A cascade of two travelling waves for the two-layer model of “gravitational fingering”

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Based on joint work with Sergey Tikhomirov, Yalchin Efendiev :

“Propagating terrace in a two-tubes model of gravitational fingering”, 2024, ArXiv: 2401.05981

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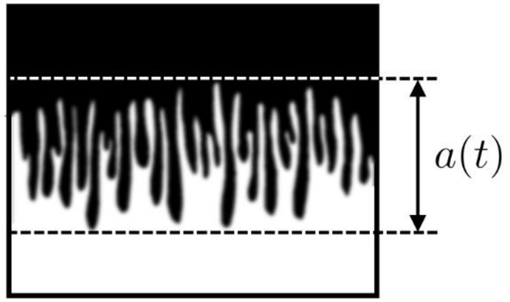
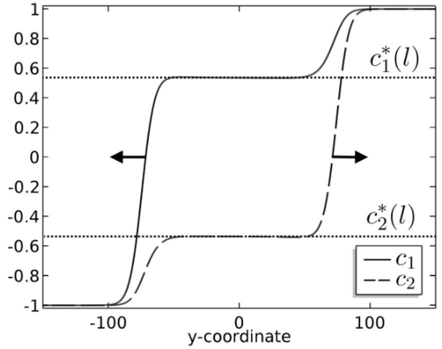
26 a 30 de agosto de 2024
João Pessoa - Brasil

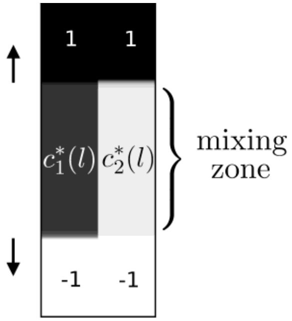
Outline

1. Motivation
   Miscible displacement in porous media
   - viscous fingering
   - gravitational fingering

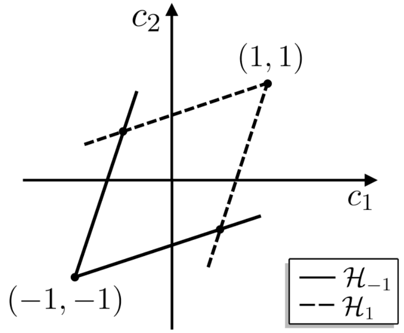


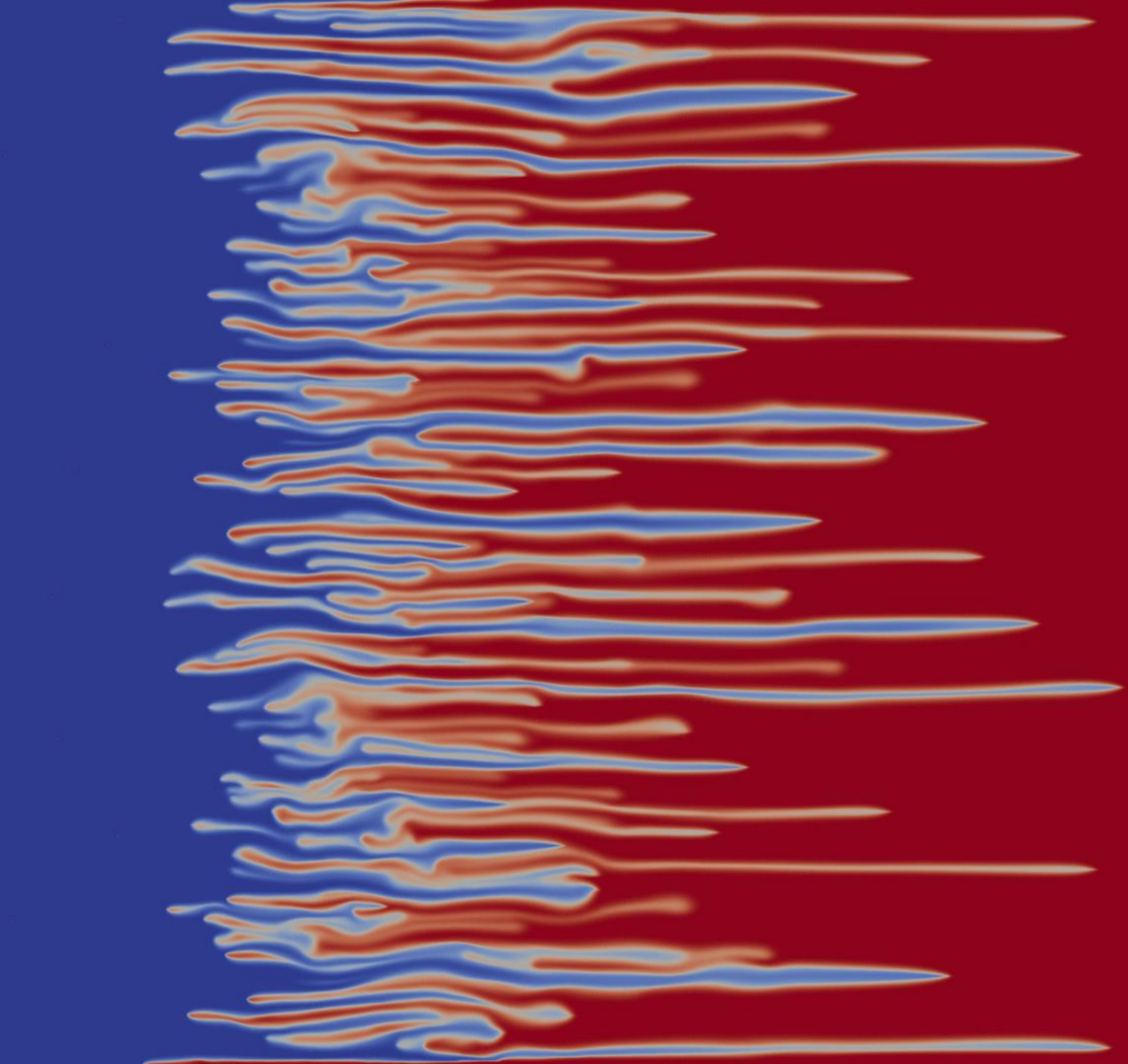
2. Problem statement
   - Two-tubes model
   - Main theorem





3. Sketch of proof:
   - traveling waves
   - slow-fast systems





“Miscible displacement in porous media”
Credit: Pavlov Dmitrii, St. Petersburg State University

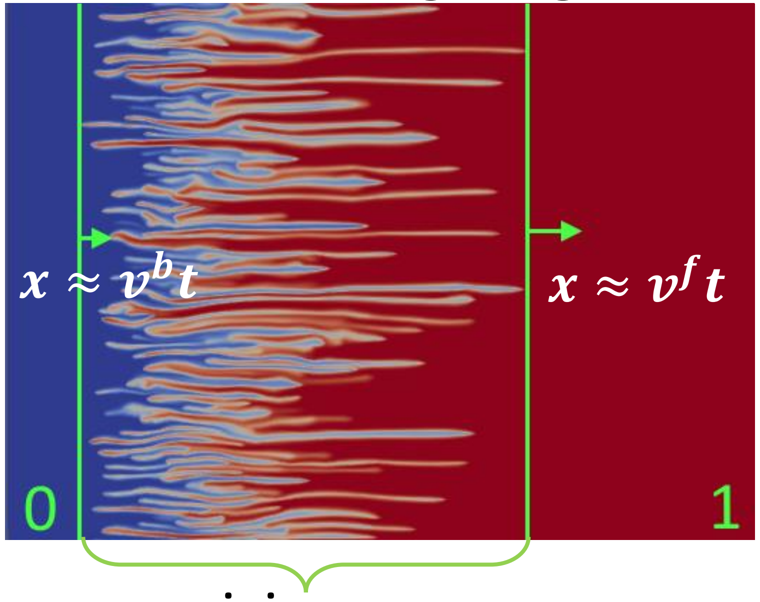
Homsy , 1987 “Viscous Fingering in Porous Media”

Viscous fingering phenomenon

water (blue color)
polymerized water (red color) 1

Incompressible Porous Medium eq – IPM, 2D (Two formulations)

Viscous fingering



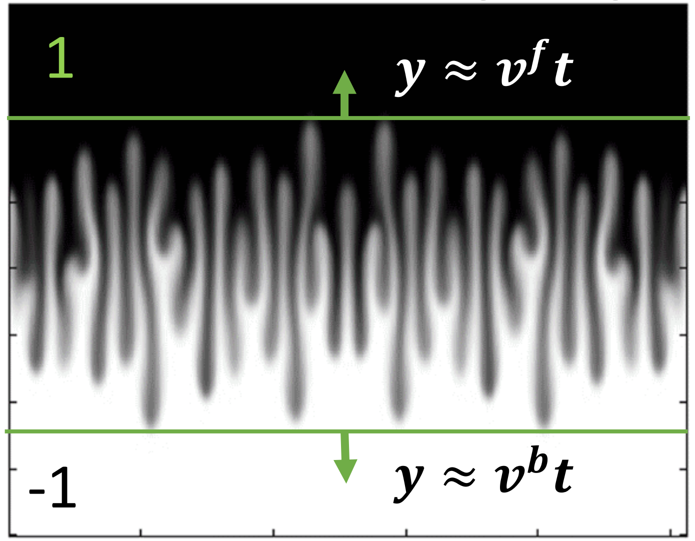
$$c_t + div(uc) = \varepsilon \cdot \Delta c$$

$$div(u) = 0$$

(viscosity) $\quad u = -m(c)\, K\, \nabla p$

(gravity) $\quad u = -\nabla p - (0, c)$

Gravitational fingering



$c = c(t,x,y)$ – concentration
$u = u(t,x,y)$ – velocity
$p = p(t,x,y)$ – pressure

$\varepsilon \geq 0$ – diffusion
$m(c)$ – mobility
K – permeability

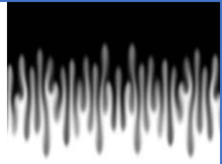
- many laboratory and numerical experiments show *linear growth of the mixing zone* [1], [2]

Question: how to find speeds v^b and v^f of propagation?

[1] Nijjer J., Hewitt D., and Neufeld J. The dynamics of miscible viscous fingering from onset to shutdown. Journal of Fluid Mechanics, 2018.

*[2] Bakharev, F., Enin, A., Groman, A., Kalyuzhnyuk, A., Matveenko, S., **Petrova, Y.**, Starkov, I. and Tikhomirov, S., Velocity of viscous fingers in miscible displacement: Comparison with analytical models. Journal of Computational and Applied Mathematics, 2022.*

IPM: $\varepsilon = 0$ (without diffusion)



Active scalar:

$$\boxed{\begin{array}{l} c_t + u \cdot \nabla c = 0 \\ u = A(c) \end{array}}$$

$$u = \nabla^{\perp}(-\Delta)^{-1}\partial_1 c \qquad \text{(Biot-Savart law)}$$

Discontinuous initial data: free boundary problem (Muskat problem) – ill-posed for unstable stratification

2011 - A. Córdoba, D. Córdoba, F. Gancedo (Annals of Mathematics)

"Interface evolution: the Hele-Shaw and Muskat problems"

Existence: smooth initial data

2007 – D. Cordoba, F. Gancedo, R. Orive (JMP): local well-posedness for initial data H^s

global solution vs finite-time blow-up? open

2017 – T. Elgindi (ARMA): global solution for small perturbations of $c = -y$

2023 – S. Kiselev, Y. Yao (ARMA): if solutions stay "smooth" for all times, then there is blow-up at $t = +\infty$

Uniqueness: non-uniqueness of weak solutions – by convex integration

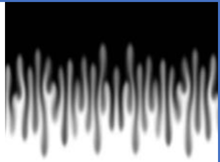
2011 – D. Córdoba, D. Faraco, F. Gancedo (ARMA)

2012 – L. Szekelyhidi Jr.

…and many others…

IPM: $\varepsilon > 0$ (with diffusion)

Question: Are those estimates sharp?

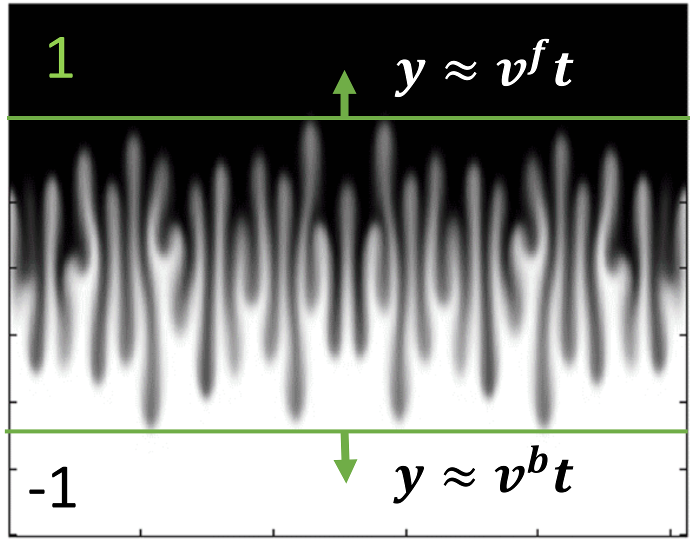


Estimates on the growth:

2005 – F. Otto, G. Menon. Proved estimates

- Full model (IPM) $v^f \leq 2$
- Simplified model (TFE) $v^f \leq 1$

(gravity)	Bofetta: $v^f \approx 0.67$
(viscosity)	empirical models
	numerics: NOT sharp



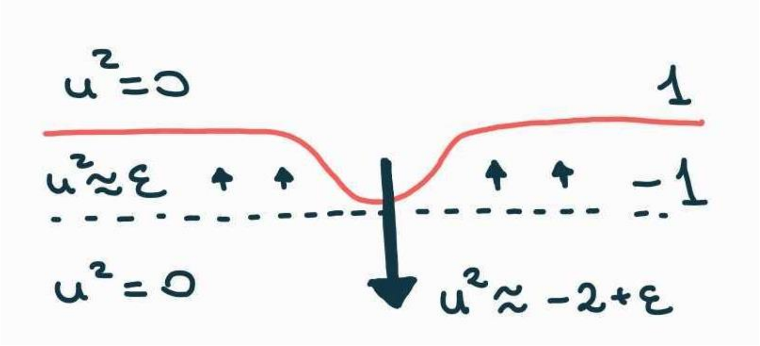
Transverse Flow Equilibrium = TFE

$$p(t,x,y) \approx p(t,y)$$

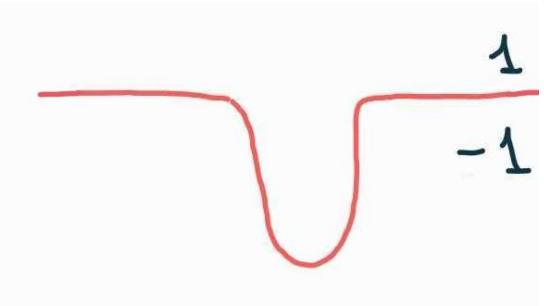
$$c_t + u \cdot \nabla c = \varepsilon \Delta c$$
$$div(u) = 0$$
$$u = (u^1, u^2), \quad u^2 = \bar{c} - c$$

Why fingers appear? It is a hair-trigger effect!

No flow

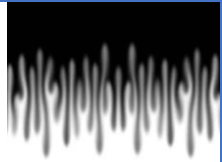


Velocity u changes due to concentration c

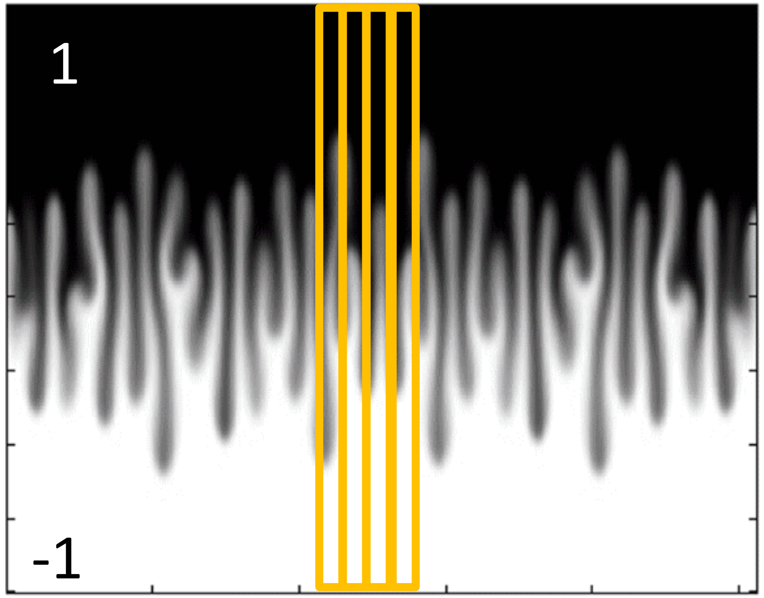


Concentration c changes due to velocity u

IDEA: semi-discrete model of gravitational fingering

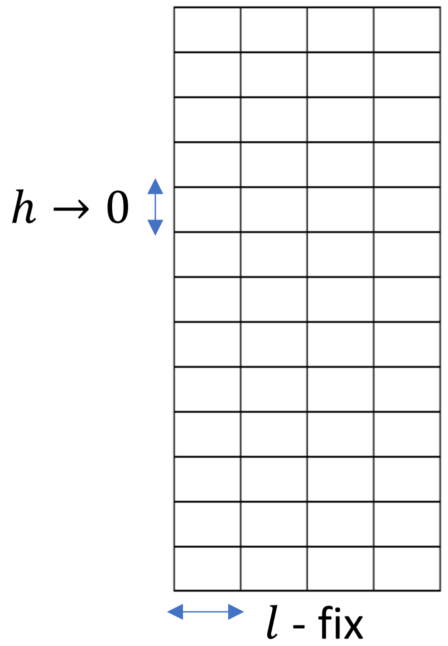


- Discretize in horizontal direction
- Take n tubes, $n = 2,3,4, \ldots$



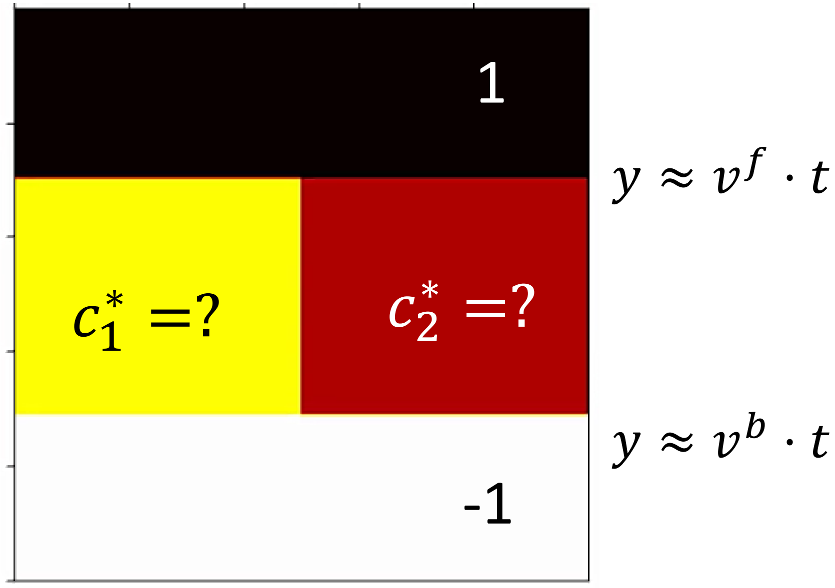
Tubes (layer, lane,…) models:

Limit of
numerical scheme



- Finite volume
- Upwind

- For simplicity, $n = 2$



We observe two traveling waves:
$c(y,t) = c(y - vt)$

1995 — Y. Yortsos “A theoretical analysis of vertical flow equilibrium”
2019 — A. Armiti-Juber, C. Rohde “On Darcy- and Brinkman-type models for two-phase flow in asympt. flat domains”
2006 — J.C. Da Mota, S. Schecter “Combustion fronts in a porous medium with two layers”
2019 — H. Holden, N. Risebro “Models for dense multilane vehicular traffic”

Two-tubes model

1. Original equation on c: $c_t + div(uc) - \Delta c = 0$

   Two-tubes equations on c:

$$\partial_t c_1 + \partial_y (u_1 c_1) - \partial_{yy} c_1 = \boxed{-B}$$
$$\partial_t c_2 + \partial_y (u_2 c_2) - \partial_{yy} c_2 = \boxed{+B}$$

2. Original equation on p: $u = -\nabla p - (0, c)$

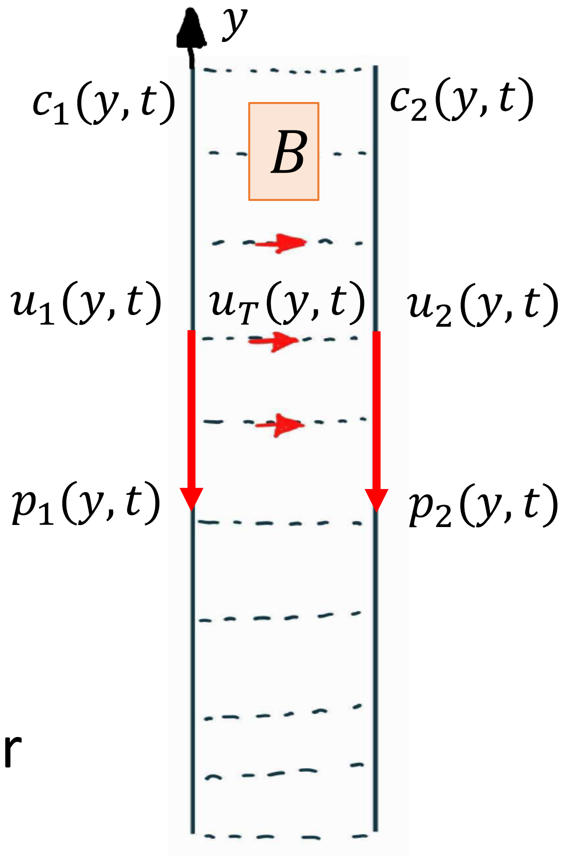
   Two-tubes equations on p:

$$u_1 = -\partial_y p_1 - c_1$$
$$u_2 = -\partial_y p_2 - c_2$$
$$u_T = -\frac{p_2 - p_1}{l}$$

3. Original equation on u: $div(u) = 0$

   Two-tubes equations on u:

$$\partial_y u_1 + \frac{u_T}{l} = 0$$



l - parameter

$$\boxed{B} = \begin{cases} \dfrac{u_T}{l} \cdot c_1, & u_T > 0, \\ \dfrac{u_T}{l} \cdot c_2, & u_T < 0 \end{cases}$$

Two-tubes model

1. Original equation on c: $$c_t + div(uc) - \Delta c = 0$$
   Two-tubes equations on c:

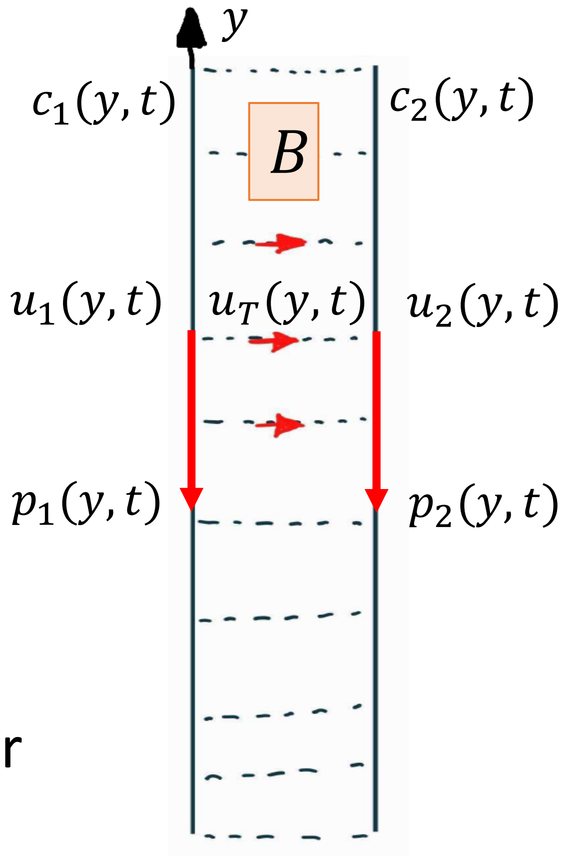
$$\partial_t c_1 + \partial_y(u_1 c_1) - \partial_{yy} c_1 = \boxed{-B}$$
$$\partial_t c_2 + \partial_y(u_2 c_2) - \partial_{yy} c_2 = \boxed{+B}$$

2. Original equation on p: $$u = -\nabla p - (0, c)$$
   Two-tubes equations on p:

$$u_1 = -\partial_y p_1 - c_1$$
$$u_2 = -\partial_y p_2 - c_2$$
$$\boxed{\frac{u_T}{l}} = -\frac{p_2 - p_1}{l^2}$$

3. Original equation on u: $$div(u) = 0$$
   Two-tubes equations on u:

$$\partial_y u_1 + \boxed{\frac{u_T}{l}} = 0$$



l - parameter

$$\boxed{B} = \begin{cases} \boxed{\frac{u_T}{l}} \cdot c_1, & u_T > 0, \\ \boxed{\frac{u_T}{l}} \cdot c_2, & u_T < 0 \end{cases}$$

Main result

Questions?

$$
(*)\quad \begin{cases}
\partial_t c_1 + \partial_y(u_1 c_1) - \partial_{yy} c_1 = -B \\
\partial_t c_2 + \partial_y(u_2 c_2) - \partial_{yy} c_2 = B \\
u_1 = -\partial_y p_1 - c_1 \\
u_2 = -\partial_y p_2 - c_2 \\
\partial_y u_1 = -\partial_y u_2 = \dfrac{p_2 - p_1}{l^2}
\end{cases}
$$

$$
B = \begin{cases}
-\partial_y u_1 \cdot c_1, & \partial_y u_1 < 0, \\
+\partial_y u_2 \cdot c_2, & \partial_y u_1 > 0
\end{cases}
$$

Remark: $\lim\limits_{l\to 0} c_1^*(l) = -0.5$ $\quad \lim\limits_{l\to 0} v^b(l) = -0.25$

$\lim\limits_{l\to 0} c_2^*(l) = +0.5$ $\quad \lim\limits_{l\to 0} v^f(l) = +0.25$

Theorem (Efendiev, P., Tikhomirov, 2024, arXiv: 2401.05981)

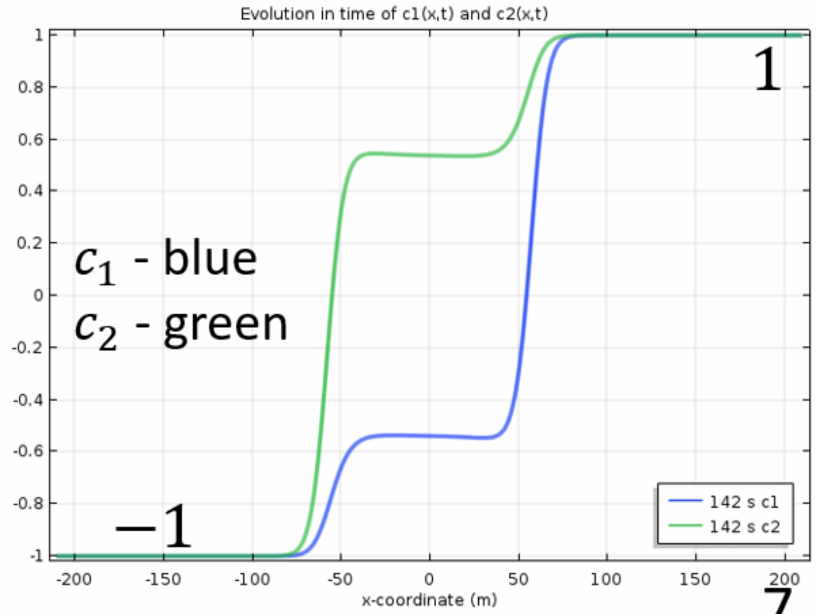
Consider a two-tube model with gravity $(*)$.

Then for all *$l > 0$ sufficiently small* there exists $c_1^*(l), c_2^*(l)$ such that there exist two traveling waves (TW):

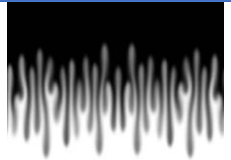
TW1 with speed $v^b(l)$: $(-1,-1) \rightarrow (c_1^*,(l)\ c_2^*(l))$

TW2 with speed $v^f(l)$: $(c_1^*,(l)\ c_2^*(l)) \rightarrow (1,1)$.

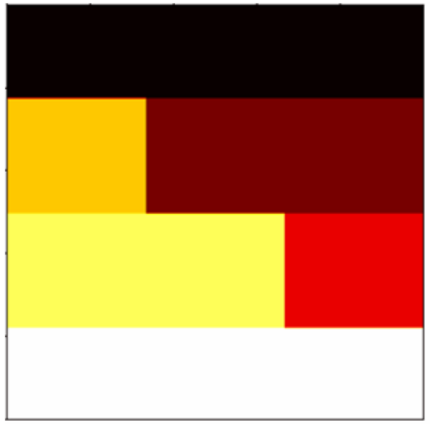
As $t \to \infty$ we observe:



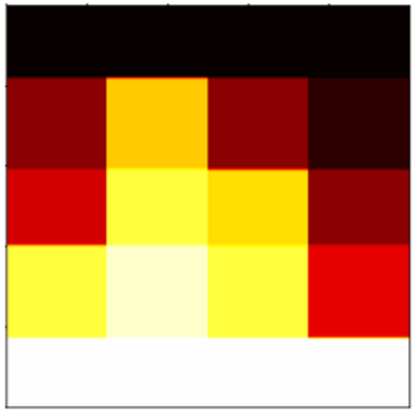
Many tubes: numerics



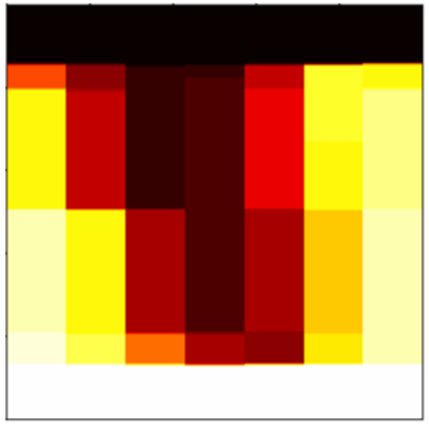
3 tubes

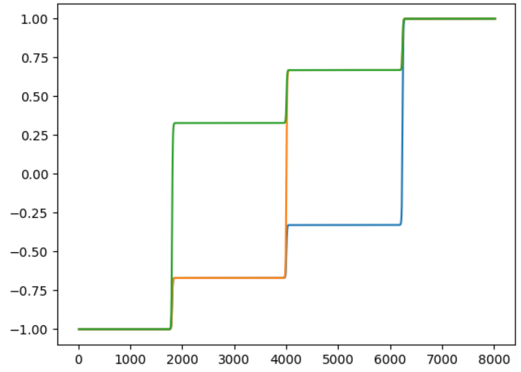


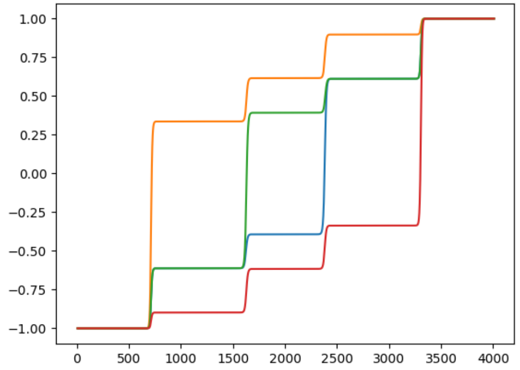
4 tubes

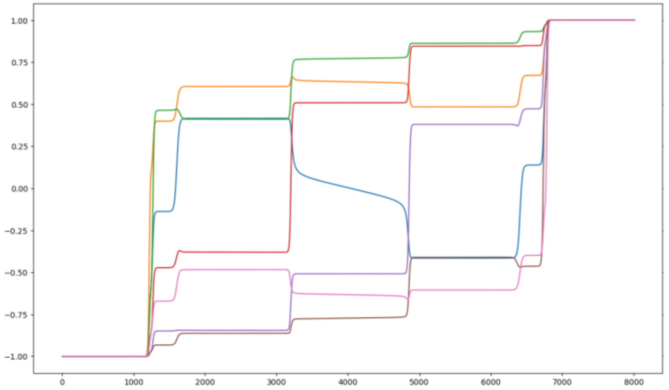


7 tubes









Questions: (open)

(1) explain the structure of “asymptotic solutions” for n tubes and study their stability

(2) find speed of growth of the mixing zone

(3) understand the behaviour as $n \to \infty$. Do we approximate 2-dim IPM?

(4) can we repeat this story for the many tubes viscous fingering model?

Scheme of proof: step 1 - TW

Travelling wave (TW) ansatz with fixed v:

$$c_1(t,y) = c_1(y-vt)$$
$$c_2(t,y) = c_2(y-vt)$$

$$u_1(t,y) = u_1(y-vt)$$
$$u_2(t,y) = u_2\,(y-vt)$$

$$p_1(t,y) = p_1(y-vt)$$
$$p_2(t,y) = p_2(y-ct)$$

With condition at $+\infty$:

$$c_1(+\infty) = 1$$
$$c_2(+\infty) = 1$$
$$u_1(+\infty) = 0$$
$$u_2(+\infty) = 0$$
$$(p_1 - p_2)(+\infty) = 0$$

System of ODEs in $\mathbb{R}^6$:

$$\boxed{\begin{cases} \dot{X} = F_v(X,Y) \\ l \cdot \dot{Y} = AY - BX \end{cases}}$$

Here:

- $X = \begin{pmatrix} c_1 \\ c_2 \\ \partial_\xi c_1 \\ \partial_\xi c_2 \end{pmatrix} \in \mathbb{R}^4, \quad Y = \begin{pmatrix} u_1 \\ p_1 - p_2 \end{pmatrix} \in \mathbb{R}^2$
- $A = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}, \qquad B \in M^{2\times 4}, \qquad l \ll 1$

Observation:

for $l \to 0$ this system has a special ``slow-fast'' structure.

Key tool: geometric singular perturbation theory (GSPT)

by Fenichel (JDE, 1979)

Scheme of proof: step 2 – cascade of 2 TW

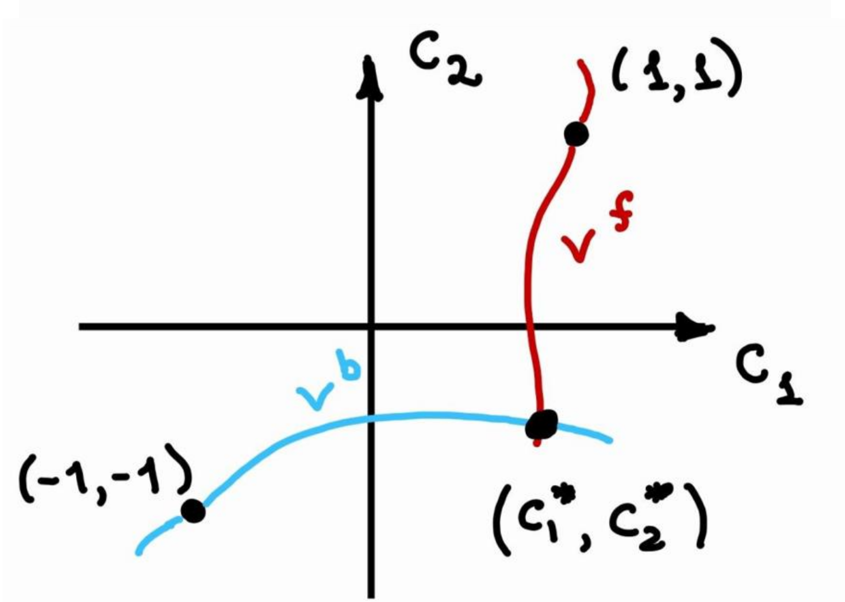
1) For each $v^f \in \mathbb{R}$ we find all points s.t. there exists a TW:

$$(c_1\,, c_2) \;\rightarrow\; (1,1)$$

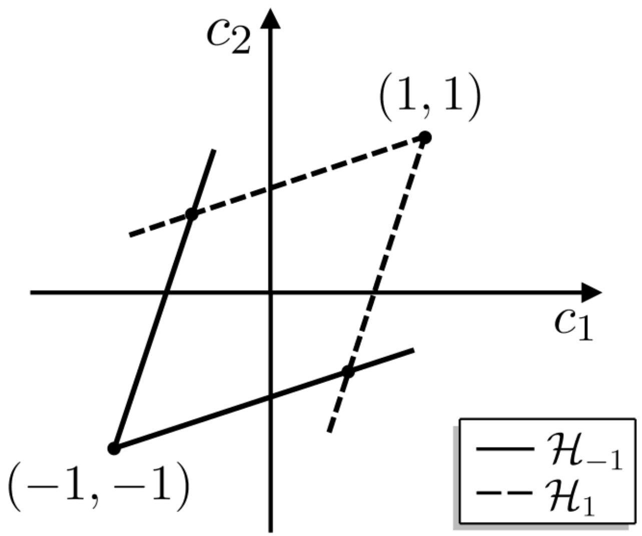
2) For each $v^b \in \mathbb{R}$ we find all points s.t. there exists a TW:

$$(-1,-1) \rightarrow (c_1\,, c_2)$$

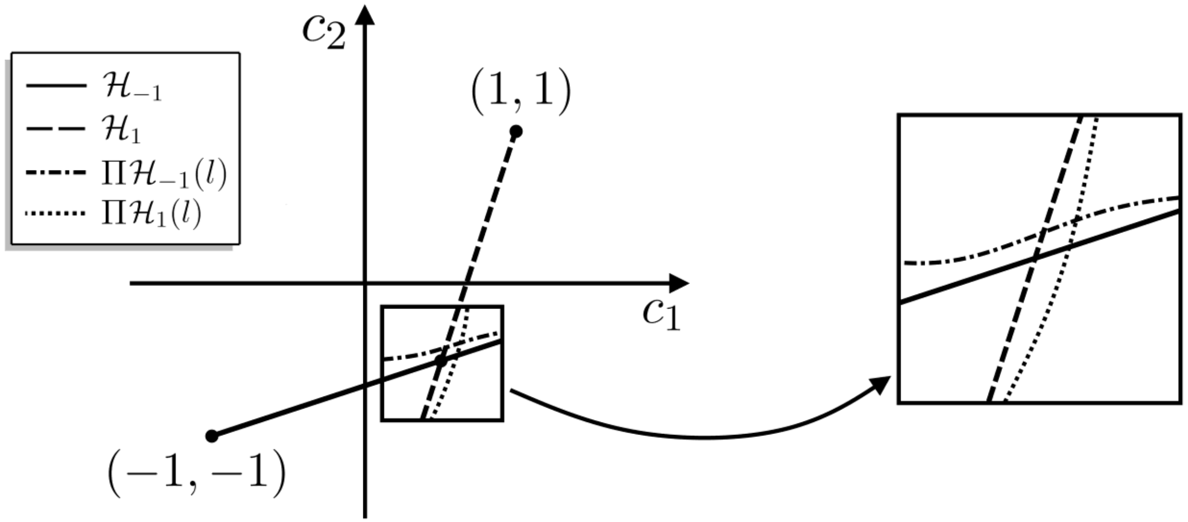
3) Find the intersection points of these two curves



$l = 0$ – these curves are just straight lines



$0 < l \ll 1$– perturbation argument



But what is a singular limit $l = 0$?

$$
(**)\begin{cases}
\partial_t c_1 + \partial_y(u_1 c_1) - \partial_{yy} c_1 = -B \\
\partial_t c_2 + \partial_y(u_2 c_2) - \partial_{yy} c_2 = B \\
\boxed{u_1 = \dfrac{c_2 + c_1}{2} - c_1 = \bar{c} - c_1} \\
B = \begin{cases} -\partial_y u_1 \cdot c_1, & \partial_y u_1 < 0, \\ +\partial_y u_2 \cdot c_2, & \partial_y u_1 > 0 \end{cases}
\end{cases}
$$

$l = 0$ corresponds to the two-tubes TFE equations (**) !!!

NB: (**) can be seen a hyperbolic system in non-conservative form (for fixed choice of B):

$$C_t + A(C)C_y = 0$$

$$C(0, y) = \begin{cases} (+1, +1), & y > 0 \\ (-1, -1), & y \leq 0 \end{cases}$$

- Use vanishing viscosity criteria to define admissible shocks
- For 2 and 3 tubes this is a “Temple-like” system (rarefaction and shock curves coincide and are linear)

Question: (open) TFE as a limit of IPM as $k_y/k_x \to \infty$? Can we use the connection to prove the linear growth in 2D IPM?

2D IPM model

$$c_t + u \cdot \nabla c = \varepsilon \, \Delta c$$
$$div\, u = 0$$
$$u = -\begin{pmatrix} k_x & 0 \\ 0 & k_y \end{pmatrix} \nabla p - (0, c)$$

$$\frac{k_y}{k_x} \to \infty$$

2D TFE model

$$c_t + u \cdot \nabla c = \varepsilon \, \Delta c$$
$$div\, u = 0$$
$$u = (u^x, u^y)$$
$$u^y = \bar{c} - c$$

Thank you for your attention!   Muito obrigada!   ¡Muchas gracias!

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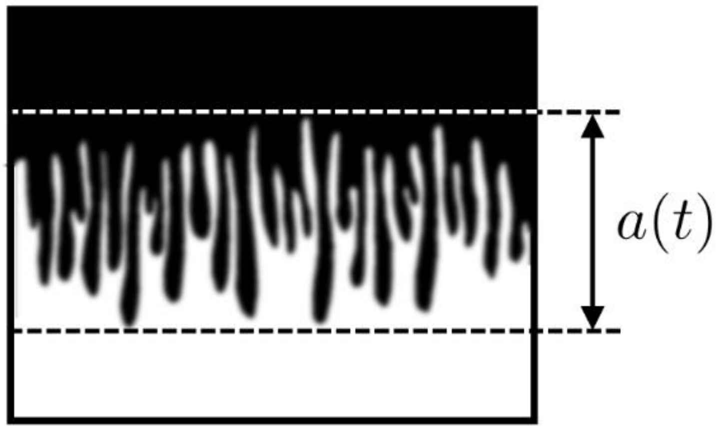

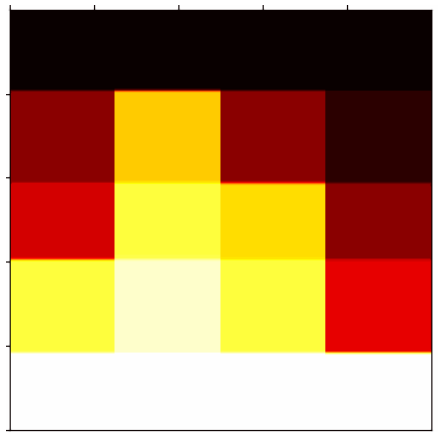

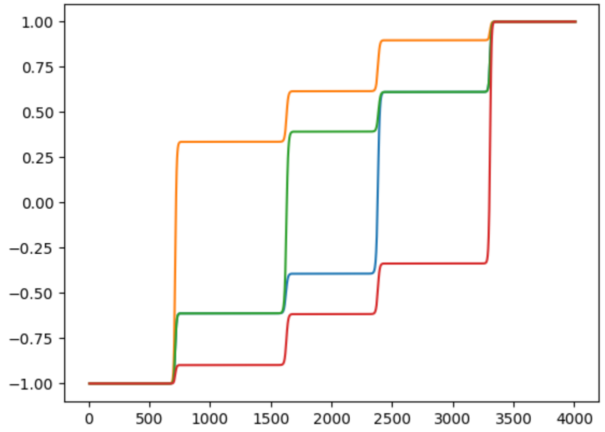

For more details see arXiv:2401.05981   (two-tubes model)

Any questions, comments and ideas are very welcome!

References Muito obrigada pela atenção!

Own works on the topic of the talk:

1. **Yu. Petrova**, S. Tikhomirov, Ya. Efendiev, *“Propagating terrace in a two-tubes model of gravitational fingering”* Submitted. ArXiv: 2401.05981; 2024.
2. Bakharev, F., Enin, A., Groman, A., Kalyuzhnyuk, A., Matveenko, S., **Petrova, Yu.**, Starkov, I. and Tikhomirov, S., “*Velocity of viscous fingers in miscible displacement: Comparison with analytical models”.* Journal of Computational and Applied Mathematics, 402, p.113808; 2022.
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