Looking for exact mixing velocities in miscible displacement: two-tube approach

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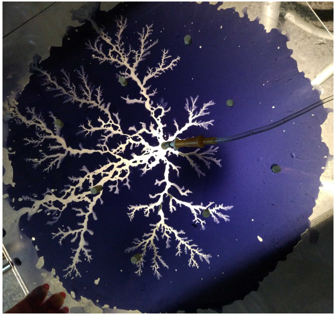
Based on discussions with Sergey Tikhomirov, Yalchin Efendiev, Aleksandr Enin, Fedor Bakharev, Dmitry Pavlov, Sergey Matveenko, Nikita Rastegaev etc

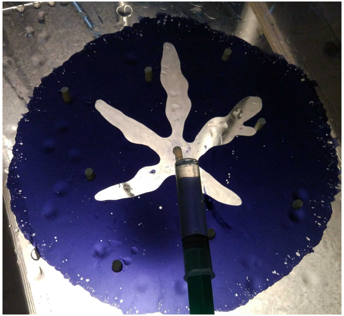
Workshop «Nonlinear PDEs and Modelling» at Chebyshev Lab organized by Gabriel Lame Chair Jean-Michel Roquejoffre

3 December 2021

Story: our experiments with Hele-Shaw cell

Beginning of the project with GazpromNeft





But...

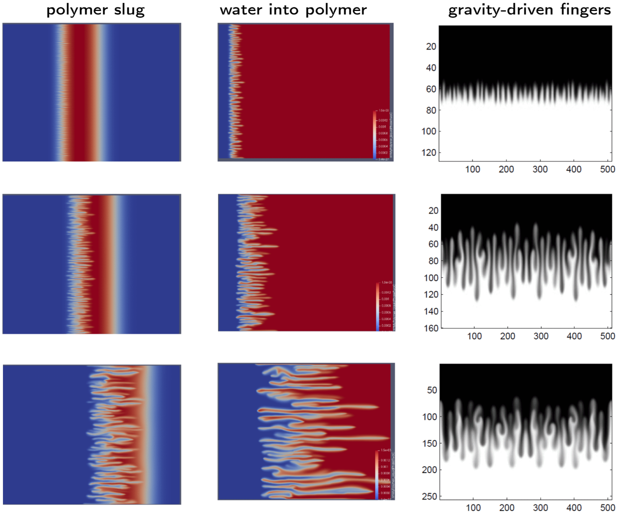
- we investigate the displacement of fluids in porous medium
- “similar” instabilities occur

Story: our numerical experiments in porous media

Video of fingers formation

Motivation and main question

- water flooding;
- chemical flooding
- cause problems for oil recovery



Main question of interest:
rigorous bounds on velocities of mixing zone propagation

Peaceman vs TFE model

Basic equations:

(conservation of species) $c_t + \vec{u} \cdot \nabla c = \varepsilon \cdot \Delta c,$

(incompressibility condition) $\mathrm{div}(\vec{u}) = 0.$

Peaceman model:	TFE model:
$\vec{u} = -m(c)\nabla p$	$\vec{u} = (u^1, u^2), \qquad u^1 = -\frac{m(c)}{\overline{m}(x,t)}$ $\overline{m}(x,t)$ is the average mobility over the transverse direction to the flow
• strong non-locality due to the presence of pressure • ¿no rigorous result?	• weaker non-locality (depends only on vertical line, not the whole space) • Felix Otto (2006) rigorously proved the linear bounds on velocities

Hypothesis: models show similar behavior when $p(x,y,t) \approx p(x,t)$ (“moderate” times)

TFE estimates

- $c \in [0,1]$, thus
$$\frac{m(c)}{m(0)} \leqslant |u^1| = \frac{m(c)}{\overline{m}(x,t)} \leqslant \frac{m(c)}{m(1)}$$
- Consider two 1-dimensional equations:
$$c_t^{\max} + \frac{m(c^{\max})}{m(0)} c_x^{\max} = \varepsilon c_{xx}^{\max}$$
$$c_t^{\min} + \frac{m(c^{\min})}{m(0)} c_x^{\min} = \varepsilon c_{xx}^{\min}$$
Their solutions are super and subsolutions for 2-dim TFE model.
- In fact, they correspond to travelling waves connecting states 0 and 1 with velocities
$$v^f = \frac{\int_0^1 m(c)\,dc}{m(1)} \qquad v^b = \frac{\int_0^1 m(c)\,dc}{m(0)}$$

What is known for Peaceman model?

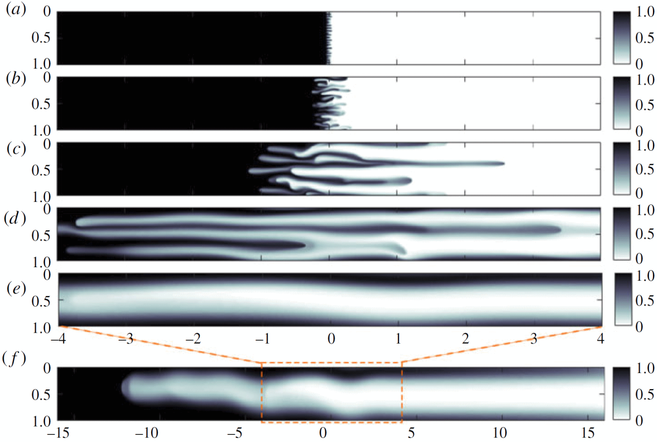
Detailed numerical analysis by Nijjer, Hewit, Neufield (Cambridge), 2018

The dynamics of miscible viscous fingering from onset to shutdown

$h(t)$ — mixing zone width

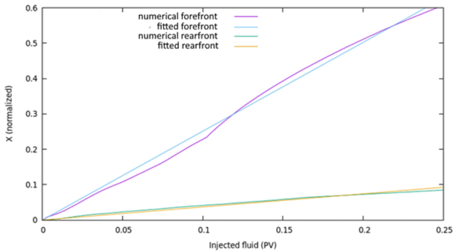
Three flow regimes:

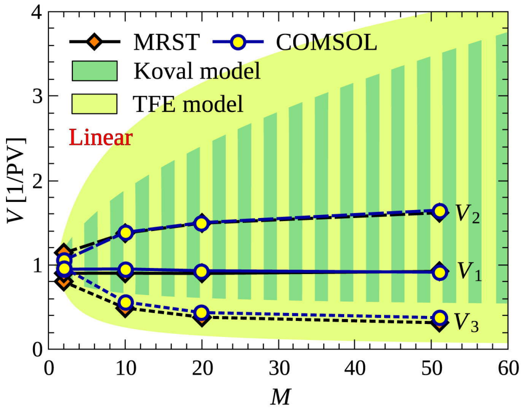
1. early-time
   $h \sim C_1\sqrt{t}$
2. intermediate-time
   $h \sim C_2 t$
3. late-time
   $h \sim C_3$



Peaceman model: numerical experiments

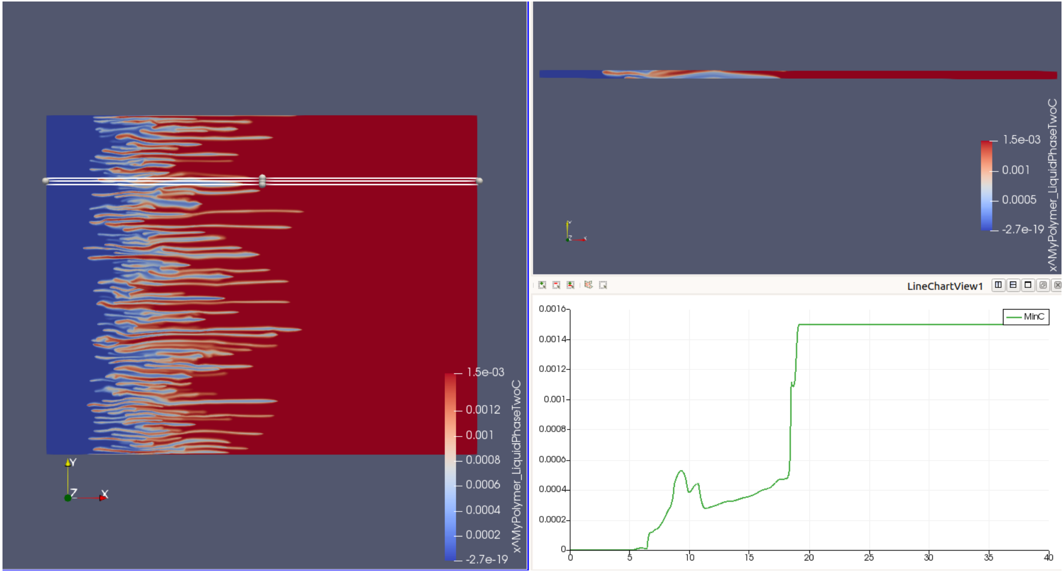
- linear growth of the “fastest finger”
- comparison to TFE model
- TFE too pessimistic



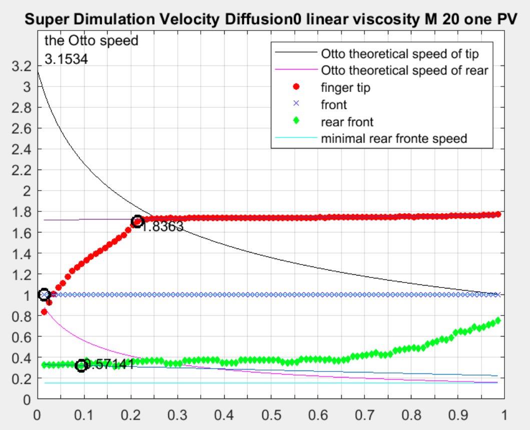


Experiments by Dmitry Pavlov (DuMuX), Ivan Starkov (COMSOL) and Sergey Matveenko (MRST)

Peaceman model: numerical experiments (show video)



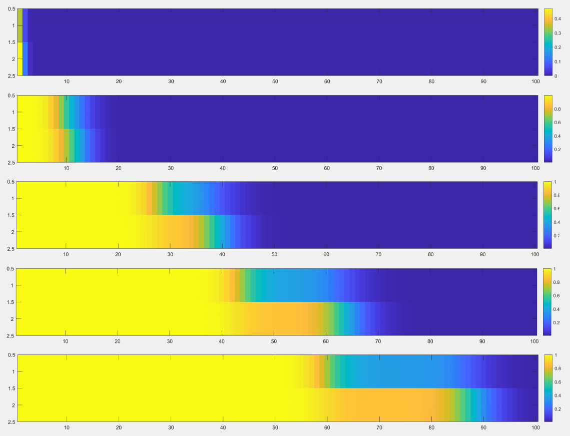
Peaceman model: velocities at the tip of the finger



Idea: looks like a travelling wave connecting some intermediate concentration c^* and 1

Numerical experiments: 2 tubes “speaking” to each other

Finite-volume scheme, simulations by Yalchin Efendiev

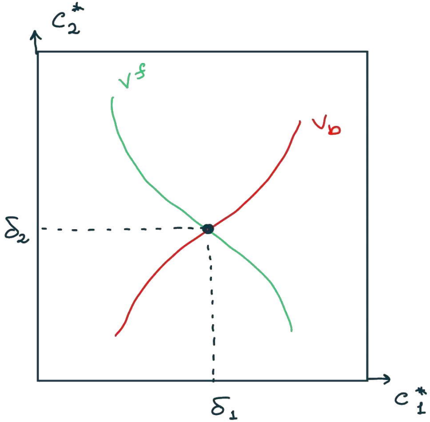


Result of experiments — seems that there are two travelling waves running at different speeds and exists some intermediate concentration c^*

Two-tube approach: general scheme

Aim: to find intermediate concentrations δ_1 and δ_2 such that there exists:

- a travelling wave between $(0, 0)$ and (δ_1, δ_2) with some velocity v^b
- a travelling wave between (δ_1, δ_2) and $(1, 1)$ with some velocity v^f



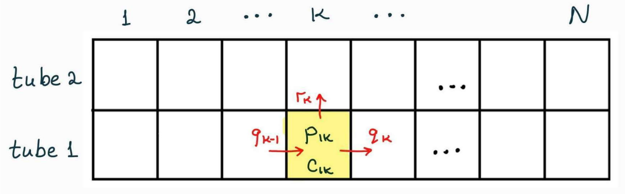
Two-tube approach: discrete and continuous settings

Discrete case

- system of 2 ODEs and 1 algebraic equation
- Travelling wave solutions: system of ODEs with delay

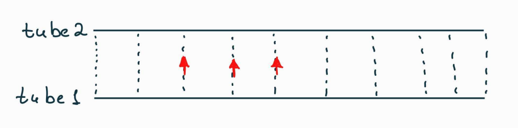
Unknowns:

- $c_{1k}(t), c_{2k}(t)$ — concentrations
- p_{1k}, p_{2k} — pressures
- $q_k(t), r_k(t)$ — velocities

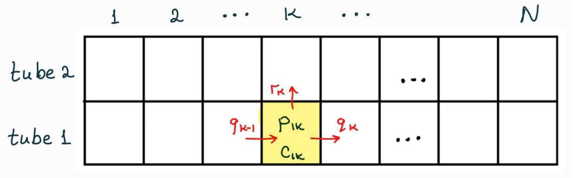


Continuous case

- two coupled conservation laws
- system of 4 ODEs for travelling wave solutions



Two-tube approach: discrete setting



Unknowns:

- $c_{1k}(t), c_{2k}(t)$ — concentrations
- p_{1k}, p_{2k} — pressures
- $q_k(t), r_k(t)$ — velocities

(pretending to be physicists — all terms up to $\pm$)

(conservation of species)
$$\frac{d}{dt}c_{1k} = -q_k c_{1k} + q_{k-1}c_{1,k-1} - r_k c_{1k}$$
$$\frac{d}{dt}c_{2k} = -(2-q_k)c_{2k} + (2-q_{k-1})c_{2,k-1} + r_k c_{1k}$$

(incompressibility condition)
$$q_k + r_k - q_{k-1} = 0$$

(Darcy's law)
$$q_k = -\frac{p_{1k} - p_{1,k+1}}{\mu(c_{1k}, c_{1,k+1})}$$

(pressure law)
$$(p_{2k} - p_{1k}) + (p_{1k} - p_{1,k-1}) + \\ + (p_{1,k-1} - p_{2,k-1}) + (p_{2,k-1} - p_{2k}) = 0$$

Looking for travelling wave solutions in discrete setting

Looking for travelling waves solutions:

$$c_{1k}(t) = f_1(k - v^b t) = f_1(\xi), \qquad c_{2k}(t) = f_2(k - v^b t) = f_2(\xi), \qquad q_k = g(k - v^b t).$$

We get system of ODEs (backward-forward)

$$\begin{aligned}
-v^b f_1' &= F_1(f_1(\xi), f_1(\xi - 1), g(\xi), g(\xi - 1)), \\
-v^b f_2' &= F_2(f_2(\xi), f_2(\xi - 1), g(\xi), g(\xi - 1)), \\
0 &= F_3(g(\xi), g(\xi - 1), g(\xi + 1), f_{1,2}(\xi), f_{1,2}(\xi - 1), f_{1,2}(\xi + 1))
\end{aligned}$$

...too difficult....

Two-tube approach: derivation continuous from discrete

Two coupled conservation laws

$$\begin{aligned} \frac{\partial c_1}{\partial t} + q\frac{\partial c_1}{\partial x} &= 0, \\ \frac{\partial c_2}{\partial t} + (2-q)\frac{\partial c_1}{\partial x} - (c_2 - c_1)\frac{\partial q}{\partial x} &= 0. \end{aligned} \tag{1}$$

For TFE model velocity u is defined as follows

$$q = \frac{2\mu(c_2, c_2)}{\mu(c_1, c_1) + \mu(c_2, c_2)}$$

In general, $q = G(c_1, c_2)$.
Here $\mu(c_1, c_2)$ is the viscosity function — some mean between $\mu(c_1)$ and $\mu(c_2)$.

¿Problem? The system (1) is not in the divergence form usual conservation laws

$$U_t + (F(U))_x = 0$$

Looking for solution being two travelling waves

Add simplest diffusion terms:

$$\frac{\partial c_1}{\partial t} + q\frac{\partial c_1}{\partial x} = \varepsilon_1 \frac{\partial^2 c_1}{\partial x^2},$$

$$\frac{\partial c_2}{\partial t} + (2-q)\frac{\partial c_1}{\partial x} - (c_2 - c_1)\frac{\partial q}{\partial x} = \varepsilon_2 \frac{\partial^2 c_2}{\partial x^2}.$$

We are looking for travelling wave solutions for the back front:

$$c_1(x,t) = f_1(x - v^b t) = f_1(\xi), \qquad c_2(x,t) = f_2(x - v^b t) = f_2(\xi).$$

We get (because of non-divergence form we cannot easily get rid of 1 derivative!):

$$-v_b \frac{\partial f_1}{\partial \xi} + G(f_1, f_2)\cdot \frac{\partial f_1}{\partial \xi} = \varepsilon_1 \frac{\partial^2 f_1}{\partial \xi^2},$$

$$-v_b \frac{\partial f_2}{\partial \xi} + (2 - G(f_1, f_2))\cdot \frac{\partial f_2}{\partial \xi} - (f_2 - f_1)\left(G_1' \frac{\partial f_1}{\partial \xi} + G_2' \frac{\partial f_2}{\partial \xi}\right) = \varepsilon_2 \frac{\partial^2 f_2}{\partial \xi^2}.$$

Dynamical system for a travelling wave solution

Let us introduce $g_1 = \frac{\partial f_1}{\partial \xi}$ and $g_2 = \frac{\partial f_2}{\partial \xi}$. Using functions g_1 and g_2 we get a system of 4 ODEs:

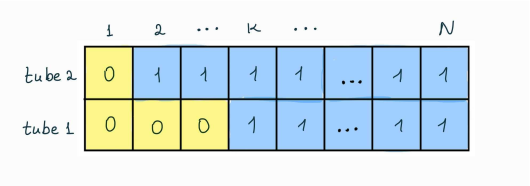
$$
\begin{aligned}
\frac{\partial f_1}{\partial \xi} &= g_1, \\
\frac{\partial g_1}{\partial \xi} &= \varepsilon_1^{-1}\big(-v_b + G(f_1, f_2)\big) g_1, \\
\frac{\partial f_2}{\partial \xi} &= g_2, \\
\frac{\partial g_2}{\partial \xi} &= \varepsilon_2^{-1}(-v_b + 2 - G(f_1, f_2) - (f_2 - f_1) G_2') \cdot g_2 - \varepsilon_2^{-1} G_1'(f_2 - f_1) \cdot g_1.
\end{aligned}
$$

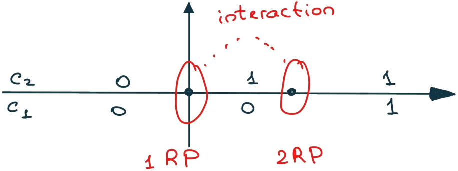
Question: how to determine (δ_1, δ_2) for any v^b such that there exists a trajectory

$$
\begin{aligned}
&f_1(-\infty) = 0, \qquad f_2(-\infty) = 0, \qquad f_1(\infty) = \delta_1, \qquad f_2(\infty) = \delta_2, \\
&g_1(-\infty) = 0, \qquad g_2(-\infty) = 0, \qquad g_1(\infty) = 0, \qquad g_2(\infty) = 0?
\end{aligned}
$$

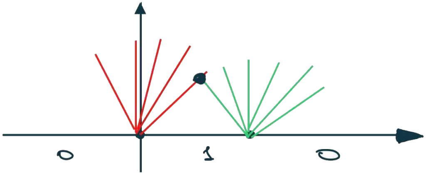
Two coupled conservation laws: another idea

Consider two Riemann problems that interact with each other!





(hahaha — welcome back to slugs — very similar)



Thank you for your attention!

Any comments? Ideas? Questions?

List of things to do (besides checking all arithmetic and $\pm$):

1. careful numerical modelling of the 2-dimensional TFE model and comparison to Peaceman model. Are they really close up to some time?
2. experiment with two-tube TFE numerical model: are there really two travelling waves? Analyze the pairs of intermediate concentrations (δ_1, δ_2) for different viscosity functions (viscosity ratios).
3. analysis of dynamical system of 4-th order for a travelling wave solution: determine functions $v^b \to (\delta_1, \delta_2)$ and $v^f \to (\delta_1, \delta_2)$
4. try approach with two interacting Riemann problems
5. ...

Any volunteer?