On solutions of a Riemann problem for chemical flooding model

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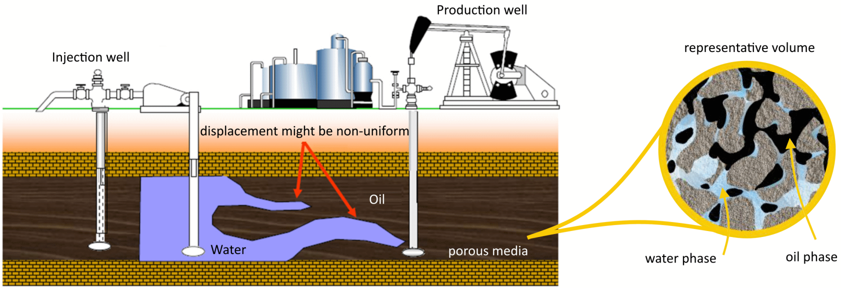
Motivation

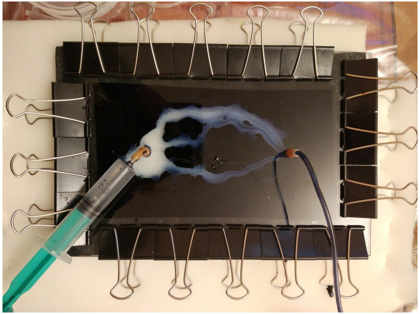
We are interested in the mathematical model of oil recovery. Some features:

- Porous media (averaged models of flow)
- Relatively small speeds (≈ 1 meter per day): Navier-Stokes $\to$ Darcy's law
- Multiphase flow: oil, water, gas.
- Unknown variables: $s(t, x)$ — the averaged water saturation in small volume
- Applications to EOR (enhanced oil recovery) methods: chemical, thermal, gas etc

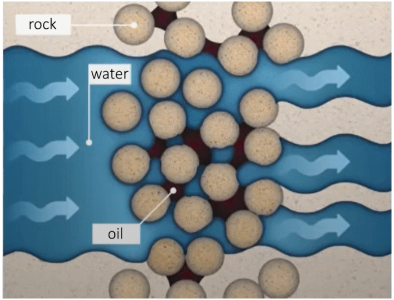


- Collaboration with Russian petroleum company GazpromNeft (2018–2021)

Problems: macroscopic and microscopic sweep efficiency



- happens due to very viscous oil or inhomogeneous media



- local entrapment of oil in pores due to high capillary pressure

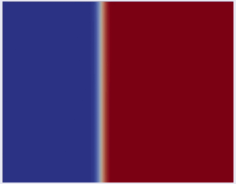
Possible solution

- Inject gas (CO_2, natural) to decrease the oil viscosity
- Add chemicals (polymer) to increase the water viscosity
- Add chemicals (surfactant) that reduce the surface tension etc

Fundamental research: two main directions

1-dim in spatial variable

- Stable displacement



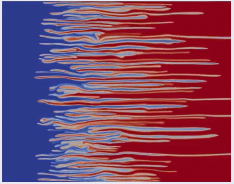
- main question: find an exact solution to a Riemann problem
- hyperbolic conservation laws

$$s_t + f(s,c)_x = 0,$$
$$(cs + a(c))_t + (cf(s,c))_x = 0.$$

Example: chemical flooding model

2-dim (or 3-dim) in spatial variable

- Unstable displacement



- source of instability: water and oil/polymer have different viscosities
- viscous fingering phenomenon

$$c_t + u \cdot \nabla c = \varepsilon \Delta c,$$
$$\text{div}(u) = 0,$$
$$u = -\nabla p / \mu(c).$$

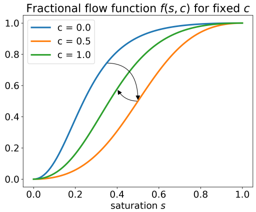
Example: Peaceman model

Problem statement

Chemical flooding can be described as the system of conservation laws ($x \in \mathbb{R}, t > 0$):

$$\begin{aligned} s_t + f(s,c)_x &= 0, && \text{(conservation of water)} \\ (cs + a(c))_t + (cf(s,c))_x &= 0. && \text{(conservation of chemical)} \end{aligned} \tag{1}$$

- $s = s(x,t)$ — water phase saturation;
- $c = c(x,t)$ — concentration of a chemical agent in water;
- $f(s,c)$ — fractional flow function (usually S-shaped);
- $a(c)$ — adsorption of a chemical agent on a rock (usually increasing, concave).



Initial data:

$$(s,c)\big|_{t=0} = \begin{cases} (1,1), & \text{if } x \leqslant 0, \\ (0,0), & \text{if } x > 0, \end{cases} \tag{2}$$

Aim:

Find a solution to initial-value problem (1)–(2) when f depends non-monotonically on c.

Hyperbolic systems of conservation laws

$$G(u)_t + F(u)_x = 0 \tag{3}$$

Here

- $G(u)$ — accumulation function (conserved quantities)
- $F(u)$ — flux function (flux of conserved quantities)

Simplest example: wave equation

$$y_{tt} - c^2 y_{xx} = 0 \qquad \text{(J. d'Alambert, 1750)}$$

can be rewritten as a system of two first-order equations on the state-vector $u = \begin{pmatrix} y_x \\ y_t \end{pmatrix}$

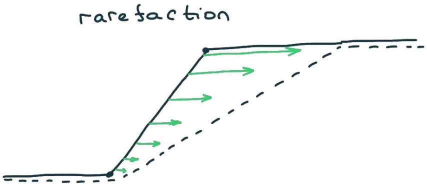
$$u_t + Du_x = 0, \qquad \text{with} \qquad D = \begin{pmatrix} 0 & -1 \\ -c^2 & 0 \end{pmatrix}$$

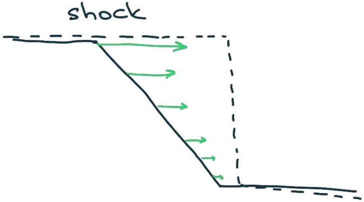
- eigenvalues $\lambda_1 = c$ and $\lambda_2 = -c$ are real, the system is hyperbolic. Solutions are two waves propagating at velocities λ_1 and λ_2.

Hyperbolic systems of conservation laws

$$u_t + \left(\frac{u^2}{2}\right)_x = 0 \qquad \text{(Burger's equation, 1948)}$$

- Due to non-linearity of the flux velocity of the wave $\lambda(u) = u$ depends on state u
- So the wave can spread (rarefaction wave) or concentrate (shock wave)





$$u_t + (f(u))_x = 0 \qquad \text{(Buckley-Leverett equation)}$$

- Typical f is an S-shaped function and solution is rarefaction wave + shock
- Gelfand, Oleinik

Riemann problem (1858)

- Riemann solved the initial-value problem with data having a single jump

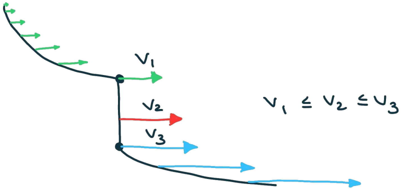
$$u\big|_{t=0} = \begin{cases} u^L, & x \leqslant 0; \\ u^R, & x > 0. \end{cases}$$

- took advantage of the scale invariance of the equations and the data:

$$u(\alpha\, x, \alpha\, t) = u(x,t) \quad \text{for all } \alpha > 0$$

- solution to a Riemann problem is important because:
  - often it appears in a long-term behavior of Cauchy problem
  - helps to prove the existence of solutions to Cauchy problem (Glimm's method)
  - helps to construct numerical solution (Godunov method)

Any solution to a Riemann problem consists of a sequence of rarefaction or shock waves (and constant states) that are compatible by speeds



Shock waves: RH condition and admissibility criteria

- discontinuous solutions are defined in the sense of distributions (weak form)
- for a shock wave from u^- to u^+ moving with velocity v, the weak condition amounts to the following Rankine-Hugoniot condition (RH)

$$-v\,G(u^-) + F(u^-) = -v\,G(u^+) + F(u^+) \tag{RH}$$

- RH means conservation: what flows into left side flows out of the right side
- Problems from the perspectives of both mathematics and physics:
  - if all RH solutions are allowed, a Riemann problem has multiple solutions
  - some RH solutions violate physical principles
- Vanishing viscosity criteria: consider a diffusive system of conservation laws

$$G(u)_t + F(u)_x = \varepsilon\,[B(u)\,u_x]_x\,, \qquad \varepsilon \to 0$$

Traveling wave solutions of diffusive system (Hopf, 1948)

- $u(x,t) = \hat{u}(\xi)$ with $\xi := x - v\,t$ for a fixed shock velocity v
- reduction to first-order system of ordinary differential equations:

$$\varepsilon\, B(\hat{u})\, \hat{u}_\xi = -v\left[G(\hat{u}) - G(u^-)\right] + F(\hat{u}) - F(u^-)$$

- u^- and u^+ are fixed points and we look for an orbit connecting them

$$\hat{u}(-\infty) = u^-, \qquad \hat{u}(+\infty) = u^+$$

- diffusive terms cause a shock wave to have a thin, smooth internal structure as a result of balancing nonlinear focusing and diffusive spreading
- traveling wave solution approaches the jump discontinuity in L^1 as $\varepsilon \to 0^+$

Historical overview

$$s_t + f(s,c)_x = \varepsilon_c (A(s,c)s_x)_x,$$
$$(cs + a(c))_t + (cf(s,c))_x = \varepsilon_c (cA(s,c)s_x)_x + \varepsilon_d c_{xx}.$$

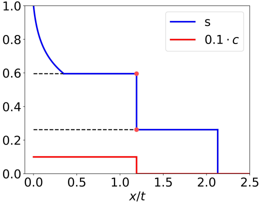
- ε_c — dimensionless capillary pressure
- ε_d — dimensionless diffusion term

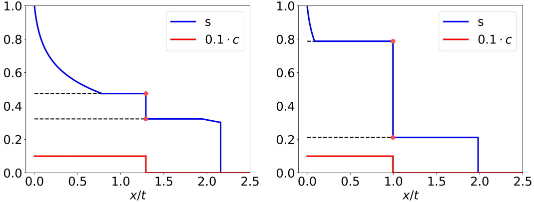
Polymer flooding [Johansen'88]

- monotone dependence of $f(s,c)$ on c
- unique solution as $\varepsilon_c, \varepsilon_d \to 0$

Surfactant flooding [Bakharev, P. et al'21]

- non-monotone dependence of $f(s,c)$ on c
- the solution depends on the ratio of $\varepsilon_d/\varepsilon_c$





When $f(s,c)$ is non-monotone in c, multiple vanishing viscosity solutions are possible. Examples can be found in Shen (2017). See also Entov-Kerimov (1986) on non-rigorous consideration of the model.

Restrictions on f and a

(F1) $f \in C^2([0,1]^2)$; $f(0,c) = 0$; $f(1,c) = 1$;

(F2) $f_s(s,c) > 0$ for $s \in (0,1)$, $c \in [0,1]$;
$f_s(0,c) = f_s(1,c) = 0$;

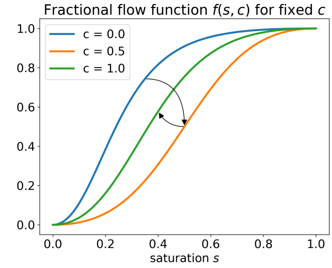
(F3) f is S-shaped in s;

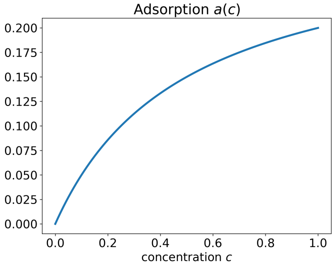
(F4) f is non-monotone in c:
$\forall s \in (0,1)\, \exists c^*(s) \in (0,1)$:

- $f_c(s,c) < 0$ for $0 < s < 1$, $0 < c < c^*(s)$;
- $f_c(s,c) > 0$ for $0 < s < 1$, $c^*(s) < c < 1$;

(A) A is bounded from zero and infinity;

(a) $a \in C^2$, $a(0) = 0$, a is strictly increasing and concave.





Travelling wave dynamical system

$$
\begin{aligned}
s_t + f(s,c)_x &= \varepsilon_c (A(s,c)s_x)_x, \\
(cs + a(c))_t + (cf(s,c))_x &= \varepsilon_c (cA(s,c)s_x)_x + \varepsilon_d c_{xx}.
\end{aligned}
$$

Searching for travelling wave solutions $s = s(\xi)$, $c = c(\xi)$, $\xi := \varepsilon_c^{-1}(x - vt)$ with boundary conditions

$$
s(\pm\infty) = s^{\pm}, \qquad c(-\infty) = 1, \qquad c(+\infty) = 0,
$$

we arrive at

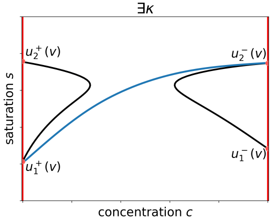
$$
\begin{aligned}
A(s,c)s_\xi &= f(s,c) - v(s + d_1), \\
\kappa c_\xi &= v(d_1 c - d_2 - a(c)).
\end{aligned} \tag{4}
$$

- Here $\kappa = \varepsilon_d / \varepsilon_c$;
- Note that $u^{\pm}$ are fixed points of dynamical system (4). Here $u^+ = (s^+, 0)$ and $u^- = (s^-, 1)$;
- We are only interested in the trajectories connecting two saddle points (or saddle-nodes) due to compatibility of speeds condition

Main result

Consider a dynamical system under assumptions (F1)–(F4), (A), (a):

$$\begin{aligned} s_\xi &= f(s,c) - v(s+d_1), \\ \kappa c_\xi &= v(d_1 c - d_2 - a(c)). \end{aligned}$$



Theorem (Bakharev, Enin, P.,Rastegaev, 2021)

There exist $0 < v_{\min} < v_{\max} < \infty$, *such that for every* $\kappa = \varepsilon_d/\varepsilon_c \in (0,+\infty)$, *there exist unique*

- *points* $s^-(\kappa) \in [0,1]$ *and* $s^+(\kappa) \in [0,1]$;
- *velocity* $v(\kappa) \in [v_{\min}, v_{\max}]$,

such that there exists a travelling wave, connecting two saddle points $u^-(\kappa) = (s^-(\kappa), 1)$ *and* $u^+(\kappa) = (s^+(\kappa), 0)$ *with velocity* $v(\kappa)$. *Moreover,* $v(\kappa)$ *is monotone and continuous;* $v(\kappa) \to v_{\min}$ *as* $\kappa \to \infty$; $v(\kappa) \to v_{\max}$ *as* $\kappa \to 0$.

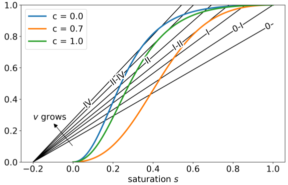
Scheme of proof

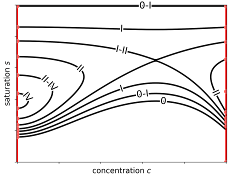
The Theorem can be divided into simpler statements:

- $\forall v \in [v_{\min}, v_{\max}] \quad \exists! \kappa(v)$: there is a saddle-to-saddle travelling wave with $\kappa(v)$.
- $\kappa(v)$ is continuous.
- $\nexists v_1 \neq v_2 : \kappa(v_1) = \kappa(v_2)$, thus $\kappa(v)$ is monotone.
- $\kappa(v) \to \infty$ as $v \to v_{\min}$.
- $\kappa(v) \to \kappa_{crit} \geqslant 0$ as $v \to v_{\max}$.
- When $\kappa < \kappa_{crit}$ and $v = v_{\max}$ there is a saddle to saddle-node travelling wave

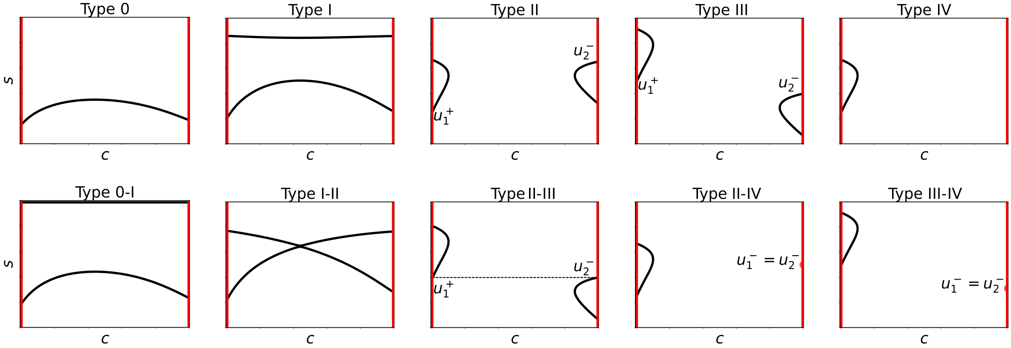
$\kappa(v)$ is monotone and continuous thus there exists an inverse function satisfying the Theorem.

Main ingredient of proof: phase portraits classification





Phase portrait evolution as v grows: Type 0 $\rightarrow$ Type I $\rightarrow$ $\rightarrow$ Type II $\rightarrow$ Type IV



Literature

Own works:

1. F. Bakharev, A. Enin, Yu. Petrova, N. Rastegaev, Impact of dissipation ratio on vanishing viscosity solutions of the Riemann problem for chemical flooding model. arXiv:2111.15001.

Other works:

1. Johansen, T. and Winther, R., 1988. The solution of the Riemann problem for a hyperbolic system of conservation laws modeling polymer flooding. SIAM journal on mathematical analysis, 19(3), pp.541-566.
2. Shen, W., 2017. On the uniqueness of vanishing viscosity solutions for Riemann problems for polymer flooding. Nonlinear Differential Equations and Applications NoDEA, 24(4), pp.1-25.
3. Entov, V.M. and Kerimov, Z.A., 1986. Displacement of oil by an active solution with a nonmonotonic effect on the flow distribution function. Fluid Dynamics, 21(1), pp.64-70.

Thank you for your attention!

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