Toy model of viscous fingering: two-tube approach

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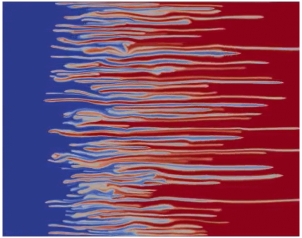
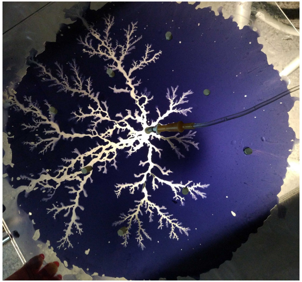
What is viscous fingering?

- Saffmann-Taylor instability (1958)
- Peaceman model: one-phase miscible displacement in porous media

$$
\begin{aligned}
c_t + \operatorname{div}(uc) &= \varepsilon \Delta c, && \text{(conservation of species)}\\
\operatorname{div}(u) &= 0, && \text{(incompressibility condition)}\\
u &= -\frac{\nabla p}{\mu(c)}. && \text{(Darcy's law)}
\end{aligned}
$$

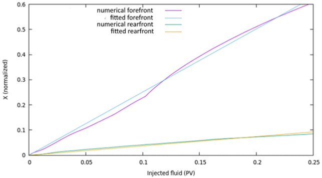
- main source of instability (and non-linearity): viscosity $\mu(c)$; mobility $m(c) = 1/\mu(c)$
- related phenomena: Hele-Shaw cell, gravity-driven fingers
- Main question of interest: rigorous bounds on velocities of mixing zone propagation





Peaceman model: numerical experiments

- linear growth of the "fastest finger"

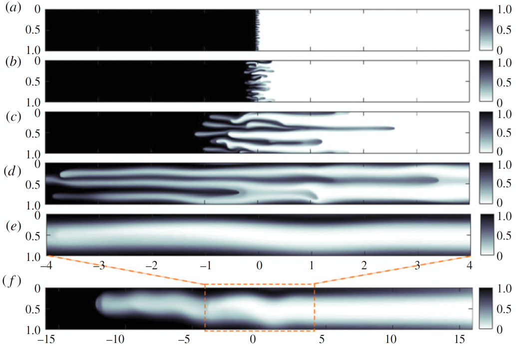


The dynamics of miscible viscous fingering from onset to shutdown

- Nijjer, Hewit, Neufield (Cambridge), 2018: $h(t)$ — mixing zone width

  Three flow regimes:

  1. early-time
     $h \sim C_1\sqrt{t}$
  2. intermediate-time
     $h \sim C_2 t$
  3. late-time
     $h \sim C_3$



Peaceman vs Transeverse Flow Equilibrium (TFE) model

(conservation of species) $\quad c_t + u \cdot \nabla c = \varepsilon \cdot \Delta c,$

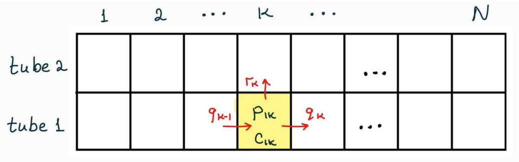
(incompressibility condition) $\quad \mathrm{div}(u) = 0.$

Peaceman model:	TFE model:
$u = -m(c)\nabla p$	$u = (u^1, u^2), \qquad u^1 = \frac{m(c)}{\overline{m}(x,t)}$ $\overline{m}(x,t)$ is the average mobility over the transverse direction to the flow
• strong non-locality due to the presence of pressure • ¿no rigorous result?	• weaker non-locality (depends only on vertical line, not the whole space) • Felix Otto (2006) rigorously proved the linear bounds on velocities

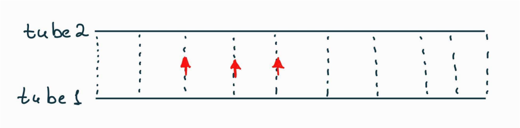
Hypothesis: models show similar behavior when $p(x, y, t) \approx p(x, t)$ (“moderate” times)

Toy model of viscous fingering: two tubes

Discrete setting



Continuous setting

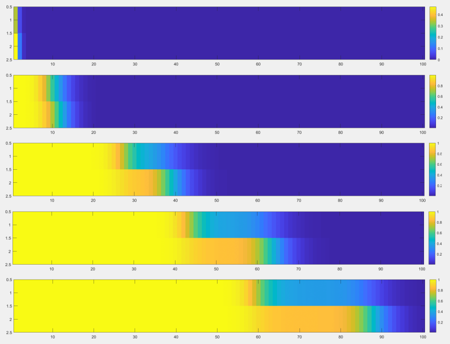


- two tubes that "speak with each other"
- initially slightly more fluid in one tube than in the other

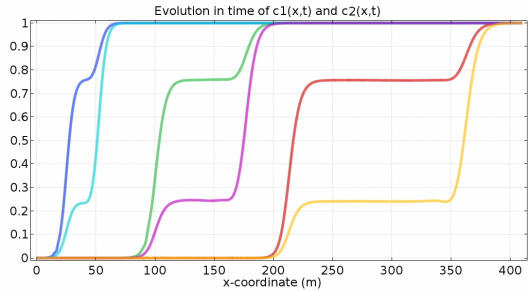
Question: how the fingers will evolve?

Toy model of viscous fingering: preview of the main idea

Discrete setting



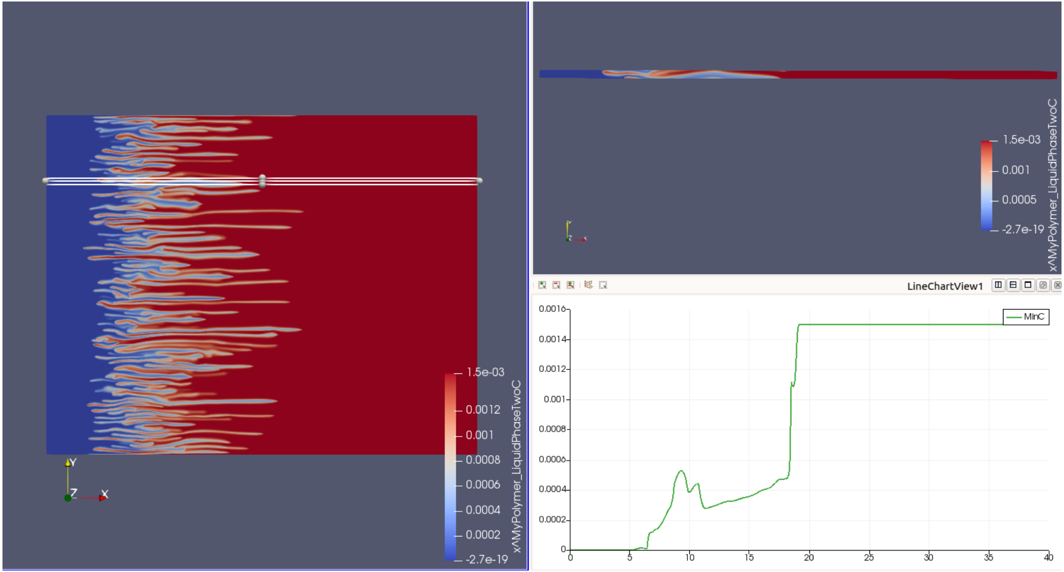
Continuous setting



Result of experiments: cascade of two travelling waves (TW)

$$(0,0) \xrightarrow{\mathrm{TW}_1} (c_1^*, c_2^*) \xrightarrow{\mathrm{TW}_2} (1,1)$$

Peaceman model: numerical experiments (show video)



Toy model of viscous fingering

14/04/2022
Notes from Yulia Petrova

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§0. Intro.

System of eqs:

I. (conservation of species): $c_t + \operatorname{div}(uc) = \varepsilon \Delta c$

II. (incompressibility cond.): $\operatorname{div}(u) = 0$

For velocity "u" we consider 2 models:

IIIa. (Darcy's law) $u = -m(c)\nabla p$

IIIb. (TFE model) $u = \dfrac{m(c)}{\overline{m(c)}}$, where $\overline{m(c)}(x,t) = \int_0^1 m(c(x,y,t))\,dy$

<u>Note</u>: I + II + IIIa is usually called Peaceman model.

<u>Here</u>: $c = c(x,y,t) \in [0,1]$ – concentration of species

$m(c)$ – mobility (typical $m(c) = e^{-Rc}\frac{1}{\mu_1}$

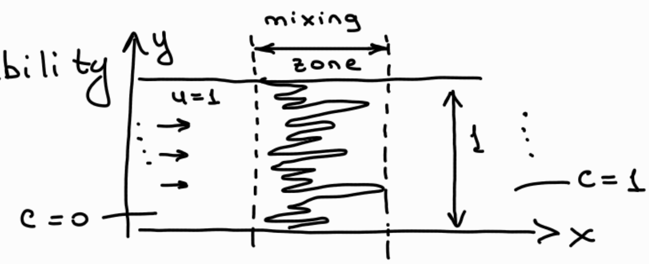
$x \in \mathbb{R}$, $y \in [0,1]$, $t \in \mathbb{R}_+$

$p = p(x,y,t)$ – pressure; $u = u(x,y,t)$ – velocity

$$\boxed{M = \frac{\mu_2}{\mu_2} = e^{-R} \text{ ratio of viscosities}}$$

$$\boxed{\varepsilon = \frac{1}{Pe} \to 0}$$

Observe: instability



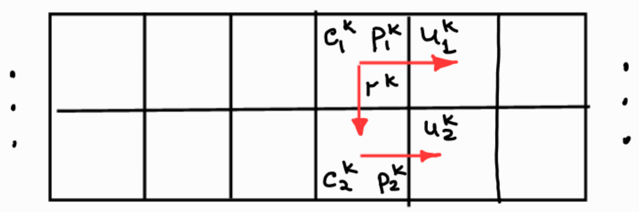
Aim: rigorously explain linear growth of the mixing zone

Two-tubes approach.

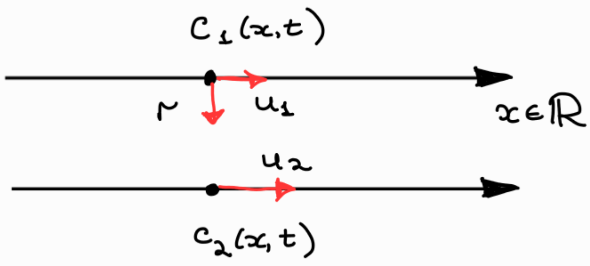
§1. Problem statement. Derivation of the model.

Discrete (in x)

$k \in \mathbb{Z}$



Continuous (in x)



Uknowns:

- $c_{1,2}^k(t)$ – concentrations in 1,2 tube $\longleftrightarrow$ $c_{1,2}(x,t)$
- $u_{1,2}^k(t)$ – velocity in 1,2 tube $\longleftrightarrow$ $u_{1,2}(x,t)$
- $r^k(t)$ – velocity between tubes $\longleftrightarrow$ $r(x,t)$
- $p_{1,2}^k(t)$ – pressure in 1,2 tube $\longleftrightarrow$ $p_{1,2}(x,t)$

$$\mathrm{I.}\quad \frac{d}{dt} c_1^k = u_1^{k-1} c_1^{k-1} - u_1^k c_1^k \boxed{- r^k c_1^k}$$

$$\frac{d}{dt} c_2^k = u_2^{k-1} c_2^{k-1} - u_2^k c_2^k \boxed{+ r^k c_2^k}$$

$$\mathrm{II.}\quad \begin{cases} u_1^k - u_1^{k-1} + r^k = 0 \\ u_2^k - u_2^{k-1} - r^k = 0 \end{cases} \Rightarrow \quad \begin{aligned} u_1^k + u_2^k = u_1^{k-1} + u_2^{k-1} = 2 \\ r^k = u_1^{k-1} - u_1^k \end{aligned}$$

$$\mathrm{III\,b.}\quad u_1(c_1, c_2) = \frac{2m(c_1)}{m(c_1)+m(c_2)} = 2 - u_2$$

width of cells is $\delta \to 0$

$\Longrightarrow$

after change of variables

$$\mathrm{I.}\quad \partial_t c_1 = -\partial_x(c_1 u_1) + c_1 \partial_x u_1$$
$$\partial_t c_2 = -\partial_x(c_2 u_2) - c_1 \partial_x u_1$$

$$\mathrm{II} + \mathrm{III\,b}:\quad u_1 = \frac{2m(c_1)}{m(c_1)+m(c_2)}$$

$$u_2 = 2 - u_1$$

Taking into account that fluid may flow not only from 1st tube to 2nd, but vice versa as well, we get:

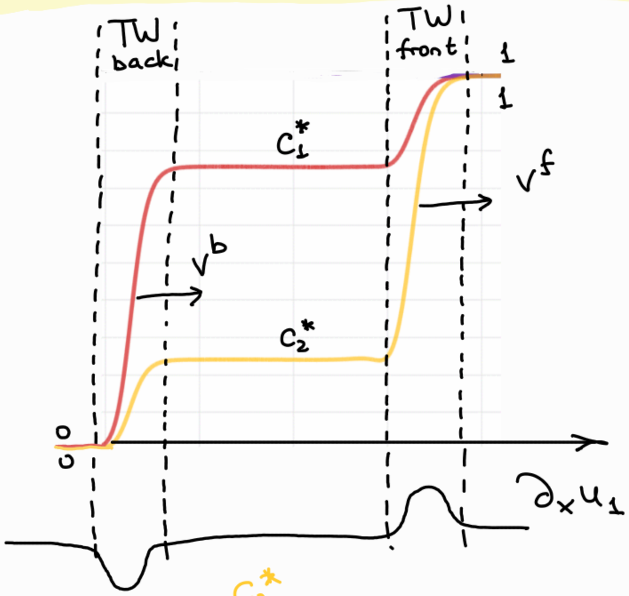
$$\boxed{\begin{aligned} \partial_t c_1 &= -\partial_x(c_1 u_1) - \boxed{(-1)^{1,2} c_{1,2}\, \partial_x u_1} + \varepsilon \partial_{xx} c_1 \\ \partial_t c_2 &= -\partial_x(c_2 u_2) + \boxed{(-1)^{1,2} c_{1,2}\, \partial_x u_1} + \varepsilon \partial_{xx} c_2 \end{aligned}} \qquad (2)$$

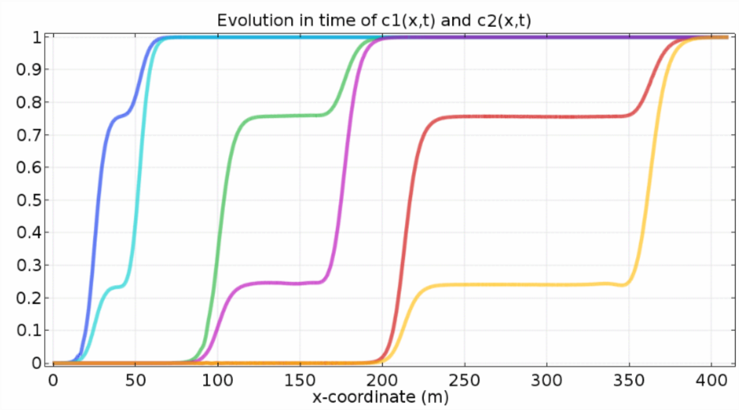
$$= \begin{cases} -c_1 \partial_x u_1 & \text{if } \partial_x u_1 < 0 \\ c_2 \partial_x u_1 & \text{if } \partial_x u_2 > 0 \end{cases}$$

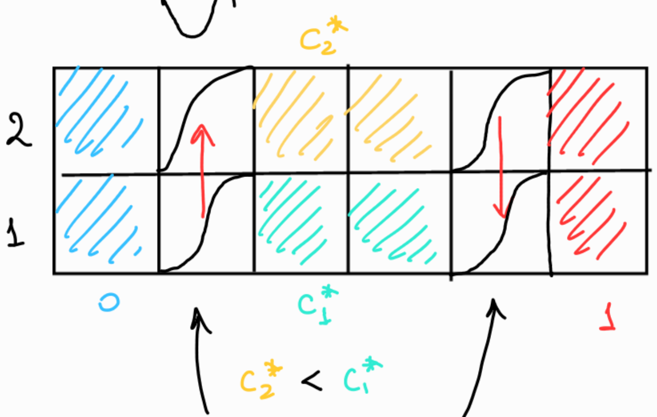
For simplicity consider TFE: $u_1 = \dfrac{2m(c_1)}{m(c_1)+m(c_2)} = 2 - u_2$

<u>NB</u>: consider Darcy's law here would be interesting and could even be easier because the system is more physically meaningful.

§2. Numerical solution: cascade of travelling waves (TW)







<u>3 zones are formed</u>:

- in both tubes concentrations 0
- in both tubes concentrations 1
- $\exists\ (c_1^*, c_2^*)$:
  in 1st tube concentration is c_1^*
  in 2nd tube concentration is c_2^*

And 2 transition zones between $(0,0)$ and (c_1^*, c_2^*); (c_1^*, c_2^*) and $(1,1)$

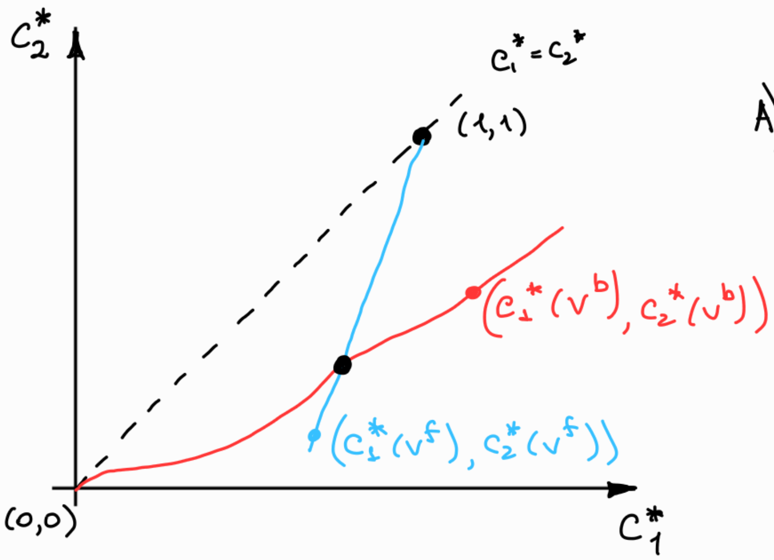
TW back : travelling wave with velocity $v^b \in (0,1)$ between $(0,0)$ and (c_1^*, c_2^*)

TW front : travelling wave with velocity $v^f \in (1,+\infty)$ between (c_1^*, c_2^*) and $(1,1)$

So we would like to prove the following structure:

$\exists\, (c_1^*, c_2^*)$: $\quad (0,0) \xrightarrow[\text{back}]{TW} (c_1^*, c_2^*) \xrightarrow[\text{front}]{TW} (1,1)$

§3. General scheme of proof of the existence of cascade of travelling waves.



Hypothesis :

A) $\forall\, v^b \in (0,1)$ $\exists!\, (c_1^*(v^b), c_2^*(v^b))$ there exists a travelling wave

$$(0,0) \xrightarrow{TW} (c_1^*(v^b), c_2^*(v^b))$$

Corresponds to red point

All such points $(c_1^*(v^b), c_2^*(v^b))$ form a red curve

B) $\forall\, v^f \in (1,+\infty)$ $\exists!\, (c_1^*(v^f), c_2^*(v^f))$ there exists a travelling wave

$$(c_1^*(v^f), c_2^*(v^f)) \xrightarrow{TW} (1,1)$$

Corresponds to blue point.

All such points $(c_1^*(v^f), c_2^*(v^f))$ form a blue curve.

If red and blue curves intersect in 1 point – – this will give us intermediate concentrations (c_1^*, c_2^*) and velocities of travelling waves v^b and v^f.

No proofs yet...

P.S. If system 2 was a system of conservation laws, these two curves would be just Hugoniot loci defined by Rankine-Hugoniot condition.

§4. Travelling wave dynamical system

Let's look for solution being a travelling wave

$$c_{1,2} = c_{1,2}\left(\frac{x - v^b t}{\varepsilon}\right), \quad v^b \text{ – velocity}, \quad \text{such that}$$

$$c_{1,2}(-\infty) = 0 \quad \text{and} \quad c_{1,2}(+\infty) = c_{1,2}^* .$$

After some modifications we get:

$$\boxed{\begin{cases} \dot{c}_1 = g_1 \\ \dot{g}_1 = (u(c_1, c_2) - v^b)\, g_1 \\ \dot{c}_2 = (u(c_1, c_2) - v^b)\, c_1 + (2 - u(c_1, c_2) - v^b)\, c_2 - g_1 \end{cases}} \quad (3)$$

Fixed points:

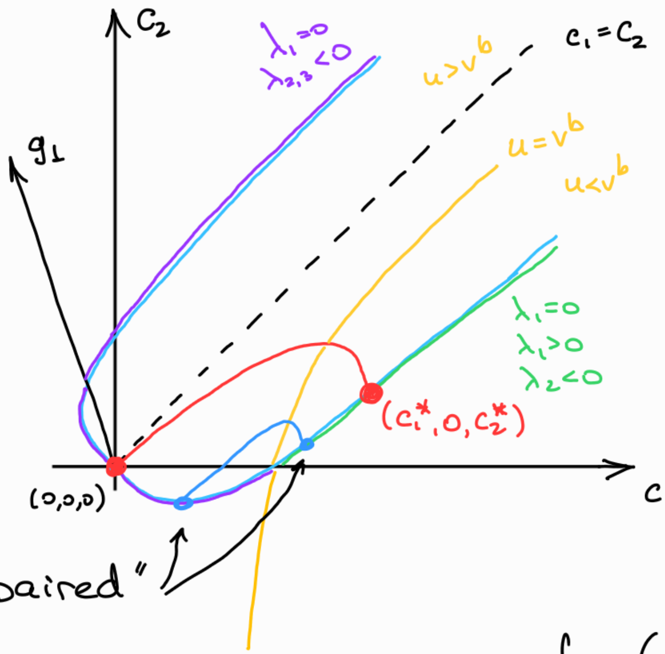
$$\begin{cases} g_1 = 0 \\ (u - v^b)\, c_1 + (2 - u - v^b)\, c_2 = 0 \end{cases}$$

We are looking for

$$(c_1^*(v^b),\ c_2^*(v^b))$$

such that there exists a trajectory from

$$(0,0,0) \longrightarrow (c_1^*, 0, c_2^*)$$



Hypothesis: points are "paired"

Ideas how to prove the existence of (c_1^*, c_2^*) and understand the dependence $c_1^*(v^b)$ and $c_2^*(v^b)$?

¿ Construct Lyapunov function ?

§5. Open questions and directions of research:

1) Existence and uniqueness of global solutions for TFE model (literature review, Yudovich theory works?)

2) Study the limit as $\varepsilon \to 0$ (spontaneous stochasticity?)

3) Careful numerical modelling of TFE model and comparison "Peaceman" and "TFE" – are they close in some sense?

4) 2-tube model with Darcy's law – numerically we observe similar cascade of 2 travelling waves – prove