Mulheres at IMPA: fluid dynamics session

Yulia Petrova [1,2]

[1] IMPA, Instituto de Matematica
Pura e Aplicada, Rio de Janeiro, Brazil

[2] St Petersburg State University,
Chebyshev Lab, Russia

https://yulia-petrova.github.io/

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Two words about me...

- I grew up in a small city Ukhta in the north of Russia, ≈ 100 thousand citizens
- At school participated and sometimes wined various town and republic Olympiads: maths, physics, chemistry, informatics
- At 17 years moved to St Petersburg, Russia, to study at University

Professional experience

2008–2013 | MSc in Mathematics, SPbSU, Russia. **Department of PDEs**

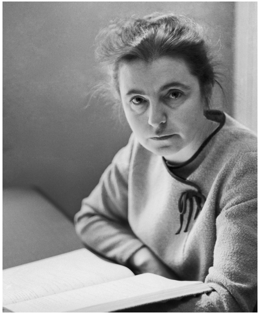
2013–2018 | PhD, SPbSU, Russia. "Exact L_2-small ball asymptotics for finite-dimensional perturbations of Gaussian processes". **Probability theory & spectral theory**

2017–2021 | Researcher at Chebyshev Laboratory, SPbSU, Russia. Participant of industrial projects with petroleum company "Gazprom-Neft". **Fluid dynamics**

2021–2023 | Postdoc at IMPA. **Fluid dynamics.** Working with Prof. Dan Marchesin.

Great women in maths from Russia


Sofia Kovalevskaya


Olga Ladyzhenskaya


Nina Uraltseva


Olga Oleinik

NB: biographies of women mathematicians:
https://mathwomen.agnesscott.org/women/women.htm

Motion of fluids inspires..!

- Euler equation (1755): ideal fluid

$$\begin{aligned} \partial_t u + (u \cdot \nabla) u &= \nabla p, \\ \nabla \cdot u &= 0. \end{aligned}$$

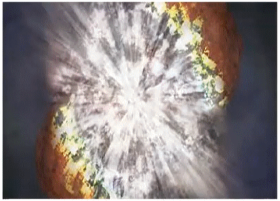
- Navier-Stokes equation (1845): adds viscosity

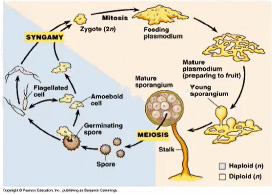
$$\partial_t u + (u \cdot \nabla) u - \nu \Delta u = \nabla p$$

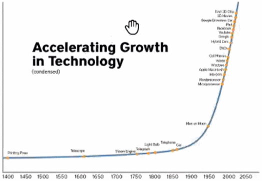
See more photos: Van Dyke "Album of fluid motion"

Fluid dynamics (PDEs) often deal with “singularities”

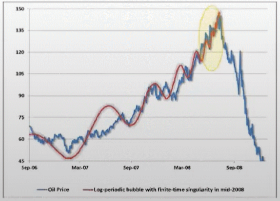
Dramatic phenomena. Explosion, catastrophe, abrupt change.





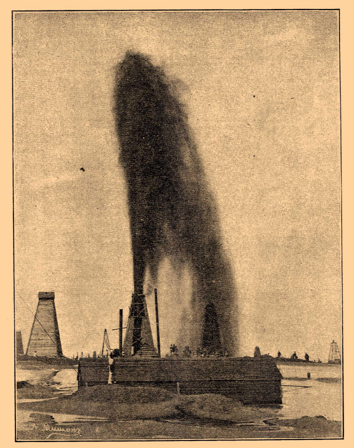


A shock wave, a transition from laminar flow to turbulence, or stock market crashes can also be linked to singularities in modelling equations.



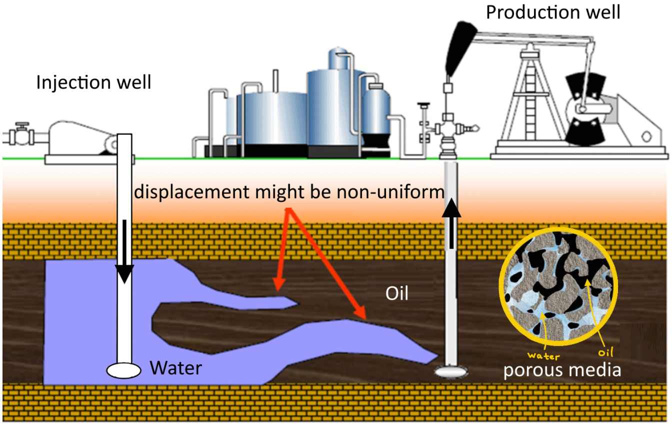
I am interested in mathematics of oil recovery

How oil was recovered in the beginning? (Baku, 1857)

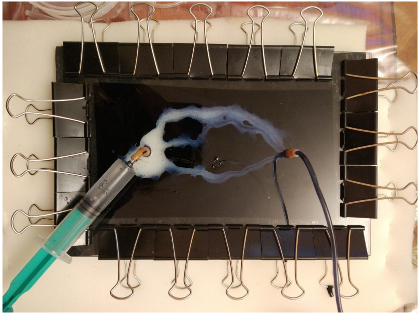


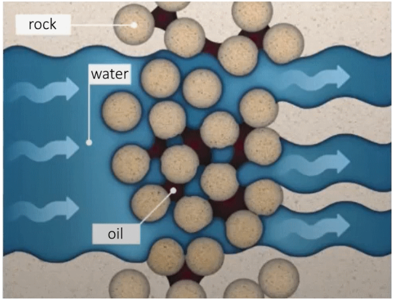
Нефтяной фонтанъ Горнаго товарищества, бившій въ сентябрѣ 1887 г. (Балаханы у Баку).

How oil is recovered now?



Problems: macroscopic and microscopic sweep efficiency





- happens due to very viscous oil or inhomogeneous media

- local entrapment of oil in pores due to high capillary pressure

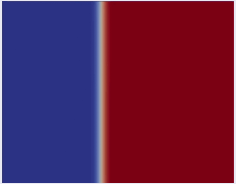
Possible solution

- Inject gas (CO_2, natural) to decrease the oil viscosity
- Add chemicals (polymer) to increase the water viscosity
- Add chemicals (surfactant) that reduce the surface tension etc

Fundamental research: two main directions

1-dim in spatial variable

- Stable displacement



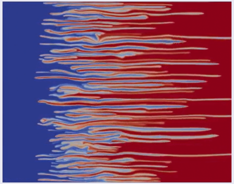
- main question: find an exact solution to a Riemann problem
- hyperbolic conservation laws

$$
\begin{aligned}
s_t + f(s,c)_x &= 0,\\
(cs + a(c))_t + (cf(s,c))_x &= 0.
\end{aligned}
$$

Example: chemical flooding model

2-dim (or 3-dim) in spatial variable

- Unstable displacement



- source of instability: water and oil/polymer have different viscosities
- viscous fingering phenomenon

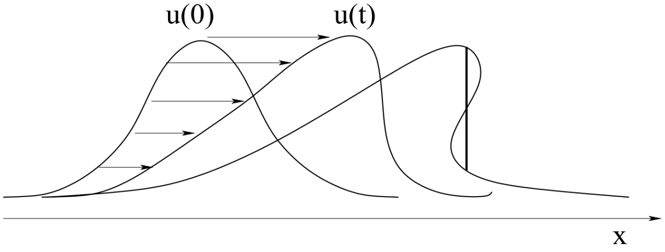
$$
\begin{aligned}
c_t + u \cdot \nabla c &= \varepsilon \Delta c,\\
\mathrm{div}(u) &= 0,\\
u &= -\nabla p / \mu(c).
\end{aligned}
$$

Example: Peaceman model

Hyperbolic systems of conservation laws

$$u_t + \left(\frac{u^2}{2}\right)_x = 0$$ (Burger's equation, 1948)

- non-linearity implies wave speed $\lambda(u) = u$ depends on state u
- So the wave can spread (rarefaction wave) or concentrate (shock wave)



$$u_t + (f(u))_x = 0$$ (Buckley-Leverett equation, water flooding)

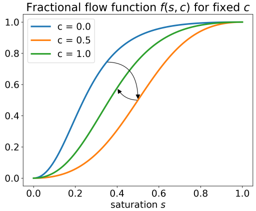
- existence, uniqueness was establiblished by Olga Oleinik (1957)

Problem statement

Chemical flooding can be described as the system of conservation laws ($x \in \mathbb{R}, t > 0$):

$$\begin{aligned} s_t + f(s,c)_x &= 0, \qquad \text{(conservation of water)} \\ (cs + a(c))_t + (cf(s,c))_x &= 0. \qquad \text{(conservation of chemical)} \end{aligned} \tag{1}$$

- $s = s(x,t)$ — water phase saturation;
- $f(s,c)$ — fractional flow function (usually S-shaped);
- $c = c(x,t)$ — concentration of a chemical agent in water;
- $a(c)$ — adsorption of a chemical agent on a rock (usually increasing, concave).



Initial data:
$$(s,c)\big|_{t=0} = \begin{cases} (1,1), & \text{if } x \leqslant 0, \\ (0,0), & \text{if } x > 0, \end{cases} \tag{2}$$

Aim:

Find a solution to initial-value problem (1)–(2) when f depends non-monotonically on c.

NB: Non-monotone dependence appears in surfactant flooding, low salinity water flooding etc

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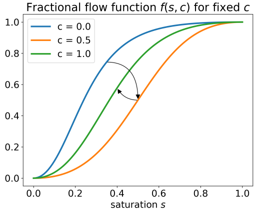

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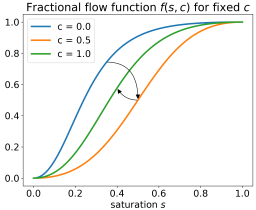

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Aim:

Find a solution to initial-value problem (1)–(2) when f depends non-monotonically on c.

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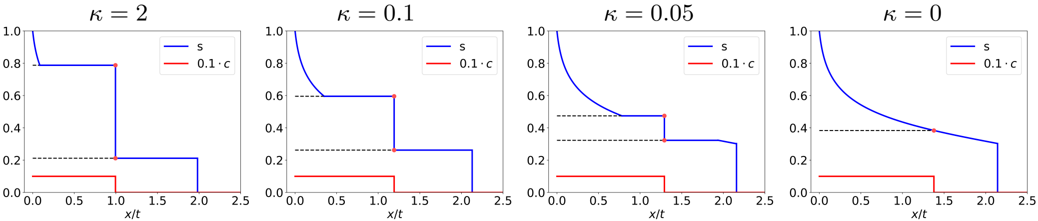
Result

- weak solutions $\Rightarrow$ non-uniqueness of solutions to a Riemann problem
- use vanishing viscosity criterion — add small diffusion/capillary terms

$$s_t + f(s,c)_x = \varepsilon_c (A(s,c) s_x)_x,$$
$$(cs + a(c))_t + (cf(s,c))_x = \varepsilon_c (cA(s,c)s_x)_x + \varepsilon_d c_{xx}.$$

- $f(s,c)$ monotone in $c \Rightarrow$ uniqueness of vanishing viscosity solution (1988)
- Interesting idea: $f(s,c)$ non-monotone in $c \Rightarrow$ exist multiple vanishing viscosity solutions, depending on ratio $\kappa = \varepsilon_d / \varepsilon_c$



NB: the appeared shock is known as undercompressive (transitional).

Undercompressive shocks

- Appear in various contexts:
  - three-phase flows
  - thin film theory
- depend heavily on the diffusion terms
- take solution in the form of a travelling wave

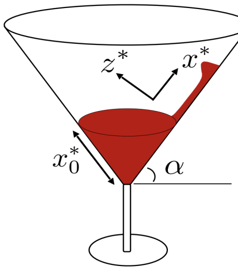
$$s = s(\xi) = s(x - vt) \qquad c = c(\xi) = c(x - vt)$$

get the dynamical system on (s, c)

$$\boxed{\begin{aligned} s_\xi &= f(s,c) - v(s + d_1), \\ c_\xi &= v(d_1 c - d_2 - a(c))/\kappa. \end{aligned}}$$

Undercompressive shocks correspond to saddle-to-saddle heteroclinic orbits

"Tears of wine"

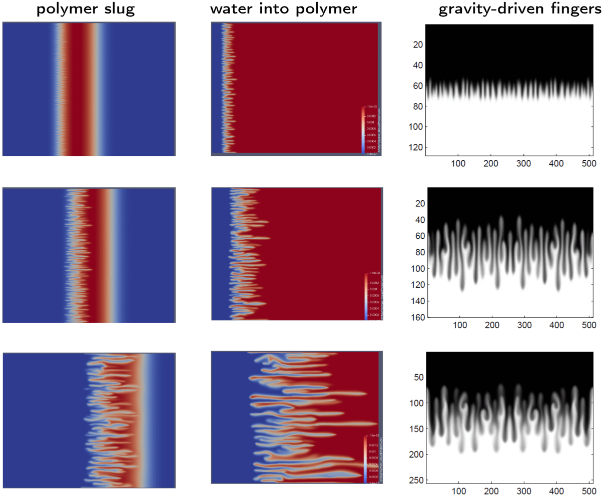


Two-dimensional problems

Show video

2d problems: motivation and main question

- water flooding;
- chemical flooding;
- cause problems for oil recovery.



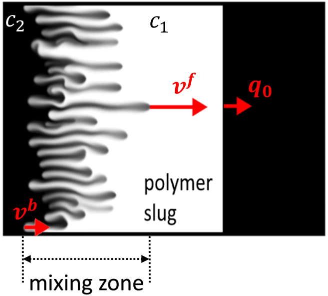
Main questions of interest:

1. what are rigorous bounds on velocities of mixing zone propagation?
2. how to calculate the optimal size of the polymer slug?

Velocities of the mixing zone

One-phase miscible displacement
(Peaceman model)

$$
\begin{aligned}
c_t + u \cdot \nabla c &= \varepsilon \Delta c, \\
\mathrm{div}(u) &= 0, \\
u &= -\frac{\nabla p}{\mu(c)}.
\end{aligned}
$$



Numerically and experimentally we observe:

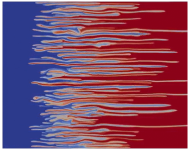
- q_0 — velocity of the stable front
- v_f — velocity of the front end of the mixing zone is constant
- v_b — velocity of the back end of the mixing zone is constant

Open problem: rigorously prove the linear growth of the mixing zone

Toy model of viscous fingering (work in progress)

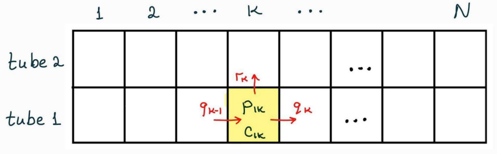
Discrete case

- system of $2N$ ODEs and N algebraic equations



Unknowns:

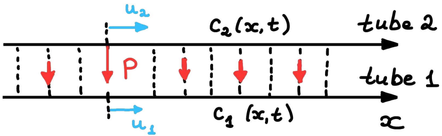
- $c_{1k}(t), c_{2k}(t)$ — concentrations
- p_{1k}, p_{2k} — pressures
- $q_k(t), r_k(t)$ — velocities



Continuous case

- two coupled advection-diffusion eqs

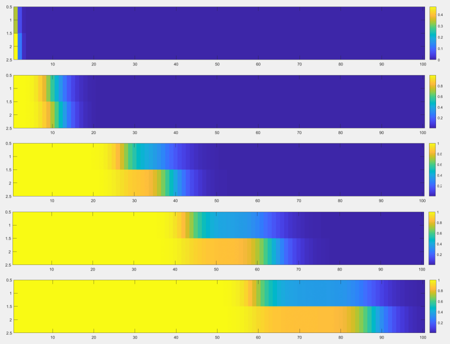
$$\partial_t c_1 = -\partial_x(u_1 c_1) + \partial_x u_{1,2} \cdot c_{1,2} + \varepsilon \partial_{xx} c_1,$$
$$\partial_t c_2 = -\partial_x(u_2 c_2) - \partial_x u_{1,2} \cdot c_{1,2} + \varepsilon \partial_{xx} c_2.$$



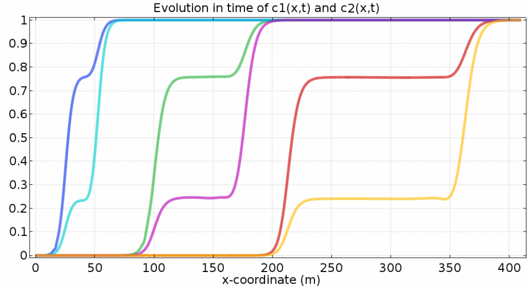
Work in progress with S. Tikhomirov, Y. Efendiev.

Toy model of viscous fingering: numerical experiments

Discrete setting



Continuous setting



Result of experiments: cascade of two travelling waves (TW)

$$(0,0) \xrightarrow{\text{TW}_1} (c_1^*, c_2^*) \xrightarrow{\text{TW}_2} (1,1)$$

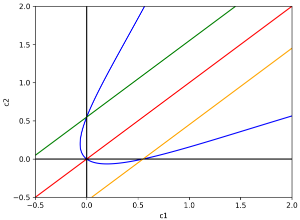
Toy model of viscous fingering: theoretical approach

For a travelling wave $c_1 = c_1(x - vt)$ and $c_2 = c_2(x - vt)$ we have a dynamical system.

Travelling wave dynamical system

$$\begin{aligned} c_1' &= g, \\ g' &= (u - v)g, \\ c_2' &= (u - v)c_1 + (2 - u - v)c_2 - g. \end{aligned}$$

Here $u = u(c_1, c_2)$, depends on $\mu(c_1)$ and $\mu(c_2)$.



Want to prove

For all $v \in \mathbb{R}_+$ there exists a unique point (c_1^*, c_2^*): there exists a trajectory $U = U(\xi)$:

$$U(-\infty) = (0, 0, 0) \qquad U(+\infty) = (c_1^*, 0, c_2^*)$$

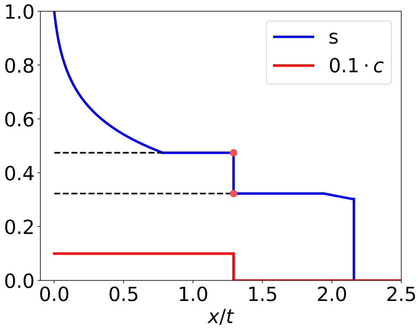
How to quantify the dependence $c_1^* = c_1^*(v)$ and $c_2^* = c_2^*(v)$?

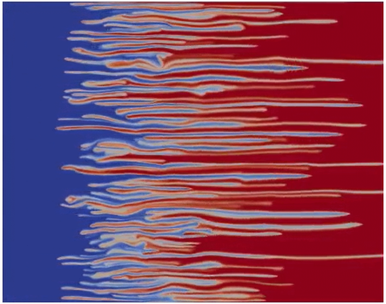
Here $U = (c_1, g, c_2)$.

Thank you for your attention!

yulia.petrova@impa.br

https://yulia-petrova.github.io/





PEACE for Russia and Ukraine (and world)

These mathematicians will never prove a theorem because of the war...

Yuliia Zdanovska (Kiev)

Konstantin Olmezov (MIPT, Moscow)

Many Ukrainian mathematicians are under bomb attacks in Ukraine.
Many Russian mathematicians are under political pressure in Russia.