On chemical flooding models: Riemann problem solutions and viscous fingering

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IMPA

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Joint work with

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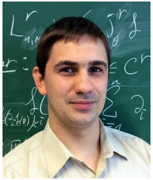


Sergey Tikhomirov

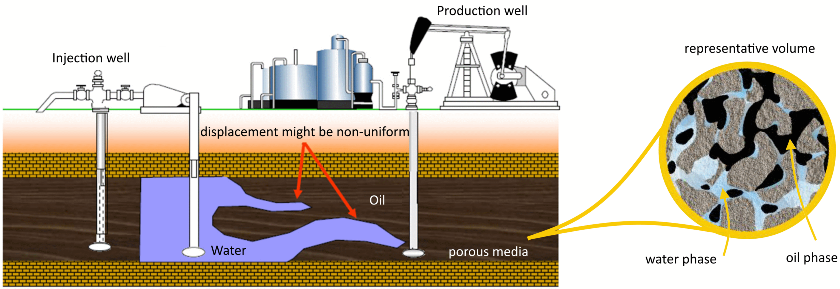
Yalchin Efendiev

- Collaboration with Russian petroleum company GazpromNeft (2018–2021)
- The talk is based on:
    - F. Bakharev, A. Enin, Yu. Petrova, N. Rastegaev "Impact of dissipation ratio on vanishing viscosity solutions of the Riemann problem for chemical flooding model" See arXiv:2111.15001
    - Ongoing research with S. Tikhomirov and Ya. Efendiev "A cascade of two travelling waves in a two-tube model of viscous and gravitational fingering"

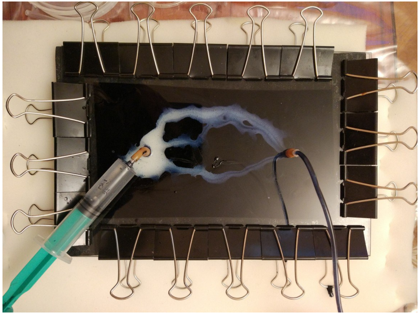
Motivation

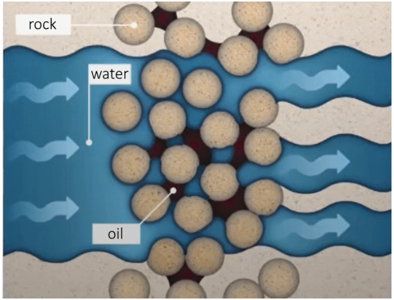
We are interested in the mathematical model of oil recovery. Some features:

- Porous media (averaged models of flow)
- Unknown variables: $s(t, x)$ — the averaged water saturation in small volume
  $1 - s$ — the average oil saturation in representative volume
- Relatively small speeds (≈ 1 meter per day): Navier-Stokes $\to$ Darcy's law
- Multiphase flow: oil, water, gas.
- Applications to EOR (enhanced oil recovery) methods: chemical, thermal, gas etc



Problems: macroscopic and microscopic sweep efficiency





- happens due to very viscous oil or inhomogeneous media
- local entrapment of oil in pores due to high capillary pressure

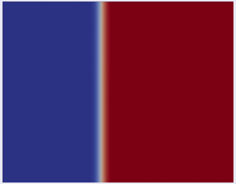
Possible solution

- Inject gas (CO_2, natural) to decrease the oil viscosity
- Add chemicals (polymer) to increase the water viscosity
- Add chemicals (surfactant) that reduce the surface tension etc

Fundamental research: two main directions

1-dim in spatial variable

- Stable displacement



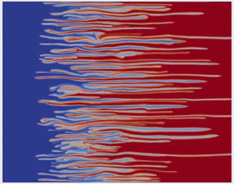
- main questions: find an exact solution to a Riemann problem; analyze stability
- hyperbolic conservation laws

$$s_t + f(s,c)_x = 0,$$
$$(cs + a(c))_t + (cf(s,c))_x = 0.$$

Example: chemical flooding model

2-dim (or 3-dim) in spatial variable

- Unstable displacement



- source of instability: water and oil/polymer have different viscosities
- viscous fingering phenomenon

$$c_t + u \cdot \nabla c = \varepsilon \Delta c,$$
$$\text{div}(u) = 0,$$
$$u = -\nabla p / \mu(c).$$

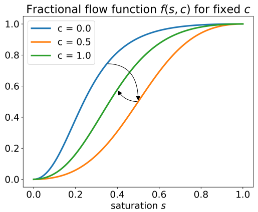
Example: Peaceman model

Problem statement

Chemical flooding can be described as the system of conservation laws ($x \in \mathbb{R}, t > 0$):

$$\begin{aligned} s_t + f(s,c)_x &= 0, && \text{(conservation of water)} \\ (cs + a(c))_t + (cf(s,c))_x &= 0. && \text{(conservation of chemical)} \end{aligned} \tag{1}$$

- $s = s(x,t)$ — water phase saturation;
- $f(s,c)$ — fractional flow function (usually S-shaped);
- $c = c(x,t)$ — concentration of a chemical in water;
- $a(c)$ — adsorption of a chemical on a rock (usually increasing, concave).



Initial data:

$$(s,c)\big|_{t=0} = \begin{cases} (s^L, c^L), & \text{if } x \leqslant 0, \\ (s^R, c^R), & \text{if } x > 0, \end{cases} \tag{2}$$

Aim:

Find a solution to initial-value problem (1)–(2) when f depends non-monotonically on c.

NB: Non-monotone dependence appears in surfactant flooding, low salinity water flooding etc

Preview of results

- weak solutions $\Rightarrow$ non-uniqueness of solutions to a Riemann problem

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- use vanishing viscosity criterion — add small diffusion/capillary terms

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- $f(s,c)$ monotone in $c \Rightarrow$ uniqueness of vanishing viscosity solution (1988)

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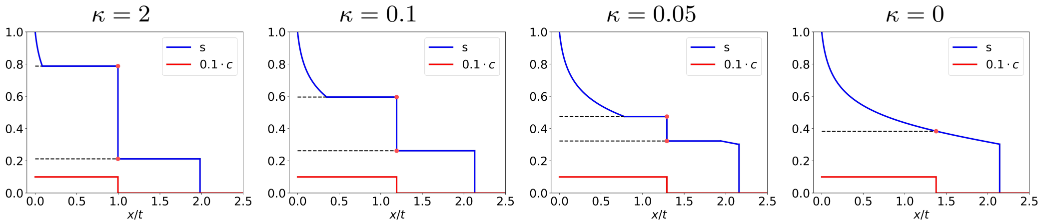
$$

- $f(s,c)$ monotone in c $\Rightarrow$ uniqueness of vanishing viscosity solution (1988)
- Main idea of the 1st part of the talk: $f(s,c)$ non-monotone in c $\Rightarrow$ exist multiple vanishing viscosity solutions, depending on ratio $\kappa = \varepsilon_d/\varepsilon_c$



NB: the appeared shock is known as undercompressive (transitional).

Hyperbolic systems of conservation laws*

$$G(u)_t + F(u)_x = 0 \tag{3}$$

- $G(u)$ — accumulation function (conserved quantities)
- $F(u)$ — flux function (flux of conserved quantities)

Simplest example: wave equation

$$y_{tt} - c^2 y_{xx} = 0 \qquad \text{(J. d'Alambert, 1750)}$$

can be rewritten as a system of two first-order equations on the state-vector $u = \begin{pmatrix} y_x \\ y_t \end{pmatrix}$

$$u_t + Du_x = 0, \qquad \text{with} \qquad D = \begin{pmatrix} 0 & -1 \\ -c^2 & 0 \end{pmatrix}$$

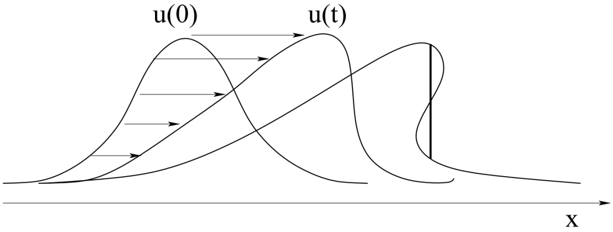
- eigenvalues $\lambda_1 = c$ and $\lambda_2 = -c$ are real, the system is hyperbolic. Solution contain two wave modes that propagate at the velocities λ_1 and λ_2.

* For more detailes see e.g. "Hyperbolic conservation laws: an illustrated tutorial" by Alberto Bressan.

Hyperbolic systems of conservation laws

$$u_t + \left(\frac{u^2}{2}\right)_x = 0$$ (Burger's equation, 1948)

- non-linearity implies wave speed $\lambda(u) = u$ depends on state u
- So the wave can spread (rarefaction wave) or concentrate (shock wave)



$$u_t + (f(u))_x = 0$$ (Buckley-Leverett equation)

- existence, uniqueness was established by Olga Oleinik (1957)

Riemann problem (1858)

- Riemann solved the initial-value problem with data having a single jump

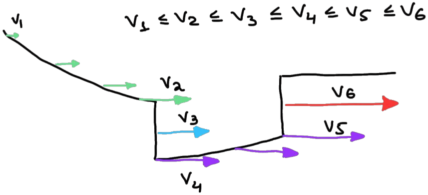
$$u\big|_{t=0} = \begin{cases} u^L, & x \leqslant 0; \\ u^R, & x > 0. \end{cases}$$

- took advantage of the scale invariance of the equations and the data:

$$u(\alpha\, x, \alpha\, t) = u(x,t) \quad \text{for all } \alpha > 0$$

- solution to a Riemann problem is important because:
    - it appears in a long-term behavior of Cauchy problem
    - helps to prove the existence of solutions to Cauchy problem (Glimm's method)
    - helps to construct numerical solution (Godunov method)

Any solution to a Riemann problem consists of a sequence of rarefaction or shock waves (and constant states) that are compatible by speeds

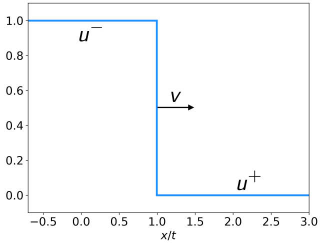


Shock waves: RH condition and admissibility criteria

- discontinuous solutions are defined in the sense of distributions (weak form)
- for a shock wave from u^- to u^+ moving with velocity v, the weak condition amounts to the following Rankine-Hugoniot condition (RH)

$$-v\,G(u^-) + F(u^-) = -v\,G(u^+) + F(u^+) \qquad \text{(RH)}$$

- RH means conservation: what flows into left side flows out of the right side



- exist various admissibility criteria: entropy, Lax, Liu, vanishing viscosity

Travelling wave solutions of diffusive system (Hopf, 1948)

- Vanishing viscosity criteria: consider a diffusive system of conservation laws

$$G(u)_t + F(u)_x = \varepsilon \left[B(u)\, u_x\right]_x , \qquad \varepsilon \to 0$$

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$$G(u)_t + F(u)_x = \varepsilon \left[B(u)\, u_x\right]_x\,, \qquad \varepsilon \to 0$$

- $u(x,t) = \hat{u}(\xi)$ with $\xi := x - v\,t$ for a fixed shock velocity v
- reduction to first-order system of ordinary differential equations:

$$\varepsilon\, B(\hat{u})\, \hat{u}_\xi = -v\left[G(\hat{u}) - G(u^-)\right] + F(\hat{u}) - F(u^-)$$

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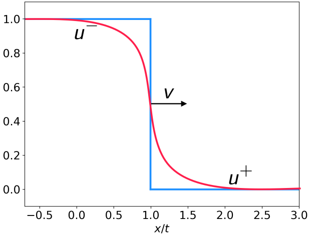
- u^- and u^+ are fixed points and we look for an orbit connecting them

$$\hat{u}(-\infty) = u^-, \qquad \hat{u}(+\infty) = u^+$$

- travelling wave solution approaches the jump discontinuity in L^1 as $\varepsilon \to 0^+$



Historical review: zero adsorption

$$\begin{aligned} s_t + f(s,c)_x &= 0, \\ (sc)_t + (cf(s,c))_x &= 0. \end{aligned} \qquad (s,c)\big|_{t=0} = \begin{cases} (s_L, c_L), & \text{if } x \leqslant 0, \\ (s_R, c_R), & \text{if } x > 0, \end{cases} \tag{4}$$

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Can be rewritten in a more symmetric form:

$$\begin{aligned} s_t + (sg(s,b))_x &= 0, \\ b_t + (bg(s,b))_x &= 0. \end{aligned}$$

$b = sc$ — total amount of chemicals

$g = f/s$ — new flux function

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Interesting properties:

- has coordinate system of Riemann invariants
- rarefaction and shock curves coincide

Problems:

- non-strictly hyperbolic
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Work in progress with D. Marchesin, B. Plohr: justification of Isaacson-Glimm admissibility criterion & construction of a solution for the case when $f(s,c)$ is non-monotone in c

Historical review: non-zero adsorption

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When $f(s,c)$ non-monotone in c, multiple vanishing viscosity solutions are possible.

- 2017 — W. Shen, gave some examples in Lagrangian coordinates. See also
- 1986 — Entov, Kerimov, non-rigorous consideration of the non-monotone case.

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Proposition (Johansen-Winther, 1988 (JW))

The solution $c(x,t)$ is monotone in x for any $t > 0$. If $a(c)$ is concave, then

- *is constant if $c_L = c_R$;*
- *contains several rarefaction waves if $c_L < c_R$;*
- *contains exactly one shock wave if $c_L > c_R$:*

$$(1,1) \xrightarrow{c=1} u^- \xrightarrow{c\text{-shock}} u^+ \xrightarrow{c=0} (0,0)$$

Dissipative system

To define a shock wave between u^- and u^+ we consider dissipative system:

$$
\begin{gathered}
s_t + f(s,c)_x = \varepsilon_c (A(s,c)s_x)_x,\\
(cs+\alpha)_t + (cf(s,c))_x = \varepsilon_c (cA(s,c)s_x)_x + \varepsilon_d c_{xx},\\
\alpha_t = \varepsilon_r^{-1}(a(c)-\alpha).
\end{gathered}
$$

- ε_c — dimensionless capillary pressure
- ε_d — dimensionless diffusion term
- ε_r — dimensionless relaxation time
- $A(s,c)$ — capillary pressure term
- $\alpha = \alpha(x,t)$ — dynamic adsorption

We consider two particular cases:

Capillarity and Diffusion

$$
\begin{aligned}
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Capillarity and Dynamic Adsorpsion

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* Bedrikovetsky, Pavel. Mathematical theory of oil and gas recovery: with applications to ex-USSR oil and gas fields. Vol. 4. Springer Science & Business Media, 1993.

Vanishing viscosity admissibility criteria

Admissibility criteria

A shock between $u^+ = (s^+, c^+)$ and $u^- = (s^-, c^-)$ is admissible if it could be obtained as a limit of smooth travelling wave solutions of the dissipative system as $\varepsilon_{c,d} \to 0$

$$\begin{aligned} s_t + f(s,c)_x &= \varepsilon_c (A(s,c)s_x)_x, \\ (cs + a(c))_t + (cf(s,c))_x &= \varepsilon_c (cA(s,c)s_x)_x + \varepsilon_d c_{xx}. \end{aligned} \tag{6}$$

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Searching for travelling wave solutions $s = s(\xi)$, $c = c(\xi)$, $\xi := \varepsilon_c^{-1}(x - vt)$ with boundary conditions $s(\pm\infty) = s^\pm$, $c(-\infty) = 1$, $c(+\infty) = 0$, we arrive at

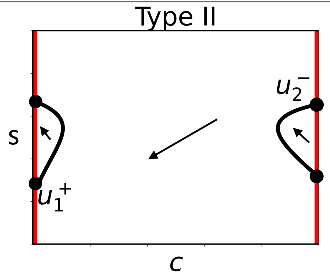
$$\boxed{\begin{aligned} s_\xi &= A^{-1}(s,c)(f(s,c) - v(s + d_1)), \\ c_\xi &= v\kappa^{-1}(d_1 c - d_2 - a(c)). \end{aligned}} \tag{7}$$

- Parameters: $\kappa = \varepsilon_d / \varepsilon_c$ and v. $d_1 = a(1)$ and $d_2 = 0$ — some constants.
- We are interested for which values of (v, κ) there exist orbits connecting two saddle points $u^\pm$ (or saddle-nodes) due to compatibility of speeds condition

Main result

Consider a dynamical system under technical restrictions on functions f, a, A:

$$s_\xi = A^{-1}(s,c)(f(s,c) - v(s+d_1)),$$
$$c_\xi = v\kappa^{-1}(d_1 c - d_2 - a(c)).$$



Theorem (Bakharev, Enin, P.,Rastegaev, 2021, arxiv:2111.15001)

There exist $0 < v_{\min} < v_{\max} < \infty$, such that for every $\kappa = \varepsilon_d/\varepsilon_c \in (0,+\infty)$, there exist unique

- *points $s^-(\kappa) \in [0,1]$ and $s^+(\kappa) \in [0,1]$;*
- *velocity $v(\kappa) \in [v_{\min}, v_{\max}]$,*

such that there exists a travelling wave, connecting two saddle points $u^-(\kappa) = (s^-(\kappa), 1)$ and $u^+(\kappa) = (s^+(\kappa), 0)$ with velocity $v(\kappa)$. Moreover, $v(\kappa)$ is monotone and continuous; $v(\kappa) \to v_{\min}$ as $\kappa \to \infty$; $v(\kappa) \to v_{\max}$ as $\kappa \to 0$.

Solution construction algorithm

1. From κ we calculate $v(\kappa)$ (binary search).
2. From v we determine $s^-(v)$ and $s^+(v)$ via Rankine-Hugoniot condition.
3. Construct waves $(1,1) \to (s^-(v), 1)$ and $(s(v), 0) \to (0, 0)$.

Example: "boomerang" model:

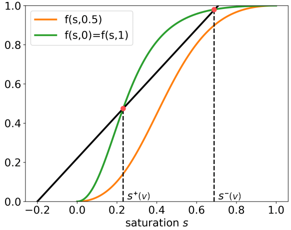


Figure 1: Flux functions

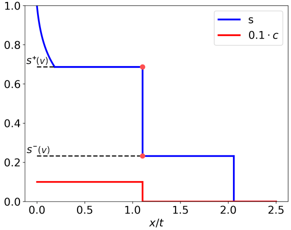


Figure 2: Solution s and c

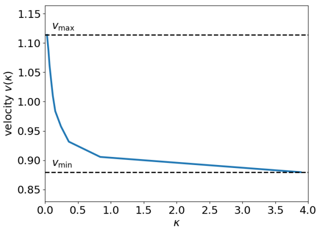


Figure 3: Function $v(\kappa)$

Possible future directions

- Existence and uniqueness:
  - to any Riemann problem (in progress)
    NB: appear transitional rarefaction waves
  - to a Cauchy problem
    NB: standard techniques (Glimm scheme and Temple functional) do not work due to non-linear resonance
- Stability and convergence
  - asymptotic stability for the travelling waves solutions?
  - does the whole solution of the dissipative system converge to a solution of non-dissipative system?
- Generalizations:
  - connection between the zero and non-zero adsorption models (in progress with D. Marchesin and B. Plohr)
  - three-phase flow with chemicals (water with chemicals, oil and gas):
    NB: travelling wave dynamical system will become three-dimensional, thus the analysis will be more complex.

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    NB: standard techniques (Glimm scheme and Temple functional) do not work due to non-linear resonance
- Stability and convergence
  - asymptotic stability for the travelling waves solutions?
  - does the whole solution of the dissipative system converge to a solution of non-dissipative system?
- Generalizations:
  - connection between the zero and non-zero adsorption models (in progress with D. Marchesin and B. Plohr)
  - three-phase flow with chemicals (water with chemicals, oil and gas):
    NB: travelling wave dynamical system will become three-dimensional, thus the analysis will be more complex.

Questions?

2-dim problems

Show video

System of equations

1. Transport of species ($\varepsilon \geq 0$)

$$c_t + u \cdot \nabla c = \varepsilon \Delta c$$

2. Incompressibility condition

$$\text{div}(u) = 0$$

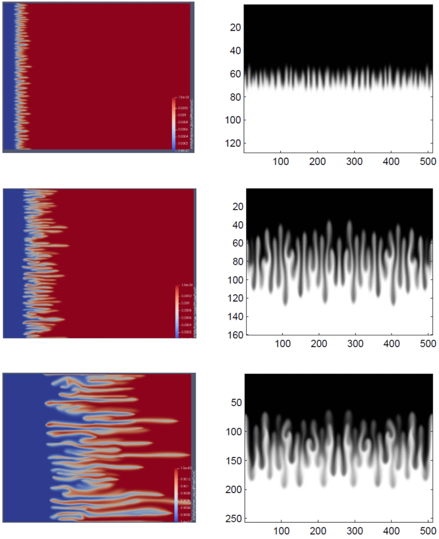
3a. Darcy's law (viscosity-driven)

$$u = -k \cdot m(c) \cdot \nabla p$$

3b. Darcy's law (gravity-driven)

$$u = -\nabla p + (0, c)$$

viscosity-driven fingers gravity-driven fingers



NB: 1 + 2+ 3b for $\varepsilon = 0$ is known as IPM (incompressible porous media equation)

Questions of interest

1. Well-posedness:
   - existence of a global solution vs finite-time blow-up:
     active scalar: $u = A(c)$ — singular integral operator (like in SQG)

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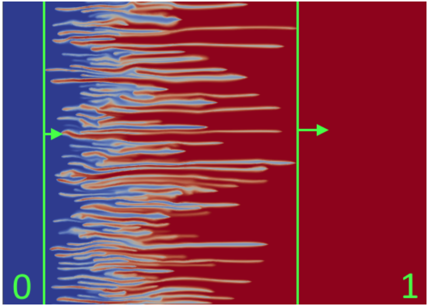

* Nijjer J., Hewitt D., and Neufeld J. The dynamics of miscible viscous fingering from onset to shutdown. Journal of Fluid Mechanics 837 (2018): 520-545.

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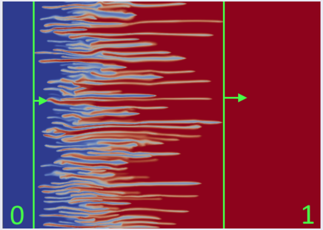
     - F. Otto, G. Menon'2006 — for a simplified model of Darcy's law

Transverse flow equilibrium model (F. Otto, G. Menon'2006)

- get rid of pressure

$$u = -\frac{m(c)}{\operatorname{avg}_y(m(c))}$$

- used techniques of sub / supersolutions
- estimated the mixing zone by two travelling waves connecting 0 to 1



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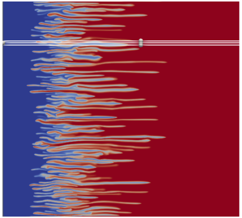
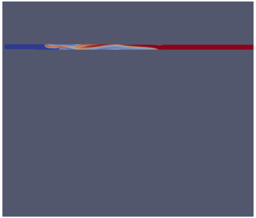
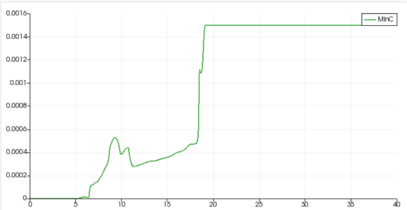
Our observation from numerical modelling:

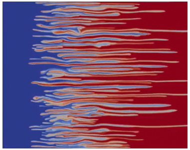
There exists some intermediate concentration that makes fingers move slower (for $\varepsilon > 0$)







Toy model of viscous fingering (work in progress)

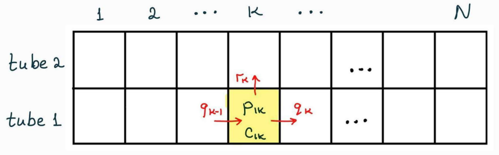


Discrete case

- system of $2N$ ODEs and N algebraic equations

Unknowns:

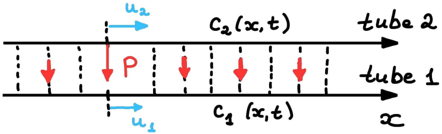
- $c_{1k}(t), c_{2k}(t)$ — concentrations
- p_{1k}, p_{2k} — pressures
- $q_k(t), r_k(t)$ — velocities



Continuous case

- two coupled advection-diffusion eqs

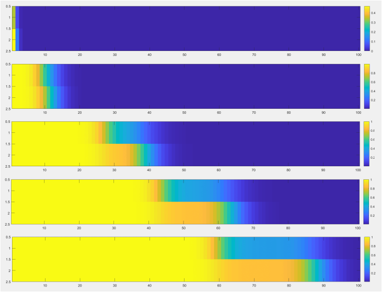
$$\partial_t c_1 = -\partial_x(u_1 c_1) + P + \varepsilon \partial_{xx} c_1,$$
$$\partial_t c_2 = -\partial_x(u_2 c_2) - P + \varepsilon \partial_{xx} c_2.$$



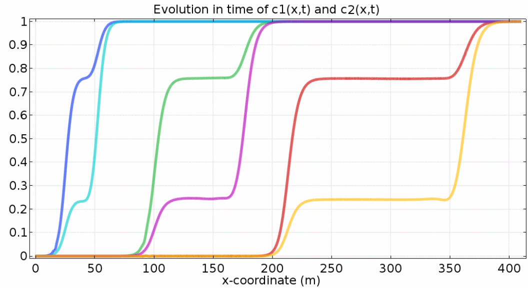
Flow between tubes: $P = (-1)^{1,2}\partial_x u_{1,2} c_{1,2}$. Work in progress with S. Tikhomirov,Y. Efendiev.

"Toy models" of fingering: numerical experiments

Discrete setting



Continuous setting



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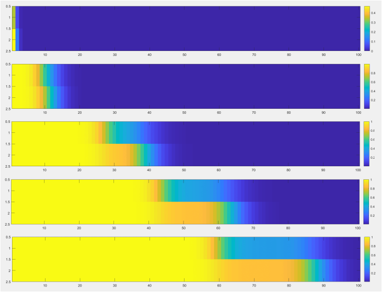

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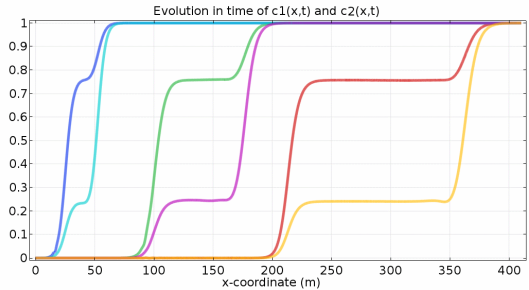
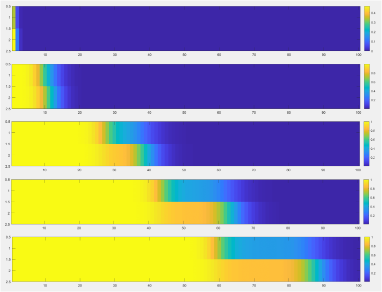


Result of experiments: cascade of two travelling waves: $(0,0) \xrightarrow{\mathrm{TW}_1} (c_1^*, c_2^*) \xrightarrow{\mathrm{TW}_2} (1,1)$

"Toy models" of fingering: numerical experiments

Discrete setting



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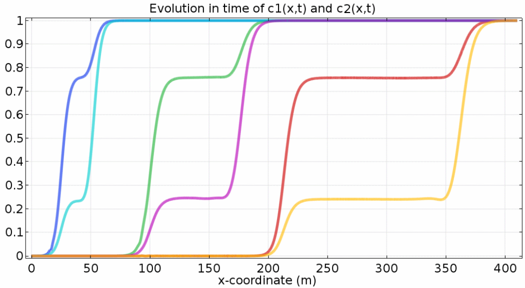


Result of experiments: cascade of two travelling waves: $(0,0) \xrightarrow{\mathrm{TW}_1} (c_1^*, c_2^*) \xrightarrow{\mathrm{TW}_2} (1,1)$

(simplified gravitational fingering) $u_1 = 0.5(c_2 - c_1) = -u_2$

(simplified viscous fingering) $u_1 = \dfrac{2m(c_1)}{m(c_1)+m(c_2)} = 2 - u_2$

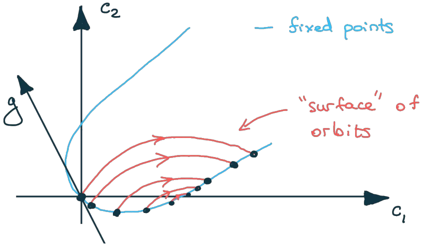
(Darcy's law with pressure) Biot-Savart Law (non-local)

"Toy models" of fingering: theoretical approach

For a travelling wave $c_1 = c_1(x - vt)$ and $c_2 = c_2(x - vt)$ we have a dynamical system.

Travelling wave dynamical system

$$c_1' = g_1,$$
$$g_1' = (u_1 - v)g_1,$$
$$c_2' = (u_1 - v)c_1 + (u_2 - v)c_2 - g_1.$$



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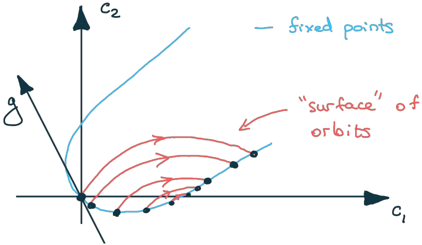


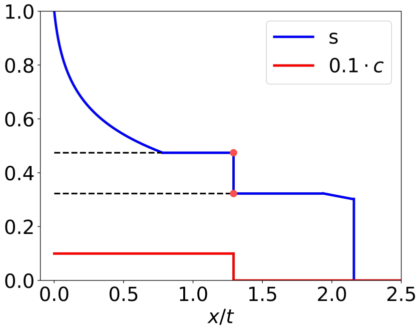
Challenges:

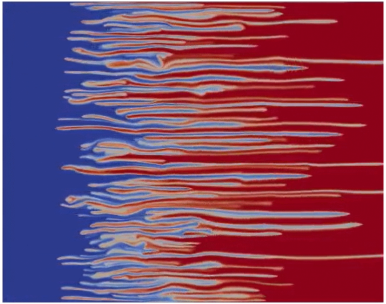
- Prove the existence of a cascade of two travelling waves in 2-tube model under Darcy's law. Is this configuration stable?
- Can one calculate the exact velocities and intermediate concentrations?
- Do these concentrations correspond to ones observed in 2-dim modelling?

Thank you for your attention!

yulia.petrova@impa.br

https://yulia-petrova.github.io/





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Own works:

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2. Yu. Petrova, D. Marchesin, B. Plohr, On admissibility criteria for contact discontinuities in Glimm-Isaacson model arising in chemical flooding. Work in progress. See slides.
3. Ya. Efendiev, S. Tikhomirov, Yu. Petrova, A cascade of two travelling waves in a two-tube model of viscous and gravitational fingering. Work in progress.

Other works:

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Restrictions on f and a

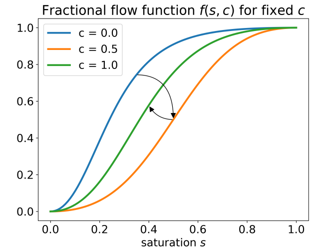
(F1) $f \in C^2([0,1]^2)$; $f(0,c) = 0$; $f(1,c) = 1$;

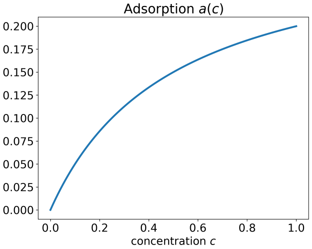
(F2) $f_s(s,c) > 0$ for $s \in (0,1)$, $c \in [0,1]$;
$f_s(0,c) = f_s(1,c) = 0$;

(F3) f is S-shaped in s;

(F4) f is non-monotone in c:
$\forall s \in (0,1)\, \exists c^*(s) \in (0,1)$:

- $f_c(s,c) < 0$ for $0 < s < 1$, $0 < c < c^*(s)$;
- $f_c(s,c) > 0$ for $0 < s < 1$, $c^*(s) < c < 1$;





(A) A is bounded from zero and infinity;

(a) $a \in C^2$, $a(0) = 0$, a is strictly increasing and concave.

NB: assumptions are just the simplest possible ones. Could be generalised.

Scheme of proof

The Theorem can be divided into simpler statements:

- $\forall v \in [v_{\min}, v_{\max}] \quad \exists! \kappa(v)$: there is a saddle-to-saddle travelling wave with $\kappa(v)$.
- $\kappa(v)$ is continuous.
- $\nexists v_1 \neq v_2 : \kappa(v_1) = \kappa(v_2)$, thus $\kappa(v)$ is monotone.
- $\kappa(v) \to \infty$ as $v \to v_{min}$.
- $\kappa(v) \to \kappa_{crit} \geqslant 0$ as $v \to v_{max}$.
- When $\kappa < \kappa_{crit}$ and $v = v_{\max}$ there is a saddle to saddle-node travelling wave

$\kappa(v)$ is monotone and continuous thus there exists an inverse function satisfying the Theorem.

Phase portrait, fixed points, isoclines

We consider travelling wave dynamical system:

$$\begin{aligned} A(s,c)s_\xi &= f(s,c) - \mathbf{v}(s+d_1), \\ \kappa c_\xi &= \mathbf{v}(d_1 c - d_2 - a(c)), \end{aligned}$$

Isoclines:

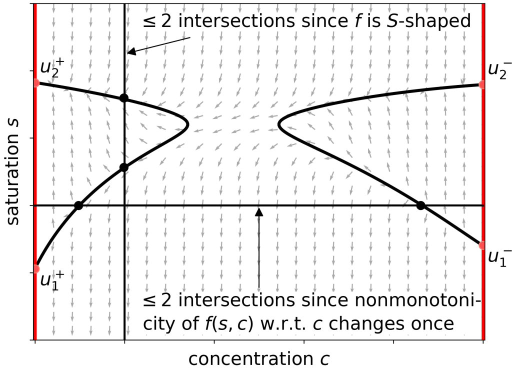
red lines are $d_1 c - d_2 - a(c) = 0$,

black lines are $f(s,c) - v(s+d_1) = 0$.

Fixed points:

u_1^+ and u_2^- — saddle points;

u_2^+ — attractor; u_1^- — repeller



Aim:

find all pairs (κ, v) for which there exists a trajectory from saddle point u_2^- to saddle point u_1^+

Nullcline configurations: main and intermediate classes

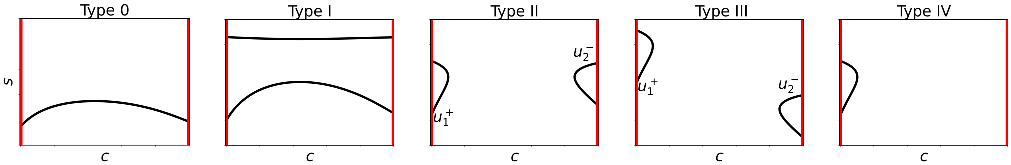


Figure 4: Five wide classes of nullcline configurations

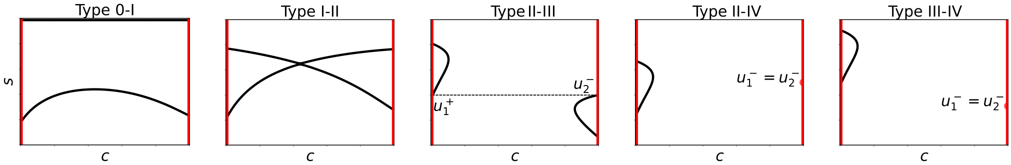
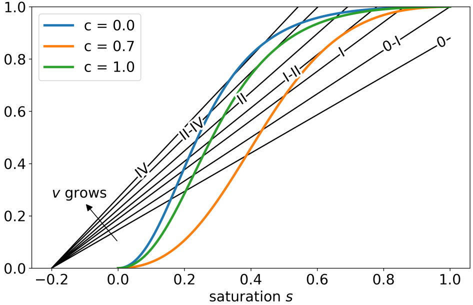


Figure 5: Intermediate types of nullcline configurations, appearing under assump. (F1)–(F4)

- Only Type II nullcline configuration has saddle-to-saddle connections.
- Type I-II corresponds to $v_{\min}$.
- Type II-III or Type II-IV correspond to $v_{\max}$.

Nullcline configurations: monotone dependence on v

black lines $f(s,c) = v(s + d_1)$



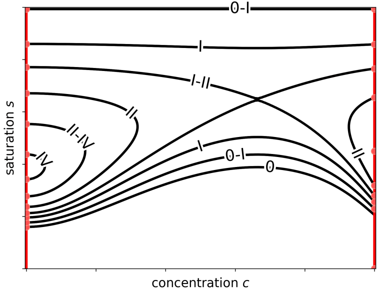


Figure 6: nullcline configuration evolution as v grows: Type 0 $\rightarrow$ Type I $\rightarrow$ Type II $\rightarrow$ Type IV

Nullcline configurations: bad cases

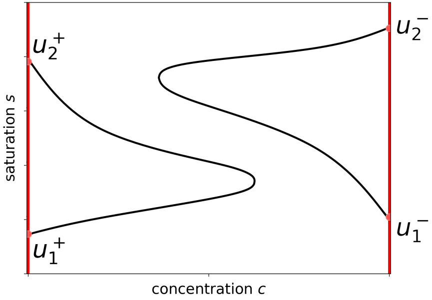


Figure 7: If f is not S-shaped.

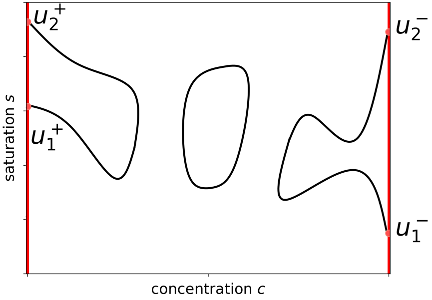


Figure 8: If non-monotonicity is more complex.

We believe that the similar result is true without conditions (F3)–(F4).

Type II configuration: for every v there exist κ

$$\boxed{\frac{s_\xi}{c_\xi} = \kappa \cdot \frac{v^{-1} f(s,c) - (s + d_1)}{A(s,c)(d_1 c - d_2 - a(c))}}$$

Used property: continuous and monotonous dependence of trajectories on κ.

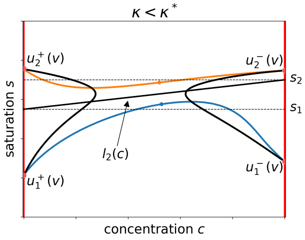


Figure 9: $\kappa << 1$.

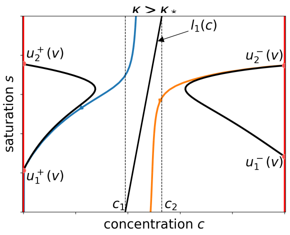


Figure 10: $\kappa >> 1$.

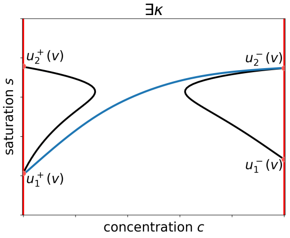


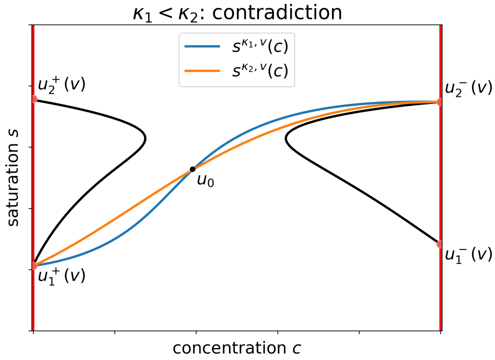
Figure 11: $\exists\kappa$.

Type II config.: $\kappa(v)$ is unique for every $v \in (v_{\min}, v_{\max})$.

If there are $\kappa_1 < \kappa_2$ for one v, then the corresponding trajectories must intersect, which leads to a contradiction.
The slope

$$\frac{s_\xi}{c_\xi} = \kappa \cdot \frac{v^{-1} f(s,c) - (s + d_1)}{A(s,c)(d_1 c - d_2 - a(c))}$$

is positive at the intersection point (s, c), so it strictly increases when κ increases.



NB: this property might be lost for more complex nullcline configurations.

Type II configuration: monotonicity of $\kappa(v)$

If $\kappa(v_1) = \kappa(v_2)$ for $v_1 < v_2$, then the corresponding trajectories must intersect, which leads to a contradiction. The slope

$$\boxed{\frac{s_\xi}{c_\xi} = \kappa \cdot \frac{v^{-1} f(s,c) - (s + d_1)}{A(s,c)(d_1 c - d_2 - a(c))}}$$

is positive at the intersection point (s, c), so it strictly increases when v increases.