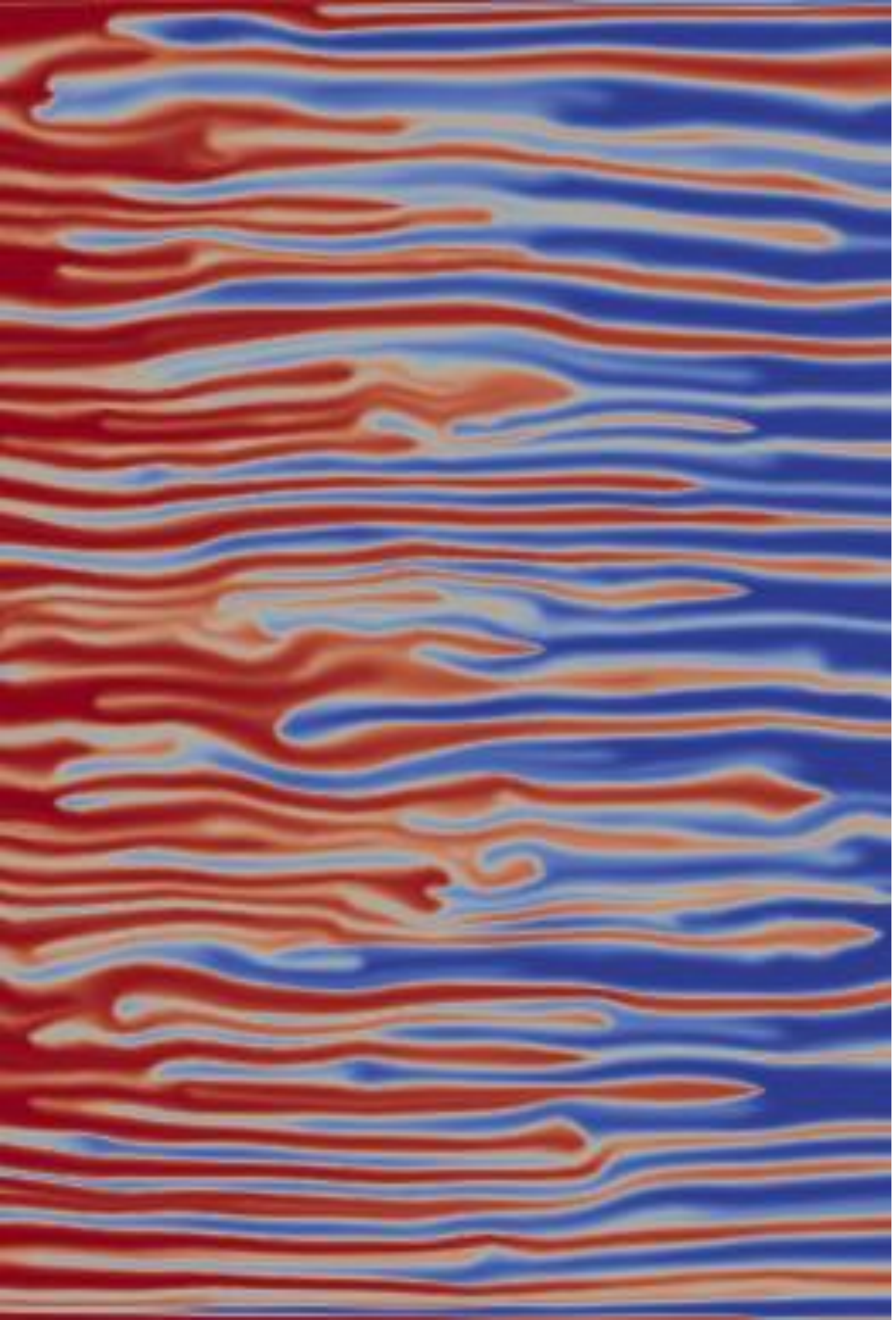

Viscous fingering: theory and applications

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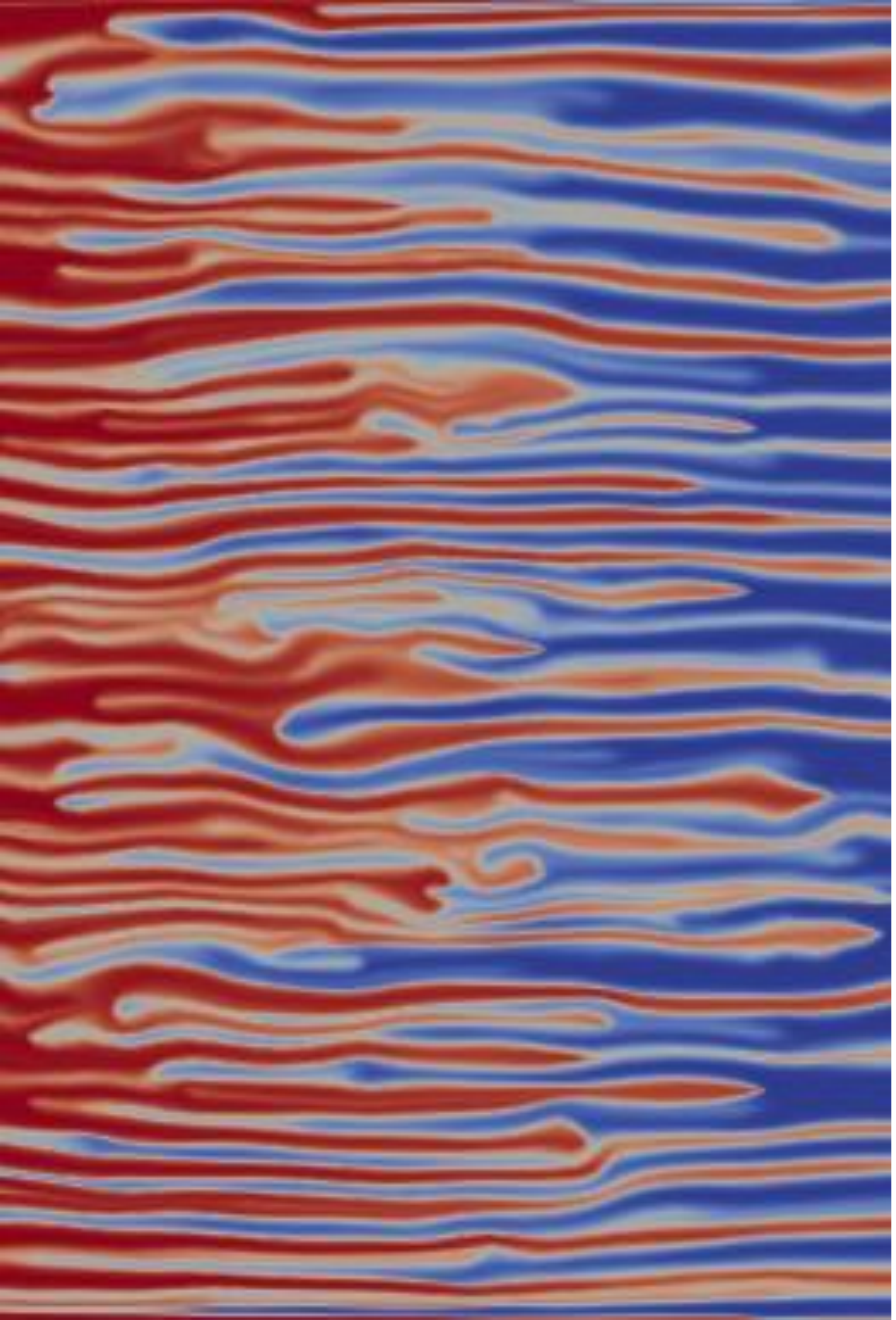
Matemática PUC-Rio

Seminário g.t.p

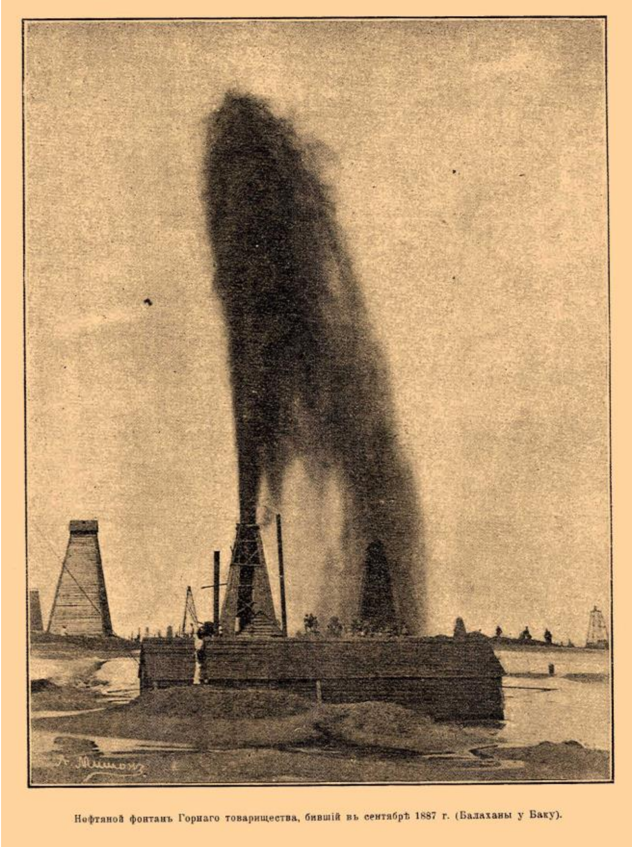


Outline

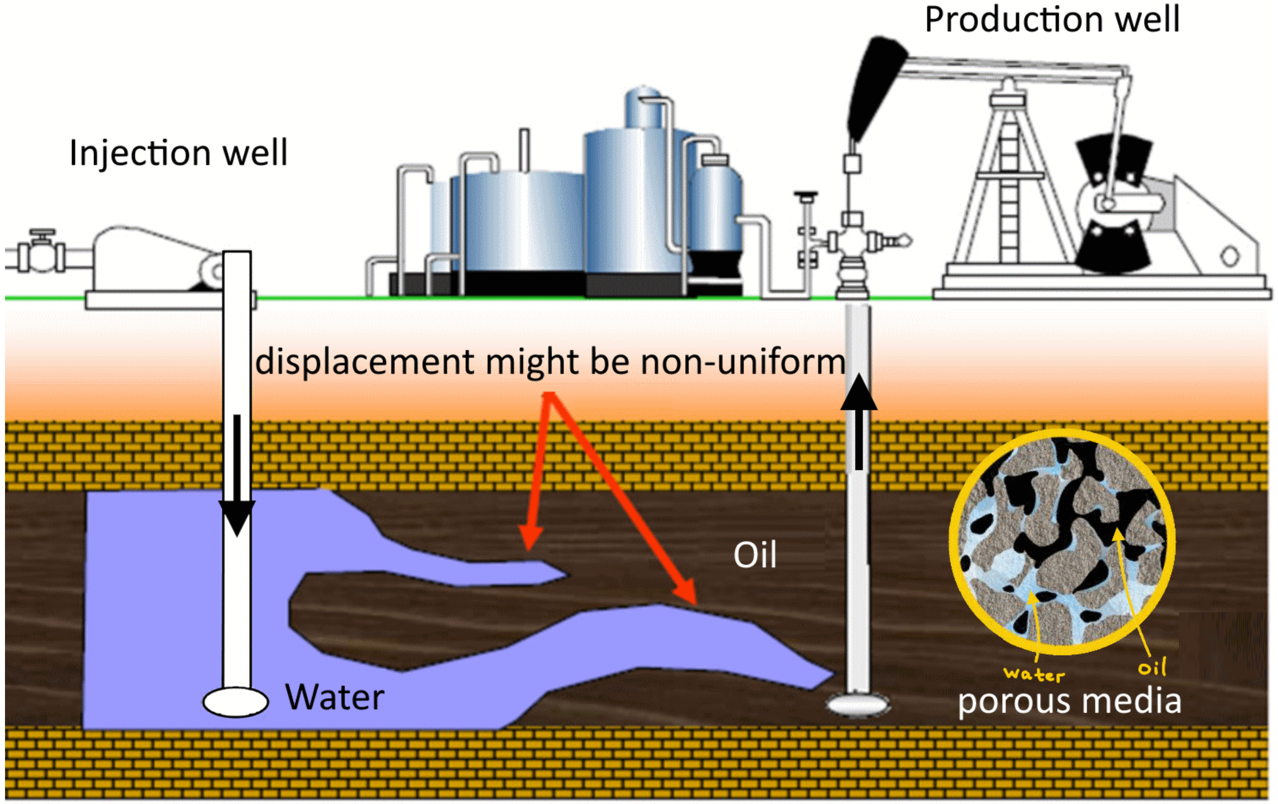
1. General phenomenon:
   - Viscous fingers
   - Applications
2. Mathematical models:
   - DLA (probability)
   - IPM (partial differential equations)
3. Theory:
   - Derivation of IPM
   - Well-posedness and dynamics
4. “Toy” model (multi-tubes model)

Oil recovery

How oil was recovered in the beginning?
(Baku, 1857)

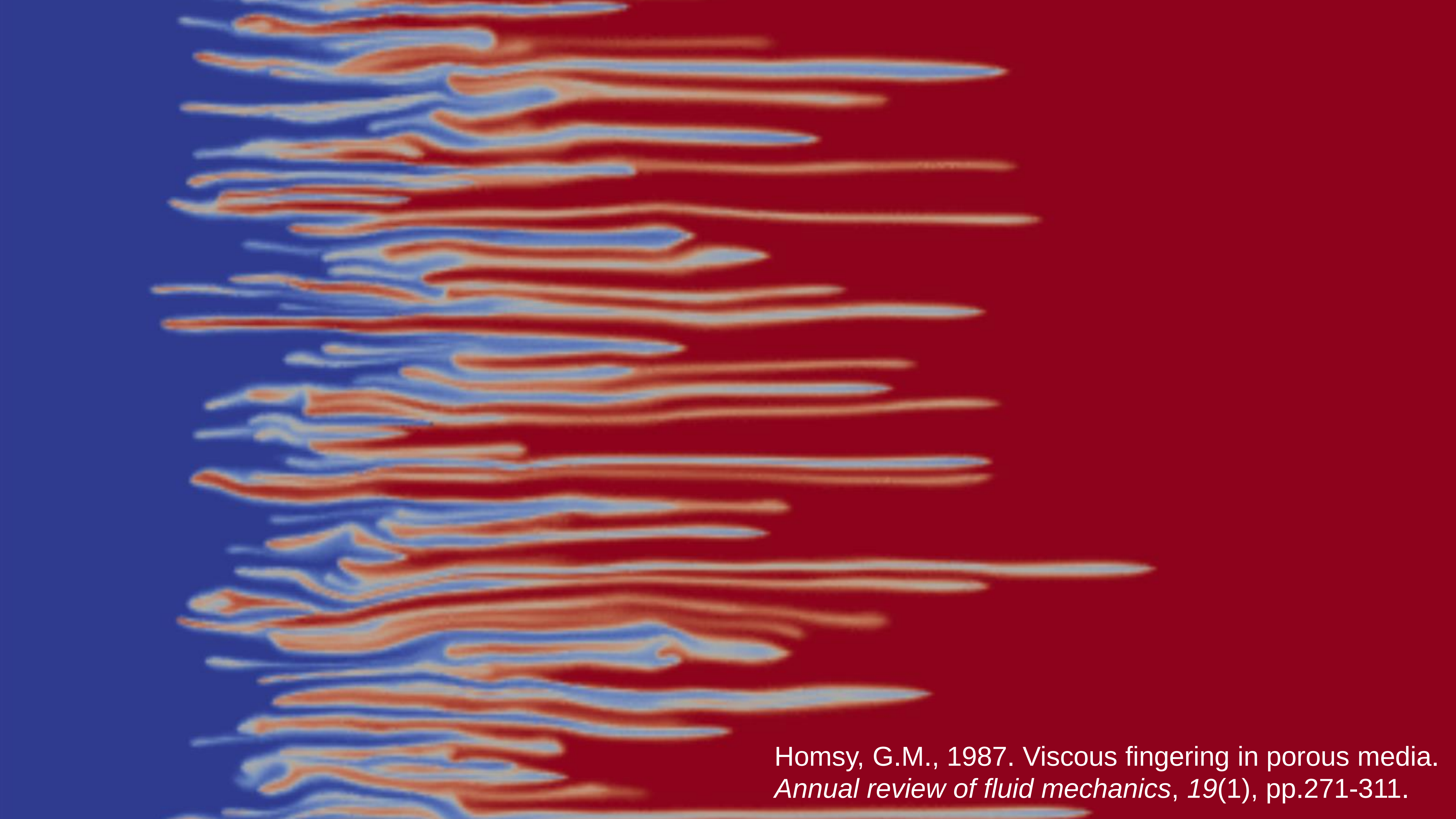


How oil is recovered now?



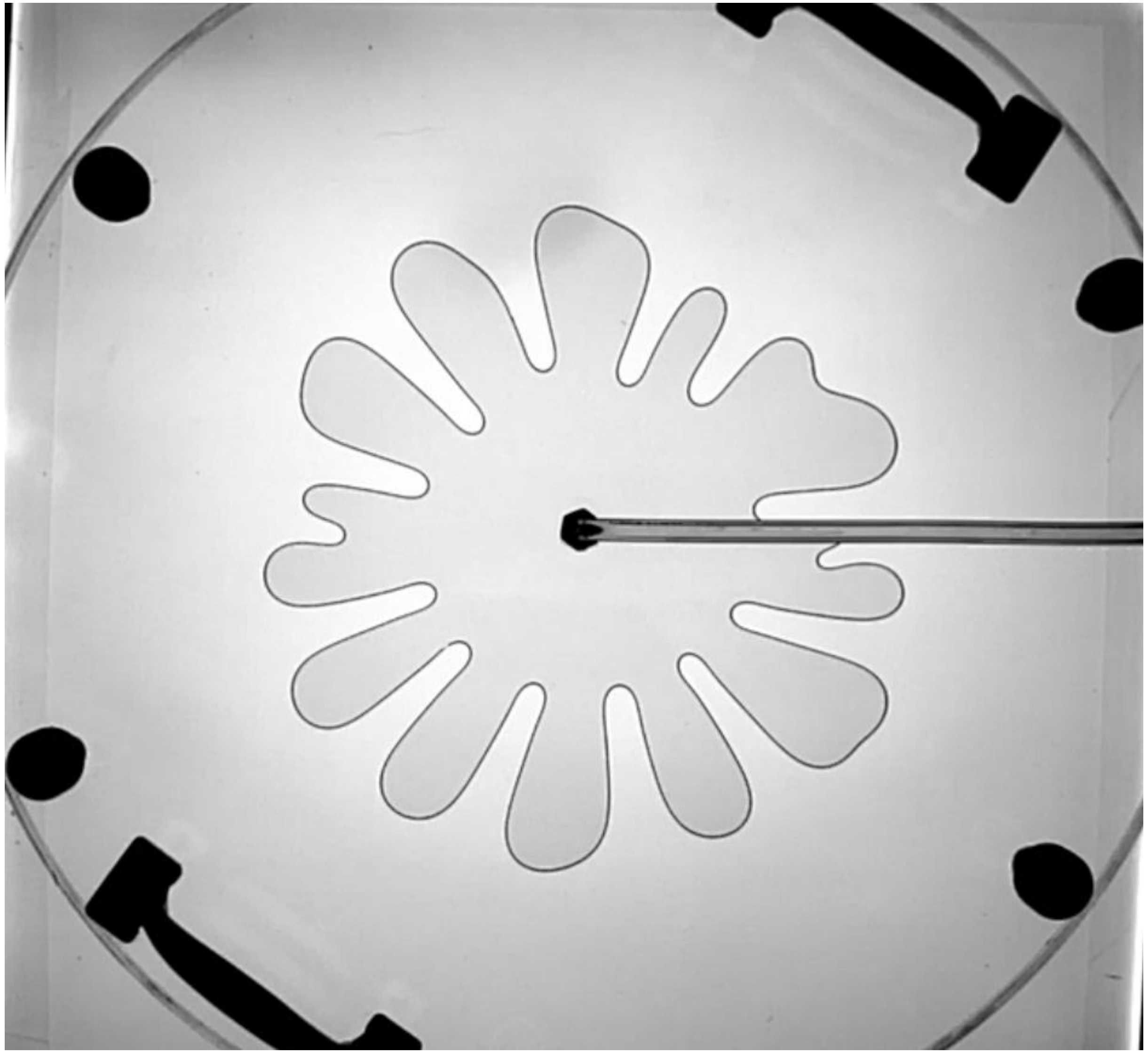
Interesting fact

First oil in Russia was found close to Ukhta (my native city! Now the city of oil and gas industry)



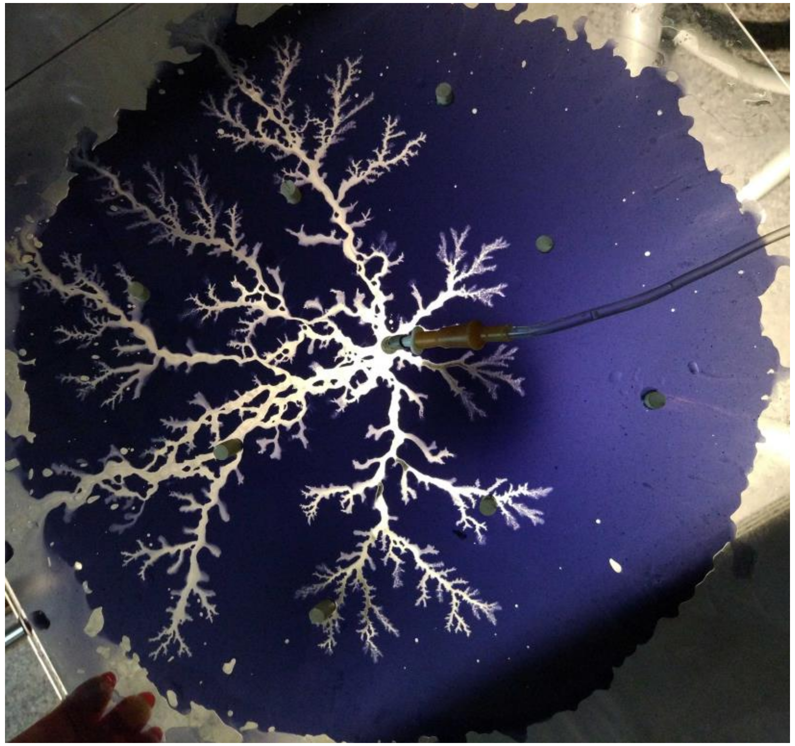
Homsy, G.M., 1987. Viscous fingering in porous media. *Annual review of fluid mechanics*, *19*(1), pp.271-311.

Hele-Shaw cell (1898)



Henry Selby Hele-Shaw
(1854-1951)
English mechanical and automobile engineer

Hele-Shaw cell

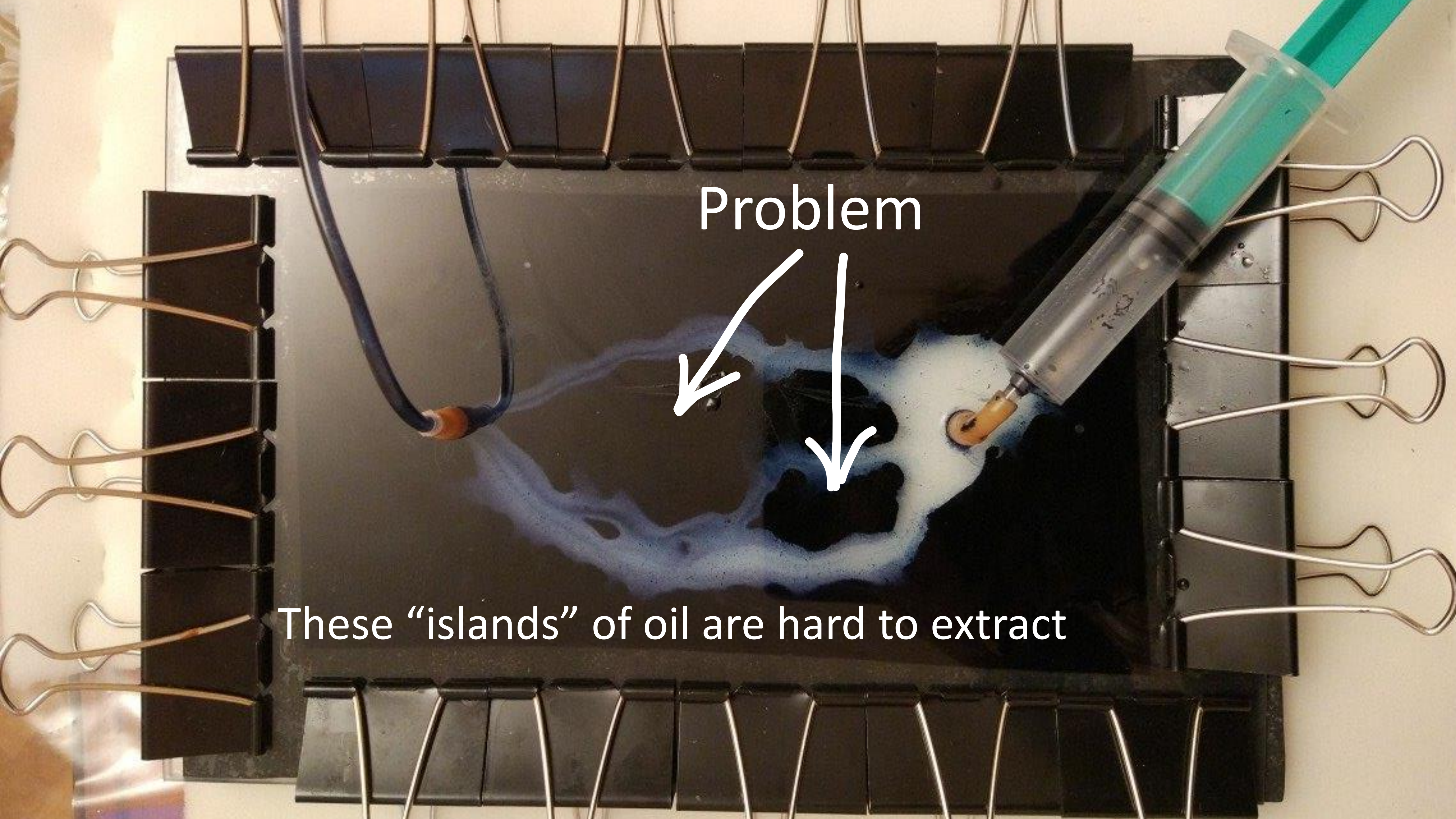


Two miscible fluids (no capillary pressure)

Two immiscible fluids (with capillary pressure)

Reason for instability – different viscosities (mobilities)

Photos from experiments at Chebyshev Laboratory, St Petersburg, Russia

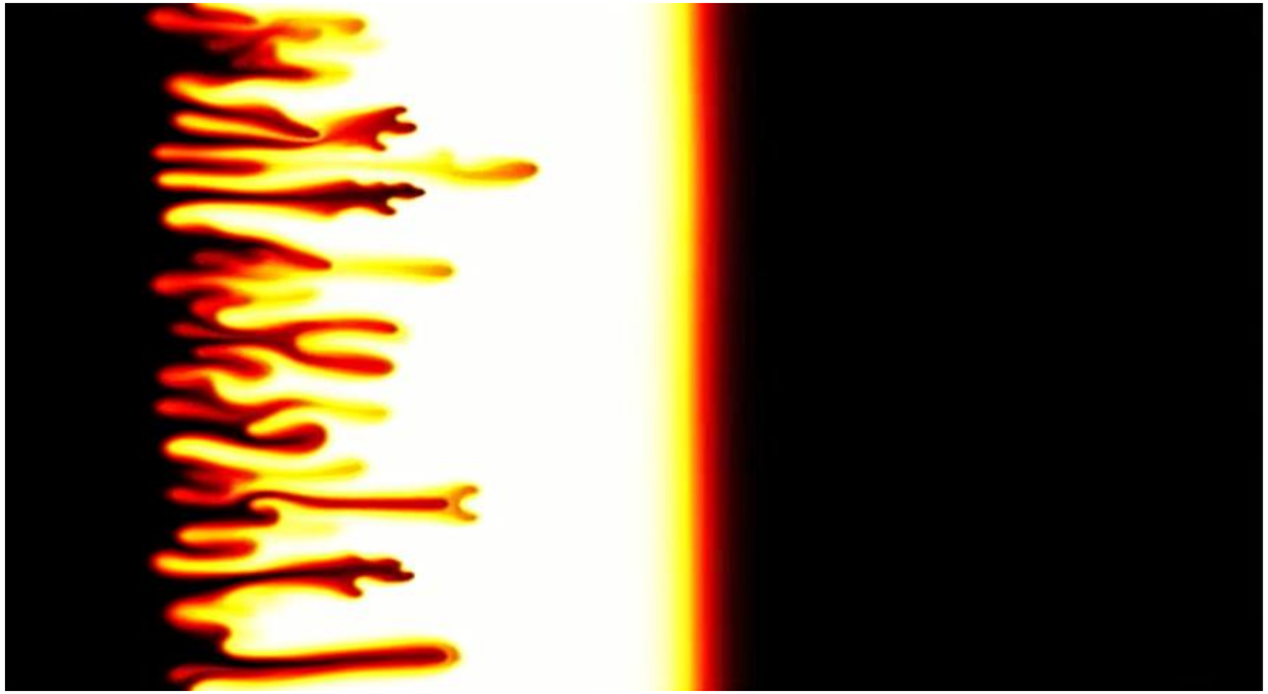

Problem
These “islands” of oil are hard to extract

An engineer tries to prevent viscous fingering….

Enhanced Oil Recovery (EOR) methods:

- Chemical – inject polymers, surfactants
- Thermal – inject hot water or steam
- …

Injection of polymer:



Questions of interest:

Optimize the injection scheme:

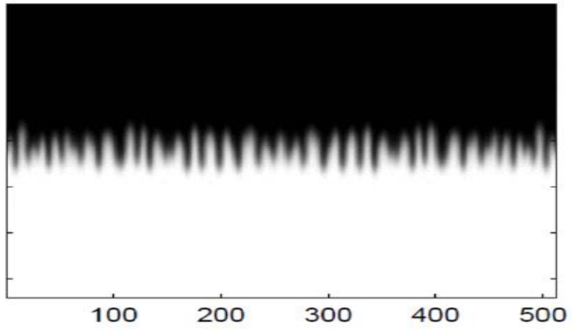
- what to inject?
- for how long?

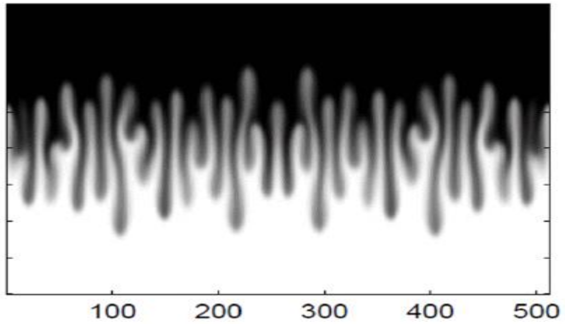
What size of polymer slug?

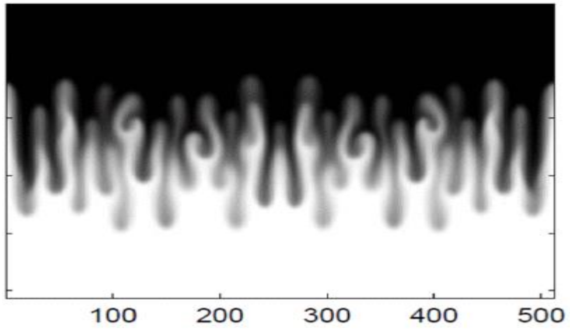
How fast the mixing zone is growing?

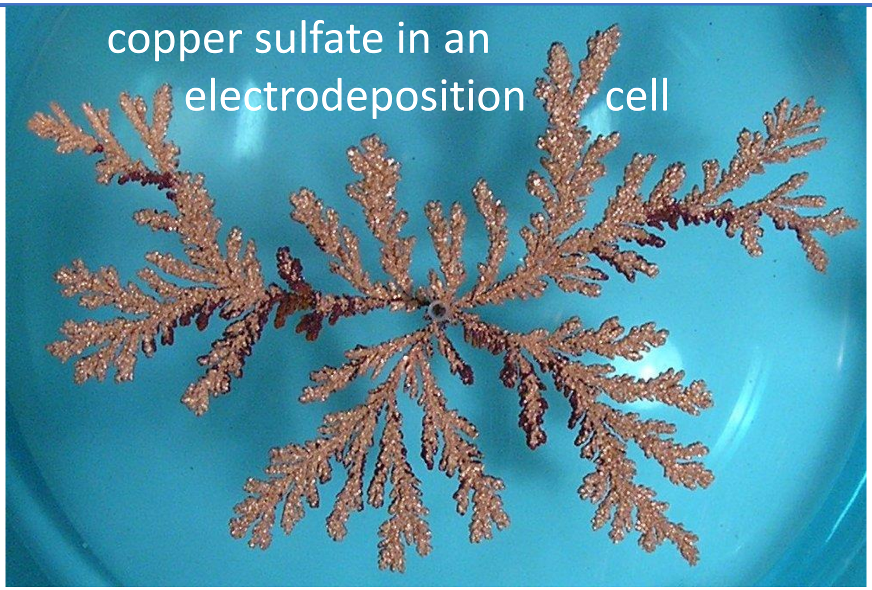
Related phenomena

Gravitational fingering
due to different densities

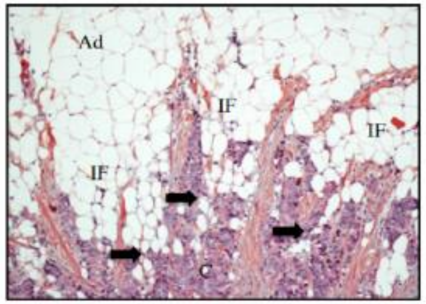


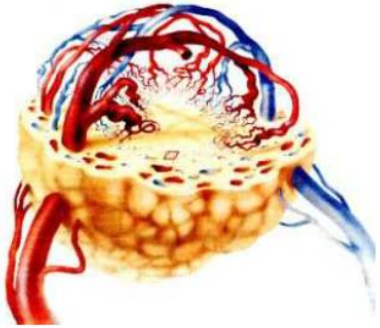




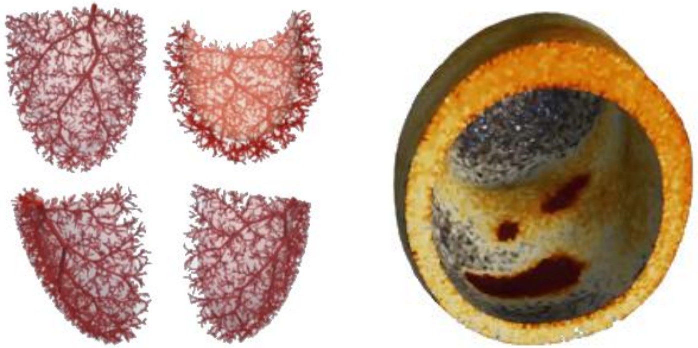


Biological processes





Tumor growth
(cells with different mobilities)

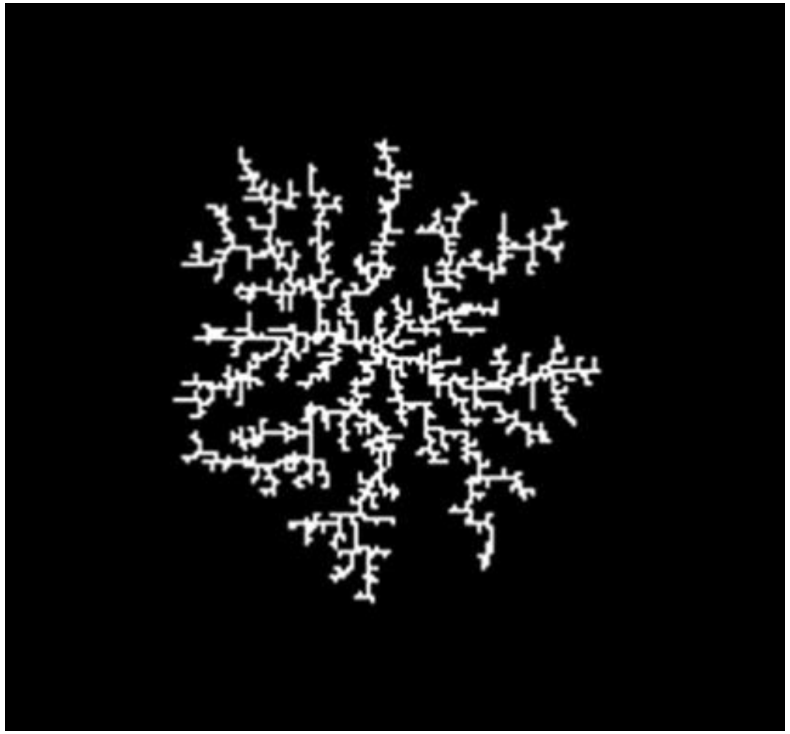


Cardiac perfusion
(cells and organs can be seen as a porous media)

Math models

PROBABILITY

- stochastic processes
- interacting particle systems



Diffusion-limited aggregation (DLA)

PARTIAL DIFFERENTIAL EQUATIONS (PDEs)

- Well-posedness
- Dynamics of solution (as time goes to infty)

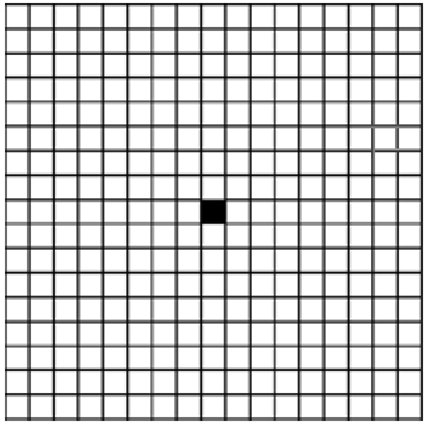
$$c_t + div(uc) = \varepsilon \Delta c$$
$$div\,(u) = 0$$
$$u = -\nabla p - (0, c)$$

c – concentration
u – velocity
p – pressure

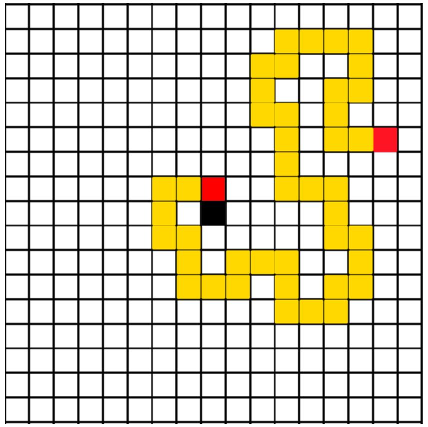
Incompressible porous medium eq (IPM)

DLA = Diffusion limited aggregation

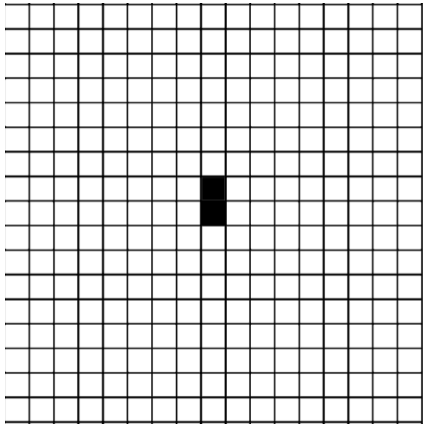
Step 1:
Fix a particle



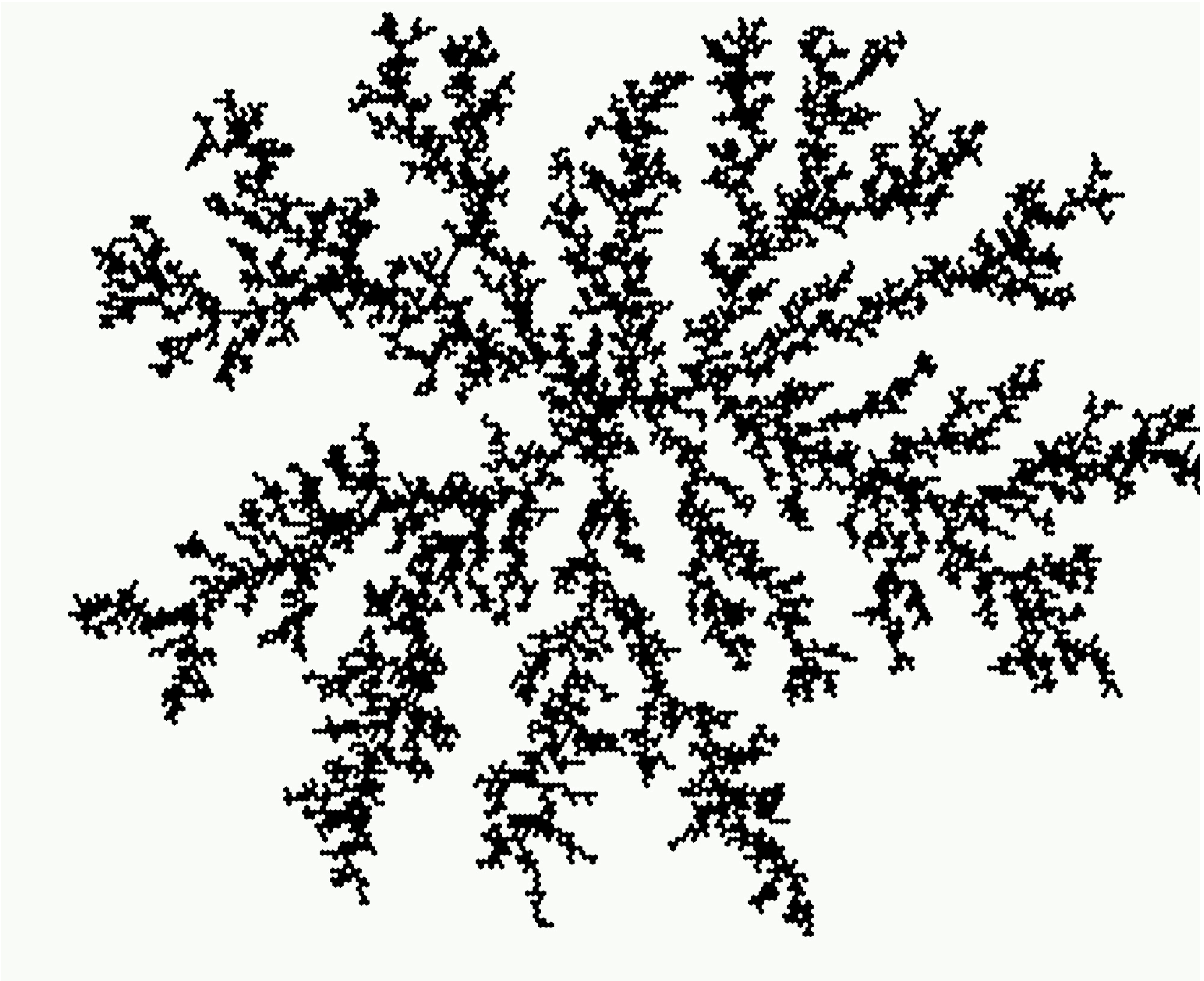
Step 2:
Take at random a particle from “far away”. Let it do random motion until the moment it “touches” one of the black particles

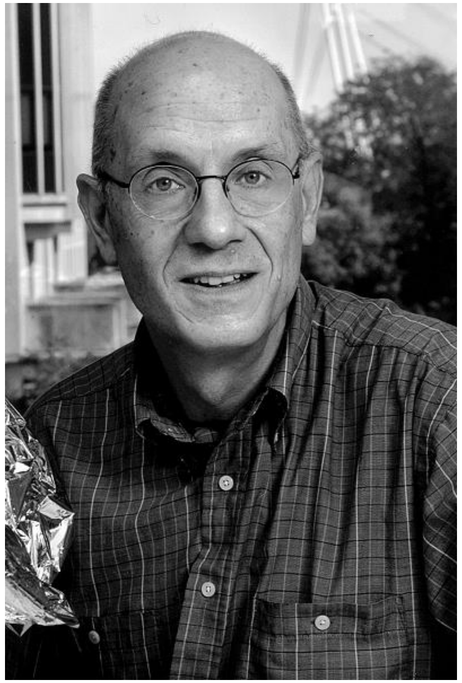


Step 3:
at the point of touch, the red particle stops and becomes black



DLA = Diffusion limited aggregation (1981)



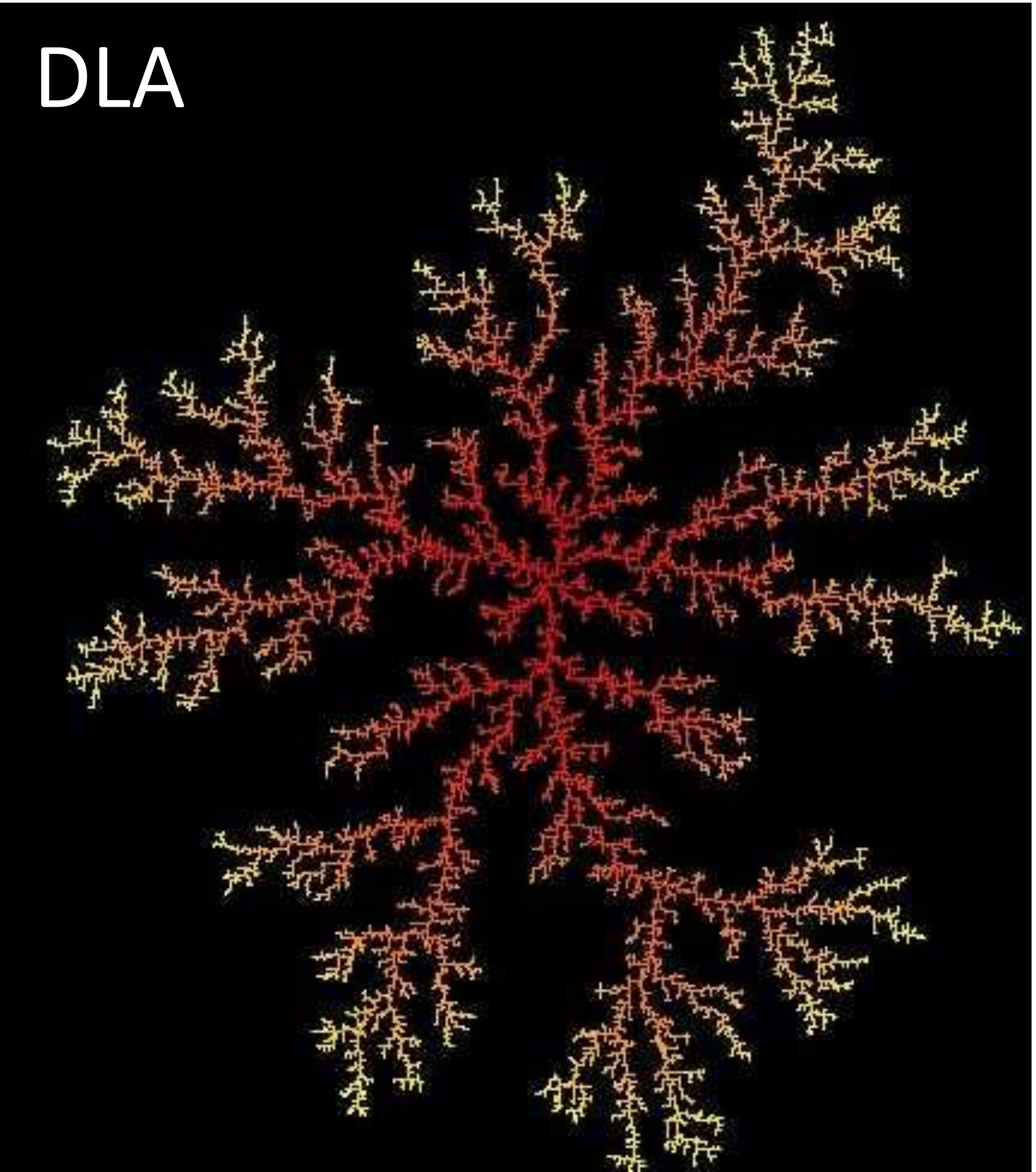


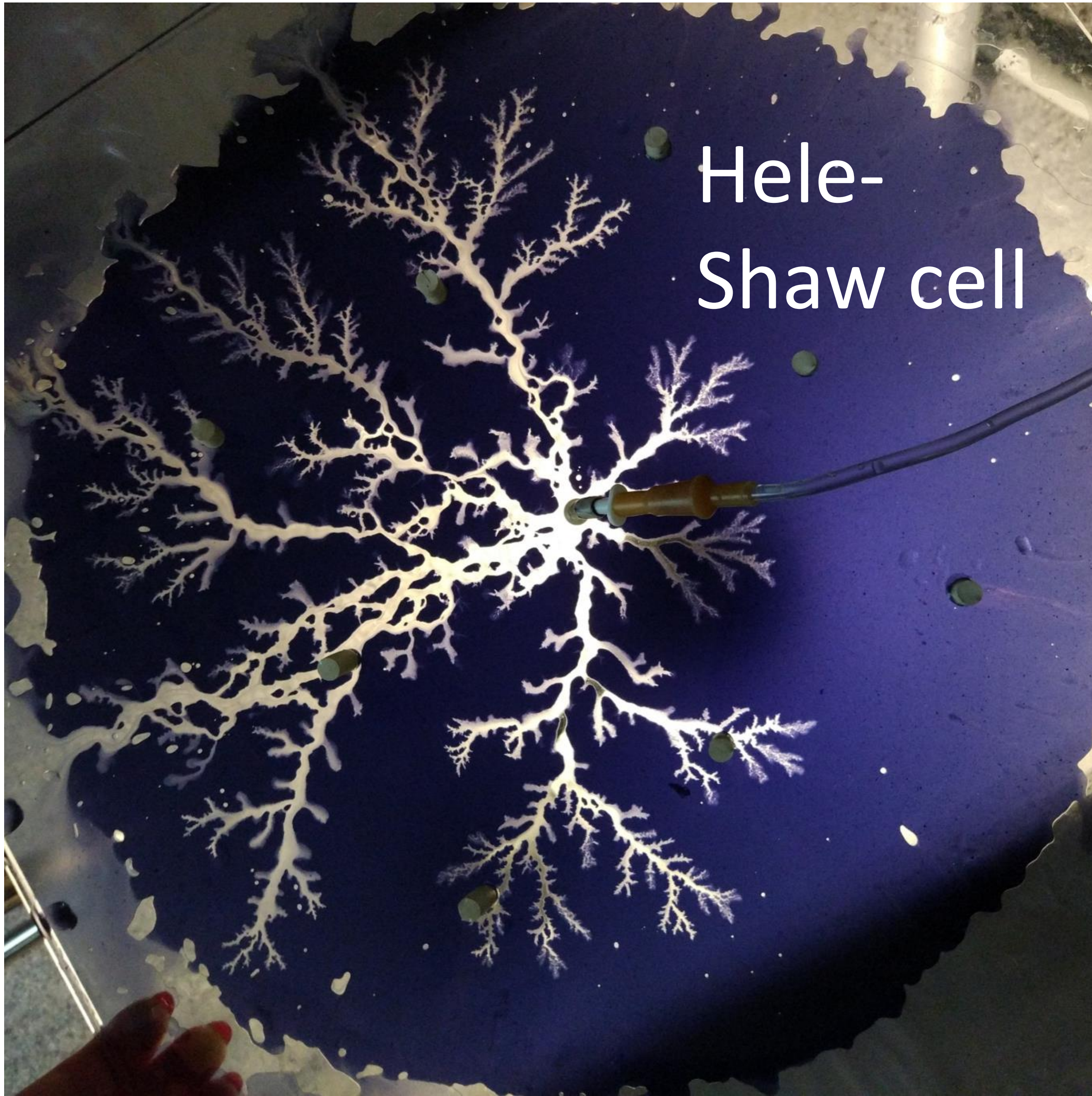
Thomas Witten   Leonard Sander

American theoretical physicists

- Fractal structure:
  fractal dimension is 1.71
- Scale-invariance
- Multifractality
- Harmonic measures

Looks very similar!





IPM = Incompressible Porous Medium equation

- Often called Saffman-Taylor instability
- 2d case
- Let us consider gravitational fingering

c – concentration of heavy fluid
u – velocity
p – pressure

$$c_t + div\,(uc) = \varepsilon\,\Delta c \qquad \text{(transport equation)}$$

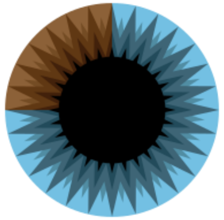
$$div\,(u) = 0 \qquad \text{(incompressibility condition)}$$

$$u = -\nabla p - (0, c) \qquad \text{(Darcy's law)}$$

Let's ''dive'' into this system and understand where it comes from!

$div\ (u) = 0$

Who has never seen @3blue1brown videos?

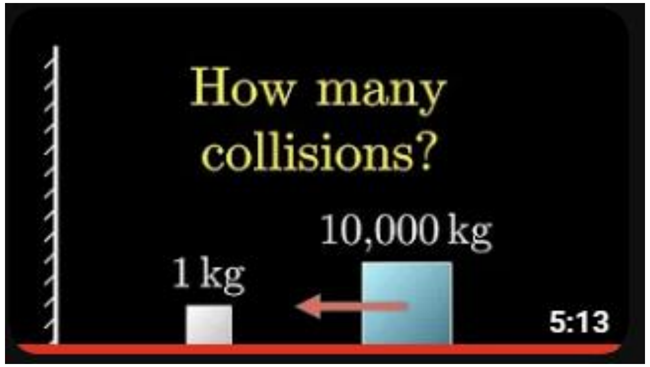


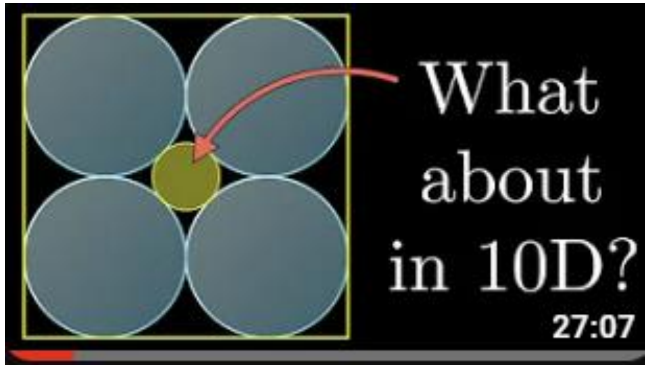
See this video (starting from 2:16 till 4:31):
“Divergence and curl: The language of Maxwell's equations, fluid flow, and more”

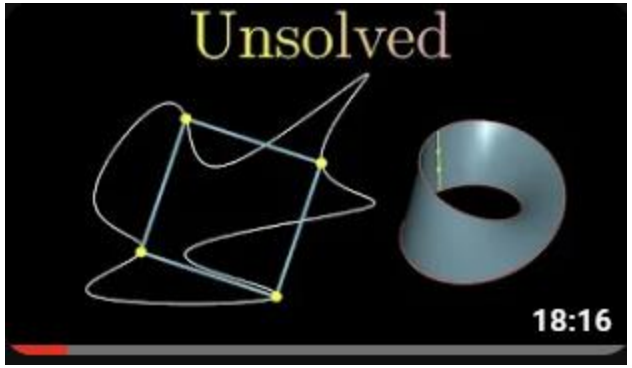
There are many incredibly beautiful math explainers made by Grant Sanderson (@3blue1brown)!

Grant Sanderson

- How do colliding blocks compute PI?
- Thinking outside the 10-dimensional box
- Who cares about topology? (Inscribed rectangle problem)





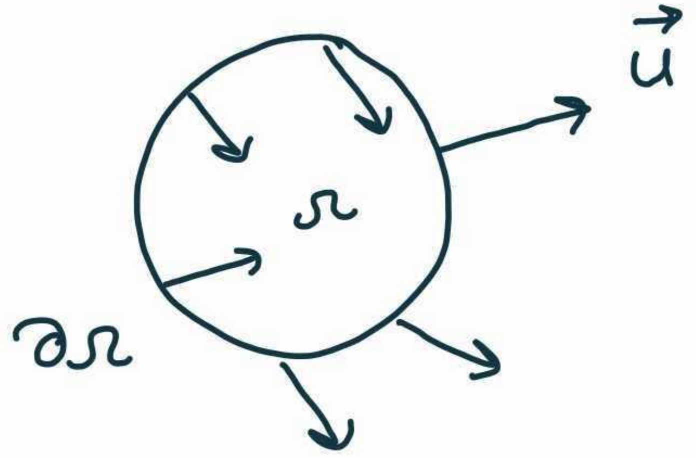


$$div\ (u) = \frac{\partial u^1}{\partial x_1} + \frac{\partial u^2}{\partial x_2} = 0$$

Conservation law $\Rightarrow$ $div\ (u) = 0$

$$0 = \int_{\partial\Omega} u \cdot n\, dS = \iint_{\Omega} div(u)\, dx$$

Green's theorem



$$c_t + div\,(uc) = 0$$

The amount of EACH fluid that "flows in" equals to the amount of fluid that "flows out"

$\int_\Omega c\,dx$ – total mass in a small volume Ω

uc – flux of mass at the boundary $\partial\Omega$

$$\frac{\partial}{\partial t}\int_\Omega c\,dx = -\int_{\partial\Omega}(uc)\cdot n\,dS = -\iint_\Omega div(uc)\,dx$$

Green's theorem

This is just a conservation of mass of the fluid with concentration c!

$$u = -\nabla p - (0, c)$$

Darcy's law

c – concentration
u – velocity
p – pressure

Simplest case

$$u = -\nabla p$$

$$div(u) = 0$$

$$\Delta p = 0$$

With gravity

$$u = -\nabla p - (0, c)$$

$$u = F(c)$$

(Biot-Savart law)

With viscosity

$$u = -\frac{k}{\mu(c)} \nabla p$$

k – permeability
$\mu(c)$ – viscosity

$$u = \tilde{F}(c)$$

Why instabilities occur?

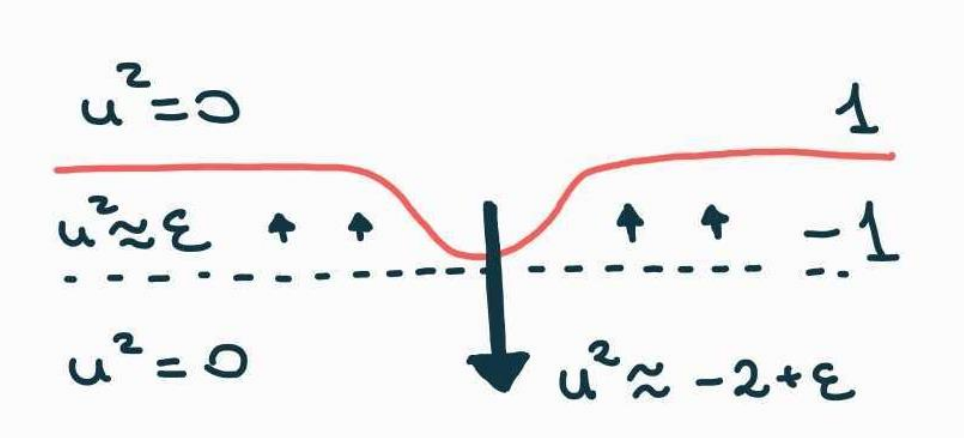
Active scalar $\begin{cases} c_t + div\,(uc) = 0 \\ u = F(c) \end{cases}$

Here $F(c)$ is some (non-local) operator

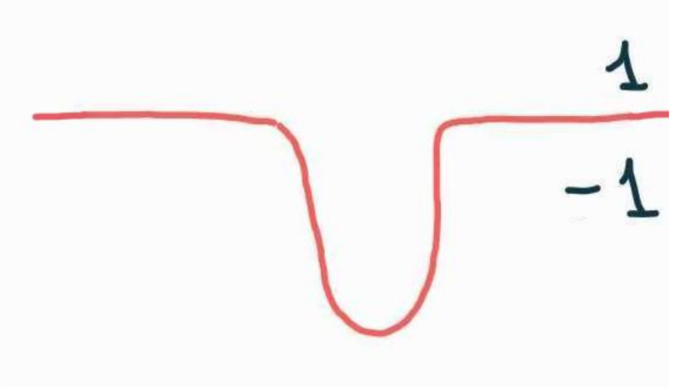
Simplified model $\begin{cases} c_t + div\,(uc) = 0 \\ div\,(u) = 0 \\ u = (u^1, \bar{c} - c) \end{cases}$

It is a hair-trigger effect!

No flow



Velocity u changes due to concentration c



Concentration c changes due to velocity u

Questions of interest

1. Well-posedness: local: yes (2007) global: UNSOLVED PROBLEM

The best result (up to Jan 2023):

Kiselev, A. and Yao, Y., 2023. Small scale formations in the incompressible porous media equation. Archive for Rational Mechanics and Analysis, 247(1), p.1.

"Informally" (only conditional result):

Let solution stay smooth for all $t > 0$ (in an "appropriate" Sobolev space). Then at least as $t \to \infty$ the Sobolev norm blows-up.

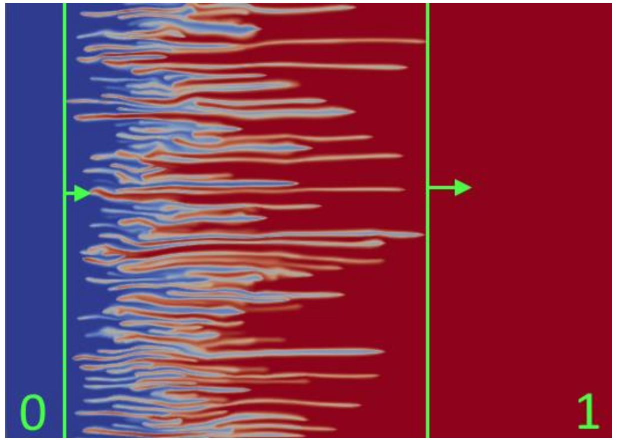
2. Dynamics of mixing zone: $\boxed{\varepsilon > 0}$

- many laboratory and numerical experiments show linear growth of the mixing zone

$$c_t + div\,(uc) = \boxed{\varepsilon \Delta c}$$

$$div\,(u) = 0$$

$$u = -\nabla p - (0, c)$$



- What is the EXACT speed of growth? UNSOLVED PROBLEM

Simplified model with gravity: comparison theorem

- Simpl. model: assumption $p(x,y) \sim p(y), \quad p_y(x,y) \sim p_y(y)$

$$c_t + u \cdot \nabla c = \varepsilon\, \Delta c$$
$$div\,(u) = 0$$
$$u = (u^1, \bar{c} - c)$$

- Consider 1d equations (viscous Burgers equation)

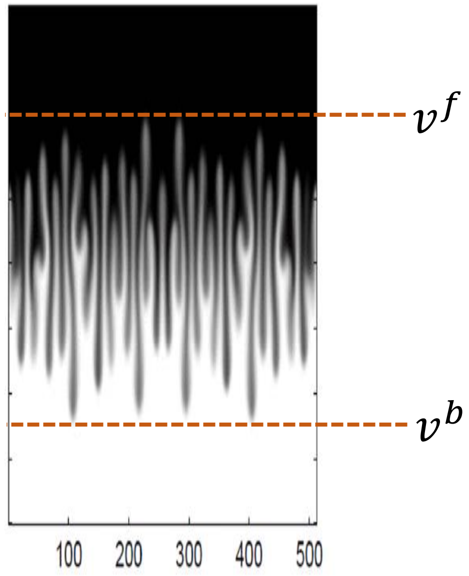
$$c_t^{max} + (1 - c^{max})c_y^{max} = \varepsilon\,(c^{max})_{yy}$$

$$c_t^{min} + (-1 - c^{min}) \cdot c_y^{min} = \varepsilon\,\left(c^{min}\right)_{yy}$$

Comparison theorem (Otto-Menon, 2005)

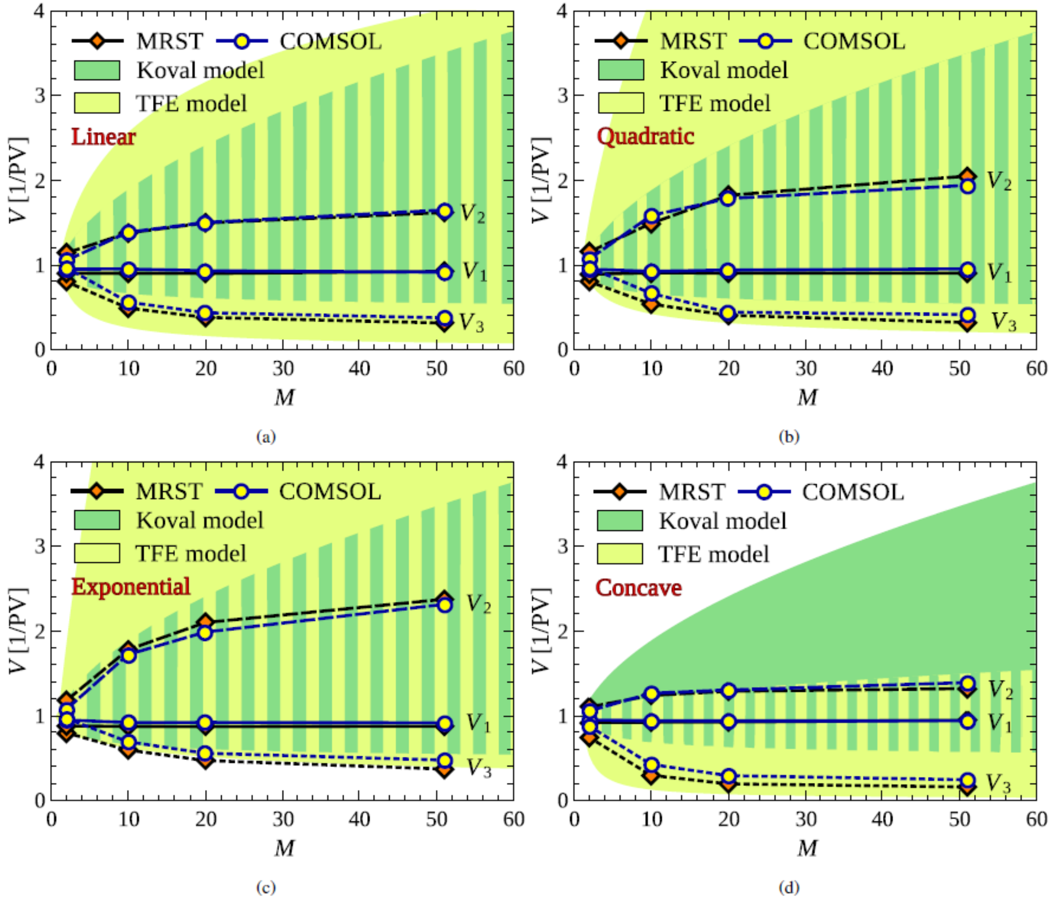
- If $c(0,x,y) < c^{max}(0,y)$ then $c(t,x,y) \le c^{max}(t,y)$
- If $c(0,x,y) > c^{min}(0,y)$ then $c(t,x,y) \ge c^{min}(t,y)$

1. It gives upper bound for the faster finger
$$v^f \le 1$$
2. It gives upper bound for the back front
$$v^b \ge -1$$
3. Estimate is sharp if
    1. There is no transverse flow
    2. Drop of concentration on a finger tip is -1 -> +1
4. Numerics shows that estimate is far from sharp
5. We want to get better estimate

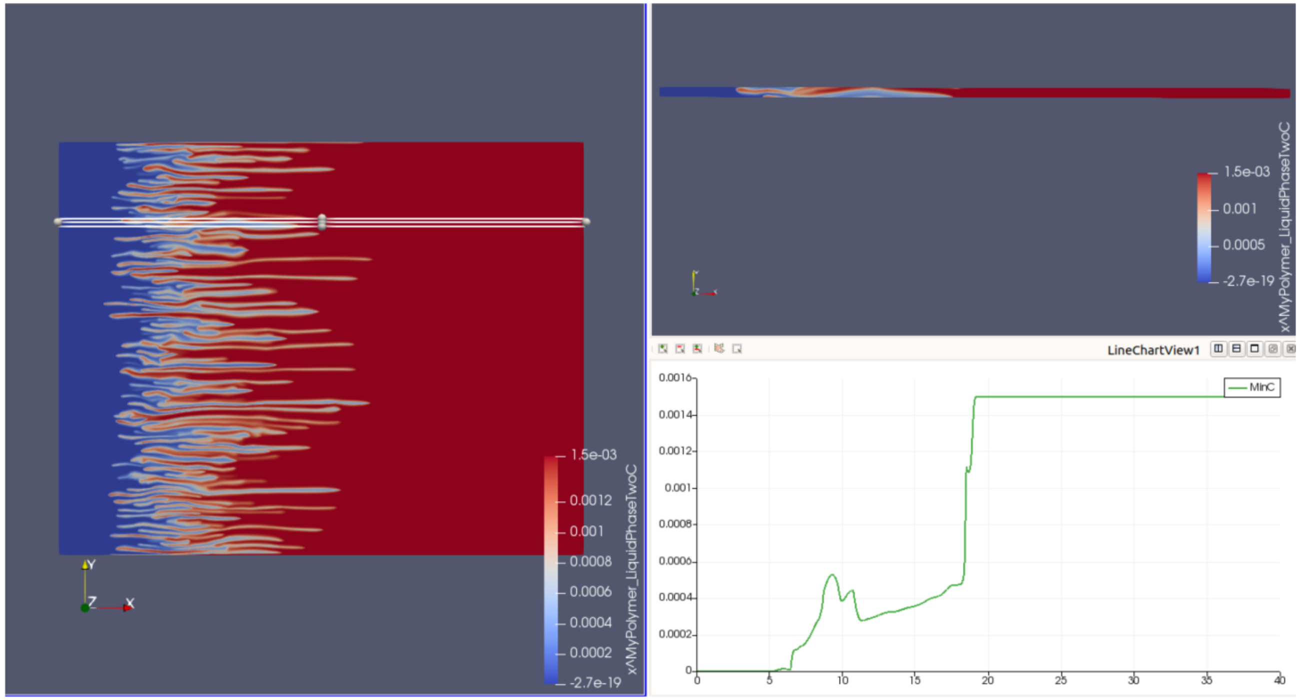


Numerics for viscous fingers

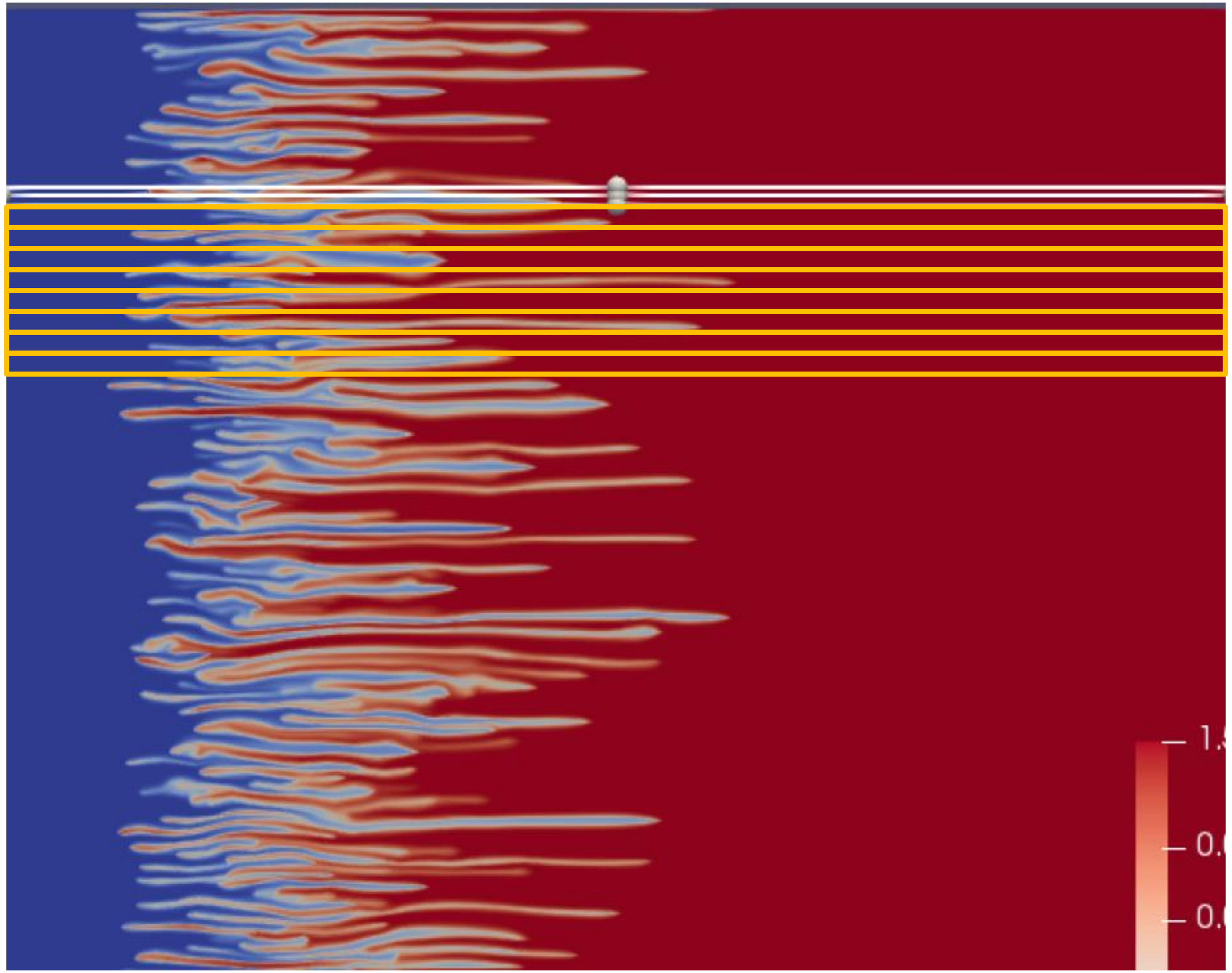
F. Bakharev, A. Enin, A. Groman, A. Kalyuzhnyuk,
S. Matveenko, **Yu. Petrova**, I. Starkov, S. Tikhomirov
“Velocity of viscous fingers in miscible displacement: Comparison with analytical models”
Journal of Computational and Applied Mathematics, 2022



Possible mechanism: intermediate concentration



Multi-tubes model



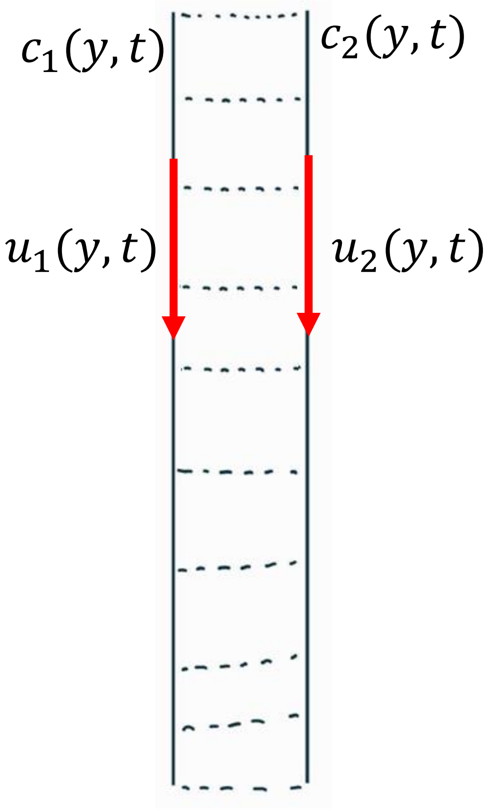
Two-tubes model (with gravity)

Original equations

$$c_t + div(uc) = \varepsilon \, \Delta c$$
$$div\, u = 0$$

Two-tube equations

$$\partial_t c_1 + \partial_y (u_1 c_1) - \varepsilon \, \partial_{yy} c_1 = 0$$
$$\partial_t c_2 + \partial_y (u_2 c_2) - \varepsilon \, \partial_{yy} c_2 = 0$$



Two-tubes model (with gravity)

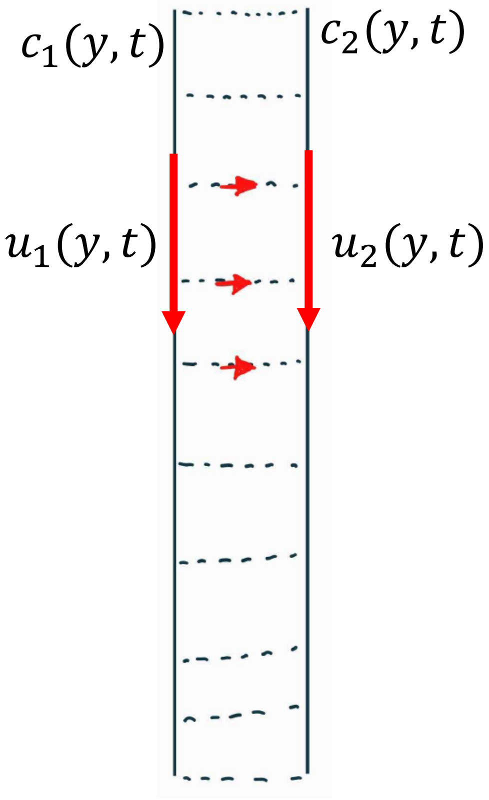
Original equations

$$c_t + div(uc) = \varepsilon\, \Delta c$$
$$div\, u = 0$$

Two-tube equations: inclusion of transverse flow

$$\partial_t c_1 + \partial_y(u_1 c_1) - \varepsilon\, \partial_{yy} c_1 = \boxed{-(-1)^{1,2}\partial_y u_{1,2} \cdot c_{1,2}}$$
$$\partial_t c_2 + \partial_y(u_2 c_2) - \varepsilon\, \partial_{yy} c_2 = \boxed{(-1)^{1,2}\partial_y u_{1,2} \cdot c_{1,2}}$$

$$(-1)^{1,2}\partial_y u_{1,2} \cdot c_{1,2} = \begin{cases} -\partial_y u_1 \cdot c_1, & \partial_y u_1 < 0, \\ +\partial_y u_2 \cdot c_2, & \partial_y u_1 > 0 \end{cases}$$



Model for velocities is different for IPM and simplified model:

- Simplified: $u = \bar{c} - c, \quad u_1 = \frac{c_1 + c_2}{2} - c_1, \qquad u_2 = \frac{c_1 + c_2}{2} - c_2$
- IPM: we need to introduce pressure (not in this lecture)

Initial condition:

$$c_{1,2}(y,0) = -1, y < 0$$
$$c_{1,2}(y,0) = +1, y > 0$$

Main result (Simplified model, gravity-driven fingers)

Theorem (Efendiev, P., Tikhomirov, 2022+)

Consider a two-tube model with gravity.
Then there exists unique (up to swap) c_1^*, c_2^* such that simplified two-tubes system has travelling waves

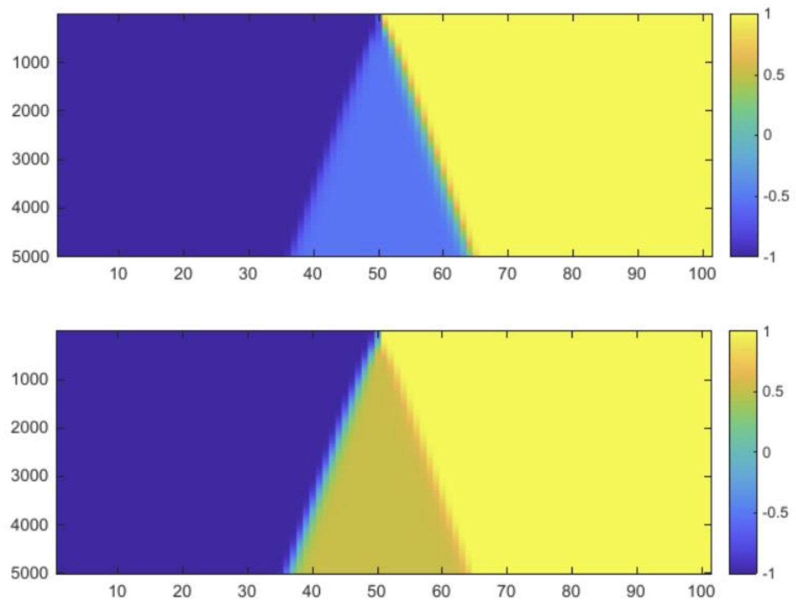
$$(-1,-1) \to (c_1^*, c_2^*) \to (1,1)$$

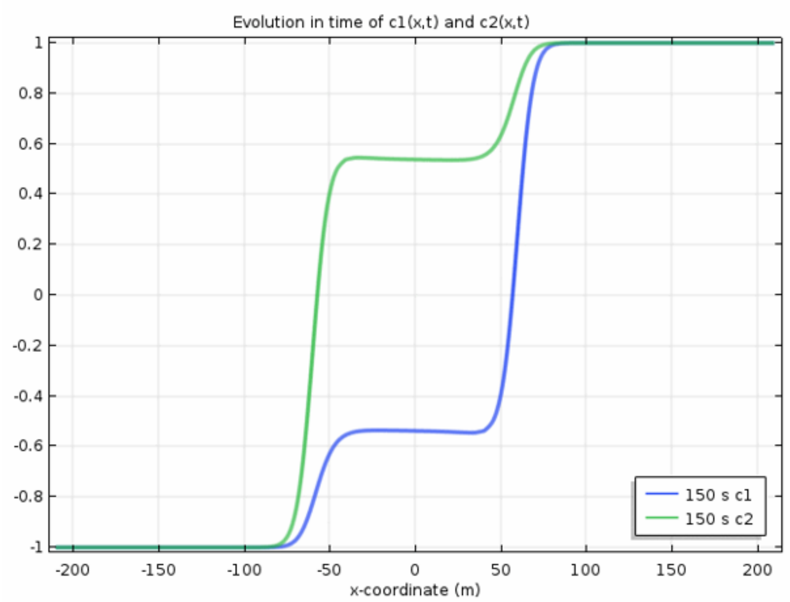
Moreover,

$$c_1^* = -\frac{1}{2}, \quad c_2^* = \frac{1}{2},$$
$$v^b = -\frac{1}{4}, \quad v^f = \frac{1}{4}.$$

Including in the system cross-flow automatically creates intermediate concentration

time

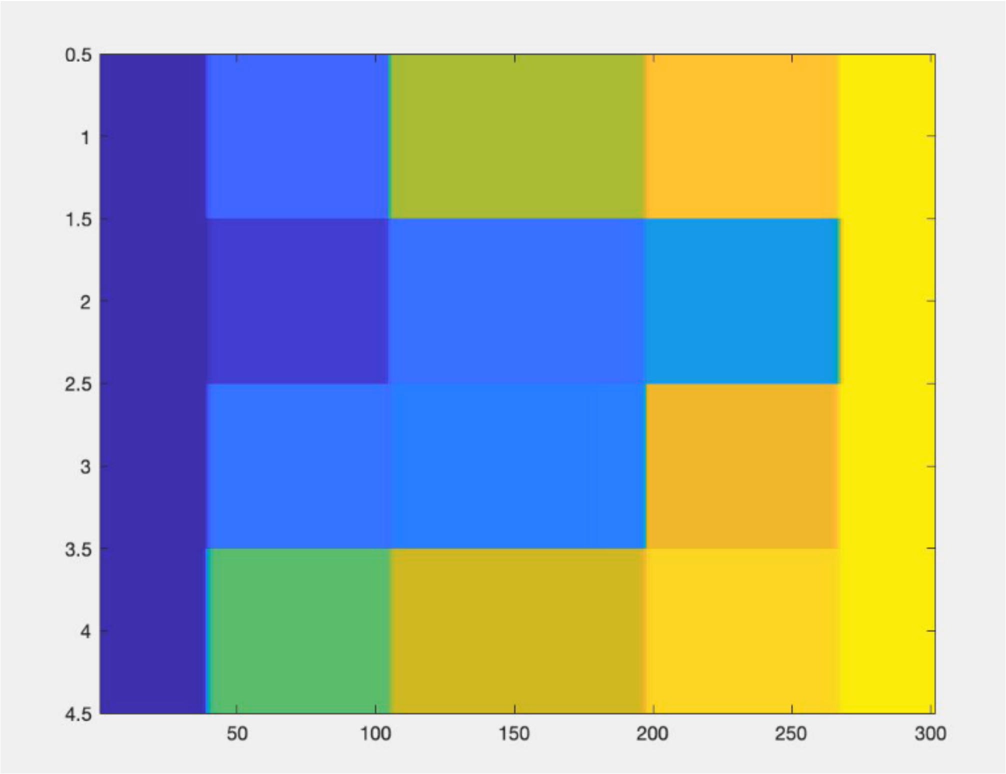




What’s next?

1. How to obtain similar results for “viscous” model?

2. Does the n-tube model posses a system of n travelling waves?
   How to determine their constant states?
   Can we go to the limit as the number of tubes $n \to \infty$?



Take-home message

1. If not yet, watch @3blue1brown YouTube Channel for beautiful math explanations

2. Conservation law $\Rightarrow$ $div(u) = 0$, where u is the flux

3. If you are interested in mathematics of shock/travelling waves there is a course at PUC-Rio this semester:

   "Shock waves in conservation laws and reaction-diffusion equations"

References

Thank you very much!

Own works:

1. Bakharev, F., Enin, A., Groman, A., Kalyuzhnyuk, A., Matveenko, S., Petrova, Y., Starkov, I. and Tikhomirov, S., 2022. Velocity of viscous fingers in miscible displacement: Comparison with analytical models. Journal of Computational and Applied Mathematics, 402, p.113808.
2. Efendiev Ya., Petrova Yu., Tikhomirov S., 2022+, A cascade of two travelling waves in a two-tube model of gravitational fingering. In preparation.

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3. Menon, G. and Otto, F., 2005. Dynamic scaling in miscible viscous fingering. Communications in mathematical physics, 257, pp.303-317.
4. Homsy, G.M., 1987. Viscous fingering in porous media. Annual review of fluid mechanics, 19(1), pp.271-311.

DLA and Hele-Shaw:

1. Witten, T.A. and Sander, L.M., 1983. Diffusion-limited aggregation. Physical review B, 27(9), p.5686.
2. Hele-Shaw, H.S., 1898. Flow of water. Nature, 58(1509), pp.520-520.

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2. A. Castro, D. Cordoba and D. Lear, Global existence of quasi-stratified solutions for the confined IPM equation, Arch. Ration. Mech. Anal. 232 (2019), no. 1, 437–471.
3. T. Elgindi, On the asymptotic stability of stationary solutions of the inviscid incompressible porous medium equation, Arch. Ration. Mech. Anal. 225 (2017), no. 2, 573–599.

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