Small ball asymptotics for detrended Green Gaussian processes of arbitrary order

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Problem statement

We are interested in the sharp L_2-small ball asymptotics for $(n-1)$-th order detrended Green Gaussian processes

$$X_n(t) := X(t) - \sum_{i=0}^{n-1} a_i t^i,$$

where a_i are determined by relations

$$\int_0^1 t^i X_n(t)\,dt = 0,\ i = 0 \dots n-1.$$

Here $X(t)$, $t \in [0,1]$, is a Gaussian process, $\mathbb{E}X = 0$, covariance function $G(s,t)$ is the Green function for a boundary value problem:

$$Lu := (-1)^p u^{(2p)} = \lambda u \ + \ \text{ some boundary conditions.}$$

Problem: find the asymptotics of $\mathbb{P}\{\|X_n\| < \varepsilon\}$ as $\varepsilon \to 0$.

Known results

The case $n = 1$ corresponds to the centered process.
Our problem was considered earlier in the following cases:

Case	**Process**	**Author**
$n = 1$ $p = 1$	centered Brownian bridge and Wiener process	2005 — E. Orsingher, Ya. Nikitin 2006 — P. Deheuvels
$n = 2$ $p = 1$	detrended Brownian motion	2012 — X. Ai, W. Li
$\forall n$ $p = 1$	m-th order detrended Brownian motion	2014 — X. Ai, W. Li

We deal with arbitrary $n, p \in \mathbb{N}$ under the assumption $n > 2p$. In this case the process X_n does not depend on the original boundary conditions.

The Karhunen–Loève (KL) expansion:

$$X_n(t) \stackrel{d}{=} \sum_{k\geq 1} \xi_k \sqrt{\mu_k} y_k(t),$$

here ξ_k is a sequence of i.i.d. $N(0,1)$ random variables, μ_k are eigenvalues and $y_k(t)$ are eigenfunctions of the integral operator with kernel $G_n(s,t)$ — the covariance function of X_n, that is:

$$\mu_k u(t) = \int_0^1 u(s) G_n(s,t)\, ds. \tag{1}$$

So

$$\|X_n\|_2^2 = \int\limits_0^1 X_n^2(t)\, dt \stackrel{d}{=} \sum_{k=1}^{\infty} \mu_k \xi_k^2.$$

Therefore the problem can be formulated as follows:

$$\text{Find: } \mathbb{P}\Big\{ \sum_{k=1}^{\infty} \mu_k \xi_k^2 < \varepsilon^2 \Big\} \text{ as } \varepsilon \to 0.$$

The Wenbo Li principle (Li 1992, Gao et al 2003)

Let $X(t)$, $\tilde{X}(t)$ be two Gaussian processes with zero mean and covariance functions $G(s,t)$ and $\tilde{G}(s,t)$. Let μ_k and $\tilde{\mu}_k$ be positive eigenvalues of integral operators with kernels $G(s,t)$ and $\tilde{G}(s,t)$, respectively. If $\prod \tilde{\mu}_k/\mu_k < \infty$ then

$$\mathbb{P}\Big\{\|X\|_2 < \varepsilon\Big\} \sim \mathbb{P}\Big\{\|\tilde{X}\|_2 < \varepsilon\Big\} \cdot \left(\prod_{k=1}^{\infty} \frac{\tilde{\mu}_k}{\mu_k}\right)^{1/2}, \qquad \varepsilon \to 0. \quad (2)$$

Let's notice that

$$G_n(s,t) = \mathbb{E}\Big(X(s) - \sum_{i=1}^{n-1} a_i s^i\Big)\Big(X(t) - \sum_{i=1}^{n-1} a_i t^i\Big) = G(s,t) + \mathcal{P}_n(s,t).$$

Here $\mathcal{P}_n(s,t)$ is a polynomial of degree $(n-1)$ in each variable. So we can rewrite (1) as:

$$\mu_k u(t) = \int_0^1 u(s)(G(s,t) + \mathcal{P}_n(s,t))\, ds. \tag{3}$$

Applying operator L to this equality we obtain:

$$(-1)^p u^{(2p)}(t) = \lambda u(t) + \mathcal{P}_{n-2p}(t), \tag{4}$$

$$\int\limits_0^1 t^i u(t)\, dt = 0, \quad i = 0 \dots n-1, \tag{5}$$

Here $\mathcal{P}_{n-2p}(t)$ is a polynomial of degree $(n-2p-1)$ with unknown coefficients, $\lambda = \lambda_k^{(n,p)} := \mu_k^{-1}$.

The equivalent problem

Consider the following eigenvalue problem:

$$(-1)^p y^{(2n)}(t) = \lambda y^{(2n-2p)}(t) \tag{6}$$

$$y^{(j)}(0) = y^{(j)}(1) = 0, \quad j = 0 \dots n-1. \tag{7}$$

Note that the smallest eigenvalue of the problem (6)-(7) gives the sharp constant in the embedding theorem $\overset{\circ}{W}{}_2^n(0,1) \hookrightarrow \overset{\circ}{W}{}_2^{n-p}(0,1)$.

Lemma

The eigenvalue problems (4)-(5) *and* (6)-(7) *are equvalent, i.e. have solutions for the same* λ. *Moreover, if* $u(t)$ *is a solution of* (4)-(5) *and* $y(t)$ *is a solution of* (6)-(7), *then* $u(t) = y^{(n)}(t)$.

Problem (6)-(7) was solved by Janet for $n \in \mathbb{Z}_+$ and $p = 1$ in 1931. For arbitrary p the answer was only formulated without proof and in implicit terms.

Equation on eigenvalues

Without loss of generality we can assume that the eigenfunction is odd or even. If $y(t)$ is an even solution of the equation (6):

$$(-1)^p y^{(2n)}(t) - \lambda y^{(2n-2p)}(t) = 0,$$

then

$$(-1)^p (y')^{(2n-2)}(t) - \lambda (y')^{(2n-2-2p)}(t) = C$$

and the constant $C = 0$, as the left hand side is odd.
So eigenvalue, corresponding to even solution of the equation (6) with parameters (n, p), equals to an eigenvalue, corresponding to odd solution of the equation (6) with parameters $(n-1, p)$. That's why we can restrict ourselves to consider only odd solutions and the equation will be of the form

$$\Delta_{n,p}(\lambda) \cdot \Delta_{n-1,p}(\lambda) = 0.$$

Equation on eigenvalues

$\Delta_{n,p}(\lambda)$ is the following determinant

$$\begin{vmatrix} \zeta_0^{(2n-2p+1)/2} J_{(2n-2p+1)/2}(\zeta_0) & \dots & \zeta_{p-1}^{(2n-2p+1)/2} J_{(2n-2p+1)/2}(\zeta_{p-1}) \\ \zeta_0^{(2n-2p+3)/2} J_{(2n-2p+3)/2}(\zeta_0) & \dots & \zeta_{p-1}^{(2n-2p+3)/2} J_{(2n-2p+3)/2}(\zeta_{p-1}) \\ \dots & \dots & \dots \\ \zeta_0^{(2n-1)/2} J_{(2n-1)/2}(\zeta_0) & \dots & \zeta_{p-1}^{(2n-1)/2} J_{(2n-1)/2}(\zeta_{p-1}) \end{vmatrix}$$

Here $J_k(x)$ are Bessel functions of the first kind,

$$\zeta_k = \frac{1}{2} \sqrt[2p]{|\lambda|} e^{i\pi k/p}, \quad k = 0 \dots p-1.$$

Using asymptotics of Bessel functions we get the asymptotics

$$\mu_k = \left(\pi k + \frac{2n-p-1}{2} + O\Big(\frac{1}{k}\Big)\right)^{-2p}.$$

As an approximation to μ_k we can take

$$\tilde{\mu}_k := \left(\pi k + \frac{2n-p-1}{2}\right)^{-2p}.$$

The small ball asymptotics for the case

$$\mu_k = \Big(\theta(k+\delta)\Big)^{-2p}$$

was already considered in the works of the following authors.

Case	**Author**
$\theta = 1,\ p = 1,\ \delta > -1$	1992 — W. Li
$\theta = 1,\ p > 1,\ \delta = 0$	1998 — T. Dunker, M. A. Lifshits, W. Linde
$\theta > 0,\ p > 1,\ \delta > -1$	2003 — Ya. Yu. Nikitin, A. I. Nazarov

Final result

So using Li's principle, finally, we obtain **sharp** small ball asymptotics

$$\mathbb{P}\{\|X_n\| < \varepsilon\} \sim C\varepsilon^{\gamma} \exp\Big(-\frac{2p-1}{2\big(2p\sin\big(\frac{\pi}{2p}\big)\big)^{\frac{2p}{2p-1}}} \cdot \varepsilon^{-\frac{2}{2p-1}}\Big),$$

where $\gamma = \frac{1-2np+p^2}{2p-1}$ and

$$C = \frac{2^{2p(\gamma-1)} \cdot p^{1+\frac{\gamma}{2}} \sin^{\frac{1+\gamma}{2}}\big(\frac{\pi}{2p}\big)}{\pi^{\frac{3p+1}{2}}(2p-1)^{\frac{1}{2}}\Big|\mathfrak{V}\Big(-1, e^{\frac{i\pi}{p}}, \dots, e^{\frac{i\pi(p-1)}{p}}\Big)\Big|} \cdot$$

$$\cdot \frac{\Gamma^{-\frac{1}{2}}(n+\frac{1}{2})\Gamma^{-\frac{1}{2}}(n-\frac{p}{2}+\frac{1}{2})}{\prod\limits_{j=1}^{p-1}\Gamma(n-p+j+\frac{1}{2})}.$$

Here $\mathfrak{V}(x_0, \dots, x_{p-1})$ is Vandermonde determinant.

Thank you for your attention!