Two tubes model of miscible displacement: travelling waves and normal hyperbolicity

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75 anos impa
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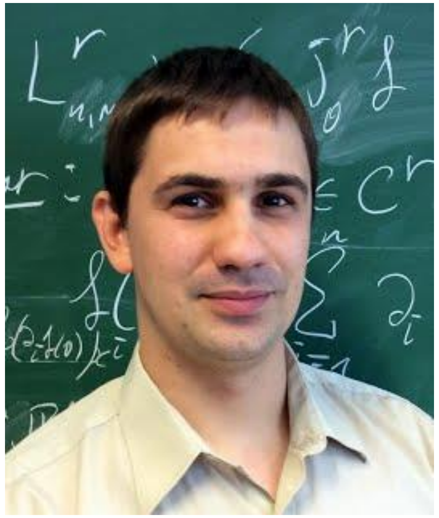

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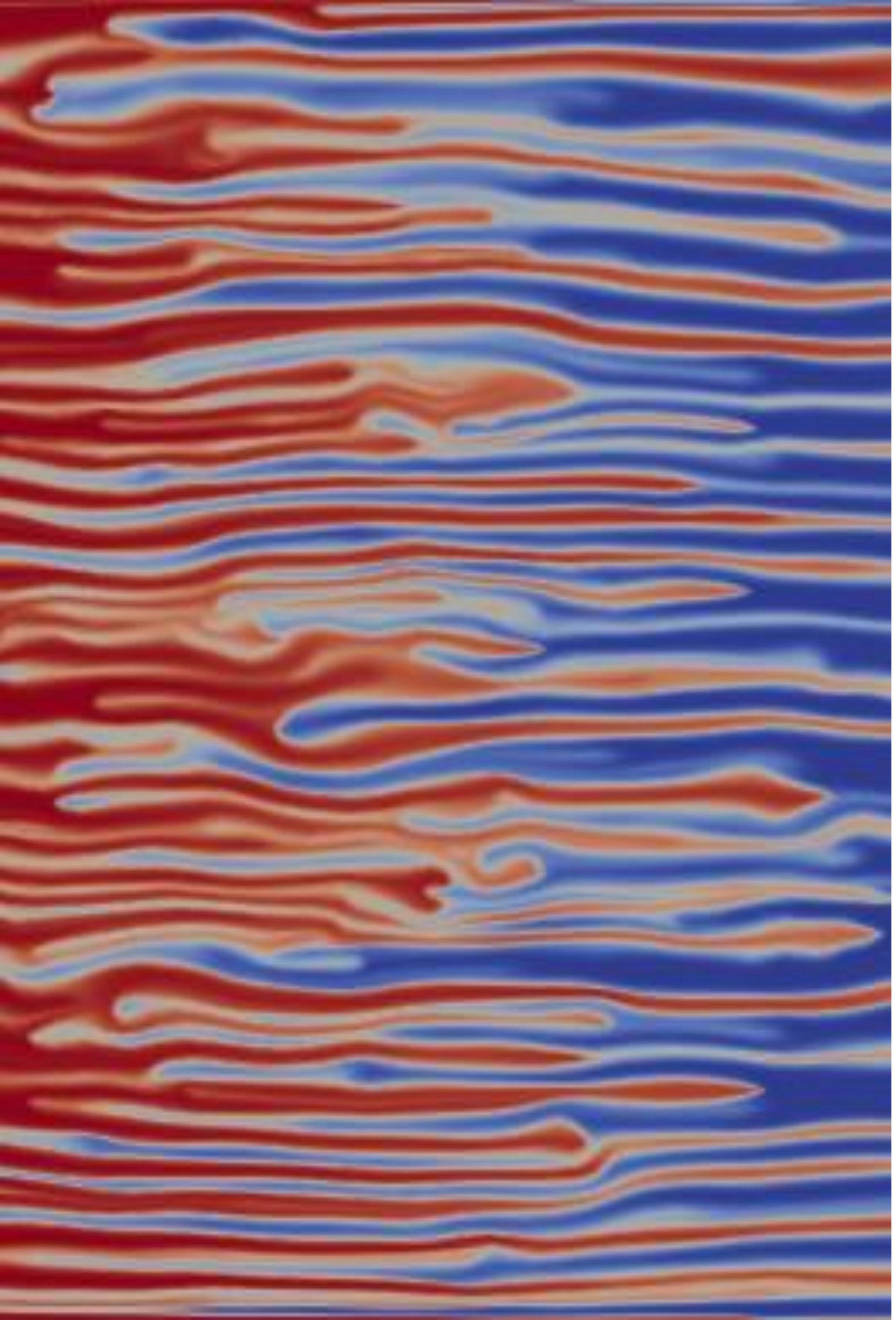
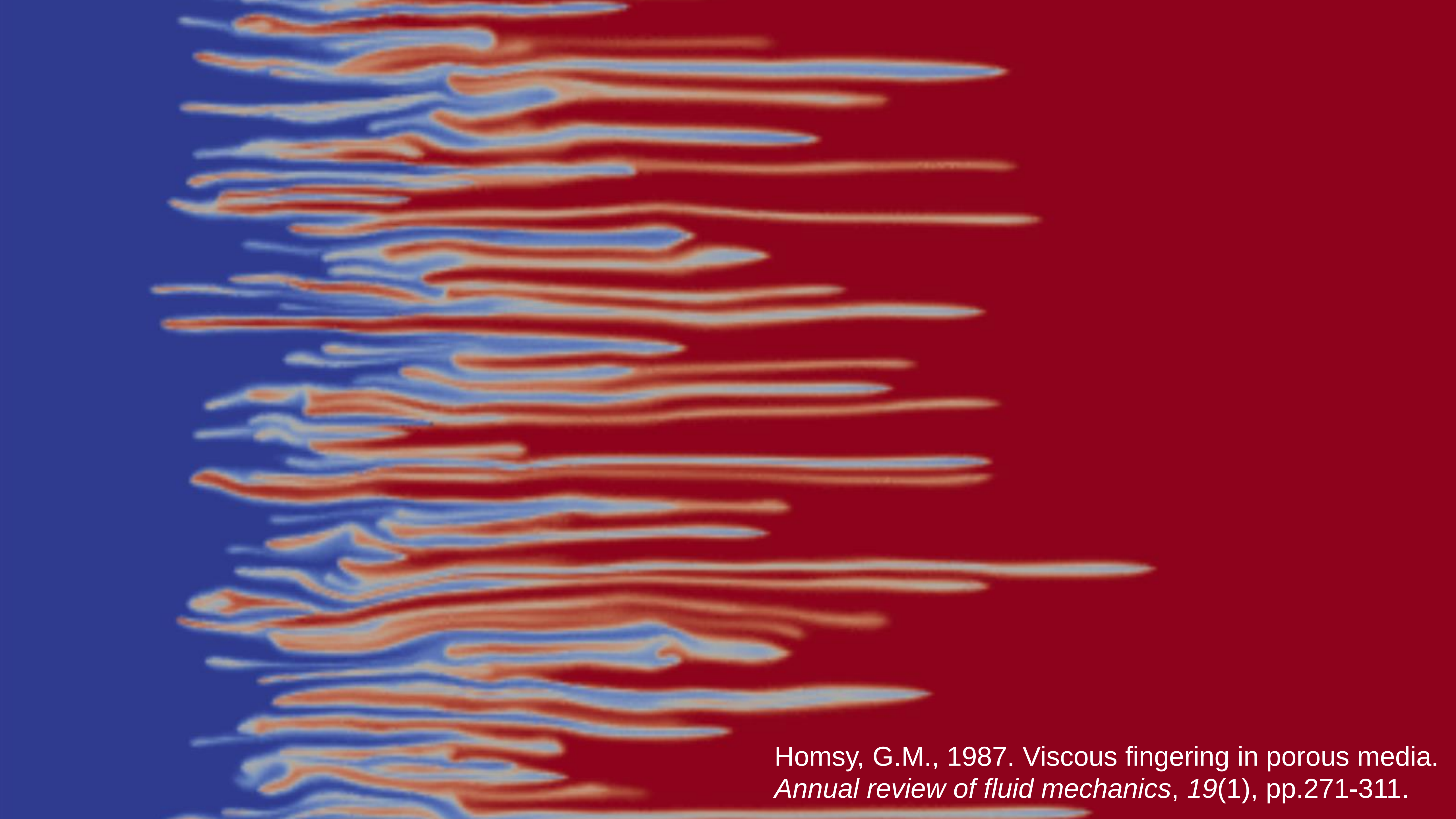
13th Americas Conference on Diff. Equations and Nonlinear Analysis and ICMC Summer Meeting on Differential Equations - 2023 Chapter



Outline

1. General phenomenon
   - Viscous fingers
   - Gravitational fingers
2. Motivation of the statement of the problem
   - Why we believe that our setting is important
   - Introduce the “toy model”
3. Theorem and Conjectures

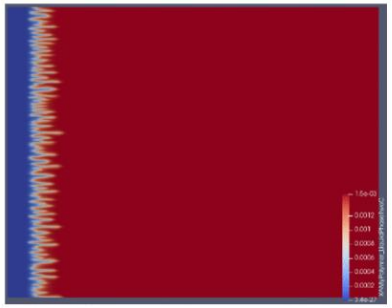
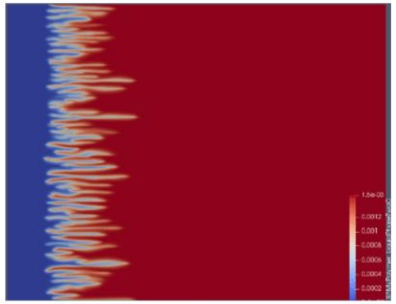
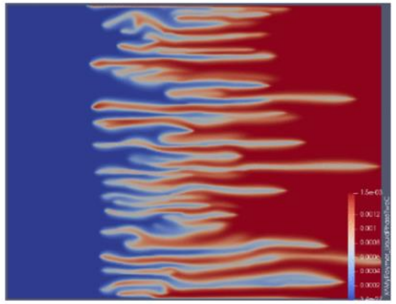


Homsy, G.M., 1987. Viscous fingering in porous media.
Annual review of fluid mechanics, *19*(1), pp.271-311.

Two settings (Peaceman model)

1. Viscosity-driven fingers

$$c_t + u \cdot \nabla c = \ \varepsilon\, \Delta c$$
$$div\, u = 0$$
$$u = -k \cdot m(c) \nabla p$$

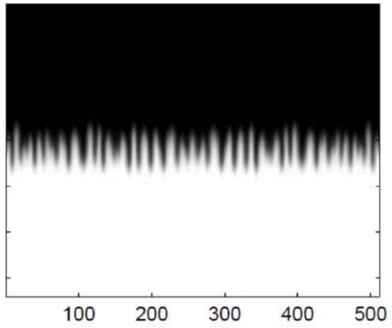




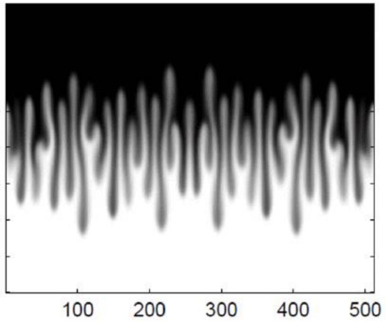
- c – concentrations of viscous spices (transport equation) $c \in [0, 1]$
- u – velocity of fluid (incompressibility condition)
- p – pressure
  velocity is defined by Darcy law and mobility of liquid $m(c)$;
  $m(c)$ – decreasing function, e.g.
  $m(c) = \mathrm{e}^{-\mathrm{a}c}$

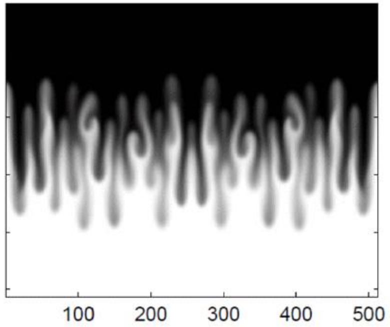
We did a lot of numerical simulations.
Motivation of statement of the problem.

2. Gravity-driven fingers

$$c_t + u \cdot \nabla c = \ \varepsilon\, \Delta c$$
$$div\, u = 0$$
$$u = -\nabla p - (0, c)$$







- c – concentrations of heavy spices (transport equation) $c \in [-1, 1]$
- u – velocity of fluid (incompressibility condition)
- p – pressure.

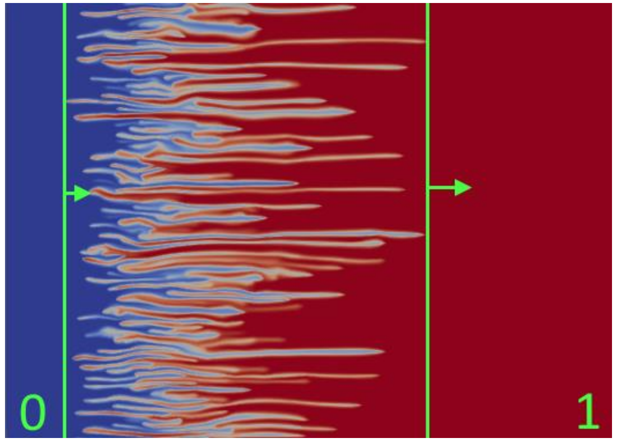
  velocity is defined by Darcy law and gravitation

We have some theorems for "toy model"

Questions of interest

1. Well-posedness:
   - $\varepsilon = 0$: incompressible porous medium (IPM)
     active scalar: $u = A(c)$ – singular integral operator (like in SQG)
     - existence of a global solution vs finite-time blow-up:
       A. Castro, D. Cordoba, D. Lear (2018), T. Elgindi (2017), A. Kiselev, Y. Yao (2023)
     - non-uniqueness of solutions (convex integration technique):
       D. Córdoba, D. Faraco, F. Gancedo (2011), R. Shvydkoy (2011), L. Szekelyhidi Jr. (2012)
   - related: generalized Buckley-Leverett equation – N. Chemetov, W. Neves (2014)
     Muskat problem & Hele-Shaw (free boundary) – A. Cordoba, D. Cordoba, F. Gancedo (2011) etc.

2. Dynamics of mixing zone:
   - many laboratory and numerical experiments show linear growth of the mixing zone [1]
   - the only mathematically rigorous result on estimates of speed of the linear growth
     - Simplified model of Darcy's law:
       Transverse flow equilibrium (TFE) [2]



[1] Nijjer J., Hewitt D., and Neufeld J. The dynamics of miscible viscous fingering from onset to shutdown. Journal of Fluid Mechanics 837 (2018): 520-545.
[2] Menon, G. and Otto, F., 2006. Diffusive slowdown in miscible viscous fingering. Communications in Mathematical Sciences, 4(1), pp.267-273.

TFE model and comparison theorem (gravity-driven)

- TFE model: assumption $p(x,y) \sim p(y), \quad p_y(x,y) \sim p_y(y)$

$$\begin{aligned} c_t + u \cdot \nabla c &= \varepsilon\, \Delta c \\ div\, u &= 0 \\ u &= (u^x, u^y) \\ u^y &= \bar{c} - c \end{aligned}$$

- Consider 1d equations (viscous Burgers equation)

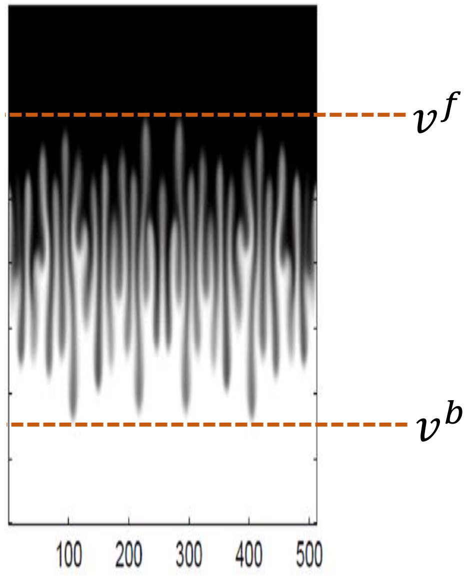
$$c_t^{max} + (1 - c^{max}) c_y^{max} = \varepsilon\, (c^{max})_{yy}$$

$$c_t^{min} + (-1 - c^{min}) \cdot c_y^{min} = \varepsilon\, \left(c^{min}\right)_{yy}$$

Comparison theorem (Otto-Menon, 2005)

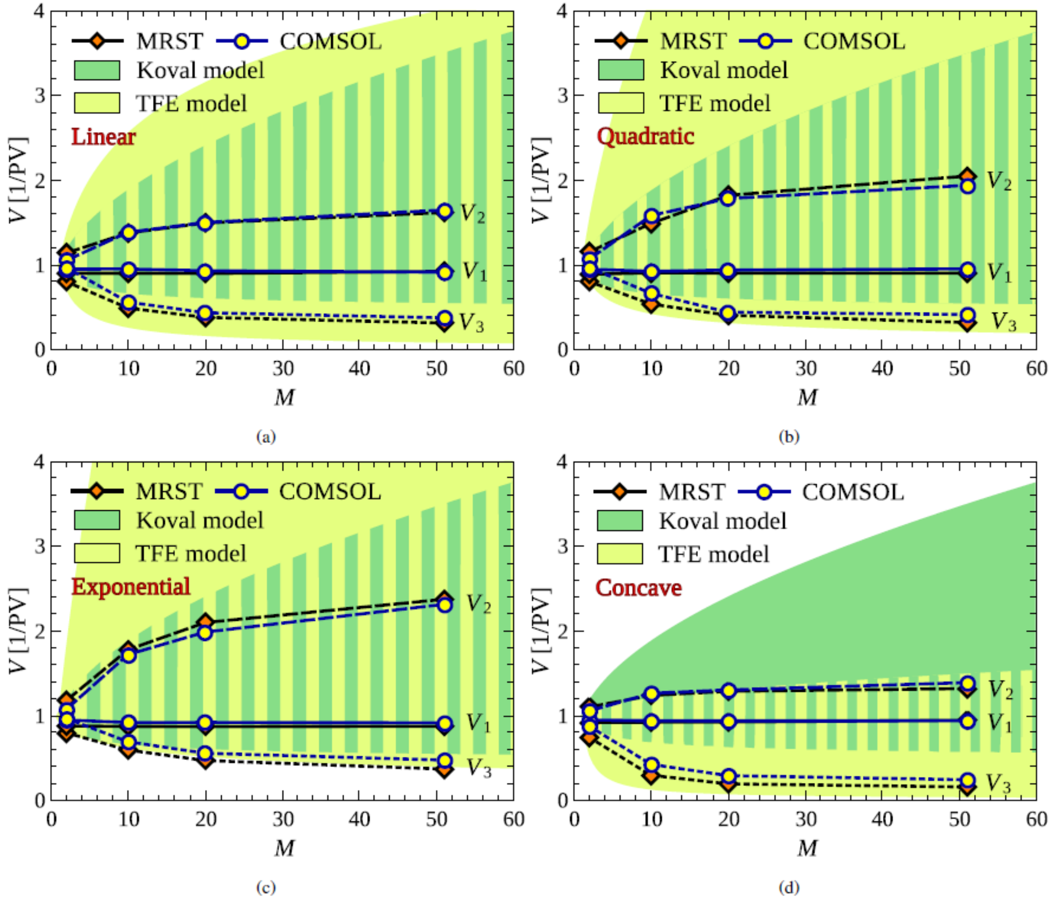
- If $c(0,x,y) < c^{max}(0,y)$ then $c(t,x,y) \leq c^{max}(t,y)$
- If $\mathrm{c}(0,x,y) > c^{min}(0,y)$ then $c(t,x,y) \geq c^{min}(t,y)$

1. It gives upper bound for the faster finger
   $$v^f \leq 1$$
2. It gives upper bound for the back front
   $$v^b \geq -1$$
3. Estimate is sharp if
   1. There is no transverse flow
   2. Drop of concentration on a finger tip is -1 -> +1
4. Numerics shows that estimate is far from sharp
5. We want to get better estimate

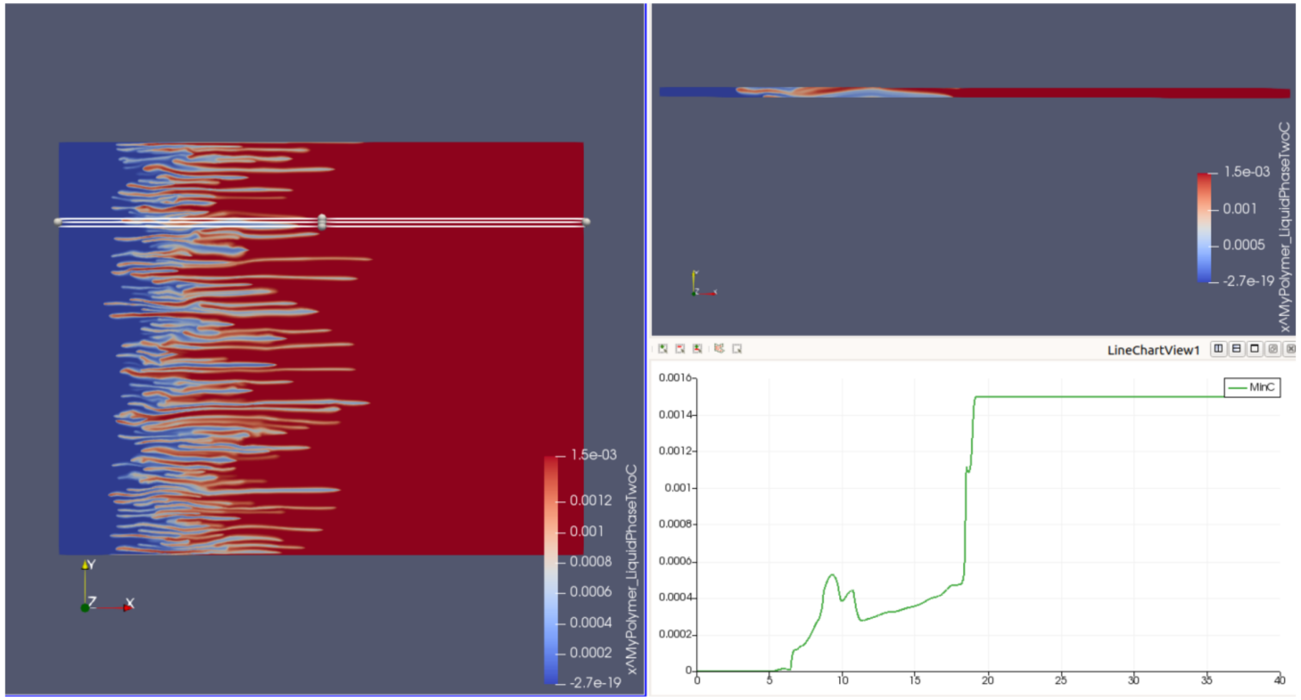


Numerics for viscous fingers

F. Bakharev, A. Enin, A. Groman, A. Kalyuzhnyuk,
S. Matveenko, **Yu. Petrova**, I. Starkov, S. Tikhomirov
“Velocity of viscous fingers in miscible displacement: Comparison with analytical models”
Journal of Computational and Applied Mathematics, 2022



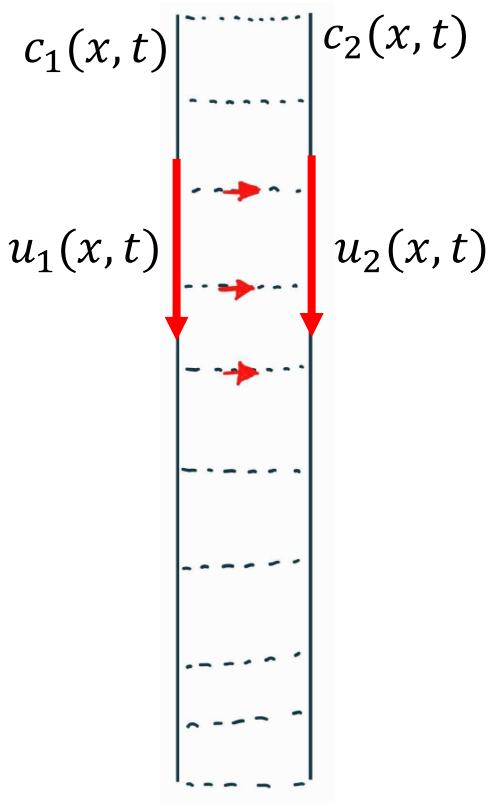
Possible mechanism: intermediate concentration



Two-tubes (two-layer) model

Original equations

$$c_t + div(uc) = \varepsilon\, \Delta c$$
$$div\, u = 0$$



Inclusion of transverse flow

$$\partial_t c_1 + \partial_x(u_1 c_1) - \varepsilon \partial_{xx} c_1 = -\boxed{(-1)^{1,2}\partial_x u_{1,2} \cdot c_{1,2},}$$
$$\partial_t c_2 + \partial_x(u_2 c_2) - \varepsilon \partial_{xx} c_2 = \boxed{(-1)^{1,2}\partial_x u_{1,2} \cdot c_{1,2},}$$

$$(-1)^{1,2}\partial_x u_{1,2} \cdot c_{1,2} = \begin{cases} -\partial_x u_1 \cdot c_1, & \partial_x u_1 < 0, \\ +\partial_x u_2 \cdot c_2, & \partial_x u_1 \geq 0. \end{cases}$$

Model for velocities is different for Peaceman and TFE:

- TFE: $u = \bar{c} - c$, $\quad u_1 = \frac{c_1 + c_2}{2} - c_1, \qquad u_2 = \frac{c_1 + c_2}{2} - c_2$
- Peaceman: we need to introduce pressure, we will do this later

Initial condition:

$$c_{1,2}(x,0) = -1, x < 0$$
$$c_{1,2}(x,0) = +1, x > 0$$

Main result (TFE model, gravity-driven fingers)

Theorem (Efendiev, P., Tikhomirov, 2022+)

Consider a two-tube model with gravity.
Then there exists unique (up to swap) c_1^*, c_2^* such that TFE two-tubes system has travelling waves

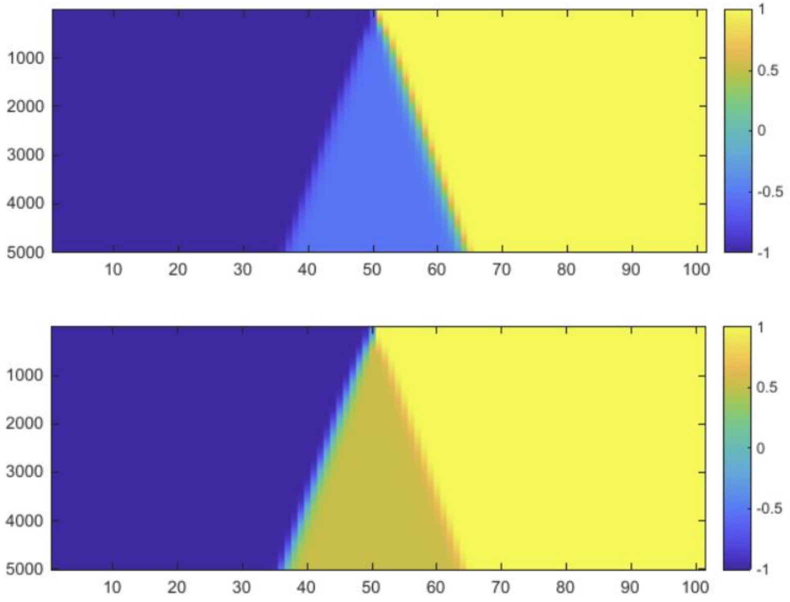
$$(-1,-1) \to (c_1^*, c_2^*) \to (1,1)$$

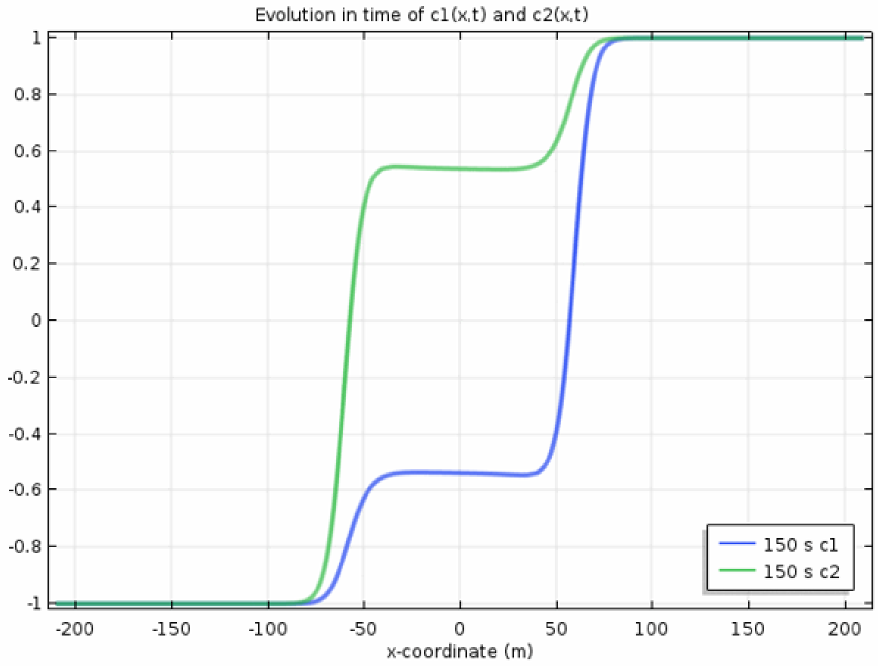
Moreover,

$$c_1^* = -\frac{1}{2}, \quad c_2^* = \frac{1}{2},$$
$$v^b = -\frac{1}{4}, \quad v^f = \frac{1}{4}.$$

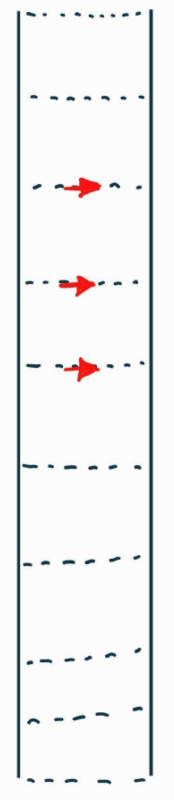
Including in the system cross-flow automatically creates intermediate concentration

time





Travelling waves. Equations.



Original system:

$$\partial_t c_1 + \partial_x (u_1 c_1) - \varepsilon \partial_{xx} c_1 = -(-1)^{1,2} \partial_x u_{1,2} \cdot c_{1,2},$$

$$\partial_t c_2 + \partial_x (u_2 c_2) - \varepsilon \partial_{xx} c_2 = (-1)^{1,2} \partial_x u_{1,2} \cdot c_{1,2},$$

$$(-1)^{1,2} \partial_x u_{1,2} \cdot c_{1,2} = \begin{cases} \boxed{-\partial_x u_1 \cdot c_1, \quad \partial_x u_1 < 0,} \\ +\partial_x u_2 \cdot c_2, \quad \partial_x u_1 \geq 0. \end{cases}$$

Travelling wave ansatz:

$$\xi = x - vt, \qquad c_{1,2}(x,t) = c_{1,2}(\xi),$$

$$c_{1,2}(\pm\infty) = c_{1,2}^{\pm}$$

4d system:

$$\dot{c}_1 = g_1,$$

$$\dot{g}_1 = g_1 (u_1 - v),$$

$$\dot{c}_2 = g_2,$$

$$\dot{g}_2 = (u_2 - v) g_2 + (c_1 - c_2) \dot{u}_1.$$

Conservation laws – 3d dynamical system:

$$\dot{c}_1 = g_1,$$

$$\dot{g}_1 = g_1 (u_1 - v),$$

$$\dot{c}_2 = -v(c_1 + c_2 - c_1^+ - c_2^+) + (u_1 c_1 + u_2 c_2 - u_1^+ c_1^+ - u_2^+ c_2^+) - g_1.$$

Connection between $c_{1,2}^{\pm}$ and v:
(Rankine-Hugoniot condition)

$$v [c_1 + c_2] \Big|_{-\infty}^{+\infty} = [u_1 c_1 + u_2 c_2] \Big|_{-\infty}^{+\infty}.$$

TFE velocity model:

$$u_1 = \frac{c_1 + c_2}{2} - c_1, \qquad u_2 = \frac{c_1 + c_2}{2} - c_2$$

Travelling waves. Phase portrait.

Substitute $u_{1,2}$, get:

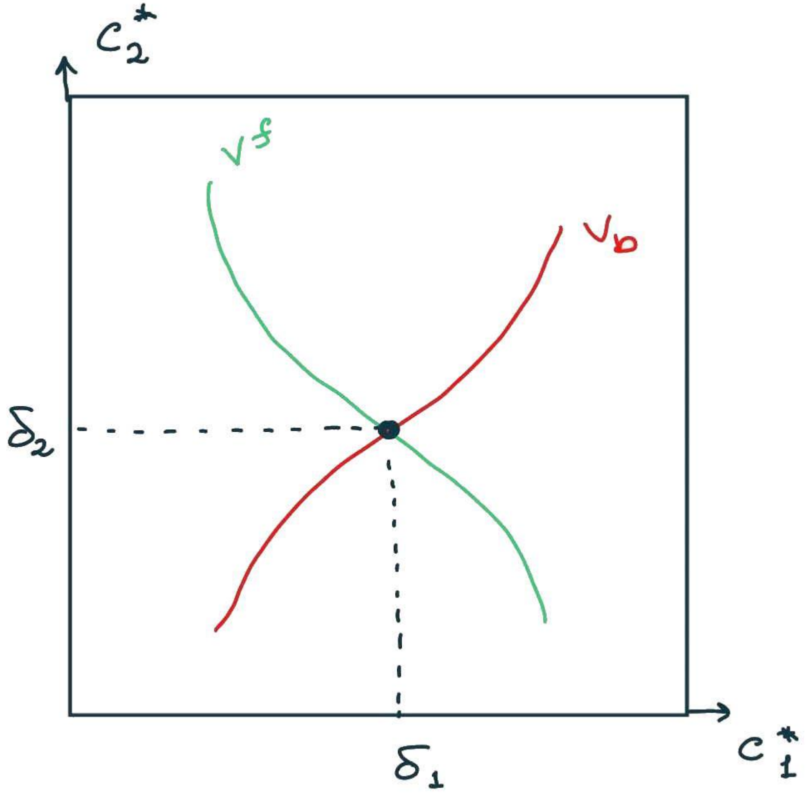
$$\dot{c}_1 = g_1,$$
$$\dot{g}_1 = g_1 \left(\frac{c_2 - c_1}{2} - v \right),$$
$$\dot{c}_2 = -v(c_1 + c_2 - c_1^+ - c_2^+) - \frac{1}{2}((c_1 - c_2)^2 - (c_1^+ - c_2^+)^2) - g_1.$$

Rankine-Hugoniot condition

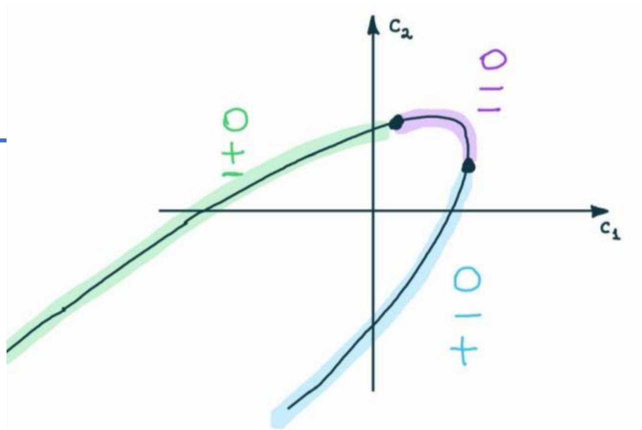
$$v(c_2^- + c_1^- - c_2^+ - c_1^+) = -\frac{1}{2}((c_1^- - c_2^-)^2 - (c_1^+ - c_2^+)^2).$$

3-dim dynamical system on (c_1, g_1, c_2)
Fix: (c_1^-, c_2^-) or (c_1^+, c_2^+)
Parameter: v

- For each v expected a travelling wave
- This generates a curve of possible c_1, c_2
- We apply this procedure for travelling wave to $(+1, +1)$ and from $(-1, -1)$



Two tubes. Invariant surface.

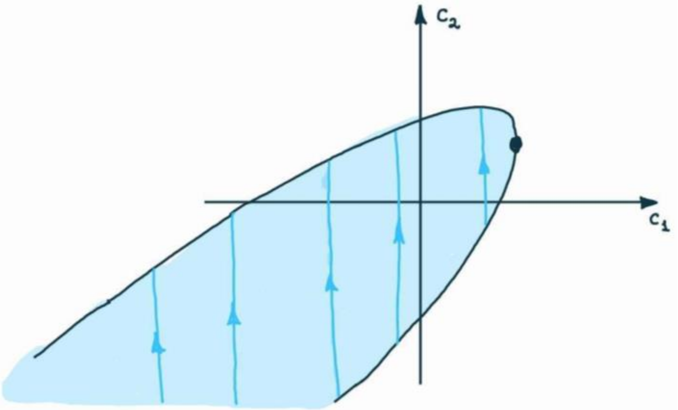


Equations

$$\dot{c}_1 = g_1,$$
$$\dot{g}_1 = g_1\left(\frac{c_2 - c_1}{2} - v\right),$$
$$\dot{c}_2 = -v(c_1 + c_2 - c_1^+ - c_2^+) - \frac{1}{2}((c_1 - c_2)^2 - (c_1^+ - c_2^+)^2) - g_1.$$

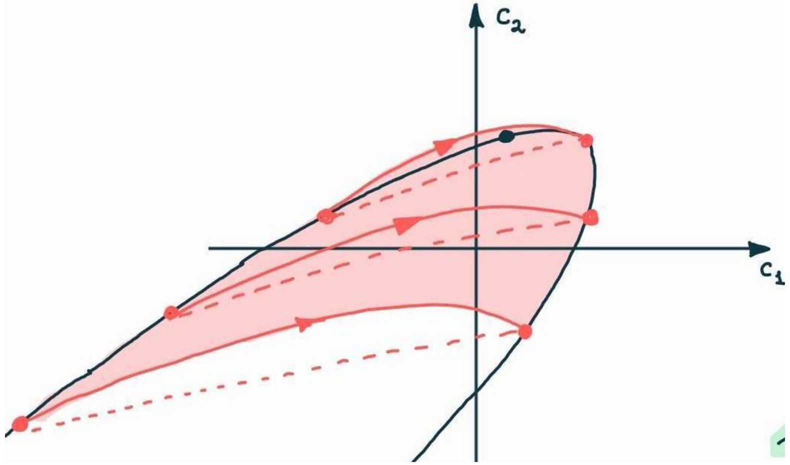
Travelling wave speed

$$v(c_2^- + c_1^- - c_2^+ - c_1^+) = -\frac{1}{2}((c_1^- - c_2^-)^2 - (c_1^+ - c_2^+)^2).$$



There exists 2dim invariant surface

$$g_1 = \tfrac{3}{4}(-v(c_2 + c_1 - c_2^+ - c_1^+) - \tfrac{1}{2}((c_1 - c_2)^2 - (c_1^+ - c_2^+)^2)),$$



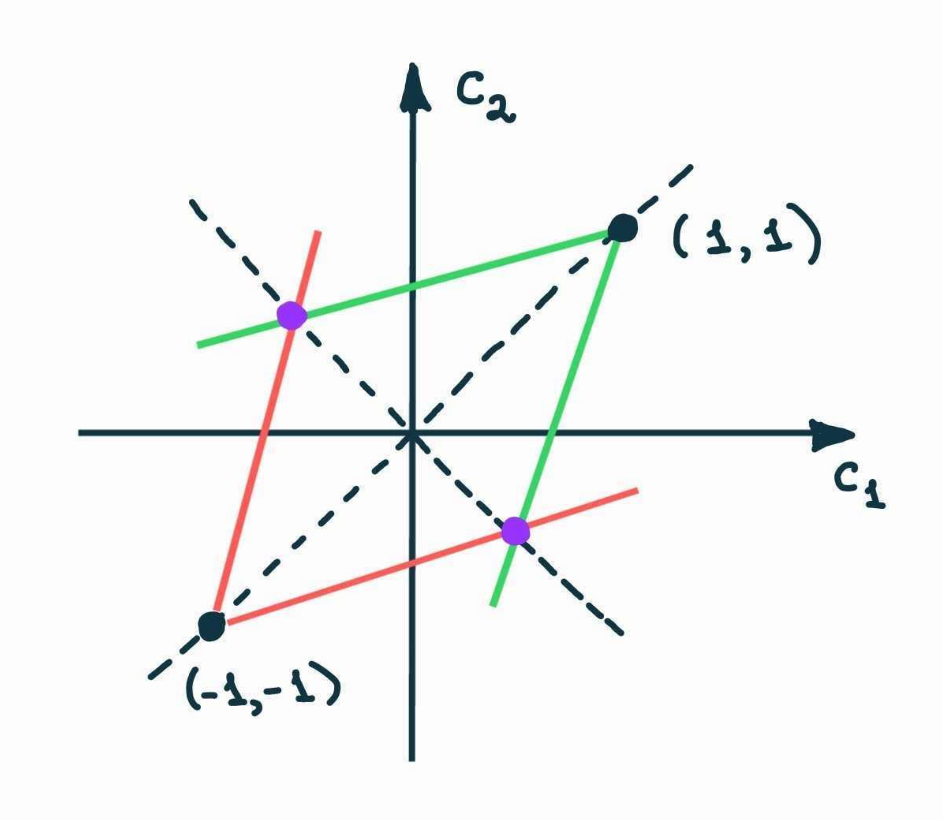
On all (for any $c_{1,2}^+$) heteroclinic holds

$$3(c_2 - c_2^+) = c_1 - c_1^+,$$

We have solved our “heteroclinic” problem analytically

Finally answer.

Admissible curves on the plane



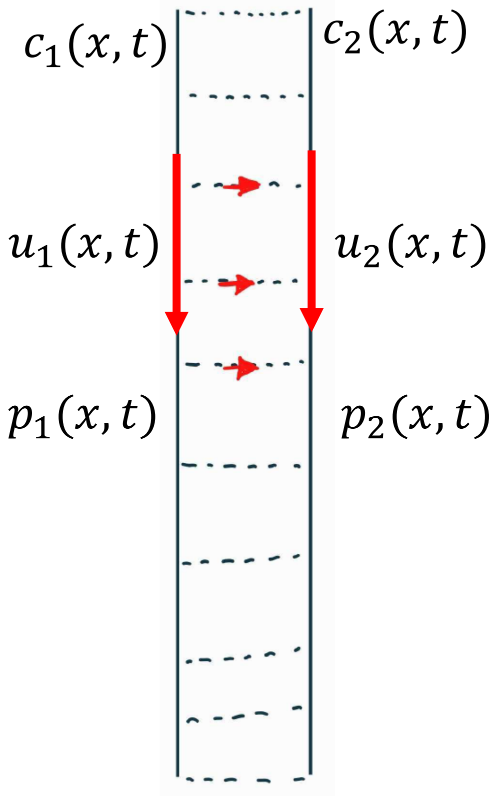
Speed and concentration

$$v^b = -\frac{1}{4}$$
$$v^f = \frac{1}{4}$$
$$c_1^* = -1/2$$
$$c_2^* = 1/2$$

Two-tubes model. Peaceman.

Original equations

$$c_t + div(uc) = \varepsilon\, \Delta c$$
$$div\, u = 0$$



$$\partial_t c_1 + \partial_x (u_1 c_1) - \varepsilon \partial_{xx} c_1 = -(-1)^{1,2} \partial_x u_{1,2} \cdot c_{1,2},$$
$$\partial_t c_2 + \partial_x (u_2 c_2) - \varepsilon \partial_{xx} c_2 = (-1)^{1,2} \partial_x u_{1,2} \cdot c_{1,2},$$

$$(-1)^{1,2} \partial_x u_{1,2} \cdot c_{1,2} = \begin{cases} -\partial_x u_1 \cdot c_1, & \partial_x u_1 < 0, \\ +\partial_x u_2 \cdot c_2, & \partial_x u_1 \geq 0. \end{cases}$$

Velocity model for Peaceman: add p_1 and p_2

$$\text{(Darcy's law in each tube)} \qquad u_1 = -\partial_x p_1 - c_1, \qquad u_2 = -\partial_x p_2 - c_2,$$
$$\text{(Darcy's law between tubes)} \qquad \partial_x u_1 = (p_2 - p_1)/l, \qquad \partial_x u_2 = -(p_2 - p_1)/l.$$

$$q = p_2 - p_1$$

Travelling wave for Peaceman.

Equations for travelling waves for Peaceman

$$\boxed{\begin{aligned} \dot c_1 &= g_1, \\ \dot g_1 &= g_1(u_1 - v), \\ \dot c_2 &= -v(c_1 + c_2 - c_1^+ - c_2^+) + u_1(c_1 - c_2) - u_1^+(c_1^+ - c_2^+) - g_1, \end{aligned}}$$

"Pressure part"

$$q = p_2 - p_1$$

$$\dot q = u_2 - u_1 + c_2 - c_1,$$

$$\dot u_1 = q/l,$$

$$\dot u_2 = -q/l.$$

Proper rescaling

$$q/\sqrt{l} \to q \qquad \sqrt{l} = \delta$$

$$\boxed{\begin{aligned} \delta\, \dot q &= -2u_1 + c_2 - c_1, \\ \delta\, \dot u_1 &= q. \end{aligned}}$$

Equations for TFE

$$\dot c_1 = g_1,$$

$$\dot g_1 = g_1(u_1 - v),$$

$$\dot c_2 = -v(c_1 + c_2 - c_1^+ - c_2^+) + u_1(c_1 - c_2) - u_1^+(c_1^+ - c_2^+) - g_1,$$

$$u_1 = \frac{c_1 + c_2}{2} - c_1 = \frac{c_2 - c_1}{2},$$

Corresponds to formal limit $\delta \to 0$

Statement (based on normal hyperbolicity):
For small enough δ the "Peaceman system" has invariant 3-dimensional manifold with dynamics close to "TFE system "

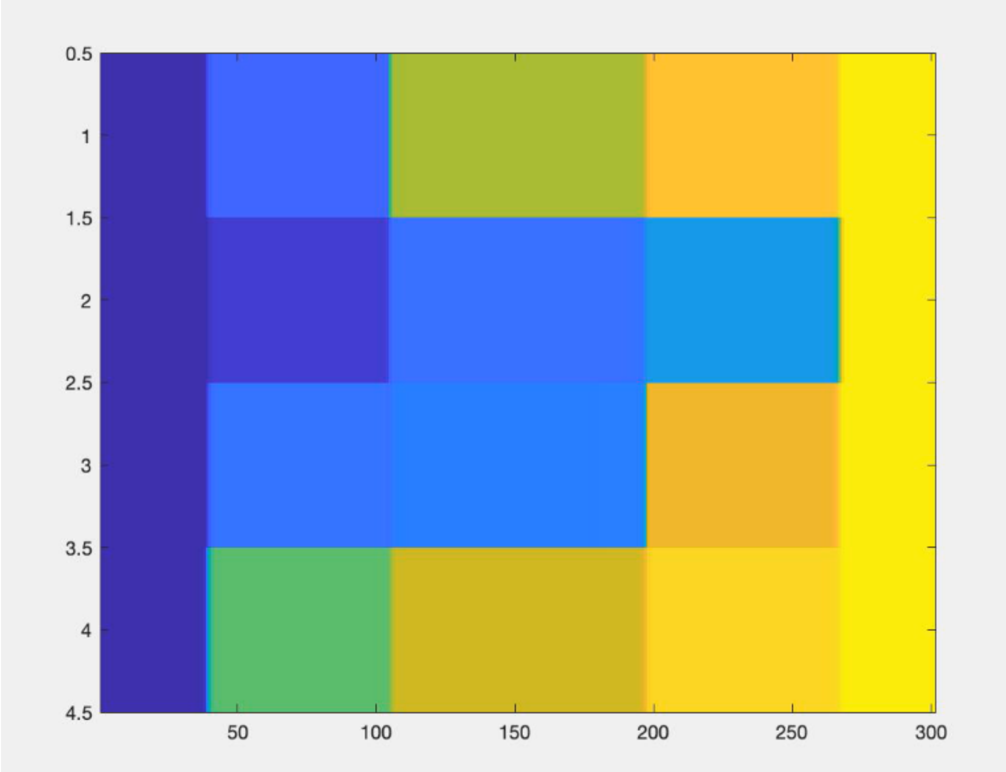
Conjecture 1 (in progress)
For $l \to 0$ we have $c_{1,2}^*(l, v^b) \to c_{1,2}^*(v^b)$

Conjecture 2 (in progress)
For $l \to 0$ we have $c_{1,2}^*(l) \to c_{1,2}^*$

What’s next?

1. How to obtain similar results for viscosity-driven fingers?

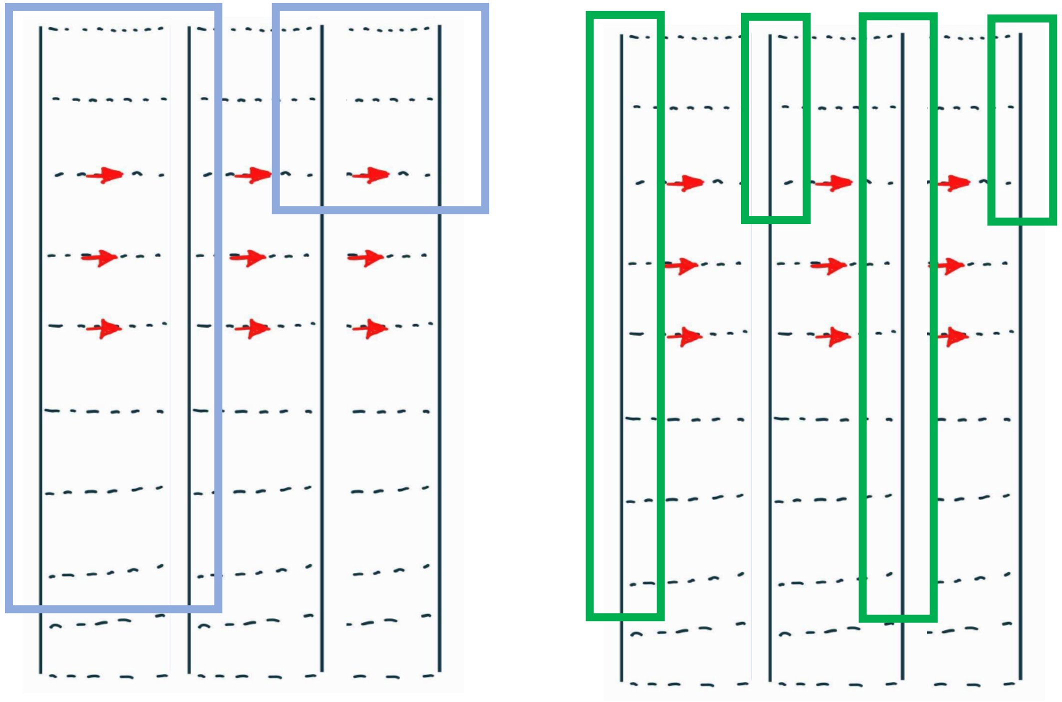
2. Does the n-tube model posses a system of n travelling waves?
   How to determine their constant states?
   Can we go to the limit as the number of tubes $n \to \infty$?



What's next?

3. Otto-Menon suggested that
   after time t fingers have length $\sim \sqrt{t}$
   What is the mechanism of merging of fingers?

   4-tubes model. What is more stable:
   - Two thin fingers?
   - One thick finger?



4. TFE as a limit of Peaceman when $\frac{k_y}{k_x} \to \infty$?
   Can we use the connection to prove the linear growth in Peaceman model?

Peaceman model

$$c_t + u \cdot \nabla c = \varepsilon \, \Delta c$$
$$div \, u = 0$$
$$u = -\begin{pmatrix} k_x & 0 \\ 0 & k_y \end{pmatrix} \nabla p - (0, c)$$

$$\frac{k_y}{k_x} \to \infty$$

TFE model

$$c_t + u \cdot \nabla c = \varepsilon \, \Delta c$$
$$div \, u = 0$$
$$u = (u^x, u^y)$$
$$u^y = \bar{c} - c$$

References

Muito obrigada!

Own works:

1. Bakharev, F., Enin, A., Groman, A., Kalyuzhnyuk, A., Matveenko, S., Petrova, Y., Starkov, I. and Tikhomirov, S., 2022. Velocity of viscous fingers in miscible displacement: Comparison with analytical models. Journal of Computational and Applied Mathematics, 402, p.113808.
2. Efendiev Ya., Petrova Yu., Tikhomirov S., 2022+, A cascade of two travelling waves in a two-tube model of gravitational fingering. In preparation.

Other references:

Dynamics of viscous fingering:

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2. Córdoba, A., Córdoba, D. and Gancedo, F., 2011. Interface evolution: the Hele-Shaw and Muskat problems. Annals of mathematics, pp.477-542.