Two tubes model of miscible displacement: travelling waves and normal hyperbolicity

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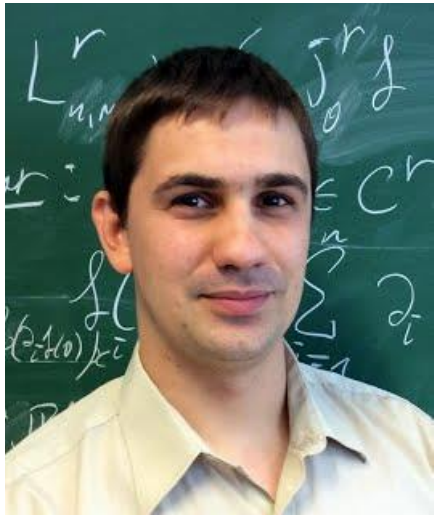

Sergey Tikhomirov
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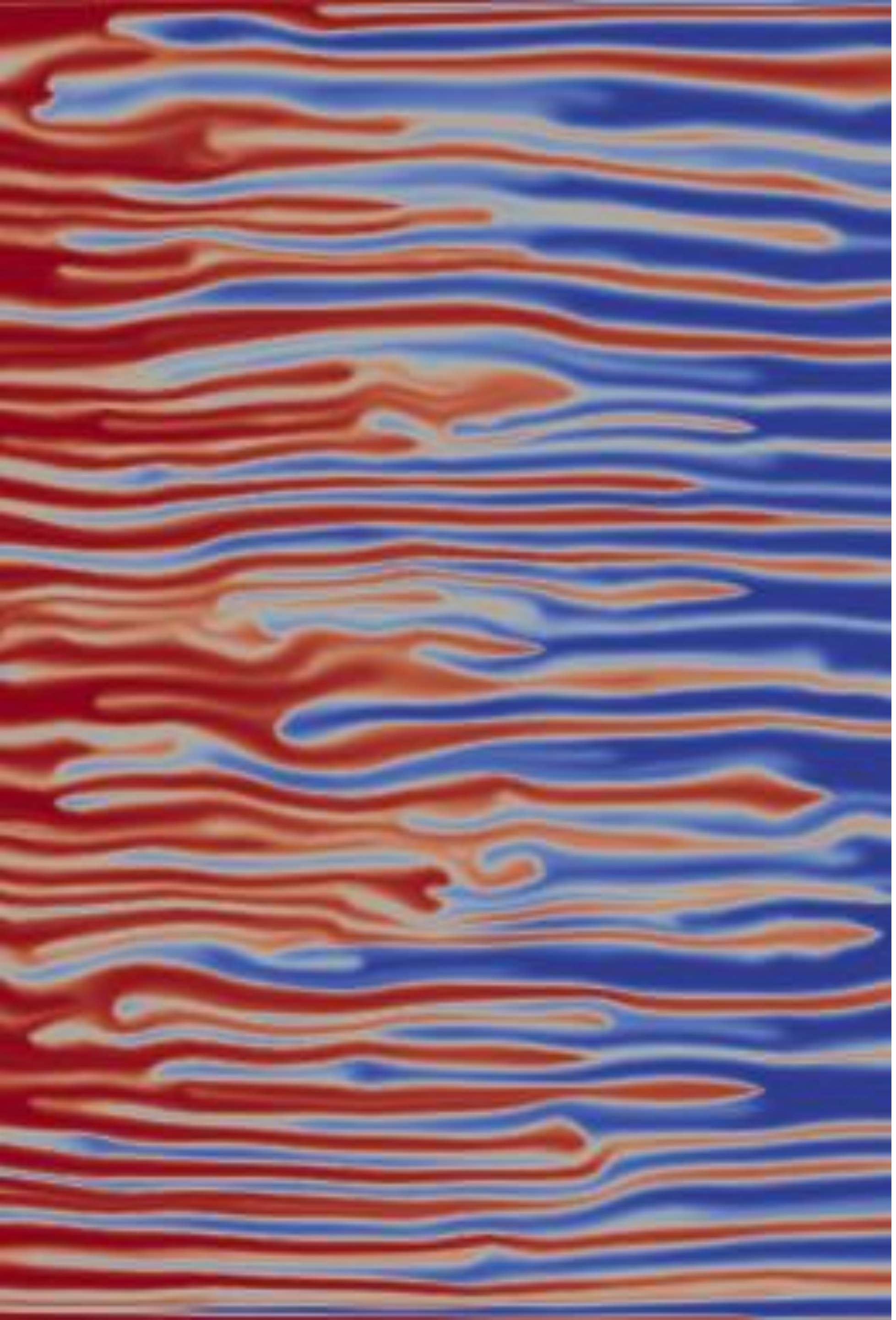
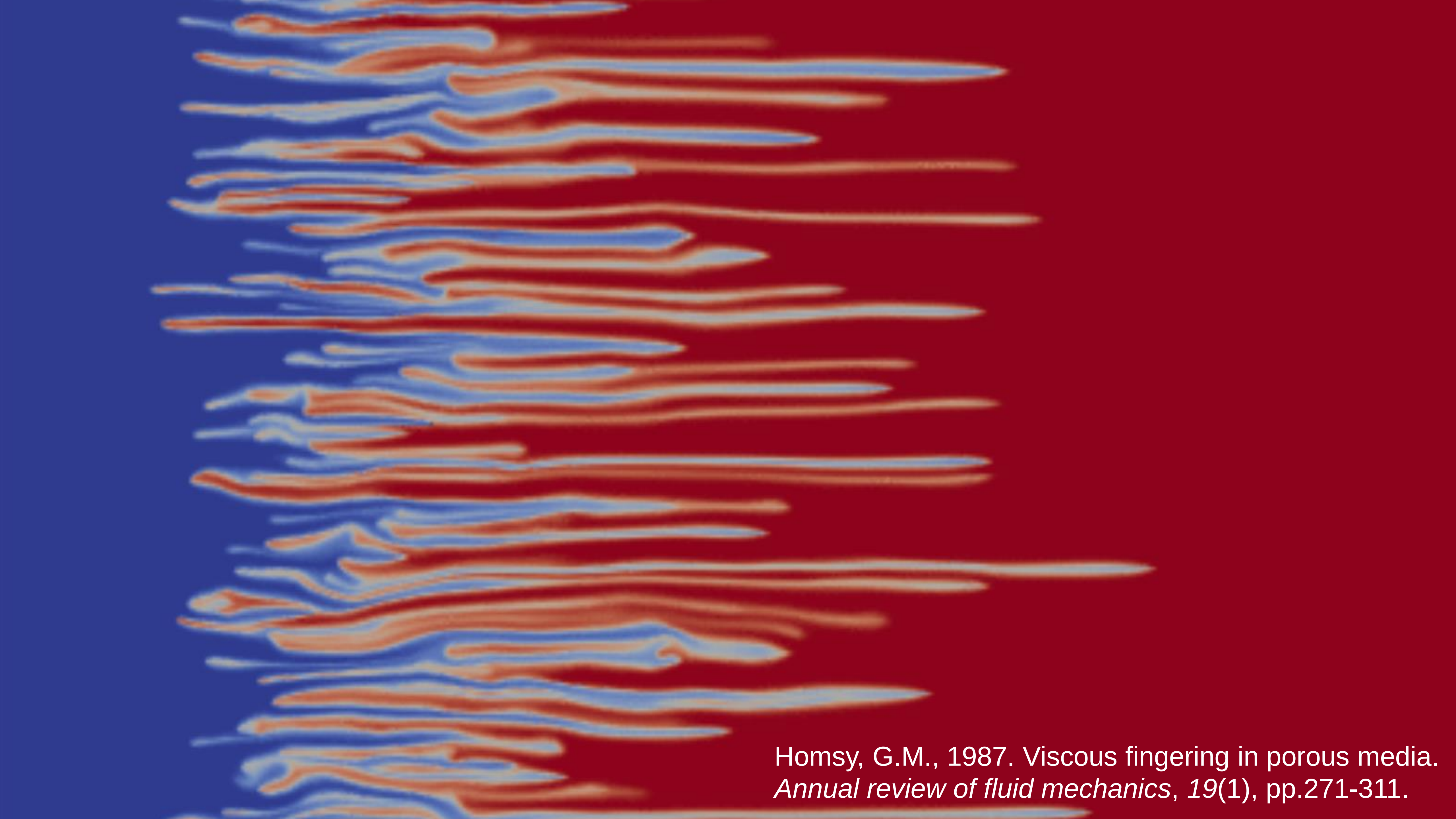
Multiscale Analysis and Methods
for Quantum and Kinetic Problems
IMS at NUS

Yalchin Efendiev
Texas A&M
USA



Outline

1. General phenomenon
   - Viscous fingers
   - Gravitational fingers
2. Motivation of the statement of the problem
   - Why we believe that our setting is important
   - Introduce the “toy model”
3. Theorem and Conjectures

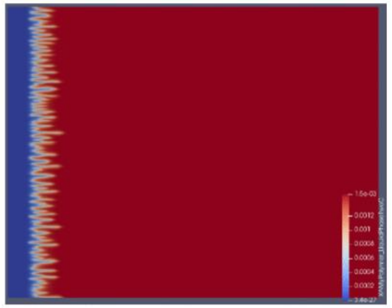


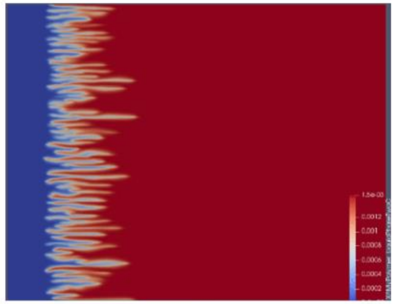
Homsy, G.M., 1987. Viscous fingering in porous media. *Annual review of fluid mechanics*, *19*(1), pp.271-311.

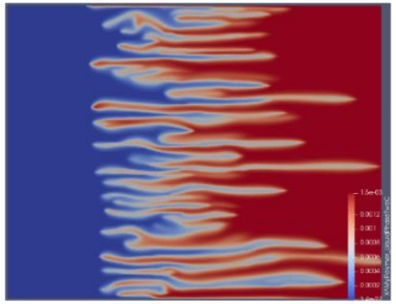
Two settings (Incompressible porous medium eqs - IPM)

1. Viscosity-driven fingers: 2d

$$c_t + u \cdot \nabla c = \varepsilon\, \Delta c$$
$$div\, u = 0$$
$$u = -k \cdot m(c) \nabla p$$





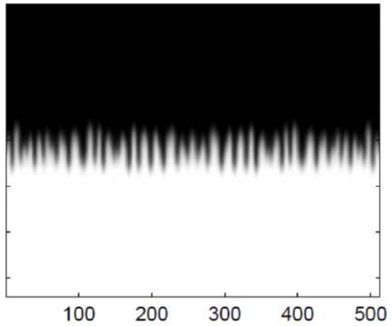


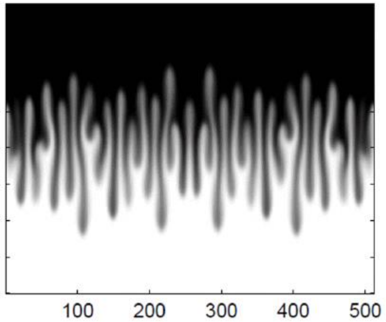
- c – concentrations of viscous spices (transport equation) $c \in [0, 1]$
- u – velocity of fluid (incompressibility condition)
- p – pressure
  velocity is defined by Darcy law and mobility of liquid $m(c)$;
  $m(c)$ – decreasing function, e.g.
  $$m(c) = \mathrm{e}^{-\mathrm{ac}}$$

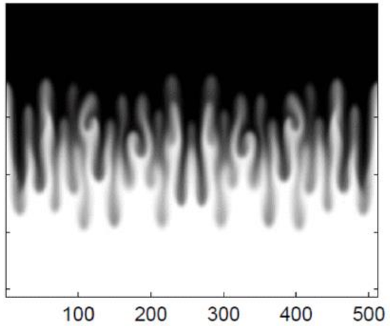
We did a lot of numerical simulations.
Motivation of statement of the problem.

2. Gravity-driven fingers: 2d

$$c_t + u \cdot \nabla c = \varepsilon\, \Delta c$$
$$div\, u = 0$$
$$u = -\nabla p - (0, c)$$







- c – concentrations of heavy spices (transport equation) $c \in [-1, 1]$
- u – velocity of fluid (incompressibility condition)
- p – pressure.

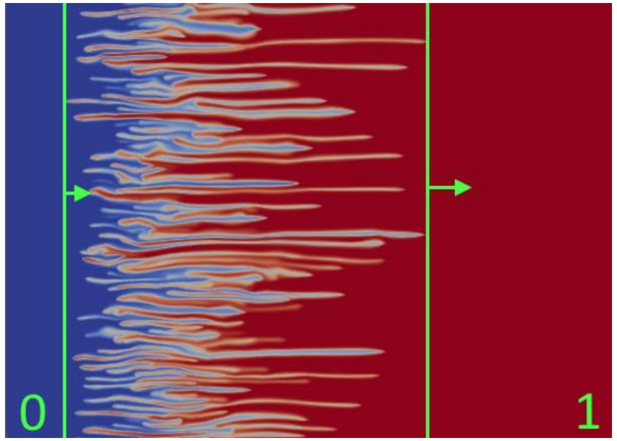
  velocity is defined by Darcy law and gravitation

We have some theorems for “toy model”

Questions of interest

1. Well-posedness:
   - $\varepsilon = 0$: incompressible porous medium (IPM)
     active scalar: $u = A(c)$ – singular integral operator (like in SQG)
     - existence of a global solution vs finite-time blow-up:
       A. Castro, D. Cordoba, D. Lear (2018), T. Elgindi (2017), A. Kiselev, Y. Yao (2023)
     - non-uniqueness of solutions (convex integration technique):
       D. Córdoba, D. Faraco, F. Gancedo (2011), R. Shvydkoy (2011), L. Szekelyhidi Jr. (2012)
   - related: generalized Buckley-Leverett equation – N. Chemetov, W. Neves (2014)
     Muskat problem & Hele-Shaw (free boundary) – A. Cordoba, D. Cordoba, F. Gancedo (2011) etc.

2. Dynamics of mixing zone: $\varepsilon > 0$
   - many laboratory and numerical experiments show linear growth of the mixing zone [1]
   - the only mathematically rigorous result on estimates of speed of the linear growth
     - Simplified model of Darcy's law:
       Transverse flow equilibrium (TFE) [2]



[1] Nijjer J., Hewitt D., and Neufeld J. The dynamics of miscible viscous fingering from onset to shutdown. Journal of Fluid Mechanics 837 (2018): 520-545.
[2] Menon, G. and Otto, F., 2006. Diffusive slowdown in miscible viscous fingering. Communications in Mathematical Sciences, 4(1), pp.267-273.

TFE model and comparison theorem (gravity-driven)

- TFE model: assumption $p(x,y) \sim p(y), \quad p_y(x,y) \sim p_y(y)$

$$\begin{aligned} c_t + u \cdot \nabla c &= \varepsilon\, \Delta c \\ div\, u &= 0 \\ u &= (u^x, u^y) \\ u^y &= \bar{c} - c \end{aligned}$$

- Consider 1d equations (viscous Burgers equation)

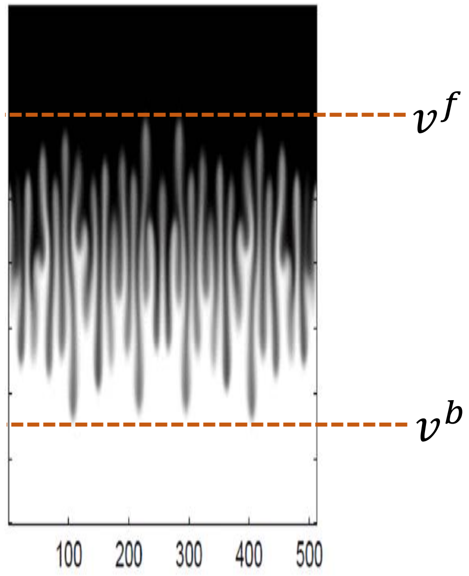
$$c_t^{max} + (1 - c^{max}) c_y^{max} = \varepsilon\, (c^{max})_{yy}$$

$$c_t^{min} + (-1 - c^{min}) \cdot c_y^{min} = \varepsilon \left(c^{min}\right)_{yy}$$

Comparison theorem (Otto-Menon, 2005)

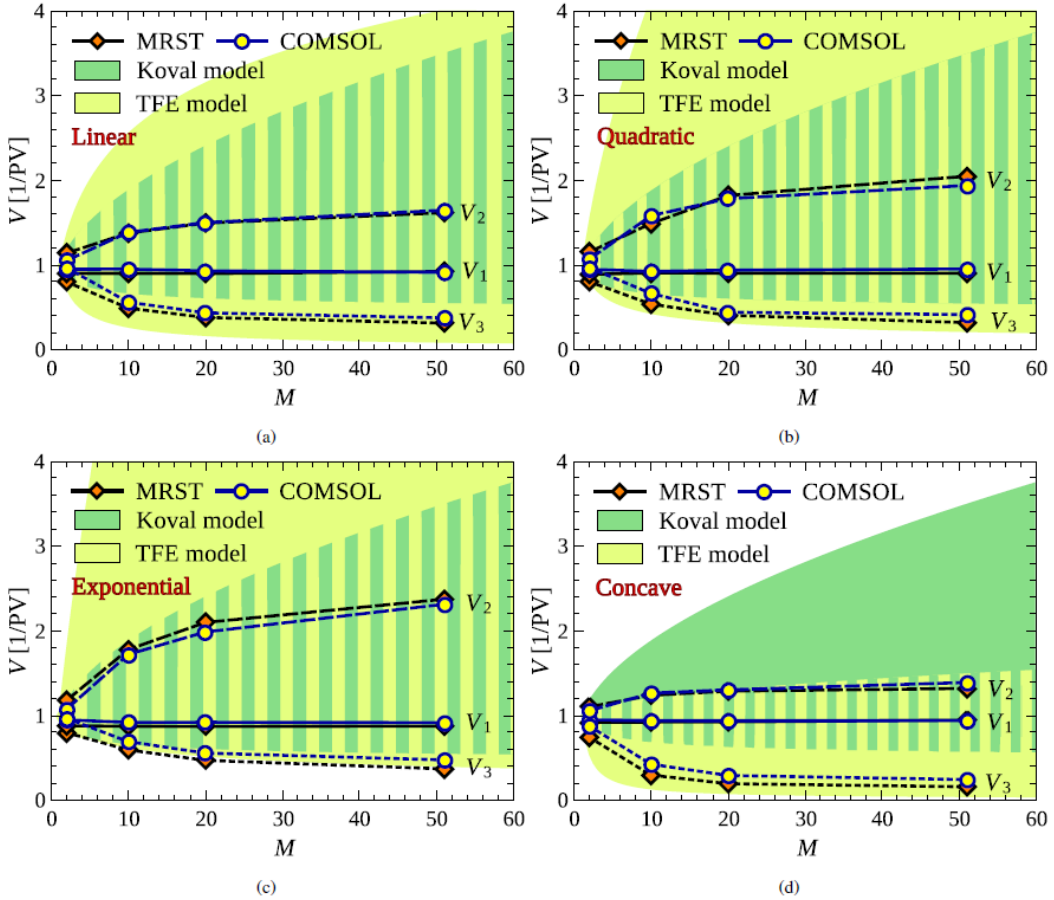
- If $c(0,x,y) < c^{max}(0,y)$ then $c(t,x,y) \leq c^{max}(t,y)$
- If $\mathrm{c}(0,x,y) > c^{min}(0,y)$ then $c(t,x,y) \geq c^{min}(t,y)$

1. It gives upper bound for the faster finger
$$v^f \leq 1$$
2. It gives upper bound for the back front
$$v^b \geq -1$$
3. Estimate is sharp if
   1. There is no transverse flow
   2. Drop of concentration on a finger tip is -1 -> +1
4. Numerics shows that estimate is far from sharp
5. We want to get better estimate

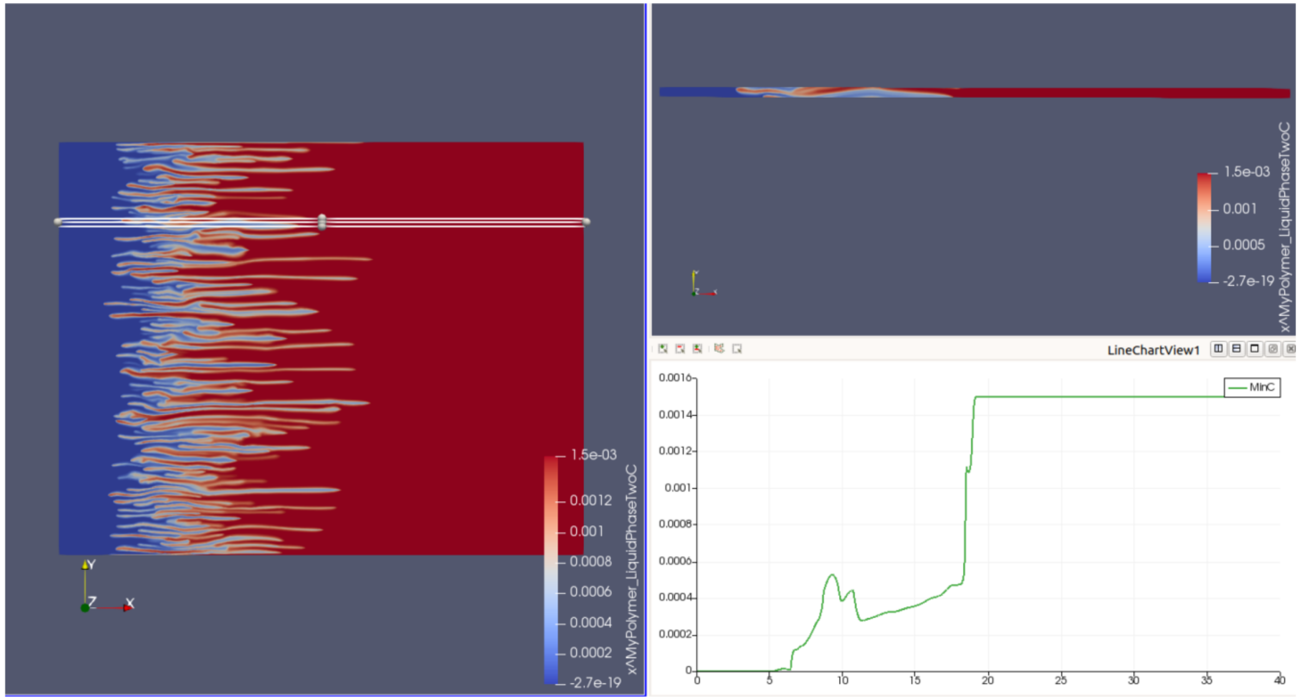


Numerics for viscous fingers

F. Bakharev, A. Enin, A. Groman, A. Kalyuzhnyuk,
S. Matveenko, **Yu. Petrova**, I. Starkov, S. Tikhomirov
"Velocity of viscous fingers in miscible displacement: Comparison with analytical models"
Journal of Computational and Applied Mathematics, 2022



Possible mechanism: intermediate concentration



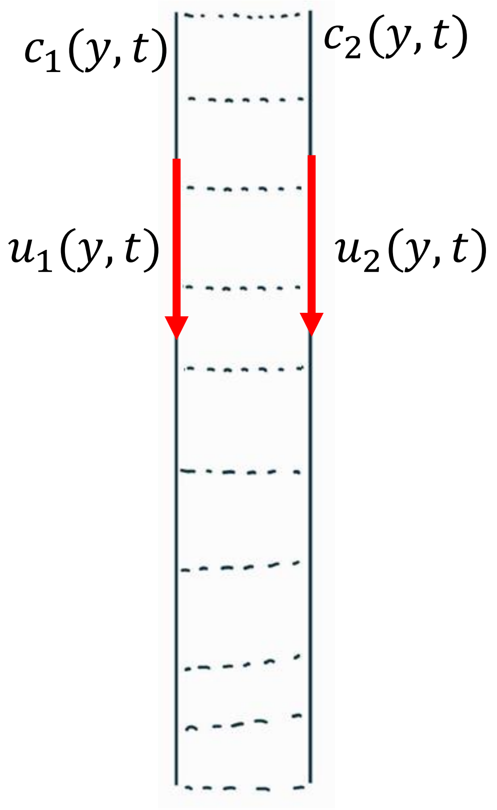
Two-tubes model (with gravity)

Original equations

$$c_t + div(uc) = \varepsilon \, \Delta c$$
$$div \, u = 0$$

Two-tube equations

$$\partial_t c_1 + \partial_y (u_1 c_1) - \varepsilon \, \partial_{yy} c_1 = 0$$
$$\partial_t c_2 + \partial_y (u_2 c_2) - \varepsilon \, \partial_{yy} c_2 = 0$$



Two-tubes model (with gravity)

Original equations

$$c_t + div(uc) = \varepsilon \, \Delta c$$
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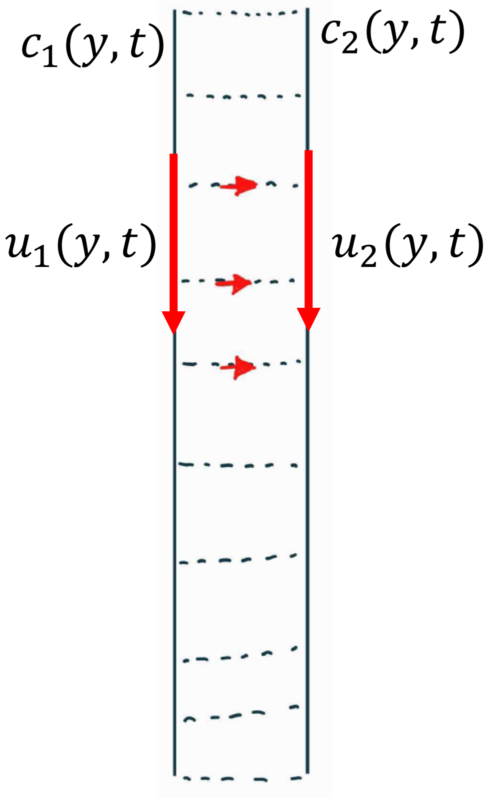


Two-tube equations: inclusion of transverse flow

$$\partial_t c_1 + \partial_y(u_1 c_1) - \varepsilon \, \partial_{yy} c_1 = \boxed{-(-1)^{1,2} \partial_y u_{1,2} \cdot c_{1,2}}$$
$$\partial_t c_2 + \partial_y(u_2 c_2) - \varepsilon \, \partial_{yy} c_2 = \boxed{(-1)^{1,2} \partial_y u_{1,2} \cdot c_{1,2}}$$

$$(-1)^{1,2} \partial_y u_{1,2} \cdot c_{1,2} = \begin{cases} -\partial_y u_1 \cdot c_1, & \partial_y u_1 < 0, \\ +\partial_y u_2 \cdot c_2, & \partial_y u_1 > 0 \end{cases}$$

Model for velocities is different for IPM and TFE:

- TFE: $u = \bar{c} - c, \quad u_1 = \frac{c_1 + c_2}{2} - c_1, \quad u_2 = \frac{c_1 + c_2}{2} - c_2$
- IPM: we need to introduce pressure, we will do this later

Initial condition:

$$c_{1,2}(y,0) = -1, y < 0$$
$$c_{1,2}(y,0) = +1, y > 0$$

Main result (TFE model, gravity-driven fingers)

Theorem (Efendiev, P., Tikhomirov, 2022+)

Consider a two-tube model with gravity.
Then there exists unique (up to swap) c_1^*, c_2^* such that TFE two-tubes system has travelling waves

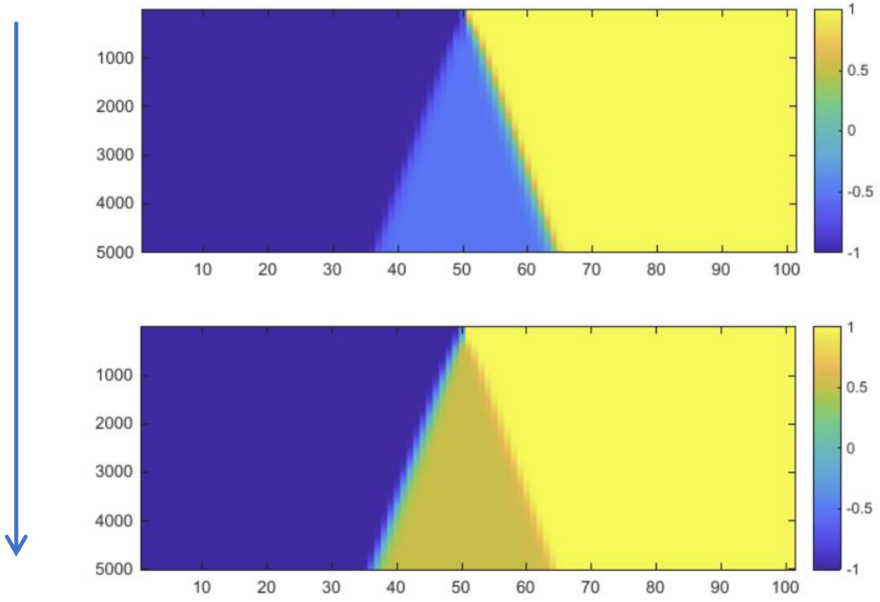
$$(-1,-1) \to (c_1^*, c_2^*) \to (1,1)$$

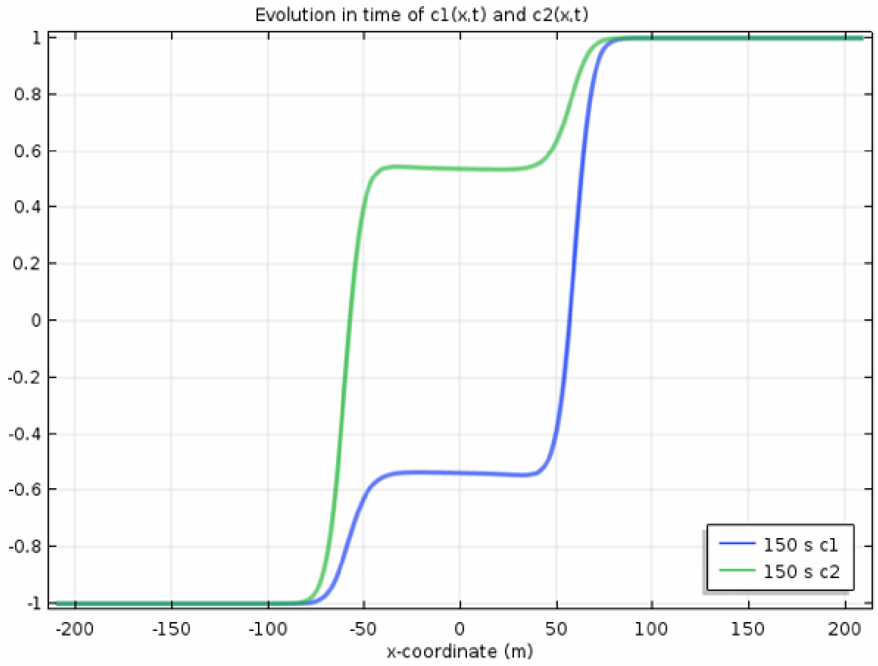
Moreover,

$$c_1^* = -\frac{1}{2}, \quad c_2^* = \frac{1}{2},$$
$$v^b = -\frac{1}{4}, \quad v^f = \frac{1}{4}.$$

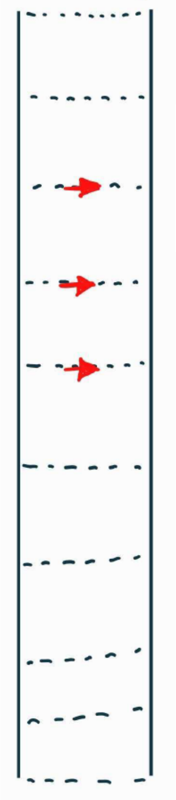
Including in the system cross-flow automatically creates intermediate concentration

time





Travelling waves. Equations.



Original system:

$$\partial_t c_1 + \partial_y (u_1 c_1) - \varepsilon\, \partial_{yy} c_1 = -(-1)^{1,2} \partial_y u_{1,2} \cdot c_{1,2}$$
$$\partial_t c_2 + \partial_y (u_2 c_2) - \varepsilon\, \partial_{yy} c_2 = (-1)^{1,2} \partial_y u_{1,2} \cdot c_{1,2}$$

$$(-1)^{1,2} \partial_y u_{1,2} \cdot c_{1,2} = \begin{cases} \boxed{-\partial_y u_1 \cdot c_1, \qquad \partial_y u_1 < 0,} \\ +\partial_y u_2 \cdot c_2, \qquad \partial_y u_1 > 0 \end{cases}$$

Travelling wave ansatz:

$$\xi = y - vt, \qquad c_{1,2}(y,t) = c_{1,2}(\xi),$$
$$c_{1,2}(\pm\infty) = c_{1,2}^{\pm}$$

4d system:

$$\dot{c}_1 = g_1,$$
$$\dot{g}_1 = g_1(u_1 - v),$$
$$\dot{c}_2 = g_2,$$
$$\dot{g}_2 = (u_2 - v) g_2 + (c_1 - c_2)\dot{u}_1.$$

Conservation laws – 3d dynamical system:

$$\dot{c}_1 = g_1,$$
$$\dot{g}_1 = g_1(u_1 - v),$$
$$\dot{c}_2 = -v(c_1 + c_2 - c_1^{+} - c_2^{+}) + (u_1 c_1 + u_2 c_2 - u_1^{+} c_1^{+} - u_2^{+} c_2^{+}) - g_1.$$

Connection between $c_{1,2}^{\pm}$ and v:
(Rankine-Hugoniot condition)

$$v[c_1 + c_2]\Big|_{-\infty}^{+\infty} = [u_1 c_1 + u_2 c_2]\Big|_{-\infty}^{+\infty}.$$

TFE velocity model:

$$u_1 = \frac{c_1 + c_2}{2} - c_1, \qquad u_2 = \frac{c_1 + c_2}{2} - c_2$$

Travelling waves. Phase portrait.

Substitute $u_{1,2}$, get:

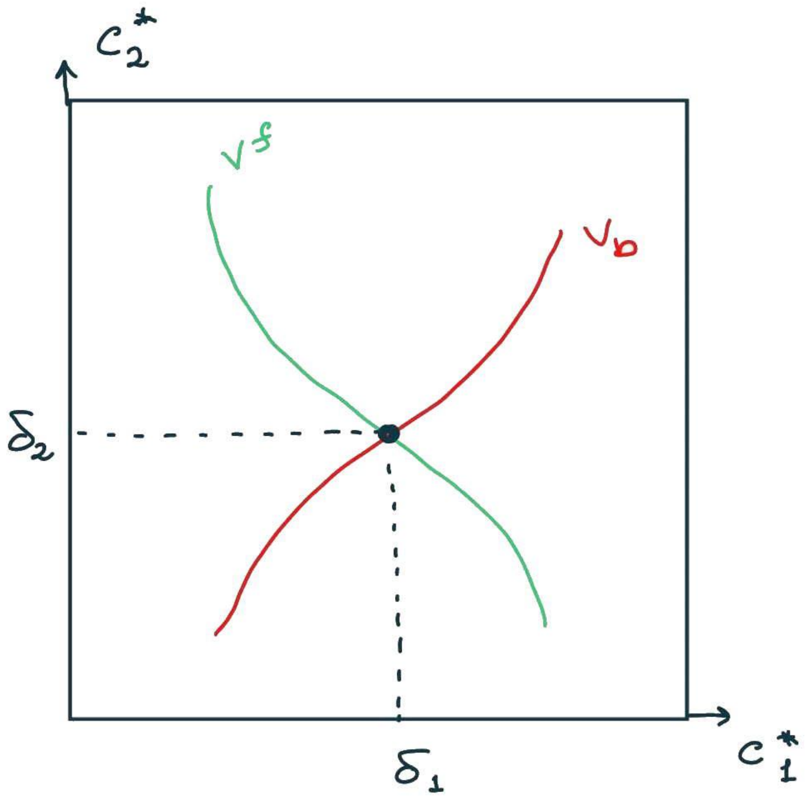
$$\dot{c}_1 = g_1,$$
$$\dot{g}_1 = g_1\left(\frac{c_2 - c_1}{2} - v\right),$$
$$\dot{c}_2 = -v(c_1 + c_2 - c_1^+ - c_2^+) - \frac{1}{2}((c_1 - c_2)^2 - (c_1^+ - c_2^+)^2) - g_1.$$

Rankine-Hugoniot condition

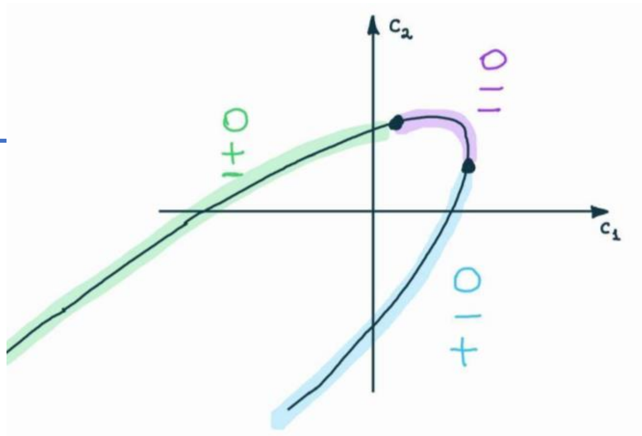
$$v(c_2^- + c_1^- - c_2^+ - c_1^+) = -\frac{1}{2}((c_1^- - c_2^-)^2 - (c_1^+ - c_2^+)^2).$$

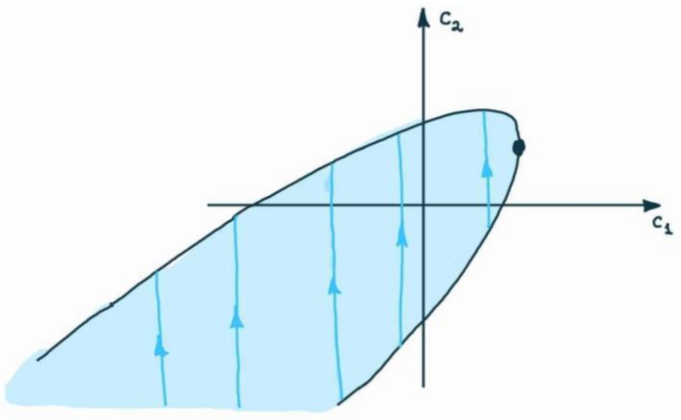
3d dynamical system on (c_1, g_1, c_2)
Fix: (c_1^-, c_2^-) or (c_1^+, c_2^+)
Parameter: v

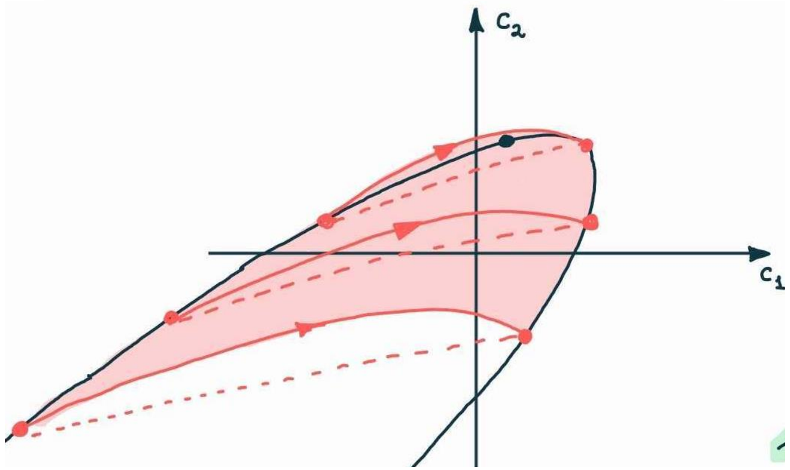
- For each v expected a travelling wave
- This generates a curve of possible c_1, c_2
- We apply this procedure for travelling wave to $(+1, +1)$ and from $(-1, -1)$



Two tubes. Invariant surface.







Equations

$$\dot{c}_1 = g_1,$$

$$\dot{g}_1 = g_1 \left(\frac{c_2 - c_1}{2} - v \right),$$

$$\dot{c}_2 = -v(c_1 + c_2 - c_1^+ - c_2^+) - \frac{1}{2}((c_1 - c_2)^2 - (c_1^+ - c_2^+)^2) - g_1.$$

Travelling wave speed

$$v(c_2^- + c_1^- - c_2^+ - c_1^+) = -\frac{1}{2}((c_1^- - c_2^-)^2 - (c_1^+ - c_2^+)^2).$$

There exists 2dim invariant surface

$$g_1 = \frac{3}{4}(-v(c_2 + c_1 - c_2^+ - c_1^+) - \frac{1}{2}((c_1 - c_2)^2 - (c_1^+ - c_2^+)^2)),$$

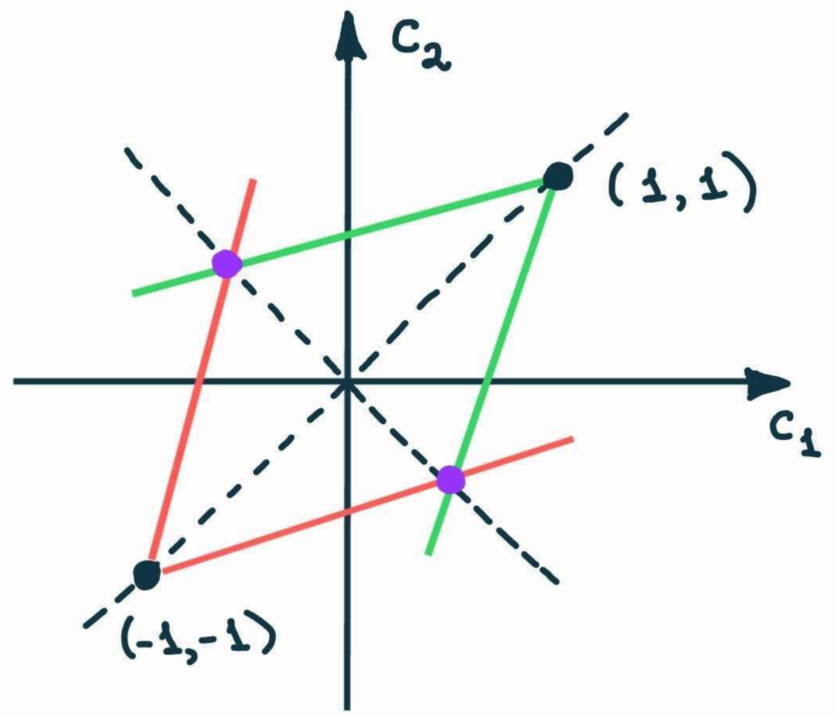
On all (for any $c_{1,2}^+$) heteroclinic holds

$$3(c_2 - c_2^+) = c_1 - c_1^+,$$

We have solved our “heteroclinic” problem analytically

Finally answer.

Admissible curves on the plane



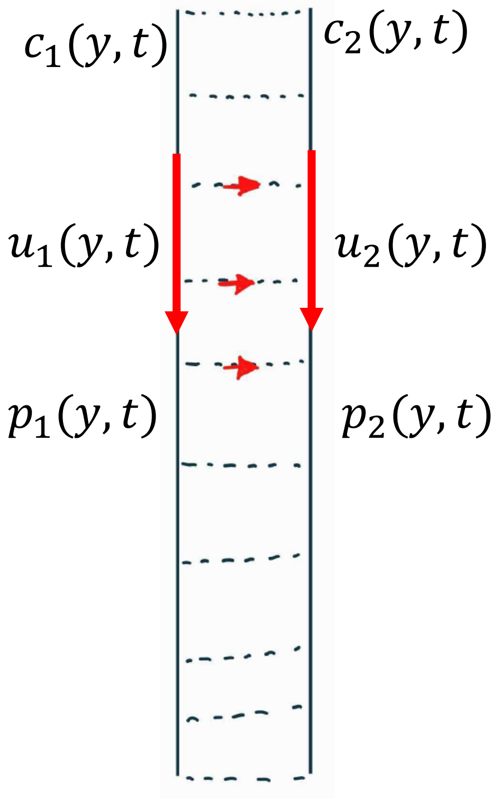
Speed and concentration

$$v^b = -\frac{1}{4}$$
$$v^f = \frac{1}{4}$$
$$c_1^* = -1/2$$
$$c_2^* = 1/2$$

Two-tubes model. IPM.

Original equations

$$c_t + div(uc) = \varepsilon\,\Delta c$$
$$div\,u = 0$$



$$\partial_t c_1 + \partial_y(u_1 c_1) - \varepsilon\,\partial_{yy} c_1 = -(-1)^{1,2}\partial_y u_{1,2}\cdot c_{1,2}$$
$$\partial_t c_2 + \partial_y(u_2 c_2) - \varepsilon\,\partial_{yy} c_2 = (-1)^{1,2}\partial_y u_{1,2}\cdot c_{1,2}$$

$$(-1)^{1,2}\partial_y u_{1,2}\cdot c_{1,2} = \begin{cases} -\partial_y u_1\cdot c_1, & \partial_y u_1 < 0, \\ +\partial_y u_2\cdot c_2, & \partial_y u_1 > 0 \end{cases}$$

Velocity model for IPM: add p_1 and p_2

(Darcy's law in each tube) $\quad u_1 = -\partial_y p_1 - c_1, \qquad u_2 = -\partial_y p_2 - c_2,$

(Darcy's law between tubes) $\quad \partial_y u_1 = \frac{p_2 - p_1}{l}, \qquad \partial_y u_2 = -\frac{p_2 - p_1}{l}.$

Parameter l – distance between tubes

Main result (IPM model, gravity-driven fingers)

Conjecture (Efendiev, P., Tikhomirov, 2022+)

Consider a two-tube model with gravity.
For small enough $l > 0$ there exists unique (up to swap) $c_1^*(l), c_2^*(l)$ such that IPM two-tubes system has travelling waves

$$(-1,-1) \to (c_1^*, (l)\ c_2^*(l)) \to (1,1)$$

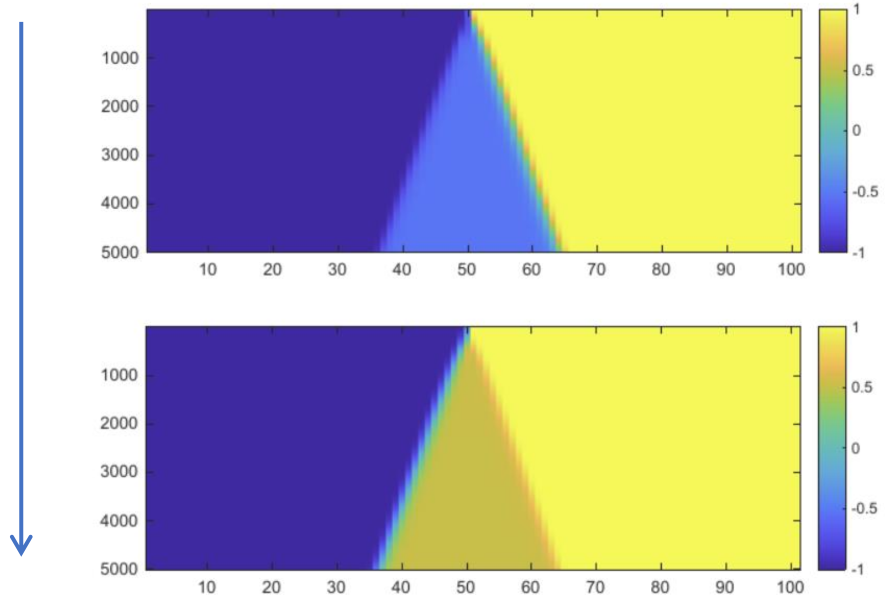
Moreover,

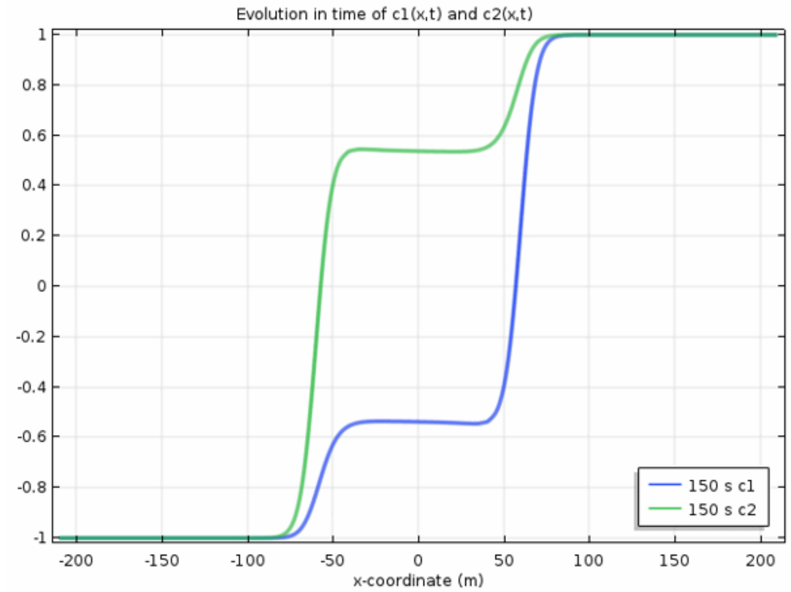
$$c_1^*(l) \to c_1^*, \quad c_2^*(l) \to c_2^* \quad \text{as} \quad l \to 0.$$
$$v^b(l) \to v^b, \quad v^f(l) \to v^f \quad \text{as} \quad l \to 0.$$

$l = 0$: travelling wave dyn system for IPM coincides with the system for TFE

Main technical ingredient: geometric singular perturbation theory and normal hyperbolicity

time

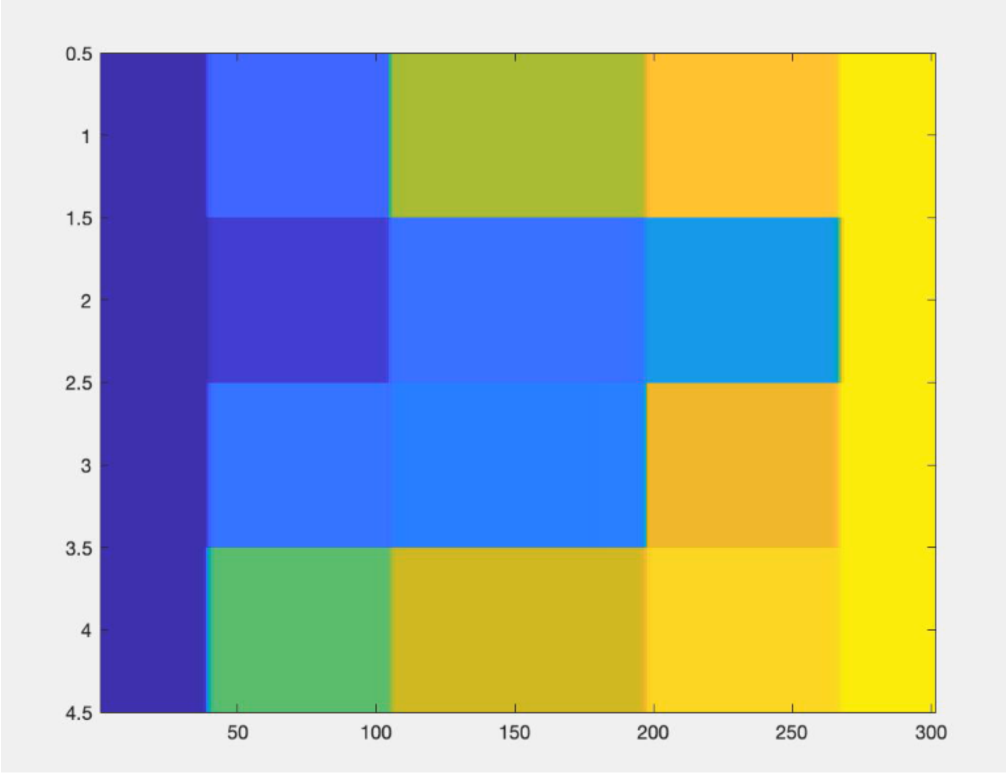




What’s next?

1. How to obtain similar results for “viscous” model?

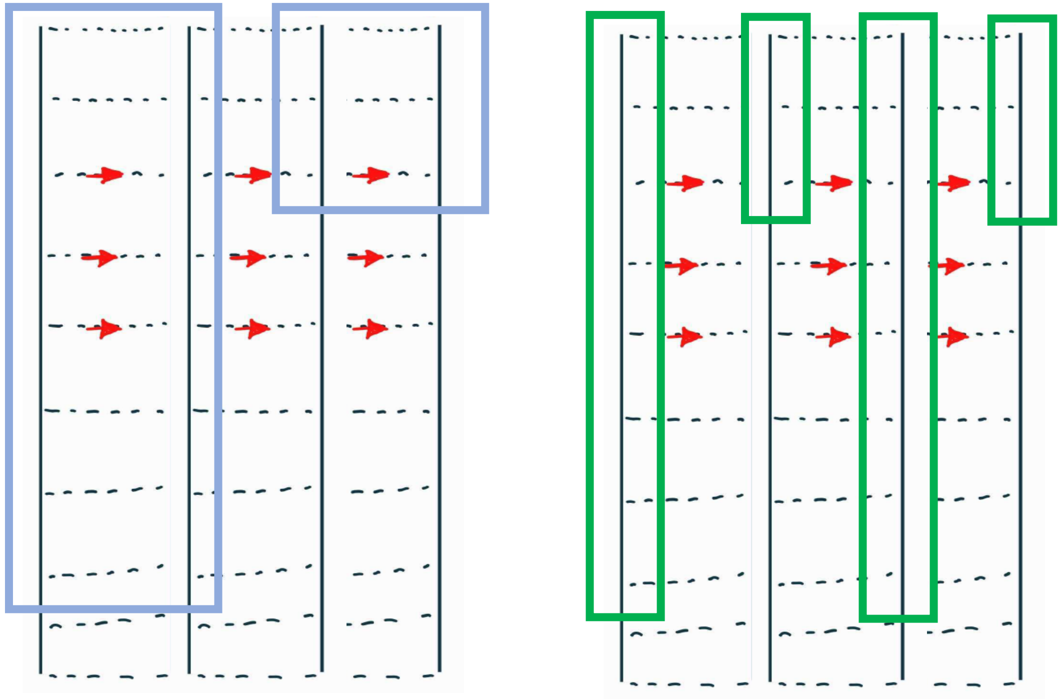
2. Does the n-tube model posses a system of n travelling waves?
   How to determine their constant states?
   Can we go to the limit as the number of tubes $n \to \infty$?



What's next?

3. Otto-Menon suggested that
   after time t fingers have length $\sim \sqrt{t}$
   What is the mechanism of merging of fingers?

   4-tubes model. What is more stable:
   - Two thin fingers?
   - One thick finger?



4. TFE as a limit of IPM when $\frac{k_y}{k_x} \to \infty$?
   Can we use the connection to prove the linear growth in IPM?

Peaceman model

$$c_t + u \cdot \nabla c = \varepsilon \, \Delta c$$
$$div\, u = 0$$
$$u = -\begin{pmatrix} k_x & 0 \\ 0 & k_y \end{pmatrix} \nabla p - (0, c)$$

$$\frac{k_y}{k_x} \to \infty$$

TFE model

$$c_t + u \cdot \nabla c = \varepsilon \, \Delta c$$
$$div\, u = 0$$
$$u = (u^x, u^y)$$
$$u^y = \bar{c} - c$$

References

Own works:

1. Bakharev, F., Enin, A., Groman, A., Kalyuzhnyuk, A., Matveenko, S., Petrova, Y., Starkov, I. and Tikhomirov, S., 2022. Velocity of viscous fingers in miscible displacement: Comparison with analytical models. Journal of Computational and Applied Mathematics, 402, p.113808.
2. Efendiev Ya., Petrova Yu., Tikhomirov S., 2022+, A cascade of two travelling waves in a two-tube model of gravitational fingering. In preparation.

Other references:

Dynamics of viscous fingering:

1. Nijjer J., Hewitt D., and Neufeld J. The dynamics of miscible viscous fingering from onset to shutdown. Journal of Fluid Mechanics 837 (2018): 520-545.
2. Menon, G. and Otto, F., 2006. Diffusive slowdown in miscible viscous fingering. Communications in Mathematical Sciences, 4(1), pp.267-273.
3. Menon, G. and Otto, F., 2005. Dynamic scaling in miscible viscous fingering. Communications in mathematical physics, 257, pp.303-317.
4. Homsy, G.M., 1987. Viscous fingering in porous media. Annual review of fluid mechanics, 19(1), pp.271-311.

References

Thank you very much!

Well-posedness for IPM:

1. Kiselev, A. and Yao, Y., 2023. Small scale formations in the incompressible porous media equation. Archive for Rational Mechanics and Analysis, 247(1), p.1.
2. A. Castro, D. Cordoba and D. Lear, Global existence of quasi-stratified solutions for the confined IPM equation, Arch. Ration. Mech. Anal. 232 (2019), no. 1, 437–471.
3. T. Elgindi, On the asymptotic stability of stationary solutions of the inviscid incompressible porous medium equation, Arch. Ration. Mech. Anal. 225 (2017), no. 2, 573–599.

Non-uniqueness for IPM:

1. D. Cordoba, D. Faraco and F. Gancedo, Lack of uniqueness for weak solutions of the incompressible porous media equation, Arch. Ration. Mech. Anal. 200 (2011), no. 3, 725–746.
2. Shvydkoy, R.: Convex integration for a class of active scalar equations. J. Am. Math. Soc. 24(4), 1159–1174 (2011).
3. L. Szekelyhidi, Jr. Relaxation of the incompressible porous media equation, Ann. Sci. de l’Ecole Norm. Superieure (4) 45 (2012), no. 3, 491–509.

Related:

1. Chemetov, N. and Neves, W., 2013. The generalized Buckley–Leverett system: solvability. Archive for Rational Mechanics and Analysis, 208, pp.1-24.
2. Córdoba, A., Córdoba, D. and Gancedo, F., 2011. Interface evolution: the Hele-Shaw and Muskat problems. Annals of mathematics, pp.477-542.