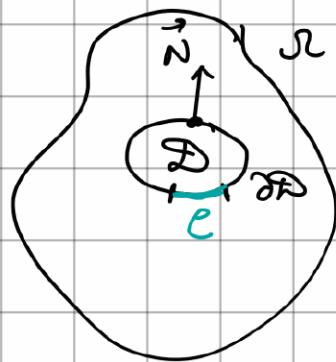


## Lecture 5: Conservation & Balance laws

Plan:

1. General definition
2. Example 1: fluid dyn (conservation of mass)
3. Example 2: scalar conservation law

### (1) Balance law



$D \subset \mathbb{R}$  with Lipschitz boundary (smooth)

$N$  - normal vector towards the exterior of the domain  $D$

$$\text{[production in } D] = \text{flux through the boundary of } \partial D$$

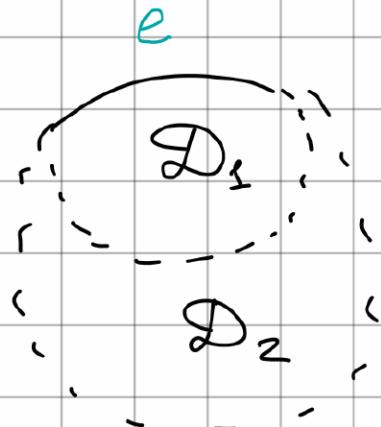
- production in  $D$  is some measure (Radon)  $P$

- flux

$$F_D(e) = \int_E q_D(x) dS(x)$$

$$P(D) = \int_{\partial D} q_D(x) dS(x) \quad (*)$$

Assume:



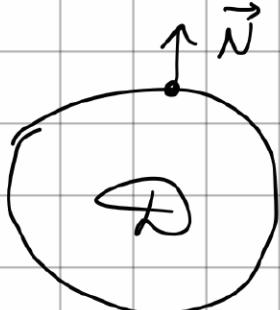
$$q_{D_1}(x) = q_{D_2}(x)$$

$$\forall x \in E$$

Take-home  
(Tuesday)

$$\boxed{\operatorname{div} A = 0}$$

Miracles : (1)  
Consequences of (\*) :



$$\exists a_{\mathcal{D}}(x) = q_{\mathcal{D}}(x)$$

$\forall x \in \mathcal{R}$

for any  $\mathcal{D} \subset \mathcal{R}$  s.t.

$\mathcal{D}$  has  $\vec{N}$  as a normal vector at  $x$ .

$$(2) \quad \exists \vec{A}(x) : \mathcal{R} \rightarrow \mathbb{R}^d :$$

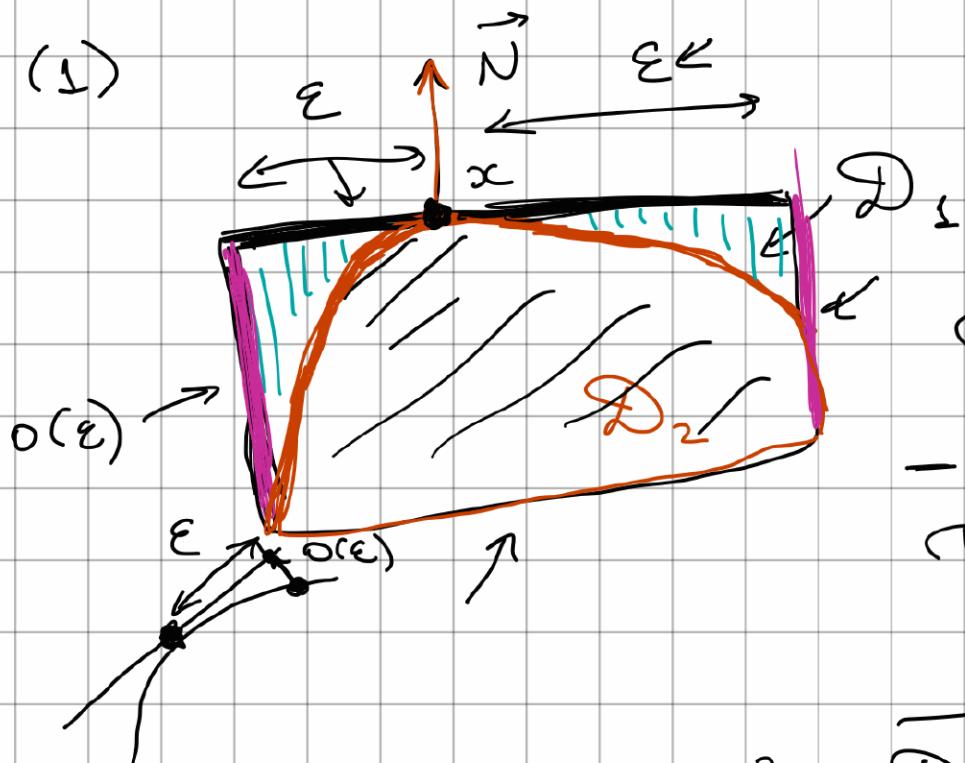
$$a_{\vec{N}}(x) = \vec{A}(x) \cdot \vec{N}$$

$$(3) \quad \exists \text{PDE} : \operatorname{div} \vec{A} = P$$

$$P(\mathcal{D}) = \int_{\partial \mathcal{D}} q_{\mathcal{D}}(x) dS(x)$$

$$\frac{\text{"}}{\vec{A}(x) \cdot \vec{N}}$$

(1)



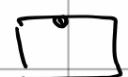
$$\epsilon \rightarrow 0$$

?

$$q_{\mathcal{D}_1}(x) = q_{\mathcal{D}_2}(x)$$

(\*)

$$P(\mathcal{D}_1) = \int q_{\mathcal{D}_1}(x) dS$$



$$P(\mathcal{D}_2) = \int q_{\mathcal{D}_2}(x) dS$$



$$\int \sim O(\epsilon)$$

||

$$\epsilon^2 \sim \overline{P(\mathcal{D}_1)} = \int q_{\mathcal{D}_1}(x) dS$$

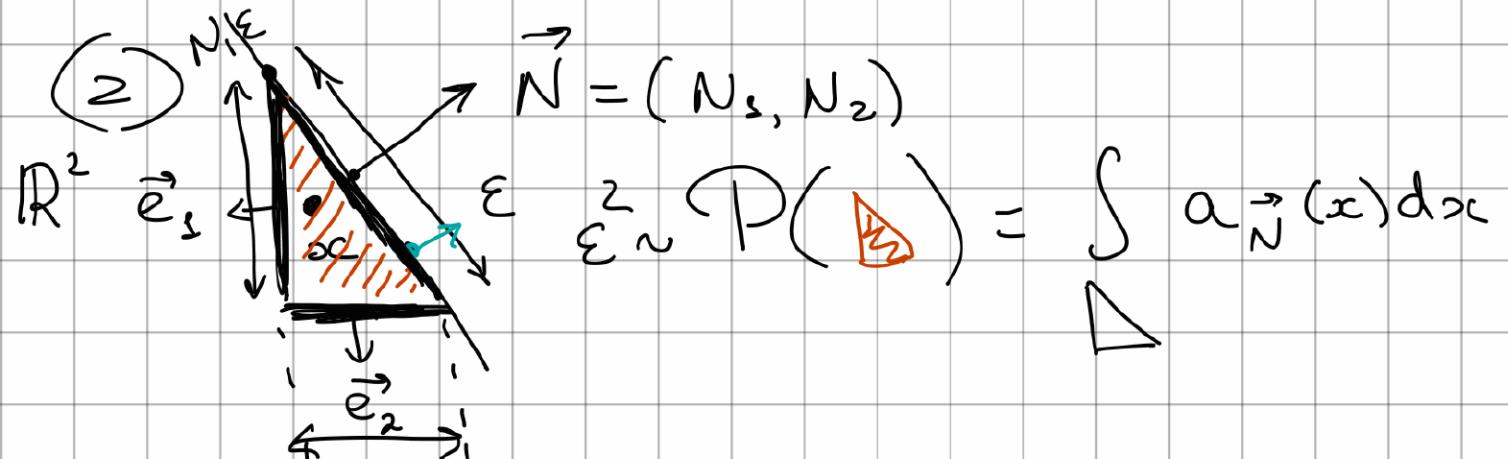


$$\Rightarrow \underline{\int q_{\mathcal{D}_1}(x) dS(x)} = \int q_{\mathcal{D}_2}(x) dS(x)$$

$$- \int q_{\mathcal{D}_2}(x) dS$$



$$\Rightarrow \exists a_{\vec{N}}(x) = q_{\vec{N}}(x) \quad \underline{\text{Cauchy}}$$



$$a_{\vec{N}}(x) \cdot \underline{\epsilon} = a_{e_1}(x) \cdot \underline{N_1 \epsilon} +$$

$$+ a_{e_2}(x) \cdot \underline{N_2 \epsilon}$$

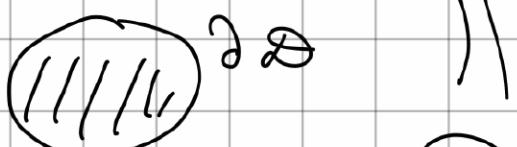
$$\Rightarrow a_{\vec{N}}(x) = a_{e_1}(x) N_1 + a_{e_2}(x) N_2$$

$$\Rightarrow a_{\vec{N}}(x) = \vec{A}(x) \cdot \vec{N}$$

$$A(x) = (a_{e_1}, a_{e_2})$$

$$(3) \int_{\partial D} \vec{A}(x) \cdot \vec{N} dS(x) = \int_D \text{div}(\vec{A}) dx$$

$\uparrow$  Green-Gauss theorem



$$P = \int_D p(x) dx$$

$$\Rightarrow \text{div}(\vec{A}) = P \quad - \text{balance law}$$

$$\boxed{\text{div}(\vec{A}) = 0} \quad - \text{conservation law}$$

# Dafermos

Example 1 : Fluid flow, continuum mechanics

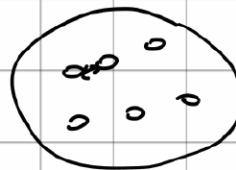
different scales

1. atoms / molecules

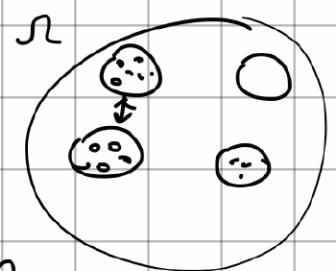
o

2. representative

small volume



3. domain  
(macroscale)



- Eulerian vs. Lagrangian point of view

Eulerian :  $(x, t) \mapsto x$

- velocity :  $u(x, t) = (u_1, \dots, u_d) : \mathbb{R}^d \times [0, \infty) \rightarrow \mathbb{R}^d$   
has units  $\left[ \frac{\text{L}}{\text{T}} \right]_{\mathbb{R}^d}$
- density :  $\rho(x, t) : \mathbb{R}^d \times [0, \infty) \rightarrow \mathbb{R}$   
with units  $\left[ \frac{\text{M}}{\text{L}^d} \right]$
- pressure :  $p(x, t) : \mathbb{R}^d \times [0, \infty) \rightarrow \mathbb{R}$   
with units  $\left[ \text{ML}^{-d+2} \text{T}^{-2} \right]$

Lagrangian : particles,  $a \in \mathbb{R}^d$   
trajectories of particles

flow map

$X(t, a) = (X_1, \dots, X_d)$  - position  
of particle  $a$  at time  $t$

$$(**) \left\{ \begin{array}{l} \partial_t X(t, a) = u(t, X(t, a)) \\ X(0, a) = a \end{array} \right. \leftarrow \text{ODE}$$

ODE theory (Cauchy-Lipschitz theorem) :

$u \in C_t \text{Lip}_x \Rightarrow \exists! \text{ solution to } (**)$

$X(t, \cdot)$  - is  $C^1$ -diffeo :  $\mathbb{R}^d \rightarrow \mathbb{R}^d$

Define inverse:  $A(t, x) : [0, +\infty) \times \mathbb{R}^d \rightarrow \mathbb{R}^d$

$$A(t, X(t, a)) = a \quad X(t, A(t, a)) = a$$

$\forall x, a \in \mathbb{R}^d$

"back-to-labels" map ( $a$  - "labels")

Incompressibility condition: " $\operatorname{div} u = 0$ "

Take  $V \subset \mathbb{R}$  - volume of fluid

$$V(t) = X(t, V) = \{X(t, a) : a \in V\}$$



Def.: velocity field is called incompressible if

$$\rightarrow |V(t)| = |V| \quad \uparrow \text{Lebesgue measure of } V$$

Lemma:  $u \in C_t \operatorname{Lip}_x$

$u$  is incompressible  $\Leftrightarrow \operatorname{div} u = 0$  ( $u$  is divergence-free)

Proof:



$$V(t) = \int_{V(t)} \mathbf{f} \cdot d\mathbf{x} ; \quad a \in V \subset \mathbb{R}^d$$

$$\int_{V(t)} \mathbf{f}(x, t) d\mathbf{x} = \int_V \mathbf{f}(X(t, a), t) \cdot \det(\nabla_a X) da$$

$$\rightarrow J(t, a) = \sum_{i_1, \dots, i_d=1}^d \varepsilon_{i_1 \dots i_d} \frac{\partial X_{i_1}}{\partial a_1} \dots \frac{\partial X_{i_d}}{\partial a_d}$$

$$\underline{\text{Exercise: }} \partial_t J(t, a) = J(t, a) \cdot (\operatorname{div} a)(t, X(t, a))$$

Corollary :  $J(t, \alpha) = \underline{1} \Leftrightarrow \frac{\operatorname{div}(u)}{\int_0^t (J(s, \alpha)) ds} = 0$

 $\rightarrow J(t, \alpha) = J(0, \alpha) \cdot e^{\int_0^t \operatorname{div}(u) ds}$

$$J(0, \alpha) = \underline{1}$$

$$J(t, \alpha) = J(0, \alpha) \quad \forall t \Rightarrow \operatorname{div}(u) = 0$$

$$V(t) = \int_{V(t)} \underline{d}x = \int_V J(t, \alpha) d\alpha = \int_V d\alpha = V$$

L

$$\text{if } \operatorname{div}(u) = 0$$

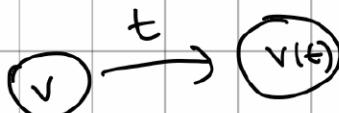
## Transport equation

Let's  $f(t, x) : [0, \infty) \times \mathbb{R}^d \rightarrow \mathbb{R}$  - scalar

Eulerian:  $\partial_t f$  - change of  $f$  at  $(t, x)$

Lagrangian:  $\partial_t f(t, X(t, \alpha)) =: D_t f$  - convective derivative

$$\partial_t f + u \cdot \nabla f$$



Thm (transport thm):

$u$  - velocity field,  $u \in C^1$ ;  $f$  be  $C^1$

$V(t)$  is pushforward of  $V$  by the flow map  $X(t, \alpha)$

$$\frac{d}{dt} \left( \int_{V(t)} f(x, t) dx \right) = \int_V (\partial_t f + \operatorname{div}(fu))(t, x) dx$$

Proof:

$$\int_{V(t)} f(x, t) dx = \int_V f(X(t, \alpha), t) \underline{J(t, \alpha) da}$$

$$\begin{aligned}
 \frac{d}{dt} \left( \underset{V(t)}{\int} f(x,t) dx \right) &= \underset{V}{\int} (\partial_t f)(x(t,a), t) J(t,a) \\
 &\quad + \underset{V}{\int} f(x(t,a), t) \cdot \partial_t J(t,a) da = \\
 &= \underset{V}{\int} \left( \partial_t f + \underbrace{u \cdot \nabla f + f \cdot \operatorname{div}(u)}_{\operatorname{div}(uf)} \right) (x(t,a), t) \cdot J(t,a) da \\
 &= \underset{V(t)}{\int} \left( \partial_t f + \operatorname{div}(fu) \right) dx \quad \blacksquare
 \end{aligned}$$

Conservation of mass:  $g(x,t)$

$$m(t, V) = \underset{V}{\int} g(x,t) dx$$

$$\frac{d}{dt} m(t, V(t)) = 0$$

Thm: conservation of mass is equivalent to the following integral eq:

$$\underset{V(t)}{\int} (g_t + \operatorname{div}(gu)) dx = 0$$

If  $g_t$  and  $\operatorname{div}(gu)$  are C, then

$$g_t + \operatorname{div}(gu) = 0$$

Remark: scalar transport eq

Proof:  $\int_0^t = \frac{d}{dt} m(t, X(t, a)) = \int_{\mathbb{R}^n} g(x, t) dx$

$$= \int_{V(t)} \underbrace{(g_t + \operatorname{div}(gu))}_{\text{are continuous}} dx.$$

$$f(x) = \lim_{\varepsilon \rightarrow 0} \frac{1}{|B_\varepsilon(x)|} \int_{B_\varepsilon(x)} f(y) dy$$

$\Rightarrow g_t + \operatorname{div}(gu) = 0.$  ■

Remark:  $0 = g_t + \operatorname{div}(gu) = g_t + u \cdot \nabla g + \operatorname{div}(u)g$

incompressibility  $\Rightarrow \operatorname{div}(u) = 0$  □

Next time

$$g_t + (gu)_x = 0$$

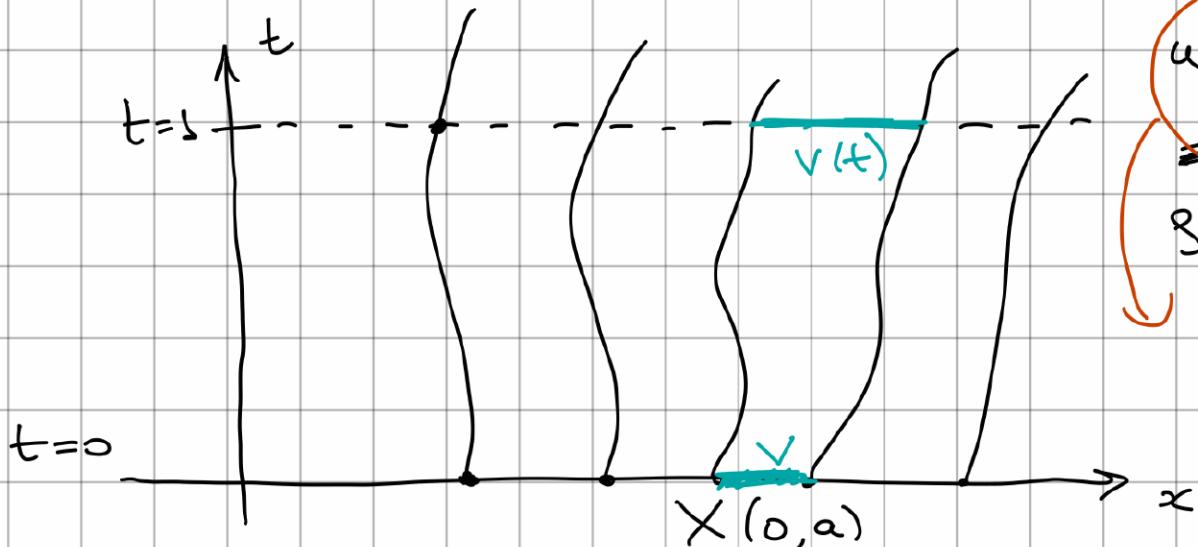
$$D_t g = g_t + u \cdot \nabla g = 0$$

$$u = u(g)$$

$$u(g) = \frac{g}{2} \Rightarrow \text{Burgers}$$

$$\frac{d}{dt} (g(t, X(t, a))) = 0$$

$$\Rightarrow g(t, X(t, a)) = \text{const.} \quad \leftarrow$$



## Lecture 6

Last time: Balance laws:  $\operatorname{div} A = P$

Conservation laws:  $\operatorname{div} A = 0$

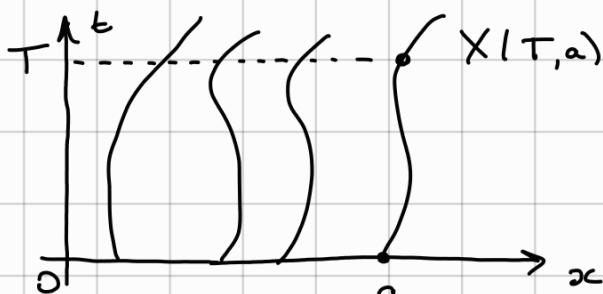
Example 1: fluid flow:  $a \rightarrow X(t, a)$  - flow map under velocity field

$u(x, t)$   $u(t, x)$  - velocity field  $\in \mathbb{R}^d$   
 $s(t, x)$  - density

$$\begin{cases} \partial_t X = u(X, t) \\ X(0, a) = a \end{cases}$$

• Conservation of mass = scalar transport equation

$$\partial_t s + \operatorname{div}(su) = 0$$

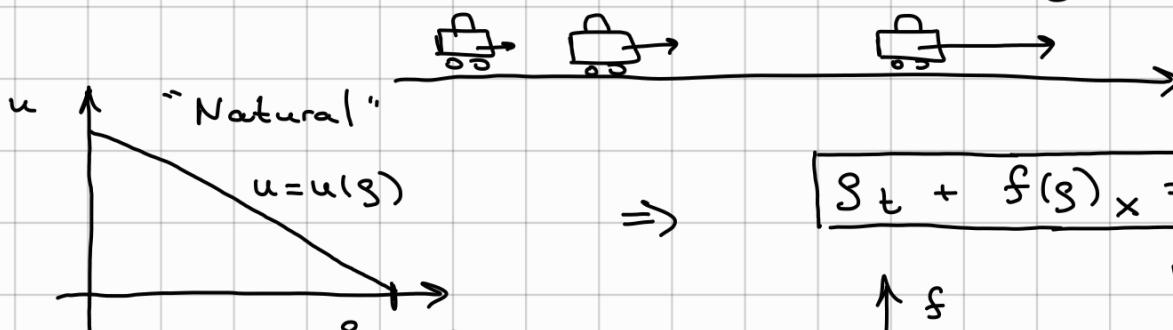


trajectories do not intersect for given  $u \in C^1_t \operatorname{Lip}_x$

Rmk:  $\begin{cases} \operatorname{div} u = 0 \\ \partial_t s + \operatorname{div}(su) = 0 \end{cases}$

$\Rightarrow s(t, X(t, a)) = \text{const}$   
density is conserved along the trajectory for incompressible flow

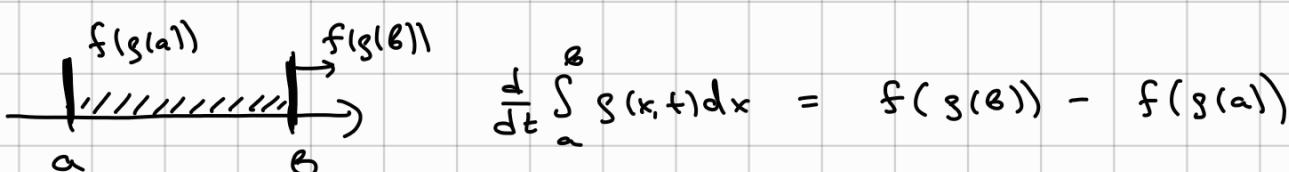
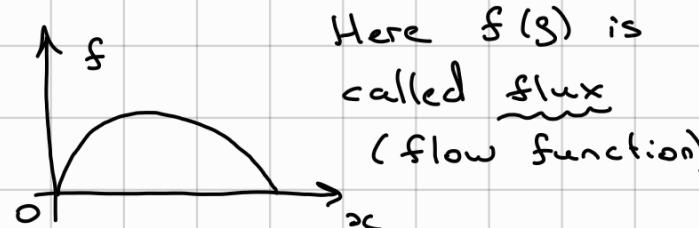
Example 2: traffic flow: cars choose their velocity depending on "density" of cars nearby



$s_m$  - density of cars corresp. to "bumper-to-bumper"

$$s_t + f(s)_x = 0$$

scalar conservation law



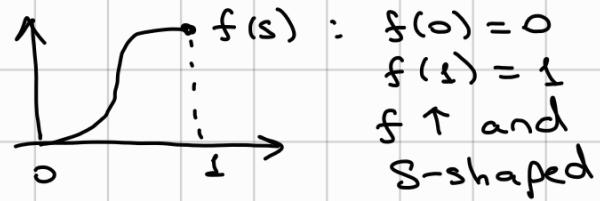
Rmk: i) taking  $u(s) = \frac{s}{2} \Rightarrow$  Burgers eq:  $s_t + \left(\frac{s^2}{2}\right)_x = 0$   
We will analyze it in detail today.

2) for oil recovery the simplest 1-dim model for displacement water-oil is again

$$s_t + (f(s))_x = 0 \quad \text{for}$$

$s$ - water saturation

$f(s)$ - fractional flow function



- One can easily create more sophisticated models such as: take drivers anticipation into account

If a driver observe an upstream increase in the density, they show a tendency to brake slightly

$$u - v(s) \sim -g_x$$

The simplest law:  $u = v(s) - \varepsilon g_x$ ,  $0 < \varepsilon \ll 1$

which leads to the "weakly" parabolic eq:

$$s_t + f(s)_x = \varepsilon (g g_x)_x$$

Example 3: wave equation!  $u_{tt} - c^2 u_{xx} = 0$

$$\operatorname{div}(u_t, -c^2 u_x) = 0$$

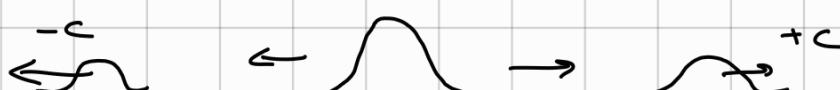
Consider  $\mathbf{U} = \begin{pmatrix} u_x \\ u_t \end{pmatrix} \Rightarrow \mathbf{U}_t + A \mathbf{U}_x = 0$

$$A = \begin{pmatrix} 0 & -1 \\ -c^2 & 0 \end{pmatrix}$$

Indeed, this is just:  $\begin{cases} u_{xt} - u_{tx} = 0 \\ u_{tt} - c^2 u_{xx} = 0 \end{cases}$

Eigenvalues of  $A$ :  $\det \begin{vmatrix} 0 - \lambda & -1 \\ -c^2 & 0 \end{vmatrix} = \lambda^2 - c^2$ ,  $\lambda_{\pm} = \pm c$

They correspond to propagation modes:



This is general fact that we will see in the future:

$$\mathbf{U} \in \mathbb{R}^d, F: \mathbb{R}^d \rightarrow \mathbb{R}^d \quad \mathbf{U}_t + (F(\mathbf{U}))_x = 0 \quad \text{- system of conservation laws}$$

Then for "smooth" solutions we have:

$$\mathbf{U}_t + \underbrace{F'(\mathbf{U}) \cdot \mathbf{U}_x}_{\text{eigenvalues of this matrix}} = 0$$

eigenvalues of this matrix play an important role!

If they are real, they correspond to velocity of propagation of waves.

Example 4: isentropic (=constant entropy) gas dynamics  
( $p$ -system)

in Lagrangian coordinates :

$$\begin{cases} v_t - u_x = 0 \\ u_t + p(u)_x = 0 \end{cases} \Rightarrow v_{tt} + p(v)_{xx} = 0$$

Rmk:  $v_t = u_x \Rightarrow$  (in a simply connected regions)  
 $\exists \Phi : v = \Phi_x$   
 $u = \Phi_t$

$$\Rightarrow \Phi_{tt} + (p(\Phi_x))_x = 0$$

$$\Phi_{tt} + p'(\Phi_x) \cdot \Phi_{xx} = 0 \quad -\text{nonlinear wave equation}$$

And many other examples :

- conservation of mass
- conservation of momentum

$$\Rightarrow \begin{cases} \partial_t u + u \cdot \nabla u = \nabla p + f \\ \operatorname{div}(u) = 0 \end{cases}$$

This is Euler equations for ideal fluid  
(1755, second PDE !)

- Navier - Stokes eqs (1845): adds viscosity

$$\partial_t u + (u \cdot \nabla) u \rightarrow \Delta u = \nabla p$$

- gas dynamics

- electromagnetism (Maxwell eqs)

- magneto-hydrodynamics (M.H.D.) - motion of fluid in the presence of electromagnetic field  
(think of a Sun)

Etc .....

Burgers equation

$$u_t + \left( \frac{u^2}{2} \right)_x = 0$$

$$u_t + u \cdot u_x = 0$$

Observation 1: if  $u \in C^1$  for all  $t > 0$ , then  
in  $\infty$  for all  $t > 0$ .



$$u(x(s), t(s)) = \text{const}$$

$$u_t \cdot t_s + u_x \cdot x_s = 0$$

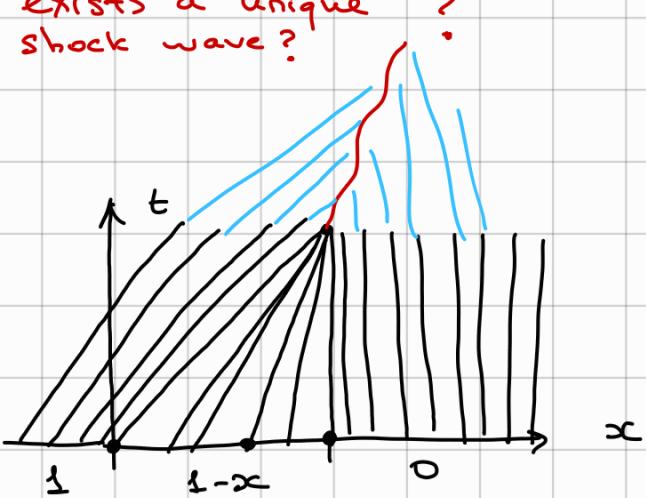
$$\begin{cases} t_s = 1 \\ x_s = u \end{cases} \Rightarrow u = \text{const on straight lines}$$

$$x = x_0 + u_0(x_0)t$$

If  $u \in C^1$  for  $\forall t > 0$ , then characteristics should not intersect  $\Rightarrow u_0(x_1) < u_0(x_2)$  if  $x_1 < x_2 \Rightarrow u_0$  is non-decreasing ( $u(x, 0)$ )  $\Rightarrow u(x, t)$  is non-decreasing in  $x$

Exercise 2 from list 1:

exists a unique  
shock wave?



$$x = x_0 + (1-x_0)t$$

$$t = 1 : x = 1$$

At  $t = 1$  there is a blow-up

Rmk: In general scalar conservation law:

$$u_t + f(u)_x = 0$$

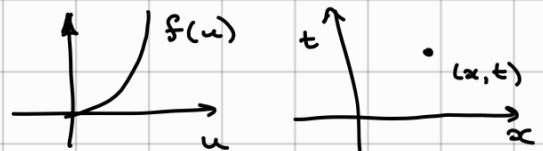
$$u_t + \underline{f'(u) \cdot u_x} = 0$$

Characteristics are  $x = x_0 + f'(u_0(x_0))t$

$u \in C^1 \wedge t > 0 \Rightarrow f'(u_0(x_1)) < f'(u_0(x_2))$  if  $x_1 < x_2$ , otherwise characteristics will intersect that leads to a blow-up!

So no matter how smooth  $f$  and  $u_0$  are, the solution  $u(x, t)$  must become discontinuous This is a purely non-linear phenomenon!!!

- Assume  $f \in C^2$  and  $f'' > 0$



$$u_0(x - t f'(u(x,t))) = u(x,t)$$

$$u_t = u'_0 \cdot (-f'(u(x,t)) - t f''(u(x,t)) \cdot u_x)$$

$$u_t (1 + t f'' u'_0) = -u'_0 f'$$

$$u_t = -\frac{u'_0 f'}{1 + t f'' u'_0}$$

Analogously,  $u_x = \frac{u'_0}{1 + t f'' u'_0}$

If  $u'_0 \geq 0$  (and  $f'' > 0$ )  $u_t$  and  $u_x$  stay bounded.

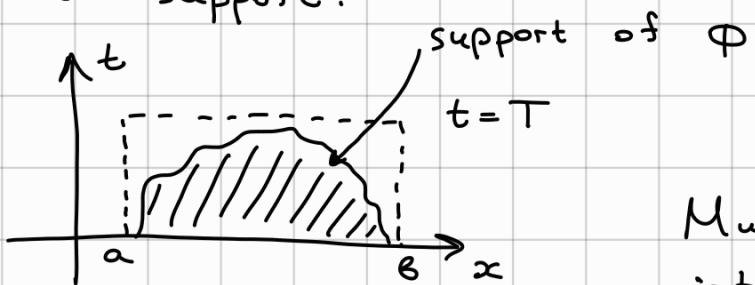
If  $u'_0 < 0$ , then  $u_t$  and  $u_x$  become unbounded as  $1 + t f'' u'_0$  tends to 0.

So we need a notion of weak solution!

Weak solutions to conservation laws

$$\begin{cases} u_t + f(u)_x = 0 \\ u|_{t=0} = u_0(x) \end{cases} \quad (\star)$$

Let  $u$  be a classical solution and  $\varphi \in C^1$  with compact support:



$\text{supp } (\varphi) \subset D = [a, B] \times [0, T]$   
that is  $\varphi$  is zero  
at  $x=a$ ,  $x=B$ ,  $t=T$

Multiply  $(\star)$  by  $\varphi$  and  
integrate over  $\mathbb{R} \times \mathbb{R}_+$

$$\begin{aligned} \iint_{t>0} (u_t + f(u)_x) \varphi \, dx dt &= \iint_D (u_t + f(u)_x) \varphi \, dx dt = \\ &= \iint_{a \leq x \leq B, 0 \leq t \leq T} (u_t + f(u)_x) \varphi \, dx dt = \int_a^B u \cdot \varphi \Big|_0^T dx - \iint_{a \leq x \leq B, 0 \leq t \leq T} u \cdot \varphi_t \, dx \\ &\quad + \int_0^T \int_a^B f(u) \cdot \varphi \Big|_a^B dt - \int_0^T \int_a^B f(u) \cdot \varphi_x \, dx dt = \end{aligned}$$

$$= - \int_0^T u_0(x) \varphi(x) dx - \iint_{\Omega \times (0,T)} (u \varphi_t + f(u) \varphi_x) dx dt$$

$$\Rightarrow \iint_{\Omega \times [0,T]} (u \varphi_t + f(u) \varphi_x) dx dt + \int_{t=0} u_0(x) \varphi(x) dx = 0 \quad (2)$$

$u \in C^1$  and satisfies (1)  $\Rightarrow u$  satisfies (2)

But in (2)  $u$  not necessarily needs to be  $C^1$ .  
It can be measurable / bounded.

Definition: A bounded measurable function  $u(x,t)$  is called a weak solution of IVP:

$$u_t + f(u)_x = 0, \quad u|_{t=0} = u_0(x) \quad \uparrow \text{bounded/meas.}$$

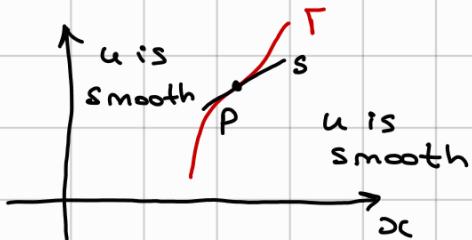
provided that

$$(2) \quad \iint_{\Omega \times [0,T]} (u \varphi_t + f(u) \varphi_x) dx dt + \int_{t=0} u_0 \varphi dx = 0$$

for all  $\varphi \in C_0^\infty$  ( $\varphi$  is  $C^\infty$  with compact supp)

Rmk: it is clear that if  $u$  is in fact  $C^1$ , then the original eq. is true:  $u_t + f(u)_x = 0$

Lemma (Rankine-Hugoniot condition)



Let  $\Gamma$  be a smooth curve across which  $u$  has a jump discontinuity. Take  $P \in \Gamma$  and

$$u_e = \lim_{(x,t) \rightarrow P} u \quad \text{from "the left"}$$

$$u_r = \lim_{(x,t) \rightarrow P} u \quad \text{from "the right"}$$

Let the tangent line of  $\Gamma$  at  $P$  have the slope

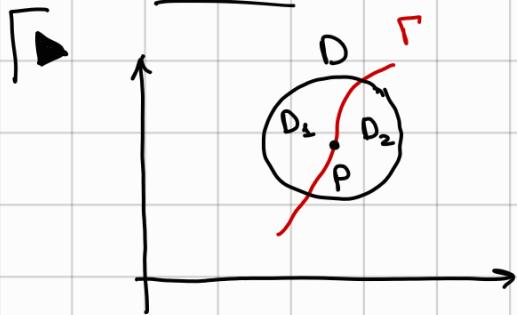
$$s = \frac{dx}{dt}. \quad \text{Then: (3)} \quad \boxed{s \cdot (u_e - u_r) = f(u_e) - f(u_r)}$$

Often a jump across the shock is denoted:

$[g(u)] = g(u_e) - g(u_r)$ , thus we have  $S[u] = [f]$

This is called the Rankine-Hugoniot condition

Proof :



Let  $D$  be a small ball centered at  $P$  and let  $\Gamma$  devide  $D$  into two regions  $D_1$  and  $D_2$  (see fig)

Let  $\varphi \in C_0^1$  on  $D$  and consider

$$0 = \iint_D (u\varphi_t + f(u)\varphi_x) dx dt = \iint_{D_1} + \iint_{D_2}$$

Divergence theorem :  
(Green-Gauss theorem)

$$\int_Q P dx + Q dy = \iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy$$

$$\iint_{D_1} u\varphi_t + f(u)\varphi_x dx dt = \iint_{D_2} (u\varphi)_t + (f(u)\varphi)_x dx dt =$$

↑  
 $D_2$

as  $u \in C^1(D_2)$  and  $u_t + f(u)_x = 0$

$$= \int_{t_1}^{t_2} \int f(u) \varphi dt - u \varphi dx = \int_{t_1}^{t_2} f(u) \varphi dt - u \varphi dx =$$

~~F~~  $t_2$   $t_1$

$$= \int_{t_1}^{t_2} [f(u_e) \varphi(u_e) - u_e \cdot \varphi(u_e) \cdot s] dt$$

Similarly,

$$\iint_{D_1} u\varphi_t + f(u)\varphi_x dx dt = - \int_{t_1}^{t_2} (f(u_r) - s u_r) \varphi(u_r) dt$$

minus because of orientation of  $\partial D_2$ : ~~F~~

Combining together:

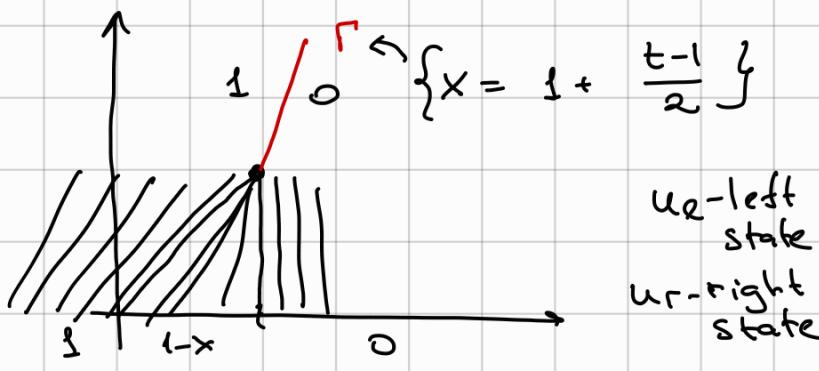
$$0 = \int_{t_1}^{t_2} ([f] - s[u]) \varphi(u_e) dt$$

Since  $\varphi$  was arbitrary, we get  $[f] - s[u] = 0$  relation (3). ■

Example: Burgers eq :

$$s = \frac{[u^2] /_0^L}{[u] /_0^L} = \frac{1}{2}$$

in general  $s = \frac{u_e + u_r}{2}$



$u_e$ -left state

$u_r$ -right state

Lecture 7 | Scalar conservation law:  $\begin{cases} u_t + (f(u))_x = 0 \\ u: \mathbb{R}^+ \rightarrow \mathbb{R} - \text{bounded, measurable} \\ f: \mathbb{R} \rightarrow \mathbb{R}, f \in C^2, f'' > 0 \text{ on the convex hull of values of } u_0 \end{cases} \quad (*)$

We understand solutions in weak sense:

$$\underset{t>0}{\int} \int (u \varphi_t + f(u) \varphi_x) dx dt + \int_{t=0}^{t=0} u_0 \varphi dx = 0 \quad (**)$$

for every test function  $\varphi \in C_0^1$ .

We want to prove theorems on  $\exists$ ,  $!$  and asymptotic behavior of solutions to  $(*)$ . From exercise session 1 we remember that we need some extra conditions for this

Thm1 ( $\exists$ ):

Let  $u_0 \in L^\infty(\mathbb{R})$ ,  $f \in C^2(\mathbb{R})$ ,  $f'' > 0$  on  $\{u: |u| \leq \|u_0\|_\infty\}$ .

Then there exists a solution with the following properties:

a)  $|u(x,t)| \leq \|u_0\|_\infty = M$ ,  $(x,t) \in \mathbb{R} \times \mathbb{R}^+$

b)  $\exists E > 0$  (which depends on  $M$ ,  $\mu = \min \{f''(u) : |u| \leq M\}$  and  $A = \max \{|f'(x)| : |u| \leq \|u_0\|_\infty\}$ ) such

such that  $\forall a > 0, t > 0, x \in \mathbb{R}$

$$\frac{u(x+a, t) - u(x, t)}{a} < \frac{E}{t} \quad (E)$$

c)  $u$  is stable and depends continuously on  $u_0$ : if  $v_0 \in L^\infty(\mathbb{R})$  with  $\|v_0\|_\infty \leq \|u_0\|_\infty$  and  $v$  is the corresponding constructed solution of  $(*)$  with initial data  $v_0$ , then for  $\forall x_1, x_2 \in \mathbb{R}$  with  $x_1 < x_2$  and  $\forall t > 0$

$$\int_{x_1}^{x_2} |u(x, t) - v(x, t)| dx \leq \int_{x_1-At}^{x_2+At} |u_0(x) - v_0(x)| dx.$$

Remarks //  $\exists$ )

Thm2 (!): Let  $f \in C^2$ ,  $f'' > 0$  and let  $u$  and  $v$  be 2 solutions of  $(*)$  satisfying condition (E). Then  $u = v$  almost everywhere in  $t > 0$ .

Rmk: we call the solution from Thm1 (that is satisfied)  $\Gamma_1$

may be there exist more solutions which do not satisfy cond. (E) or (a)

- 2) property (a) is not valid for systems! Sup-norm of solution can increase! It is non-trivial to prove the bounds on the sup-norm. locally
- 3) Cond. (E) implies some regularity:  $u$  is of bounded total variation (for  $\forall t$  as a function of  $x$ )  
 Indeed, let  $c_1$  be a constant such that  $c_1 > \frac{E}{t}$  and let  $v = u - c_1 x$ . Then  
 $v(x+a, t) - v(x, t) = u(x+a, t) - u(x, t) - c_1 a < a \left( \frac{E}{t} - c_1 \right) < 0$   
 Thus,  $v$  is a non-decreasing function, and  $v$  is a function of local bounded total variation.  
 Since  $c_1 x$  is also of bounded total variation, then  $u$  is of local bounded total variation.  
 $(\Rightarrow$  countable number of jump discontinuities)

- 4) finite speed of propagation:

$$v = v_0 \equiv 0 \Rightarrow \int_{x_1}^{x_2} |u(x, t)| dx \leq \int_{x_1 - At}^{x_2 + At} |u_0(x)| dx$$

Before proving thm 1 and 2 let us understand better condition (E). Let us give some equivalent formulations and interpretations.

Lemmat: a) A smooth solution  $u(x, t)$  satisfies condition (E)

b) If  $u$  has a discontinuity at point  $x_0$ : (but is smooth to the left and to the right of  $x_0$ )  
 $\lim_{x \rightarrow x_0^-} u(x, t) = u_L$  and  $\lim_{x \rightarrow x_0^+} u(x, t) = u_R$  and  
 $satisfies condition (E) \Rightarrow u_L > u_R.$

(discontinuities can be only down).

Proof:

► a) Indeed, let us write:

$$u(x, t) = u_0(x - t f'(u(x, t)))$$

$$u_x = u'_0 \cdot (1 - t f''(u_x)) \Rightarrow u_x = \frac{u'_0}{1 + t f''(u'_0)}$$

If  $u$  is smooth for  $\forall t > 0$ , then  $u'_0 > 0$ .

$$\text{Then } u_x \leq \frac{u'_0}{t f''(u'_0)} = \frac{E}{t} \text{ for } E = \frac{1}{\inf f''}.$$

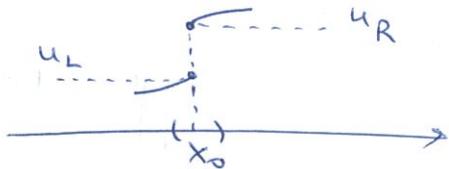
Using Lagrange theorem:  $\frac{u(x+a, t) - u(x, t)}{a} = u_x(\xi, t)$ ,  
 for some  $\xi \in (x, x+a)$ , and (a) is proved

(b) Either  $u_L > u_R$  or  $u_L < u_R$  (the case  $u_L = u_R$  is not a discontinuity).

• For  $u_L < u_R$  the converse of cond. (E) is true:

$$\forall E > 0 \exists x, a > 0, t : \frac{u(x+a, t) - u(x, t)}{a} > \frac{E}{t}.$$

Indeed, fix  $E$  and take small enough neighbourhood of  $x_0$  such that



• for  $x \in (x_0 - \delta, x_0)$   $|u - u_L| \leq \varepsilon = \frac{u_R - u_L}{4}$

• for  $x \in (x_0, x_0 + \delta)$   $|u - u_R| \leq \varepsilon = \frac{u_R - u_L}{4}$ .

This means that for  $\forall x_1 \in (x_0 - \delta, x_0)$  and  $x_2 \in (x_0, x_0 + \delta)$   $u(x_2) - u(x_1) \geq \frac{u_R - u_L}{2}$ .

Fix  $t$  and take <sup>small</sup>  $a$ :

$$x_2 - x_1 = a$$

$$x_1 \in (x_0 - \delta, x_0)$$

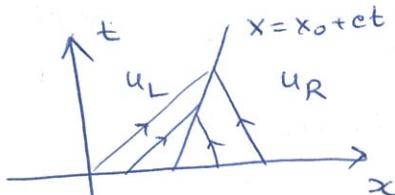
$$x_2 \in (x_0, x_0 + \delta)$$

$$\frac{u(x_2) - u(x_1)}{a} \cancel{\leq \frac{E}{t}}$$
  

$$\frac{u_R - u_L}{2a} = \frac{E}{t}$$

• For  $u_L > u_R$   $\frac{u(x+a, t) - u(x, t)}{a} \leq 0$ , thus  $\forall E > 0$  is ok

Lemma 2 (Remark):  $u$  satisfies condition (E) and is a shock wave solution  $u = \begin{cases} u_L, & x < ct \\ u_R, & x > ct \end{cases}$  then  $f'(u_L) > c > f'(u_R)$  (Lax condition)



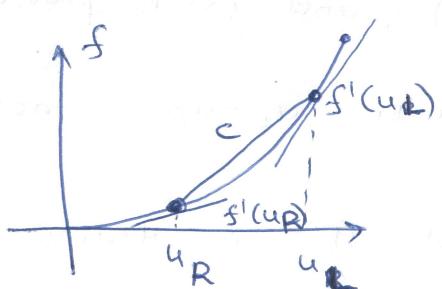
"Characteristics come to the line of discontinuity"

The converse situation corresponds to the case "information" appears on the discontinuity,



in some sense where "the information" appears on the discontinuity, which is not

We will generalize the Lax condition to the case of systems.



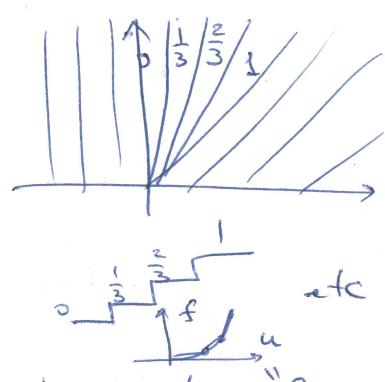
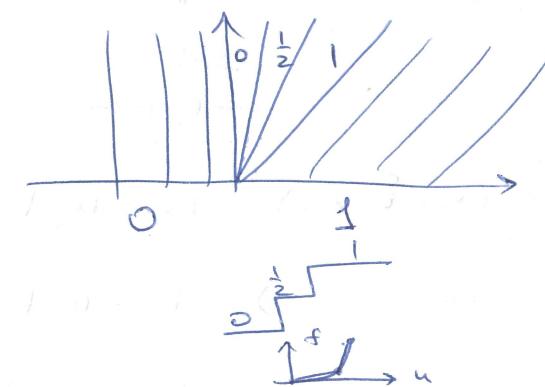
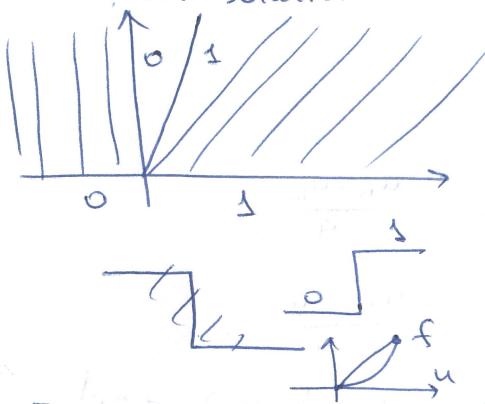
Indeed,  $f'' > 0 \Rightarrow$  (see picture)

$$f'(u_L) > c = \frac{f(u_L) - f(u_R)}{u_L - u_R} > f'(u_R)$$

Remark (on Liu criterion) "internal stability of shock"

Remember the situation with Burgers equation:

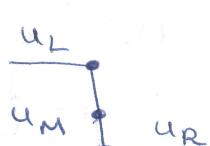
1st solution



In some sense if we divide the shock into "2 smaller" shocks, they could have a tendency of either glueing into 1 shock (some kind of stability) or going further one from another (instability)

Condition (E)  $\Rightarrow$  this kind of internal stability of a shock, more precisely the inequalities

$$c(u_L, u_R) = \frac{f(u_R) - f(u_L)}{u_R - u_L} \leq c(u_L, u_M) = \frac{f(u_L) - f(u_M)}{u_L - u_M}$$



$$c(u_M, u_R) = \frac{f(u_R) - f(u_M)}{u_R - u_M} \quad \text{if } u_M \in (\min(u_L, u_R), \max(u_L, u_R))$$

If  $u_M \rightarrow u_L$  we have Lax condition.

Vanishing viscosity criterion for shock waves.

We think of equation  $u_t + (f(u))_x = 0$  as a first approximation to the following parabolic eq

$$u_t + (f(u))_x = \varepsilon u_{xx}, \quad \varepsilon > 0 \quad (\text{P})$$

small regularizing term

Rmk 1: it is well-known (and we see in future when dealing with reaction-diffusion equations) that solutions of (P) are very regular (no shocks) "opposite"

Rmk 2: equation (P) is a combination of 2 effects

$\rightarrow u_t + (f(u))_x = 0 \rightsquigarrow$  creates shocks:  $\overbrace{\quad}^{\text{shock}} \rightarrow \overbrace{\quad}^{\text{shock}}$

$\rightarrow u_t = \varepsilon u_{xx} \rightsquigarrow$  "smooths":  $\overbrace{\quad}^{\text{"smooths"}} \rightarrow \overbrace{\quad}^{\text{"smooths"}}$

As a consequence of this confrontation there exist very special solutions, called travelling waves (TW) such that:

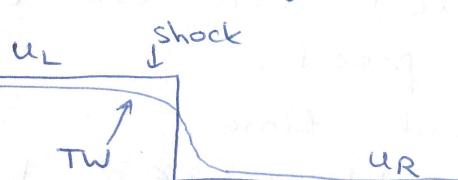
$$u(x, t) = v(x - ct) \quad \text{TW}$$

for  $c \in \mathbb{R}$  and  $v$  - some smooth profile.

They look like "smoothed" shocks!!!

This motivates the following definition:

Defn(vanishing viscosity criterion for shock waves):  
A shock wave is an entropy solution if is a limit<sup>in  $L^\infty$</sup>  of a travelling wave solution of (P)  
 $f \in C^2, f'' > 0$   
as  $\varepsilon \rightarrow 0$ .



Lemma: a shock wave is an entropy solution in sense of defn, iff  $u_L > u_R$ .

Proof: Let's look for travelling wave solutions for eq. (P):  $v\left(\frac{x-ct}{\varepsilon}\right)$ :  $v(-\infty) = u_L, v(+\infty) = u_R$

$$-cv' + (f(v))' = \varepsilon v''$$

Integrate  $\int_{-\infty}^{+\infty}$ :  $-c(v_R - v_L) + f(u_R) - f(u_L) = 0$

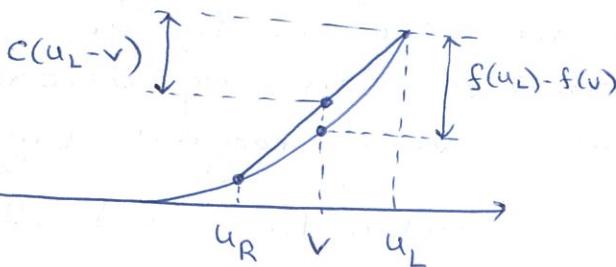
Interesting feature: it is exactly RH condition  
OK, let us integrate  $\int_{-\infty}^{+\infty}$ :  $-c(v(\xi) - u_L) + (f(v(\xi))) - f(u_L) = v'(\xi) \int_{-\infty}^{+\infty}$

ODE)  $v' = f(v) - f(u_L) - c(v - u_L) = F(v)$   
 Note that RHS  $F(u_L) = 0$  and  $F(u_R) = 0$  (due to RH!)

Thus  $u_L$  and  $u_R$  are two fixed points of this ODE

Consider 2 cases:  $u_L > u_R$  and  $u_L < u_R$ .

$$F(v) < 0$$



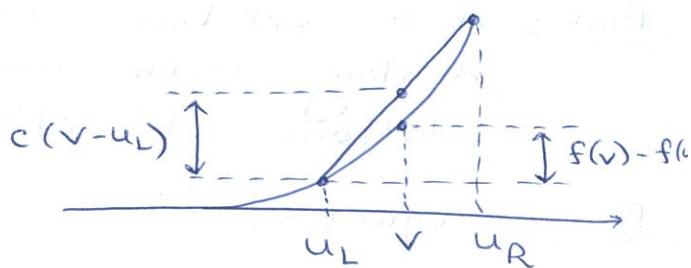
In this case:  $F(v) < 0 \quad \forall v \in (u_R, u_L)$

And there exists a solution  $v$  of ODE:

$$v(-\infty) = u_L; v(+\infty) = u_R$$

L

$$F(v) < 0$$



In this case:  $F(v) < 0 \quad \forall v \in (u_L, u_R)$

And there DOES NOT exist a solution  $v$  of ODE

$$v(-\infty) = u_L; v(+\infty) = u_R$$

Lecture 8: Scalar conservation law :  $\begin{cases} u_t + (f(u))_x = 0 \\ u|_{t=0} = u_0(x) \end{cases} \quad (*)$

- $u: \mathbb{R} \times \mathbb{R}_+ \rightarrow \mathbb{R}$  - bounded, measurable

$f: \mathbb{R} \rightarrow \mathbb{R}$ ,  $f \in C^2$ ,  $f'' > 0$ . As we will see it is enough to define  $f$  on the convex hull of values  $u_0$

We understand solutions in weak sense:

$$\int_0^\infty \left[ u \varphi_t + f(u) \varphi_x \right] dx dt + \int_{t=0} u_0 \varphi dx = 0 \quad (**)$$

for every test function  $\varphi \in C_0^\infty$ .

Define  $M := \|u_0\|_\infty$ ,  $A := \max_{|u| \leq M} |f'(u)|$ ,  $\mu := \min_{|u| \leq M} f''(u)$

Today we will start proving theorem on existence.

Thm 1 (E): Let  $u_0 \in L_\infty(\mathbb{R})$ ;  $f \in C^2(\mathbb{R})$ ,  $f'' > 0$  on  $\{u: |u| \leq M\}$

There exists a solution with the following properties

(a)  $|u(x,t)| \leq M$ ,  $(x,t) \in \mathbb{R} \times \mathbb{R}_+$

(b)  $\exists E = E(M, \mu, A) > 0$  such that  $\forall x_0, \forall t > 0$

$$\frac{|u(x+a, t) - u(x, t)|}{a} < \frac{E}{t} \quad (E)$$

"entropy" cond.

(c)  $u$  is stable and depends continuously on  $u_0$ :  
 if  $v_0 \in L_\infty(\mathbb{R})$  with  $\|v_0\|_\infty \leq \|u_0\|_\infty$  and  $v$  is the corresponding constructed solution of (f)  
 with initial data  $v_0$ , then for  $\forall x_1, x_2 \in \mathbb{R}$   
 with  $x_1 < x_2$  and  $t > 0$

$$\int_{x_1}^{x_2} |u(x, t) - v(x, t)| dx \leq \int_{x_1-At}^{x_2+At} |u_0(x) - v_0(x)| dx \quad (S)$$

"stability"

How to prove this theorem?

There exist (at least) 5 approaches:

- Calculus of variations and Hamilton-Jacobi theory
- Vanishing viscosity method
- Non-linear semigroup theory

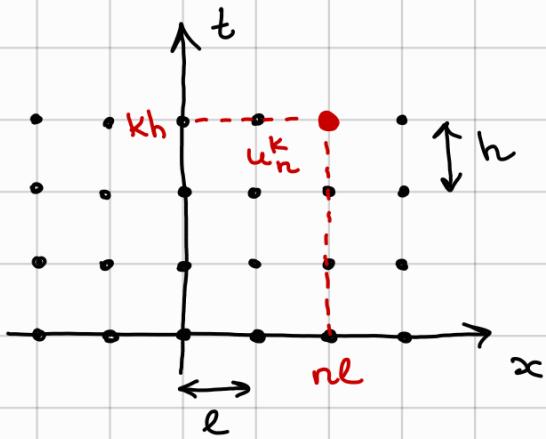
(d) Method of characteristics

(e) Finite-difference method

We will follow Smoller (Chapter 16) and use (e).

Here is the scheme of the proof:

Step 2: discretization in space and time



$$x_n = nl, n \in \mathbb{Z} \quad l = \Delta x > 0$$

$$t_k = kh, k \in \mathbb{N} \cup \{0\} \quad h = \Delta t > 0$$

$$u_n^k = u(nl, kh)$$

Consider a finite-difference (explicit) scheme:

(D) 
$$u_n^{k+1} = \frac{u_{n+1}^k + u_{n-1}^k}{2} - \frac{h}{2e} \cdot (f(u_{n+1}^k) - f(u_{n-1}^k))$$

$$u_n^0 = u_0(nl), n \in \mathbb{Z}$$

In what follows we will always assume:

$$\frac{Ah}{e} \leq 1$$

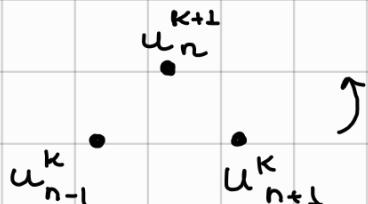
(CFL condition)

Courant-Friedrichs-Lowy

It is important for the stability of the numerical scheme and tells that the time step  $h$  should be small enough.

First, we will formulate and prove some properties of solutions  $u_n^k$  of a discrete eq. (D):

(1a) solution exists (evident!)



(1b) if  $|u_n^0| \leq M$ , then  $|u_n^k| \leq M \quad \forall k \in \mathbb{N}$   
(boundedness)

$$(1c) \exists E = E(M, A, \mu) > 0 : \frac{u_n^k - u_{n-2}^k}{2\ell} \leq \frac{E}{kh} \quad (\text{E-disc})$$

discrete entropy condition

NB: the discrete entropy condition is a natural consequence of a finite difference approximation (D).

(1d) local bounded variation:  $\forall X > 0$  and  $Kh > 0$   
 $\exists c(X, \alpha)$  (but independent of  $h$  and  $\ell$ ):

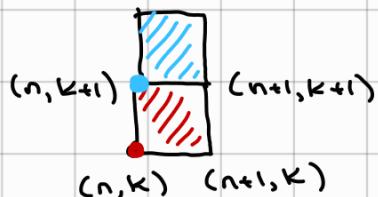
$$\sum_{|nl| \leq X/\ell} |u_{n+2}^k - u_n^k| \leq C \quad \text{and some other...}$$

Step 2: We will prove convergence as  $h, \ell \rightarrow 0$ .

Consider  $U_{h,e}(x, t) = u_n^k$  if

$$nl \leq x \leq (n+s)\ell$$

$$kh \leq t \leq (k+s)h$$



We will prove that there exists

subsequence  $U_{h_i, e_i}$  of  $U_{h,e}$  such that  $U_{h_i, e_i} \rightarrow u(x,t)$   
- some measurable function

Step 3: We will prove that this limiting function satisfies integral equality (\*\*)  
and all properties of theorem on  $\mathbb{Z}$ .

Proof of the theorem 1.

Lemma 1 (boundedness of  $u_n^k$ ):  $|u_n^k| \leq M$ ,  $\forall n \in \mathbb{Z}, k \in \mathbb{N}$

This is an exercise 2 from list 3.

Lemma 2 (discrete entropy condition)

If  $C = \min \left( \frac{\mu}{2}, \frac{A}{4M} \right)$ , then

$$\frac{u_n^k - u_{n-2}^k}{2\ell} \leq \frac{E}{kh} \quad \text{where } E = \frac{1}{C}.$$

Proof:

Let  $z_n^k = \frac{u_n^k - u_{n-2}^k}{2\ell}$  and first let us prove  
some recurrent relation for  $z_n^{k+1}$  of the form  
"  $z_n^{k+1} = A z_{n+1}^k + B z_{n-1}^k + C$ "

$$z_n^{k+1} = \frac{1}{2} [z_{n+1}^k + z_{n-1}^k] - \frac{h}{(2e)^2} (f(u_{n+1}^k) - f(u_{n-1}^k)) + \frac{h}{(2e)^2} (f(u_{n-1}^k) - f(u_{n-3}^k))$$

Notice that due to  $f \in C^2$  we can write

$$f(u_{n+1}^k) = f(u_{n-1}^k) + f'(u_{n-1}^k)(u_{n+1}^k - u_{n-1}^k) + f''(\theta_1) \frac{(u_{n+1}^k - u_{n-1}^k)^2}{2}$$

for some  $\theta_1$  between  $u_{n+1}^k$  and  $u_{n-1}^k$

$$= f(u_{n-1}^k) + f'(u_{n-1}^k) \cdot 2l \cdot z_{n+1}^k + f''(\theta_1) \cdot \frac{(2l)^2}{2} (z_{n+1}^k)^2$$

Analogously,

$$f(u_{n-3}^k) = f(u_{n-1}^k) - f'(u_{n-1}^k) \cdot 2l \cdot z_{n-1}^k + f''(0_2) \frac{(2l)^2}{2} (z_{n-1}^k)^2$$

Thus,

$$\text{Thus, } z_n^{k+1} = z_{n+1}^k \cdot \left[ \frac{1}{2} - \frac{h}{2e} f'(u_{n-1}^k) \right] + z_{n-1}^k \left[ \frac{1}{2} + \frac{h}{2e} f'(u_{n-1}^k) \right] - \frac{h}{2} \cdot \left[ f''(Q_1) \cdot (z_{n+1}^k)^2 + f''(Q_2) \cdot (z_{n-1}^k)^2 \right]$$

Note that  $A + B = 1$  and  $A, B \geq 0$

Define  $\tilde{z}_n^k = \max \{ z_{n-1}^k, z_{n+1}^k, 0 \}$

If  $\tilde{z}_n^k = 0 \quad \forall n$ , then (E-disc) is true since

$$z_{jk}^k \leq \hat{z}_{jk}^k = 0 \leq \frac{1}{\sum_{j=1}^J \prod_{k=1}^K f_j^{(k)}}, \quad \forall F > 0.$$

Thus, w.l.o.g. we can assume  $\tilde{z}_n^k \neq 0$ . Suppose

$\tilde{z}_n^k = z_{n+1}^k$  (the other case is similar)

$$\begin{aligned} z_{n+1}^{k+1} &\leq z_{n+1}^k \cdot [A + B] - h \frac{M}{2} \left( z_{n+1}^k \right)^2 \leq \\ &\leq z_n^k + h.c \left( z_n^k \right)^2. \end{aligned}$$

Notice that

$$z_n^k \leq |z_n^k| \leq \frac{M}{\ell} \leq \frac{M}{A h} \leq \frac{M}{n} \cdot \frac{1}{4MC} = \frac{1}{4ch}$$

$\uparrow$   
**CFL**

$$c \leq \frac{A}{4M}$$

$$\text{Let } M^k = \max_{n \in \mathbb{Z}} \left\{ \tilde{x}_n^k \right\} \geq 0.$$

Let  $\varphi(y) = y - c \cdot h \cdot y^2$ . Since  $\varphi' = 1 - 2chy$ ,  $\varphi$  is

increasing if  $y \leq \frac{1}{2ch}$ . But we have  
 $\tilde{z}_n^k \leq M^k \leq \frac{1}{4ch} < \frac{1}{2ch}$ .

So that  $\varphi(\tilde{z}_n^k) \leq \varphi(M^k)$  and we have

$$\tilde{z}_n^k - ch(\tilde{z}_n^k)^2 \leq M^k - ch(M^k)^2$$

Thus,  $\tilde{z}_n^{k+1} \leq M^k - ch(M^k)^2 \quad \forall n \in \mathbb{Z}$ .

It follows that

$$M^{k+1} \leq M^k - ch(M^k)^2 \quad (M)$$

Claim:  $M^k = \frac{1}{ch k + 1/\mu_0}$ .

Suppose we have proven claim. Let us see how it helps to prove lemma 2. Indeed,

$$z_n^k \leq M^k \leq \frac{1}{ch k + 1/\mu_0} \leq \frac{1}{ch k} = \frac{E}{h k}, E = \frac{1}{c}.$$

Proof of claim: first - intuition why such estimate could be true

Inequality (M) for  $M^k$  is a discrete analog of ODE inequality:  $\varphi' \leq -ch \varphi^2$

if it was an equality  $\varphi' = -ch \varphi^2$ , then the solution is:

$$\frac{d\varphi}{\varphi^2} = -ch dt$$

$$-\frac{1}{\varphi} = -cht + C_1$$

$$\varphi(t) = \frac{1}{cht - C_1} \quad \text{and with } \text{ic } \varphi(0) = \varphi_0$$

we will have

$$\varphi(t) = \frac{1}{cht + 1/\varphi_0}$$

So one can try to prove

$$\varphi(t) \leq \frac{1}{cht + 1/\varphi_0}$$

Second, let us make the formal proof.

We will do it by induction.

Base :  $k=0$  : - clear :  $M^0 = \frac{1}{1/\mu^0} = M^0$ .

$k > 0$  : suppose that

$$M^k = \frac{1}{ch k + 1/\mu^0}$$

and we want to prove that

$$M^{k+1} = \frac{1}{ch(k+1) + 1/\mu^0}$$

We have:  $\frac{1}{M^k} \geq ch k + \frac{1}{\mu^0}$ , so

$$1 - ch M^k \geq 1 - ch k M^k \geq \frac{M^k}{\mu^0} \geq 0.$$

Thus  $1 - (ch M^k)^2 \geq 0$ .

We have  $M^{k+1} \leq M^k (1 - ch M^k)$ , so that

$$\frac{M^{k+1}}{1 - ch M^k} \leq M^k \leq \frac{M^k}{1 - (ch M^k)^2}$$

and thus  $M^{k+1} \leq \frac{M^k}{1 + ch M^k} = \frac{1}{ch + 1/\mu^k} \leq \frac{1}{ch(k+1) + 1/\mu^0}$  q.e.d. ■

L

Lemma (space estimate) : For any  $X > 0$  and  $kh \geq \alpha > 0$ , there is a constant  $C = C(X, \alpha, N)$  (but independent on  $h, e$ ) such that:

$$\sum_{|nl| \leq X/e} |u_{n+2}^k - u_n^k| \leq C$$

Proof :

► Set  $v_n^k = u_n^k - c_1 nl$ , where  $c_1$  is chosen so large that  $E/\alpha < c_1$ . Then

$$\begin{aligned} v_{n+2}^k - v_n^k &= u_{n+2}^k - u_n^k - 2c_1 l \leq \frac{2E}{kh} - 2c_1 l \leq \\ &\leq 2E \left( \frac{E}{\alpha} - c_1 \right) < 0, \text{ so } v_n^k \text{ is decreas. inn} \end{aligned}$$

Thus

$$\begin{aligned} \sum_{|nl| \leq X/e} |u_{n+2}^k - u_n^k| &\leq \sum |v_{n+2}^k - v_n^k| + \sum 2c_1 l = \\ &= - \sum (v_{n+2}^k - v_n^k) + 2c_1 l \left( \frac{2X}{e} + 1 \right) \leq 4M + 2c_1 X + c_2 X \end{aligned}$$

L telescopic sum  $\leq 4(M + c_1 X)$

q.e.d. ■

Lecture 9 : We continue proving theorem on existence of entropy solution for scalar conslaw.

Lemma 4 (time estimate -  $u_n^k$  are  $L^2$  locally Lipschitz in  $k$ )

If  $h/e \geq \delta > 0$  and  $h, e \leq 1$ , then exists  $L > 0$  (independent of  $h, e$ ) such that

if  $k > p$ , where  $(k-p)$  is even and  $p\hbar \geq 2 > 0$ , then

$$\sum_{|n| \leq X/e} |u_n^k - u_n^p| e \leq L(k-p) h$$

A similar estimate holds if  $(k-p)$  is odd.

Proof:

Let us express  $u_n^k$  in terms of  $u_n^p$  where  $(k-p)$  is even.

$$u_n^k = \frac{1}{2} (u_{n+1}^{k-1} + u_{n-1}^{k-1}) - \frac{h}{2e} f'(Q) (u_{n+1}^{k-1} - u_{n-1}^{k-1}) =$$

$$= u_{n+1}^{k-1} \left( \frac{1}{2} - \frac{h}{2e} f'(Q) \right) + u_{n-1}^{k-1} \left( \frac{1}{2} + \frac{h}{2e} f'(Q) \right)$$

$$\text{or } u_n^k = a_{n+1}^{k-1} u_{n+1}^{k-1} + a_{n-1}^{k-1} u_{n-1}^{k-1}, \text{ where } a_{n+1}^{k-1} + a_{n-1}^{k-1} = 1 \text{ and } a_{n+1}^{k-1}, a_{n-1}^{k-1} \geq 0.$$

Applying this to  $u_{n-1}^{k-1}$  and  $u_{n+1}^{k-1}$  gives a formula:

$$u_n^{k+1} = A u_{n+2}^{k-1} + B u_n^{k-1} + C u_{n-2}^{k-1}$$

where  $A, B, C \geq 0$ ,  $A+B+C=1$ .

$$\text{Hence, } |u_n^{k+1} - u_n^{k-1}| \leq A |u_{n+2}^{k-1} - u_n^{k-1}| + C |u_{n-2}^{k-1} - u_n^{k-1}|$$

Multiplying this by  $\Delta x = e$  and summing, we

get :

$$\sum_{|n| \leq X/e} |u_n^{k+1} - u_n^{k-1}| \Delta x \leq C \Delta x$$

↑  
Lemma 3

Now if  $(k-p)$  is even, we can do this operation several times and using the triangle inequality, we get:

$$\sum_{|n| \leq X/e} |u_n^k - u_n^p| \Delta x \leq \sum_{i=p}^{k-2} \sum_{|n| \leq X/e} |u_n^{i+2} - u_n^i| \Delta x \leq (k-p) C \Delta x \leq$$

$$\leq \frac{\Delta t}{\delta} (k-p)c = L(k-p) h \quad \text{for } L = \frac{c}{\delta}, h = \Delta t$$

Lemma 5 (stability): Let  $u_n^k$  and  $v_n^k$  be solutions to the finite-difference scheme (D) corresponding to the initial conditions  $u_n^0$  and  $v_n^0$ , respectively, where

$$\sup_{n \in \mathbb{Z}} |u_n^0| \leq M \quad \text{and} \quad \sup_{n \in \mathbb{Z}} |v_n^0| \leq M$$

Then, if  $k > 0$ ,

$$\sum_{|n| \leq N} |u_n^k - v_n^k| \cdot \ell \leq \sum_{|n| \leq N+k} |u_n^0 - v_n^0| \cdot \ell$$

Proof:

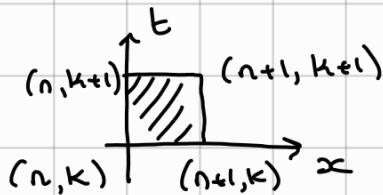
►  $w_n^k = u_n^k - v_n^k$ . From (D) we have

$$\begin{aligned} w_n^{k+1} &= u_n^{k+1} - v_n^{k+1} = \frac{u_{n+1}^k + u_{n-1}^k}{2} - \frac{h}{2e} (f(u_{n+1}^k) - f(u_{n-1}^k)) \\ &\quad - \frac{v_{n+1}^k + v_{n-1}^k}{2} + \frac{h}{2e} (f(v_{n+1}^k) - f(v_{n-1}^k)) = \\ &= \frac{w_{n+1}^k + w_{n-1}^k}{2} - \frac{h}{2e} (f(u_{n+1}^k) - f(v_{n+1}^k)) \\ &\quad + \frac{h}{2e} (f(u_{n-1}^k) - f(v_{n-1}^k)) = \\ &= w_{n+1}^k \underbrace{\left[ \frac{1}{2} - \frac{h}{2e} f'(Q_1) \right]}_{\geq 0 \text{ due to CFL}} + w_{n-1}^k \underbrace{\left[ \frac{1}{2} + \frac{h}{2e} f'(Q_2) \right]}_{\geq 0} \end{aligned}$$

Now proceed by induction.

$$\begin{aligned} \sum_{|n| \leq N} |w_n^{k+1}| &\leq \sum_{|n| \leq N} |w_{n+1}^k| \cdot A_{n+1}^k + \sum_{|n| \leq N} |w_{n-1}^k| \cdot B_{n-1}^k = \\ &= \sum_{m=N}^{N+1} |w_m^k| A_m^k + \sum_{m=-N}^{N-1} |w_m^k| \cdot B_m^k \leq \\ &\leq \sum_{|m| \leq N+1} |w_m^k| A_m^k + \sum_{|m| \leq N+1} |w_m^k| \cdot B_m^k \leq \sum_{|m| \leq N+1} |w_m^k| \end{aligned}$$

Step 2: Rather than define  $u_n^k$  in mesh points let us continue  $u_n^k$  as a piecewise constant function in the upper half plane.



$$U_{h,e}(x,t) = u_n^k \text{ if } nl \leq x \leq (n+1)l \\ kh \leq t \leq (k+1)h$$

So we have a family of functions  $\{U_{h,e}\}$  and would like to choose a convergent subsequence  $U_{h_i, e_i}$  as  $h_i, l_i \rightarrow 0$   $i \rightarrow \infty$ .

Lemma 6 (convergence : the set of functions  $\{U_{h,e}\}$  is compact in the topology of  $L_1$ -convergence on compacta)

There exists a subsequence  $\{U_{h_i, e_i}\}_{i \in \mathbb{N}}$  which converges to a measurable function  $u(x,t)$  in the sense that for  $\forall X > 0, t > 0, T > 0$  both

$$\int_{|x| \leq X} |U_{h_i, e_i}(x,t) - u(x,t)| dx \rightarrow 0 \text{ as } h_i, l_i \rightarrow 0$$

and

$$\int_0^T \int_{|x| \leq X} |U_{h_i, e_i}(x,t) - u(x,t)| dx dt \rightarrow 0.$$

Furthermore, the function  $u(x,t)$  satisfies :

- (a)  $\sup_{\substack{x \in \mathbb{R} \\ t > 0}} |u(x,t)| \leq M$ ; (b) inequality (S)  
(stability)

Proof :

► First, take  $t = \text{const}$  and consider  $U_{h,e}(x,t)$  as functions of  $x$ . By Lemma 1 and Lemma 3 the set of functions  $\{U_{h,e}\}$  is bounded and have uniformly bounded total variation on each bounded interval in  $x$ .

Helly's theorem (simple version):

A uniform bounded sequence of monotone, real functions admits a convergent subsequence.

Helly's theorem (generalized version):

A uniform bounded sequence of BV<sub>loc</sub> (locally of bounded variation) real functions admits a convergent subsequence on every compact set.

Rmk: a function of BV<sub>loc</sub> can be written as a sum of increasing and decreasing functions (on each compact interval). This is why the generalized version of the Helly's theorem is true.

So by Helly's theorem on each interval

we have a convergent subsequence  $\{U'_{h,e}\}$ .

By a standard diagonal process we can construct a subsequence  $\{U''_{h,e}\}$  from  $\{U'_{h,e}\}$  which converges at every  $x \in \mathbb{R}$  for this particular  $t = \text{const} > 0$ .

Second, take  $\{t_m\}_{m=1}^{\infty}$  - a countable and dense subset of  $(0, T)$ , e.g.  $\mathbb{Q} \cap (0, T)$ .

For  $t=t_1$  we have  $\{U'_{h_1,e_1}\}$  a convergent subsequence.

For  $t=t_2$  take a convergent sub.  $\{U'_{h_2,e_2}\}$  from  $\{U'_{h_1,e_1}\}$  etc. So we have:

$$t=t_1: U'_{h_1,e_1}$$

$$t=t_2: U'_{h_2,e_2}$$

$$t=t_3: U'_{h_3,e_3}$$

$$t=t_4: U'_{h_4,e_4}$$

$$U'_{h_2,e_2}$$

$$U'_{h_3,e_3}$$

$$U'_{h_4,e_4}$$

$$U'_{h_1,e_1}$$

$$U'_{h_3,e_3}$$

$$U'_{h_4,e_4}$$

$$U'_{h_1,e_1}$$

$$U'_{h_2,e_2}$$

$$U'_{h_3,e_3}$$

$$U'_{h_4,e_4}$$

$$U'_{h_4,e_4}$$

$$U'_{h_1,e_1}$$

$$U'_{h_2,e_2}$$

$$U'_{h_3,e_3}$$

$$U'_{h_4,e_4}$$

$$U'_{h_1,e_1}$$

...

By a standard diagonal process, we can choose a subsequence  $\{U_{h_i,e_i}\}$  which converges for all  $\{t_m\}_{m=1}^{\infty}$  and all  $x \in \mathbb{R}$ .

Third, we want to show that there is a convergence for all  $t \in (0, T)$ . So that in the limit we indeed obtain a function defined in the strip  $0 < t < T$ .

Let  $U_i = U_{\epsilon_i, h_i}$  and we want to show that

$$I_{ij} = \int_{-X}^X |U_i(x, t) - U_j(x, t)| dx \rightarrow 0 \quad \forall i, j \rightarrow \infty$$

i.e. that  $\{U_i\}$  is a Cauchy sequence in  $L_1(|x| \leq X)$

For  $t \in (0, T)$  we find a subsequence  $\{t_{m_s}\} \subset \{t_m\}$  such that  $t_{m_s} \rightarrow t$  as  $s \rightarrow \infty$ . Let  $\tau_s = t_{m_s}$ . Then

$$\begin{aligned} I_{ij}(t) &\leq \int_{-X}^X |U_i(x, t) - U_i(x, \tau_s)| dx + \int_{-X}^X |U_i(x, \tau_s) - U_j(x, \tau_s)| dx \\ &\quad + \int_{-X}^X |U_j(x, t) - U_j(x, \tau_s)| dx =: I_1 + I_2 + I_3 \end{aligned}$$

For  $t = \tau_s$  we have a convergence of  $U_i$ , thus for  $s$  large enough we have  $I_2 < \varepsilon/3$

Let's estimate  $I_1$ :

$$\begin{aligned} I_1 &= \int_{-X}^X |U_i(x, [\frac{t}{h_i}] h_i) - U_i(x, [\frac{\tau_s}{h_i}] h_i)| dx = \\ &= \sum_{\substack{(n+1)\ell_i \\ |n| < \frac{X}{\epsilon_i} + 1}} \int_{n\ell_i}^{(n+1)\ell_i} |U_i(x, [\frac{t}{h_i}] h_i) - U_i(x, [\frac{\tau_s}{h_i}] h_i)| dx = \\ &= \sum_{\substack{|n| < \frac{X}{\epsilon_i} + 1}} |u_n^{[t/h_i]} - u_n^{[\tau_s/h_i]}| \quad \ell_i = \lfloor h_i \rfloor \left[ \frac{t}{h_i} \right] - \left[ \frac{\tau_s}{h_i} \right] \uparrow \\ &\quad \text{Lemma 4} \end{aligned}$$

$$\leq \lfloor |t - \tau_s| \rfloor < \frac{\varepsilon}{3} \quad \text{for } s \text{ large enough.}$$

Analogously,  $I_3 < \frac{\varepsilon}{3}$ . Thus  $I_{ij} \leq 3 \cdot \frac{\varepsilon}{3} = \varepsilon$ .

We have proved pointwise limit for every  $t \in (0, T)$ , that is  $\exists u(x, t) \in L_1(|x| \leq X)$  (in part, measurable)

Fourth, let us show that  $I_{ij} \rightarrow 0$  uniformly in  $t$ ,  $0 \leq t \leq T$ . Indeed, fix  $\varepsilon > 0$ . Choose finite subset  $\mathcal{F} \subset \{t_m\}$  such that if  $0 \leq t \leq T$  there is a  $t_m \in \mathcal{F}$  such that  $L(t-t_m) < \frac{\varepsilon}{3}$ . Then we choose  $i, j$  so large that  $I_{ij} < \frac{\varepsilon}{3}$  for all  $t_m \in \mathcal{F}$  (it is possible because  $\mathcal{F}$  is finite). This reasoning gives us the desired uniformity int.

Fifth, using uniform convergence, we have

$$\forall \tau \in (0, T] \quad \int_{\tau}^T I_{ij} dt \rightarrow 0.$$

Now we write

$$\int_0^T = \int_0^\tau + \int_\tau^T : \quad \underbrace{\int_\tau^T}_{< \frac{\varepsilon}{2} \text{ for } i, j \text{ suffic. large}} + \underbrace{\int_0^\tau}_{< \frac{\varepsilon}{2} \text{ if } 8N\chi_\tau < \varepsilon} |U_i - U_j| dx dt = \iint_{\Omega \times (0, T)} |U_i - U_j| dx dt + \int_{\Omega \times [\tau, T]} |U_i - U_j| dx dt < \varepsilon$$

That means

$$\int_0^T I_{ij} dt \rightarrow 0 \quad \text{as } i, j \rightarrow +\infty.$$

Sixth, since local convergence in  $L_1$  implies pointwise convergence a.e. of a subsequence, we see

$$|U_i| \leq M \Rightarrow |u| \leq M$$

and Lemma 5  $\Rightarrow$  (S)

Step 3: Let us show that the limiting function  $u(x, t)$ , indeed, satisfies the properties from thm.

Lemma 7 (entropy inequality) :  $u$  satisfies (E).

Proof:

It is sufficient to show that if  $(x_1 - x_2) > 2l_i$  and  $t > h_i$  then

$$\frac{U_i(x_1, t) - U_i(x_2, t)}{x_1 - x_2} < \frac{2E}{t - h_i}.$$

Let  $x_1 > x_2$  and note that

$$U_i(x_j, t) = U_i(x_j - \gamma_j, [\frac{t}{h_i}] h_i) \quad j=1,2$$

for some  $0 \leq \gamma_j < l_j$ . Thus,

$$\frac{U_i(x_1, t) - U_i(x_2, t)}{x_1 - x_2} = \frac{1}{x_1 - x_2} \sum_{\substack{\text{over all integers in the interval } [x_2 - \gamma_2, x_1 - \gamma_1]}} (u_n^k - u_{n-2}^k) \quad \text{for } k = [\frac{t}{h_i}]$$

Using Lemma 2, we have

$$\begin{aligned} \frac{U_i(x_1, t) - U_i(x_2, t)}{x_1 - x_2} &\leq \frac{E(x_1 - \gamma_1 - x_2 + \gamma_2)}{[\frac{t}{h_i}] h_i (x_1 - x_2)} \leq \frac{E(x_1 - \gamma_1 - x_2 + \gamma_2)}{(t - h_i)(x_1 - x_2)} \\ &= \frac{E}{t - h_i} + \frac{E(\gamma_2 - \gamma_1) < l_i}{(t - h_i)(x_1 - x_2)} < \frac{2E}{t - h_i} \blacksquare \end{aligned}$$

Lecture 10 : Let's finish proving theorem on  $\exists$  of entropy solution

Reminder : Scalar conservation law :  $\begin{cases} u_t + (f(u))_x = 0 \\ u|_{t=0} = u_0(x) \end{cases} \quad (*)$

- $u : \mathbb{R} \times \mathbb{R}_+ \rightarrow \mathbb{R}$  - bounded, measurable

$f : \mathbb{R} \rightarrow \mathbb{R}$ ,  $f \in C^2$ ,  $f'' > 0$ . As we will see it is enough to define  $f$  on the convex hull of values  $u_0$ .

We understand solutions in weak sense :

$$\iint_{t>0} [u\varphi_t + f(u)\varphi_x] dx dt + \int_{t=0} u_0 \varphi dx = 0 \quad (**)$$

for every test function  $\varphi \in C_0^\infty$ .

Lemma 8 (last lemma)

Let  $U_i$  be a convergent subsequence from Lemma 6.

We know that  $U_i \rightarrow u(x,t)$ ,  $i \rightarrow +\infty$ , and  $\forall x \in \mathbb{R}$

$$\int_{-x}^x |U_i(x,0) - u_0(x)| dx \rightarrow 0.$$

Then  $u$  satisfies (\*\*), i.e.  $u$  is a weak solution of (\*).

Proof.

► Rewrite (D) in such a form:

$$\frac{u_n^{k+1} - u_n^k}{h} - \frac{u_{n+1}^k - 2u_n^k + u_{n-1}^k}{2e^2} \cdot \frac{e^2}{h} + \frac{f(u_{n+1}^k) - f(u_{n-1}^k)}{2e} = 0$$

Multiply this equality by  $\varphi_n^k = \varphi(nl, kh)$  and get

$$\frac{\varphi_n^{k+1} u_n^{k+1} - \varphi_n^k u_n^k}{h} - u_n^{k+1} \frac{\varphi_n^{k+1} - \varphi_n^k}{h} + \frac{e^2}{h} \cdot u_n^k \cdot \frac{2\varphi_n^k - \varphi_{n+1}^k - \varphi_{n-1}^k}{e^2}$$

$$+ \frac{\varphi_{n+1}^k u_n^k - \varphi_n^k u_{n-1}^k}{2h} + \frac{\varphi_{n-1}^k u_n^k - \varphi_n^k u_{n+1}^k}{2h} +$$

$$+ \frac{\varphi_{n+1}^k f(u_{n+1}^k) - \varphi_{n-1}^k f(u_{n-1}^k)}{2e} - f(u_{n+1}^k) \frac{\varphi_{n+1}^k - \varphi_n^k}{2e}$$

$$- f(u_{n-1}^k) \frac{\varphi_n^k - \varphi_{n-1}^k}{2e} = 0$$

Since  $\varphi \in C_0^\infty$  has compact support, we may assume  
 $\varphi_n^k = 0$  if  $k \geq [\frac{T}{h}]$

Multiply this equality by  $hl$  and sum over  $n \in \mathbb{Z}$ ,  $k \in \mathbb{N} \cup \{\infty\}$ .

$$\sum_{k,n} \frac{\varphi_n^{k+1} u_n^{k+1} - \varphi_n^k u_n^k}{h} = - \sum_n \varphi_n^\infty u_n^\infty \quad (\text{telescopic sum})$$

$$\sum_{k,n} \frac{\varphi_{n+1}^k u_n^k - \varphi_n^k u_{n-1}^k}{2h} = 0 \quad \text{and} \quad \sum_{k,n} \frac{\varphi_{n-1}^k u_n^k - \varphi_n^k u_{n+1}^k}{2h} = 0$$

Thus,

$$-h \sum_n \varphi_n^\infty u_n^\infty + hl \left[ \sum_{k,n} \left[ -u_n^{k+1} \frac{\varphi_n^{k+1} - \varphi_n^k}{h} - \frac{l^2}{2h} \frac{\varphi_{n+1}^k + \varphi_{n-1}^k - 2\varphi_n^k}{2e} \right] - \sum_{k,n} f(u_{n+1}^k) \frac{\varphi_{n+1}^k - \varphi_n^k}{2e} - \sum_{k,n} f(u_{n-1}^k) \frac{\varphi_n^k - \varphi_{n-1}^k}{2e} \right] = 0$$

Instead of a sum for  $u_n^k$  we can write integral for  $U_{h,e}$

$$\begin{aligned} & - \int_{t=0} U_{h,e} \varphi + \delta_1 - \iint_{t \geq 0} U_{h,e} \varphi_t + \delta_2 - \frac{l^2}{2h} \iint_{t \geq 0} U_{h,e} \varphi_{xx} \\ & + \delta_3 - \iint_{t \geq 0} f(U_{h,e}) \varphi_x + \delta_4 = 0 \end{aligned}$$

where  $\delta_i \rightarrow 0$  as  $h, l \rightarrow 0$ . Replace  $U_{h,e}$  by  $U_i$ :

$$- \int_{t=0} U_i \varphi - \iint_{t \geq 0} U_i \varphi_t - \frac{l_i^2}{2h_i} \iint_{t \geq 0} U_i \varphi_{xx} - \iint_{t \geq 0} f(U_i) \varphi_x = \delta(h_i, l_i)$$

$\downarrow i \rightarrow \infty$   
0

$l_i \rightarrow 0$ ,  $\frac{l_i^2}{h_i}$  is bounded;  $\frac{l_i^2}{h_i} \rightarrow 0$ ;  $U_i \rightarrow u$  in  $L^2$ -loc

$$\Rightarrow \iint_{t \geq 0} U_i \varphi_t - \frac{l_i^2}{2h_i} \iint_{t \geq 0} U_i \varphi_{xx} \rightarrow \iint_{t \geq 0} u \varphi_t$$

By choice of initial values:  $\int_{t=0}^T U_i \varphi \rightarrow \int_{t=0}^T u_0 \varphi$

Also,  $\left| \sum_{t \geq 0} (f(u_i) - f(u)) \varphi_x \right| \leq \| \varphi_x \|_\infty \sum |f(u_i) - f(u)|$   
 $\leq \| \varphi_x \|_\infty \sum |f'(s)| \cdot |u_i - u| \rightarrow 0$   
 D:  $\varphi \neq 0$

And we have:

$$\sum_{t \geq 0} f(u_t) \varphi_x \rightarrow \sum_{t \geq 0} f(u) \varphi_x.$$

We have proved (\*\*\*) for  $\forall \varphi \in C_0^3$ .

$C_0^3 \subset C_0^\infty$  is a dense subset, then (\*\*\*\*) are also true for  $\varphi \in C_0^\infty$ . ■

Now, let's prove the theorem on uniqueness.

Thm 2 (!): Let  $f \in C^2$ ,  $f'' > 0$ .

Let  $u, v$  be 2 solutions of (\*\*\*) satisfying entropy condition (E):  $\exists E \forall a > 0, t > 0, x \in \mathbb{R}$

$$\frac{u(x+a) - u(x)}{a} < \frac{E}{t}. \quad (\text{E})$$

Then  $u = v$  almost everywhere in  $t > 0$ .

Rmk 1: we call such a solution - an entropy sol.

Rmk 2: If we had a linear operator, then the main idea of the proof could be as follows (we will adapt this idea to non-linear)  
 Let  $H$  be a Hilbert space.

$A: H \rightarrow H$ ,  $\mathcal{Z}(A) = \{g \in H : A(g) = 0\}$  - null space

$R(A) = \{f \in H : \exists g \in H : A(g) = f\}$  - range of  $A$

$A^*$  is the adjacent operator:

$$\langle Ax, y \rangle = \langle x, A^*y \rangle$$

$R(A^*)$  is the orthogonal complement of  $\gamma(A)$

Fact:  $R(A^*) \oplus \gamma(A) = H$

The "bigger" is  $R(A^*)$ , the "smaller" is  $\gamma(A)$ .

That means that if there exist sufficiently many solutions to the adjoint equation, then the null space of  $A$  is zero  $\Rightarrow A$  has a unique solution!

If  $Ax = Ay$  we can choose  $w: A^*w = x-y$ :

$$\|x-y\| = \langle x-y, x-y \rangle = \langle x-y, A^*w \rangle = \langle Ax-Ay, w \rangle = 0$$

$\Rightarrow x=y$  (idea of Holgram ~1901)

But we have a nonlinear eq!

Let us adapt this idea.

Proof of thm 2.

► Let  $u, v$  be 2 solutions of (\*\*).

In order to prove that  $u=v$  a.e. in  $t>0$  it suffices to show that  $\forall \varphi \in C_0^1$ :

$$\iint_{t>0} (u-v) \varphi = 0.$$

$t>0$

What we know? Let  $\psi \in C_0^1$ , then

$$(1) \quad \iint_{t>0} [u \psi_t + f(u) \psi_x] dx dt + \int_{t=0} u_0 \varphi dx = 0$$

$$(2) \quad \iint_{t>0} [v \psi_t + f(v) \psi_x] dx dt + \int_{t=0} v_0 \varphi dx = 0$$

Subtract (1) - (2) and we get:

$$\iint_{t>0} (u-v) \left[ \psi_t + \underbrace{\frac{f(u)-f(v)}{u-v} \cdot \psi_x} \right] dx dt = 0$$

$$=: F(x, t)$$

$$\iint_{t>0} (u-v) [\psi_t + F \psi_x] dx dt = 0$$

?  $\varphi \in C_0^1$

Now if for  $\forall \varphi \in C_0'$  we could solve the linear (adjoint!) equation and have a solution  $\psi \in C_0'$ , we could conclude that  $u=v$  a.e.

However, there is an obstruction to this approach : "velocity field"  $F$  is not smooth (not even continuous), so it is not clear why solution  $\psi \in C_0'$ .

To struggle this difficulty, one can approximate  $u$  and  $v$  by smooth functions and solve corresponding linear eqs :

$$(M) \quad \psi_t^m + F_m \psi_x^m = \varphi, \quad F_m = \frac{f(u_m) - f(v_m)}{u_m - v_m}$$

$$\begin{aligned} \text{Then } \iint_{t \geq 0} (u-v) \varphi &= \iint_{t \geq 0} (u-v) [\psi_t^m + F_m \psi_x^m] = \\ &= -\underbrace{\iint_{t \geq 0} (u-v) [\psi_t^m + F \psi_x^m]}_{=0} + \iint_{t \geq 0} (u-v) [\psi_t^m + F_m \psi_x^m] = \\ &= \iint_{t \geq 0} (u-v) \cdot [F_m - F] \cdot \psi_x^m \end{aligned}$$

If  $F_m \rightarrow F$  locally in  $L_1$ ,  $\psi_x^m$  is bounded (independently of  $m$ ), then we could pass to the limit and get  $=0$ .

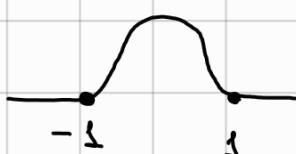
So our plan is :

(1) approximate  $u, v$  by smooth functions  $u_m, v_m$  such that  $u_m \rightarrow u$   $v_m \rightarrow v$   $F_m \rightarrow F$  locally in  $L_1$

(2) show that for  $\forall \varphi \in C_0^1$  there exists  $\psi \in C^\infty$   
- a solution of  $\psi_t^n + F_n \psi_x^n = \varphi$  and it's derivative  
 $\psi_x^n$  is bounded (independently of  $n$ )  
We will use entropy ineq. (E) HERE!

Step (2): One of the classical ideas to get  
a "smoother" function from any function  
 $u$  is to use convolution with "good kernel."

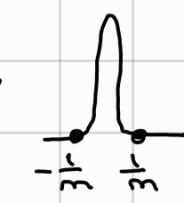
Consider  $\omega(x)$  the standard "hat" function (bump)



$$\omega(x) = \begin{cases} e^{-\frac{1}{|x|^2-1}}, & |x| \leq 1 \\ 0, & |x| \geq 1 \end{cases}$$

$\omega_m = c_m \omega(mx)$  is a "hat"

on the interval  $[-\frac{1}{m}, \frac{1}{m}]$



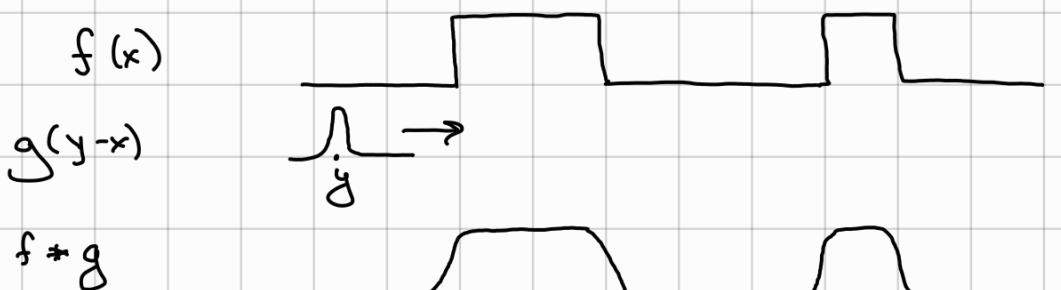
Properties:

- 1)  $\omega_m \in C^\infty(\mathbb{R})$  (exercise)
- 2)  $\omega_m \geq 0$ ,  $\text{supp } (\omega_m) = [-\frac{1}{m}, \frac{1}{m}]$
- 3)  $\int_{\mathbb{R}} \omega_m = 1$
- 4)  $\omega_m \xrightarrow[m \rightarrow \infty]{} \delta(x)$

Let  $u_m = u * \omega_m$  and  $v_m = v * \omega_m$ , where

$$(f * g)(y) = \int_{\mathbb{R}} f(x) g(y-x) dx - \text{convolution}$$

Have in mind such a picture



See blue & brown about convolution

$f * g$  at point  $y$   
is just averaging  
of  $f$  in a small  
neighbourhood of  
point  $y$ .

Properties of  $u_m = u * \omega_m$ :

- (a)  $u \in L^{\frac{r}{\alpha}}_{loc} \Rightarrow u_m \in C^{\infty}$
- (b)  $u_m \rightarrow u$  in  $L^{\frac{r}{\alpha}}_{loc}$
- (c)  $F_m \rightarrow F$  in  $L^{\frac{r}{\alpha}}_{loc}$ .

First studied by Kurt Otto Friedrichs (1944) and Sergey Sobolev (1938)

Proof:

► a)

$$u_m(y) = \int_{\mathbb{R}} u(x) \omega_m(y-x) dx$$

$$\frac{u_m(y+h) - u_m(y)}{h} = \int_{\mathbb{R}} u(x) \cdot \frac{\omega_m(y+h-x) - \omega_m(y-x)}{h} dx$$

↓ Lebesgue theorem  
 $\int_{\mathbb{R}} u(x) \cdot \frac{\partial}{\partial y} \omega_m(y-x) dx$  etc

and  $\omega_m \in C^{\infty}$

$$\begin{aligned} b) \quad u_m - u &= \int_{\mathbb{R}} \omega_m(y-x) [u(x) - u(y)] dx = \\ &= \int_{-\infty}^{\infty} \int_{\mathbb{R}} \omega_m(z) [u(y+z) - u(y)] dz = \\ &= \int_{-\frac{1}{m}}^{\frac{1}{m}} \int_{\mathbb{R}} \omega_m(z) [u(y+z) - u(y)] dz \\ \Rightarrow \int_K |u_m - u| dy &\leq \int_K dy \int_{-\frac{1}{m}}^{\frac{1}{m}} \omega_m(z) [u(y+z) - u(y)] dz \\ &\leq \int_{-\frac{1}{m}}^{\frac{1}{m}} \omega_m(z) dz \cdot \sup_{|z| < \frac{1}{m}} \int_K [u(y+z) - u(y)] dy \\ &\qquad\qquad\qquad \downarrow m \rightarrow \infty \\ &\qquad\qquad\qquad 0 \\ \Rightarrow u_m &\rightarrow u \text{ in } L^{\frac{r}{\alpha}}_{loc}. \end{aligned}$$

c) Write  $F_m$  as follows:

$$F_m(x, t) = \frac{f(u_m) - f(v_m)}{u_m - v_m} = \frac{1}{u_m - v_m} \int_{u_m}^{v_m} f'(s) ds = \frac{1}{\Theta} \int_0^1 f'(\theta u_m + (1-\theta)v_m) d\theta$$

Analogously,  $F(x, t) = \int_0^t f'(u\theta + v(1-\theta)) d\theta$ .

Let  $c := \max_{|u| \leq M} |f''(u)|$ . Then

$$F - F_m = \int_0^t \left[ f'(u\theta + (1-\theta)v) - f'(u_m\theta + (1-\theta)v_m) \right] d\theta =$$

$$= \int_0^t f''(\xi) [\theta(u-u_m) + (1-\theta)(v-v_m)] d\theta, \text{ where}$$

$\xi$  is between  $\theta u + (1-\theta)v$  and  $\theta u_m + (1-\theta)v_m$ .

Due to estimates  $|u|, |v|, |u_m|, |v_m| \leq M$ , we have  $|\xi| \leq M$ .

Thus,

$$|F(x, t) - F_m(x, t)| \leq c \int_0^t [\theta |u-u_m| + (1-\theta) |v-v_m|] d\theta \leq$$

$$\leq c (|u-u_m| + |v-v_m|)$$

Then for any compact set  $K$  in  $\{t \geq 0\}$

$$\iint_K |F(x, t) - F_m(x, t)| \leq c \iint_K |u-u_m| + c \cdot \iint_K |v-v_m| \rightarrow 0$$

$\downarrow$        $\downarrow$

]

■

Lecture 11: Let's finish proving uniqueness of entropy sol.

Reminder: Scalar conservation law :  $\begin{cases} u_t + (f(u))_x = 0 \\ u|_{t=0} = u_0(x) \end{cases} \quad (*)$

•  $u: \mathbb{R} \times \mathbb{R}_+ \rightarrow \mathbb{R}$  - bounded, measurable

$f: \mathbb{R} \rightarrow \mathbb{R}$ ,  $f \in C^2$ ,  $f'' > 0$ .

We understand solutions in weak sense :

$$\iint_{t>0} [u\varphi_t + f(u)\varphi_x] dx dt + \int_{t=0} u_0 \varphi dx = 0 \quad (**)$$

for every test function  $\varphi \in C_0^1$ .

Thm 2 (!):

Let  $u, v$  be 2 solutions of  $(**)$ , satisfying entropy condition  $(E): \exists E \forall a > 0, t > 0, x \in \mathbb{R}$

$$\frac{u(x+a) - u(x)}{a} < \frac{E}{t}. \quad (E)$$

Then  $u=v$  almost everywhere in  $t > 0$ .

Proof:

► Our plan is as follows:

① We want to show that  $\forall \varphi \in C_0^1$  :

$$\iint_{t>0} (u-v)\varphi = 0 \quad [\Rightarrow u=v \text{ a.e.}]$$

From  $(**)$  we have  $\iint_{t>0} (u-v)[\psi_t + F(x,t)\psi_x] = 0$   
 $\forall \psi \in C_0^1$

$$\text{for } F(x,t) = \frac{f(u(x,t)) - f(v(x,t))}{u(x,t) - v(x,t)}.$$

So if  $\forall \varphi \in C_0^1 \exists \psi \in C_0^1$  such that

$$\psi_t + F(x,t)\psi_x = \varphi \quad - \text{we would be done!}$$

Unfortunately this is not true as  $u, v$  can be discontinuous and  $F$  is not necessarily smooth

We need to use a PDE trick - "smoothen"  $u, v$

② Consider

$$u_m = u * \omega_m \in C^\infty ; u_m \xrightarrow{L^1} u$$

$$v_m = v * \omega_m \in C^\infty ; v_m \xrightarrow{L^1} v$$

$$F_m = \frac{f(u_m) - f(v_m)}{u_m - v_m} ; F_m \xrightarrow{L^1} F$$

We have identity: fix  $\varphi \in C_0^1$ : it is enough to prove

$$\iint_{t \geq 0} (u - v) \varphi = \iint_{t \geq 0} (u - v) [F_m - F] \cdot \psi_x^m \xrightarrow[m \rightarrow \infty]{} 0$$

where  $\psi_m$  is the solution of the equation:

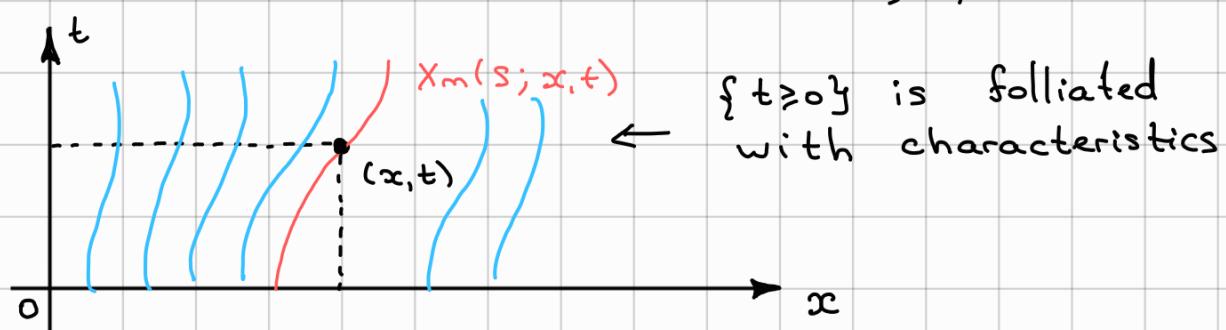
$$(M_1) \begin{cases} \psi_t^m + F_m(x, t) \psi_x^m = \varphi \\ \psi^m(x, T) = 0 \end{cases}$$

Here we may choose  $T$  so big such that  $\varphi(x, t) = 0$  for  $t \geq T$ .

Notice that as  $F_m$  at least  $C^1$ , we obtain that the characteristic ODE:

$$\begin{cases} \frac{dx_m}{ds} = F_m(x_m, s) \\ x_m|_{s=t} = x \end{cases}$$

has a unique solution  $x_m(s)$ . It will be important for us the initial point  $(x, t)$ , so we will denote such solution  $x_m(s; x, t)$ .



Lemma (solution to inhomogeneous transport equation)

The solution of the problem  $(M_1)$

is given by:

$$\psi^m(x, t) = \int_T^t \varphi(x_m(s; x, t), s) ds.$$

Proof: This is once again Duhamel principle!

Indeed, let's check directly:

$$\psi_t^m = \underbrace{\varphi(x_m(t; x, t), t)}_{\varphi(x, t)} + \int_T^t \frac{d}{dt} \varphi(x_m(s; x, t), s) ds$$

$$\psi_x^m = \int_T^t \frac{d}{dx} \varphi(x_m(s; x, t), s) ds$$

$$\text{Thus, } \psi_t^m + F \psi_x^m = \varphi + \int_T^t \left[ \frac{d}{dt} + F \frac{d}{dx} \right] \varphi(x_m(s; x, t), s) ds$$

$= 0$

Indeed,  $\left[ \frac{d}{dt} + F \frac{d}{dx} \right]$  is the derivative along the characteristics. But if we move the starting point  $(x, t)$  along characteristics, the function  $\varphi$  does not change  $\Rightarrow \left[ \frac{d}{dt} + F \frac{d}{dx} \right] \varphi(x_m(s; x, t), s) = 0 \quad \forall s$ .

Corollary of Lemma:  $\psi^m \in C_0^1(t \geq 0)$ .

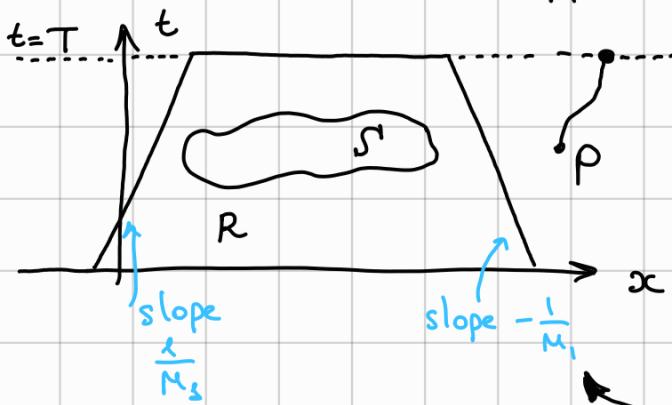
Proof:

$$\rightarrow \text{By Lemma } \psi^m = \int_T^t \varphi(x_m(s; t, x), s) ds$$

$$\text{As } \varphi \in C^1(t \geq 0) \Rightarrow \psi^m \in C^1(t \geq 0).$$

Why  $\psi^m$  has a compact support?

Let  $S$  be support of  $\varphi$  (as  $\varphi \in C_0^\infty$ )



$$\text{As } F_m(x, t) = \int_0^1 f'(\theta u_m + (1-\theta)v_m) d\theta$$

$$\Rightarrow |F_m| < M_S \quad (\text{as } f \in C^2)$$

Consider a trapezoid  $R$  as on the figure:

(a)  $S \subset R$

(b)  $R$  is bounded by four lines:  $t=0$ ,  $t=T$

$$\text{and } t = -\frac{1}{M_1} x + \text{const}_1; \quad t = \frac{1}{M_1} x + \text{const}_2$$

Let's show that  $\psi^m \equiv 0$  out of  $R$ :

1.  $\psi^m = 0$  for  $t \geq T$  because  $\phi \equiv 0$  there

2. Take  $P = (x_1, t_1) \notin R$ ,  $t_1 < T$ .

$x_m(s; x_1, t_1) \notin R \quad \forall s \Rightarrow x_m(T; x_1, t_1) \notin R$

$\Rightarrow \phi(x_m(s; x_1, t_1), s) = 0 \quad \forall s \Rightarrow \psi^m = 0$ . ■

Lemma (boundedness of  $|\psi_x^m|$ )

$\exists C$  (independent of  $m$ ):

$$|\psi_x^m| < C$$

Proof:

► The main ingredient of proof is the **entropy condition**:  $\forall a > 0, t > 0$

$$\frac{u(x+a, t) - u(x, t)}{a} < \frac{E}{t}.$$

We see that close to  $t=0$  the entropy condition spoils ( $\frac{E}{t} \rightarrow \infty$  as  $t \rightarrow 0$ ).

Let  $\alpha > 0$  be arbitrary. Then for  $\forall t \geq \alpha$  the function  $u(x, t) - \frac{Ex}{\alpha}$  is non-increasing in  $x$ : □

$$u(x+a, t) - \frac{E(x+a)}{\alpha} - u(x, t) + \frac{Ex}{\alpha} \leq \frac{Ea}{t} - \frac{Ea}{\alpha} = Ea\left(\frac{1}{t} - \frac{1}{\alpha}\right)$$

In what follows we will consider 2 cases:

(a)  $t \geq \alpha$

(b)  $0 \leq t \leq \alpha$

Case  $t \geq \alpha$

Claim:  $\frac{\partial u_m}{\partial x} \leq \frac{E}{\alpha}$ ;  $\frac{\partial v_m}{\partial x} \leq \frac{E}{\alpha}$

and  $\exists K = K_\alpha : \frac{\partial F_m}{\partial x} \leq K_\alpha$

► Indeed, the function  $w_n * (u - \frac{Ex}{\alpha}) = u_m - \frac{Ew_n * x}{\alpha}$  is also non-increasing (and smooth)

$$\Rightarrow \frac{\partial}{\partial x} \left( u_m - \frac{Ew_n * x}{\alpha} \right) = \frac{\partial u_m}{\partial x} - \frac{E}{\alpha} \leq 0$$

Analogously,  $\frac{\partial v_m}{\partial x} \leq \frac{E}{2}$ .

$$\begin{aligned}\frac{\partial F_m}{\partial x} &= \int_0^1 f''(\theta u_m + (1-\theta)v_m) \left[ \theta \frac{\partial u_m}{\partial x} + (1-\theta) \frac{\partial v_m}{\partial x} \right] d\theta \\ \Rightarrow \frac{\partial F_m}{\partial x} &\leq \int_0^1 f''(\theta u_m + (1-\theta)v_m) \left[ \theta \frac{E}{2} + (1-\theta) \frac{E}{2} \right] d\theta \\ &= \frac{E}{2} \int_0^1 f''(\theta u_m + (1-\theta)v_m) d\theta\end{aligned}$$

Therefore,

$$\frac{\partial F_m}{\partial x} \leq K_2 = \frac{E}{2} \max_{|u| \leq M} f''(u).$$

L

Let's use this to prove  $\left| \frac{\partial \psi^m}{\partial x} \right| \leq C, t \geq d$

$$\frac{\partial \psi^m}{\partial x} = \int_T^t \underbrace{\frac{\partial \varphi}{\partial x_m} \cdot \frac{\partial x_m}{\partial x}}_{\text{is bounded}} (s; x, t) ds$$

Let's examine  $\frac{\partial x_m}{\partial x}$ .

For convenience, denote  $a_m(s) = \frac{\partial x_m}{\partial x}(s; x, t)$

Here  $(x, t)$  - some fixed point in  $\{t \geq 0\}$ .

Notice  $x_m(t; x, t) = x$

$$\Rightarrow a_m(t) = \frac{\partial x_m}{\partial x} = 1.$$

How  $a_m(s)$  is changing with  $s$ ?

$$\frac{\partial a_m}{\partial s} = \frac{\partial}{\partial s} \frac{\partial x_m}{\partial x} = \frac{\partial}{\partial x} \frac{\partial x_m}{\partial s} = \frac{\partial}{\partial x} F_m(x_m, s) =$$

$$= \frac{\partial}{\partial x} F_m(x_m(s; x, t), s) = \frac{\partial F_m}{\partial x} \cdot \frac{\partial x_m}{\partial x} =$$

$$= \frac{\partial F_m}{\partial x} \cdot a_m \Rightarrow$$

$$\boxed{\frac{\partial a_m}{\partial s} = \frac{\partial F_m}{\partial x} \cdot a_m}$$

We can solve it:  $a_m(s) = \exp \left( \int_t^s \frac{\partial F_m}{\partial x}(x_m(z), z) dz \right)$

Since we have  $d \leq t \leq s \leq T$

$$\left| \frac{\partial x_m}{\partial x} \right| = |a_m(s)| = a_m(s) \leq e^{K_2(s-t)} \leq e^{K_2(T-d)}$$

$$\text{Thus, } \left| \frac{\partial \psi^m}{\partial x} \right| \leq \int_0^t \left| \frac{\partial \Phi}{\partial x} \right| \cdot \left| \frac{\partial x_m}{\partial x} \right| ds \leq \\ \leq (T-\alpha) \cdot C_1 \cdot e^{K_2(T-\alpha)} =: C$$

The most important is that  $C$  does not depend on  $m$ !

Case  $0 \leq t \leq \alpha$

Consider the total variation of  $\psi^m$  as a function of  $x$  for each fixed  $t > 0$ .

$$V_t(\psi^m) = \int_{\mathbb{R}} \left| \frac{\partial \psi^m}{\partial x} \right| dx$$

As  $\psi^m \in C_0^1$  and for  $t \geq \alpha$   $\left| \frac{\partial \psi^m}{\partial x} \right| \leq C$  we have

$$V_t(\psi^m) \leq C_\alpha, \quad t \geq \alpha$$

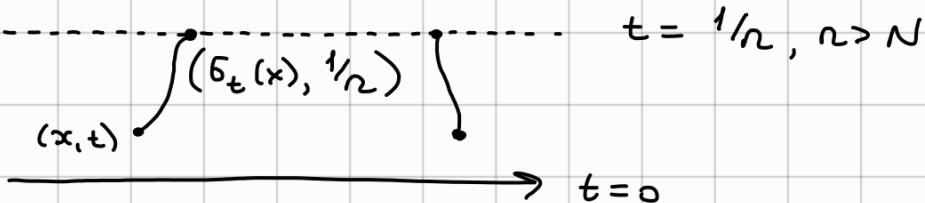
$\tau$  does not depend on  $m$ .

Rmk: let's show that  $\exists N \forall n > N$

$$V_t(\psi^m) \leq C_{1/n} \quad \forall t : 0 < t < \frac{1}{n} < \frac{1}{N}$$

Since  $\Phi$  has a compact support in  $\{t > 0\}$ , there exists  $N$ :  $\Phi(x, t) = 0$  if  $t < 1/N$ .

Thus,  $\psi_t^m + F_m \psi_x^m = 0$  if  $t < 1/N$



Let  $\delta_t : \mathbb{R} \rightarrow \mathbb{R}$  - bijection that takes  $\psi^m$  at time  $t$  as initial condition and sends it to solution  $\psi^m$  at time  $t = 1/n$ . As  $\psi^m$  is constant along characteristics, it is clear that

$$\sum_{k=1}^{p-1} |\psi^m(x_{k+1}, t) - \psi^m(x_k, t)| = \sum_{k=1}^{p-1} |\psi^m(\delta_t(x_{k+1}), \frac{1}{n}) -$$

for any finite sequence  $x_1 < x_2 < \dots < x_p$   $|\psi^m(\delta_t(x_k), \frac{1}{n})| \leq$

$$\leq V_{1/n}(\psi^m) \leq C_{1/n}.$$

Let's complete the proof of thm 2.

Fix  $\epsilon > 0$  - arbitrary. Take  $N$  from Rmk above.

Choose  $d > 0$  so small s.t.  $d < \frac{1}{n} \leq \frac{1}{N}$  and

$$4M M_1 C_{1,n} d < \frac{\epsilon}{2}.$$

For this  $d$  choose  $\tilde{M}$  so large that

$$\underset{t>d}{\iint} |u-v| \cdot |F_m - F| \cdot |\psi_x^m| < \frac{\epsilon}{2} \quad \text{if } m \geq \tilde{M}$$

This can be done since  $|u-v| \leq 2M$ ,  $|\frac{\partial \psi^m}{\partial x}| \leq k_2$  and  $F_m \rightarrow F$  in  $L^2_{\text{loc}}$ .

Then  $\left| \underset{t \geq 0}{\iint} (u-v) \varphi \right| \leq \underset{t \geq d}{\iint} + \underset{t < d}{\iint} \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$

↑  
see below

Now since  $d < \frac{1}{n} \leq \frac{1}{N}$

$$\underset{t < d}{\iint} |u-v| \cdot |F_m - F| \cdot |\psi_x^m| \leq 2M \cdot 2M_1 \underset{t < d}{\iint} |\psi_x^m| = 4MN \underset{\mathbb{R}}{\iint} |\psi_x^m|$$

$$= 4MN \underset{0}{\int} V_t(\psi^m) dt \leq 4MN C_{1,n} d < \frac{\epsilon}{2}$$

Thus,  $\underset{t \geq 0}{\iint} (u-v) \varphi = 0 \quad \forall \varphi \in C'_0 \Rightarrow u=v \text{ a.e.}$  ■