Exact L_2-small ball probabilities for finite-dimensional perturbations of Gaussian processes: spectral method

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Joint work with
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- Nazarov A. I., **Petrova Yu. P.** (2015). The small ball asymptotics in Hilbertian norm for the Kac–Kiefer–Wolfowitz processes. Probab. Theory and Applicat., 60(3), pp. 482--505.
- **Petrova Yu. P.** (2017) Exact L_2-small ball asymptotics for some Durbin processes. Zapiski POMI, 466, pp. 211--233.
- **Petrova Yu. P.** (Work in progress) The L_2-small ball asymptotics for finite dimensional perturbations of Gaussian processes

Outline: small deviations for Gaussian processes

1. Introduction: basic notions and main tools
   - Small deviation probability
   - How to use Hilbert structure?
   - Short historical review

2. Problem statement and motivation

3. Main results: finite-dimensional perturbations
   - One-dimensional perturbations (Alexander Nazarov'2009)
   - Finite-dimensional perturbations (Yulia Petrova'2018)
   - Examples orthogonal to general theorems

Introduction: basic notions and main tools

Basic notion: small deviation probability

$X(t)$, $t \in (0,1)$, — Gaussian process, $\mathbb{E}X(t) \equiv 0$, $G_X(s,t) = \mathbb{E}X(s)X(t)$.

Definition

To find the asymptotics of small deviation probability of the process $X(t)$ *in* L_2*-norm means to find the asymptotics:*

$$\mathbb{P}\left(\|X\|_2 < \varepsilon\right) = \mathbb{P}\left(\int_0^1 (X(t))^2\,dt < \varepsilon^2\right), \qquad \varepsilon \to 0 \tag{1}$$

$$\mathbb{P}\left(\|W\|_2 < \varepsilon\right) \sim \frac{4}{\sqrt{\pi}}\,\varepsilon\,\exp\bigl(-\frac{1}{8}\,\varepsilon^{-2}\bigr)$$

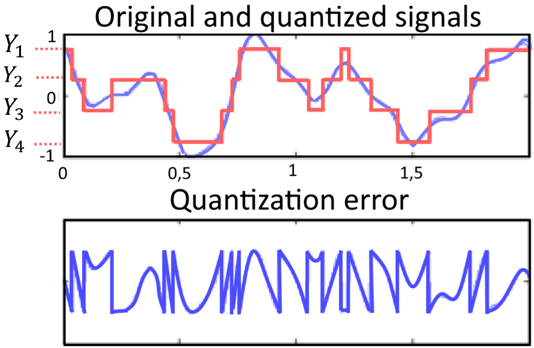
«Typical» answer:

$$\mathbb{P}\left(\|X\|_2 < \varepsilon\right) \sim D \cdot \varepsilon^{C} \cdot \exp\bigl(-B\varepsilon^{-A}\bigr)$$

A, B — *Logarithmic* asymptotics; A, B, C, D — *Exact* asymptotics

Applications of small deviation probabilities

Signal processing: quantization (discretization) of random vectors



X — random vector (signal)

$Y = \{Y_j\}_{j=1}^n$ «dictionary»

Minimize «quantization error»:

$$\mathbb{E} \min_{1 \leqslant i \leqslant n} \rho(X, Y_i)$$

If you know $(\varepsilon \to 0)$ $\Longrightarrow$ You can estimate $(n \to \infty)$

$$\psi(\varepsilon) = -\ln \mathbb{P}(\|X\| < \varepsilon) \qquad \mathbb{E} \min_{1 \leqslant i \leqslant n} \rho(X, Y_i)$$

Hilbert structure $\Longrightarrow$ spectral problem

Karhunen–Loève expansion (KL-expansion):
(due to K. Karhunen'1947, M. Loève'1948)

$$X(t) \stackrel{d}{=} \sum_{k=1}^{\infty} \sqrt{\mu_k}\, u_k(t)\, \xi_k$$

- ξ_k, $k \in \mathbb{N}$, — iid standard normal r.v.
- $u_k(t)$, μ_k — orthonormal eigenfunctions and positive eigenvalues of the covariance operator $\mathbb{G}_X$:

$$\mu_k u_k = \mathbb{G}_X u_k \qquad \Longleftrightarrow \qquad \mu_k\, u_k(t) = \int_0^1 G_X(s,t)\, u_k(s)\, ds.$$

The small deviation problem ($\varepsilon \to 0$):

$$\mathbb{P}(\|X\|_2 < \varepsilon) = \mathbb{P}\left(\sum_{k=1}^{\infty} \mu_k \xi_k^2 < \varepsilon^2\right).$$

Main idea: all information about the process is contained in the spectrum.

What is already known?

1974 — G. Sytaya: implicit solution of the problem in terms of Laplace transform of $\sum \mu_k \xi_k^2$

from 1974 — V. M. Zolotarev, J. Hoffmann-Jorgensen , L. Shepp, R. Dudley, Il. A. Ibragimov, M. A. Lifshits,. . . :
simplification under different assymptions

1998 — T. Dunker, M. A. Lifshits, W. Linde (DLL):
Rather simple formulas for

$$\mathbb{P}\left(\sum \mu_k \xi_k^2 < \varepsilon^2\right) \qquad \text{when}$$

- μ_k — decreasing, logarifmically convex
- $\mu_k = k^{-d}, \quad d > 0,$ — polynomial decreasing
- $\mu_k = A^{-k}, \quad A > 0,$ — exponential decreasing

Useful fact: Wenbo Li principle

Let $\widehat{\mu}_k \approx \mu_k$ be some approximation.
Question: How the following small deviation probabilities are related

$$\mathbb{P}\left(\sum \mu_k \xi_k^2 < \varepsilon^2\right) \text{ and } \mathbb{P}\left(\sum \widehat{\mu}_k \xi_k^2 < \varepsilon^2\right)?$$

Theorem (The Wenbo Li principle 1992, Gao et al. 2003)

Let μ_k, $\widehat{\mu}_k$ — two summable sequences. If

$$\prod_{k=1}^{\infty} \frac{\widehat{\mu}_k}{\mu_k} < \infty, \tag{2}$$

then as $\varepsilon \to 0$

$$\mathbb{P}\left(\sum_{k=1}^{\infty} \mu_k \xi_k^2 < \varepsilon^2\right) \sim \mathbb{P}\left(\sum_{k=1}^{\infty} \widehat{\mu}_k \xi_k^2 < \varepsilon^2\right) \cdot \left(\prod \frac{\widehat{\mu}_k}{\mu_k}\right)^{1/2}$$

«good» approx. $\widehat{\mu}_k$ for μ_k + Wenbo Li principle + DLL theorem = small ball probability

Spectral theory helps probability

beg. XX	—	G. Birkgoff, Ya.D. Tamarkin: spectral asymptotics for ODE
fin. 60-s	—	M.Sh. Birman, M.Z. Solomyak: spectral asymptotics for integral operators
2004	—	A.I. Nazarov, Ya.Yu. Nikitin: «Green» processes — the processes which covariance operator is the inverse for ODE operator (with some boundary conditions).
from 2004	—	many works from scientists from Saint-Petersburg strongly using spectral theory of operators

Problem statement and motivation

Problem statement

$X_0(t)$ — Gaussian process, $\mathbb{E}X_0(t) \equiv 0$, $G_0(s,t) = \mathbb{E}X_0(s)X_0(t)$,
μ_k^0 — eigenvalues of the covariance operator $\mathbb{G}_0$, $\mu_k^0 > 0$

$$\mathbb{P}\left(\|X_0\|_2 < \varepsilon\right) \text{ — known}$$

$X(t)$ — finite-dimensional perturbation of rank m of the process $X_0(t)$:

$$\mathbb{E}X(t) \equiv 0 \qquad G_X(s,t) = G_0(s,t) + \vec{\psi}^{\,T}(s) \cdot D \cdot \vec{\psi}(t) \tag{3}$$

- $\vec{\psi}(t) = (\psi_1(t), \ldots, \psi_m(t))^T$
- $D \in M_{m\times m}$ — symmetric (w.l.o.g.)
- μ_k — eigenvalues of the covariance operator $\mathbb{G}_X$, $\mu_k > 0$

What is the relation between $\mathbb{P}\left(\|X_0\|_2 < \varepsilon\right)$ and $\mathbb{P}\left(\|X\|_2 < \varepsilon\right)$?

The motivating example: limit Durbin processes

A sample $x_1, \ldots, x_n \sim F(x, \theta)$
$\theta = (\theta_1, \ldots, \theta_m)$ — parameters of the distribution. Let's consider:

$$F_n^0(t) = \{\text{number of } x_i \colon F(x_i, \theta_0) \leqslant t\}, \quad \theta_0 = \text{fix}.$$

Then $\quad n^{1/2}\big[F_n^0(t) - t\big] \xrightarrow{w} B(t). \quad$ Let's consider:

$$\hat{F}_n(t) = \{\text{number of } x_i \colon F(x_i, \hat{\theta}_n) \leqslant t\}, \quad \hat{\theta}_n \text{ — estimated from the data}$$

Then $n^{1/2}\big[\hat{F}_n(t) - t\big] \xrightarrow{w} B(t) + \ldots$ — perturbation of Brownian bridge.
It is a Gaussian process with zero mean and covariance function:

$$G(s,t) = G_B(s,t) - \vec{\psi}^{\,T}(s)\, S^{-1}\, \vec{\psi}(t)$$

- $G_B(s,t) = \min(s,t) - st$
- $S_{ij} = \mathbb{E}\left(\frac{\partial}{\partial\theta_i} \ln(f(x,\theta)) \frac{\partial}{\partial\theta_j} \ln(f(x,\theta))\right)\Big|_{\theta=\theta_0}$ — Fisher information
- $\psi_j(t) = \dfrac{\partial F}{\partial \theta_j}\Big|_{\theta=\theta_0}, \quad \theta_0$ — fixed vector of parameters

Kac-Kiefer-Wolfowitz processes (KKW)

Important example: test for normality, $x_1, \ldots, x_n \sim F(x, \theta)$

$$f(x,\theta) = \frac{1}{\beta\sqrt{2\pi}} \exp\left(-\frac{1}{2}\Big(\frac{x-\alpha}{\beta}\Big)^2\right); \qquad F(x,\theta) = \int_{-\infty}^{x} f(y,\theta)\,dy$$

$\widehat{\alpha}$ estimated, $\beta = 1$:

$$G_1(s,t) = G_B(s,t) - \psi_1(s)\psi_1(t), \qquad \psi_1(t) = f_{st}(F_{st}^{-1}(t))$$

$\alpha = 0$, $\widehat{\beta}$ estimated:

$$G_2(s,t) = G_B(s,t) - \psi_2(s)\psi_2(t), \qquad \psi_2(t) = \psi_1(t) \cdot \frac{F_{st}^{-1}(t)}{\sqrt{2}}$$

$\widehat{\alpha}$, $\widehat{\beta}$ estimated:

$$G_3(s,t) = G_B(s,t) - \psi_1(s)\psi_1(t) - \psi_2(s)\psi_2(t)$$

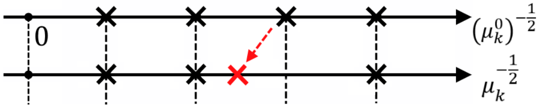
Problem: small deviation probabilities for KKW processes?

Main results:
finite-dimensional perturbations

$$G_X(s,t) = G_0(s,t) + D\psi(s)\psi(t), \qquad D \in \mathbb{R}$$

The simplest case: $\psi(t)$ — eigenfunction of the integral operator $\mathbb{G}_0$

What happens if we change D?



Decrease $D \downarrow$

$\mu_k^0 = \mu_k,\ k \to \infty$

$$G_X(s,t) = G_0(s,t) + D\psi(s)\psi(t), \qquad D \in \mathbb{R}$$

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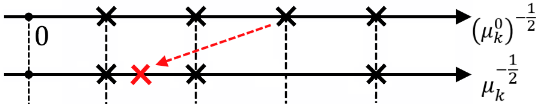


Decrease $D \downarrow$

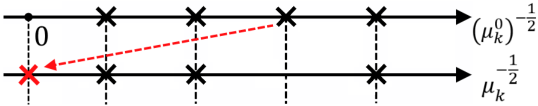
$\mu_k^0 = \mu_k,\ k \to \infty$

I. One-dimensional perturbation: first observation

$$G_X(s,t) = G_0(s,t) + D\psi(s)\psi(t), \qquad D \in \mathbb{R}$$

The simplest case: $\psi(t)$ — eigenfunction of the integral operator $\mathbb{G}_0$

What happens if we change D?



D — critical

$\mu_k^0 = \mu_{k-1}, \ k \to \infty$

Similar effect can be observed in a more general situation (when $\psi(t)$ is not necessarily the eigenfunction)

I. One-dimensional perturbation: spectral main theorem

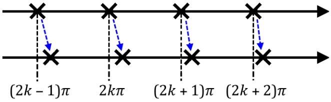
Let $\varphi(t) = \mathbb{G}_0^{-1}\psi(t)$, and

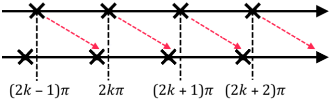
$$Q := \langle \mathbb{G}_0 \varphi, \varphi \rangle < \infty \quad \Leftrightarrow \quad \psi \in \mathrm{Im}(\mathbb{G}_0^{1/2})$$

Theorem (Alexander Nazarov '2009)

There exists $D_{crit} = -1/Q$ such that:

1. *If* $D > D_{crit}$, *then* $\prod_{k=1}^{\infty} \frac{\mu_k^0}{\mu_k} < +\infty$
2. *If* $D = D_{crit}$, $\psi \in \mathrm{Im}(\mathbb{G}_0)$, *then* $\prod_{k=2}^{\infty} \frac{\mu_k^0}{\mu_{k-1}} < +\infty$





Note: Let's call $\psi \in \mathrm{Im}(\mathbb{G}_0)$ a «good» perturbation ($\Leftrightarrow \varphi \in L_2[0,1]$)

I. One-dimensional perturbation: probabilistic main theorem

Let $\varphi(t) = \mathbb{G}_0^{-1}\psi(t)$, and

$$Q := \langle \mathbb{G}_0\varphi, \varphi\rangle < \infty \quad \Leftrightarrow \quad \psi \in \mathrm{Im}(\mathbb{G}_0^{1/2})$$

Theorem (Alexander Nazarov '2009)

There exists $D_{crit} = -1/Q$ *such that:*

1. *(non-critical)* *If* $D > D_{crit}$, *then as* $\varepsilon \to 0$

$$\mathbb{P}\left(\|X\|_2 < \varepsilon\right) \sim \frac{\mathbb{P}\left(\|X_0\|_2 < \varepsilon\right)}{|1 + QD|}.$$

2. *(critical)* *If* $D = D_{crit}$, $\varphi \in L_2[0,1]$, *then as* $\varepsilon \to 0$

$$\mathbb{P}\left(\|X\|_2 < \varepsilon\right) \sim \frac{\sqrt{Q}}{\|\varphi\|_2} \cdot \sqrt{\frac{2}{\pi}} \cdot \int_0^{\varepsilon^2} \frac{d}{dt}\mathbb{P}(\|X_0\|_2 < t) \cdot \frac{dt}{\sqrt{\varepsilon^2 - t^2}}$$

II. Finite-dimensional perturbations: spectral main theorem

$$G_X(s,t) = G_0(s,t) + \vec{\psi}^{\,T}(s) \cdot D \cdot \vec{\psi}(t),$$
$$\vec{\psi}(t) = (\psi_1(t), \dots, \psi_m(t))^T, \qquad D \in M_{m\times m}$$

Let $\varphi_j(t) = \mathbb{G}_0^{-1}\psi_j(t)$, and

$$Q := \langle \mathbb{G}_0 \vec{\varphi}, \vec{\varphi}^{\,T} \rangle < \infty \quad \Leftrightarrow \quad \psi_j \in \mathrm{Im}(\mathbb{G}_0^{1/2})$$

Theorem (Yulia Petrova '2018)

1. *If* $(Q^{\,T} D + E_m) > 0,$ *then* $\prod_{k=1}^{\infty} \frac{\mu_k^0}{\mu_k} < +\infty.$
2. *If* $\mathrm{rank}(Q^{\,T} D + E_m) = m - s,$ $\psi_j \in \mathrm{Im}(\mathbb{G}_0),$ *then*

$$\prod_{k=s+1}^{\infty} \frac{\mu_k^0}{\mu_{k-s}} < +\infty.$$

II. Finite-dimensional perturbations: probabilistic theorem

Let $\varphi_j(t) = \mathbb{G}_0^{-1}\psi_j(t)$, and $Q := \langle \mathbb{G}_0\vec{\varphi}, \vec{\varphi}^{\,T}\rangle < \infty \quad \Leftrightarrow \quad \psi_j \in \mathrm{Im}(\mathbb{G}_0^{1/2})$.

Theorem (Yulia Petrova '2018)

1. *(non-critical)* *If* $(Q^{\,T}D + E_m) > 0$, *then as* $\varepsilon \to 0$

$$\mathbb{P}\left(\|X\|_2 < \varepsilon\right) \sim \frac{\mathbb{P}\left(\|X_0\|_2 < \varepsilon\right)}{\det(Q^T D + E_m)}.$$

2. *(critical)* *If* $(Q^{\,T}D + E_m) \equiv 0$, $\psi_j \in \mathrm{Im}(\mathbb{G}_0)$, *then as* $r \to 0$

$$\mathbb{P}\left(\|X\|_2 < \sqrt{r}\right) \sim \sqrt{\frac{\det(Q)}{\det\left(\int_0^1 \vec{\varphi}(t)\,\vec{\varphi}^{\,T}(t)\,dt\right)}} \cdot \left(\sqrt{\frac{2}{\pi}}\right)^m \cdot$$

$$\cdot \int_0^r \int_0^{r_1} \dots \int_0^{r_{m-1}} \frac{d^m}{dr_m^m}\mathbb{P}(\|X_0\|_2 < r_m)\frac{dr_m \dots dr_1}{\sqrt{(r-r_1)\cdot \ldots \cdot (r_{m-1}-r_m)}}.$$

Main tools in proof

1. Fredholm determinants:

$$F_0(z) = \prod_{k=1}^{\infty} \left(1 - \frac{z}{1/\mu_k^0}\right), \qquad F(z) = \prod_{k=1}^{\infty} \left(1 - \frac{z}{1/\mu_k}\right), \quad z \in \mathbb{C}$$

2. Jensen's theorem:

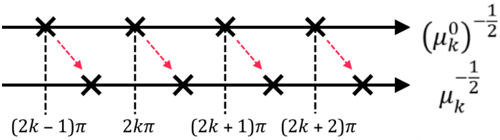
$$\prod_{j=1}^{\infty} \frac{\mu_j^0}{\mu_j} = \lim_{|z|\to\infty} \exp\left(\frac{1}{2\pi}\int_0^{2\pi} \ln\left|\frac{F(z)}{F_0(z)}\right| d\arg(z)\right).$$

Example 1: «bad» critical perturbation

$G_0(s,t) = \min(s,t) - st, \quad \psi(t) = t\ln(t)$, then

$$G(s,t) = G_0(s,t) - \psi(s)\psi(t), \text{ critical perturbation.}$$

Writing down the equation on the eigenalues μ_k and solving it directly, we get:



$$(\mu_k^0)^{-1/2} = \pi k; \qquad \mu_k^{-1/2} = \pi k + \frac{\pi}{2} + O\left(\frac{1}{k}\right), \quad k \to \infty$$

P.S. This case corresponds to Durbin process when testing exponentiality

Example 2: Kac-Kiefer-Wolfowitz processes (KKW)

Important example: test for normality — no general theorem works

X_1	$\widehat{\alpha}$ estimated, $\beta = 1$	$\psi_1(t) = f(F^{-1}(t))$	critical, not «good»
X_2	$\alpha = 0$, $\widehat{\beta}$ estimated	$\psi_2(t) = \psi_1(t) \cdot \frac{F^{-1}(t)}{\sqrt{2}}$	critical, not «good»
X_3	$\widehat{\alpha}$, $\widehat{\beta}$ estimated:	$\psi_1(t), \psi_2(t)$	critical, not «good»

Example 2: Kac-Kiefer-Wolfowitz processes (KKW)

Straightforward solution — equation on eigenvalues μ_k:

$$\mu_k u(t) = \int\limits_0^1 G_0(s,t)u(s)\,ds - f(F^{-1}(t)) \int\limits_0^1 f(F^{-1}(s))u(s)\,ds$$

Apply $-\dfrac{d^2}{dt^2}$:

$$-\mu_k u''(t) = u(t) + \frac{1}{f(F^{-1}(t))} \int\limits_0^1 f(F^{-1}(s))u(s)\,ds, \quad u(0) = u(1) = 0$$

Let $\omega_k := \mu_k^{-1/2}$. So the solution is

$$u(t) = c_0 \cos(\omega_k t) + c_1 \sin(\omega_k t) + c_2 \eta(t, \omega_k)$$

Theorem: Kac-Kiefer-Wolfowitz processes

Theorem (A.Nazarov, Yu.Petrova'2015)

$$X_1: \qquad \omega_{2k-1} = 2\pi k + \frac{\pi}{\ln(k)} + O\left(\frac{\ln(\ln(k))}{\ln^2(k)}\right), \qquad \omega_{2k} = 2\pi k.$$

$$\mathbb{P}\Big\{\|X_1\| < \varepsilon\Big\} \sim C \cdot \varepsilon^{-1} \cdot \ln^{\frac{1}{2}}\left(\frac{1}{\varepsilon}\right) \cdot \exp\Big(-\frac{1}{8\varepsilon^2}\Big)$$

$$X_2: \qquad \omega_{2k-1} = 2\pi k - \pi, \qquad \omega_{2k} = 2\pi k + \pi + O\left(\frac{1}{\ln^2(k)}\right)$$

$$\mathbb{P}\Big\{\|X_2\| < \varepsilon\Big\} \sim \frac{2\sqrt{2}}{\pi^{3/2}}\,\varepsilon^{-1} \exp\Big(-\frac{1}{8\varepsilon^2}\Big)$$

Example 3: general Durbin processes

Theorem (Yulia Petrova '2018)

All Durbin processes are critical.

However, perturbations are «often» not «good». We considered Durbin processes when testing for distributions with parameters $\theta = (\alpha, \beta)$:

- Laplace $$F(x,\theta) = \begin{cases} \frac{1}{2}\exp(\frac{x-\alpha}{\beta}), & x \leqslant \alpha; \\ 1 - \frac{1}{2}\exp(-\frac{x-\alpha}{\beta}), & x > \alpha. \end{cases}$$
- logistic $$F(x,\theta) = \left(1 + \exp(-\tfrac{x-\alpha}{\beta})\right)^{-1}.$$
- Gumbel $$F(x,\theta) = \exp\left(-\exp\left(-\tfrac{x-\alpha}{\beta}\right)\right).$$
- Gamma $$F(x,\theta) = \begin{cases} \int\limits_0^{x/\beta} \frac{y^{\alpha-1}e^{-y}}{\Gamma(\alpha)}dy, & x \geqslant 0; \\ 0, & x < 0. \end{cases}$$

Note: all perturbations, but X_1 for logistic dist, are «bad»

Example 3: Gumbel distribution

Theorem (Yulia Petrova '2017)

For Durbin process $X(t)$ when testing for Gumbel distribution

$$G(s,t) = G_0(s,t) - \psi(t)\psi(s), \qquad \psi(t) = C\, t \ln(t) \cdot \ln(-\ln(t))$$

the asymptotics of corresponding eigenvalues is the following

$$\mu_k^{-1/2} = \pi k + \frac{\pi}{2} + (-1)^k \cdot 2 \operatorname{arctg}\Big(\frac{1}{\ln(\ln(k)) + 1}\Big) - \frac{1}{\ln(k)\ln(\ln(k))} + \\ + O\Big(\frac{1}{\ln(k)(\ln(\ln(k)))^2}\Big).$$

And asymptotics of small ball probabilities

$$\mathbb{P}\Big\{\|X\| < \varepsilon\Big\} \sim C \cdot \ln^{-1}(\ln(\varepsilon^{-1})) \cdot \varepsilon^{-1} \cdot \exp\big(-\frac{1}{8\varepsilon^2}\big)$$

Spectral asymptotics $\longleftrightarrow$ small ball probabilities

Theorem (A. Nazarov, Yu. Petrova '2015)

If $\hat{\mu}_k = (\vartheta(k + \delta + F(k)))^{-2}$. *Then we have, as* $\varepsilon \to 0$,

$$\mathbb{P} \sim C \cdot \exp\left(\frac{1}{2} \cdot F_{-1}(\varepsilon^{-2})\right) \cdot \varepsilon^{-2\delta} \cdot \exp\left(-\left(\frac{\pi}{2\vartheta}\right)^2 \cdot \frac{\varepsilon^{-2}}{2}\right),$$

where $F(t)$ *is a slowly-varying function at infinity,* $F(t) \to 0$ *as* $t \to \infty$,

$$F_{-1}(t) = \int_1^t \frac{F(x)}{x}\, dx.$$

Note: $\exp\left(\frac{1}{2} \cdot F_{-1}(t)\right)$ is also a slowly-varying function as $t \to \infty$.

Example 3: logistic, Gumbel distributions etc.

Theorem (Yulia Petrova '2017)

Small deviations probabilities for some Durbin processes:

LOG 1	$\frac{2\sqrt{15}}{\sqrt{\pi}} \cdot \varepsilon^{-2} \exp\left(-\frac{1}{8\varepsilon^2}\right)$
LOG 2	$\frac{4\sqrt{3+\pi^2}}{3\sqrt{2}\,\pi^{3/2}} \cdot \varepsilon^{-1} \exp\left(-\frac{1}{8\varepsilon^2}\right)$
LOG 3	$\frac{4\sqrt{15(3+\pi^2)}}{3\,\pi^{3/2}} \cdot \varepsilon^{-3} \exp\left(-\frac{1}{8\varepsilon^2}\right)$
GUM 1	$\frac{4}{\pi^{3/2}} \cdot \varepsilon^{-1} \exp\left(-\frac{1}{8\varepsilon^2}\right)$
GUM 2	$C \cdot \ln^{-1}(\ln(\varepsilon^{-1})) \cdot \varepsilon^{-1} \exp\left(-\frac{1}{8\varepsilon^2}\right)$
GUM 3	$C \cdot \exp\left(2\pi \ln^2(\ln(\varepsilon^{-1}))\right) \cdot \varepsilon^{-2} \exp\left(-\frac{1}{8\varepsilon^2}\right)$

Thank you for your attention!

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Using standard methods we get the equation on the eigenvalues $\mu = \omega^{-2}$:

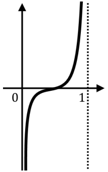
$$P(\mathcal{S}(\omega), \mathcal{C}(\omega), \mathcal{I}(\omega)) = 0, \qquad \omega \to \infty,$$

where

$$\mathcal{C}(\omega) = \int_0^{\frac{1}{2}} F_{st}^{-1}(t) \cos(\omega t)\, dt \qquad \mathcal{S}(\omega) = \int_0^{\frac{1}{2}} F_{st}^{-1}(t) \sin(\omega t)\, dt$$

$$\mathcal{I}(\omega) = \int_0^{\frac{1}{2}} \int_0^{\tau} F_{st}^{-1}(t) F_{st}^{-1}(\tau) \sin(\omega \tau) \cos(\omega t)\, dt\, d\tau$$

Here $F_{st}(t)$ — standart normal distribution function
$F_{st}^{-1}(t) \sim -\sqrt{-2\ln(t)}, \quad t \to 0,$
$F_{st}^{-1}(t)$ has singularity at $t = 0$



Slowly varying functions = SVF

Definition

Function $V(t)$ is called SVF at infinity, if it doesn't change sign on some $[A, \infty)$, $A > 0$, and for any $\lambda > 0$

$$\lim_{t \to \infty} \frac{V(\lambda t)}{V(t)} = 1.$$

Function $V(t)$ is called SVF at zero, if $V(1/t)$ is SVF at infinity.
For example, $\ln^{\alpha}(t)$, $\alpha \in \mathbb{R}$.

Note: $F^{-1}(t)$ has the following properties:

- $V_0(t) := F^{-1}(t)$, $V_{n+1}(t) := tV_n'(t)$, $n \geq 0$, are SVF at zero.
- $F^{-1}\left(\frac{1}{2}\right) = 0$.

Note: for any SVF at zero: $tV'(t) = o(V(t))$ when $t \to 0$.
So $\forall n \geq 0 \quad V_{n+1}(t) = o(V_n(t))$.

Asymptotics of integrals

Let

- $V_0(t)$ and $V_{n+1}(t) = t \cdot V_n'(t)$ be SVF at zero.
- $V_0(\frac{1}{2}) = 0$.

Theorem (A.Nazarov, Yu.Petrova'2015)

As $\omega \to \infty$:

$$\mathcal{C} = \int\limits_0^{\frac{1}{2}} V(t)\cos(\omega t)\,dt = \frac{1}{\omega}\sum_{k=1}^{N} c_k V_k\Big(\frac{1}{\omega}\Big) + R_N, \tag{4}$$

where

$$|R_N| \le C(V,N) \cdot \frac{\left|V_{N+1}(\frac{1}{\omega})\right|}{\omega}.$$

Example: $\int\limits_0^{1/2} \sqrt{-\ln(2t)}\cos(\omega t)\,dt = \frac{\pi}{2\ln^{1/2}(2\omega)} - \frac{\gamma\pi}{2\ln^{3/2}(2\omega)} + O\left(\frac{1}{\ln^{5/2}(\omega)}\right)$

Asymptotics of integrals

Theorem (A.Nazarov, Yu.Petrova'2015)

$$\int_0^{\frac{1}{2}}\int_0^{\tau} V(t)V(\tau)\sin(\omega\tau)\cos(\omega t)\,dt\,d\tau =$$

$$= \frac{1}{2\omega}\int_0^{\frac{1}{2}} V^2(t)\,dt + \sum_{n=2}^{N}\sum_{\substack{k+m=n\\ k,m\geq 1}} a_{k,m}\frac{V_k(\frac{1}{\omega})V_m(\frac{1}{\omega})}{\omega^2} + R_N,$$

where $|R_N| \leq C(V,N)\sum_{\substack{i+j=N+1\\ i,j\geq 1}}\frac{|V_i(\frac{1}{\omega})V_j(\frac{1}{\omega})|}{\omega^2}.$

Thank you again!