On the impact of diffusion ratio on vanishing viscosity solutions of Riemann problems for chemical flooding models

Yulia Petrova

IMPA, Brasil, and Chebyshev Lab, Russia

Joint work with Fedor Bakharev, Aleksandr Enin and Nikita Rastegaev

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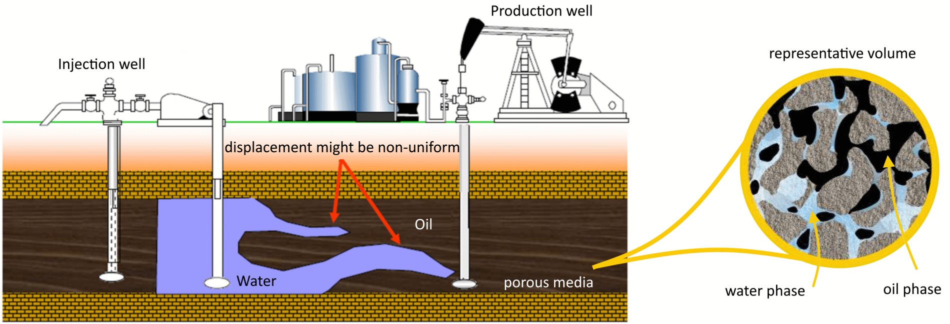
Motivation

We are interested in the mathematical model of oil recovery. Some features:

- Porous media (averaged models of flow)
- Relatively small speeds ($\approx$ 1 meter per day): Navier-Stokes $\to$ Darcy's law
- Multiphase flow: oil, water, gas.
- Unknown variables: $s(t, x)$ — the averaged water saturation in small volume
- Applications to EOR (enhanced oil recovery) methods: chemical, thermal, gas etc

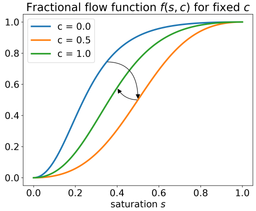


Problem statement

Chemical flooding can be described as the system of conservation laws ($x \in \mathbb{R}$, $t > 0$):

$$\begin{aligned} s_t + f(s,c)_x &= 0, \\ (cs + a(c))_t + (cf(s,c))_x &= 0. \end{aligned} \tag{1}$$

- $s = s(x,t)$ — water phase saturation;
- $c = c(x,t)$ — concentration of a chemical agent in water;
- $f(s,c)$ — fractional flow function (usually S-shaped);
- $a(c)$ — adsorption of a chemical agent on a rock (usually increasing, concave).



Initial data:
$$(s,c)\big|_{t=0} = \begin{cases} (1,1), & \text{if } x \leqslant 0, \\ (0,0), & \text{if } x > 0, \end{cases} \tag{2}$$

Aim:

Find a solution to initial-value problem (1)–(2) when f depends non-monotonically on c.

Hyperbolic systems of conservation laws

$$G(u)_t + F(u)_x = 0$$

Here

- $G(u)$ — accumulation function (conserved quantities)
- $F(u)$ — flux function (flux of conserved quantities)

Simplest example: wave equation

$$y_{tt} - c^2 y_{xx} = 0 \qquad \text{(J. d'Alambert, 1750)}$$

can be rewritten as a system of two first-order equations on the state-vector $u = \begin{pmatrix} y_x \\ y_t \end{pmatrix}$

$$u_t + Du_x = 0, \qquad \text{with} \qquad D = \begin{pmatrix} 0 & -1 \\ -c^2 & 0 \end{pmatrix}$$

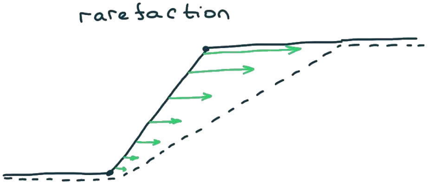
- eigenvalues $\lambda_1 = c$ and $\lambda_2 = -c$ are real, the system is hyperbolic. Solutions are two waves propagating at velocities λ_1 and λ_2.

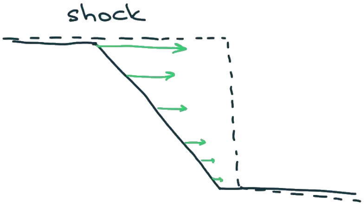
Hyperbolic systems of conservation laws

Burger's equation (1948)

$$u_t + \left(\frac{u^2}{2}\right)_x = 0.$$

- Due to non-linearity of the flux velocity of the wave $\lambda(u) = u$ depends on state u
- So the wave can spread (rarefaction wave) or concentrate (shock wave)





Riemann problem (1858)

- Riemann solved the initial-value problem with data having a single jump

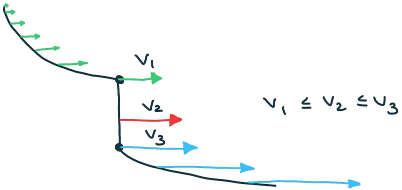
$$u\big|_{t=0} = \begin{cases} u^L, & x \leqslant 0; \\ u^R, & x > 0. \end{cases}$$

- took advantage of the scale invariance of the equations and the data:

$$u(\alpha\, x, \alpha\, t) = u(x, t) \quad \text{for all } \alpha > 0$$

- solution to a Riemann problem is important because:
  - often it appears in a long-term behavior of Cauchy problem
  - helps to prove the existence of solutions to Cauchy problem (Glimm's method)
  - helps to construct numerical solution (Godunov method)

Any solution to a Riemann problem consists of a sequence of rarefaction or shock waves (and constant states) that are compatible by speeds



Shock waves: RH condition and admissibility criteria

- discontinuous solutions are defined in the sense of distributions (weak form)
- for a shock wave from u^- to u^+ moving with velocity v, the weak condition amounts to the following Rankine-Hugoniot condition (RH)

$$-v\,G(u^-) + F(u^-) = -v\,G(u^+) + F(u^+) \qquad \text{(RH)}$$

- RH means conservation: what flows into left side flows out of the right side
- Problems from the perspectives of both mathematics and physics:
  - if all RH solutions are allowed, a Riemann problem has multiple solutions
  - some RH solutions violate physical principles
- Vanishing viscosity criteria: consider a diffusive system of conservation laws

$$G(u)_t + F(u)_x = \varepsilon\,[B(u)\,u_x]_x\,, \qquad \varepsilon \to 0$$

Traveling wave solutions of diffusive system (Hopf, 1948)

- $u(x,t) = \hat{u}(\xi)$ with $\xi := x - v\,t$ for a fixed shock velocity v
- reduction to first-order system of ordinary differential equations:

$$\varepsilon\, B(\hat{u})\, \hat{u}_\xi = -v\left[G(\hat{u}) - G(u^-)\right] + F(\hat{u}) - F(u^-)$$

- u^- and u^+ are fixed points and we look for an orbit connecting them

$$\hat{u}(-\infty) = u^-, \qquad \hat{u}(+\infty) = u^+$$

- diffusive terms cause a shock wave to have a thin, smooth internal structure as a result of balancing nonlinear focusing and diffusive spreading
- traveling wave solution approaches the jump discontinuity in L^1 as $\varepsilon \to 0^+$

Reduced problem: find an admissible shock wave

$$\begin{aligned} s_t + f(s,c)_x &= 0, \\ (sc + a(c))_t + (cf(s,c))_x &= 0. \end{aligned} \qquad (s,c)\big|_{t=0} = \begin{cases} (1,1), & \text{if } x \leqslant 0, \\ (0,0), & \text{if } x > 0, \end{cases}$$

Proposition (Johansen-Winther, 1988 (JW))

There exists $u^- = (s^-, 1)$ and $u^+ = (s^+, 0)$ such that the solution to a Riemann problem has the following structure:

$$(1,1) \xrightarrow{c=1} u^- \xrightarrow{\text{c jumps from 1 to 0}} u^+ \xrightarrow{c=0} (0,0). \tag{3}$$

Historical review:

- JW considered $f(s,c)$ monotone in c. Found a unique vanishing viscosity solution.
- When $f(s,c)$ is non-monotone in c, multiple vanishing viscosity solutions are possible. Examples can be found in Shen (2017); see also Entov-Kerimov (1986) on non-rigorous consideration of the non-monotone case.

Dissipative system

To define a shock wave between u^- and u^+ we consider dissipative system:

$$
\begin{gathered}
s_t + f(s,c)_x = \varepsilon_c (A(s,c) s_x)_x, \\
(cs + \alpha)_t + (cf(s,c))_x = \varepsilon_c (cA(s,c) s_x)_x + \varepsilon_d c_{xx}, \\
\alpha_t = \varepsilon_r^{-1} (a(c) - \alpha).
\end{gathered}
$$

- ε_c — dimensionless capillary pressure
- ε_d — dimensionless diffusion term
- ε_r — dimensionless relaxation time

- $A(s,c)$ — capillary pressure term
- $\alpha = \alpha(x,t)$ — dynamic adsorption

We consider two particular cases:

Capillarity and Diffusion

$s_t + f(s,c)_x = \varepsilon_c (A(s,c) s_x)_x,$

$(cs + a(c))_t + (cf(s,c))_x = \varepsilon_c (cA(s,c) s_x)_x + \varepsilon_d c_{xx},$

Capillarity and Dynamic Adsorpsion

$s_t + f(s,c)_x = \varepsilon_c (A(s,c) s_x)_x.$

$(cs + \alpha)_t + (cf(s,c))_x = \varepsilon_c (cA(s,c) s_x)_x,$

$\alpha_t = \varepsilon_r^{-1} (a(c) - \alpha).$

Restrictions on f and a

(F1) $f \in C^2([0,1]^2)$; $f(0,c) = 0$; $f(1,c) = 1$;

(F2) $f_s(s,c) > 0$ for $s \in (0,1)$, $c \in [0,1]$;
$f_s(0,c) = f_s(1,c) = 0$;

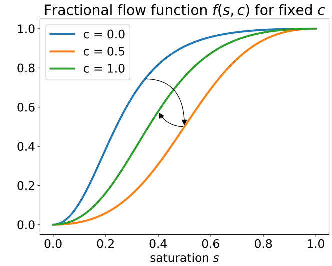
(F3) f is S-shaped in s;

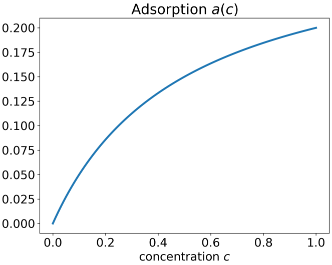
(F4) f is non-monotone in c:
$\forall s \in (0,1)\, \exists c^*(s) \in (0,1)$:

- $f_c(s,c) < 0$ for $0 < s < 1$, $0 < c < c^*(s)$;
- $f_c(s,c) > 0$ for $0 < s < 1$, $c^*(s) < c < 1$;

(A) A is bounded from zero and infinity;

(a) $a \in C^2$, $a(0) = 0$, a is strictly increasing and concave.





Travelling wave dynamical system

$$s_t + f(s,c)_x = \varepsilon_c (A(s,c)s_x)_x,$$
$$(cs + a(c))_t + (cf(s,c))_x = \varepsilon_c (cA(s,c)s_x)_x + \varepsilon_d c_{xx}.$$

Searching for travelling wave solutions $s = s(\xi)$, $c = c(\xi)$, $\xi := \varepsilon_c^{-1}(x - vt)$ with boundary conditions

$$s(\pm\infty) = s^\pm, \qquad c(-\infty) = 1, \qquad c(+\infty) = 0,$$

we arrive at

$$\begin{aligned} A(s,c)s_\xi &= f(s,c) - v(s + d_1), \\ \kappa c_\xi &= v(d_1 c - d_2 - a(c)). \end{aligned} \tag{4}$$

- Here $\kappa = \varepsilon_d / \varepsilon_c$;
- Note that $u^\pm$ are fixed points of dynamical system (4);
- We are only interested in the trajectories connecting two saddle points (or saddle-nodes) due to compatibility of speeds in

$$(1,1) \to u^- \xrightarrow{c\text{-shock}} u^+ \to (0,0).$$

Main result

Consider a dynamical system under assumptions (F1)–(F4), (A), (a):

$$\begin{aligned} A(s,c)s_\xi &= f(s,c) - v(s+d_1), \\ \kappa c_\xi &= v(d_1 c - d_2 - a(c)). \end{aligned}$$

Theorem (Bakharev, Enin, P.,Rastegaev, 2021)

There exist $0 < v_{\min} < v_{\max} < \infty$, such that for every $\kappa = \varepsilon_d/\varepsilon_c \in (0, +\infty)$, there exist unique

- *points $s^-(\kappa) \in [0,1]$ and $s^+(\kappa) \in [0,1]$;*
- *velocity $v(\kappa) \in [v_{\min}, v_{\max}]$,*

such that there exists a travelling wave, connecting two saddle points $u^-(\kappa) = (s^-(\kappa), 1)$ and $u^+(\kappa) = (s^+(\kappa), 0)$ with velocity $v(\kappa)$. Moreover, $v(\kappa)$ is monotone and continuous; $v(\kappa) \to v_{\min}$ as $\kappa \to \infty$; $v(\kappa) \to v_{\max}$ as $\kappa \to 0$.

Scheme of proof

The Theorem can be divided into simpler statements:

- $\forall v \in [v_{\min}, v_{\max}] \quad \exists! \kappa(v)$: there is a saddle-to-saddle travelling wave with $\kappa(v)$.
- $\kappa(v)$ is continuous.
- $\nexists v_1 \neq v_2 : \kappa(v_1) = \kappa(v_2)$, thus $\kappa(v)$ is monotone.
- $\kappa(v) \to \infty$ as $v \to v_{min}$.
- $\kappa(v) \to \kappa_{crit} \geqslant 0$ as $v \to v_{max}$.
- When $\kappa < \kappa_{crit}$ and $v = v_{\max}$ there is a saddle to saddle-node travelling wave

$\kappa(v)$ is monotone and continuous thus there exists an inverse function satisfying the Theorem.

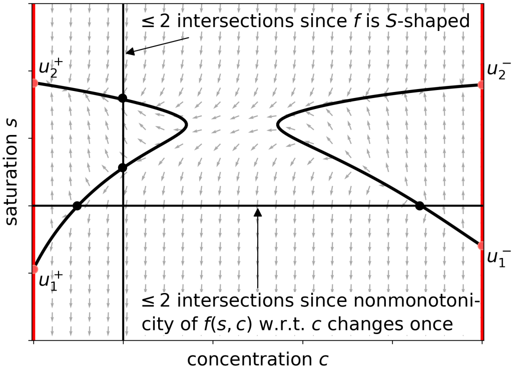
Phase portraits

In order to study the existence of solutions of the dynamical system

$$\begin{aligned} A(s,c)s_\xi &= f(s,c) - v(s+d_1), \\ \kappa c_\xi &= v(d_1 c - d_2 - a(c)), \end{aligned}$$

we draw phase portraits in (s,c) plane:
red lines are $d_1 c - d_2 - a(c) = 0$,
black lines are $f(s,c) - v(s+d_1) = 0$.

Here u_1^+ and u_2^- are saddle points.



Aim: find all pairs (κ, v) for which there exists a trajectory from u_2^- to u_1^+.

Phase portraits types. Main and intermediate classes

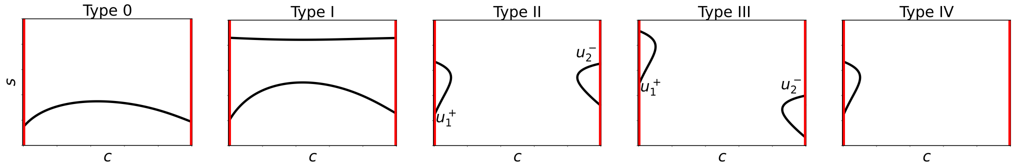


Figure 1: Five wide classes of phase portraits

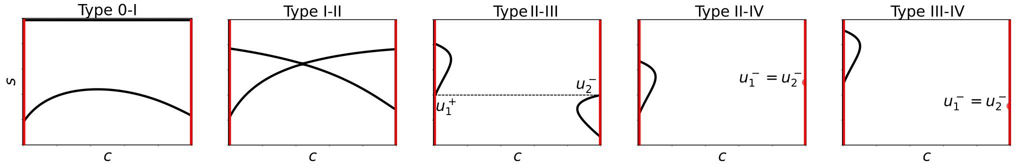


Figure 2: Intermediate types of phase portraits, appearing under the assumptions (F1)–(F4)

- Only Type II phase portrait has saddle-to-saddle connections.
- Type I-II corresponds to $v_{\min}$.
- Type II-III or Type II-IV correspond to $v_{\max}$.

Phase portraits: monotone dependence on v

black lines $f(s, c) = v(s + d_1)$

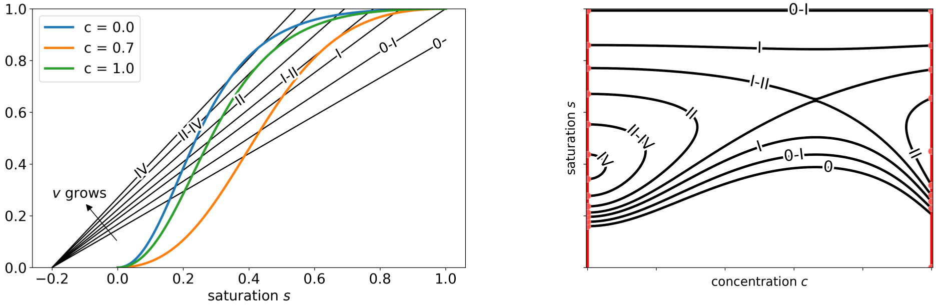


Figure 3: Phase portrait evolution as v grows: Type 0 $\to$ Type I $\to$ Type II $\to$ Type IV

Phase portraits: bad cases

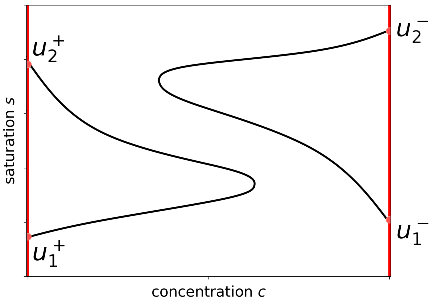


Figure 4: If f is not S-shaped.

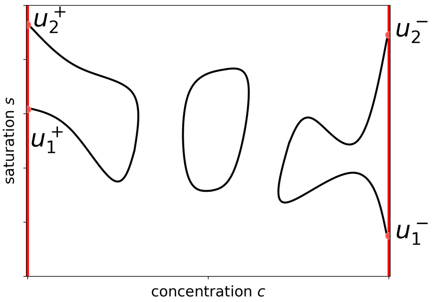


Figure 5: If non-monotonicity is more complex.

We believe that the similar result is true without conditions (F3)–(F4).

Basic properties of trajectories

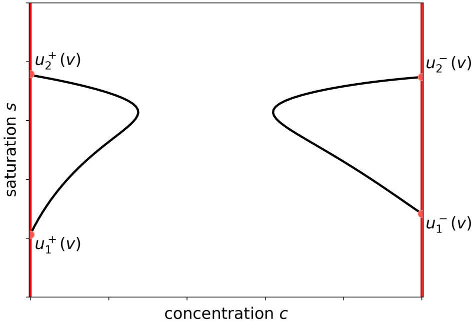
Lemma

If the slope s_ξ / c_ξ is positive for some point (s, c) then it strictly increases when κ or v increases.

Proposition

For Type II phase portrait:

- $\exists!$ *trajectory leaving* $u_2^-(v)$;
- $\exists!$ *trajectory entering* $u_1^+(v)$;
- *they depend continuously on* κ *and* v;
- *they depend monotonously on* κ *and* v *in some vicinity of the critical point.*



Type II portrait: for every v there exist κ

$$A(s,c)s_\xi = f(s,c) - v(s+d_1),$$
$$\kappa c_\xi = v(d_1 c - d_2 - a(c)),$$

Used property: continuous and monotonous dependence of trajectories on κ.

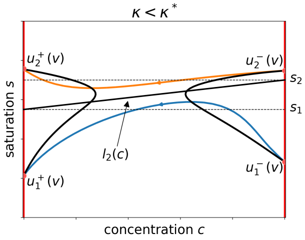


Figure 6: $\kappa << 1$.

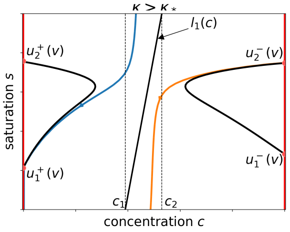


Figure 7: $\kappa >> 1$.

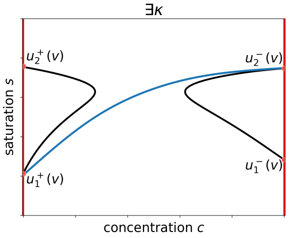


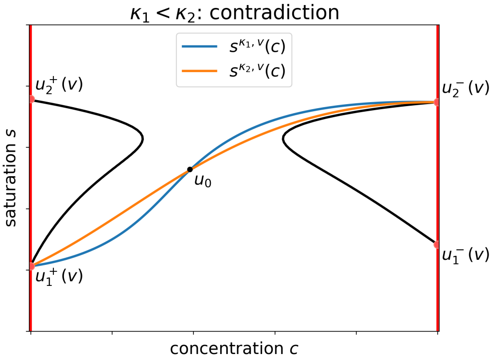
Figure 8: $\exists\kappa$.

Type II portrait: $\kappa(v)$ is unique for every $v \in (v_{\min}, v_{\max})$.

If there are $\kappa_1 < \kappa_2$ for one v, then the corresponding trajectories must intersect, which leads to a contradiction.
The slope

$$s_\xi / c_\xi = \kappa \frac{v^{-1} f(s,c) - (s + d_1)}{A(s,c)(d_1 c - d_2 - a(c))}$$

is positive at the intersection point (s,c), so it strictly increases when κ increases.



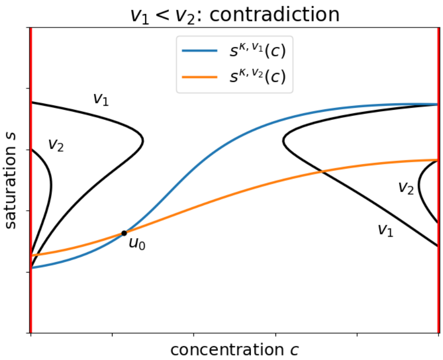
NB: this property might be lost for more complex phase portraits.

Type II portrait: monotonicity of $\kappa(v)$

If $\kappa(v_1) = \kappa(v_2)$ for $v_1 < v_2$, then the corresponding trajectories must intersect, which leads to a contradiction. The slope

$$s_\xi / c_\xi = \kappa \frac{v^{-1} f(s,c) - (s + d_1)}{A(s,c)(d_1 c - d_2 - a(c))}$$

is positive at the intersection point (s, c), so it strictly increases when v increases.

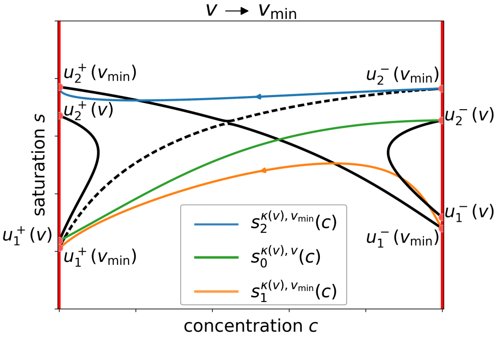


$\kappa \to +\infty$ as $v \to v_{\min}$

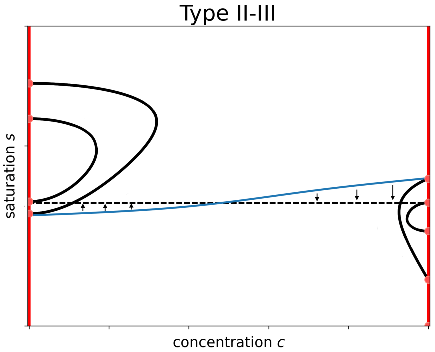
For any finite κ:

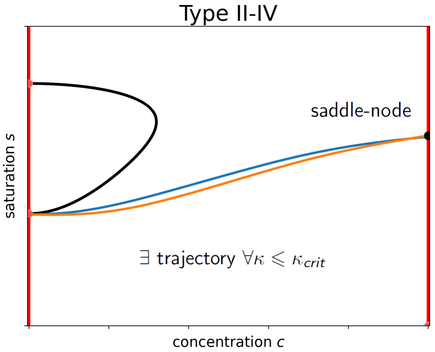
- green orbit is between blue and orange;
- blue orbit is upper than - - - -;
- orange orbit is lower than - - - - .

When $v \to v_{\min}$ the limits of green, blue and orange orbits coincide, which can not happen for any finite κ.



$\kappa \to 0$ as $v \to v_{\max}$ | $\kappa \to \kappa_{crit}$ as $v \to v_{\max}$





Solution construction algorithm

1. From κ we calculate $v(\kappa)$ (binary search).
2. From v we determine $s^-(v)$ and $s^+(v)$ via Rankine-Hugoniot condition.
3. Construct waves $(1,1) \to (s^-(v),1)$ and $(s(v),0) \to (0,0)$.

Example: "boomerang" model:

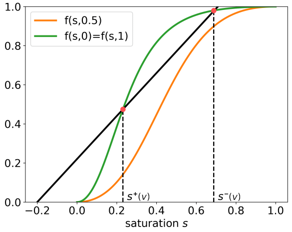


Figure 9: Flux functions

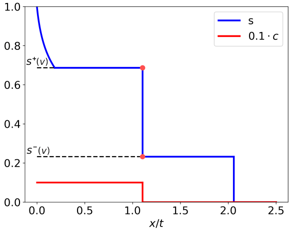


Figure 10: Solution s and c

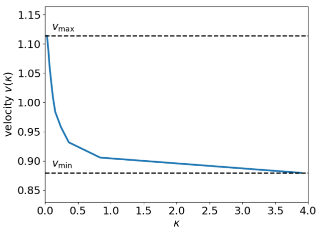


Figure 11: Function $v(\kappa)$

Possible directions for future research

- System with three small parameters $\varepsilon_c, \varepsilon_d, \varepsilon_r \to 0$
- General classes of f and a
- Construct solution to any Riemann problem
- Asymptotic stability as $t \to \infty$:
  is it true that a solution of a Cauchy problem with some «good» initial data tends to a solution of a Riemann problem?

Merci pour votre attention!

Questions? Comments? Remarks?