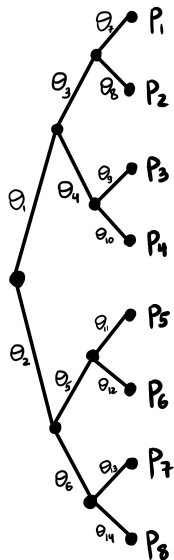


# Decomposable context-specific models

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(joint work with Eliana Duarte and Julian Vill)

CEG Workshop at the University of Warwick  
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# Staged trees

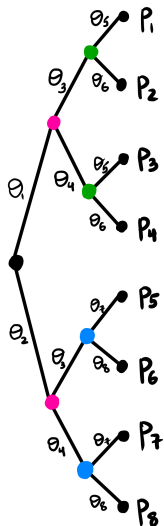


We start with a staged tree  $\mathcal{T}$ . The leaves of  $\mathcal{T}$  represent all possible outcomes of a sequence of events.

Let  $\Theta_{\mathcal{T}}$  denote the parameter space of conditional probability tables. Let  $\mathcal{M}_{(\mathcal{T}, \Theta)}$  denote all probability distributions that factor according to  $(\mathcal{T}, \Theta_{\mathcal{T}})$ .

$\mathcal{M}_{(\mathcal{T}, \Theta)} = \Delta_7$ , because all labels are distinct!

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$$\mathcal{M}_{(\mathcal{T}, \Theta)} = \{(p_1, \dots, p_8) \in \Delta_7 :$$

$$\begin{aligned} p_1 p_4 &= p_3 p_2, p_4 p_6 = p_2 p_8, p_3 p_6 = p_1 p_8, \\ p_4 p_5 &= p_2 p_7, p_3 p_5 = p_1 p_7, p_5 p_8 = p_7 p_6 \}, \end{aligned}$$

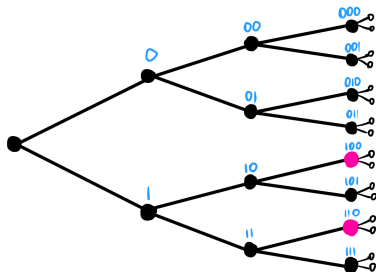
because some vertices are in the same stage!

# CStrees

A *CStree* is a staged tree such that:

- only vertices in the same level can be in the same stage,
- $\theta(x_1 \cdots x_{k-1} \rightarrow x_1 \cdots x_{k-1} x_k) = \theta(y_1 \cdots y_{k-1} \rightarrow y_1 \cdots y_{k-1} x_k)$  whenever the nodes  $x_1 \cdots x_{k-1}$  and  $y_1 \cdots y_{k-1}$  are in the same stage,
- for every stage  $S \subseteq L_{k-1}$ , there is some context  $X_C = x_C$  such that

$$S = \bigcup_{y \in \mathcal{R}_{[k-1] \setminus C}} \{x_C y\}$$



For the pink stage, the *stage-defining context* is  $\{X_1 X_3 = 10\}$ .

# Algebraic approach to CStrees

Every CStree model has an associated *algebraic variety*. The staged tree model we saw earlier

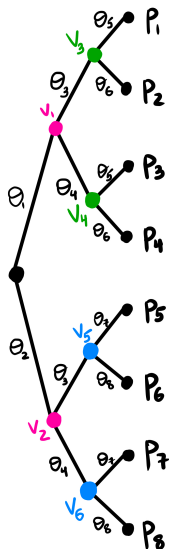
$$\begin{aligned}\mathcal{M}_{(\mathcal{T}, \Theta)} = \{ (p_1, \dots, p_8) \in \Delta_7 : & p_1 p_4 - p_3 p_2 = 0, p_4 p_6 - p_2 p_8 = 0, \\ & p_3 p_6 - p_1 p_8 = 0, p_4 p_5 - p_2 p_7 = 0, \\ & p_3 p_5 - p_1 p_7 = 0, p_5 p_8 - p_7 p_6 = 0 \}\end{aligned}$$

is also a CStree model. Note it is defined by binomial equations!  
In general, consider the ring homomorphism

$$\psi_{\mathcal{T}}: \mathbb{R}[p_x: x \in \mathcal{R}] \rightarrow \mathbb{R}[\Theta_{\mathcal{T}}], \quad p_x \mapsto \prod_{e \in E(r \rightarrow x)} \theta(e).$$

A CStree model is the vanishing set of the ideal  $\ker(\psi_{\mathcal{T}})$  inside  $\Delta_{|\mathcal{R}|-1}$ .  
When the kernel is generated by binomials, the model is *toric*.

# Interpolating polynomial



For any vertex  $v = x_1 \cdots x_{k-1}$  in a CStree we define the *interpolating polynomial* in  $\mathbb{R}[\theta(e) : e \in E]$ :

$$t(v) := \sum_{z \in \mathcal{R}_{[p] \setminus [k-1]}} \left( \prod_{e \in E(v \rightarrow vz)} \theta(e) \right).$$

$$t(v_3) = \theta_5 + \theta_6$$

$$t(v_4) = \theta_5 + \theta_6$$

$$t(v_1) = \theta_3(\theta_5 + \theta_6) + \theta_4(\theta_5 + \theta_6)$$

# CStrees and DAGs

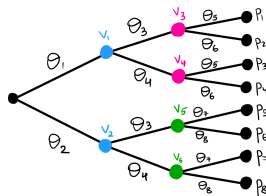
Duarte and Solus showed that any directed acyclic graph (DAG) model is also a CStree model. Thus, to any DAG  $G$ , we may associate a CStree  $\mathcal{T}_G$ .

## Theorem (Duarte, Solus)

*The DAG  $G$  is perfect if and only if  $\mathcal{T}_G$  is balanced.*

**Balanced:** for any  $v, w$  in the same stage and for every pair of children  $v_i, v_j$  and  $w_i, w_j$  with  $\theta(v \rightarrow v_i) = \theta(w \rightarrow w_i)$  &  $\theta(v \rightarrow v_j) = \theta(w \rightarrow w_j)$ , we have  $t(v_i)t(w_j) = t(w_i)t(v_j)$ .

**Perfect:** parents of every vertex form a clique.



$$t(v_3)t(v_6) = t(v_4)t(v_5)$$

$$(\theta_5 + \theta_6)(\theta_7 + \theta_8) = (\theta_7 + \theta_8)(\theta_5 + \theta_6)$$

$$\implies \text{balanced!}$$

perfect:

$$X_1 \rightarrow X_2 \rightarrow X_3$$

$$X_2 \leftarrow X_1 \rightarrow X_3$$

non-perfect:

$$X_1 \rightarrow X_3 \leftarrow X_2$$

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*Perfect:* parents of every vertex form a clique.

## Theorem (Duarte, Görden)

*A staged tree  $(\mathcal{T}, \Theta_{\mathcal{T}})$  is balanced if and only if  $\mathcal{M}_{(\mathcal{T}, \Theta)}$  is toric.*



# CStrees and DAGs

Not every CStree represents a DAG. However, to any CStree we may associate a **collection** of DAGs [Duarte and Solus]. Let  $\mathcal{T}$  be a CStree.

- For any stage  $S \subset L_{k-1}$ , distributions in  $\mathcal{M}_{\mathcal{T},\Theta}$  entail CSI statements

$$X_k \perp\!\!\!\perp X_{[k-1]\setminus C} | X_C = x_C$$

where  $C$  is the set of all indices  $\ell$  such that all elements in  $S$  have the same outcome for  $X_\ell$ . Here  $C$  is a (stage-defining) *context*.

- Let  $\mathcal{J}(\mathcal{T})$  be all the CSI statements implied by these statements.
- By the absorption axiom, there exists a collection of contexts  $\mathcal{C}_{\mathcal{T}} = \{X_C = x_C\}$  such that for any  $X_A \perp\!\!\!\perp X_B | X_S, X_C = x_C \in \mathcal{J}(\mathcal{T})$  there is no  $T \subseteq C$  such that  $X_A \perp\!\!\!\perp X_B | X_{S \cup T}, X_{C \setminus T} = x_{C \setminus T} \in \mathcal{J}(\mathcal{T})$ . These are the *minimal contexts* of  $\mathcal{T}$ .
- To each such  $X_C = x_C \in \mathcal{C}_{\mathcal{T}}$ , we can associate a *minimal context DAG*  $G_{X_C=x_C}$  via the I-MAP.

# CStrees and DAGs

**Main question:** can we relate the *balanced* property of CStrees to the *perfect* property of the associated minimal context DAGs?

**Initial conjecture:** A CStree is balanced if and only if all of its minimal context DAGs are perfect.

# CStrees and DAGs

**Main question:** can we relate the *balanced* property of CStrees to the *perfect* property of the associated minimal context DAGs?

**Initial conjecture:** A CStree is balanced if and only if all of its minimal context DAGs are perfect. **FALSE!**

**Proposition (A., Duarte, Vill)**

*If  $\mathcal{T}$  is a CStree with only perfect minimal context DAGs, it is balanced.*

Let  $p$  be the total number of random variables in  $\mathcal{T}$ .

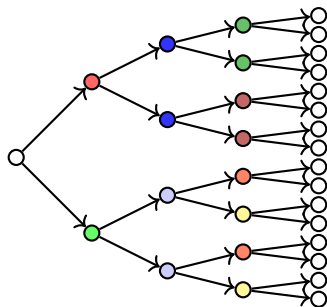
**Theorem (A., Duarte, Vill)**

*A CStree with  $p = 3$  is balanced if and only if all of its minimal context DAGs are perfect.*

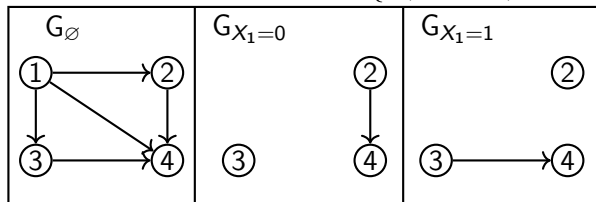
For  $p \geq 4$ , there are balanced CStrees with non-perfect minimal context DAGs.

# Counterexample

Let  $\mathcal{T} =$



The minimal contexts of  $\mathcal{T}$  are  $\{\emptyset, X_1 = 0, X_1 = 1\}$ .



$G_{\emptyset}$  is not perfect!!

## Balanced vs. perfect: saturated statements

We can still describe a balanced CStree by a finite collection of perfect DAGs using **ALGEBRA**.

We say the CSI statement  $X_A \perp\!\!\!\perp X_B | X_S, X_C = x_C$  is *saturated* if  $A \cup B \cup C \cup S = [p]$ .

Let  $\text{Sat}(\mathcal{T})$  be the set of all saturated CSI statements in  $\mathcal{J}(\mathcal{T})$ .

### Theorem (A., Duarte, Vill)

*If  $\mathcal{T}$  is a balanced CStree, then  $\ker(\psi_{\mathcal{T}})$  is generated by the binomials associated to all saturated CSI statements in  $\mathcal{J}(\mathcal{T})$ , i.e.*

$$\ker(\psi_{\mathcal{T}}) = I_{\text{Sat}(\mathcal{T})}.$$

Proof uses inductive construction and toric fiber products!

## Balanced vs. perfect: moralization

A *moralization* of a DAG  $G$  is another DAG  $G^m$  obtained from  $G$  by “marrying” any two “unmarried” parents of each vertex.



To any DAG  $G$ , we may associate an ideal generated by polynomials that entail all *d-separation statements* that hold in  $G$ . Let  $\text{Sat}(G)$  denote the set of all saturated *d-separation* (CSI) statements implied by  $G$ .

### Proposition (Sullivant)

*A saturated CSI statement is implied by  $G$  if and only if it is implied by  $G^m$ .*

# Balanced vs. perfect: moralization

## Theorem (A., Duarte, Vill)

A CStree  $\mathcal{T}$  is balanced if and only if

$$\ker(\psi_{\mathcal{T}}) = \sum_{X_C = x_C \in \mathcal{C}_{\mathcal{T}}} I_{\text{Sat}}(G_{X_C = x_C}^m).$$

In particular, every balanced CStree can **algebraically** be described by a finite collection of perfect DAGs.

*Thanks!*