# Vanishing adsorption admissibility criterion for contact discontinuities in the polymer model


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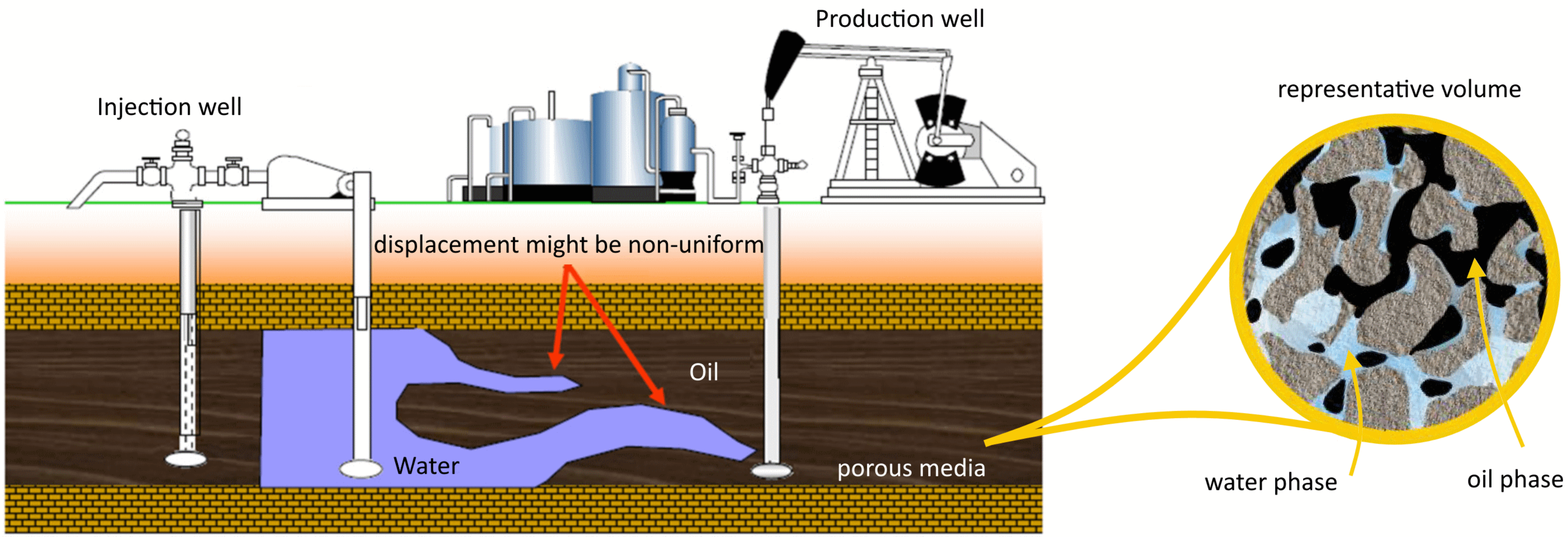
The talk is based on:

- **Y. Petrova**, B. Plohr, and Marchesin D. Vanishing adsorption admissibility criterion for contact discontinuities in the polymer model. arXiv:2211.10326.
- F. Bakharev, A. Enin, **Y. Petrova**, and N. Rastegaev. Impact of dissipation ratio on vanishing viscosity solutions of the Riemann problem for chemical flooding model. Journal of Hyperbolic Differential Equations, 20:1–26, 2023.

# Motivation: enhanced oil recovery (EOR)

We are interested in the mathematical model of oil recovery.

- porous media (averaged models of flow)
- unknown variables: $s \in [0,1]$ - water saturation, $1 - s$ - oil concentration
- relatively small speeds ($\approx 1$ meter per day): Navier-Stokes → Darcy's law
- multiphase flow: oil, water, gas
- applications to EOR methods: thermal, gas, chemical flooding



# Two main directions of investigation

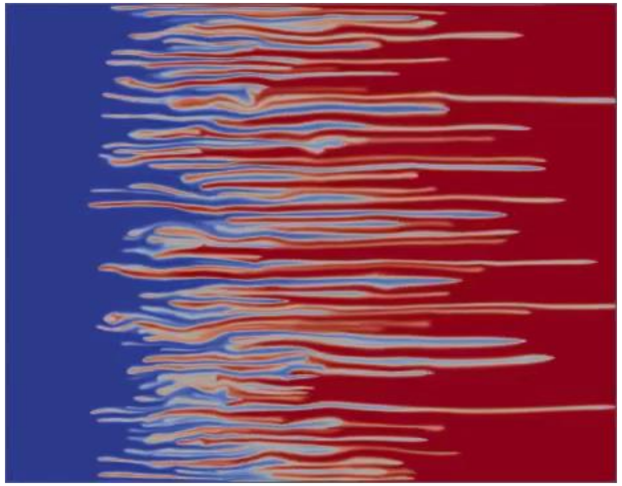
*Stable displacement* (1-dim)

- hyperbolic conservation laws
- main question: find an exact solution for a Riemann problem

$$
\begin{aligned}
s_t + \quad f(s,c)_x &= 0 \\
(sc)_t + \left(cf(s,c)\right)_x &= 0
\end{aligned}
$$

Example: polymer model

*Unstable displacement* (2-dim)



- viscous fingering phenomenon
- source of instability: water and oil/polymer have different viscosities

$$
c_t + div(uc) = \Delta c
$$
$$
div(u) = 0, \qquad u = -m(c)\,\nabla p
$$

Example: incompressible porous media equation (IPM)

# Glimm-Isaacson model (KKIT)*

Two-phase oil-water flow with *polymer* in the water (1980)

$$\begin{aligned} s_t + \quad f(s,c)_x &= 0 \\ (cs)_t + \big(cf(s,c)\big)_x &= 0 \end{aligned}$$

- $s \in [0,1]$ – water saturation
- $c \in [0,1]$ – polymer concentration in water
- $f(s,c)$ – fractional flow function: affected by polymer
  - S-shaped in $s$ (for fixed $c$)

Case 1:
$f_c' < 0$ (monotone in $c$)

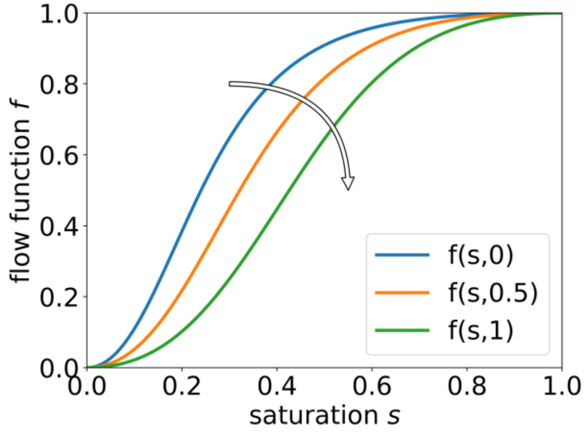
Case 2:
$f$ changes monotonicity once

Initial data: $(s,c)(x,0) = \begin{cases} (s_L, c_L), & x \le 0 \\ (s_R, c_R), & x \ge 0 \end{cases}$

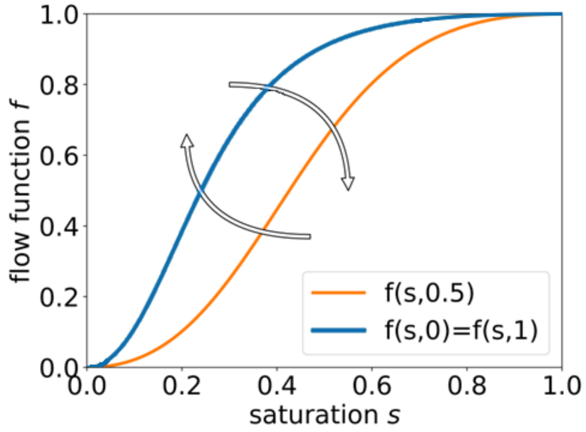
Question: find an exact solution $s(x,t)$ and $c(x,t)$ to any Riemann problem

NB: $s(x,t) = s(x/t)$ – self-similar

Case 1:




Case 2:




* KKIT = Keyfitz, Kranzer, Isaacson, Temple

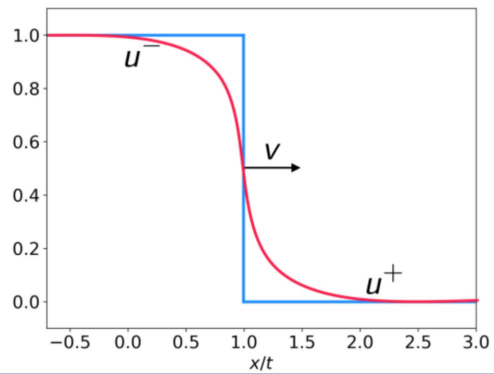
# Main idea

Polymer model (1980' E. Isaacson)

$$\begin{aligned} s_t + \quad f(s,c)_x &= 0 \\ (sc)_t + \big(cf(s,c)\big)_x &= 0 \end{aligned} \qquad (M_0)$$

- Contact discontinuities (linearly degenerate) ⇒ non-uniqueness of solutions
- Vanishing viscosity criterion helps? Directly no…
- Existing admissibility criteria need to be justified from "physical" perspective

Aim: select unique physically admissible weak solution




Main idea: add small physical effect – adsorption of polymer on the rock (1987' T. Johansen, R. Winther)

$$\begin{aligned} s_t \quad + \quad f(s,c)_x &= 0 \\ (sc + \alpha\, a(c))_t + \big(cf(s,c)\big)_x &= 0 \end{aligned} \qquad (M_\alpha)$$

Here $\alpha > 0$ - small, $a(c)$ – strictly concave

- Contact discontinuities ⇝ rarefaction and shock waves ⇝ vanishing viscosity criteria is applicable

*Vanishing adsorption criterion*: the admissible contacts for $M_0$ are the $L^1_{loc}$ limits of a family of admissible solutions for a Riemann problem for $M_\alpha$ as $\alpha \to 0$.

# Algorithm to find a solution $U = (s, c)$ for $2 \times 2$ systems

$$\boxed{U_t + F(U)_x = 0}$$

$$U(x,0) = \begin{cases} U_L, & x \leq 0 \\ U_R, & x \geq 0 \end{cases}$$

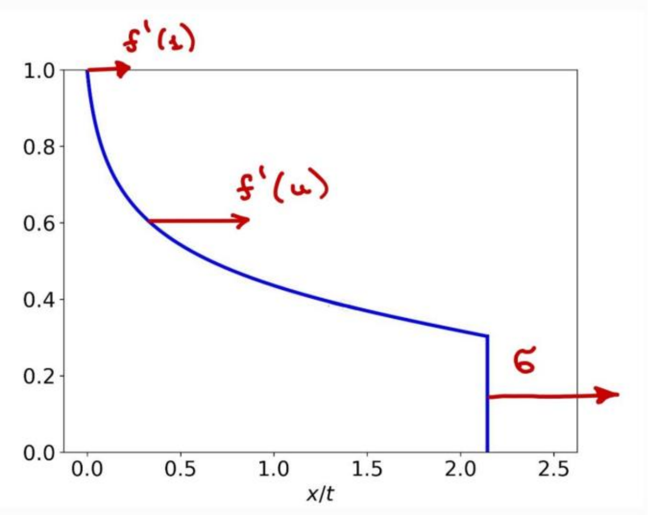
Eigenvalues of $DF(U)$: $\lambda_1(U) < \lambda_2(U)$

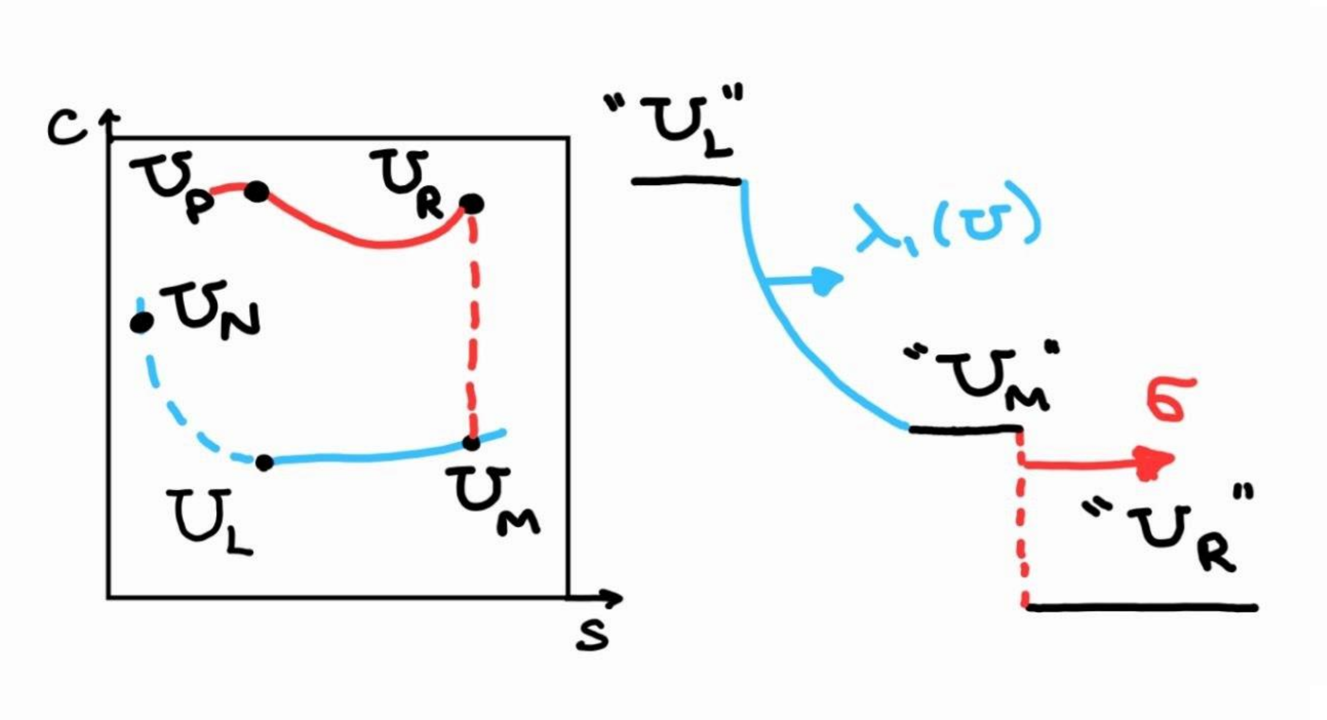
Take $U_L$ and construct a “slow wave curve” corresponding to $\lambda_1(U)$

Take $U_R$ and construct a “fast wave curve” corresponding to $\lambda_2(U)$

Intersection of these two wave curves gives a solution

$$u_t + f(u)_x = 0$$

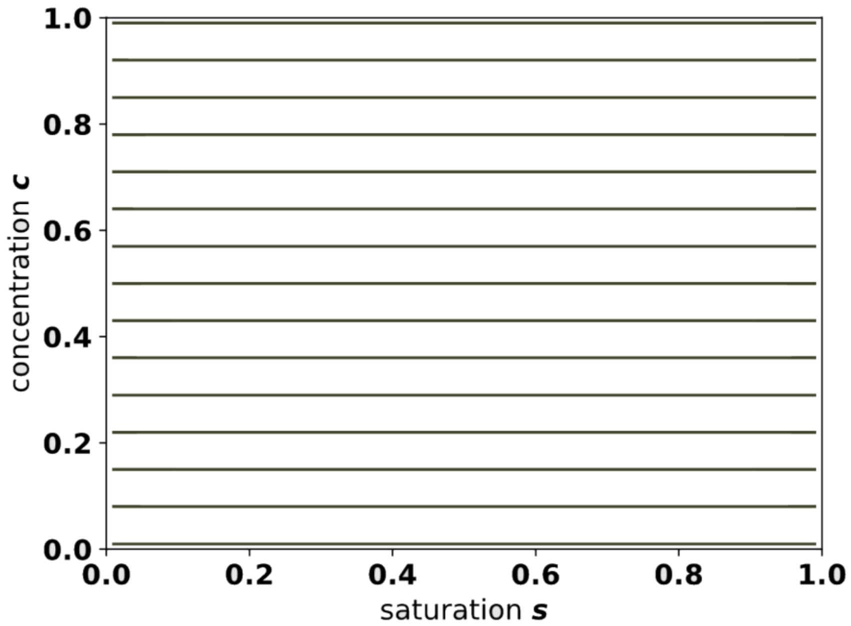


# Characteristic families: s and c-waves

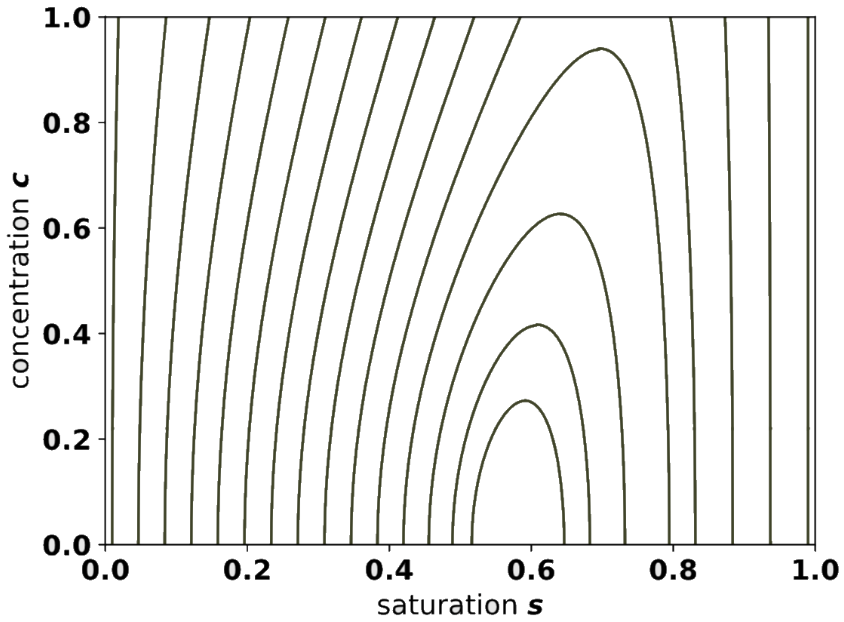
$s$-waves

- $\lambda^s = f_s'$
- Solve the Buckley-Leverett equation $c = const$
- Riemann invariant $c = const$
- "line" family



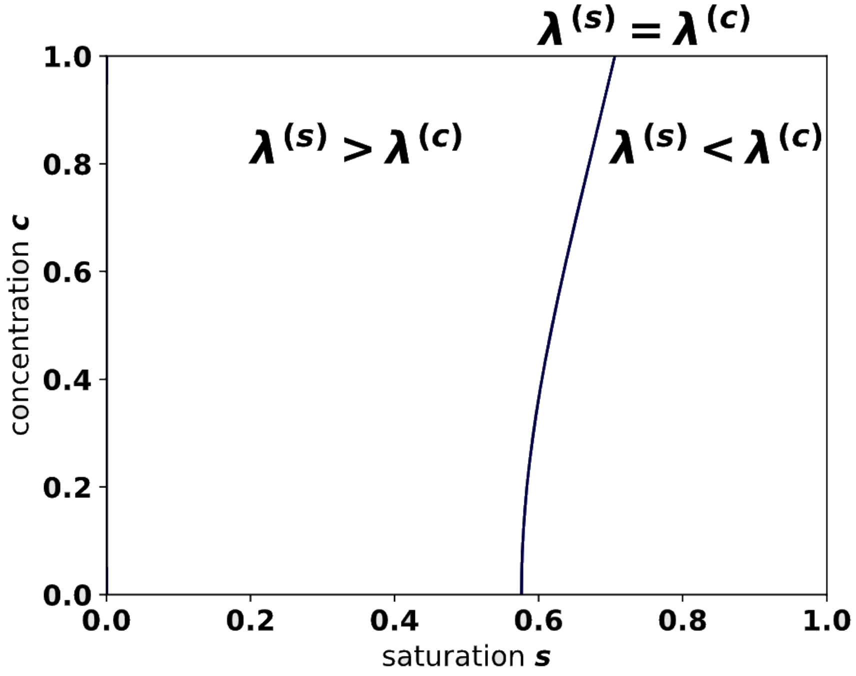

$c$-waves

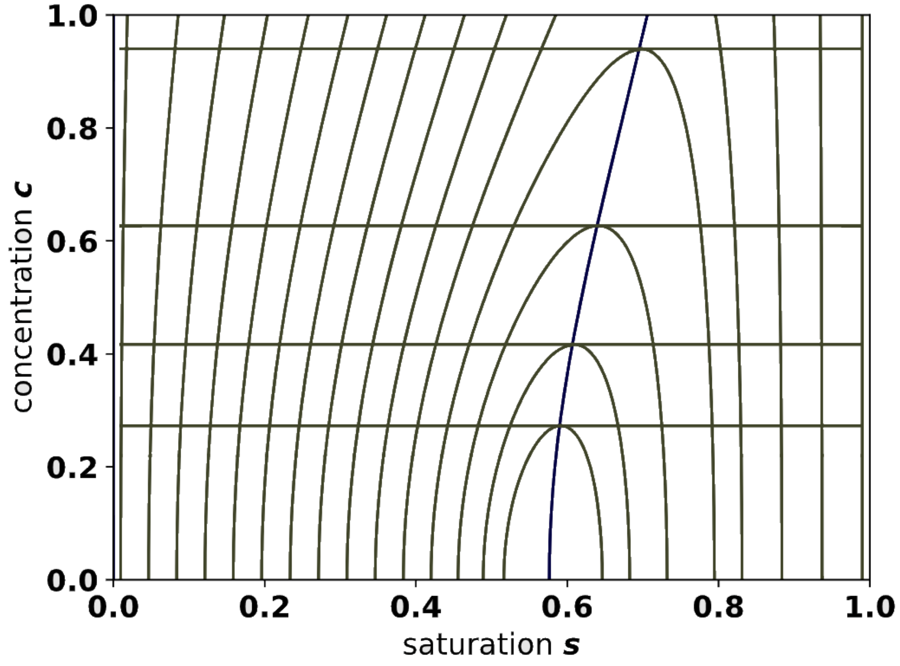
- $\lambda^c = f/s$
- Are contact discontinuities
- Riemann invariant $f/s = const$
- "contact" family




- For both families, the rarefaction and shock curves coincide! But in a different way (Temple'1983)
- Any solution for a Riemann problem is a combination of $s$ and $c$ waves

# Non-strictly hyperbolic system

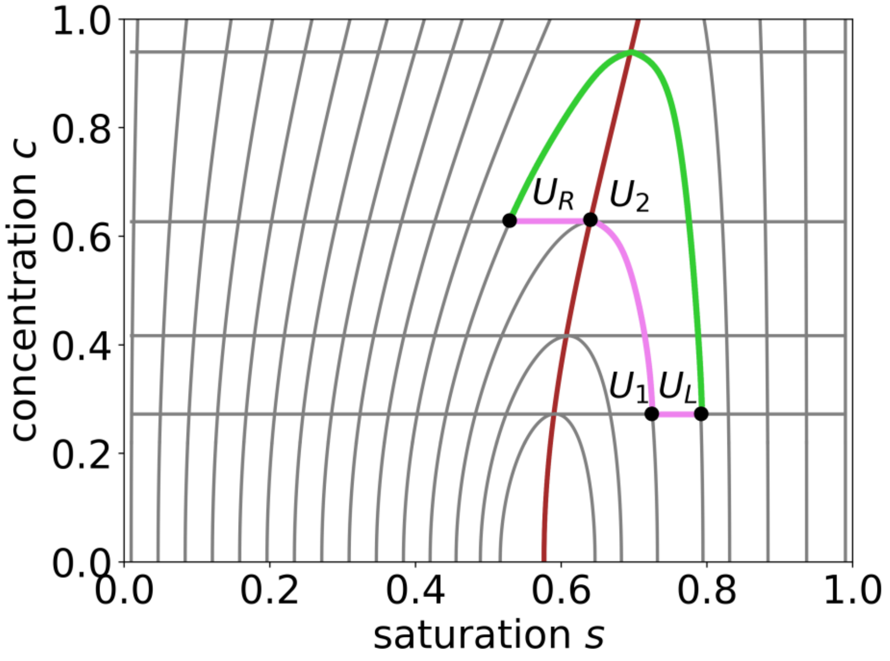


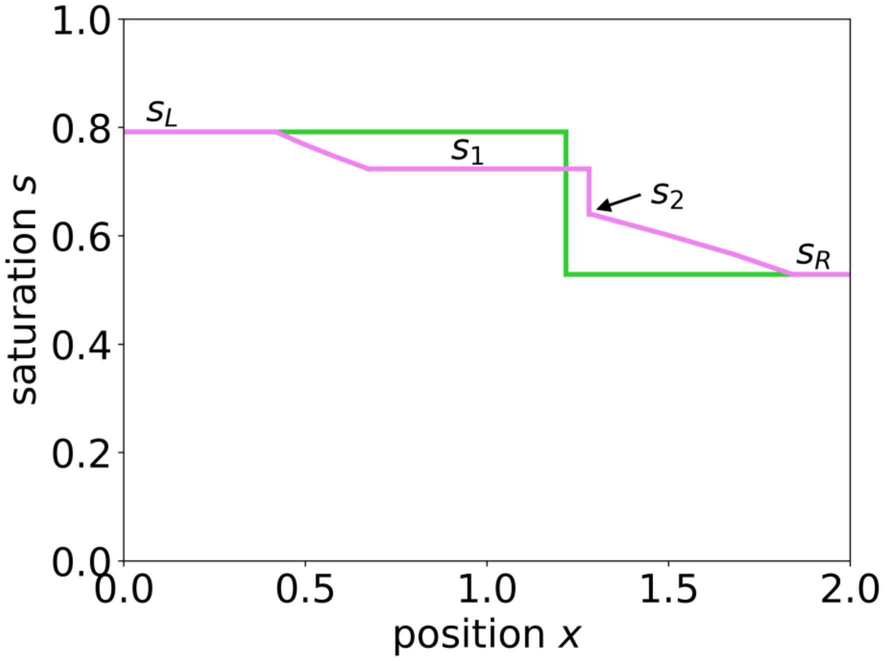




The coordinate system of wave curves is singular and the wave speeds coincide on a co-dimension one curve (coincidence locus): $\lambda^s = f_s' = f/s = \lambda^c$

$s$ and $c$ waves are tangent on coincidence curve

# Non-uniqueness of solutions




A contact discontinuity between $U_-$ and $U_+$ is admissible if and only if:

Criterion 1 (Isaacson): either $U_-, U_+ \in \{\lambda^s \geq \lambda^c\}$ or $U_-, U_+ \in \{\lambda^s \leq \lambda^c\}$

Criterion 2 (de Souza-Marchesin): $c$ is continuous and monotone along the sequence of contact curves, connecting $U_-$ and $U_+$

*What is the (physical) motivation of these criteria?*

# Main result

Polymer model

$$\begin{aligned} s_t \quad + \quad f(s,c)_x &= 0 \\ (sc + \alpha\, a(c))_t + \big(cf(s,c)\big)_x &= 0 \end{aligned} \qquad (M_\alpha)$$

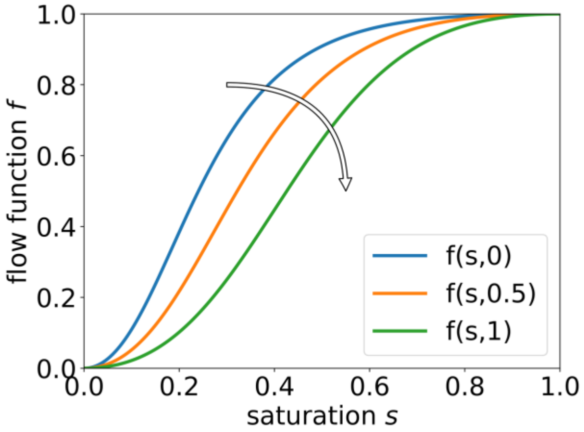
Criterion 3: vanishing adsorption (Petrova-Marchesin-Plohr):

A contact discontinuity between $U_-$ and $U_+$ for $M_0$ is admissible if and only if it is the $L^1_{loc}$ limit of a family of admissible solutions for a Riemann problem for $M_\alpha$ as $\alpha \to 0$.

**Theorem 1 (P., Marchesin, Plohr, arxiv:2211.10326)**

If $f$ satisfies the monotonicity assumption $f_c{}' < 0$,
then the set of admissible Riemann solutions for $M_0$ is the same for criteria 1, 2 and 3.

Case 1:

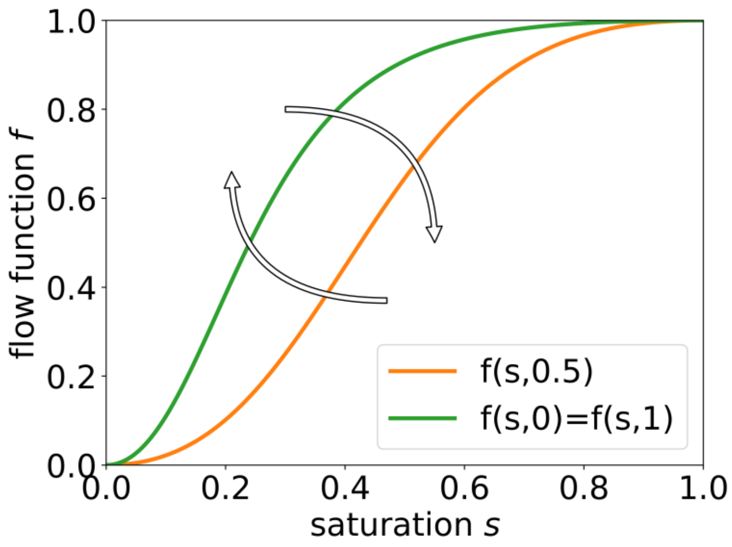



Corollary: any solution to a Riemann problem for $M_0$ exists and is unique.

Question: what happens when $f$ is non-monotone in $c$?

# What happens when $f$ is non-monotone in $c$?

$$\begin{aligned} s_t + \quad f(s,c)_x &= 0 \\ (sc)_t + \left(cf(s,c)\right)_x &= 0 \end{aligned}$$

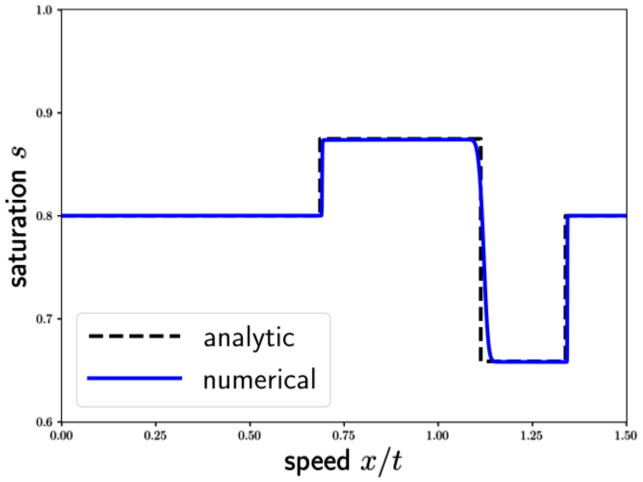


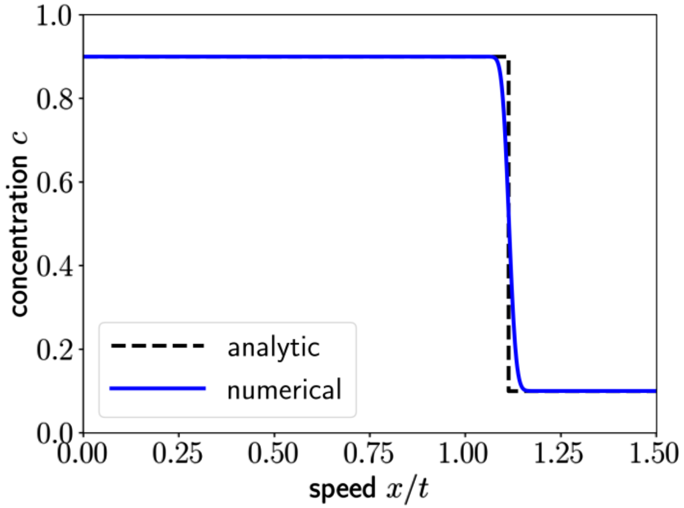

Example: “boomerang”

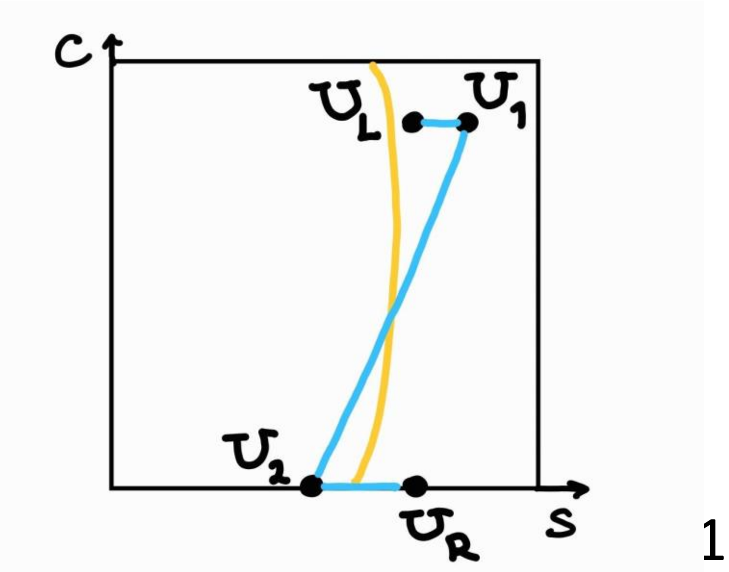
$$f(s,c) = \frac{s^2}{s^2+\mu(c)(1-s)^2} \qquad \text{with} \qquad \mu(c) = 1 + 4c(1-c)$$

Riemann problem: $(s,c)(x,0) = \begin{cases} (0.8, 1), & x \le 0 \\ (0.8, 0), & x \ge 0 \end{cases}$

Results of numerical modelling:

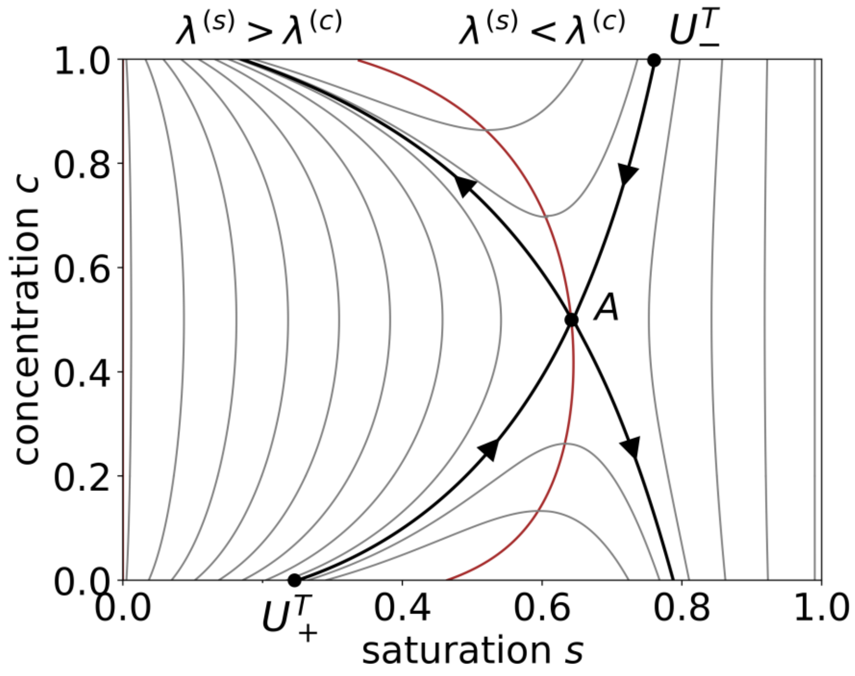


# …a more careful look…

$$\begin{aligned} s_t + \quad f(s,c)_x &= 0 \\ (sc)_t + \big(cf(s,c)\big)_x &= 0 \end{aligned}$$




Let's call the corresponding contact discontinuity undercompressive (it does NOT satisfy the Lax admissibility criterion)

Is this contact discontinuity admissible by the existing criteria?

Criteria 1: NO
Criteria 2: YES
Criteria 3: YES

**Theorem 2 (P., Marchesin, Plohr, 2022, arxiv:2211.10326)**

The undercompressive contact discontinuities satisfy the vanishing adsorption admissibility criterion

# Main step in proof of Thm 2

$$\begin{aligned} s_t \quad + \quad f(s,c)_x &= 0 \\ (sc + \alpha\, a(c))_t + \big(cf(s,c)\big)_x &= 0 \end{aligned}$$

Add diffusion terms $\Downarrow$

$$\begin{aligned} s_t \quad + \quad f(s,c)_x &= \varepsilon_1 s_{xx} \\ (sc + \alpha\, a(c))_t + \big(cf(s,c)\big)_x &= \varepsilon_1 (c s_x)_x + \varepsilon_2 c_{xx} \end{aligned}$$

$\varepsilon_1, \varepsilon_2 \to 0,\qquad k = \dfrac{\varepsilon_1}{\varepsilon_2}$

Travelling wave ansatz $\Downarrow$ $\quad s = s(x - \sigma t) = s(\xi),\qquad c = c(x - \sigma t) = c(\xi)$

$$\begin{cases} \alpha \cdot s_\xi = f - \sigma(s + d_1) \\ c_\xi = \dfrac{\varepsilon_1}{\varepsilon_2} \cdot \sigma \cdot (d_1 c - d_2 - a(c)) \end{cases}$$

**Theorem 3 (Bakharev, Enin, P., Rastegaev, 2023, JHDE)**

For any $k = \frac{\varepsilon_1}{\varepsilon_2} > 0$ there exists $s_-(k)$ and $s_+(k)$ and velocity $\sigma(k)$ such that there exists a travelling wave, connecting two saddle points $(s_-(k), 1)$ and $(s_+(k), 0)$ with velocity $\sigma(k)$.

# References Muito obrigada pela sua atenção!

**Own works:**
1. F. Bakharev, A. Enin, Y. Petrova, and N. Rastegaev. Impact of dissipation ratio on vanishing viscosity solutions of the Riemann problem for chemical flooding model. Journal of Hyperbolic Differential Equations, 20:1–26, 2023.
2. Y. Petrova, B. Plohr, and Marchesin D. Vanishing adsorption admissibility criterion for contact discontinuities in the polymer model. arXiv:2211.10326.

**Other references:**

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5. Keyfitz, B.L. and Kranzer, H.C., 1980. A system of non-strictly hyperbolic conservation laws arising in elasticity theory. Archive for Rational Mechanics and Analysis, 72(3), pp.219-241.
6. E. L. Isaacson, Global solution of a Riemann problem for a non-strictly hyperbolic system of conservation laws arising in enhanced oil recovery, Rockefeller University, New York, NY, preprint (1981).