Questions for exam on the course: "Shock waves in conservation laws and reaction-diffusion equations"

Last update: 02/05/2023.

Part 1: Around wave equation.

1. Wave equation: "physical" derivation (balls and springs).
2. Wave equation: derivation from general principles.
3. D'Alambert's formula for 1D wave equation, and well-posedness of Cauchy problem on real line.
4. Inhomogeneous wave equation. Duhamel principle.
5. Mixed initial-boundary value problem for wave equation: existence and uniqueness of solution.
6. Mixed initial-boundary value problem for wave equation: solution by a Fourier series.

Part 2: Conservation and balance laws.

7. Fluid flow: Eulerian vs. Lagrangian point of view; flow map; incompressibility condition.
8. Fluid flow: scalar transport equation, conservation of mass.
9. Scalar conservation law. Weak form of solution. Rankine-Hugoniot condition.
10. Burgers equation: blow-up in finite time, explicit solutions to different Riemann problems, multiplicity of solutions, definition of entropy solution, irreversibility.
11. Scalar conservation law with convex flux function: various interpretations of entropy condition (Lax, Liu, vanishing viscosity).
12. Scalar conservation law with convex flux function: theorem on existence of entropy solution. Lemmas 1 and 2 describing properties for discrete approximation (boundedness, entropy condition).
13. Scalar conservation law with convex flux function: theorem on existence of entropy solution. Lemmas 3, 4 and 5 describing properties for discrete approximation (space and time estimates, stability).
14. Scalar conservation law with convex flux function: theorem on existence of entropy solution. Lemma 6 on convergence and properties of the limiting solution.
15. Scalar conservation law with convex flux function: theorem on existence of entropy solution. Lemmas 7 and 8 on properties of the limiting solution.
16. Scalar conservation law with convex flux function: uniqueness of entropy solution. General plan of proof without technical details.
17. Scalar conservation law with convex flux function: uniqueness of entropy solution. Proof that $|\psi_x^m|$ is bounded using the entropy condition.
18. Scalar conservation law with convex flux function: solution to a Riemann problem for two cases ($u_l < u_r$ and $u_l > u_r$).
19. Systems of conservation laws: weak solution, Rankine–Hugoniot condition, notion of hyperbolic and strictly hyperbolic systems, examples.
20. Systems of conservation laws: notion of genuinely nonlinear and linearly degenerate characteristic family; simple waves. Theorem on existence of k-rarefaction wave.
21. Systems of conservation laws: notion of shock curves (Hugoniot locus). Theorem on structure of shock waves (property (iii) without proof). Notion of Lax admissibility criteria for shocks.
22. Systems of conservation laws: notion of k-contact discontinuity. Theorem on linear degeneracy (shock and rarefaction curves coincide). Example (linear wave equation).
23. Systems of conservation laws: theorem on local solvability of a Riemann problem for strictly hyperbolic systems (each characteristic family is genuinely nonlinear or linearly degenerate).