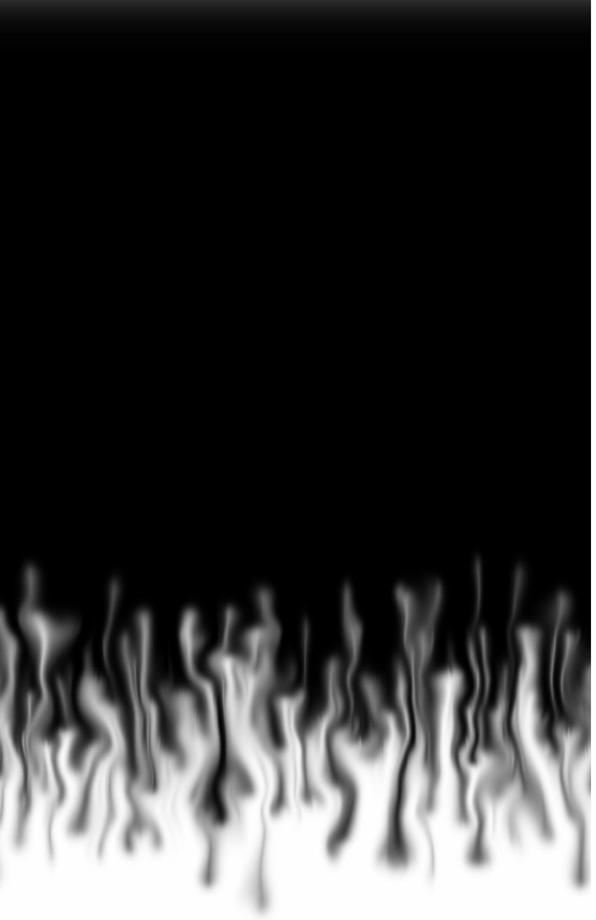


Ondas viajantes na instabilidade gravitacional do movimento de fluidos: uma abordagem de sistemas dinâmicos

Yulia Petrova

IME-USP

yulia-petrova.github.io

7 Maio 2025

Sergey Tikhomirov
(PUC-Rio)

Based on:
Propagating terrace in a two-tubes model of gravitational fingering

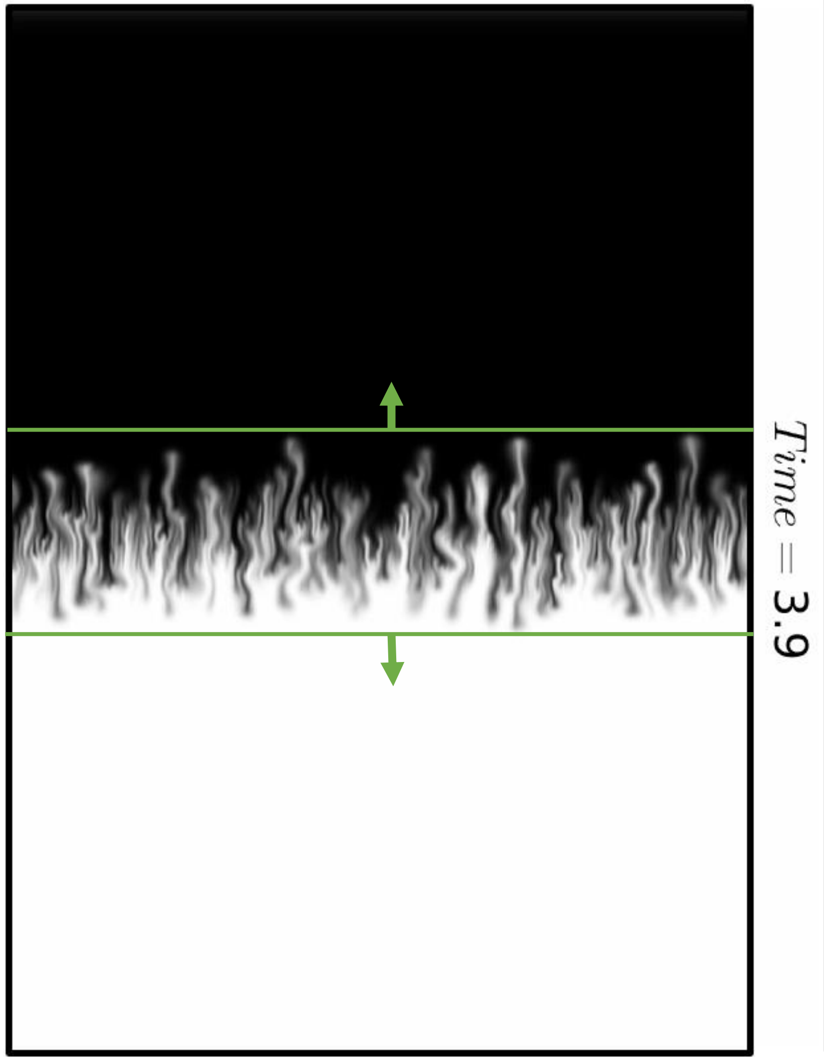
ArXiv: 2401.05981
SIMA, 2025

Yalchin Efendiev
(Texas A&M)

Gravitational fingering (unstable displacement of fluids)

- Miscible displacement
- porous media
  (averaged models of flow)
- Relatively small speeds
  Navier Stokes $\longrightarrow$ Darcy's law

Formulation: PDEs
Proof: Dynamical Systems



Heavy fluid

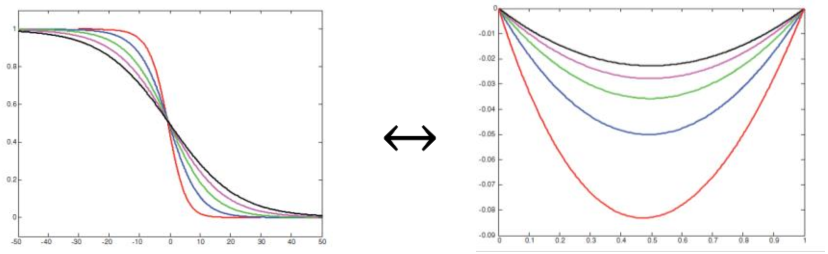
Mixing zone

Light fluid

Credit: Nicolas Valade, INRIA

Outline

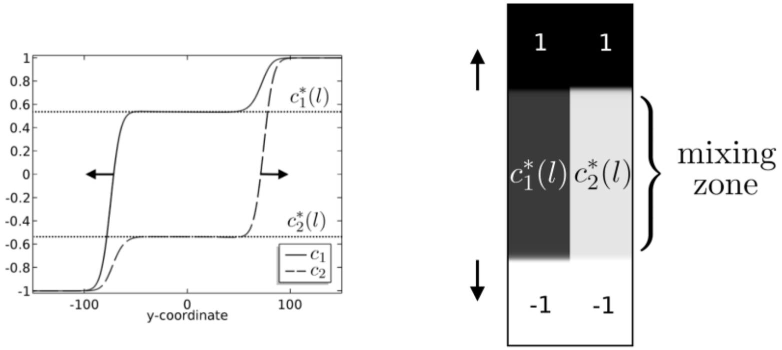
1. Travelling waves in PDEs & Heteroclinic orbits in Dynamical Systems



2. Slow-fast systems
   - Geometric singular perturbation theory (Fenichel)
   - Normal hyperbolicity

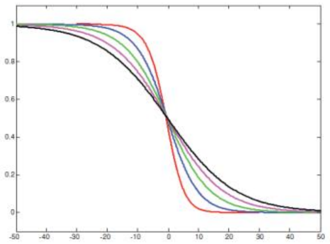
$$\begin{cases} \dot{X} &= F\,(X, Y, \varepsilon) \\ \varepsilon \cdot \dot{Y} &= G(X, Y, \varepsilon) \end{cases}$$

3. Application to the problem in fluid dynamics:
   - Problem statement
   - Scheme of proof
   - Open problems



Outline

1. Travelling waves in PDEs & Heteroclinic orbits in Dynamical Systems



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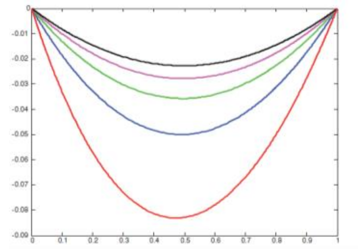

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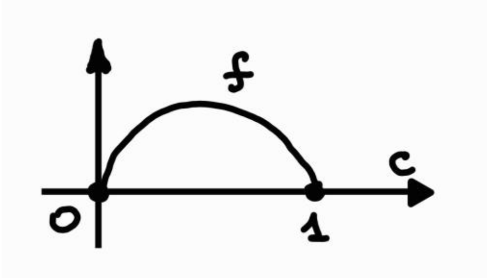
Example: population dynamics

- spreading of animals
- $c(x,t) \in [0,1]$ – density of population of mosquitoes
- propagation due to reproduction and diffusion

Aedes aegypti
(yellow fever mosquito)

Reaction-diffusion equation:
$$c_t = \Delta c + f(c)$$

1. Reproduction:
$$c_t = f(c)$$



2. Diffusion:
$$c_t = \Delta c$$

Aim: understand the behaviour of $c(x,t)$ as $t \to \infty$

Fisher-KPP equation (for $x \in \mathbb{R}$)

"Invasion" occurs!

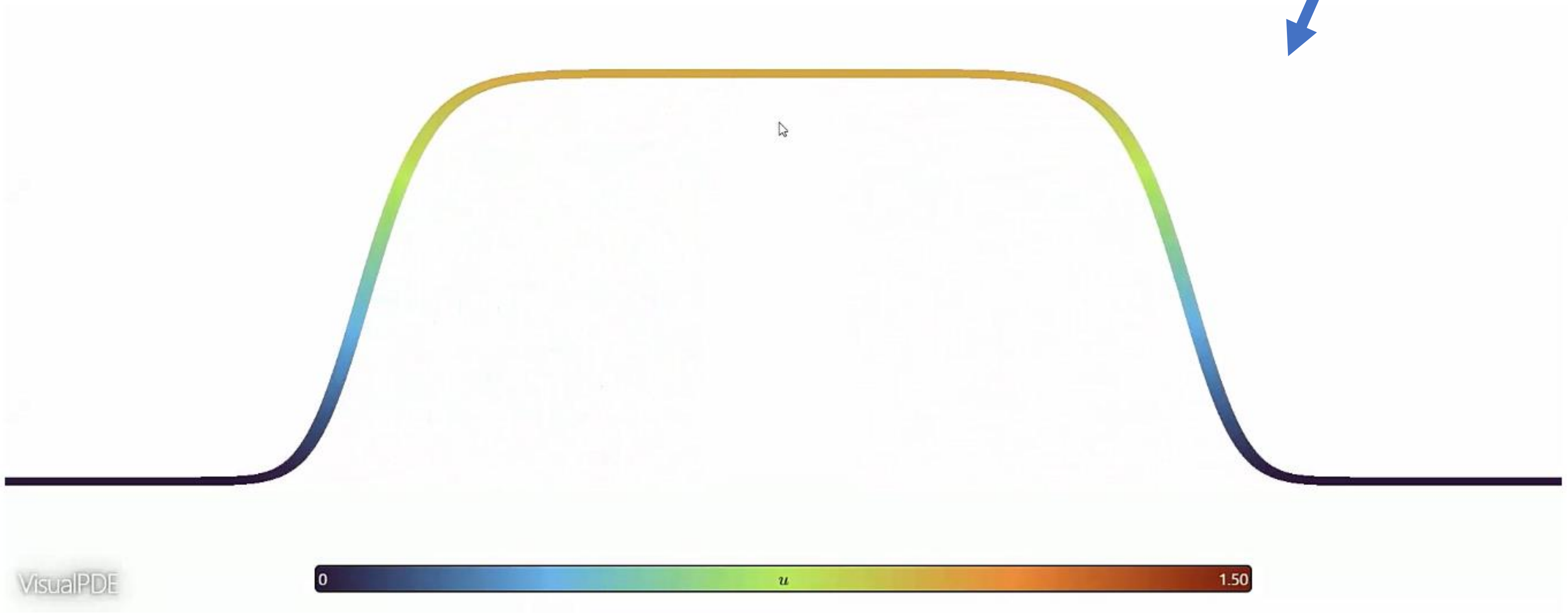
$$c_t = c_{xx} + c(1-c)$$

Question: how to find a speed v of propagation?

Travelling Wave Solution

$$c(t,x) = c(x - vt)$$

$$c(-\infty) = 1$$
$$c(+\infty) = 0$$



Credit from https://visualpde.com/
It is a fun - enjoy!

1. Fisher, R.A., 1937. The wave of advance of advantageous genes. Annals of eugenics, 7(4), pp.355-369.
2. A.N. Kolmogorov, I.G. Petrovskii and N.S. Piskunov, Etude de l'équation de la diffusion avec croissance de la quantité de matière et son application à un problème biologique, Bulletin Universite d'Etat a Moscou, Serie Internationale, section A 1, 1937, 1–26.

Uspekhi Mat. Nauk 46:1 (1991), 225–232 Russian Math. Surveys 46:1 (1991), 271–278

A mathematical trivium

V.I. Arnol'd

How can the standard of training of a mathematician be measured? Neither a list of courses nor their syllabuses determine the standard. The only way to determine what we have actually taught our students is to list the problems which they should be able to solve as a result of their instruction.

I am not talking about difficult kinds of problems, but about the simple questions which form the strictly essential minimum. There need not necessarily be many of these problems, but we must insist that the students are able to solve them. I.E. Tamm used to tell the story that having fallen into the hands of the Makhnovtsy during the Civil War, he said under interrogation that he taught in the physics and mathematics faculty. He owed his life to the fact that he could solve a problem in the theory of series, which was put to him as a test of his veracity. Our students should be prepared for such ordeals!

82. For what values of the velocity c does the equation $u_t = u-u^2+u_{xx}$ have a solution in the form of a travelling wave $u = \varphi(x-ct)$, $\varphi(-\infty) = 1$, $\varphi(\infty) = 0$, $0 \leqslant u \leqslant 1$?

Travelling wave solution $\leftrightarrow$ Heteroclinic orbit

1-dim Fisher KPP equation:

$$c_t = c_{xx} + c(1-c)$$

Travelling Wave Solution

$$c(t,x) = c(x - vt)$$

with
$$c(-\infty) = 1$$
$$c(+\infty) = 0$$

$$-vc' = c'' + c(1-c)$$

$$\begin{cases} c' = w \\ w' = -vw - c(1-c) \end{cases}$$

$$\left.\begin{aligned} (c,w)(-\infty) &= (1{,}0) \\ (c,w)(+\infty) &= (0{,}0) \end{aligned}\right\} \text{Fixed points}$$

Take-home message 1:

FINDING TRAVELLING WAVE SOLUTIONS FOR PDE $\Longleftrightarrow$ (sometimes) FINDING HETEROCLINIC (HOMO-) ORBITS IN DYNAMICAL SYSTEM

… looking for heteroclinic orbits …

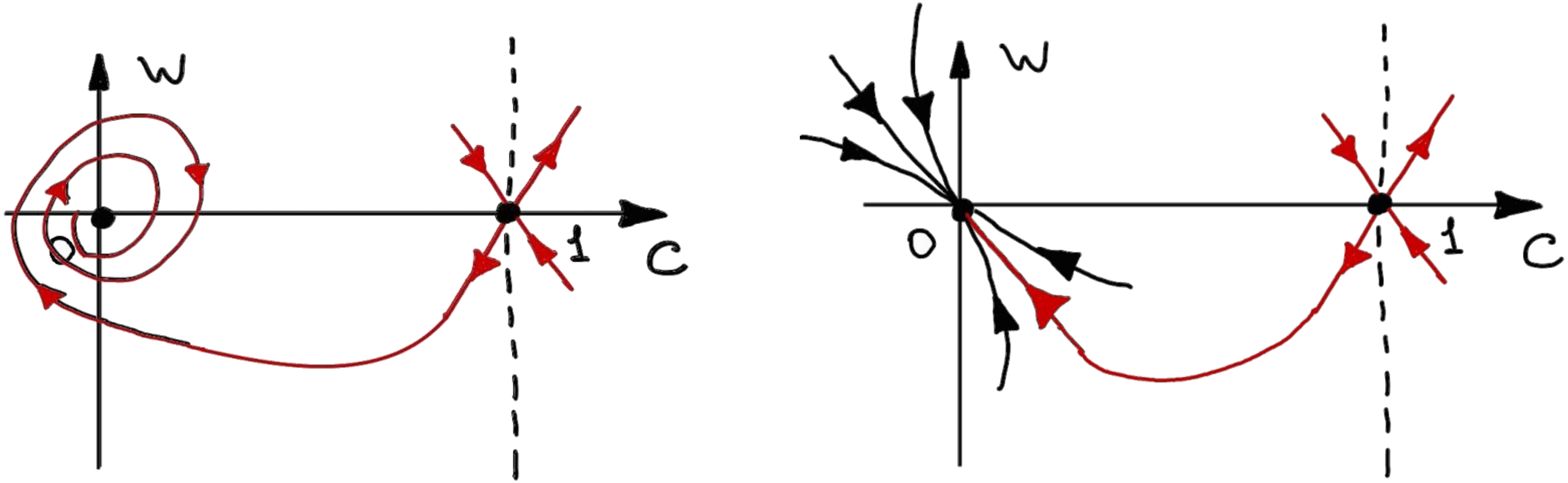
$c \in [0,1]$ – population density

$v \in \mathbb{R}$ – speed

$$\begin{cases} c' = w \\ w' = -vw - c(1-c) \end{cases}$$

$(c,w)(-\infty) = (1,0)$ – saddle point

$(c,w)(+\infty) = (0,0)$



$v \in (0,2)$ $\qquad$ $v \in [2,+\infty)$

No orbit with restriction $c \in [0,1]$

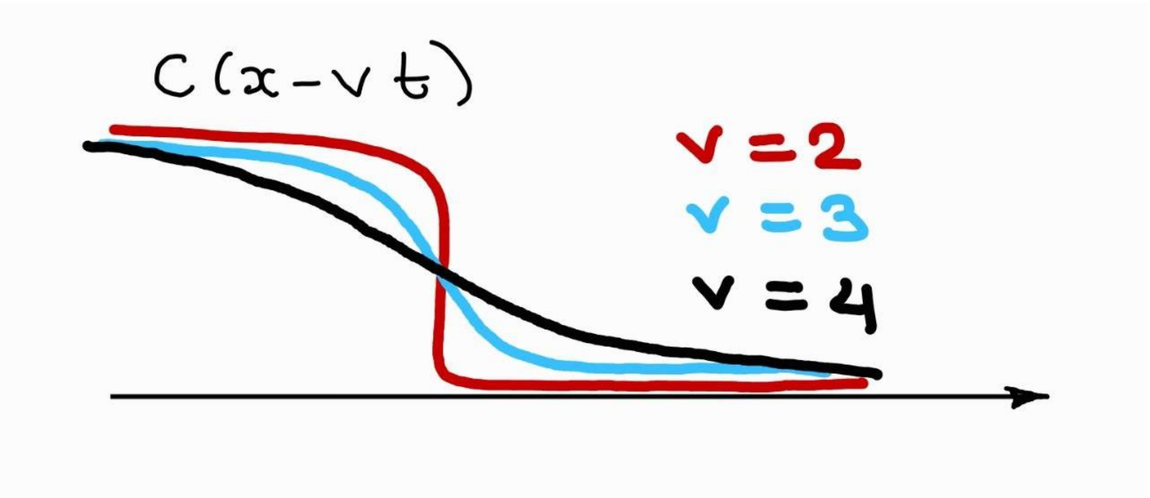
For any $v \in [2,+\infty)$ there exists an orbit

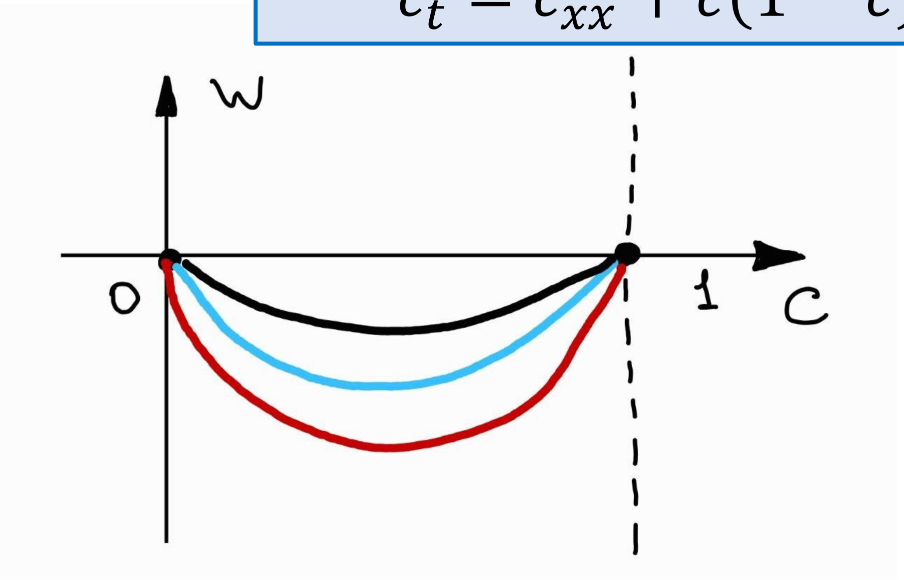
Travelling wave solution (TW) $\leftrightarrow$ Heteroclinic orbit

- There exists a family of TW parametrized by speed $v \in [2, \infty)$

1-dim Fisher KPP equation:

$$c_t = c_{xx} + c(1-c)$$





- If initial data has compact support, then the limiting TW has speed $v = 2$ (the minimal speed)
  Proof of convergence of solution to the TW is a PDE stuff…

Take-home message 1:

Questions?

FINDING TRAVELLING WAVE SOLUTIONS FOR PDES	$\Leftrightarrow$	FINDING HOMO-/HETEROCLINIC ORBITS IN DYNAMICAL SYSTEMS

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   - Normal hyperbolicity

$$\begin{cases} \dot{X} = F\,(X, Y, \varepsilon) \\ \varepsilon \cdot \dot{Y} = G(X, Y, \varepsilon) \end{cases}$$

3. Application to the problem in fluid dynamics:
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Slow-fast systems: example

Slow system

$$\begin{cases} \dot{x} &= -x \\ \varepsilon \cdot \dot{y} &= x^2 - y \end{cases}$$

Fast system

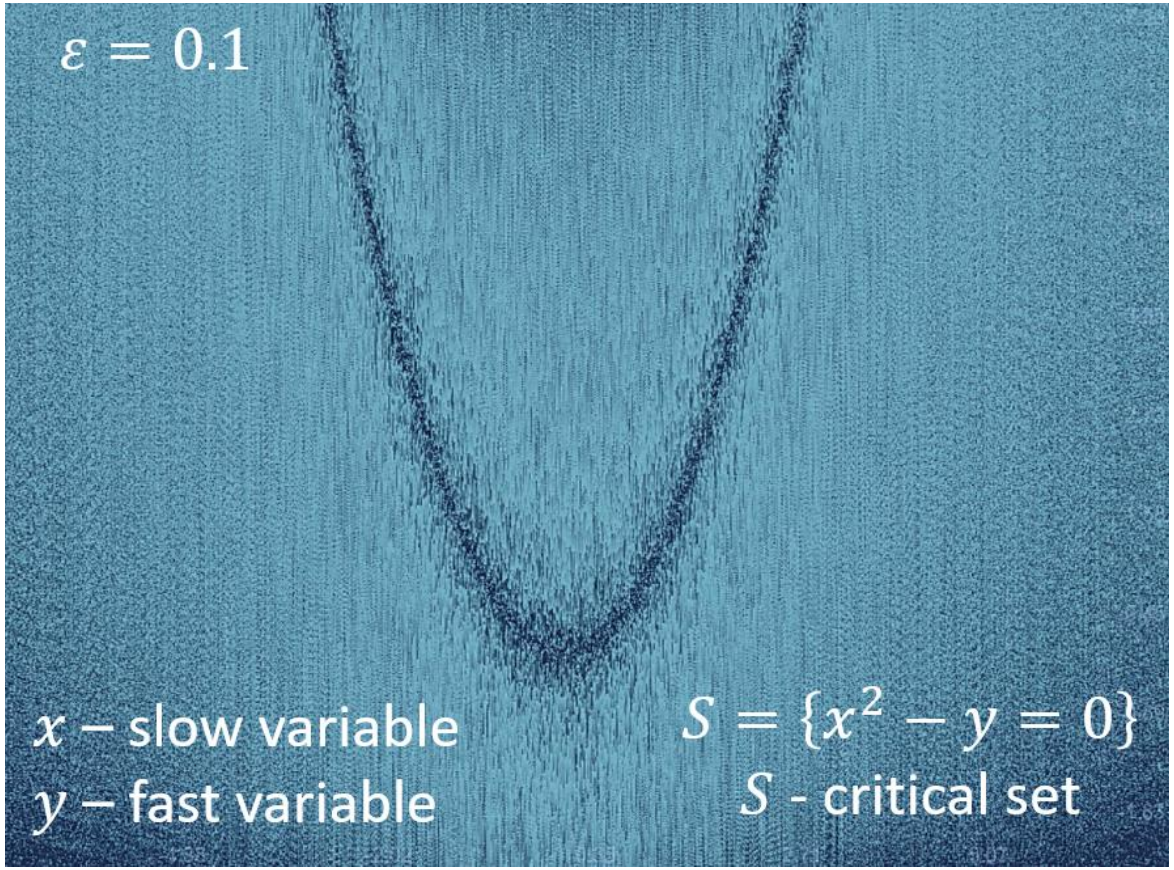
$$\begin{cases} x' &= \varepsilon \cdot (-x) \\ y' &= \quad x^2 - y \end{cases}$$

$$t = \varepsilon \cdot s$$

$$0 < \varepsilon \ll 1$$

Formally
$\varepsilon \to 0$

Formally
$\varepsilon \to 0$



Reduced slow system

$$\begin{cases} \dot{x} &= -x \\ 0 &= x^2 - y \end{cases}$$

Reduced fast system

$$\begin{cases} x' &= 0 \\ y' &= x^2 - y \end{cases}$$

Geometric singular perturbation theory (GSPT)

Slow system (t – slow time)

$$\begin{cases} \dot{X} &= F(X,Y,\varepsilon) \\ \varepsilon \cdot \dot{Y} &= G(X,Y,\varepsilon) \end{cases}$$

Fast system (s – fast time)

$$\begin{cases} X' &= \varepsilon \cdot F(X,Y,\varepsilon) \\ Y' &= G(X,Y,\varepsilon) \end{cases}$$

$t = \varepsilon \cdot s$

$0 < \varepsilon \ll 1$

Formally $\varepsilon \to 0$

Reduced slow system

$$\begin{cases} \dot{X} &= F(X,Y,0) \\ 0 &= G(X,Y,0) \end{cases}$$

Formally $\varepsilon \to 0$

Reduced fast system

$$\begin{cases} X' &= 0 \\ Y' &= G(X,Y,0) \end{cases}$$

$S = \{G(X,Y,0) = 0\}$ – critical set

empty or consists of isolated points
(regular perturbation problem)

contains a differentiable manifold
(singular perturbation problem)

Normally hyperbolic manifolds (“fast-slow” version)

$$\begin{cases} X' = \varepsilon \cdot F(X,Y,\varepsilon) \\ Y' = G(X,Y,\varepsilon) \end{cases}$$

$(X,Y) \in \mathbb{R}^m \times \mathbb{R}^n$, $F(X,Y,\varepsilon), G(X,Y,\varepsilon)$ – smooth

$S = \{(X,Y) \in \mathbb{R}^{m+n}: \; G(X,Y,0) = 0\}$ – critical manifold

Definition: A smooth compact manifold $S_0 \subset S$ is called *normally hyperbolic* if
the $n \times n$ matrix $DG_Y(X,Y,0)$ is hyperbolic for all $(X,Y) \in S_0$, i.e.
for all eigenvalues λ_i of the matrix $DG_Y(X,Y,0)$, we have $Re(\lambda_i) \neq 0$.

In particular, S_0 is called:

- attracting, if all eigenvalues of $DG_y(X,Y,0)$ have negative real part
- repelling, if all eigenvalues of $DG_y(X,Y,0)$ have positive real part
- of saddle-type, if it is neither attracting nor repelling

Take-home message 2:

Normal hyperbolicity of critical manifold $\Rightarrow$ ``nice'' perturbation

Fenichel, N., 1979. Geometric singular perturbation theory for ordinary differential equations. Journal of differential equations, 31(1), pp.53-98.

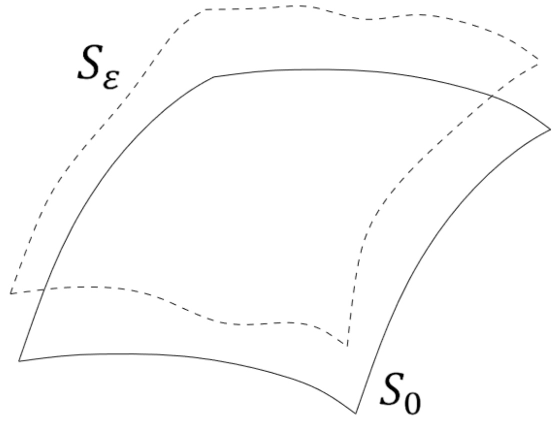
Fenichel's theorem (``fast-slow'' version)

Let S_0 be a compact normally hyperbolic submanifold (possibly with boundary) of the critical manifold $S = \{G(X, Y, 0) = 0\}$ of the system

$$\begin{cases} X' = \varepsilon \cdot F(X, Y, \varepsilon) \\ Y' = G(X, Y, \varepsilon) \end{cases}$$

and that $F, G \in C^r$ $(r \geq 1)$.

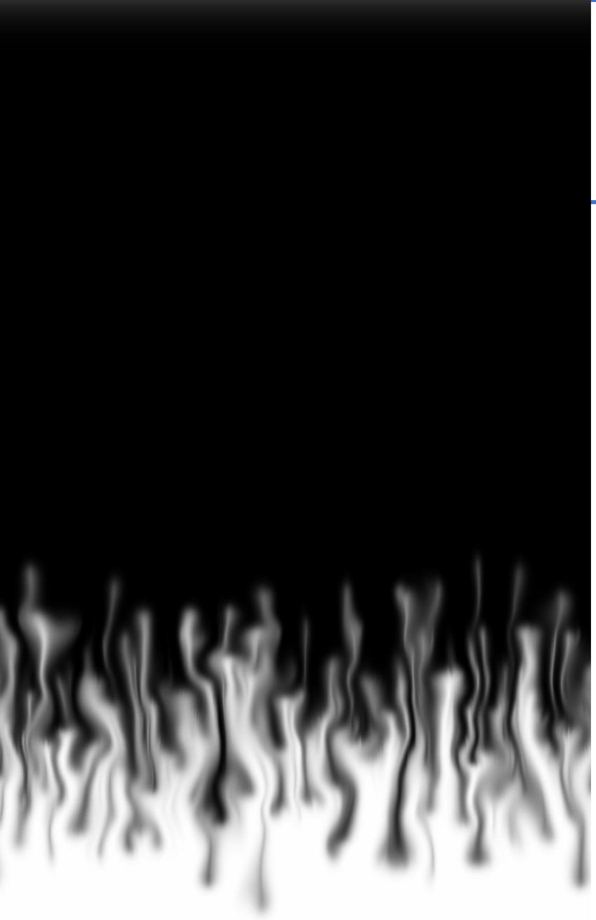
Then for $\varepsilon > 0$ sufficiently small, the following hold:



(F1) There exists a locally invariant manifold S_ε:

- diffeomorphic to S_0
- C^r-smooth and normally hyperbolic
- has Hausdorff distance $O(\varepsilon)$ from S_0 (as $\varepsilon \to 0$).

(F2) There exist local stable manifold $W^s_{loc,\varepsilon}(S_\varepsilon)$ and local unstable manifold $W^u_{loc,\varepsilon}(S_\varepsilon)$, which are C^{r-1} in ε.



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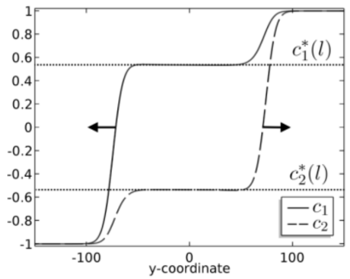
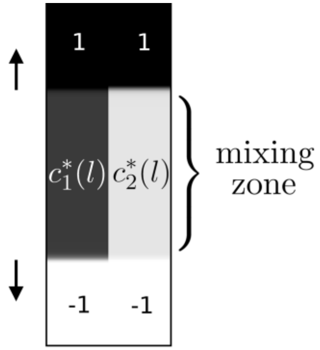




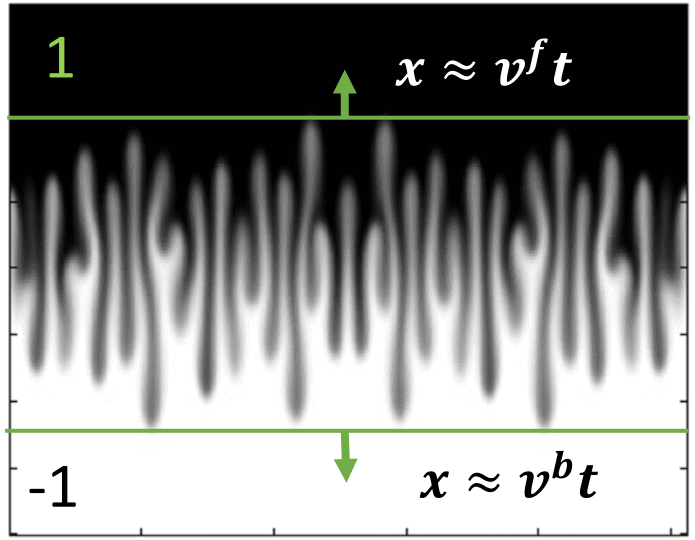
Instability due to gravity in fluid dynamics

PDEs (“black box”):

$$c_t + div(uc) = \Delta c$$
$$div(u) = 0$$
$$u = -\nabla p - (0, c)$$

- c – concentration
- u – velocity
- p – pressure

Gravitational fingering



mixing zone

- many laboratory and numerical experiments show *linear growth of the mixing zone*

Question: how to find speeds v^b and v^f of propagation?

Open problem…

Instability due to viscosity in fluid dynamics

PDEs ("black box"):

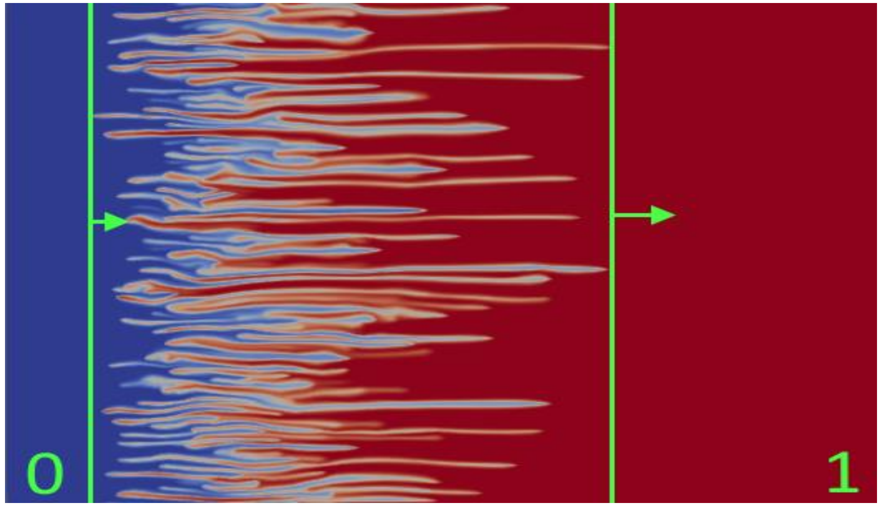
$$c_t + div(uc) = \Delta c$$

$$div(u) = 0$$

$$u = -m(c)\nabla p$$

- c – concentration
- u – velocity
- p – pressure
- $m(c)$ - mobility

Viscous fingering



Applications in petroleum industry

- many laboratory and numerical experiments show *linear growth of the mixing zone*

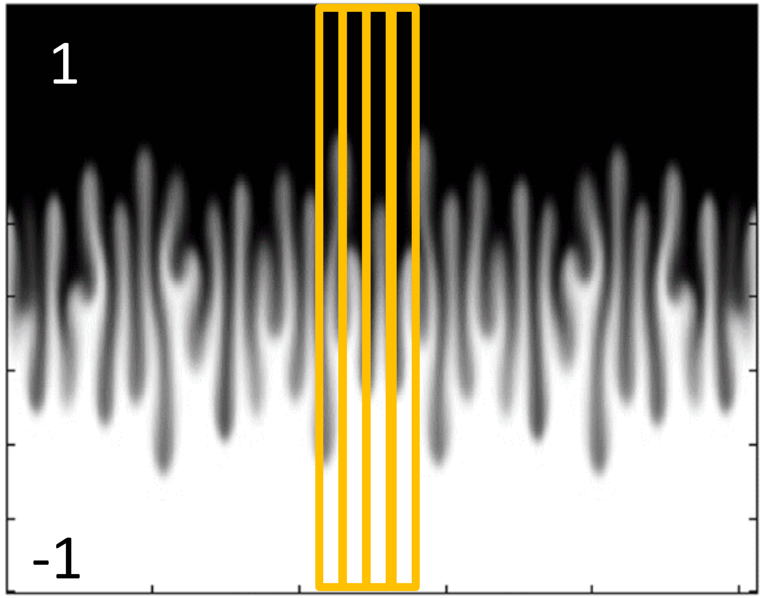
Question: how to find speeds v^b and v^f of propagation?

Open problem...

- *Bakharev F., Enin A., Groman, A., Kalyuzhnyuk, A., Matveenko, S.,* ***Petrova, Y.****, Starkov, I., Tikhomirov, S., 2022.*
Velocity of viscous fingers in miscible displacement: Comparison with analytical models. Journal of Computational and Applied Mathematics, 402.
- *Bakharev F., Enin A., Matveenko S., Pavlov D.,* ***Petrova Y.****, Rastegaev N., Tikhomirov S., 2024.*
Velocity of viscous fingers in miscible displacement: Intermediate concentration. Journal of Computational and Applied Mathematics, 451.

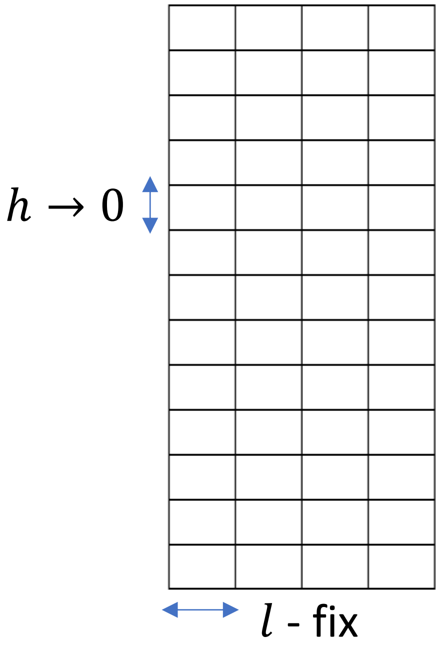
IDEA: semi-discrete model of gravitational fingering

- Discretize in horizontal direction
- Take n tubes, $n = 2,3,4, \dots$



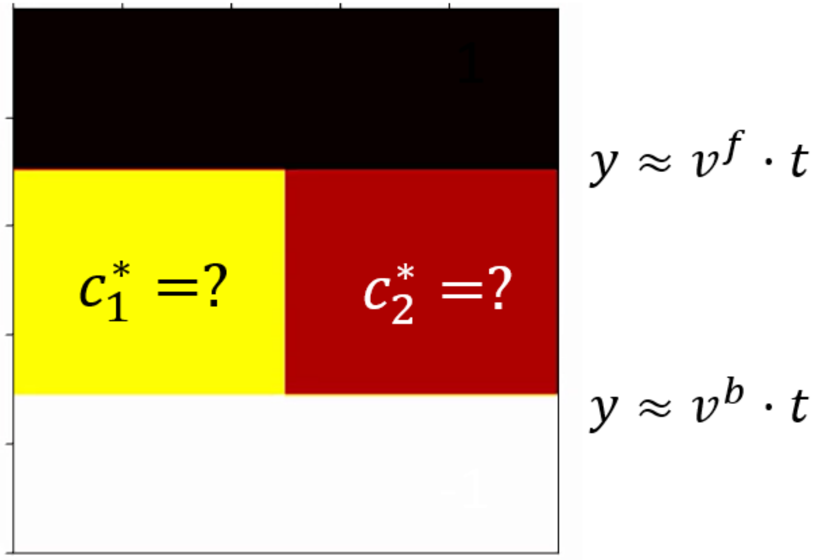
Tubes (layer, lane,…) models:

Limit of
numerical scheme



- Finite volume
- Upwind

- For simplicity, $n = 2$



We observe two traveling waves:
$c(y,t) = c(y - vt)$

1995 — Y. Yortsos “A theoretical analysis of vertical flow equilibrium”
2006 — J.C. Da Mota, S. Schecter “Combustion fronts in a porous medium with two layers”
2019 — A. Armiti-Juber, C. Rohde “On Darcy- and Brinkman-type models for two-phase flow in asympt. flat domains”
2019 — H. Holden, N. Risebro “Models for dense multilane vehicular traffic”

Two-tubes model

1. Original equation on c: $c_t + div(uc) - \Delta c = 0$

   Two-tubes equations on c:

$$\partial_t c_1 + \partial_y(u_1 c_1) - \partial_{yy} c_1 = \boxed{-B}$$
$$\partial_t c_2 + \partial_y(u_2 c_2) - \partial_{yy} c_2 = \boxed{+B}$$

2. Original equation on p: $u = -\nabla p - (0, c)$

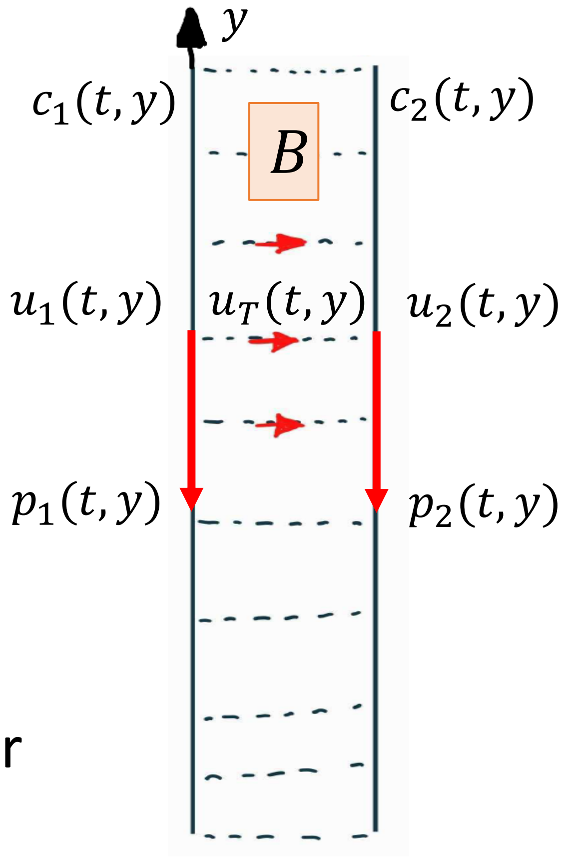
   Two-tubes equations on p:

$$u_1 = -\partial_y p_1 - c_1$$
$$u_2 = -\partial_y p_2 - c_2$$
$$u_T = -\frac{p_2 - p_1}{l}$$

3. Original equation on u: $div(u) = 0$

   Two-tubes equations on u:

$$\partial_y u_1 + \frac{u_T}{l} = 0$$



l - parameter

$$\boxed{B} = \begin{cases} \frac{u_T}{l} \cdot c_1, & u_T > 0, \\ \frac{u_T}{l} \cdot c_2, & u_T < 0 \end{cases}$$

Two-tubes model

1. Original equation on c: $c_t + div(uc) - \Delta c = 0$

   Two-tubes equations on c:

$$\partial_t c_1 + \partial_y(u_1 c_1) - \partial_{yy} c_1 = \boxed{-B}$$
$$\partial_t c_2 + \partial_y(u_2 c_2) - \partial_{yy} c_2 = \boxed{+B}$$

2. Original equation on p: $u = -\nabla p - (0, c)$

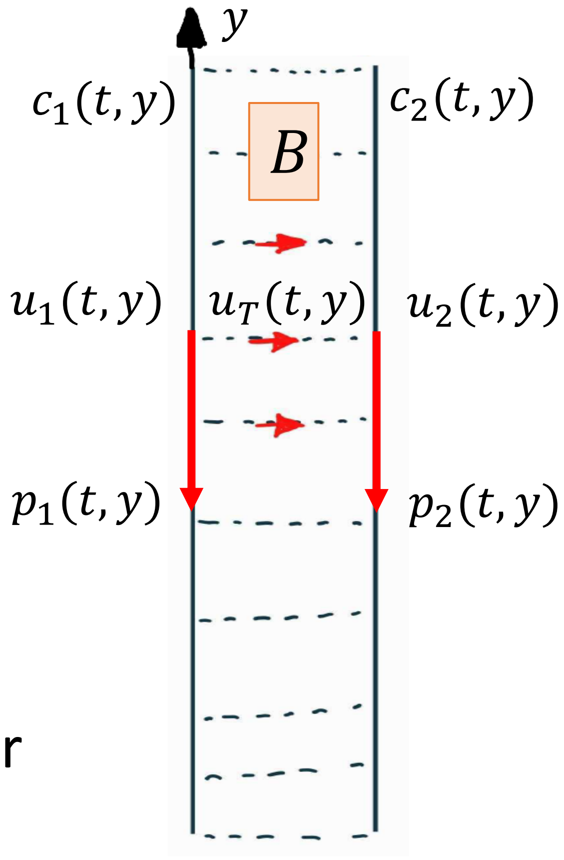
   Two-tubes equations on p:

$$u_1 = -\partial_y p_1 - c_1$$
$$u_2 = -\partial_y p_2 - c_2$$
$$\boxed{\frac{u_T}{l}} = -\frac{p_2 - p_1}{l^2}$$

3. Original equation on u: $div(u) = 0$

   Two-tubes equations on u:

$$\partial_y u_1 + \boxed{\frac{u_T}{l}} = 0$$



l - parameter

$$\boxed{B} = \begin{cases} \boxed{\frac{u_T}{l}} \cdot c_1, & u_T > 0, \\ \boxed{\frac{u_T}{l}} \cdot c_2, & u_T < 0 \end{cases}$$

Main result

$$
(*)\begin{cases}
\partial_t c_1 + \partial_y(u_1 c_1) - \partial_{yy} c_1 = -B \\
\partial_t c_2 + \partial_y(u_2 c_2) - \partial_{yy} c_2 = B \\
u_1 = -\partial_y p_1 - c_1 \\
u_2 = -\partial_y p_2 - c_2 \\
\partial_y u_1 = -\partial_y u_2 = \dfrac{p_2 - p_1}{l^2}
\end{cases}
$$

$$
B = \begin{cases} -\partial_y u_1 \cdot c_1, & \partial_y u_1 < 0, \\ +\partial_y u_2 \cdot c_2, & \partial_y u_1 > 0 \end{cases}
$$

Remark: $\lim_{l\to 0} c_1^*(l) = -0.5$ $\quad \lim_{l\to 0} v^b(l) = -0.25$

$\lim_{l\to 0} c_2^*(l) = +0.5$ $\quad \lim_{l\to 0} v^f(l) = +0.25$

As $t \to \infty$ we observe:

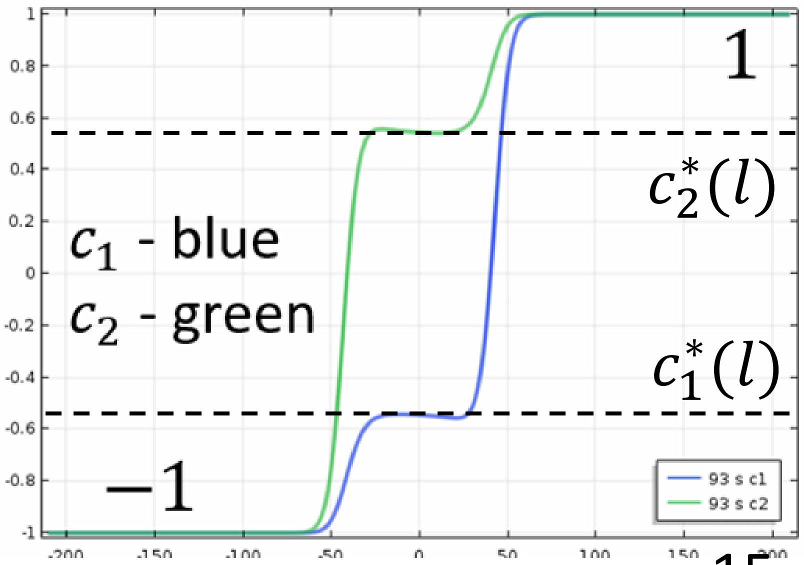
Theorem (Efendiev, P., Tikhomirov, 2025, SIMA)

Consider a two-tube model with gravity $(*)$.

Then for all *$l > 0$ sufficiently small* there exists $c_1^*(l), c_2^*(l)$ such that there exist two travelling waves (TW):

TW1 with speed $v^b(l)$: $\quad (-1,-1) \quad \to \quad (c_1^*, (l)\ c_2^*(l))$

TW2 with speed $v^f(l)$: $\quad (c_1^*, (l)\ c_2^*(l)) \quad \to \quad (1,1).$



Scheme of proof

Step 1: structure of the set of traveling wave (TW) solutions

$$c_1(t,y) = c_1(y - vt)$$
$$c_2(t,y) = c_2(y - vt)$$

$$u_1(t,y) = u_1(y - vt)$$
$$u_2(t,y) = u_2(y - vt)$$

$$p_1(t,y) = p_1(y - vt)$$
$$p_2(t,y) = p_2(y - ct)$$

$v \in \mathbb{R}$ – speed of traveling wave

Theorem

For sufficiently small $l > 0$ and for each v close to $\frac{1}{4}$ there exists a TW:

$$(c_1^*, c_2^*, u_1^*, u_2^*, p_1^* - p_2^*) \to (1,1,0,0,0)$$

Similarly,

$$(-1,-1,0,0,0) \to (c_1^{**}, c_2^{**}, u_1^{**}, u_2^{**}, p_1^{**} - p_2^{**})$$

Step 2: existence of a propagating terrace of two traveling waves

- Find a common intermediate state $(c_1, c_2, u_1, u_2, p_1 - p_2)$ for traveling waves above

Step 2: propagating terrace of two traveling waves

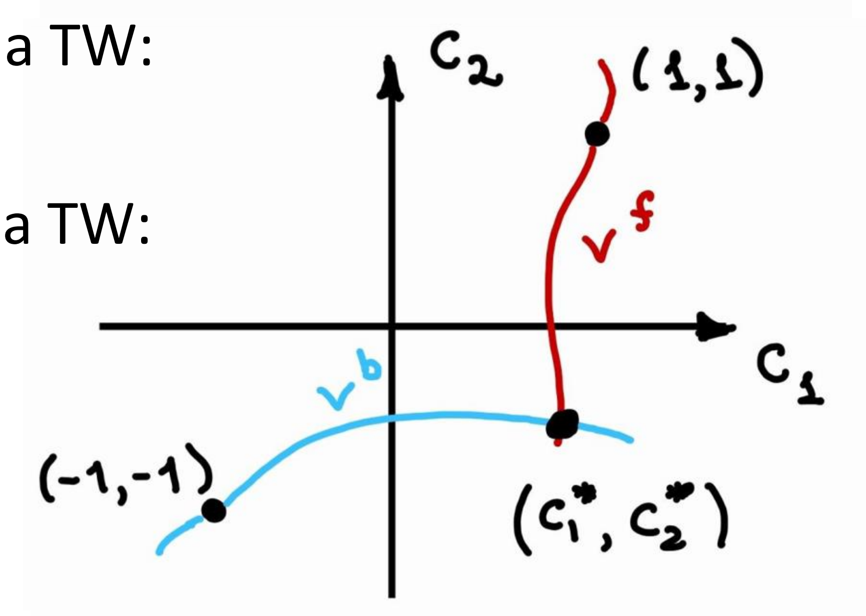
1) For each $v^f \in I_f \subset \mathbb{R}$ we find all points s.t. there exists a TW:

$$(c_1, c_2) \rightarrow (1,1)$$

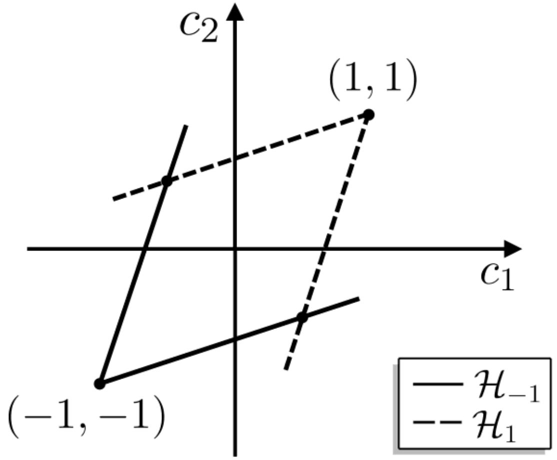
2) For each $v^b \in I_b \subset \mathbb{R}$ we find all points s.t. there exists a TW:

$$(-1,-1) \rightarrow (c_1, c_2)$$

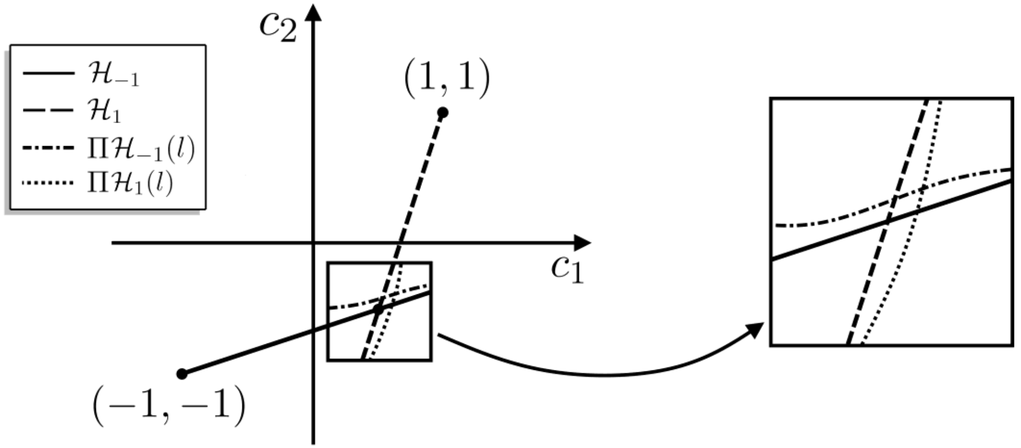
3) Find the intersection points of these two curves



$l = 0$ – these curves are just straight lines



$0 < l \ll 1$– perturbation argument



Step 1: traveling wave ansatz

Fix the sign $\partial_y u_1 < 0$

$$\begin{cases} \partial_t c_1 + \partial_y(u_1 c_1) - \partial_{yy} c_1 = +\partial_y u_1 \cdot c_1 \\ \partial_t c_2 + \partial_y(u_2 c_2) - \partial_{yy} c_2 = -\partial_y u_1 \cdot c_1 \\ u_1 = -\partial_y p_1 - c_1 \\ u_2 = -\partial_y p_2 - c_2 \\ \partial_y u_1 = +\dfrac{p_2 - p_1}{l^2} \\ \partial_y u_2 = -\dfrac{p_2 - p_1}{l^2} \end{cases}$$

Denote $q = \frac{p_2 - p_1}{l}$, $\varepsilon = l$. Also $u_1 = -u_2$

$$\begin{cases} \boxed{\begin{aligned} \dot{c_1} &= d_1 \\ \dot{d_1} &= (-v + u_1) d_1 \\ \dot{c_2} &= d_2 \\ \dot{d_2} &= (-v - u_1) d_2 - \dot{u_1}(c_2 - c_1) \end{aligned}} \\ \boxed{\begin{aligned} \varepsilon\, \dot{u_1} &= q \\ \varepsilon\, \dot{q} &= 2u_1 + c_1 - c_2 \end{aligned}} \end{cases}$$

Slow-fast system!

Travelling wave ansatz with fixed v:

$$c_1(t,y) = c_1(y - vt) \qquad u_1(t,y) = u_1(y - vt) \qquad p_1(t,y) = p_1(y - vt)$$
$$c_2(t,y) = c_2(y - vt) \qquad u_2(t,y) = u_2\,(y - vt) \qquad p_2(t,y) = p_2(y - ct)$$

$$c_1(+\infty) = 1 \qquad u_1(+\infty) = 0 \qquad (p_1 - p_2)(+\infty) = 0$$
$$c_2(+\infty) = 1 \qquad u_2(+\infty) = 0$$

Step 1: traveling wave ansatz

Fix the sign $\partial_y u_1 < 0$

$$\begin{cases} \partial_t c_1 + \partial_y(u_1 c_1) - \partial_{yy} c_1 = +\partial_y u_1 \cdot c_1 \\ \partial_t c_2 + \partial_y(u_2 c_2) - \partial_{yy} c_2 = -\partial_y u_1 \cdot c_1 \\ u_1 = -\partial_y p_1 - c_1 \\ u_2 = -\partial_y p_2 - c_2 \\ \partial_y u_1 = +\dfrac{p_2 - p_1}{l^2} \\ \partial_y u_2 = -\dfrac{p_2 - p_1}{l^2} \end{cases}$$

$\Rightarrow$

System of ODEs in $\mathbb{R}^6$, $\varepsilon = l \ll 1$:

$$\boxed{\begin{cases} \dot{X} = F_v(X, Y) \\ \varepsilon \cdot \dot{Y} = AY - BX \end{cases}}$$

- $X = (c_1 \;\; c_2 \;\; d_1 \;\; d_2)^T \in \mathbb{R}^4$
- $Y = \begin{pmatrix} u_1 \\ p_1 - p_2 \end{pmatrix} \in \mathbb{R}^2$
- $A = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}$, $\quad B \in M^{2\times 4}$

Travelling wave ansatz with fixed v:

$$c_1(t,y) = c_1(y - vt) \qquad u_1(t,y) = u_1(y - vt) \qquad p_1(t,y) = p_1(y - vt)$$
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$$c_1(+\infty) = 1 \qquad u_1(+\infty) = 0 \qquad (p_1 - p_2)(+\infty) = 0$$
$$c_2(+\infty) = 1 \qquad u_2(+\infty) = 0$$

GSPT in action

$$\begin{cases} \dot{X} = F_v(X,Y) \\ \varepsilon \cdot \dot{Y} = AY - BX \end{cases}$$

- $X \in \mathbb{R}^4, Y \in \mathbb{R}^2$

We have:

- Critical manifold:

$$S = \{(X,Y): \quad Y = A^{-1}BX\}, \qquad \dim S = 4$$

- $K \subset S$ (compact) is normally hyperbolic as the matrix

$$A = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix} \quad \text{has eigenvalues } \lambda_{\pm} = \pm\sqrt{2}$$

Thus, by Fenichel's theorem:

- For any compact submanifold $K \subset S$ there exists a locally invariant manifold $K_\varepsilon \subset \mathbb{R}^6$

$$K_\varepsilon = \{(X,Y): \quad Y = A^{-1}BX + \varepsilon h(X,\varepsilon)\} \qquad \text{for some smooth function } h$$

Result: 6-dim system on (X,Y) $\Rightarrow$ 4-dim system on X on K_ε:

$$\dot{X} = F_v(X, A^{-1}BX + \varepsilon h(X,\varepsilon))$$

The unperturbed case ($\varepsilon = 0$)

We have a perturbation problem ($X \in \mathbb{R}^4$):

$\varepsilon > 0$: $\dot{X} = F_v(X, A^{-1}BX + \varepsilon \cdot h(X,\varepsilon))$

$\varepsilon = 0$: $\dot{X} = F_v(X, A^{-1}BX)$

$$\begin{cases} \dot{a} = r \\ \dot{b} = s \\ \dot{r} = -vr - \dfrac{a}{2}(s - r) \\ \dot{s} = -vs - ra \end{cases}$$

Fixed points:
$\{r = s = 0\}$

4-dim …we can find all heteroclinic orbits explicitly when $\varepsilon = 0$!…

3-dim Obs 1: there is no b in the right hand side!

2-dim Obs 2: there are 2-dim invariant manifolds: $\{s = 2r\}$

1-dim Obs 3: inside these invariant manifolds holds:

$$r = -\left(v + \frac{a}{2}\right)^2 + r_0, \qquad r_0 \in \mathbb{R}$$

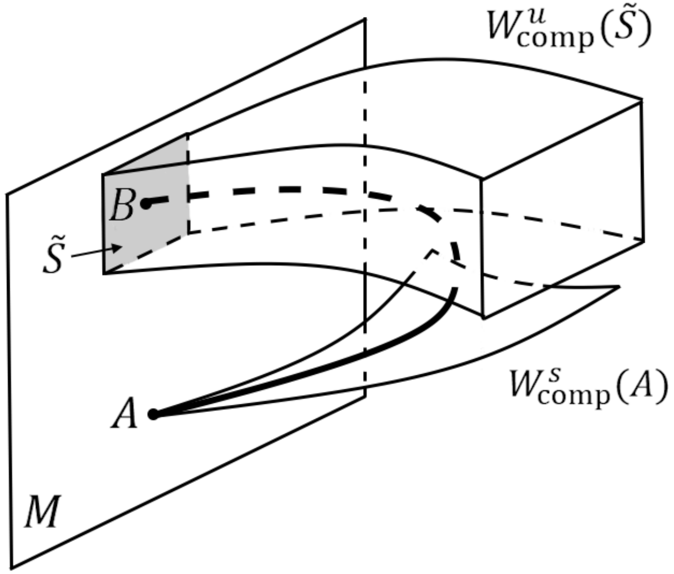
The perturbed case ($\varepsilon > 0$)

We have a perturbation problem ($X \in \mathbb{R}^4$):

$$\dot{X} = F_v(X, A^{-1}BX + \varepsilon \cdot h(X, \varepsilon))$$

$\varepsilon = 0$:

- we can find all heteroclinic orbits explicitly
- heteroclinic orbits can be represented as a transverse intersection of stable and unstable manifolds



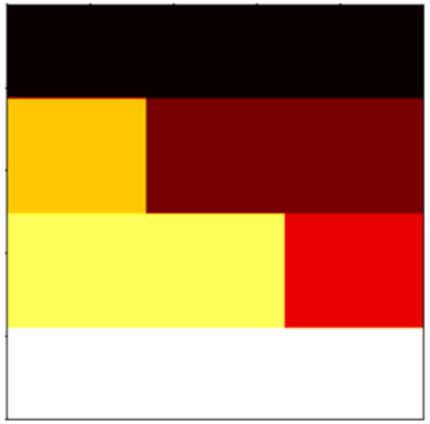
$\varepsilon > 0$:

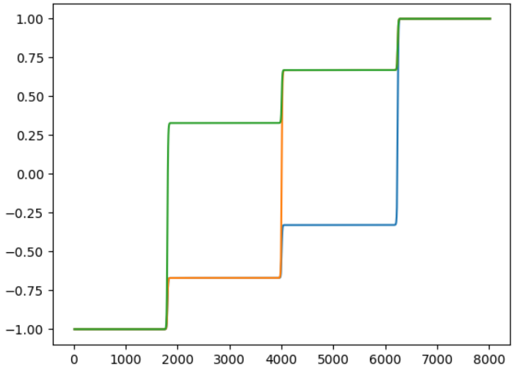
- 1970 – M. Hirsh, C. Pugh, M. Shub, Invariant manifolds:
  The invariant manifolds change continuously in ε, thus the intersection of manifolds persists under perturbations

Proof is finished.

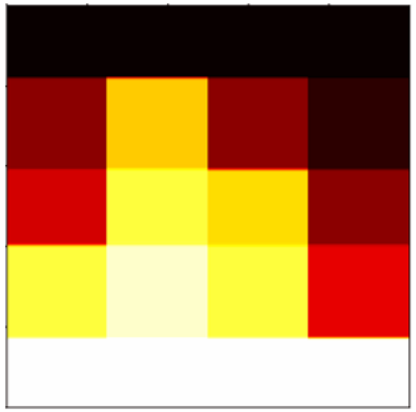
Many tubes: numerics

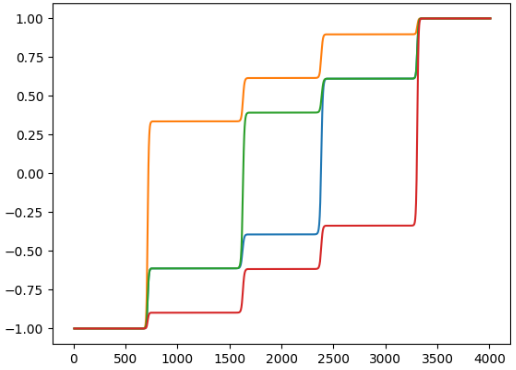
3 tubes



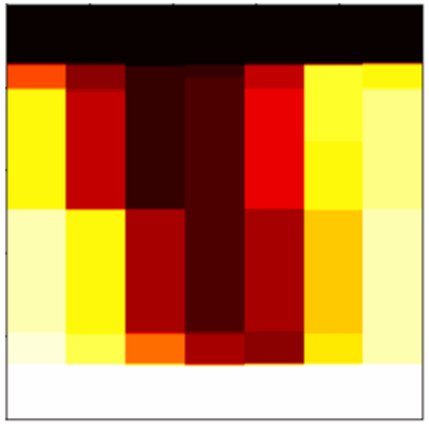


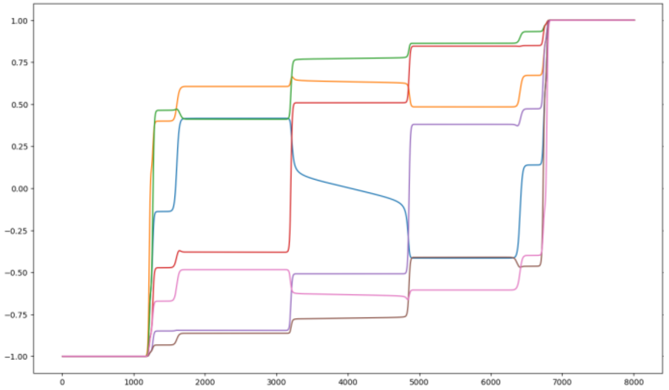
4 tubes





7 tubes





Questions: (open)

(1) explain the structure of "asymptotic solutions" for n tubes and study their stability

(2) find speed of growth of the mixing zone

(3) understand the behaviour as $n \to \infty$. Do we approximate 2-dim IPM?

(4) can we repeat this story for the many tubes viscous fingering model?

Open questions

1. How to prove the existence of hetero-clinic orbits for ε not necessarily small?

$$\begin{cases} \dot{c_1} = d_1 \\ \dot{d_1} = (-v + u_1)d_1 \\ \dot{c_2} = d_2 \\ \dot{d_2} = (-v - u_1)d_2 - \dot{u_1}(c_2 - c_1) \\ \varepsilon\, \dot{u_1} = q \\ \varepsilon\, \dot{q} = 2u_1 + c_1 - c_2 \end{cases}$$

[numerical evidence]

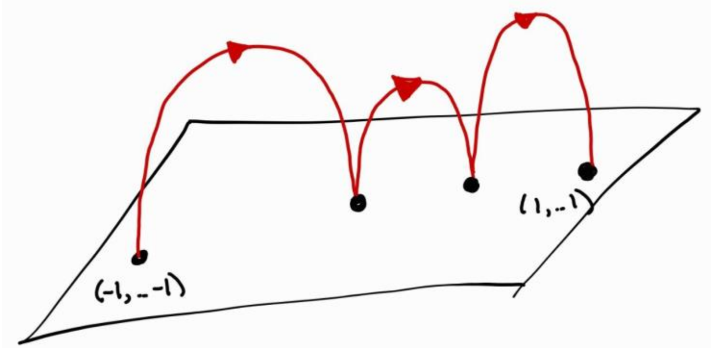
2. n tubes: traveling wave dynamical system has Lipshitz continuous RHS (but not C^1)

$$(**)\begin{cases} \dot{c_i} = d_i \\ \dot{d_i} = -vd_i + \frac{1}{n}\sum_{j=1}^{n}(c_j - c_i)\cdot\left(d_i + (d_i - d_j)^-\right) \end{cases}$$

Here $a^- = \begin{cases} 0, & a > 0 \\ a, & a \leq 0 \end{cases}$ $\quad c = (c_1 \;\ldots\; c_n)^T$, $\quad d = (d_1 \;\ldots d_n)^T$

[We do not know if Fenichel's and HPS theory works]

3. n tubes: are there algorithms that find numerically a chain of heteroclinics?



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