Asymptotics of eigenvalues for some integro-differential operators

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Problem statement

We are interested in sharp asymptotics of eigenvalues of integral operators with kernels

$$G_1(s,t) = G(s,t) - h_1(s)h_1(t), \tag{1}$$
$$G_2(s,t) = G(s,t) - h_2(s)h_2(t), \tag{2}$$
$$G_3(s,t) = G(s,t) - h_1(s)h_1(t) - h_2(s)h_2(t), \tag{3}$$

where $G(s,t) = \min(s,t) - st$ is the Green function of boundary value problem $Lu := -u'' = \lambda u$, $u(0) = u(1) = 0$, and

$$h_1(t) = \phi(\Phi^{-1}(t)), \qquad h_2(t) = \phi(\Phi^{-1}(t))\frac{\Phi^{-1}(t)}{\sqrt{2}},$$

where

$$\phi(s) = \frac{1}{\sqrt{2\pi}}\exp\Big(-\frac{s^2}{2}\Big), \quad \Phi(t) = \int\limits_{-\infty}^{t} \phi(s)\,ds$$

are the probability density and the distribution function of standard normal distrubition, respectively.

Motivation

Let $X(t)$, $t \in [0,1]$, be a random Gaussian process. It is of statistical interest the behaviour of $\mathbb{P}(\|X\|_2 < \varepsilon)$ when $\varepsilon \to 0$.

The Wenbo Li principle, 1992

Let $X(t)$, $\tilde{X}(t)$ be two Gaussian processes with zero mean and covariance functions $G(s,t)$ and $\tilde{G}(s,t)$. Let λ_k and $\tilde{\lambda}_k$ be positive eigenvalues of integral operators with kernels $G(s,t)$ and $\tilde{G}(s,t)$, respectively. If $\prod \tilde{\lambda}_k/\lambda_k < \infty$ then

$$\mathbb{P}\Big\{\|X\|_2 < \varepsilon\Big\} \sim \mathbb{P}\Big\{\|\tilde{X}\|_2 < \varepsilon\Big\} \cdot \left(\prod_{k=1}^{\infty} \frac{\tilde{\lambda}_k}{\lambda_k}\right)^{1/2}, \qquad \varepsilon \to 0. \quad (4)$$

Gaussian processes with covariance functions (**??**)-(**??**) firstly appeared in the work of Kac, Kiefer, Wolfowitz (1955).
Asymptotics we are looking for should be «sharp» in the sense that $\prod \tilde{\lambda}_k/\lambda_k < \infty$,where $\tilde{\lambda}_k$ are approximated values of eigenvalues λ_k.

Motivation

A.I. Nazarov considered a family of processes

$$X_{\varphi,\alpha}(t) = X(t) - \alpha\psi(t) \int X(s)\varphi(s)\,ds,$$

where $\varphi \in L_{1,loc}$, $\psi(t) = \int G(s,t)\varphi(s)\,ds$, and

$$q := \int \psi(s)\varphi(s)\,ds = \int\int G(s,t)\varphi(s)\varphi(t)\,ds\,dt < +\infty.$$

Their covariance functions

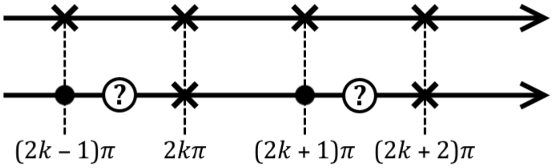
$$G_{\varphi,\alpha}(s,t) = G(s,t) + Q\psi(s)\psi(t), \quad Q = q\alpha^2 - 2\alpha.$$

Theorem (A.I. Nazarov, 2009)

1. (non-critical case) If $\alpha \neq 1/q$, then $\prod_{k=1}^{\infty} \frac{\lambda_k}{\tilde{\lambda}_k} < +\infty$.
2. (critical case) If $\alpha = 1/q$, $\varphi \in L_2$, then $\prod_{k=1}^{\infty} \frac{\lambda_{k+1}}{\tilde{\lambda}_k} < +\infty$.

Equation for eigenvalues

Cases (**??**) and (**??**) are critical. But corresponding $\varphi(s) \notin L_2$.
Therefore the previous theorem is *not* valid for this case.
Perturbation h_1 is even, so only odd eigenvalues «move».



The corresponding integro-differential eigenvalue problem is:

$$\begin{cases} -\lambda u''(t) = u(t) + h_1''(t) \int\limits_0^1 u(s) h_1(s)\, ds, \quad t \in (0,1), \\ u(0) = u(1) = 0. \end{cases} \tag{5}$$

Using standard methods we get equation for $\omega_{2k-1}^{(1)} := \left(\lambda_{2k-1}^{(1)}\right)^{-1/2}$:

$$D_1(\omega) := \frac{2\sin\left(\frac{\omega}{2}\right)}{\omega} \cdot \mathcal{C}_1^2 + \frac{\cos\left(\frac{\omega}{2}\right)}{\omega^2} - \frac{4\cos\left(\frac{\omega}{2}\right)}{\omega} \cdot \mathcal{I}_1 = 0, \qquad (6)$$

where

$$\mathcal{C}_1 = \int_0^{\frac{1}{2}} \Phi^{-1}(t)\cos(\omega t)\,dt,$$

$$\mathcal{I}_1 = \int_0^{\frac{1}{2}}\int_0^{t} \Phi^{-1}(t)\Phi^{-1}(s)\sin(\omega t)\cos(\omega s)\,ds\,dt.$$

Analogously, we obtain equation on $\omega_{2k}^{(2)} := \left(\lambda_{2k}^{(2)}\right)^{-1/2}$ with integrals $\mathcal{C}_2$, $\mathcal{I}_2$ of the same type as $\mathcal{C}_1$, $\mathcal{I}_1$.

Slowly varying functions = SVF

Definition

Function $F(t)$ is called SVF at infinity, if it doesn't change sign on some $[A, \infty)$, $A > 0$, and for any $\lambda > 0$

$$\lim_{t\to\infty} \frac{F(\lambda t)}{F(t)} = 1.$$

Function $F(t)$ is called SVF at zero, if $F(1/t)$ is SVF at infinity. For example, $\ln^{\alpha}(t)$, $\alpha \in \mathbb{R}$.

Note: $\Phi^{-1}(t)$ has the following properties:

- $F_0(t) := \Phi^{-1}(t)$, $F_{n+1}(t) := tF_n'(t)$, $n \geq 0$, are SVF at zero.
- $\Phi^{-1}\left(\frac{1}{2}\right) = 0$.

Note: for any SVF at zero: $tF'(t) = o(F(t))$ when $t \to 0$.
So $\forall n \geq 0 \quad F_{n+1}(t) = o(F_n(t))$.

Asymptotics of integrals

We obtain asymptotic expansion of integrals with SVF $F(t)$, satisfying above mentioned two properties.

Theorem (cos)

When $\omega \to \infty$:

$$\mathcal{C} := \int\limits_0^{\frac{1}{2}} F(t)\cos(\omega t)\,dt = \sum_{k=1}^{N} c_k^{cos}\frac{F_k(\frac{1}{\omega})}{\omega} + R_N^{cos}, \tag{7}$$

where

$$|R_N^{cos}| \leq C(F,N)\cdot\frac{\left|F_{N+1}(\frac{1}{\omega})\right|}{\omega}.$$

Asymptotics of integrals

Theorem (sincos)

$$\mathcal{I} := \int_0^{\frac{1}{2}}\int_0^{\tau} F(t)F(\tau)\sin(\omega\tau)\cos(\omega t)\,dt\,d\tau =$$

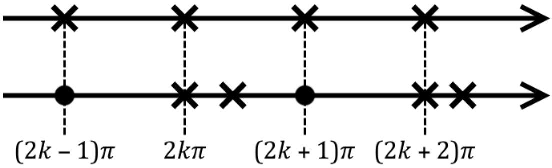
$$= \frac{1}{2\omega}\int_0^{\frac{1}{2}} F^2(t)\,dt + \sum_{n=2}^{N}\sum_{\substack{k+m=n\\ k,m\geq 1}} a_{k,m}\frac{F_k(\frac{1}{\omega})F_m(\frac{1}{\omega})}{\omega^2} + R_N^{sc},$$

$$\text{where} \quad |R_N^{sc}| \leq C(F,N)\sum_{\substack{i+j=N+1\\ i,j\geq 1}}\frac{|F_i(\frac{1}{\omega})F_j(\frac{1}{\omega})|}{\omega^2}. \qquad (8)$$

Asymptotic of eigenvalues

Finally, we obtain asymptotics of eigenvalues for (**??**):

$$\omega_{2k-1}^{(1)} = 2\pi k + \frac{\pi}{\ln(k)} + O\left(\frac{\ln(\ln(k))}{\ln^2(k)}\right).$$



Analogously for (**??**):

$$\omega_{2k}^{(2)} = \pi(2k+1) + O\left(\frac{1}{\ln^2(k)}\right).$$

Small deviation probability asymptotics

Sytaya, 1974: complete description, but in implicit way
Zolotarev, Ibragimov, Nazarov, Nikitin...
Dunker, Linde, Lifshits, 1998: small ball asymptotics under some general conditions on λ_k. F.e.:

$$\lambda_k = \Big(\vartheta(k+\delta)\Big)^{-2}, \quad k \to \infty.$$

So in case (**??**) using Li's principle for $\gamma_k = \Big[(2k+1)\pi/2\Big]^{-2}$ we get:

$$\mathbb{P}\Big\{\|X^{(2)}\|_2 < \varepsilon\Big\} \sim \frac{2\sqrt{2}}{\pi^{3/2}}\,\varepsilon^{-1}\exp\Big(-\frac{1}{8\varepsilon^2}\Big), \quad \varepsilon \to 0.$$

«The distortion constant» from Li's principle can be found using theorems from complex analysis.

Small deviation probability asymptotics

The situation is different in case (**??**). Here we have

$$\lambda_k = \Big(\pi\Big(k + \frac{1}{2} + \frac{1}{2\ln(k)}\Big)\Big)^{-2}, \quad k \to \infty.$$

Using DLL theorem we calculate small ball asymptotics in case

$$\lambda_k = \Big(\vartheta\Big(k + \delta + F(k)\Big)\Big)^{-2}, \quad k \to \infty,$$

where $F(k)$ is SVF and tends to 0 at infinity.
Final asymptotics in case (**??**) and (**??**) is «up to constant»:

$$\mathbb{P}\Big\{\|X^{(1)}\| < \varepsilon\Big\} \sim C \cdot \varepsilon^{-1} \cdot \ln^{\frac{1}{2}}\left(\frac{1}{\varepsilon}\right) \cdot \exp\Big(-\frac{1}{8\varepsilon^2}\Big), \qquad \varepsilon \to 0.$$

Thank you for your attention!