Spectral asymptotics in some problems with integral constraints

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Problem statement

We are interested in the asymptotics of eigenvalues of the problem

$$(-1)^p u^{(2p)}(t) = \lambda u(t) + \mathcal{P}_{n-2p}(t), \quad t \in [0,1] \tag{1}$$

$$\int_0^1 t^i u(t)\,dt = 0, \quad i = 0 \ldots n-1, \tag{2}$$

where $n, p \in \mathbf{N}$, $n > 2p$, and $\mathcal{P}_{n-2p}(t)$ is a polynomial of degree less than $(n-2p)$ with unknown coefficients.
This problem occurs in the theory of random processes, namely when we study the asymptotic behavior of small deviations for some "detrended" Gaussian processes.

The equivalent problem

Consider the following eigenvalue problem:

$$(-1)^p y^{(2n)}(t) = \lambda y^{(2n-2p)}(t) \tag{3}$$

$$y^{(j)}(0) = y^{(j)}(1) = 0, \quad j = 0 \ldots n-1. \tag{4}$$

Note that the principle eigenvalue of the problem (3)-(4) gives the sharp constant in the embedding theorem $\overset{\circ}{W}{}_2^n(0,1) \hookrightarrow \overset{\circ}{W}{}_2^{n-p}(0,1)$.

Lemma

The eigenvalue problems (1)-(2) *and* (3)-(4) *are equivalent, i.e. have solutions for the same* λ*. Moreover, if* $u(t)$ *is a solution of* (1)-(2) *and* $y(t)$ *is a solution of* (3)-(4)*, then* $u(t) = y^{(n)}(t)$*.*

Problem (3)-(4) was solved by M. Janet for $n \in \mathbb{Z}_+$ and $p = 1$ in 1931. For arbitrary p the answer was only formulated without proof and in implicit terms.

Equation on eigenvalues

Without loss of generality we can assume that the eigenfunction is odd or even (w.r.t. $t = \frac{1}{2}$). If $y(t)$ is an even solution of the eq. (3):

$$(-1)^p y^{(2n)}(t) - \lambda y^{(2n-2p)}(t) = 0,$$

then

$$(-1)^p (y')^{(2n-2)}(t) - \lambda (y')^{(2n-2-2p)}(t) = C$$

and the constant $C = 0$, as the left hand side is odd.
So eigenvalue, corresponding to even solution of the equation (3) with parameters (n, p), equals to an eigenvalue, corresponding to odd solution of the equation (3) with parameters $(n-1, p)$. That's why we can restrict ourselves to consider only odd solutions.

Every odd solution of the equation (3) is of the form:

$$y = a_0 \sin \xi_0(2t-1) + a_1 \sin \xi_1(2t-1) + \ldots + a_{p-1} \sin \xi_{p-1}(2t-1) + \\ + a_p(2t-1) + \ldots + a_{n-1}(2t-1)^{2n-2p-1},$$

here $\xi_k = \frac{1}{2}|\lambda|^{\frac{1}{2p}} e^{\frac{ik\pi}{p}}$, $k = 0 \ldots p-1$.
Substituting $y(t)$ into the boundary conditions (4), we get the equation $\Delta_{n,p}(\lambda) = 0$, where $\Delta_{n,p}(\lambda)$ is some determinant.
$\Delta_{n,p}$ as a function of $\xi_0, \ldots, \xi_{p-1}$ satisfies such an equation:

$$\frac{\partial^p}{\partial \xi_0 \ldots \partial \xi_{p-1}} \Delta_{n,p} = C \cdot \xi_0 \cdot \ldots \cdot \xi_{p-1} \cdot \Delta_{n-1,p}. \tag{5}$$

And for $n = p$

$$\Delta_{p,p} = C \begin{vmatrix} \xi_0^{1/2} J_{1/2}(\xi_0) & \ldots & \xi_{p-1}^{1/2} J_{1/2}(\xi_{p-1}) \\ \xi_0^{3/2} J_{3/2}(\xi_0) & \ldots & \xi_{p-1}^{3/2} J_{3/2}(\xi_{p-1}) \\ \ldots & \ldots & \ldots \\ \xi_0^{(2p-1)/2} J_{(2p-1)/2}(\xi_0) & \ldots & \xi_{p-1}^{(2p-1)/2} J_{(2p-1)/2}(\xi_{p-1}) \end{vmatrix}$$

Here $J_k(x)$ are Bessel functions of the first kind.

Equation on eigenvalues

Using relation (5) we get the following representation for $\Delta_{n,p}(\lambda)$

$$\begin{vmatrix} \xi_0^{(2n-2p+1)/2} J_{(2n-2p+1)/2}(\xi_0) & \dots & \xi_{p-1}^{(2n-2p+1)/2} J_{(2n-2p+1)/2}(\xi_{p-1}) \\ \xi_0^{(2n-2p+3)/2} J_{(2n-2p+3)/2}(\xi_0) & \dots & \xi_{p-1}^{(2n-2p+3)/2} J_{(2n-2p+3)/2}(\xi_{p-1}) \\ \dots & \dots & \dots \\ \xi_0^{(2n-1)/2} J_{(2n-1)/2}(\xi_0) & \dots & \xi_{p-1}^{(2n-1)/2} J_{(2n-1)/2}(\xi_{p-1}) \end{vmatrix}$$

The final equation will be of the form

$$\Delta_{n,p}(\lambda) \cdot \Delta_{n-1,p}(\lambda) = 0.$$

Using asymptotics of Bessel functions we get the asymptotics

$$\lambda_k = \left(\pi k + \frac{2n-p-1}{2} + O\Big(\frac{1}{k}\Big)\right)^{2p}. \qquad (6)$$

Application to small ball probabilities

We apply the asymptotic formula (6) to calculate sharp L_2-small ball asymptotics as $\varepsilon \to 0$ of $\mathbb{P}\{\|X_n(t)\|_{L_2[0,1]} < \varepsilon\}$ for Gaussian process

$$X_n(t) := X(t) - \sum_{i=0}^{n-1} a_i t^i,$$

where a_i are determined by relations

$$\int_0^1 t^i X_n(t)\, dt = 0,\ i = 0 \dots n-1.$$

Here $X(t)$, $t \in [0,1]$, is a Gaussian process, $\mathbb{E}X = 0$, covariance function $G(s,t) = \mathbb{E}X(s)X(t)$ is the Green function for a boundary value problem:

$$Lu := (-1)^p u^{(2p)} = \lambda u \ + \ \text{some boundary conditions.}$$

Known results

The case $n = 1$ corresponds to the so-called centered process. Our problem was considered earlier in the following cases:

Case	Process	Author
$n = 1$ $p = 1$	centered Brownian bridge and Brownian motion	2005 — E. Orsingher, Ya. Nikitin 2006 — P. Deheuvels
$n = 2$ $p = 1$	detrended Brownian motion	2012 — X. Ai, W. Li
$\forall n$ $p = 1$	m-th order detrended Brownian motion	2014 — X. Ai, W. Li

We deal with arbitrary $n, p \in \mathbb{N}$ under the assumption $n > 2p$. In this case the process X_n does not depend on the original boundary conditions.

The Karhunen–Loève (KL) expansion:

$$X_n(t) \stackrel{d}{=} \sum_{k\geq 1} \xi_k \sqrt{\mu_k} y_k(t),$$

here ξ_k is a sequence of i.i.d. $N(0,1)$ random variables, μ_k are eigenvalues and $y_k(t)$ are eigenfunctions of the integral operator with kernel $G_n(s,t)$ — the covariance function of X_n, that is:

$$\mu_k u(t) = \int_0^1 u(s) G_n(s,t)\, ds. \tag{7}$$

So

$$\|X_n\|_2^2 = \int\limits_0^1 X_n^2(t)\, dt \stackrel{d}{=} \sum_{k=1}^{\infty} \mu_k \xi_k^2.$$

Therefore the problem can be formulated as follows:

$$\text{Find: } \mathbb{P}\Big\{ \sum_{k=1}^{\infty} \mu_k \xi_k^2 < \varepsilon^2 \Big\} \text{ as } \varepsilon \to 0.$$

The Wenbo Li principle (Li 1992, Gao et al 2003)

Let $X(t)$, $\tilde{X}(t)$ be two Gaussian processes with zero mean and covariance functions $G(s,t)$ and $\tilde{G}(s,t)$. Let μ_k and $\tilde{\mu}_k$ be positive eigenvalues of integral operators with kernels $G(s,t)$ and $\tilde{G}(s,t)$, respectively. If $\prod \tilde{\mu}_k/\mu_k < \infty$ then

$$\mathbb{P}\Big\{\|X\|_2 < \varepsilon\Big\} \sim \mathbb{P}\Big\{\|\tilde{X}\|_2 < \varepsilon\Big\} \cdot \left(\prod_{k=1}^{\infty} \frac{\tilde{\mu}_k}{\mu_k}\right)^{1/2}, \qquad \varepsilon \to 0. \quad (8)$$

Application to small ball probabilities

$$\mathbb{P}\Big\{\|X_n\|_2 < \varepsilon\Big\} \sim C\varepsilon^{\gamma} \exp\Big(-\frac{2p-1}{2(2p\sin(\frac{\pi}{2p}))^{\frac{2p}{2p-1}}}\varepsilon^{-\frac{2}{2p-1}}\Big),$$

where $\gamma = \dfrac{1-2np+p^2}{2p-1}$ and

$$C = \frac{(2p)^{1+\frac{\gamma}{2}+\frac{p}{2}} \cdot \pi^{\frac{p-1}{2}} \cdot \sin^{\frac{1+\gamma}{2}}(\frac{\pi}{2p})}{2^{p(2n-p-\frac{1}{2})}\sqrt{2p-1} \cdot \mathfrak{V}[1, e^{\frac{i\pi}{p}}, \dots, e^{\frac{i\pi(p-1)}{p}}]} \cdot \frac{\Gamma^{-\frac{1}{2}}\left(n-p+\frac{1}{2}\right)\Gamma^{-\frac{1}{2}}\left(n+\frac{1}{2}\right)}{\prod_{j=1}^{p-1}\Gamma(n-p+j+\frac{1}{2})}.$$

Here $\mathfrak{V}[x_0, \dots, x_{p-1}]$ is Vandermonde determinant.

Thank you for your attention!