Lecture 15: Reaction-diffusion equations

$u = u(t,x)$, $x \in \mathbb{R}^N$, $t > 0$, $u \in \mathbb{R}^m$

$$(*) \quad \partial_t u - \underbrace{\Delta u}_{\text{(local) diffusion}} = \underbrace{f(u)}_{\substack{\text{reaction} \\ \text{term}}}$$

- excitible medium: more generally $f = f(t,x,u)$
- Δu — comes from particles moving according to Brownian motion (in a rough way, the population tends spread out uniformly, to move towards areas where there are fewer individuals)

"Intuitive" probabilistic justification:

Let the population consist of finite number n of individuals. Consider a discrete space:

$$\{\lambda k : k \in \mathbb{Z}^N\} \subset \mathbb{R}^N, \quad \lambda > 0$$

For a given individual we denote:

$p(t,x)$ – probability that the individual is at point x at time t.

$$X_k(t,x) = \begin{cases} 1, & \text{if } k\text{-th individual is at point } x \text{ at time } t \\ 0, & \text{otherwise} \end{cases}$$

Then $$U(t,x) = \frac{1}{n} \sum_{k=1}^{n} X_k(t,x)$$ - normalized distrib. of the population

Assuming the movements of individuals are independent of each other, $U(t,x) \to p(t,x)$.

At each instant an individual can:

- move to a neighbouring point with prob. $q < \frac{1}{2n}$
- do not move with probability $1 - q \cdot 2n$

Note that the probability q does not depend on the position in time and space, nor on the previous position $\Rightarrow$ random walk $\Rightarrow$

$$p(t+\tau, \lambda k) = (1 - 2nq)\, p(t, \lambda k) + q \sum_{j=1}^{n} \left[p(t, \lambda(k+e_j)) + p(t, \lambda(k-e_j)) \right]$$

Assume that there exists a regular $p(t,x)$ for which the same relation is true for all x, t. So

$$\partial_t p + O(\tau) = \frac{q\lambda^2}{\tau} \sum_{j=1}^{n} \frac{\partial^2 p}{\partial x_j^2} + O\left(\frac{\lambda^3}{\tau}\right)$$

Now let $\lambda, \tau \to 0$ such that $\frac{q\lambda^2}{\tau} \to D \in (0, +\infty)$

Thus, we get $\partial_t p = D \cdot \Delta p$.

Examples : (1) population (ecology) dynamics : u - concentration density

$$u_t - u_{xx} = f(u)$$

For a moment forget about diffusion and consider an ODE : $u_t = f(u)$, $u(0) = u_0$

Cases : (a) $f(u) = ru$ (Malthus equation '1798)

Solution: $u(t) = u_0 e^{rt}$, $r \in \mathbb{R}$

r - growth rate, the population grows infinitely (which is not natural)

(b) $f(u) = ru\left(1 - \frac{u}{K}\right)$ (logistic equation, ~ 1838)

$r \in \mathbb{R}$, $K \in \mathbb{R}$

Explicit solution: $u(t) = \dfrac{K}{1 + \left(\frac{K}{u_0} - 1\right) e^{-rt}}$

We observe, that :

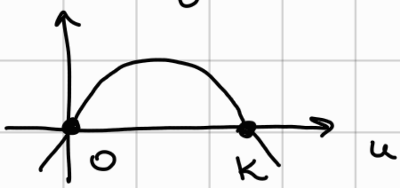
(i) whenever $u_0 > 0$, the solution is well-defined for $\forall t > 0$, $u(t) > 0$ and $u(t) \xrightarrow[t \to +\infty]{} K$

(ii) $u_0 = 0 \Rightarrow u(t) \equiv 0$

This corresponds to a more general fact that we will see later !

→ When u increases, there is a competition for resourses. Here K is called the capacity of environment

More general : monostable equations : $\dot{u} = f(t,u)$



assumptions: $f(0) = f(K) = 0$, f - Lipch in u

$f > 0$ for $u \in (0,K)$

$f < 0$ for $u \notin [0,K]$

Sometimes, there is an extra assumption: $\frac{f(u)}{u} \downarrow$

Lemma: $f: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ - continuous, loc. Lipschitz in u

(i) If $f(t,0) = 0$ $\forall t$, then if $u(0) > 0 \Rightarrow u(t) > 0$ $\forall t$

(ii) If u, v - two solutions and $u(0) > v(0)$, then $u(t) > v(t)$ (in the domain where both sol. exist)

(iii) If $u' \leq f(t, u(t))$ and $v' > f(t, v(t))$ and $u(0) \leq v(0)$, then $u(t) < v(t)$ $\forall t$.

Rmk 1: when u satisfies the differential inequality $u' \leq f(t, u(t))$ we say that u is a sub-solution; otherwise super-solution

Rmk 2: these statements are true for a single equation, but in general are not true for systems of eqs.

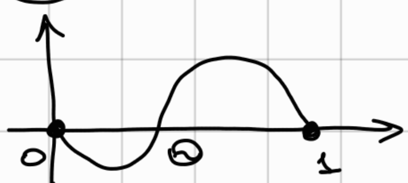
Rmk 3: items (ii) and (iii) are the so-called "comparison theorems" in this very simple setting. We will see more of them for reaction-diffusion eqs.

Here $u = 0$ is unstable equilibrium point (asymp)

$u = K$ is stable equilibrium point (asymp)

Thus, the name "monostable" (1 stable point)

c) $f(u) = u(1-u)(u-\theta)$, Bistable equations

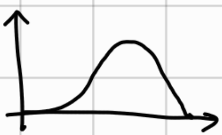


strong Allee effect

or more general assumptions:

- $f(0) = f(\theta) = f(1) = 0$
- $f > 0$ for $u \in (\theta, 1)$
- $f < 0$ for $u \in (0, \theta)$

Weak Allee effect:



monostable equation without condition $\frac{f(u)}{u}$ is decreasing

Theorem: for $u(0) \in [0,1]$ the equation admits global-in-time solution $u(t) \in [0,1]$ $\forall t \in \mathbb{R}$

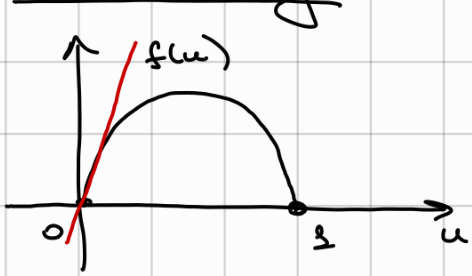
Moreover, if $u(0) < \Theta \Rightarrow u(t) \xrightarrow[t \to +\infty]{} 0$

$u(0) > \Theta \Rightarrow u(t) \to 1$

(the small population will turn off – may be not enough sexual partners or can not form big enough groups for fighting against predators)

This theorem explains the term "bistable":
$u=0$ and $u=1$ are stable equilibrium state
$u=\Theta$ – unstable equilibrium state

Concluding: we will consider 3 different $f(u)$:



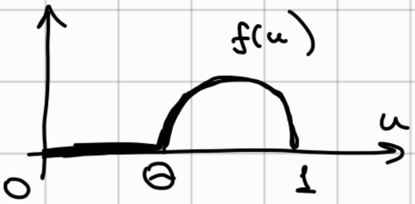
F-KPP
Fisher, Kolmogorov
Petrovskii, Piskunov (1937)

Monostable

Bistable

- monostable case with condition that $f(u)$ lies below the tangent line at $u=0$ (think of $f(u)=u(1-u)$)

There is also a case of ignition / combustion non-linearity: $f(u)=0$, $u \in [0,\Theta]$



Rmk: there is one more notion of stability: linear stability

state α is called linearly stable if $f'(\alpha)<0$

state α —//— linearly unstable if $f'(\alpha)>0$

Thm: $f \in C^1$ in the vicinity of α ($f(\alpha)=0$)

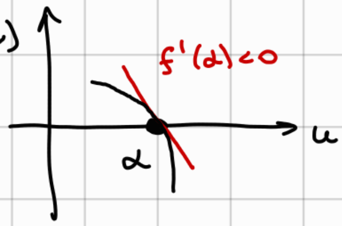
(i) If $f'(\alpha)<0$ and $u(0)$ is sufficiently close to α, then $u(t) \to \alpha$ as $t \to +\infty$

(ii) If $f'(\alpha)>0$, then no solution (except $u \equiv \alpha$) converges to α as $t \to +\infty$.

On the other hand, if $u(0)$ is close enough to α, then $u(t) \to -\alpha$ as $t \to -\infty$.

Proof:

(i)



$f(u) > 0$ for $u \in [\alpha-\varepsilon, \alpha)$

$f(u) < 0$ for $u \in (\alpha, \alpha+\varepsilon]$

$\dot{u} = f(u)$

If $u(0) < \alpha \Rightarrow u(t) < \alpha$ and $\uparrow$

If $u(0) > \alpha \Rightarrow u(t) > \alpha$ and $\downarrow$

$u(t)$ can converge only to α.

(ii)

$f(u) < 0$ for $u \in [\alpha-\varepsilon, \alpha)$

$f(u) > 0$ for $u \in (\alpha, \alpha+\varepsilon]$

$\dot{u} = f(u)$

If $u(0) < \alpha \Rightarrow \dot{u} = f(u) < 0 \Rightarrow u \downarrow$ and $u(t) < u(0) < \alpha$

If $u(0) > \alpha \Rightarrow \dot{u} = f(u) > 0 \Rightarrow u \uparrow$ and $u(t) > u(0) > \alpha$ ∎

There are many-many ways to generalize these equations:

$\Delta u \rightsquigarrow \int_\Omega K(x-y)\, u(y)\, dy$ – non-local diffusion

$\rightsquigarrow$ general (uniformly elliptic) term

$$\sum_{i,j=1}^{N} a_{ij}(t,x)\, \partial_i \partial_j u$$

with condition $\exists\ 0 < \alpha < \beta < +\infty$:

$\forall \xi \in \mathbb{R}^N, \forall t > 0, \forall x \in \Omega$ $\quad \alpha \|\xi\|^2 \le \sum a_{ij}(t,x)\, \xi_i \xi_j \le \beta \|\xi\|^2$

$f(u) \rightsquigarrow f(t,x,u)$ – depend on space x and time t

$u \in \mathbb{R} \rightsquigarrow \vec{u} \in \mathbb{R}^n$ – many species (Lotka-Volterra, predator-prey system, competitive media)

$\Omega \subset \mathbb{R}^N \rightsquigarrow$

more complex geometries

line of "fast diffusion" ("roads" in forests)

etc....

Other contexts: → combustion theory (propagation of flame, thermo-diffusive model)
→ probability (BBM - Branching Brownian Motion McKean representation)
→ statistical physics etc....

Reaction-diffusion eqs: problem statement

$$(*) \qquad \partial_t u = D\Delta u + f(t,x,u)$$

- $t \in (0,+\infty)$
- $x \in \Omega$ → $\Omega = \mathbb{R}^N$; → $\Omega \subset \mathbb{R}^N$ - bounded, connected with reg. boundary
- $D > 0$
- $u \in \mathbb{R}$ - scalar
- $f(u)$ is of one of the types above

\+ Initial condition: $u\big|_{t=0} = u_0(x) \in C(\Omega) \cap L^\infty(\Omega)$

\+ Boundary conditions:

(Neumann) $\partial_n u = 0$ for $(t,x) \in (0,+\infty) \times \partial\Omega$

(Dirichlet) $u = 0$ for — // —

(Robin) $\partial_n u + qu = 0$ for — // —

Interpretations:

Neumann: no individuals cross the boundary (in any direction) ✓

Dirichlet: exterior of Ω is extremely unfavorable so population density is zero at boundary

Robin: there is a flow of individuals entering ($q>0$) or leaving the domain ($q<0$)

We consider classical solution u which satisfies

$$(**) \begin{cases} u \in C^0([0,+\infty) \times \bar{\Omega}) \\ \partial_t u \in C^0((0,+\infty) \times \Omega) \\ \forall i \quad \partial_{x_i} u \in C^0((0,+\infty) \times \bar{\Omega}) \\ \forall i,j \quad \partial_{x_i x_j} u \in C^0((0,+\infty) \times \Omega) \end{cases}$$

and equation $(*)$, initial and one of the boundary cond.

If $\Omega = \mathbb{R}^N$ we also assume some growth cond. at infinity: $\forall T>0 \ \exists A, B > 0$:

$$|u(t,x)| \leq A e^{B|x|}, \quad x \in \mathbb{R}^N, \quad t > 0$$

What are the important topics?

① Comparison theorems: roughly speaking

if $u(0,x) \le v(0,x)$ are both solutions of (*)

then $u(t,x) \le v(t,x)$ $\forall t > 0$

Closely connected to maximum principle for parabolic PDEs.

This can be very helpful:

example 1:
$$u_t = \Delta u + u(1-u)$$
$$u(0,x) \in [0,1] \quad \forall x \in \mathbb{R}^N$$

- $u \equiv 0$ is solution and $u(0,x) \ge 0$
  $\Rightarrow u(t,x) \ge 0$
- $u \equiv 1$ is solution and $u(0,x) \le 1$
  $\Rightarrow u(t,x) \le 1$

Thus, $u(0,x) \in [0,1] \Rightarrow u(t,x) \in [0,1]$

example 2:
$$u_t = \Delta u - u^3$$
$$u|_{t=0} = u_0 \in [m, M], \quad x \in \Omega \subset \mathbb{R}^N$$

Consider $\begin{cases} \dot{v} = -v^3 \\ v(0) = m \end{cases}$ and $\begin{cases} \dot{w} = -w^3 \\ w(0) = M \end{cases}$

These are sub and supersolutions:

$$v(t) \le u(x,t) \le w(t)$$

$$-\frac{dv}{v^3} = dt \Rightarrow \frac{1}{2v^2} - \frac{1}{2m^2} = t \Rightarrow v = \left(\frac{1}{m^2} + 2t\right)^{-\frac{1}{2}}$$

$v(t) \to 0$ as $t \to \infty$

Analogously, $w(t) \to 0$ as $t \to \infty$

Thus, if u exists, then

$$\underset{\substack{\downarrow \\ 0}}{v(t)} \le u(x,t) \le \underset{\substack{\downarrow \\ 0}}{w(t)} \Rightarrow u \underset{t \to +\infty}{\longrightarrow} 0$$

→ well-posedness of $(*)$: $\exists!$, cont. dependence

→ special solutions : traveling waves (planar)

take direction $e \in \mathbb{R}^N$ and consider a solution of the form:

$$u(t,x) = \tilde{u}(\underbrace{x \cdot e}_{\text{scalar product}} - vt)$$

$\tilde{u} : \mathbb{R} \to \mathbb{R}$

v – speed of propagation

We will see that for different nonlinearities there exist travelling waves (TW)

$x \in \mathbb{R}^1$: FKPP : $\exists\, c^* : \forall c \geq c^*\ \exists$ TW

Bistable : $\exists!\, c : \exists$ TW

→ $x \in \mathbb{R}^1$: long-time behaviour as $t \to +\infty$ for some initial data (like

Heaviside) the solution u of $(*)$ "converges" to a TW

§ Maximum principle

for parabolic equations

This is an extension of the results that we have seen for ODEs. First, some definitions:

Def 1 : $u(t,x)$ is called sub-solution of $(*)$ if it satisfies $(**)$ and inequalities :

$$\partial_t u \leq \Delta u + f(t,x,u)$$

and on the boundary (if applicable) : on $\partial\Omega$

(Neumann) $\partial_n u \leq 0$; (Dirichlet) $u \leq 0$; (Robin) $\partial_n u + qu \leq 0$

If $\Omega = \mathbb{R}^N$, then $|u| \leq A e^{B|x|}$, $A, B > 0$

Analogously, $v(t,x)$ is called a super solution if all inequalities are reversed (except $|v| \le A e^{B|x|}$)

We want to prove the following theorem:

<u>Theorem</u> (comparison principle)

Let u and v be sub- and super-solutions of the reaction-diffusion eq $(*)$.

(i) If $u(0,x) \le v(0,x)$ for $x \in \bar{\Omega}$, then $u(t,x) \le v(t,x)$ for $t>0$, $x \in \bar{\Omega}$

(ii) If moreover, $u(t_0,x_0) = v(t_0,x_0)$ for some $t_0 > 0$, $x_0 \in \Omega$, then $u \equiv v$.

(iii) If Ω is bounded and the boundary condition is of Neumann or Robin type, then (ii) is true even for $x_0 \in \partial\Omega$

Note that the difference $(u-v)$ satisfies

$$\partial_t (u-v) \le \Delta(u-v) + f(t,x,u) - f(t,x,v)$$

Thanks to regularity of u, v, f we can rewrite this equation as follows: $w = u - v$

$$(1) \quad \partial_t w \le \Delta w + g(t,x) w$$

where

$$g(t,x) = \begin{cases} \dfrac{f(t,x,u) - f(t,x,v)}{u-v} & \text{if } u \ne v \\[2ex] \partial_u f(t,x,u) & \text{if } u = v. \end{cases}$$

is continuous and uniformly bdd function

So we reduced a problem to studying the linear eq (1) and showing $w \le 0$ $\forall\, t>0$, $x \in \bar{\Omega}$.

Linear problem and maximum principle

Let us consider a more general case:

(2) $\partial_t u = \Delta u + \sum b_i(t,x)\,\partial_i u + c(t,x)\,u$

Let b_i, c be uniformly bdd.

Thm 1 (weak maximum principle)

(i) Let be a sub-solution of linear eq (2).
If $u(0,x) \le 0$, then $u(t,x) \le 0 \quad \forall t > 0$

(ii) Let v be super-solution of linear eq (2).
If $v(0,x) \ge 0$, then $v(t,x) \ge 0 \quad \forall t > 0$.

because $u(x_0,t_0) = 0 \Rightarrow u \equiv 0$

Thm 2 (strong maximum principle)

(i) Let u be a subsolution of (2) and $u(0,x) \le 0$.
If $\exists\, t_0 > 0, x_0 \in \Omega: u(t_0, x_0) = 0 \Rightarrow u \equiv 0$ on $[0,t_0] \times \Omega$

(ii) Let v be a supersolution of (2) and $v(0,x) \ge 0$.
If $\exists\, t_0 > 0, x_0 \in \Omega: v(t_0, x) = 0 \Rightarrow v \equiv 0$ on $[0,t_0] \times \Omega$

(iii) If Ω is bdd, then for Neumann and Robin the same statement as in (i), (ii) are true if $x_0 \in \partial\Omega$.

Rmk: it is clear that it is sufficient to consider subsolutions. For the supersolutions just consider $v = -u$.

Proof of maximum principle:

⌐ We will prove in 2 cases: (a) Ω - bdd, Dirichlet
(b) $\Omega = \mathbb{R}^N$

First, let's prove the simple case:

Lemma: let u be a subsolution with strict ineq:

$\partial_t u - \Delta u - \sum b_i(t,x)\,\partial_i u - c(t,x)\,u < 0$, $u(0,\cdot) < 0$, $\boxed{u|_{\partial\Omega} < 0}$

$\Rightarrow u(t,x) < 0$

Proof of lemma:

$\ulcorner$ Indeed, take first time $t_0 > 0$ such that $u(x_0, t_0) = 0$ for $x_0 \in \Omega$.

At this point: $\partial_t u \geq 0$

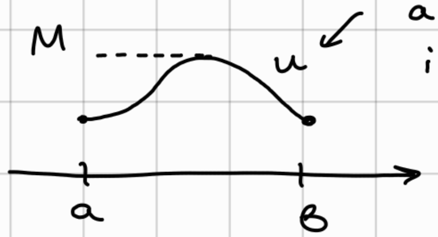
$\Delta u \leq 0$ (the local picture )

$\partial_i u = 0$ (as it is local maximum)

$u = 0$

$\llcorner$ $\Rightarrow$ $\partial_t u - \Delta u - \sum b_i \partial_i u - cu \geq 0$ (!?) $\blacksquare$

Lecture 16: Maximum principles for ODEs.



a non-constant function that achives its maximum over an interval $[a,b]$

Let $[a,b] \subset \mathbb{R}$

$u \in C^2((a,b)) \cap C^0([a,b])$

Consider a differential operator:

$$L = -\frac{d^2}{dx^2} + g\frac{d}{dx} + h$$

- g, h – bounded functions on (a,b)

Let

$$M = \max_{[a,b]} u$$

Question: how inequalities for Lu can lead to conclusions about M?

Lemma 1 (basic lemma for $h \equiv 0$):

Let $h \equiv 0$ and $Lu < 0$. Then u can equal to M only at the endpoints $x = a$ or $x = b$.

Proof:

▶ By contradiction: suppose $\exists x_0 \in (a,b): u(x_0) = M$

Then $u'(x_0) = 0$

$u''(x_0) \le 0$

$$\Rightarrow Lu\Big|_{x_0} \ge 0 \quad (!?) \quad \blacksquare$$

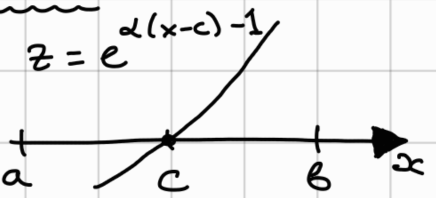
Thm 1 (one-dimensional maximum principle for $h \equiv 0$)

Let $h \equiv 0$ and $Lu \le 0$.

Then if $\exists c \in (a,b) : u(c) = M \Rightarrow u \equiv M$.

Proof:

▶



Suppose $u \not\equiv M \Rightarrow$ $\exists d \in (a,b)$ such that $u(d) < M$ (w.l.o.g. $d > c$)

We would like to construct a "barrier" $z(x)$ such that for $w = u + \varepsilon z$:

$$Lw < 0 \quad \text{on } (a, \beta)$$

and we could apply lemma 1.

Take

$$z = e^{\alpha(x-c)} - 1.$$

$z(c) = 0$, $z > 0$ for $x \in (c, \beta)$

$$Lz = (-\alpha^2 + g\alpha)\, e^{\alpha(x-c)}$$

Since g is bounded we can choose $\alpha > 0$ large enough such that $Lz < 0$

Thus, $Lw = Lu + \varepsilon Lz < 0$.

Moreover, $w(a) = u(a) + \varepsilon \underbrace{z(a)}_{<0} < u(a) \leq M$

$$w(d) = \underbrace{u(d)}_{M} + \varepsilon z(d) < M$$

by taking very small ε we can guarantee that $w(d) < M$

Thus, we have a contradiction with Lemma 1. So, $u \equiv M$. ■

<u>Rmk</u>: this idea of "adding a small barrier" is very useful and we will encounter this many times in future.
The choice of z is not unique!

<u>Thm 2</u> (one-dimensional Hopf lemma for $h \equiv 0$)
Let $h \equiv 0$ and $Lu \leq 0$.
If $u(a) = M$, then either $u'(a) < 0$ or $u \equiv M$
Similarly, if $u(\beta) = M$, then either $u'(\beta) > 0$ or $u \equiv M$

Rmk: the essense of the Hopf lemma is in strict inequality $\underline{u'(a) < 0}$. Because the non-strict inequality is straight forward: if $u(a) = M \Rightarrow u'(a) \le 0$. So if the maximum is on the boundary, this point can not be a critical point (unless $u \equiv$ constant)

Proof:

$\ulcorner$ ▶ Let $u(a) = M$ and by contradiction $\exists\, d \in (a,b) : u(d) < M$

We can use the same "barrier"

$$z = e^{\alpha(x-a)} - 1$$

and consider $w = u + \varepsilon z$.

First, $Lw < 0$ for sufficiently large α.

And $w(a) = M > w(d)$ for sufficiently small ε.

So w achieves its maximum at $x = a$.

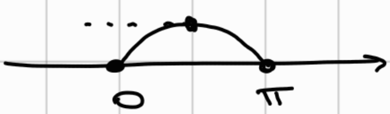
$$w'(a) = u'(a) + \varepsilon\alpha \le 0$$

$\llcorner \Rightarrow u'(a) \le -\varepsilon\alpha < 0$. ■

Interestingly, if we relax condition $h \equiv 0$, the statements are no longer valid. Consider the following counter-example:

- $Lu = -u'' - u$

  Take $Lu = 0$

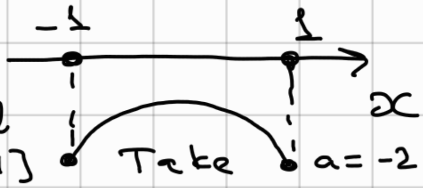


$\sin(x)$!

- $Lu = -u'' + u$, $x \in [-1, 1]$

Look for the solution of the form $u = -x^2 + a,\ a \in \mathbb{R}$

$$Lu = 2 - x^2 + a \le 0, \quad a \le x^2 - 2 \quad \forall x \in [-1,1]$$



Take $a = -2$

In these examples $h \cdot M \le 0$. If $h \cdot M \ge 0$, then everything ok!

Thm 3 (one-dimensional maximum principle for $h \not\equiv 0$)

exercise

Let $h \geq 0$ and $M \geq 0$.

If $Lu \leq 0$ on (a, b)

then u can attain maximum at some point $c \in (a, b)$ only if $u \equiv M$.

Rmk: this theorem should also work for $h \leq 0, M \leq 0$

Thm 4 (one-dimensional Hopf lemma for $h \geq 0$)

exercise

Let $h \geq 0$.

Let $Lu \leq 0$ on (a, b) and $M \geq 0$.

If $u(a) = M$, then either $u'(a) < 0$ or $u \equiv M$.

Similarly, if $u(b) = M$, then either $u'(b) > 0$ or $u \equiv M$.

Thm 5 (comparison principle)

Let $h \equiv 0$, $f \in C^1$

$$Lu \leq f(x) \qquad x \in (a, b)$$
$$Lv \geq f(x), \qquad x \in (a, b)$$

Then if $\begin{cases} u(a) \leq v(a) \\ u(b) \leq v(b) \end{cases}$, then $u(x) \leq v(x)$ $\forall x\ (a, b)$

Moreover, if $\exists x_0 :\ u(x_0) = v(x_0) \Rightarrow u \equiv v$

Proof:

$$w = u - v; \quad \left.\begin{array}{l} Lw \leq 0 \\ w(a) \leq 0 \\ w(b) \leq 0 \end{array}\right\} \Rightarrow$$

$w(x) \leq 0$ as maximum is obtained on the boundary $\begin{bmatrix} x = a \\ x = b \end{bmatrix}$

And if $w(x_0) = 0$ for some $x_0 \in (a, b) \Rightarrow w \equiv 0$ ∎

Rmk: if $f = f(x, u)$ the theorem does not easily work without any other assumptions

Rmk: The above strong max. principles say that subsolution u and supersolution v can NOT touch at a point: either $u \equiv v$ or $u < v$

This "untouchability" condition can be very helpful. Consider such an example.

Example: consider a boundary value problem:

$$(1) \quad \begin{cases} -u'' = e^u, & x \in [0, L] \\ u(0) = u(L) = 0 \end{cases}$$

One can interpret the "u" as an equilibrium temperature: conditions $u(0) = u(L) = 0$ say that we have a "cold" boundary, while e^u is the "heating term". They compete with each other and non-negative solution corresponds to an equilibrium between these two effects. We would like to show that if the length of the interval L is suff. large, then no such equilibrium is possible. The physical reason is that the cold boundary is too far from the middle of the interval so that the heating term wins.

Task: show that for large enough $L > 0$ there is no non-negative solution of (1)

Step 1: consider $w = u + \varepsilon$

$$\Rightarrow (2) \begin{cases} -w'' = e^{-\varepsilon} e^{w} \\ w(0) = w(L) = \varepsilon \end{cases}$$

Step 2: consider family of functions:

$$V_\lambda(x) = \lambda \sin\left(\frac{\pi x}{L}\right)$$

They are solutions of the following problem:

$$(3)\ \begin{cases} -V_\lambda'' = \frac{\pi^2}{L^2} V_\lambda \\ V_\lambda(0) = V_\lambda(L) = 0 \end{cases}$$

Step 3: Notice that for L large enough

$$e^{-\varepsilon} e^{s} > \frac{\pi^2}{L^2} s, \qquad \forall s > 0.$$

Thus, w as solution of (2) is a supersolution to (3): $w(0) = w(L) = \varepsilon > 0$

$$\begin{cases} -w'' \geq \frac{\pi^2}{L^2} w \\ w(0) = w(L) \geq 0 \end{cases}$$

We assume that $w \geq 0$.

Clearly, for small enough $\lambda > 0$

$$V_\lambda(x) < w(x).$$

Step 4: (Sliding method) Now start increasing λ until some $\lambda_0 > 0$ s.t. the graphs of V_λ and w "touch" at some point:

$$\lambda_0 = \sup\{\lambda > 0 : V_\lambda(x) \leq w(x),\ 0 \leq x \leq L\}$$

Look at the difference: $p = V_\lambda - w$

- $-p'' \leq \frac{\pi^2}{L^2} p \qquad p(x) \leq 0$
- $p(0) = p(L) = -\varepsilon$

In addition, $\exists\, x_0 : p(x_0) = 0$. It can not be in (a, b) because of maximum principle and it can not be on the boundary (!?)

Exercise (for interest):

Show that $\exists L_1 > 0$ so that non-negative solution of (1) exists for all $0 < L < L_1$ and does not exist for all $L > L_1$.

Exercise (for now): consider

$$\begin{cases} -u'' - cu' = f(u), & x \in [-L, L] \\ u(-L) = 1, \ u(L) = 0 \end{cases}$$

Prove that if solution exists, then it is unique and decreasing ($u' < 0$)

Hint: use sliding method for 2 solutions u and v, e.g. consider

$$v_h(x) = v(x+h)$$

- Strong maximum principle for any h with assumption $M = 0$.

Thm 6 (one-dimensional maximum principle for $h \not\equiv 0$)

Let $M = 0$.

exercise

If $Lu \leq 0$ on (a, b), then u can attain maximum at some point $c \in (a, b)$ only if $u \equiv 0$.

Rmk: no assumptions on the sign of h!

Thm 7 (comparison principle):

$f \in C^1$

$$Lu \leq f(x, u) \qquad x \in (a, b)$$
$$Lv \geq f(x, u), \quad x \in (a, b)$$

Then if $\begin{cases} u(x) \leq v(x) & \forall x \ (a, b) \\ \exists x_0 : & u(x_0) = v(x_0) \end{cases} \Rightarrow u \equiv v$

Lecture 17: Maximum principle for linear parabolic PDEs

Let us consider a linear parabolic PDE:

(1) $\partial_t u = \Delta u + \sum b_i(t,x)\,\partial_i u + c(t,x)\,u =: -Lu$

Here:
- $x \in \Omega$ (either bounded open connected set or $\mathbb{R}^N$, $N \geq 1$)
- $t \geq 0$
- $u : [0,+\infty) \times \Omega \to \mathbb{R}$ – scalar function
- coefficients b_i, c are continuous and uniformly bdd (=bounded)

Initial condition: $u(0,x) = u_0(x)$

Boundary conditions:
- Ω – bdd: (Dirichlet) $u|_{\partial\Omega} = 0$

  (Neumann) $\frac{\partial u}{\partial n}\Big|_{\partial\Omega} = 0$

  (Robin) $\frac{\partial u}{\partial n} + qu\,\Big|_{\partial\Omega} = 0$
- $\Omega = \mathbb{R}^N$: $\exists A, B > 0$: $|u| \leq A e^{B|x|}$, $x \in \Omega$

<u>Def</u>: u - subsolution of (1) if $\partial_t u + Lu \leq 0$ and either $u|_{\partial\Omega} \leq 0$ or $\frac{\partial u}{\partial n}\Big|_{\partial\Omega} \leq 0$ or $\frac{\partial u}{\partial n} + qu\,\Big|_{\partial\Omega} \leq 0$

Analogously, v - supersolution if $\partial_t v + Lv \leq 0$ —//—

<u>Thm 1</u> (weak maximum principle = weak MP)

(i) Let u be a subsolution of (1) s.t. $u(0,x) \leq 0$,
Then $\forall t > 0$ $u(t,x) \leq 0$.

(ii) Let v be a supersolution of (1) s.t. $u(0,x) \geq 0$,
Then $\forall t > 0$ $v(t,x) \geq 0$.

<u>Rmk</u>: it is clear that it is sufficient to consider subsolutions. For the supersolutions just consider $v = -u$.

Thm 2 (strong maximum principle = strong MP)

(i) Let u be a subsolution of (1) and $u(0,x) \le 0$.
If $\exists\, t_0 > 0, x_0 \in \Omega : u(t_0, x_0) = 0 \Rightarrow u \equiv 0$ on $[0,t_0] \times \Omega$

(ii) Let v be a supersolution of (2) and $v(0,x) \ge 0$.
If $\exists\, t_0 > 0, x_0 \in \Omega : v(t_0, x) = 0 \Rightarrow v \equiv 0$ on $[0,t_0] \times \Omega$

(iii) If Ω is bdd, then for Neumann and Robin b the same statement as in (i), (ii) are true if $x_0 \in \partial\Omega$.

Proof of maximum principle (weak and strong):

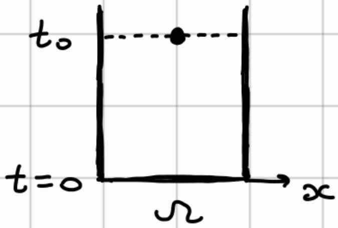
Case 1: Dirichlet boundary conditions

Lemma 1: Let $\partial_t u - Lu < 0$, $u(0,x) < 0$, $u|_{\partial\Omega} < 0$
Then $\forall\, t > 0 \quad u(t,x) < 0$.

Proof

By contradiction. Let t_0 be the first time when $\exists\, x_0 \in \Omega : u(x_0, t_0) = 0$

At this point: $\partial_t u \ge 0$



$$-Lu \le 0 \Leftarrow \begin{cases} \Delta u \le 0 \\ \partial_{x_i} u = 0 \\ u = 0 \end{cases}$$

$\Rightarrow \partial_t u + Lu \ge 0$ (!?)

Thus at $\forall\, x \in \Omega$, $t > 0$ $\quad u(t,x) < 0$ ■

Observation: take $u = e^{kt} w$ for some $K \in \mathbb{R}$
$u < 0 \iff v < 0$ and $u \le 0 \iff w \le 0$

But now w satisfies:

$$\partial_t w - \Delta w - \sum b_i \partial_i w - (c - K) w \underset{(\le)}{<} 0$$

Taking $K > \max |c|$ we can guarantee that $c - K < 0$, or taking $K < -\max|c|$ we have $c - k > 0$.

Let's take $K \geq \max |c| + 1$, and thus $c - K \leq -1$
In order not to change the notation we stay with letter "u" and consider $c \leq -1 < 0$ in (1).

Now we are ready to prove thm 1 (i).
By contradiction. Take the first moment $t_0 > 0$ s.t. $\exists\, x_0 \in \Omega$: $u(t_0, x_0) = \delta$ for some $\delta > 0$.
At this point (t_0, x_0) :

$$\left.\begin{aligned} \partial_t u &\geq 0 \\ \Delta u &\leq 0 \\ \partial_{x_i} u &= 0 \end{aligned}\right\} \Rightarrow -Lu \leq c\delta \leq -\delta$$

$$\Rightarrow \quad \partial_t u + Lu \geq \delta > 0 \quad (!?)$$

Thus, for all $x \in \Omega$, $t > 0$ $\quad u(t,x) \leq 0$.

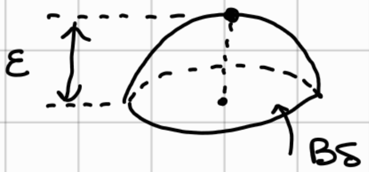
We have proven the weak MP for Dirichlet $\overset{bc}{\downarrow}$

Let's prove the strong maximum principle for Dirichlet $\overset{bc}{\downarrow}$.

<u>Lemma 2</u>: Let u be subsolution of (1) with Dirichlet and $u(0,x) < 0$ $\forall x \in \Omega \Rightarrow u(t,x) < 0$ $\forall t > 0$

<u>Proof</u>:

▶ It is enough to consider $\Omega = B_\delta(0)$.
The idea is to construct a "barrier"
Let $\quad w = u + \varepsilon(\delta^2 - |x|^2)^2 e^{-\alpha t}$



Take $\varepsilon > 0$ so small s.t. $w(0,x) < 0$. Moreover, $w\big|_{\partial B_\varepsilon} = u\big|_{\partial B_\varepsilon} \leq 0$

We can choose α such that w is a subsolution
Indeed, $\partial_i (\delta^2 - |x|^2)^2 = 2(\delta^2 - |x|^2) \cdot (-2x_i)$

$$\partial^2_{ii} (\delta^2 - |x|^2)^2 = -4(\delta^2 - |x|^2) + 8x_i^2$$

Then $(-L)(\delta^2 - |x|^2)^2 = \left(\Delta + \Sigma b_i \partial_i + c\right)(\delta^2 - |x|^2)^2 =$

$$= 8|x|^2 - 4N(\delta^2 - |x|^2) - 4b \cdot x(\delta^2 - |x|^2) + c(\delta^2 - |x|^2)^2$$

By estimating $|B(t,x)| \le \|B\|_\infty$ and $|c(t,x)| \le \|c\|_\infty$ we obtain:

$$(\partial_t + L)\, z \le \varepsilon e^{-\alpha t} \Big[-\alpha \cdot (\delta^2 - |x|^2)^2 - 8|x|^2 + 4N(\delta^2 - x^2) + $$
$$ + 4|x| \cdot \|B\|_\infty (\delta^2 - |x|^2) + \|c\|_\infty (\delta^2 - |x|^2)^2 \Big]$$

We would like : $(\partial_t + L)\, z \le 0$.

<u>Naive idea</u> : just take $\alpha > 0$ very big and then the first term $-\alpha(\delta^2 - |x|^2)^2$ will be very negative and dominate all other (positive) terms.

<u>Bad news</u> : the term $-\alpha^2(\delta^2 - |x|^2)^2$ is small close to the boundary of the $B_\delta(0)$. So the previous idea works only inside some smaller ball $B_{\delta'}(0) \subset B_\delta(0)$ $(0 < \delta' < \delta)$

<u>What to do?</u> Divide the ball into 2 parts:

(1) $B_\delta(0) \setminus B_{\delta'}(0)$

(2) $B_{\delta'}(0)$

and estimate $(\partial_t + L)\, z$ in each part separately.

(1) If δ' is close to δ, then all terms that have $(\delta^2 - |x|^2)$ are small and the dominating term is $-8|x|^2$. Take δ' such that $\forall\ x \in B_\delta(0) \setminus B_{\delta'}(0)$ the following ineq is true

$$8|x|^2 > (\delta^2 - x^2) \cdot \left[4N + 4|x| \cdot \|B\|_\infty + \|c\|_\infty \cdot (\delta^2 - |x|^2) \right]$$

Or

$$8(\delta')^2 > (\delta^2 - (\delta')^2) \cdot \left[4N + 4\delta \|B\|_\infty + \delta^2 \cdot \|c\|_\infty \right]$$

Such δ' exists as $8(\delta')^2 \approx 8\delta^2$ when $\delta' \approx \delta$ and right hand side is almost 0.

Thus, for $x \in B_\delta \setminus B_{\delta'}$: $(\partial_t + L)\, z \le -\alpha \varepsilon e^{-\alpha t} (\delta^2 - |x|^2)^2 < 0$

(2) Now take α so big such that for all $x \in B_{\delta'}(0)$ we have:

$$\alpha\,(\delta^2-|x|^2)^2 > (\delta^2-|x|^2)\left[4N + 4\cdot|x|\cdot\|b\|_\infty + \|c\|_\infty(\delta^2-|x|^2)\right]$$

Divide by $\delta^2-|x|^2$ and it is enough to have

$$\alpha\cdot(\delta^2-(\delta')^2)^2 > \delta^2\left[4N + 4\delta'\cdot\|b\|_\infty + \|c\|_\infty\delta^2\right]$$

$$\alpha > \frac{\delta^2\left[4N+4\delta'\|b\|_\infty + \|c\|_\infty\delta^2\right]}{(\delta^2-(\delta')^2)^2}$$

(remember, here δ' is already some fixed value)

Thus, for $x \in B_{\delta'}(0)$: $(\partial_t + L)z < -8\varepsilon e^{-\alpha t}|x|^2 < 0$

$$\Rightarrow (\partial_t + L)w = \underbrace{(\partial_t+L)u}_{\leq 0} + \underbrace{(\partial_t+L)\left(\varepsilon(\delta^2-|x|^2)e^{-\alpha t}\right)}_{\leq 0} \leq 0$$

$\Rightarrow$ $w \leq 0$ $\Rightarrow$ $u < w \leq 0$; q.e.d.

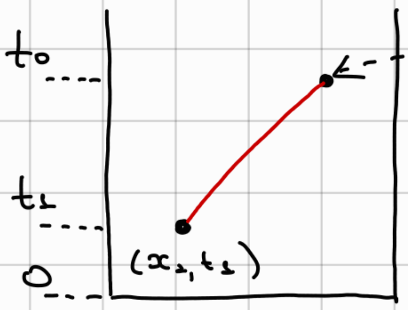
weak MP

Now let's finish proving the strong MP for (D).

Take (t_0, x_0) : $u(t_0,x_0) = 0$.

It is enough to prove that $u \equiv 0$ for $t \in [a,t_0]$, $x \in \Omega$

By contradiction, there exists a point (t_1, x_1), $t_1 < t_0$ such that $u(t_1,x_1) < 0$.



By continuity $u<0$ in $B_\delta(t_1,x_1)$ – ball in Ω

Assume that the segment connecting x_1 and x_0 in Ω lies in Ω (e.g. Ω - convex)

If necessary, take smaller δ s.t. $B_\delta(x) \subset \Omega$ for all x in this segment $[x_1, x_0]$ (this can be done by compactness of segment)

Now consider $w(t,x) = u\left(t, x + \frac{t-t_1}{t_0-t_1}(x_0-x_1)\right)$

$$\partial_t w = \partial_t u + \sum_{i=1}^{N} \partial_i u$$

Clearly, w satisfies the equation of type (1)

By previous lemma : $w(t_1,x_1) = u(t_1,x_1)$
$w(t_0,x_1) = u(t_0,x_0)$

$$w(t_1,x_1) < 0 \Rightarrow w(t_0,x_1) < 0 \Rightarrow u(t_0,x_0) < 0 \ (!?)$$

It is easy to generalize this argument for arbitrary connected domains Ω, as there

exists a path between x_1 and x_0 and this path can be approximated by segments. ∎

Both weak and strong MP for Dirichlet bc are proven (case 1)

Case 2 : Neumann and Robin bc.

Lemma 3 (Hopf lemma)

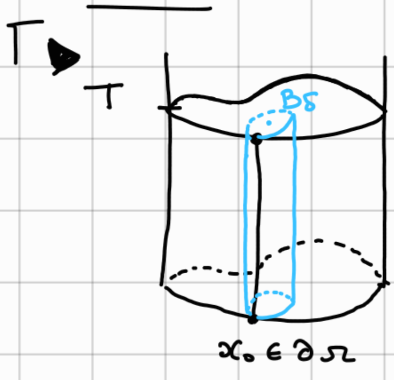
Let u be subsolution of (1) with NO boundary conditions. And let $u(t,x) < 0$ for all $t \in [0,T]$ and $x \in \Omega$.

If $u(T,x_0) = 0$ at $x_0 \in \partial\Omega$, then

$$\frac{\partial u}{\partial n}(T,x_0) > 0 .$$

Rmk : the sign $\frac{\partial u}{\partial n} \geq 0$ is clear, the important statement in lemma is STRICT inequal.

Proof :



By contradiction.

Let $\exists x_0 \in \partial\Omega$ s.t.

$$u(T,x_0) = \frac{\partial u}{\partial n}(T,x_0) = 0$$

Take a ball $B_\delta \subset \Omega$ s.t. $x_0 = \partial B_\delta \cap \partial\Omega$ (this is just some condition on regularity of $\partial\Omega$)

For simplicity we can always assume that the center of the ball B_δ is in the origin and the normal $n = (-1, 0, \dots 0)$

As $u<0$ in $\Omega \times [0,T]$, then $\forall\ 0<r<\delta$

$$\sup_{t\in[0,T]} \sup_{x\in B_r} u(t,x) < 0.$$

Consider

$$w = u + \varepsilon_1 (t-T) + \varepsilon_2 \left[e^{-\alpha|x|^2} - e^{-\alpha\delta^2} \right]$$

$\alpha, \varepsilon_1, \varepsilon_2 > 0$ will be chosen soon.

We want to prove : for domain $A := B_\delta(0) \setminus \overline{B_r(0)}$
$0<r<\delta$

$$\begin{cases} ⓐ\ \partial_t w + Lw \le 0, & x\in A,\ t\in[0,T] \\ ⓑ\ w(0,x) < 0, & x\in A \\ ⓒ\ w\big|_{\partial A}(t,x) \le 0 \end{cases}$$

Thus, by Dirichlet weak MP $\Rightarrow$ for $x\in A$ $w(T,x) \le 0$

This will be a contradiction with

$$w(T,-\delta,0\dots0) = u\Big|_{\substack{x=x_0\\ t=T}} = 0$$

$$\frac{\partial}{\partial n} w(T,\cdot) = -\partial_{x_1} w(T,-\delta,0\dots0) = -\partial_{x_1} u + \varepsilon_2 \alpha \cdot 2x_1 e^{-\alpha|x|^2}\Big|_{\dots} =$$

$$= 0 - \varepsilon_2 \cdot 2\alpha\delta \cdot e^{-\alpha\delta^2} < 0$$

Let's show ⓐ, ⓑ, ⓒ.

ⓐ $\partial_t w + Lw = \partial_t w - \Delta w - b\cdot\nabla w - cw \le$

$$\le \varepsilon_1(1+CT) - \varepsilon_2 e^{-\alpha|x|^2} \cdot \left[4\alpha^2|x|^2 - 2N\alpha - 2C\alpha|x| - C\right]$$

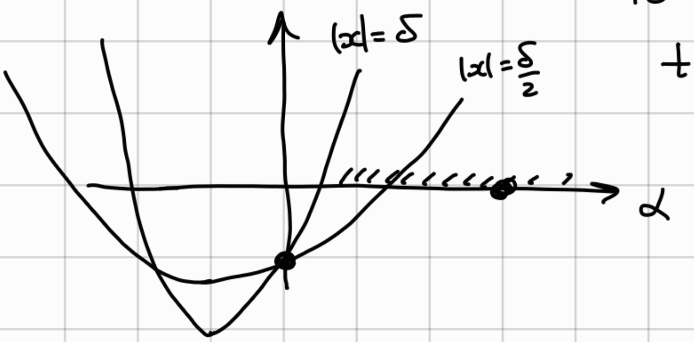
where C is $\max(\|b_i\|_\infty, \|c\|_\infty)$.

$$L\left[e^{-\alpha|x|^2} - e^{-\alpha\delta^2}\right] = \sum \frac{d}{dx_i}\left(-2\alpha x_i e^{-\alpha|x|^2}\right) + \sum b_i\left(-2\alpha x_i e^{\dots}\right)$$
$$+ c\left(e^{-\alpha|x|^2} - e^{-\alpha\delta^2}\right) = -2\alpha N e^{-\alpha|x|^2} + 4\alpha^2 \sum x_i^2 e^{-\alpha|x|^2}$$

$$+ \sum' B_i\left(-2\alpha x_i e^{-\alpha|x|^2}\right) + c\left(e^{-\alpha|x|^2} - e^{-\alpha\delta^2}\right)$$

Fix $\alpha > 0$ s.t. $4\alpha^2|x|^2 - 2N\alpha - C\alpha|x| - C \geq \alpha$

for $x \in B_\delta \setminus B_{\delta/2}$: $d_1\alpha^2 + d_2\alpha + d_3 \geq 0$



This can be done if $|x|$ is not close to 0, e.g. $|x| > \delta/2$ (that's why we take the domain A to be a ring!)

Then w is a subsolution in A if

(cond 1) $$\frac{\varepsilon_2}{\varepsilon_1} \geq \frac{(1+cT)e^{\alpha\delta^2}}{\alpha}$$

(b) $$w(0,x) = u(0,x) \underbrace{- \varepsilon_1 T + \varepsilon_2\left[e^{-\alpha|x|^2} - e^{-\alpha\delta^2}\right]}_{\leq 0 \text{ for } x \in B_\delta \setminus \overline{B_r}} \quad <0$$

(cond 2) $$\frac{\varepsilon_2}{\varepsilon_1} \leq \frac{T}{e^{-\alpha r^2} - e^{-\alpha\delta^2}}$$

If we choose r very close to δ, then

RHS of (cond 1) $<$ RHS of (cond 2)

(c) Boundary consists of 2 pieces: $\partial B_\delta, \partial B_r$

- Clear that $w(t, \partial B_\delta) = u(t,\partial B_\delta) + \varepsilon_1(t-T)$ $\forall t \in [0,T]$ $\leq 0 + \varepsilon_1(t-T) \leq 0$

- $$w(t,\partial B_r) = \underset{\overset{\wedge}{0}}{u(t,\partial B_r)} + \varepsilon_1(t-T) + \varepsilon_2\left(e^{-\alpha r^2} - e^{-\alpha\delta^2}\right)$$

It is enough to take small $\varepsilon_2 > 0$, e.g.

$$\varepsilon_2 < \frac{-\sup_{t\in[0,\infty)} u(t,\partial B_r) \neq 0}{e^{-\alpha r^2} - e^{-\alpha\delta^2}}$$

↳ Then $w(t,\partial B_r) < 0$. ■

Next time we will finish the proof of the weak MP for Neumann / Robin bc.