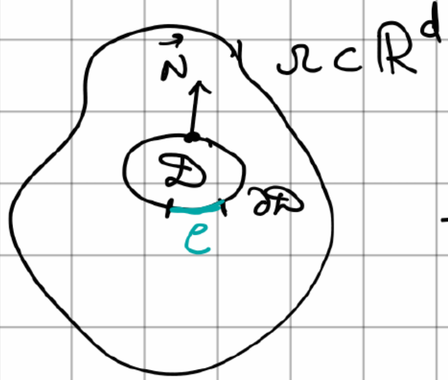
Lecture 5: Conservation & Balance laws

<u>Plan</u>: 1. General definition
2. Example 1: fluid dyn (conservation of mass)
3. Example 2: scalar conservation law

(1) <u>Balance law</u>



$\mathcal{D} \subset \Omega$ with Lipshitz boundary (smooth)

$\vec{N}$ - normal vector towards the exterior of the domain $\mathcal{D}$

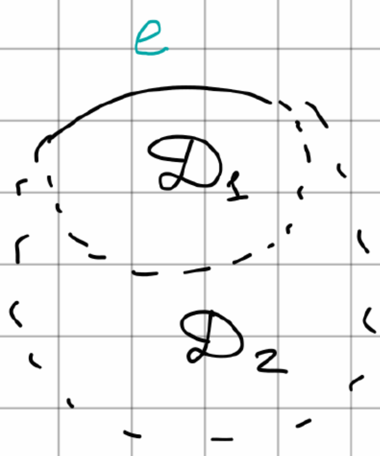
$$\boxed{\text{production in } \mathcal{D} = \text{flux through the boundary of } \partial\mathcal{D}}$$

- production in $\mathcal{D}$ is some measure (Radon) $\mathcal{P}$
- flux

$$\mathcal{F}_{\mathcal{D}}(e) = \int_{e} q_{\mathcal{D}}(x)\, dS(x)$$

$$\mathcal{P}(\mathcal{D}) = \int_{\partial\mathcal{D}} \underline{q_{\mathcal{D}}(x)}\, dS(x) \qquad (*)$$

<u>Assume</u>:



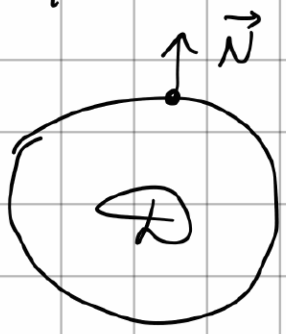
$q_{\mathcal{D}_1}(x) = q_{\mathcal{D}_2}(x)$
$\forall x \in e$

Take-home (Tuesday)

$$\boxed{\operatorname{div} A = 0}$$

Miracles:

Consequences of (*):



(1) $\exists\ a_{\vec{N}}(x) = q_{\mathcal{D}}(x)$ $\forall x \in \Omega$ for any $\mathcal{D} \subset \Omega$ s.t. $\mathcal{D}$ has $\vec{N}$ as a normal vector at x.

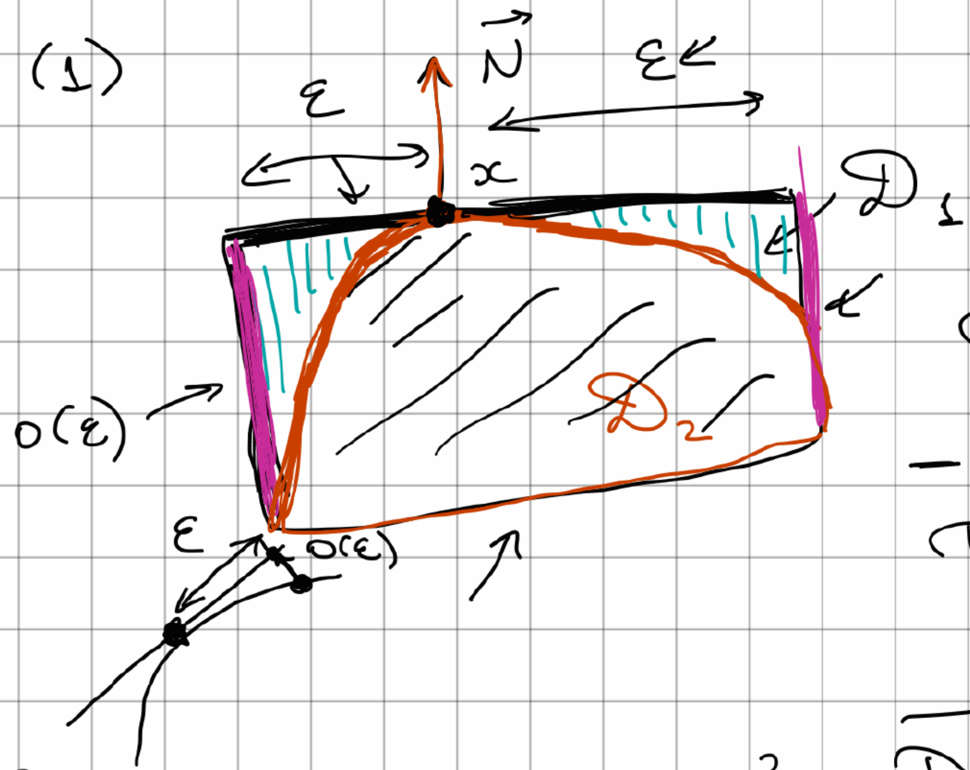
(2) $\exists\ \vec{A}(x) : \Omega \to \mathbb{R}^d$:

$$a_{\vec{N}}(x) = \vec{A}(x) \cdot \vec{N}$$

(3) $\exists$PDE: $\operatorname{div} \vec{A} = P$

$$P(\mathcal{D}) = \int_{\partial \mathcal{D}} \underbrace{q_{\mathcal{D}}(x)}_{\vec{A}(x)\cdot\vec{N}} dS(x)$$

(1)



$\varepsilon \to 0$

$$q_{\mathcal{D}_1}(x) \overset{?}{=} q_{\mathcal{D}_2}(x)$$

(*)

$$P(\mathcal{D}_1) = \int q_{\mathcal{D}_1}(x) dS$$

$-$

$$P(\mathcal{D}_2) = \int q_{\mathcal{D}_2}(x) dS$$

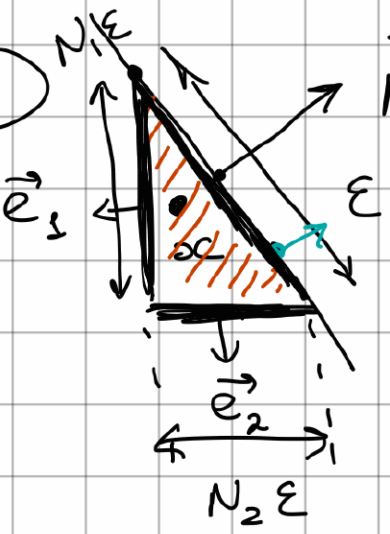
$$\varepsilon^2 \sim P(\ ) = \int q_{\mathcal{D}_1}(x) dS - \int q_{\mathcal{D}_2}(x) dS$$

$$\int \sim o(\varepsilon)$$

$$\Rightarrow \int q_{\mathcal{D}_1}(x)\, dS(x) = \int q_{\mathcal{D}_2}(x) dS(x)$$

$$\Rightarrow \quad \exists \quad a_{\vec N}(x) = q_{\mathcal{D}}(x) \qquad \underline{\text{Cauchy}}$$

(2) $\mathbb{R}^2$



$$\vec N = (N_1, N_2)$$

$$\varepsilon^2 \sim \mathcal{P}(\triangle) = \int_{\triangle} a_{\vec N}(x)\,dx$$

$$a_{\vec N}(x)\cdot \underline{\underline{\cancel{\varepsilon}}} = a_{e_2}(x)\cdot \underline{\underline{N_2\cancel{\varepsilon}}} +$$

$$+\, a_{e_1}(x)\cdot N_1\cancel{\varepsilon}$$

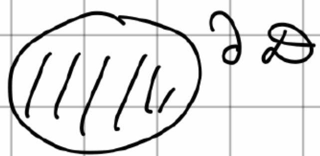
$$\Rightarrow \quad a_{\vec N}(x) = a_{e_1}(x)\,N_1 + a_{e_2}(x)\,N_2$$

$$\Rightarrow \quad a_N(x) = \vec A(x)\cdot \underline{\underline{\vec N}}$$

$$A(x) = (a_{e_1}, a_{e_2})$$

(3)
$$\int_{\partial\mathcal{D}} \vec A(x)\cdot\vec N\,dS(x) = \int_{\mathcal{D}} \operatorname{div}(\vec A)\,dx$$

Green-Gauss theorem



$$\mathcal{P} = \int_{\mathcal{D}} p(x)\,dx$$

$$\Rightarrow \quad \operatorname{div}(\vec A) = p \qquad \text{– balance law}$$

$$\boxed{\operatorname{div}(\vec A) = 0} \qquad \text{– conservation law}$$

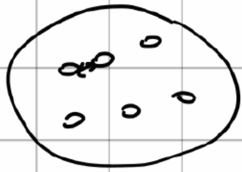
Dafermos

Example 1 : Fluid flow, continuum mechanics

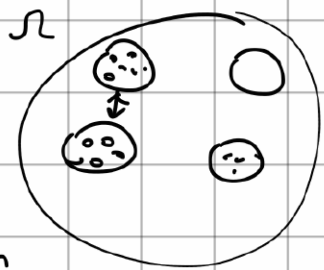
different scales

1. atoms / molecules
   o

2. representative small volume



3. domain (macroscale)



- Eulerian vs. Lagrangian point of view

Eulerian : (x,t) - fix

- velocity : $u(x,t) = (u_1, \dots, u_d) : \mathbb{R}^d \times [0,\infty) \to \mathbb{R}^d$
  has units $\left[\frac{L}{T}\right]$
- density : $\rho(x,t) : \mathbb{R}^d \times [0,+\infty) \to \mathbb{R}$
  with units $\left[\frac{M}{L^d}\right]$
- pressure : $p(x,t) : \mathbb{R}^d \times [0,+\infty) \to \mathbb{R}$
  with units $\left[M \, L^{-d+2} \, T^{-2}\right]$

Lagrangian : particles, $a \in \mathbb{R}^d$
trajectories of particles

flow map $X(t,a) = (X_1, \dots, X_d)$ - position of particle a at time t

$$(**) \begin{cases} \partial_t X(t,a) = u(t, X(t,a)) & \leftarrow \text{ODE} \\ X(0,a) = a \end{cases}$$

ODE theory (Cauchy-Lipschitz theorem) :
$u \in C_t \mathrm{Lip}_x \implies \exists!$ solution to $(**)$

$X(t,\cdot)$ – is C^1-diffeo : $\mathbb{R}^d \to \mathbb{R}^d$

Define inverse: $A(t,x) : [0,+\infty) \times \mathbb{R}^d \to \mathbb{R}^d$

$$A(t, X(t,a)) = a \qquad X(t, A(t,x)) = x$$

$$\forall x, a \in \mathbb{R}^d$$

"back-to-labels" map (a – "labels")

Incompressibility condition : "div $u = 0$"

Take $V \subset \Omega$ – volume of fluid

$$V(t) = X(t, V) = \{X(t,a) : a \in V\}$$

$$V \xrightarrow{t} V(t)$$

Def: velocity field is called incompressible if

$\rightarrow$ $|V(t)| \equiv |V|$

$\uparrow$ Lebesgue measure of V

Lemma: $u \in C_t \mathrm{Lip}_x$

u is incompressible $\iff$ div $u = 0$ (u is divergence-free)

Proof:

$$V(t) = \int_{V(t)} 1 \cdot dx \quad ; \quad a \in V \subset \mathbb{R}^d$$

$$\int_{V(t)} f(x,t)\,dx = \int_V f(X(t,a),t) \cdot \boxed{\det(\nabla_a X)}^{(t,a)} da$$

$a \mapsto X(t,a)$ $\qquad$ $= J(t,a)$

$$\rightarrow \quad J(t,a) = \sum_{i_1,\dots i_d=1}^{d} \underline{\varepsilon_{i_1 \dots i_d}} \frac{\partial X_{i_1}}{\partial a_1} \cdot \ldots \cdot \frac{\partial X_{i_d}}{\partial a_d}$$

Exercise: $\partial_t J(t,a) = J(t,a) \cdot (\operatorname{div} a)(t, X(t,a))$

Corollary : $J(t,a) \equiv 1 \iff \underline{\mathrm{div}(u) = 0}$

$$J(t,a) = J(0,a) \cdot e^{\int_0^t (\mathrm{div}\, a)(s, X(s,a))\,ds}$$

$$J(0,a) = 1$$

$$J(t,a) = J(0,a) \quad \forall t \Rightarrow \mathrm{div}(a) = 0$$

$$V(t) = \int_{V(t)} 1\,dx = \int_V J(t,a)\,da = \int_V da = V$$

iff $\mathrm{div}(a) = 0$

∎

Transport equation

Let's $f(t,x) : [0,+\infty) \times \mathbb{R}^d \to \mathbb{R}$ - scalar

Eulerian: $\partial_t f$ - change of f at (t,x)

Lagrangian: $\partial_t f(t, X(t,a)) =: D_t f$ - convective derivative

$$\partial_t f(t, X(t,a)) = \partial_t f + u \cdot \nabla f$$

$$V \xrightarrow{t} V(t)$$

Thm (transport thm):

u - velocity field, $u \in C^1$; f be C^1

$V(t)$ is pushforward of V by the flow map $X(t,a)$

$$\frac{d}{dt}\left(\int_{V(t)} f(x,t)\,dx\right) = \int_{V(t)} (\partial_t f + \mathrm{div}(fu))(t,x)\,dx$$

Proof:

$$\int_{V(t)} f(x,t)\,dx = \int_V f(X(t,a),t)\,\underline{J(t,a)}\,da$$

$$\frac{d}{dt}\left(\int_{V(t)} f(x,t)\,dx\right) = \int_V \underline{\left(D_t f\right)}\left(X(t,a),t\right) J(t,a)\,da$$

$$+ \int_V f(X(t,a),t)\cdot \underbrace{\partial_t J(t,a)}_{J(t,a)\cdot \mathrm{div}(u)}\,da =$$

$$= \int_V \Big(\partial_t f + \underbrace{u\cdot\nabla f + f\cdot \mathrm{div}(u)}_{\mathrm{div}(uf)}\Big)\left(X(t,a),t\right)\cdot J(t,a)\,da$$

$$= \int_V \Big(\partial_t f + \mathrm{div}(fu)\Big)\left(X(t,a),t\right)\cdot J(t,a)\,da =$$

$$= \int_{V(t)} \Big(\partial_t f + \mathrm{div}(fu)\Big)\,dx \qquad \blacksquare$$

Conservation of mass: $\rho(x,t)$

$$m(t,V) = \int_V \rho(x,t)\,dx$$

$$\frac{d}{dt} m(t,V(t)) = 0$$

<u>Thm</u>: conservation of mass is equivalent to the following integral eq:

$$\int_{V(t)} \left(\rho_t + \mathrm{div}(\rho u)\right) dx = 0$$

If ρ_t and $\mathrm{div}(\rho u)$ are C, then

$$\boxed{\rho_t + \mathrm{div}(\rho u) = 0}$$

<u>Rmk</u>: scalar transport eq

Proof:

$$0 = \frac{d}{dt} m(t, X(t,a)) = \frac{d}{dt} \int_{V(t)} \rho(x,t)\, dx$$

$$= \int_{V(t)} \underbrace{\left(\rho_t + \operatorname{div}(\rho u)\right)}_{\text{are continuous}} dx .$$

$$f(x) = \lim_{\varepsilon \to 0} \frac{1}{|B_\varepsilon(x)|} \int_{B_\varepsilon(x)} f(y)\, dy$$

$$\Rightarrow \quad \rho_t + \operatorname{div}(\rho u) = 0. \qquad \blacksquare$$

Rmk:

$$0 = \rho_t + \operatorname{div}(\rho u) = \rho_t + u \cdot \nabla \rho + \operatorname{div}(u) \rho$$

incompressibility $\Rightarrow$ $\operatorname{div}(u) = 0$

$$D_t \rho = \rho_t + u \cdot \nabla \rho = 0$$

$$\frac{d}{dt}\left(\rho(t, X(t,a)) \right) = 0$$

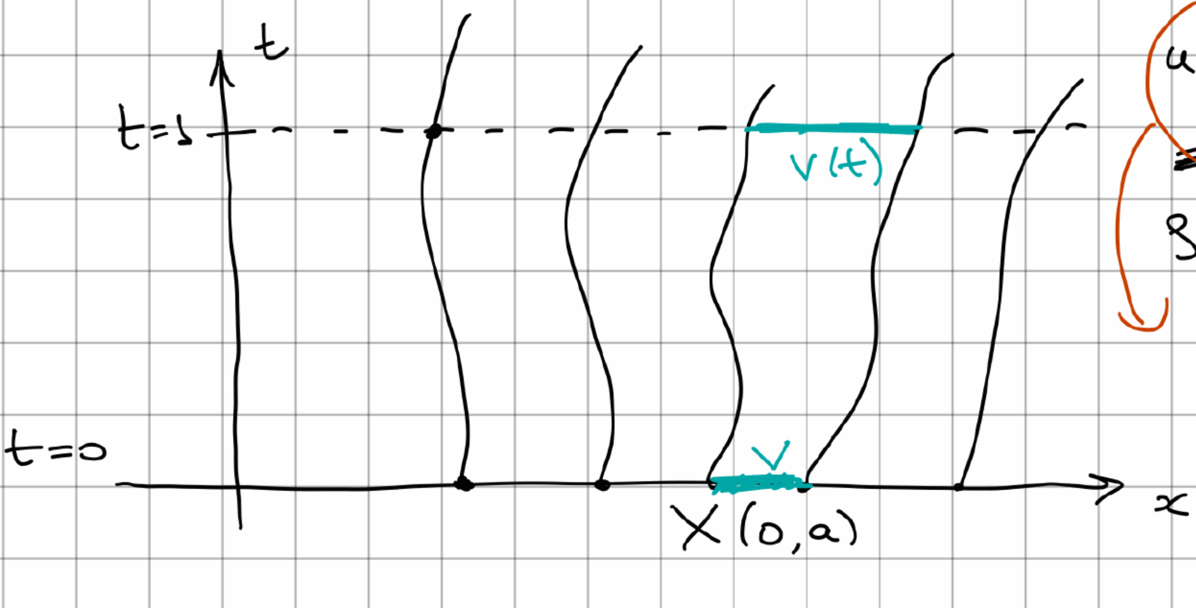
$$\Rightarrow \quad \rho(t, X(t,a)) = \text{const}.$$

Next time

$\rho_t + (\rho u)_x = 0$

$u = u(\rho)$

$u(\rho) = \frac{\rho}{2} \Rightarrow$ Burgers



$u \in C_t \mathrm{Lip}_x$

linear in ρ

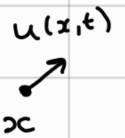
$\rho_t + u \cdot \nabla \rho = 0$

Lecture 6

Last time: Balance laws: $\operatorname{div} A = P$

Conservation laws: $\operatorname{div} A = 0$

Example 1: fluid flow: $a \to X(t,a)$ – flow map under velocity field, $a \in \mathbb{R}^d$



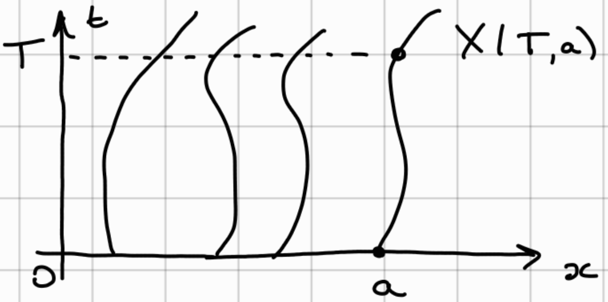
$u(t,x)$ – velocity field

$\rho(t,x)$ – density

$$\begin{cases} \partial_t X = u(X,t) \\ X(0,a) = a \end{cases}$$

• Conservation of mass = scalar transport equation

$$\boxed{\partial_t \rho + \operatorname{div}(\rho u) = 0}$$



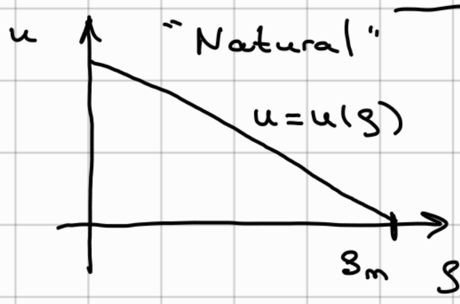
trajectories do not intersect for given $u \in C^0_t \mathrm{Lip}_x$

Rmk: $\begin{cases} \operatorname{div} u = 0 \\ \partial_t \rho + \operatorname{div}(\rho u) = 0 \end{cases}$

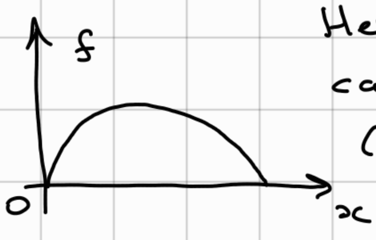
$\Rightarrow$ $\rho(t, X(t,a)) = \text{const}$

density is conserved along the trajectory for incompressible flow

Example 2: traffic flow: cars choose their velocity depending on "density" of cars nearby

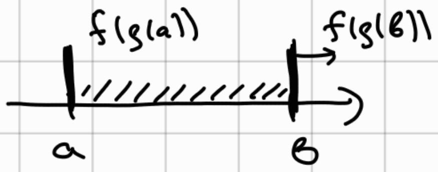


$\Rightarrow$ $\boxed{\rho_t + f(\rho)_x = 0}$ scalar conservation law



Here $f(\rho)$ is called flux (flow function)

ρ_m – density of cars corresp. to "bumper-to-bumber"



$$\frac{d}{dt}\int_a^b \rho(x,t)\,dx = f(\rho(b)) - f(\rho(a))$$

Rmk: 1) taking $u(\rho) = \frac{\rho}{2}$ $\Rightarrow$ Burgers eq: $\rho_t + \left(\frac{\rho^2}{2}\right)_x = 0$

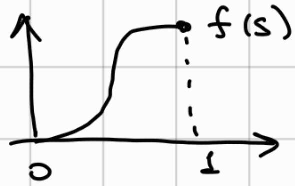
We will analyze it in detail today.

2) for oil recovery the simplest 1-dim model for displacement water-oil is again

$$S_t + (f(S))_x = 0 \quad \text{for}$$

S - water saturation

f(s) - fractional flow function



f(s) : $f(0)=0$, $f(1)=1$, $f\uparrow$ and S-shaped

- One can easily create more sophisticated models such as: take drivers anticipation into account

If a driver observe an upstream increase in the density, they show a tendency to brake slightly

$$u - V(\rho) \sim -\rho_x$$

The simplest law: $\quad u = V(\rho) - \varepsilon \rho_x, \quad 0 < \varepsilon \ll 1$

which leads to the "weakly" parabolic eq:

$$\rho_t + f(\rho)_x = \varepsilon(\rho \rho_x)_x$$

Example 3: wave equation! $\quad u_{tt} - c^2 u_{xx} = 0$

$$\operatorname{div}(u_t, -c^2 u_x) = 0$$

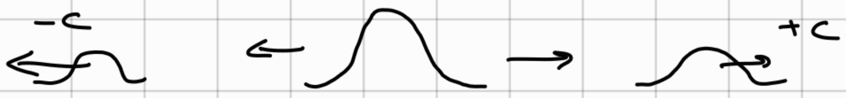
Consider $\quad U = \begin{pmatrix} u_x \\ u_t \end{pmatrix} \Rightarrow U_t + A U_x = 0$

$$A = \begin{pmatrix} 0 & -1 \\ -c^2 & 0 \end{pmatrix}$$

Indeed, this is just: $\begin{cases} u_{xt} - u_{tx} = 0 \\ u_{tt} - c^2 u_{xx} = 0 \end{cases}$

Eigenvalues of A: $\det\begin{vmatrix} 0-\lambda & -1 \\ -c^2 & -\lambda \end{vmatrix} = \lambda^2 - c^2, \quad \lambda_\pm = \pm c$

They correspond to propagation modes:



This is general fact that we will see in the future:

$U \in \mathbb{R}^d$, $F: \mathbb{R}^d \to \mathbb{R}^d$ $\quad U_t + (F(U))_x = 0$ - system of conservation laws

Then for "smooth" solutions we have:

$$U_t + \underbrace{F'(U)}\cdot U_x = 0$$

eigenvalues of this matrix play an important role !

If they are real, they correspond to velocity of propagation of waves.

Example 4 : isentropic (= constant entropy) gas dynamics
(p-system)

in Lagrangian coordinates :
$$\begin{cases} v_t - u_x = 0 \\ u_t + p(v)_x = 0 \end{cases} \Rightarrow v_{tt} + p(v)_{xx} = 0$$

Rmk : $v_t = u_x \Rightarrow$ (in a simply connected regions)
$\exists\ \varphi :\ v = \varphi_x$
$u = \varphi_t$

$$\Rightarrow \quad \varphi_{tt} + (p(\varphi_x))_x = 0$$
$$\varphi_{tt} + p'(\varphi_x)\cdot \varphi_{xx} = 0$$
– nonlinear wave equation

And many other examples :

- conservation of mass
  conservation of momentum $\Big\} \Rightarrow \begin{cases} \partial_t u + u\cdot\nabla u = \nabla p + f \\ \operatorname{div}(u) = 0 \end{cases}$

  This is Euler equations for ideal fluid
  (1755, second PDE ?)
- Navier - Stokes eqs (1845): adds viscosity
  $$\partial_t u + (u\cdot\nabla)u - \nu\Delta u = \nabla p$$
- gas dynamics
- electromagnetism (Maxwell eqs)
- magneto-hydrodynamics (M.H.D.) – motion of fluid in the presence of electromagnetic field (think of a Sun)

Etc ......

Burgers equation

$$u_t + \left(\frac{u^2}{2}\right)_x = 0 \qquad\qquad u_t + u\cdot u_x = 0$$

Observation 1: if $u \in C^1$ for all $t>0$, then u is monotonically nondecreasing in x for all $t>0$.

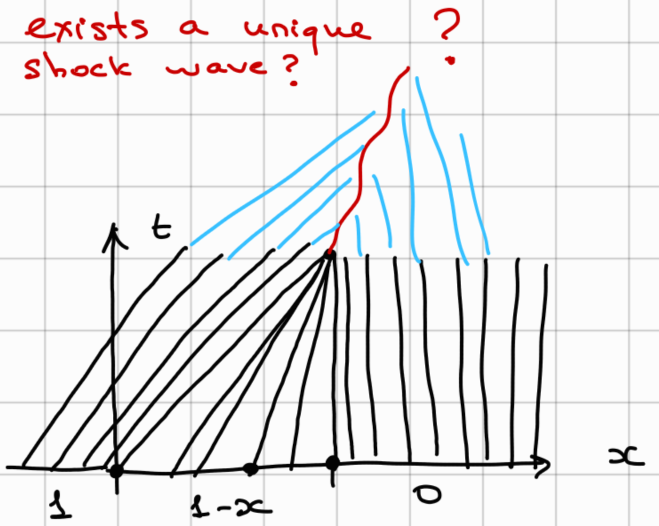
$$u\,(x(s),\ t(s)) = \text{const}$$

$$u_t \cdot t_s + u_x \cdot x_s = 0$$

$$\begin{cases} t_s = 1 \\ x_s = u \end{cases} \Rightarrow \ u = \text{const on straigh lines } x = x_0 + u_0(x_0)\,t$$

If $u \in C^1$ for $\forall\, t>0$, then characteristics should not intersect $\Rightarrow$ $u_0(x_1) < u_0(x_2)$ if $x_1 < x_2$ $\Rightarrow$ u_0 is non-decreasing $(u(x,0))$ $\Rightarrow$ $u(x,t)$ is non-decreasing in x

Exercise 2 from list 1:



$$x = x_0 + (1-x_0)\,t$$

$t=1$: $\quad x = 1$

At $t=1$ there is a blow-up

Rmk: In general scalar conservation law:

$$u_t + f(u)_x = 0$$

$$u_t + \underline{\underline{f'(u)}} \cdot u_x = 0$$

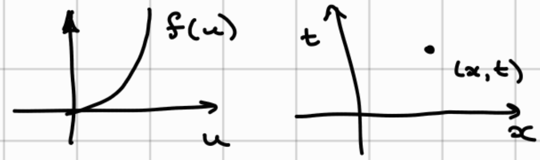
Characteristics are $x = x_0 + f'(u_0(x_0))\,t$

$u \in C^1$ $\forall t>0 \Rightarrow f'(u_0(x_1)) < f'(u_0(x_2))$ if $x_1 < x_2$, otherwise characteristics will intersect that leads to a blow-up!

So no matter how smooth f and u_0 are, the solution $u(x,t)$ must become discontinuous

This is a purely non-linear phenomenon!!!

- Assume $f \in C^2$ and $f'' > 0$



$$u_0\big(x - t f'(u(x,t))\big) = u(x,t)$$

$$u_t = u_0' \cdot \big(-f'(u(x,t)) - t f''(u(x,t)) \cdot u_t\big)$$

$$u_t\,(1 + t f'' u_0') = -u_0' f'$$

$$u_t = -\frac{u_0' f'}{1 + t f'' u_0'}$$

Analogously, $u_x = \dfrac{u_0'}{1 + t f'' u_0'}$

If $u_0' \geq 0$ (and $f'' > 0$) u_t and u_x stay bounded.

If $u_0' < 0$, then u_t and u_x become unbounded as $1 + t f'' u_0'$ tends to 0.

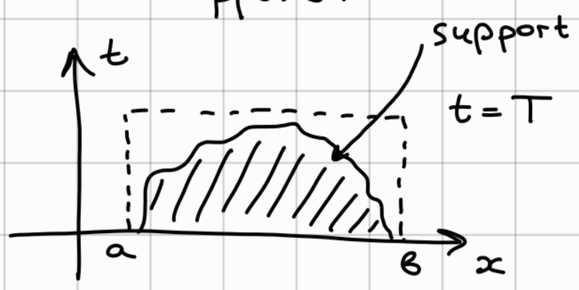
So we need a notion of weak solution!

Weak solutions to conservation laws

$$\begin{cases} u_t + f(u)_x = 0 & (1) \\ u\big|_{t=0} = u_0(x) \end{cases}$$

Let u be a classical solution and $\varphi \in C^1$ with compact support:

$\operatorname{supp}(\varphi) \subset D = [a,b] \times [0,T]$ that is φ is zero at $x = a$, $x = b$, $t = T$



Multiply (1) by φ and integrate over $\mathbb{R} \times \mathbb{R}_+$

$$\iint_{t>0} (u_t + f(u)_x)\,\varphi\,dx\,dt = \iint_D (u_t + f(u)_x)\,\varphi\,dx\,dt =$$

$$= \int_a^b \int_0^T (u_t + f(u)_x)\,\varphi\,dx\,dt = \int_a^b u \cdot \varphi \Big|_0^{\cancelto{0}{T}} dx - \int_a^b \int_0^T u \cdot \varphi_t\,dx$$

$$+ \int_0^T \cancelto{0}{f(u) \cdot \varphi \Big|_a^b}\,dt - \int_0^T \int_a^b f(u) \cdot \varphi_x\,dx\,dt =$$

$$= -\int_a^b u_0(x)\,\varphi(x)\,dx - \int_a^b\int_0^T (u\varphi_t + f(u)\,\varphi_x)\,dx\,dt$$

$$\Rightarrow \quad \iint_{t\geq 0} (u\varphi_t + f(u)\varphi_x)\,dx\,dt + \int_{t=0} u_0(x)\,\varphi(x)\,dx = 0 \qquad (2)$$

$u \in C^1$ and satisfies (1) $\Rightarrow$ u satisfies (2)

But in (2) u not necessarily needs to be C^1!

It can be measurable / bounded.

Definition: A bounded measurable function $u(x,t)$ is called a weak solution of IVP:

$$u_t + f(u)_x = 0, \quad u\big|_{t=0} = u_0(x)$$

$\uparrow$ bounded / meas.

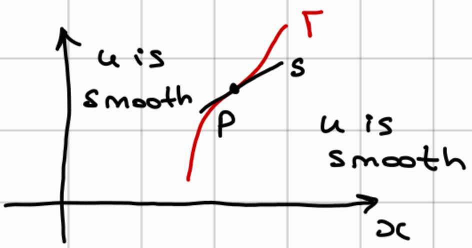
provided that

$$(2) \qquad \iint_{t\geq 0} (u\varphi_t + f(u)\varphi_x)\,dx\,dt + \int_{t=0} u_0\varphi\,dx = 0$$

for all $\varphi \in C_0^1$ (φ is C^1 with compact supp)

Rmk: it is clear that if u is in fact C^1, then the original eq. is true: $u_t + f(u)_x = 0$

Lemma (Rankine - Hugoniot condition)



Let Γ be a smooth curve across which u has a jump discontinuity. Take $P \in \Gamma$ and

$u_\ell = \lim\limits_{(x,t)\to P} u$ from "the left"

$u_r = \lim\limits_{(x,t)\to P} u$ from "the right"

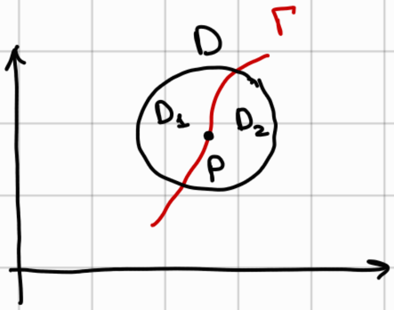
Let the tangent line of Γ at P have the slope $s = \frac{dx}{dt}$. Then: (3) $\boxed{s\cdot(u_\ell - u_r) = f(u_\ell) - f(u_r)}$

Often a jump across the shock is denoted:

$[g(u)] = g(u_\ell) - g(u_r)$, thus we have $s\,[u] = [f]$

This is called the Rankine-Hugoniot condition

Proof :



Let D be a small ball centered at P and let Γ devide D into two regions D_1 and D_2 (see fig)

Let $\varphi \in C_0^1$ on D and consider

$$0 = \iint_D (u\varphi_t + f(u)\varphi_x)\,dx\,dt = \iint_{D_1} + \iint_{D_2}$$

Divergence theorem : (Green-Gauss theorem)
$$\int_{\partial\Omega} P\,dx + Q\,dy = \iint_{\Omega}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dx\,dy$$

$$\iint_{D_1} u\varphi_t + f(u)\varphi_x\,dx\,dt = \iint_{D_1} (u\varphi)_t + (f(u)\varphi)_x\,dx\,dt =$$

as $u \in C^1(D_1)$ and $u_t + f(u)_x = 0$

$$= \int f(u)\varphi\,dt - u\varphi\,dx = \int_{t_1}^{t_2} f(u)\varphi\,dt - u\varphi\,dx =$$

$$= \int_{t_1}^{t_2}\left[f(u_e)\varphi(u_e) - u_e\cdot\varphi(u_e)\cdot s\right]dt$$

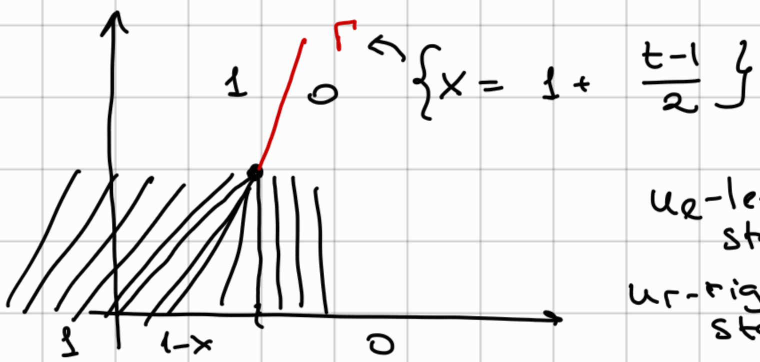
Similarly,
$$\iint_{D_1} u\varphi_t + f(u)\varphi_x\,dx\,dt = -\int_{t_1}^{t_2}\left(f(u_r) - su_r\right)\varphi(u_r)\,dt$$

minus because of orientation of ∂D_2 :

Combining together:
$$0 = \int_{t_1}^{t_2}\left([f] - s[u]\right)\varphi(u_e)\,dt$$

Since φ was arbitrary, we get $[f] - s[u] = 0$ relation (3). ■

Example: Burgers eq:



$$s = \frac{\left[\frac{u^2}{2}\right]\Big|_0^1}{[u]\Big|_0^1} = \frac{1}{2}$$

u_e - left state

u_r - right state

in general $s = \frac{u_e + u_r}{2}$

$\boxed{\text{Lecture 7}}$ Scalar conservation law: $\begin{cases} u_t + (f(u))_x = 0 \\ u\big|_{t=0} = u_0(x) \end{cases}$ $(*)$

$u: \mathbb{R}\times\mathbb{R}_+ \to \mathbb{R}$ - bounded, measurable

$f: \mathbb{R}\to\mathbb{R}$, $f\in C^2$, $f''>0$ on the convex hull of values of u_0

We understand solutions in weak sense:

$$\iint_{t>0} (u\varphi_t + f(u)\varphi_x)\,dx\,dt + \int_{t=0} u_0\varphi\,dx = 0 \qquad (**)$$

for every test function $\varphi\in C_0^1$.

We want to prove theorems on $\exists$, $!$ and asymptotic behavior of solutions to $(*)$. From exercise session 1 we remember that we need some extra conditions for this

$\boxed{\text{Thm1}}$ ($\exists$):

Let $u_0\in L_\infty(\mathbb{R})$, $f\in C^2(\mathbb{R})$, $f''>0$ on $\{u: |u|\le \|u_0\|_\infty\}$

Then there exists a solution with the following properties:

(a) $|u(x,t)| \le \|u_0\|_\infty = M$, $(x,t)\in\mathbb{R}\times\mathbb{R}_+$

(b) $\exists\, E>0$ (which depends on M, $\mu = \min\{f''(u): |u|\le M\}$ and $A = \max\{|f'(x)| : |u|\le \|u_0\|\}$) ~~such~~

such that $\forall\, a>0,\ t>0,\ x\in\mathbb{R}$

$$\frac{u(x+a,t) - u(x,t)}{a} < \frac{E}{t} \qquad (E)$$

(c) u is stable and depends continuously on u_0: if $v_0\in L_\infty(\mathbb{R})$ with $\|v_0\|_\infty \le \|u_0\|_\infty$ and v is the corresponding constructed solution of $(*)$ with initial data v_0, then for $\forall\, x_1,x_2\in\mathbb{R}$ with $x_1<x_2$ and $\forall\, t>0$

$$\int_{x_1}^{x_2} |u(x,t) - v(x,t)|\,dx \le \int_{x_1-At}^{x_2+At} |u_0(x) - v_0(x)|\,dx.$$

~~Remarks: 1)~~

$\boxed{\text{Thm2}}$ ($!$):

Let $f\in C^2$, $f''>0$ and let u and v be 2 solutions of $(*)$ satisfying condition (E). Then $u=v$ almost everywhere in $t>0$.

Rmk: we call the solution from Thm 1 (that is satisf. (E) may be there exist more solutions which do not satisfy cond. (E) or (c)

2) property (a) is not valid for systems! Sup-norm of solution can increase! It is non-trivial to prove the bounds on the sup-norm.

3) Cond. **(E)** implies some regularity : u is of locally bounded total variation (for $\forall t$ as a function of x)

Indeed, let c_1 be a constsant such that $c_1 > \frac{E}{t}$ and let $v = u - c_1 x$. Then

$$v(x+a,t) - v(x,t) = u(x+a,t) - u(x,t) - c_1 a < a\left(\frac{E}{t} - c_1\right) < 0$$

Thus, v is a non-decreasing function, and v is a function of local bounded total variation.

Since $c_1 x$ is also of bounded total variation, then u is of local bounded total variation.

($\Rightarrow$ countable number of jump discontinuities)

4) finite speed of propagation:

$$v = v_0 \equiv 0 \quad \Rightarrow \quad \int_{x_1}^{x_2} |u(x,t)|\, dx \le \int_{x_1 - At}^{x_2 + At} |u_0(x)|\, dx$$

Before proving thm 1 and 2 let us understand better condition (E). Let us give some equivalent formulations and interpetations.

Lemma 1: (a) A smooth solution $u(x,t)$ satisfies condition (E) $(\forall t)$

(b) If u has a discontinuity at point x_0: (but is smooth to the left and to the right of x_0) and $\lim_{x \to x_0 - 0} u(x,t) = u_L$ and $\lim_{x \to x_0 + 0} u(x,t) = u_R$ and satisfies condition (E) $\Rightarrow u_L > u_R$.

(discontinuities can be only down).

Proof:

(a) Indeed, let us write:

$$u(x,t) = u_0(x - t f'(u(x,t)))$$

$$u_x = u_0' \cdot (1 - t f'' u_x) \Rightarrow u_x = \frac{u_0'}{1 + t f'' u_0'}.$$

If u is smooth for $\forall t > 0$, then $u_0' > 0$.

Then $u_x \le \frac{u_0'}{t f'' u_0'} = \frac{E}{t}$ for $E = \frac{1}{\inf f''}$.

Using Lagrange theorem : $\dfrac{u(x+a,t)-u(x,t)}{a} = u_x(\xi,t)$

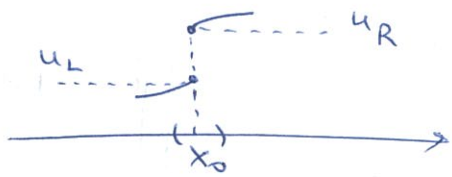
for some $\xi \in [x, x+a)$, and (a) is proved

(b) Either $u_L > u_R$ or $u_L < u_R$ (the case $u_L = u_R$ is not a discontinuity).

- For $u_L < u_R$ the converse of cond. (E) is true:

$\forall E > 0 \;\; \exists x, a > 0, t$: $\dfrac{u(x+a)-u(x)}{a} > \dfrac{E}{t}$.

Indeed, fix E and take small enough neighbourhood of x_0 such that



- for $x \in (x_0 - \delta, x_0)$ $\quad |u - u_L| \leq \varepsilon = \dfrac{u_R - u_L}{4}$
- for $x \in (x_0, x_0 + \delta)$ $\quad |u - u_R| \leq \varepsilon = \dfrac{u_R - u_L}{4}$.

This means that for $\forall\, x_1 \in (x_0 - \delta, x_0)$ and $x_2 \in (x_0, x_0 + \delta)$

$$u(x_2) - u(x_1) \geq \frac{u_R - u_L}{2}.$$

Fix t and take small a :

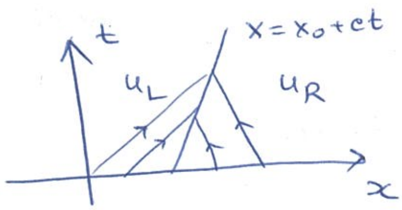
$x_2 - x_1 = a$ $\qquad \dfrac{u(x_2)-u(x_1)}{a}$

$x_1 \in (x_0 - \delta, x_0)$ $\qquad \vee\!|$

$x_2 \in (x_0, x_0 + \delta)$ $\qquad \dfrac{u_R - u_L}{2a} = \dfrac{E}{t}$.

- For $u_L > u_R$ $\quad \dfrac{u(x+a)-u(x)}{a} \leq 0$, thus $\forall E > 0$ is ok ■

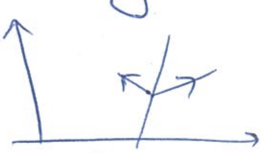
Lemma 2 (Remark) : u satisfies condition (E) and is a shock wave solution $u = \begin{cases} u_L, & x < ct \\ u_R, & x > ct \end{cases}$



then $\quad f'(u_L) > c > f'(u_R)$ $\quad$ (Lax condition)

" Characteristics come to the line of discontinuity "

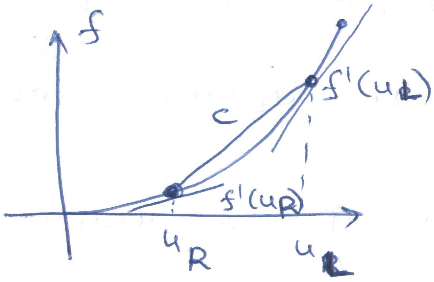
The converse situation in some sense corresponds to the case



where " the information" appears on the discontinuity - which is not

We will generalize the Lax condition to the case of systems.

Indeed, $f''>0 \Rightarrow$ (see picture)

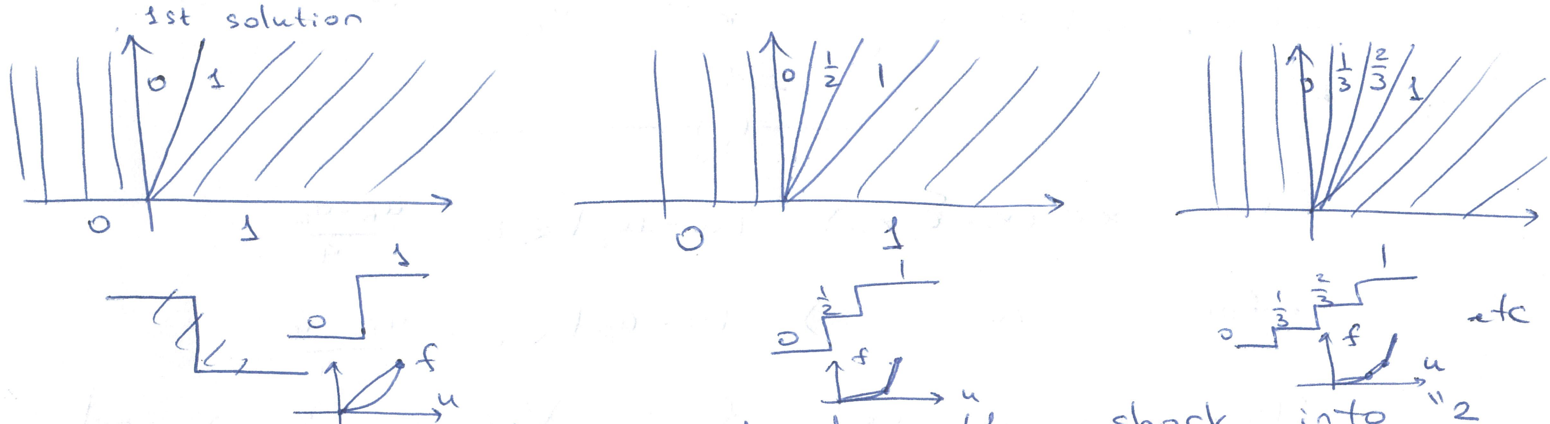


$$f'(u_L) > c = \frac{f(u_L)-f(u_R)}{u_L-u_R} > f'(u_R)$$

Remark (on Liu criterion) "internal stability of shock"

Remember the situation with Burgers equation:

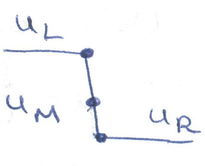
1st solution



In some sense if we divide the shock into "2 smaller" shocks, they could have a tendency of eigher gluing into 1 shock (some kind of stabilit[y]) or going further one from another (instability)

Condition (E) $\Rightarrow$ this kind of internal stability of a shock, more precisely the inequalities

$$c(u_L,u_R)=\frac{f(u_R)-f(u_L)}{u_R-u_L} \le c(u_L,u_M)=\frac{f(u_L)-f(u_M)}{u_L-u_M}$$



$$\ge c(u_M,u_R)=\frac{f(u_R)-f(u_M)}{u_R-u_M} \qquad \forall u_M \in (\min(u_L, \ldots, \max(u_L,u_{\ldots}$$

If $\begin{cases} u_M \to u_L \\ u_M \to u_R \end{cases}$ we have Lax condition.

Vanishing viscosity criterion for shock waves.

- We think of equation $u_t + (f(u))_x = 0$ as a first approximation to the following parabolic eq

$$u_t + (f(u))_x = \underbrace{\varepsilon u_{xx}}_{\text{small regularizing term}}, \quad \varepsilon > 0 \qquad (P)$$

Rmk 1: it is well-known (and we see in future when dealing with reaction-diffusion equations) that solutions of (P) are very regular (no shocks)

Rmk 2: equation (P) is a combination of 2 "opposite" effects

$\rightarrow$ $u_t + (f(u))_x = 0$ $\rightsquigarrow$ creates shocks:

$\rightarrow$ $u_t = \varepsilon u_{xx}$ $\rightsquigarrow$ "smooths":

As a consequence of this confrontation there exist very special solutions, called travelling waves (TW) such that:

$$u(x,t) = v(x - ct)$$

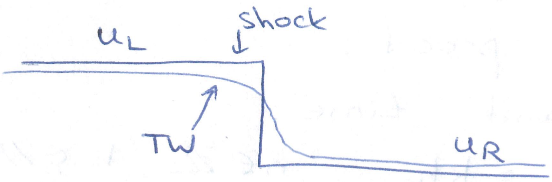
for $c \in \mathbb{R}$ and v - some smooth profile.

They look like "smoothed" shocks!!!

This motivates the following definition:

Def 1 (vanishing viscosity criterion for shock waves):
A shock wave is an entropy solution if is a limit in L^1_{loc} of a travelling wave solution of (P) as $\varepsilon \to 0$.

$f \in C^2$, $f'' > 0$



Lemma: a shock wave is an entropy solution in sense of def 1, iff $u_L > u_R$.

Proof: Let's look for travelling wave solutions for eq. (P): $v\left(\frac{x-ct}{\varepsilon}\right)$: $v(-\infty) = u_L$, $v(+\infty) = u_R$

$$-c v' + (f(v))' = \varepsilon v''$$

Integrate $\int_{-\infty}^{+\infty}$: $-c(u_R - u_L) + f(u_R) - f(u_L) = 0$

Interesting feature: it is exactly RH condition

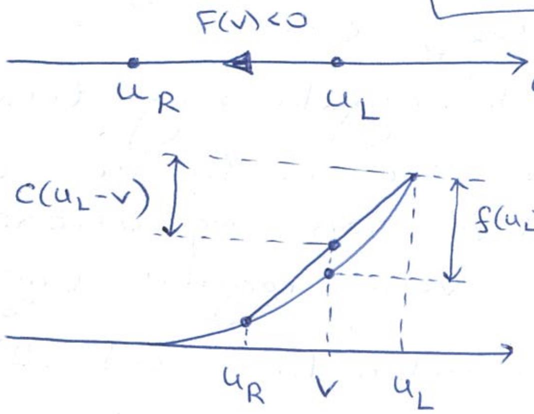
OK, let us integrate $\int_{-\infty}^{\xi}$: $-c(v(\xi) - u_L) + (f(v(\xi)) - f(u_L))$
$= v'(\xi)$

ODE) $v' = f(v) - f(u_L) - c(v - u_L) = F(v)$

Note that RHS $F(u_L) = 0$ and $F(u_R) = 0$ (due to RH!)

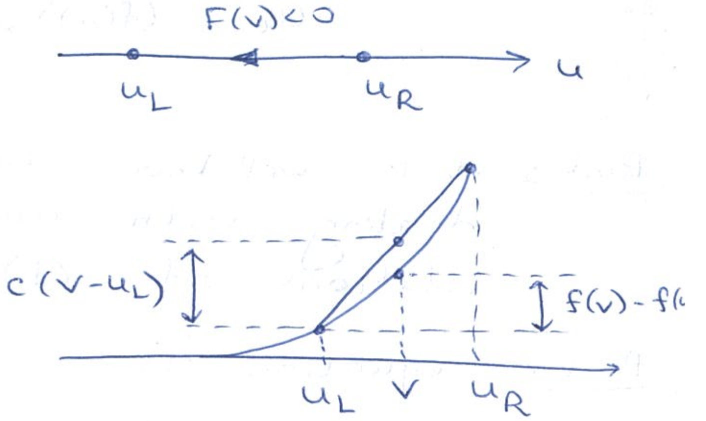
Thus u_L and u_R are two fixed points of this ODE

Consider 2 cases: $\boxed{u_L > u_R}$ and $\boxed{u_L < u_R}$.



In this case: $F(v) < 0$ $\forall v \in (u_R, u_L)$

And there exists a solution v of ODE: $v(-\infty) = u_L$; $v(+\infty) = u_R$



In this case: $F(v) < 0$ $\forall v \in (u_L, u_R)$

And there DOES NOT exist a solution v of ODE $v(-\infty) = u_L$; $v(+\infty) = u_R$ ∎

Lecture 8: Scalar conservation law: $\begin{cases} u_t + (f(u))_x = 0 \\ u|_{t=0} = u_0(x) \end{cases}$ (*)

- $u: \mathbb{R}\times\mathbb{R}_+ \to \mathbb{R}$ - bounded, measurable

  $f: \mathbb{R}\to\mathbb{R}$, $f\in C^2$, $f''>0$. As we will see it is enough to define f on the convex hull of values of u_0

We understand solutions in weak sense:

$$\iint_{t>0} [u\varphi_t + f(u)\varphi_x]\,dx\,dt + \int_{t=0} u_0\varphi\,dx = 0 \qquad (**)$$

for every test function $\varphi\in C_0^1$.

Define $M := \|u_0\|_\infty$, $A := \max\limits_{|u|\le M} |f'(u)|$, $\mu := \min\limits_{|u|\le M} f''(u)$

Today we will start proving theorem on existence.

Thm 1 ($\exists$): Let $u_0\in L_\infty(\mathbb{R})$; $f\in C^2(\mathbb{R})$, $f''>0$ on $\{u: |u|\le M\}$

There exists a solution with the following properties

(a) $|u(x,t)| \le M$, $(x,t)\in\mathbb{R}\times\mathbb{R}_+$

(b) $\exists E = E(M,\mu,A)>0$ such that $\forall a>0$, $\forall t>0$, $x\in\mathbb{R}$

$$\frac{u(x+a,t)-u(x,t)}{a} < \frac{E}{t} \qquad (E)$$

"entropy" cond.

(c) u is stable and depends continuously on u_0: if $v_0\in L_\infty(\mathbb{R})$ with $\|v_0\|_\infty \le \|u_0\|_\infty$ and v is the corresponding constructed solution of (*) with initial data v_0, then for $\forall x_1,x_2\in\mathbb{R}$ with $x_1<x_2$ and $t>0$

$$\int_{x_1}^{x_2} |u(x,t)-v(x,t)|\,dx \le \int_{x_1-At}^{x_2+At} |u_0(x)-v_0(x)|\,dx \qquad (S)$$

"stability"

How to prove this theorem?

There exist (at least) 5 approaches:

(a) Calculus of variations and Hamilton-Jacobi theory

(b) Vanishing viscosity method

(c) Non-linear semigroup theory

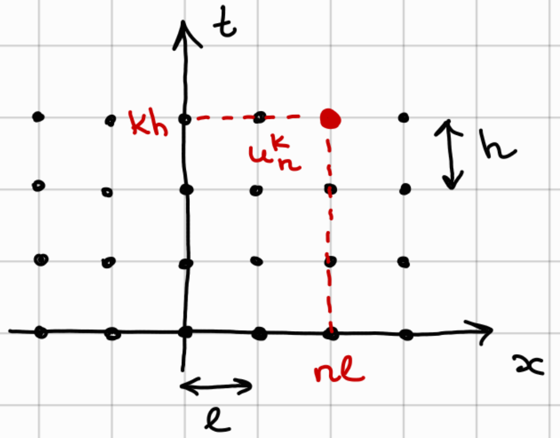
(d) Method of characteristics

(e) Finite - difference method

We will follow Smoller (Chapter 16) and use (e).

Here is the scheme of the proof :

Step 1 : discretization in space and time



$x_n = n\ell$, $n \in \mathbb{Z}$ $\qquad \ell = \Delta x > 0$

$t_k = kh$, $k \in \mathbb{N} \cup \{0\}$ $\qquad h = \Delta t > 0$

$u_n^k = u(n\ell, kh)$

Consider a finite-difference (explicit) scheme:

$$\text{(D)} \quad \boxed{u_n^{k+1} = \frac{u_{n+1}^k + u_{n-1}^k}{2} - \frac{h}{2\ell} \cdot \left(f(u_{n+1}^k) - f(u_{n-1}^k) \right)}$$

$u_n^0 = u_0(n\ell)$, $n \in \mathbb{Z}$

In what follows we will always assume :

$$\boxed{\frac{Ah}{\ell} \le 1} \qquad \text{(CFL condition)}$$

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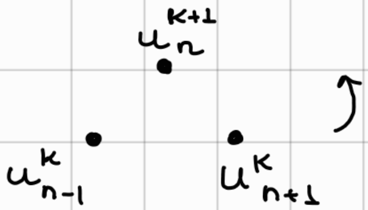
It is important for the stability of the numerical scheme and tells that the time step h should be small enough.

First, we will formulate and prove some properties of solutions u_n^k of a discrete eq. (D) :

(1a) solution exists (evident!)



(1b) if $|u_n^0| \le M$, then $|u_n^k| \le M \;\; \forall k \in \mathbb{N}$

(boundedness)

(1c) $\exists E = E(M, A, \mu) > 0 : \quad \dfrac{u_n^k - u_{n-2}^k}{2\ell} \le \dfrac{E}{kh}$ (E-disc)

discrete entropy condition

NB: the discrete entropy condition is a natural consequence of a finite difference approximation (D).

(1d) local bounded variation: $\forall X > 0$ and $kh > \alpha > 0$ $\exists\, c(X, \alpha)$ (but independent of h and ℓ):

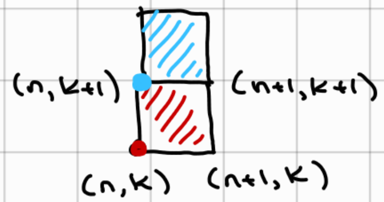
$$\sum_{|n| \le X/\ell} |u_{n+2}^k - u_n^k| \le C$$ and some other...

Step 2: We will prove convergence as $h, \ell \to 0$.

Consider $U_{h,\ell}(x,t) = u_n^k$ if

$n\ell \le x \le (n+1)\ell$

$kh \le t \le (k+1)h$



We will prove that there exists subsequence U_{h_i,ℓ_i} of $U_{h,\ell}$ such that $U_{h_i,\ell_i} \to u(x,t)$ - some measurable function

Step 3: We will prove that this limiting function satisfies integral equality (**) and all properties of theorem on $\exists$.

Proof of the theorem 1.

Lemma 1 (boundedness of u_n^k): $|u_n^k| \le M$, $n \in \mathbb{Z}$, $k \in \mathbb{N}$

This is an exercise 2 from list 3.

Lemma 2 (discrete entropy condition)

If $c = \min\left(\frac{\mu}{2}, \frac{A}{4M}\right)$, then

$$\frac{u_n^k - u_{n-2}^k}{2\ell} \le \frac{E}{kh} \quad \text{where } E = \frac{1}{c}.$$

Proof:

Let $z_n^k = \dfrac{u_n^k - u_{n-2}^k}{2\ell}$ and first let us prove some recurrent relation for z_n^{k+1} of the form

" $z_n^{k+1} = A\, z_{n+1}^k + B\, z_{n-1}^k + C$ "

$$z_n^{k+1} = \frac{1}{2}\left[z_{n+1}^k + z_{n-1}^k \right] - \frac{h}{(2\ell)^2}\left(f(u_{n+1}^k) - f(u_{n-1}^k)\right)$$
$$+ \frac{h}{(2\ell)^2}\left(f(u_{n-1}^k) - f(u_{n-3}^k)\right)$$

Notice that due to $f \in C^2$ we can write

$$f(u_{n+1}^k) = f(u_{n-1}^k) + f'(u_{n-1}^k)(u_{n+1}^k - u_{n-1}^k) + f''(\theta_1)\frac{(u_{n+1}^k - u_{n-1}^k)^2}{2}$$

for some θ_1 between u_{n+1}^k and u_{n-1}^k

$$= f(u_{n-1}^k) + f'(u_{n-1}^k) \cdot 2\ell \cdot z_{n+1}^k + f''(\theta_1) \cdot \frac{(2\ell)^2}{2}(z_{n+1}^k)^2$$

Analogously,

$$f(u_{n-3}^k) = f(u_{n-1}^k) - f'(u_{n-1}^k) \cdot 2\ell \cdot z_{n-1}^k + f''(\theta_2)\frac{(2\ell)^2}{2}(z_{n-1}^k)^2$$

Thus,

$$z_n^{k+1} = z_{n+1}^k \cdot \Big[\overbrace{\frac{1}{2} - \frac{h}{2\ell} f'(u_{n-1}^k)}^{A} \Big] + z_{n-1}^k \Big[\overbrace{\frac{1}{2} + \frac{h}{2\ell} f'(u_{n-1}^k)}^{B} \Big]$$
$$- \frac{h}{2} \cdot \left[f''(\theta_1) \cdot (z_{n+1}^k)^2 + f''(\theta_2) \cdot (z_{n-1}^k)^2 \right]$$

Note that $A + B = 1$ and $A, B \geq 0$

Define $\tilde{z}_n^k = \max\{ z_{n-1}^k, z_{n+1}^k, 0\}$

If $\tilde{z}_n^k = 0 \quad \forall n$, then (E-disc) is true since

$$z_n^k \leq \tilde{z}_n^k = 0 \leq \frac{E}{kh} \quad \forall E > 0.$$

Thus, w.l.o.g. we can assume $\tilde{z}_n^k \neq 0$. Suppose $\tilde{z}_n^k = z_{n+1}^k$ (the other case is similar)

$$z_n^{k+1} \leq z_{n+1}^k \cdot [A + B] - h\frac{\mu}{2}(z_{n+1}^k)^2 \leq$$
$$\leq \tilde{z}_n^k + h c (\tilde{z}_n^k)^2 .$$

Notice that

$$z_n^k \leq |z_n^k| \leq \frac{M}{\ell} \underset{\uparrow \text{ CFL}}{\leq} \frac{M}{Ah} \underset{\uparrow c \leq \frac{A}{4M}}{\leq} \frac{M}{h} \cdot \frac{1}{4Mc} = \frac{1}{4ch}$$

Let $M^k = \max\limits_{n \in \mathbb{Z}} \{\tilde{z}_n^k\} \geq 0$.

Let $\varphi(y) = y - c \cdot h \cdot y^2$. Since $\varphi' = 1 - 2chy$, φ is increasing if $y \le \frac{1}{2ch}$. But we have $\tilde{z}_n^k \le M^k \le \frac{1}{4ch} < \frac{1}{2ch}$.

So that $\varphi(\tilde{z}_n^k) \le \varphi(M^k)$ and we have

$$\tilde{z}_n^k - ch(\tilde{z}_n^k)^2 \le M^k - ch(M^k)^2$$

Thus, $z_n^{k+1} \le M^k - ch(M^k)^2 \quad \forall n \in \mathbb{Z}$.

It follows that

$$\boxed{M^{k+1} \le M^k - ch(M^k)^2} \quad (M)$$

<u>Claim</u>: $M^k \le \frac{1}{chk + 1/M^0}$.

Suppose we have proven claim. Let us see how it helps to prove lemma 2. Indeed,

$$z_n^k \le M^k \le \frac{1}{chk + 1/M^0} \le \frac{1}{chk} = \frac{E}{hk}, \quad E = \frac{1}{c}.$$

Proof of claim: first – intuition why such estimate could be true

Inequality (M) for M^k is a discrete analog of ODinEquality: $\varphi' \le -ch\varphi^2$

if it was an equality $\varphi' = -ch\varphi^2$, then the solution is:

$$\frac{d\varphi}{\varphi^2} = -ch\, dt$$

$$-\frac{1}{\varphi} = -cht + C_1$$

$\varphi(t) = \frac{1}{cht - C_1}$ and with ic $\varphi(0) = \varphi_0$ we will have $\varphi(t) = \frac{1}{cht + 1/\varphi_0}$.

So one can try to prove $\varphi(t) \le \frac{1}{cht + 1/\varphi_0}$.

Second, let us make the formal proof. We will do it by induction.

Base: $k=0$: – clear: $M^0 \le \frac{1}{1/M^0} = M^0$.

$k>0$: suppose that

$$M^k \le \frac{1}{chk + 1/M^0}$$

and we want to prove that

$$M^{k+1} \le \frac{1}{ch(k+1) + 1/M^0}$$

We have: $\frac{1}{M^k} \ge chk + \frac{1}{M^0}$, so

$$1 - ch M^k \ge 1 - chk\, M^k \ge \frac{M^k}{M^0} \ge 0.$$

Thus $1 - (ch M^k)^2 \ge 0$.

We have $M^{k+1} \le M^k (1 - ch M^k)$, so that

$$\frac{M^{k+1}}{1 - ch M^k} \le M^k \le \frac{M^k}{1 - (ch M^k)^2}$$

and thus

$$M^{k+1} \le \frac{M^k}{1 + ch M^k} = \frac{1}{ch + 1/M^k} \le$$

$$\le \frac{1}{ch(k+1) + 1/M^0} \quad \text{q.e.d.} \quad ■$$

<u>Lemma</u> (space estimate): For any $X>0$ and $kh \ge \alpha > 0$, there is a constant $C = C(X, \alpha, M)$ (but independent on h, e) such that:

$$\sum_{|n| \le X/e} |u_{n+2}^k - u_n^k| \le C$$

Proof:

Set $v_n^k = u_n^k - c_1 n e$, where c_1 is chosen so large that $E/\alpha < c_1$. Then

$$v_{n+2}^k - v_n^k = u_{n+2}^k - u_n^k - 2c_1 e \le \frac{2eE}{kh} - 2c_1 e \le$$

$$\le 2e\left(\frac{E}{\alpha} - c_1\right) < 0, \text{ so } v_n^k \text{ is decreas. in } n$$

Thus

$$\sum_{|n| \le X/e} |u_{n+2}^k - u_n^k| \le \sum |v_{n+2}^k - v_n^k| + \sum 2c_1 e =$$

$$= \underbrace{-\sum (v_{n+2}^k - v_n^k)}_{\text{telescopic sum}} + 2c_1 e\left(\frac{2X}{e} + 1\right) \le 4M + 2c_1 X + c_2 X$$

$$\le 4(M + c_1 X)$$

q.e.d. ■

Lecture 9: We continue proving theorem on existence of entropy solution for scalar cons law.

Lemma 4 (time estimate – u_n^k are L^1 locally Lipshits in k)

If $h/\ell \geq \delta > 0$ and $h, \ell \leq 1$, then exists $L > 0$ (independent of h, ℓ) such that

if $k > p$, where $(k-p)$ is even and $ph \geq \alpha > 0$, then

$$\sum_{|n| \leq X/\ell} |u_n^k - u_n^p| \ell \leq L(k-p)h$$

A similar estimate holds if $(k-p)$ is odd.

Proof:

▶ Let us express u_n^k in terms of u_n^p where $(k-p)$ is even.

$$u_n^k = \frac{1}{2}\left(u_{n+1}^{k-1} + u_{n-1}^{k-1}\right) - \frac{h}{2\ell} f'(\Theta)\left(u_{n+1}^{k-1} - u_{n-1}^{k-1}\right) =$$

$$= u_{n+1}^{k-1}\left(\frac{1}{2} - \frac{h}{2\ell} f'(\Theta)\right) + u_{n-1}^{k-1}\left(\frac{1}{2} + \frac{h}{2\ell} f'(\Theta)\right)$$

or $u_n^k = a_{n+1}^{k-1} u_{n+1}^{k-1} + a_{n+1}^{k-1} u_{n+1}^{k-1}$, where $a_{n+1}^{k-1} + a_{n-1}^{k-1} = 1$ and $a_{n+1}^{k-1}, a_{n-1}^{k-1} \geq 0$.

Applying this to u_{n-1}^k and u_{n+1}^k gives a formula:

$$u_n^{k+1} = A u_{n+2}^{k-1} + B u_n^{k-1} + C u_{n-2}^{k-1}$$

where $A, B, C \geq 0$, $A + B + C = 1$.

Hence, $|u_n^{k+1} - u_n^{k-1}| \leq A|u_{n+2}^{k-1} - u_n^{k-1}| + C|u_{n-2}^{k-1} - u_n^{k-1}|$

Multiplying this by $\Delta x = \ell$ and summing, we get:

$$\sum_{|n| \leq X/\ell} |u_n^{k+1} - u_n^{k-1}| \Delta x \underset{\substack{\uparrow \\ \text{lemma 3}}}{\leq} c\,\Delta x$$

Now if $(k-p)$ is even, we can do this operation several times and using the triangle inequality, we get:

$$\sum_{|n| \leq X/\ell} |u_n^k - u_n^p| \Delta x \leq \sum_{i=p}^{k-2} \sum_{|n| \leq X/\ell} |u_n^{i+2} - u_n^i| \Delta x \leq (k-p) c\,\Delta x \leq$$

$$L \le \frac{\Delta t}{\delta}(k-p)c = L(k-p)h \quad \text{for } L = \frac{c}{\delta},\ h = \Delta t \ \blacksquare$$

__Lemma 5__ (stability): Let u_n^k and v_n^k be solutions to the finite-difference scheme (D) corresponding to the initial conditions u_n^0 and v_n^0, respectively, where

$$\sup_{n\in\mathbb{Z}} |u_n^0| \le M \quad \text{and} \quad \sup_{n\in\mathbb{Z}} |v_n^0| \le M$$

Then, if $k>0$,

$$\sum_{|n|\le N} |u_n^k - v_n^k| \cdot \ell \le \sum_{|n|\le N+k} |u_n^0 - v_n^0| \cdot \ell$$

__Proof__:

Γ $w_n^k = u_n^k - v_n^k$. From (D) we have

$$w_n^{k+1} = u_n^{k+1} - v_n^{k+1} = \frac{u_{n+1}^k + u_{n-1}^k}{2} - \frac{h}{2\ell}\left(f(u_{n+1}^k) - f(u_{n-1}^k)\right)$$

$$- \frac{v_{n+1}^k + v_{n-1}^k}{2} + \frac{h}{2\ell}\left(f(v_{n+1}^k) - f(v_{n-1}^k)\right) =$$

$$= \frac{w_{n+1}^k + w_{n-1}^k}{2} - \frac{h}{2\ell}\left(f(u_{n+1}^k) - f(v_{n+1}^k)\right)$$

$$+ \frac{h}{2\ell}\left(f(u_{n-1}^k) - f(v_{n-1}^k)\right) =$$

$$= w_{n+1}^k \overbrace{\underbrace{\left[\frac{1}{2} - \frac{h}{2\ell} f'(Q_1)\right]}_{\ge 0 \text{ due to CFL}}}^{A_{n+1}^k} + w_{n-1}^k \left[\overbrace{\underbrace{\frac{1}{2} + \frac{h}{2\ell} f'(Q_2)}_{\ge 0}}^{B_{n+1}^k}\right]$$

Now proceed by induction.

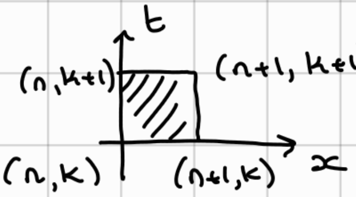
$$\sum_{|n|\le N} |w_n^{k+1}| \le \sum_{|n|\le N} |w_{n+1}^k| \cdot A_{n+1}^k + \sum_{|n|\le N} |w_{n-1}^k| \cdot B_{n+1}^k =$$

$$= \sum_{1-N}^{N+1} |w_m^k| A_m^k + \sum_{-1-N}^{N-1} |w_m^k| \cdot B_m^k \le$$

$$\le \sum_{|m|\le N+1} |w_m^k| A_m^k + \sum_{|m|\le N+1} |w_m^k| \cdot B_m^k \le \sum_{|m|\le N+1} |w_m^k| \ \blacksquare$$

q.e.d.

<u>Step 2</u>: Rather than define u_n^k in mesh points let us continue u_n^k as a piece wise constant function in the upper half plane.



$$\mathcal{U}_{h,\ell}(x,t) = u_n^k \quad \text{if} \quad \begin{array}{l} n\ell \le x \le (n+1)\ell \\ kh \le t \le (k+1)h \end{array}$$

So we have a family of functions $\{\mathcal{U}_{h,\ell}\}$ and would like to choose a convergent subsequence $\mathcal{U}_{h_i,\ell_i}$ as $h_i, \ell_i \to 0$ $i \to \infty$.

<u>Lemma 6</u> (convergence : the set of functions $\{\mathcal{U}_{h,\ell}\}$ is compact in the topology of L_1-convergence on compacta)

There exists a subsequence $\{\mathcal{U}_{h_i,\ell_i}\}$ $i \in \mathbb{N}$ which converges to a measurable function $u(x,t)$ in the sense that for $\forall X > 0, t > 0, T > 0$ both

$$\int_{|x| \le X} |\mathcal{U}_{h_i,\ell_i}(x,t) - u(x,t)| \, dx \to 0 \quad \text{as } h_i, \ell_i \to 0$$

and

$$\int_0^T \int_{|x| \le X} |\mathcal{U}_{h_i \ell_i}(x,t) - u(x,t)| \, dx \, dt \to 0 .$$

Furthermore, the function $u(x,t)$ satisfies :

(a) $\sup\limits_{\substack{x \in \mathbb{R} \\ t > 0}} |u(x,t)| \le M$; (b) inequality (S) (stability)

<u>Proof</u> :

▶ First, take $t = \text{const}$ and consider $\mathcal{U}_{h,\ell}(x,t)$ as functions of x. By Lemma 1 and Lemma 3 the set of functions $\{\mathcal{U}_{h,\ell}\}$ is bounded and have uniformly bounded total variation on each bounded interval in x.

Helly's theorem (simple version):

A uniform bounded sequence of monotone, real functions admits a convergent subsequence.

Helly's theorem (generalized version):

A uniform bounded sequence of BV_{loc} (locally of bounded variation) real functions admits a convergent subsequence on every compact set.

Rmk: a function of BV_{loc} can be written as a sum of increasing and decreasing functions (on each compact interval). This is why the generalized version of the Helly's theorem is true.

So by Helly's theorem on each interval we have a convergent subsequence $U'_{h,e}$.

By a standard diagonal process we can construct a subsequence $\{U''_{h,e}\}$ from $\{U'_{h,e}\}$ which converges at every $x \in \mathbb{R}$ for this particular $t = const > 0$.

Second, take $\{t_m\}_{m=1}^{\infty}$ - a countable and dense subset of $(0,T)$, e.g. $\mathbb{Q} \cap (0,T)$.

For $t = t_1$ we have $\{U^1_{h_i, e_i}\}$ a convergent subsequence.

For $t = t_2$ take a convergent sub. $\{U^2_{h_i, e_i}\}$ from $\{U^1_{h_i, e_i}\}$

etc. So we have:

$$\begin{array}{lllll}
t = t_1: & U^1_{h_1,e_1} & U^1_{h_2,e_2} & U^1_{h_3,e_3} & U^1_{h_4,e_4} \\
t = t_2: & U^2_{h_1,e_1} & U^2_{h_2,e_2} & U^2_{h_3,e_3} & U^2_{h_4,e_4} \\
t = t_3: & U^3_{h_1,e_1} & U^3_{h_2,e_2} & U^3_{h_3,e_3} & U^3_{h_4,e_4} \\
t = t_4: & U^4_{h_1,e_1} & U^4_{h_2,e_2} & U^4_{h_3,e_3} & U^4_{h_4,e_4}
\end{array}$$

...

By a standard diagonal process, we can choose a subsequence U_{h_i, e_i} which converges for all $\{t_m\}_{m=1}^{\infty}$ and all $x \in \mathbb{R}$.

Third, we want to show that there is a convergence for all $t \in (0,T)$. So that in the limit we indeed obtain a function defined in the strip $0<t<T$

Let $U_i = U_{\ell_i, h_i}$ and we want to show that

$$I_{ij} = \int_{-X}^{X} | U_i(x,t) - U_j(x,t) | \, dx \longrightarrow 0 \quad \forall i,j \to \infty$$

i.e. that $\{U_i\}$ is a Cauchy sequence in $L_1(|x| \le X)$

For $t \in (0,T)$ we find a subsequence $\{t_{m_s}\} \subset \{t_m\}$ such that $t_{m_s} \to t$ as $s \to \infty$. Let $\tau_s = t_{m_s}$. Then

$$I_{ij}(t) \le \int_{-X}^{X} | U_i(x,t) - U_i(x,\tau_s) | \, dx + \int_{-X}^{X} | U_i(x,\tau_s) - U_j(x,\tau_s) |$$

$$+ \int_{-X}^{X} | U_j(x,t) - U_j(x,\tau_s) | \, dx =: I_1 + I_2 + I_3$$

For $t = \tau_s$ we have a convergence of U_i, thus for s large enough we have $I_2 < \varepsilon/3$

Let's estimate I_1:

$$I_1 = \int_{-X}^{X} | U_i(x, [\tfrac{t}{h_i}] h_i) - U_i(x, [\tfrac{\tau_s}{h_i}] h_i) | =$$

$$= \sum_{|n| < \frac{X}{\ell_i} + 1} \int_{n\ell_i}^{(n+1)\ell_i} | U_i(x, [\tfrac{t}{h_i}] h_i) - U_i(x, [\tfrac{\tau_s}{h_i}] h_i) | \, dx =$$

$$= \sum_{|n| < \frac{X}{\ell_i} + 1} | u_n^{[t/h_i]} - u_n^{[\tau_s/h_i]} | \, \ell_i \underset{\text{Lemma 4}}{\le} L h_i \left| [\tfrac{t}{h_i}] - [\tfrac{\tau_s}{h_i}] \right|$$

$$\le L | t - \tau_s | < \frac{\varepsilon}{3} \quad \text{for } s \text{ large enough.}$$

Analogously, $I_3 < \frac{\varepsilon}{3}$. Thus $I_{ij} \le 3 \cdot \frac{\varepsilon}{3} = \varepsilon$.

We have proved pointwise limit for every $t \in (0,T)$, that is $\exists\, u(x,t) \in L_1(|x| \le X)$ (in part, measurable)

Fourth, let us show that $I_{ij} \to 0$ uniformly in t, $0 < \tau \le t \le T$. Indeed, fix $\varepsilon > 0$. Choose finite subset $\mathcal{F} \subset \{t_m\}$ such that if $0 \le t \le T$ there is a $t_m \in \mathcal{F}$ such that $L(t - t_m) < \frac{\varepsilon}{3}$. Then we choose i, j so large that $I_2 < \frac{\varepsilon}{3}$ for all $t_m \in \mathcal{F}$ (it is possible because $\mathcal{F}$ is finite). This reasoning gives us the desired uniformity in t.

Fifth, using uniform convergence, we have

$$\forall \tau \in (0,T] \qquad \int_\tau^T I_{ij}\, dt \to 0 .$$

Now we write $\int_0^T = \int_0^\tau + \int_\tau^T$:

$$\int_0^T \int_{-X}^X |U_i - U_j|\, dx\, dt = \underbrace{\int_0^\tau \int_{-X}^X |U_i - U_j|}_{< \varepsilon/2 \text{ if } 8MX\tau < \varepsilon} + \overbrace{\int_\tau^T \int_{-X}^X |U_i - U_j|}^{< \frac{\varepsilon}{2} \text{ for } i,j \text{ suffic. large}} < \varepsilon$$

That means $$\int_0^T I_{ij}\, dt \longrightarrow 0 \quad \text{as} \quad i,j \to +\infty .$$

Sixth, since local convergence in L_1 implies poinwise convergence a.e. of a subsequence, we see

$|U_i| \le M \implies |u| \le M$

and Lemma 5 $\implies$ (S) ■

Step 3: Let us show that the limiting function $u(x,t)$, indeed, satisfies the properties from thm 1.

Lemma 7 (entropy inequality) : u satisfies (E).

Proof:

$\ulcorner$ ▶ It is sufficient to show that if $(x_1 - x_2) > 2l_i$ and $t > h_i$ then

$$\frac{U_i(x_1,t) - U_i(x_2,t)}{x_1 - x_2} < \frac{2E}{t-h_i}.$$

Let $x_1 > x_2$ and note that

$$U_i(x_j,t) = U_i\left(x_j - \eta_j, \left[\frac{t}{h_i}\right] h_i\right) \quad j=1,2$$

for some $0 \le \eta_j < l_j$. Thus,

$$\frac{U_i(x_1,t) - U_i(x_2,t)}{x_1 - x_2} = \frac{1}{x_1 - x_2} \sum (u_n^k - u_{n-2}^k) \quad \text{for } k = \left[\frac{t}{h_i}\right]$$

over all integers in the interval $[x_2 - \eta_2, x_1 - \eta_1]$

Using Lemma 2, we have

$$\frac{U_i(x_1,t) - U_i(x_2,t)}{x_1 - x_2} \le \frac{E(x_1 - \eta_1 - x_2 + \eta_2)}{\left[\frac{t}{h_i}\right] h_i (x_1 - x_2)} \le \frac{E(x_1 - \eta_1 - x_2 + \eta_2)}{(t-h_i)(x_1-x_2)} =$$

$$= \frac{E}{t-h_i} + \frac{E\overbrace{(\eta_2 - \eta_1)}^{<l_i}}{(t-h_i)\underbrace{(x_1-x_2)}_{>2l_i}} < \frac{2E}{t-h_i} \quad ■$$

$\llcorner$

Lecture 10 : Let's finish proving theorem on $\exists$ of entropy solution

Reminder : Scalar conservation law : $\begin{cases} u_t + (f(u))_x = 0 \\ u|_{t=0} = u_0(x) \end{cases}$ $(*)$

- $u : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$ - bounded, measurable

$f : \mathbb{R} \to \mathbb{R}$, $f \in C^2$, $f'' > 0$. As we will see it is enough to define f on the convex hull of values of u_0

We understand solutions in weak sense :

$$\iint_{t>0} [u\varphi_t + f(u)\varphi_x]\,dx\,dt + \int_{t=0} u_0 \varphi\,dx = 0 \qquad (**)$$

for every test function $\varphi \in C_0^1$.

Lemma 8 (last lemma)

Let U_i be a convergent subsequence from Lemma 6. We know that $U_i \to u(x,t)$, $i \to +\infty$, and $\forall X \in \mathbb{R}$

$$\int_{-X}^{X} |U_i(x,0) - u_0(x)|\,dx \to 0 .$$

Then u satisfies $(**)$, i.e. u is a weak solution of $(*)$.

Proof.

▶ Rewrite (D) in such a form:

$$\frac{u_n^{k+1} - u_n^k}{h} - \frac{u_{n+1}^k - 2u_n^k + u_{n-1}^k}{2\ell^2} \cdot \frac{\ell^2}{h} + \frac{f(u_{n+1}^k) - f(u_{n-1}^k)}{2\ell} = 0$$

Multiply this equality by $\varphi_n^k = \varphi(n\ell, kh)$ and get

$$\frac{\varphi_n^{k+1} u_n^{k+1} - \varphi_n^k u_n^k}{h} - u_n^{k+1} \frac{\varphi_n^{k+1} - \varphi_n^k}{h} + \frac{\ell^2}{h} \cdot u_n^k \cdot \frac{2\varphi_n^k - \varphi_{n+1}^k - \varphi_{n-1}^k}{\ell^2}$$

$$+ \frac{\varphi_{n+1}^k u_n^k - \varphi_n^k u_{n-1}^k}{2h} + \frac{\varphi_{n-1}^k u_n^k - \varphi_n^k u_{n+1}^k}{2h} +$$

$$+ \frac{\varphi_{n+1}^k f(u_{n+1}^k) - \varphi_{n-1}^k f(u_{n-1}^k)}{2\ell} - f(u_{n+1}^k) \frac{\varphi_{n+1}^k - \varphi_n^k}{2\ell}$$

$$- f(u_{n-1}^k) \frac{\varphi_n^k - \varphi_{n-1}^k}{2\ell} = 0$$

Since $\varphi \in C^3_0$ has compact support, we may assume $\varphi_n^k = 0$ if $k \geq [\frac{T}{h}]$

Multiply this equality by $h\ell$ and sum over $n \in \mathbb{Z}$, $k \in \mathbb{N} \cup \{0\}$.

- $$\sum_{k,n} \frac{\varphi_n^{k+1} u_n^{k+1} - \varphi_n^k u_n^k}{h} = -\sum_n \varphi_n^0 u_n^0 \qquad \text{(telescopic sum)}$$

- $$\sum_{k,n} \frac{\varphi_{n+1}^k u_n^k - \varphi_n^k u_{n-1}^k}{2h} = 0 \quad \text{and} \quad \sum_{k,n} \frac{\varphi_{n-1}^k u_n^k - \varphi_n^k u_{n+1}^k}{2h} = 0$$

Thus,

$$-h \sum_n \varphi_n^0 u_n^0 + h\ell \Bigg[\sum_{k,n} \Big[-u_n^{k+1} \frac{\varphi_n^{k+1} - \varphi_n^k}{h} - \frac{\ell^2}{2h} \overset{u_n^k}{\vee} \frac{\varphi_{n+1}^k + \varphi_{n-1}^k - 2\varphi_n^k}{2\ell} \Big]$$
$$- \sum_{k,n} f(u_{n+1}^k) \frac{\varphi_{n+1}^k - \varphi_n^k}{2\ell} - \sum_{k,n} f(u_{n-1}^k) \frac{\varphi_n^k - \varphi_{n-1}^k}{2\ell} \Bigg] = 0$$

Instead of a sum for u_n^k we can write integral for $v_{h,\ell}$

$$-\int_{t=0} v_{h,\ell}\varphi + \delta_1 - \iint_{t\geq 0} v_{h,\ell}\varphi_t + \delta_2 - \frac{\ell^2}{2h} \iint_{t\geq 0} v_{h,\ell}\varphi_{xx}$$
$$+ \delta_3 - \iint_{t\geq 0} f(v_{h,\ell})\varphi_x + \delta_4 = 0$$

where $\delta_i \to 0$ as $h,\ell \to 0$. Replace $v_{h,\ell}$ by v_i:

$$-\int_{t=0} v_i\varphi - \iint_{t\geq 0} v_i\varphi_t - \frac{\ell_i^2}{2h_i} \iint_{t\geq 0} v_i \varphi_{xx} - \iint_{t\geq 0} f(v_i)\varphi_x = \underset{\substack{\downarrow\, i\to\infty \\ 0}}{\delta(h_i,\ell_i)}$$

$\ell_i \to 0$, $\frac{\ell_i}{h_i}$ is bounded ; $\frac{\ell_i^2}{h_i} \to 0$; $v_i \to u$ in L^1-loc

$$\Rightarrow \iint_{t\geq 0} v_i\varphi_t - \frac{\ell_i^2}{2h_i} \iint_{t\geq 0} v_i\varphi_{xx} \to \iint_{t\geq 0} u\varphi_t$$

By choice of initial values: $\int\limits_{t=0} U_i \varphi \to \int\limits_{t=0} u_0 \varphi$

Also, $\left| \iint\limits_{t\geq 0} \left(f(U_i) - f(u)\right) \varphi_x \right| \leq \|\varphi_x\|_\infty \iint\limits_{D:\,\varphi\neq 0} |f(U_i) - f(u)|$

$\leq \|\varphi_x\|_\infty \iint\limits_{D:\varphi\neq 0} |f'(\xi)| \cdot |U_i - u| \to 0$

And we have: $\iint\limits_{t\geq 0} f(U_i)\varphi_x \to \iint\limits_{t\geq 0} f(u)\varphi_x$.

We have proved $(**)$ for $\forall\, \varphi \in C_0^3$.
$C_0^3 \subset C_0^1$ is a dense subset, then $(**)$ are also true for $\varphi \in C_0^1$. ∎

Now, let's prove the theorem on uniqueness.

<u>Thm 2</u> (!): Let $f \in C^2$, $f'' > 0$.
Let u, v be 2 solutions of $(**)$, satisfying entropy condition (E): $\exists E\ \forall a > 0,\ t > 0,\ x \in \mathbb{R}$

$$\frac{u(x+a) - u(x)}{a} < \frac{E}{t}. \qquad (E)$$

Then $u = v$ almost everywhere in $t > 0$.

<u>Rmk 1</u>: we call such a solution – an entropy sol.

<u>Rmk 2</u>: If we had a linear operator, then the main idea of the proof could be as follows (we will adapt this idea to non-linear)

Let H be a Hilbert space.

$A: H \to H$, $\eta(A) = \{g \in H : A(g) = 0\}$ - null space

$R(A) = \{f \in H : \exists g \in H : A(g) = f\}$ - range of A

A^* is the adjacent operator:

$\langle Ax, y\rangle = \langle x, A^* y\rangle$

$R(A^*)$ is the orthogonal complement of $\eta(A)$

<u>Fact:</u> $R(A^*) \oplus \eta(A) = H$

The "bigger" is $R(A^*)$, the "smaller" is $\eta(A)$.

That means that if there exist sufficiently many solutions to the adjoint equation, then the null space of A is zero $\Rightarrow$ A has a unique solution !

If $Ax = Ay$ we can choose w: $A^* w = x - y$:

$\|x-y\| = \langle x-y, x-y\rangle = \langle x-y, A^* w\rangle = \langle Ax - Ay, w\rangle = 0$

$\Rightarrow x = y$ (idea of Holgrem ~ 1901)

But we have a nonlinear eq !

Let us adapt this idea.

<u>Proof of thm 2.</u>

Let u, v be 2 solutions of $(**)$.

In order to prove that $u = v$ a.e. in $t > 0$ it suffices to show that $\forall \varphi \in C_0^1$:

$$\iint_{t>0} (u-v)\,\varphi = 0 .$$

What we know? Let $\psi \in C_0^1$, then

(1) $$\iint_{t\geq 0} [u\,\psi_t + f(u)\,\psi_x]\,dx\,dt + \int_{t=0} u_0 \psi\, dx = 0$$

(2) $$\iint_{t\geq 0} [v\,\psi_t + f(v)\,\psi_x]\,dx\,dt + \int_{t=0} u_0 \psi\, dx = 0$$

Subtract (1) - (2) and we get:

$$\iint_{t\geq 0} (u-v)\left[\psi_t + \underbrace{\frac{f(u)-f(v)}{u-v}}_{=:F(x,t)} \cdot \psi_x\right] dx\,dt = 0$$

$$\iint_{t\geq 0} (u-v)\left[\psi_t + F\psi_x\right] dx\,dt = 0$$

?" $\varphi \in C_0^1$

Now if for $\forall\, \varphi \in C_0^1$ we could solve the linear (adjoint!) equation and have a solution $\psi \in C_0^1$, we could conclude that $u=v$ a.e.

However, there is an obstruction to this approach: "velocity field" F is not smooth (not even continuous), so it is not clear why solution $\psi \in C_0^1$.

To struggle this difficulty, one can approximate u and v by smooth functions and solve corresponding linear eqs:

$$(M) \qquad \psi_t^m + F_m \psi_x^m = \varphi, \quad F_m = \frac{f(u_m) - f(v_m)}{u_m - v_m}$$

Then $\iint\limits_{t\geq 0} (u-v)\varphi = \iint\limits_{t\geq 0} (u-v)\left[\psi_t^m + F_m \psi_x^m\right] =$

$$= \underbrace{-\iint\limits_{t\geq 0} (u-v)\left[\psi_t^m + F\psi_x^m\right]}_{=0} + \iint\limits_{t\geq 0} (u-v)\left[\psi_t^m + F_m \psi_x^m\right] =$$

$$= \iint\limits_{t\geq 0} (u-v)\cdot\left[F_m - F\right]\cdot \psi_x^m$$

If $F_m \to F$ locally in L_1

ψ_x^m is bounded (independently of m), then we could pass to the limit and get $=0$.

So our plan is:

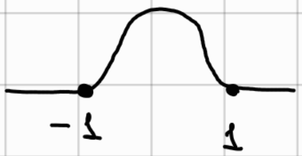
(1) approximate u, v by smooth functions u_m, v_m such that $\left.\begin{array}{l} u_m \to u \\ v_m \to v \\ F_m \to F \end{array}\right\}$ locally in L_1

(2) show that for $\forall\, \varphi \in C_0^1$ there exists $\psi^m \in C_0^1$ – a solution of $\psi^m_t + F_m \psi^m_x = \varphi$ and it's derivative ψ^m_x is bounded (independently of m)

We will use entropy ineq. (E) HERE!

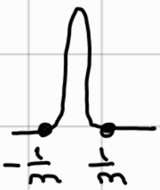
Step (1): One of the classical ideas to get a "smoother" function from any function u is to use convolution with "good kernel".

Consider $\omega(x)$ the standard "hat" function (bump)



$$\omega(x) = \begin{cases} e^{-\frac{1}{|x|^2-1}}, & |x| \leq 1 \\ 0, & |x| \geq 1 \end{cases}$$

$\omega_m = c_m\, m\, \omega(mx)$ is a "hat" on the interval $\left[-\frac{1}{m}, \frac{1}{m}\right]$



Properties:

1) $\omega_m \in C^\infty(\mathbb{R})$ (exercise)

2) $\omega_m \geq 0$, $\operatorname{supp}(\omega_m) = \left[-\frac{1}{m}, \frac{1}{m}\right]$

3) $\int_{\mathbb{R}} \omega_m = 1$

4) $\omega_m \xrightarrow[m\to\infty]{} \delta(x)$

Let $u_m = u * \omega_m$ and $v_m = v * \omega_m$, where

$$(f * g)(y) = \int_{\mathbb{R}} f(x)\, g(y-x)\, dx \quad \text{– convolution}$$

Have in mind such a picture

$f*g$ at point y is just averaging of f in a small neighbourhood of point y.

See 3blue 1brown about convolution

First studied by Kurt Otto Friedrichs (1944) and Sergey Sobolev (1938)

Properties of $u_m = u * \omega_m$:

ⓐ $u \in L^1_{loc} \Rightarrow u_m \in C^\infty$

ⓑ $u_m \to u$ in L^1_{loc}

ⓒ $F_m \to F$ in L^1_{loc}.

Proof:

a)

$$u_m(y) = \int_{\mathbb{R}} u(x)\, \omega_m(y-x)\, dx$$

$$\frac{u_m(y+h) - u_m(y)}{h} = \int_{\mathbb{R}} u(x) \cdot \frac{\omega_m(y+h-x) - \omega_m(y-x)}{h}\, dx$$

$\downarrow$ Lebesgue theorem

$$\int_{\mathbb{R}} u(x) \cdot \frac{\partial}{\partial y} \omega_m(y-x)\, dx \quad \text{etc}$$

and $\omega_m \in C^\infty$

b)

$$u_m - u = \int_{\mathbb{R}} \omega_m(y-x)\,[u(x) - u(y)]\, dx =$$

$$= \int_{\mathbb{R}} \omega_m(z)\,[u(y+z) - u(y)]\, dz =$$

$$= \int_{-\frac{1}{m}}^{1/m} \omega_m(z)\,[u(y+z) - u(y)]\, dz$$

$$\Rightarrow \int_K |u_m - u|\, dy \le \int_K dy \int_{-\frac{1}{m}}^{\frac{1}{m}} \omega_m(z)\,[u(y+z) - u(y)]\, dz$$

K - compact set

$$\le \underbrace{\int_{-\frac{1}{m}}^{1/m} \omega_m(z)\, dz}_{=1} \cdot \underbrace{\sup_{|z| < \frac{1}{m}} \int_K [u(y+z) - u(y)]\, dy}_{\xrightarrow{m \to \infty} 0}$$

$\Rightarrow u_m \to u$ in L^1_{loc}.

c) Write F_m as follows:

$$F_m(x,t) = \frac{f(u_m) - f(v_m)}{u_m - v_m} = \frac{1}{u_m - v_m} \int_{u_m}^{v_m} f'(s)\, ds = \int_0^1 f'(u_m \theta + v_m(1-\theta))\, d\theta$$

Analogously, $F(x,t) = \int_0^1 f'(u\Theta + v(1-\Theta))\,d\Theta$.

Let $c := \max_{|u|\le M} |f''(u)|$. Then

$$F - F_m = \int_0^1 \left[f'(u\Theta + (1-\Theta)v) - f'(u_m\Theta + (1-\Theta)v_m) \right] d\Theta =$$

$$= \int_0^1 f''(\xi) \left[\Theta(u-u_m) + (1-\Theta)(v-v_m) \right] d\Theta, \text{ where}$$

ξ is between $\Theta u + (1-\Theta)v$ and $\Theta u_m + (1-\Theta)v_m$.

Due to estimates $|u|, |v|, |u_m|, |v_m| \le M$, we have $|\xi| \le M$.

Thus,

$$|F(x,t) - F_m(x,t)| \le c \int_0^1 \left[\Theta |u-u_m| + (1-\Theta)|v-v_m| \right] d\Theta \le$$

$$\le c\left(|u-u_m| + |v-v_m| \right)$$

Then for any compact set K in $\{t \ge 0\}$

$$\iint_K |F(x,t) - F_m(x,t)| \le c \underbrace{\iint_K |u-u_m|}_{\downarrow\, 0} + c \cdot \underbrace{\iint_K |v-v_m|}_{\downarrow\, 0} \to 0$$

$\blacksquare$