proper boundary conditions. This class of processes is very important as it includes the Wiener process, the Brownian bridge, the Ornstein-Uhlenbeck process, their (multiply) integrated counterparts etc.

First, we recall some definitions. Let $\mathcal{L}$ be an ODO given by the differential expression

$$\mathcal{L}u := (-1)^{\ell}\left(p_\ell u^{(\ell)}\right)^{(\ell)} + \left(p_{\ell-1}u^{(\ell-1)}\right)^{(\ell-1)} + \cdots + p_0 u, \tag{14}$$

(here p_j, $j = 0,\dots,\ell$ are functions on $[0,1]$, and $p_\ell(t) > 0$) and by 2ℓ boundary conditions

$$U_\nu(u) := U_{\nu 0}(u) + U_{\nu 1}(u) = 0, \qquad \nu = 1,\dots,2\ell, \tag{15}$$

where

$$U_{\nu 0}(u) := \alpha_\nu u^{(k_\nu)}(0) + \sum_{j=0}^{k_\nu-1}\alpha_{\nu j}u^{(j)}(0),$$
$$U_{\nu 1}(u) := \gamma_\nu u^{(k_\nu)}(1) + \sum_{j=0}^{k_\nu-1}\gamma_{\nu j}u^{(j)}(1),$$

and for any index ν at least one of coefficients α_ν and γ_ν is not equal to zero.

For simplicity we assume $p_j \in W_\infty^j[0,1]$, $j = 0,\dots,\ell$. Then the domain $\mathcal{D}(\mathcal{L})$ consists of the functions $u \in W_2^{2\ell}[0,1]$ satisfying boundary conditions (15).

It is well known, see, e.g., [40, §4], [41, Chap. XIX], that the system of boundary conditions can be reduced to the ***normalized form*** by equivalent transformations. In what follows we always assume that this reduction is realized. This form is specified by the minimal sum of orders of all boundary conditions $\varkappa = \sum\limits_{\nu=1}^{2\ell} k_\nu$.

The ***Green function*** of the boundary value problem

$$\mathcal{L}u = \lambda u \quad \text{on} \quad [0,1], \qquad u \in \mathcal{D}(\mathcal{L}), \tag{16}$$

is the function $G(t,s)$ such that it satisfies the equation $\mathcal{L}G(t,s) = \delta(s-t)$ in the sense of distributions and satisfies the boundary conditions (15).[5] The existence of Green function is equivalent to the invertibility of the operator $\mathcal{L}$ with given boundary conditions, and $G(t,s)$ is the kernel of the integral operator $\mathcal{L}^{-1}$.

If the problem (16) has a zero eigenvalue corresponding to the eigenfunction φ_0 (without loss of generality, it can be assumed to be normalized in $L_2(0,1)$), then the Green function obviously does not exist. If φ_0 is unique up to a constant multiplier, then the function $G(t,s)$ is called the ***generalized Green function*** if it satisfies the equation $\mathcal{L}G(t,s) = \delta(t-s) - \varphi_0(t)\varphi_0(s)$ in the sense of distributions, subject to the boundary conditions and the orthogonality condition

$$\int\limits_0^1 G(t,s)\,\varphi_0(s)\,ds = 0, \qquad \text{for all } 0 \le t \le 1.$$

The generalized Green function is the kernel of the integral operator which is inverse to $\mathcal{L}$ on the subspace of functions orthogonal to φ_0 in $L_2(0,1)$. In a similar way, one can consider the case of multiple zero eigenvalue.

Thus, (non-zero) eigenvalues μ_k of the covariance operator of a Green Gaussian process are inverse to the (non-zero) eigenvalues λ_k of the corresponding boundary value problem (16): $\mu_k = \lambda_k^{-1}$. So, to obtain rather good asymptotics of eigenvalues μ_k one can use the powerful methods of spectral theory of ODOs, originated from the classical works of G. Birkhoff [42, 43] and J.D. Tamarkin [44, 45].

Notice that classical operations on the random processes – integration and centering – transform a Green Gaussian process to a Green one. It is easy to see that if $G(t,s)$ is the covariance function of a random process X then the covariance functions of the integrated and the centered process

$$X_1(t) = \int\limits_0^t X(s)\,ds, \qquad \overline{X}(t) = X(t) - \int\limits_0^1 X(s)\,ds$$

are, respectively,

$$G_1(t,s) = \int\limits_0^t \int\limits_0^s G(x,y)\,dydx, \tag{17}$$

$$\overline{G}(t,s) = G(t,s) - \int\limits_0^1 G(t,y)\,dy - \int\limits_0^1 G(x,s)\,dx + \int\limits_0^1 \int\limits_0^1 G(x,y)\,dydx. \tag{18}$$

[5] We stress that if G is the covariance function of a random process then $G(t,s) \equiv G(s,t)$, and therefore the problem (16) is always self-adjoint.

Theorem 4 ([46, Theorem 2.1]; [47, Theorem 3.1])**. 1**. *Let the kernel $G(t,s)$ be the Green function for the boundary value problem* (16)*. Then the integrated kernel* (17) *is the Green function for the boundary value problem*

$$\mathcal{L}_1 u := -(\mathcal{L}u')' = \lambda u \quad \text{on} \quad [0,1], \qquad u \in \mathcal{D}(\mathcal{L}_1), \tag{19}$$

where the domain $\mathcal{D}(\mathcal{L}_1)$ consists of functions $u \in W_2^{2\ell+2}[0,1]$ satisfying the boundary conditions

$$u(0) = 0; \qquad u' \in \mathcal{D}(\mathcal{L}); \qquad \mathcal{L}u'(1) = 0.$$

2. *Let the boundary value problem* (16) *have a zero eigenvalue with constant eigenfunction $\varphi_0(t) \equiv 1$, and let the kernel $G(t,s)$ be the generalized Green function of the problem* (16)*. Then the integrated kernel* (17) *is the (conventional) Green function of the boundary value problem* (19) *where the domain $\mathcal{D}(\mathcal{L}_1)$ consists of functions $u \in W_2^{2\ell+2}[0,1]$ satisfying the boundary conditions*

$$u(0) = 0; \qquad u(1) = 0; \qquad u' \in \mathcal{D}(\mathcal{L}). \tag{20}$$

3. *Let the kernel $G(t,s)$ be the Green function of the problem* (19)*–*(20)*. Then the centered kernel* (18) *is the generalized Green function of the boundary value problem*[6]

$$\overline{\mathcal{L}}_1 u := -(\mathcal{L}u')' = \lambda u \quad \text{on} \quad [0,1], \qquad u \in \mathcal{D}(\overline{\mathcal{L}}_1), \tag{21}$$

where the domain $\mathcal{D}(\overline{\mathcal{L}}_1)$ consists of the functions $u \in W_2^{2\ell+2}[0,1]$ satisfying the boundary conditions

$$u(0) - u(1) = 0; \qquad u' \in \mathcal{D}(\mathcal{L}); \qquad (\mathcal{L}u')(0) - (\mathcal{L}u')(1) = 0. \tag{22}$$

Remark 1. *Somewhat more complicated boundary value problems arise if we multiply a Green Gaussian process by a deterministic function ψ or apply the so-called online-centering [48]:*

$$\widehat{X}(t) = X(t) - \frac{1}{t}\int_0^t X(s)\,ds.$$

Theorem 5. 1 ([49, Lemma 2.1]).[7] *Let $X(t)$ be a Green Gaussian process, corresponding to the boundary value problem* (16)*, $\psi \in W_\infty^\ell[0,1]$, $\psi > 0$ on $(0,1)$. Then the covariance function of the process $\psi(t)X(t)$ is the Green function of the problem*

$$\mathcal{L}_\psi u := \psi^{-1}\mathcal{L}(\psi^{-1}u) = \lambda u \quad \text{on} \quad [0,1], \qquad \psi^{-1}u \in \mathcal{D}(\mathcal{L}).$$

2 ([52, Theorem 2]). *Let $X(t)$ be a Green Gaussian process, corresponding to the boundary value problem* (16)*. Then the covariance function of the online-centered integrated process $\widehat{X_1}(t)$ is the Green function of the problem*

$$\widehat{\mathcal{L}_1} u := t(t^{-1}\mathcal{L}(t^{-1}(tu)'))' = \lambda u \quad \text{on} \quad [0,1], \qquad u \in \mathcal{D}(\widehat{\mathcal{L}_1}).$$

where the domain $\mathcal{D}(\widehat{\mathcal{L}_1})$ is defined by $2\ell+2$ boundary conditions

$$u(0) = 0, \qquad t^{-1}(tu)' \in \mathcal{D}(\mathcal{L}), \qquad \mathcal{L}(t^{-1}(tu)')(1) = 0.$$

[6]The problem (21)–(22) obviously has a zero eigenvalue with constant eigenfunction $\varphi_0(t) \equiv 1$.

[7]For $X = W$ and $X = B$ this fact was obtained earlier in [50, Theorems 1.1 and 1.2]. Some interesting examples can be found in [51].

4.2 Exact L_2-small ball asymptotics for the Green Gaussian processes

Apparently, the first works where the eigenvalues asymptotics for covariance operators was obtained by the passage to the corresponding boundary value problem were the papers [53] by L. Beghin, Ya. Nikitin and E. Orsingher and [50] by P. Deheuvels and G. Martynov. They derived the small deviation asymptotics up to a constant for several concrete Green Gaussian processes, including some integrated ones [53] and weighted ones [50].

To formulate the general result, first, we describe the asymptotic behavior of the eigenvalues of the boundary value problem (16).

It is well known, see, e.g., [40, §4], [41, Chap. XIX], that eigenvalues λ_k of Birkhoff-regular (in particular, self-adjoint) boundary value problem for ODOs with smooth coefficients can be expanded into asymptotic series in powers of k (analogous results under more general hypotheses, as well as some additional references, can be found in [54], [55]). Taking into account the Li comparison principle (Theorem 3), it is easy to show that the L_2-small ball asymptotics up to a constant for the Green Gaussian process requires just two term spectral asymptotics for corresponding boundary value problem (with the remainder estimate).

Theorem 6 ([40, §4, Theorem 2]). *Let $p_\ell \equiv 1$. Then the eigenvalues of the problem* (16) *counted according to their multiplicities can be split into two subsequences λ_j', λ_j'', $j \in \mathbb{N}$, such that, as $j \to \infty$,*

$$\lambda_j' = \Big(2\pi j + \rho' + O(j^{-\frac{1}{2}})\Big)^{2\ell}, \qquad \lambda_j'' = \Big(2\pi j + \rho'' + O(j^{-\frac{1}{2}})\Big)^{2\ell}. \tag{23}$$

The first term of this expansion is completely determined by the principal coefficient of the operator,[8] whereas the formulas for other terms (in particular, for ρ' and ρ'') are rather complicated. However, it turns out that the exact L_2-small ball asymptotics (up to a constant) for the Green Gaussian processes depends only on the sum $\rho' + \rho''$, which is completely determined by $\varkappa$, the sum of orders of boundary conditions.

Theorem 7 ([46, Theorem 7.2]; [47, Theorem 1.2]). *Let the covariance $G_X(t,s)$ of a zero mean Gaussian process $X(t)$, $0 \le t \le 1$, be the Green function of the boundary value problem* (16) *generated by a differential expression* (14) *and by boundary conditions* (15). *Let $\varkappa < 2\ell^2$. Then, as $\varepsilon \to 0$,*

$$\mathbb{P}\{\|X\|_{2,[0,1]} \le \varepsilon\} \sim \mathcal{C}(X) \cdot \varepsilon^\gamma \exp\Big(-D_\ell \varepsilon^{-\frac{2}{2\ell-1}}\Big). \tag{24}$$

Here we denote

$$D_\ell = \frac{2\ell-1}{2}\left(\frac{\vartheta_\ell}{2\ell \sin\frac{\pi}{2\ell}}\right)^{\frac{2\ell}{2\ell-1}}, \qquad \gamma = -\ell + \frac{\varkappa+1}{2\ell-1}, \qquad \vartheta_\ell = \int_0^1 p_\ell^{-\frac{1}{2\ell}}(t)\,dt, \tag{25}$$

and the constant $\mathcal{C}(X)$ is given by

$$\mathcal{C}(X) = C_{\text{dist}}(X) \cdot \frac{(2\pi)^{\frac{\ell}{2}} \left(\frac{\pi}{\vartheta_\ell}\right)^{\ell\gamma} \left(\sin\frac{\pi}{2\ell}\right)^{\frac{1+\gamma}{2}}}{(2\ell-1)^{\frac{1}{2}} \left(\frac{\pi}{2\ell}\right)^{1+\frac{\gamma}{2}} \Gamma^\ell\left(\ell - \frac{\varkappa}{2\ell}\right)}, \tag{26}$$

[8] In general case the problem (16) can be reduced to the case $p_\ell \equiv 1$ by the independent variable transform, see [40, §4]. The expressions in brackets in (23) should be divided by $\int_0^1 p_\ell^{-\frac{1}{2\ell}}(t)\,dt$.

where $C_{\text{dist}}(X)$ *is the so-called* **distortion constant** *(cf. formula* (7)*)*

$$C_{\text{dist}}(X) := \prod_{k=1}^{\infty} \frac{\lambda_k^{\frac{1}{2}}}{\left(\frac{\pi}{\vartheta_\ell} \cdot \left[k+\ell-1-\frac{\varkappa}{2\ell}\right]\right)^{\ell}}, \tag{27}$$

and λ_k *are the eigenvalues of the problem* (16) *taken in non-decreasing order counted with their multiplicities.*

Remark 2. *If* $G_X(t,s)$ *is the generalized Green function of the problem* (16) *then formula* (24) *should be corrected. Namely, if, say, one of eigenvalues* λ_k *vanishes, the exponent* γ *in* (25) *increases by* $\frac{2\ell}{2\ell-1}$*, and* λ_k *in the product* (27) *should be properly renumbered.*

On the other hand, if $\varkappa \geq 2\ell^2$ *then formulas* (26) *and* (27) *cannot be applied directly. The correct result can be obtained if we substitute* $\varkappa \to \varkappa + \epsilon$ *and then push* $\epsilon \to 0$ *in* (26)*.*

The first step towards a general result was made in [56] by F. Gao, J. Hannig and F. Torcaso. They considered the generalized m-times integrated Wiener process $W_m^{[\beta_1,\dots,\beta_m]}(t)$. Here and later on, for a random process $X(t)$ on $[0,1]$ we denote

$$X_m^{[\beta_1,\dots,\beta_m]}(t) = (-1)^{\beta_1+\dots+\beta_m} \int_{\beta_m}^{t} \int_{\beta_{m-1}}^{t_{m-1}} \dots \int_{\beta_1}^{t_1} X(s)\, ds dt_1 \dots\, dt_{m-1} \tag{28}$$

(here $0 \leq t \leq 1$, and any β_j equals either zero or one; the usual m-times integrated process corresponds to $\beta_1 = \dots = \beta_m = 0$).

From Theorem 4 it follows that the covariance function of $W_m^{[\beta_1,\dots,\beta_m]}$ is the Green function of the boundary value problem

$$Lu := (-1)^{m+1} u^{(2m+2)} = \lambda u \quad \text{on} \quad [0,1],$$

with boundary conditions depending on parameters β_j, $j = 1, \dots, m$. The special case $W_m^{[1,0,1,\dots]}$ is called in [56] the Euler-integrated Brownian motion, since its covariance function can be expressed in terms of Euler polynomials.

By particular fine analysis, authors of [56] obtained the formula

$$\mathbb{P}\{\|W_m^{[\beta_1,\dots,\beta_m]}\|_{2,[0,1]} \leq \varepsilon\} \sim C \cdot \varepsilon^{\frac{1-k_0(2m+2)}{2m+1}} \exp\Big(-D_m \varepsilon^{-\frac{2}{2m+1}}\Big). \tag{29}$$

Here D_m is equal to the corresponding coefficient in (24) for $\ell = m+1$ and k_0 is an *unknown* integer. It was also conjectured in [56] that $k_0 = 0$ for all $m \in \mathbb{N}$ and for any choice of β_j. This conjecture was verified in [30].

Simultaneously, A.I. Nazarov and Ya.Yu. Nikitin [46] obtained the result of Theorem 7 for arbitrary Green Gaussian process under assumption that the boundary conditions (15) are ***separated***:

$$U_{\nu 0}(u) = 0, \quad \nu = 1, \dots, \ell; \qquad U_{\nu 1}(u) = 0, \quad \nu = \ell+1, \dots, 2\ell. \tag{30}$$

For general self-adjoint boundary value problems, Theorem 7 was established later in [47].

Theorem 7 provides the exact L_2-small ball asymptotics up to a constant. Methods of derivation of this constant were proposed simultaneously and independently in [57] and [58]. Although slightly different, both methods are based on complex variables techniques (the Hadamard factorization and the Jensen theorem).

Remark 3. *As an auxiliary result the following generalization of* (12) *was obtained in [46, Theorem 6.2]:*[9]

$$\mathbb{P}\Big\{\sum_{k=1}^{\infty}(\vartheta(k+\delta))^{-p}\xi_k^2 \le \varepsilon^2\Big\} \sim \mathcal{C}(\vartheta,d,\delta)\cdot\varepsilon^{\gamma}\exp\Big(-D\varepsilon^{-\frac{2}{p-1}}\Big), \tag{31}$$

where

$$D = \frac{p-1}{2}\Big(\frac{\pi}{\vartheta p\sin\frac{\pi}{p}}\Big)^{\frac{p}{p-1}}, \qquad \gamma = \frac{2-p-2\delta p}{2(p-1)},$$

while

$$\mathcal{C}(\vartheta,d,\delta) = \frac{(2\pi)^{\frac{p}{4}}\vartheta^{\frac{d\gamma}{2}}\big(\sin\frac{\pi}{p}\big)^{\frac{1+\gamma}{2}}}{(p-1)^{\frac{1}{2}}\big(\frac{\pi}{p}\big)^{1+\frac{\gamma}{2}}\Gamma^{\frac{p}{2}}(1+\delta)}.$$

Formula (31)*, together with Theorem 3 and the results of [57], gave the opportunity to derive ([59], [60]) the exact L_2-small ball asymptotics for Gaussian processes satisfying the following condition on the eigenvalues μ_k of the covariance operator:*

$$\mu_k = \frac{\big(P_1(k)\big)^{\nu_1}}{\big(P_2(k)\big)^{\nu_2}},$$

where P_1 and P_2 are polynomials, $\nu_1, \nu_2 > 0$. Several examples of such processes can be found in [61] (see also [62], [63]).

Now we give an important corollary from Theorem 7 and Part 1 of Theorem 4.

Corollary 1 ([46, Theorem 4.1])**.** *Given Green Gaussian process X on* $[0,1]$*, the parameter $\varkappa$ (the sum of orders of boundary conditions) for a boundary value problem* (16) *corresponding to any m-times integrated process $X_m^{[\beta_1,\dots,\beta_m]}$ does not depend on β_j, $j = 1,\dots,m$. Thus, the L_2-small ball asymptotics for various processes $X_m^{[\beta_1,\dots,\beta_m]}$ can differ only by a constant.*

This corollary generates a natural question: which choices of parameters β_j give extremal (maximal/minimal) constant in L_2-small ball asymptotics among all m-times integrated processes?

For $X = W$ the answer was given independently in [30, Remark 2] and [57, Theorem 4.1]. It turns out that the usual integrated Wiener process $W_m^{[0,0,0,\dots]}$ has the biggest multiplicative constant, while the Euler-integrated process $W_m^{[1,0,1,\dots]}$ has the smallest one. The same extremal properties of the processes $X_m^{[0,0,0,\dots]}$ and $X_m^{[1,0,1,\dots]}$ for the Green Gaussian processes X under some symmetry assumptions[10] were obtained in [57, Proposition 4.4], [64].

Methods established in [46], [30], [47] and [57], [58] were widely used to obtain exact small ball asymptotics to many concrete Green Gaussian processes. Several examples were considered even in the pioneer papers [57], [58]. Much more results can be found in [65], [47], [66], [67].

[9] For $p = 2$ this formula was established earlier in [29].

[10] In particular, these assumptions are fulfilled for a symmetric process, for instance, for the Brownian bridge and the Ornstein–Uhlenbeck process.

The L_2-small ball asymptotics of the weighted Green Gaussian processes are of particular interest. The first results on exact asymptotics were obtained in [50], [57], [58].

Numerous examples were considered in [49]. Also it was noticed in this paper that if the weight is sufficiently smooth and ***non-degenerate*** (i.e. bounded and bounded away from zero) then the formula (24) holds with exponent γ independent of the weight.[11] However, if the weight degenerates (vanishes or blows up) at least at one point of the segment, generally speaking, this is not the case.

In fact, the following comparison theorem holds for the Green Gaussian processes with non-degenetate weights.

Theorem 8 ([68, Theorem 2]).[12] *Let $X(t)$ be a Green Gaussian process on $[0,1]$. Suppose that the boundary conditions of corresponding boundary value problem are separated (see (30)).*

Suppose that the weight functions $\psi_1, \psi_2 \in W_\infty^\ell[0,1]$ are bounded away from zero, and

$$\int_0^1 \psi_1^{\frac{1}{\ell}}(t)\,dt = \int_0^1 \psi_2^{\frac{1}{\ell}}(t)\,dt.$$

Denote by $\varkappa_0$ and $\varkappa_1$ sums of orders of boundary conditions at zero and one, respectively: $\varkappa_0 = \sum\limits_{\nu=1}^{\ell} k_\nu$, $\varkappa_1 = \sum\limits_{\nu=\ell+1}^{2\ell} k_\nu$. Then, as $\varepsilon \to 0$,

$$\mathbb{P}\big\{\|\psi_1 X\|_{2,[0,1]} \le \varepsilon\big\} \sim \mathbb{P}\big\{\|\psi_2 X\|_{2,[0,1]} \le \varepsilon\big\} \cdot \Big(\frac{\psi_2(0)}{\psi_1(0)}\Big)^{-\frac{\ell}{2}+\frac{1}{4}+\frac{\varkappa_0}{2\ell}} \Big(\frac{\psi_2(1)}{\psi_1(1)}\Big)^{-\frac{\ell}{2}+\frac{1}{4}+\frac{\varkappa_1}{2\ell}}.$$

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[11] In contrast, the coefficient D_ℓ depends on the weight.

[12] For some concrete processes this result was established earlier by Ya.Yu. Nikitin and R.S. Pusev [69].

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