Reliability Engineering and System Safety 131 (2014) 242–250

Contents lists available at ScienceDirect

Reliability Engineering and System Safety

journal homepage: www.elsevier.com/locate/ress

Optimal number of tests to achieve and validate product reliability

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ARTICLE INFO

Available online 9 May 2014

Keywords:
Reliability validation tests
Statistical uncertainties
Reliability-based Optimal design

ABSTRACT

The reliability validation of engineering products and systems is mandatory for choosing the best cost-effective design among a series of alternatives. Decisions at early design stages have a large effect on the overall life cycle performance and cost of products. In this paper, an optimization-based formulation is proposed by coupling the costs of product design and validation testing, in order to ensure the product reliability with the minimum number of tests. This formulation addresses the question about the number of tests to be specified through reliability demonstration necessary to validate the product under appropriate confidence level. The proposed formulation takes into account the product cost, the failure cost and the testing cost. The optimization problem can be considered as a decision making system according to the hierarchy of structural reliability measures. The numerical examples show the interest of coupling design and testing parameters.

1. Introduction

Manufacturers have to prove that their products meet the target reliability with a specified confidence level. Under cost and time constraints, it is often desired to demonstrate that the reliability goal is achieved on the basis of small number of tests. Although the theory of statistics requires a large population of units to be tested up to failure, only a small number of reliability demonstration tests can be carried out in practice, due to limited cost and time [1]. In many sensitive industries, such as military and aerospace, no more than one or two units can be tested. The situation is even worst when no failure can be observed within the specified test time-span! In automotive, engine fatigue tests cannot be run on more than five units, and production cannot be stopped to wait until all testing outcomes become available. In this case, the obtained information is very limited and additional uncertainties are strongly involved in the reliability model. In such a situation, the decision-maker cannot trust on the reliability demonstration tests. Although the reliability of the distributed product remains practically unknown, the reliability engineer has to state that he/she can guarantee the target (i.e. product resistance or performance) with a certain confidence.

The problem of reliability demonstration has gained large interest in recent years, due to the increasing industrial demands, under physical, functional and time considerations [2]. Most of the works in the literature are based on the Bayesian techniques, either by using the failure free time period testing [1], or by using failure data within a Bayesian procedure in order to carry out inference analysis on product reliability [3,4]. In these approaches, the reliability and the confidence intervals have been previously stipulated by the customer. This method depends on the “success run formula”, as defined in [5]. According to these authors, the above method has the drawback that small sample tests are useful for comparative but not for absolute evaluation of high reliabilities. It is clear that the optimal test design that does not take into account the system degradation (e.g. [6]) is not suitable for the large majority of engineering systems, especially in mechanical and civil engineering. Ahmed et al. [7] have compared four reliability demonstration approaches, namely the confidence interval, the test of hypothesis, the Bayesian analysis and the compound uncertainty method. The application to case studies has shown that the compound uncertainties has the best stability and convergence rate, as it considers both aleatoric and epistemic uncertainties; this approach is therefore adopted in the present work.

Regarding product design, the Reliability-Based Design Optimization [8] has been developed in order to reduce the structural cost under reliability constraints. Two main formulations are usually considered – either minimizing the expected total cost (e.g. [9,10]) or minimizing the initial cost under reliability constraints (e.g. [11,12]); a combination of these two formulations has also been considered by several authors (e.g. [9,13]). Depending on the way in which cost is considered, the optimal design may be

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http://dx.doi.org/10.1016/j.ress.2014.04.014
0951-8320/

completely different, resulting from some incompatibility in the basic assumptions, such as conflicting needs of optimization objectives and constraints.

In parallel, the reliability validation activities are defined as the formal process of confirming through testing analysis, inspections and other engineering activities that the product reliability is met, taking into account the cost of these activities which includes energy, labor, maintenance, equipment depreciation and miscellaneous [14]. Zhang et al. [15] have integrated reliability testing and computational reliability analysis for product development, including statistical uncertainty and modeling error, in order to reach cost-efficient estimation of failure probability and life distribution. The validation may represent a significant expense that must trade off with design target value (i.e. nominal or mean resistance) [16]. In other words, performing more testing may lead to design cost reduction, and vice versa. In a recent work, Villanueva et al. [17] considered the redesign after testing to balance development costs versus performance by simultaneously designing the design and the post-test redesign rules during the initial design stage. This work aimed at minimizing the structural mass with the minimum redesign costs. To the authors' knowledge, the testing optimization has never been fully coupled with the reliability-based design of products including lifetime degradation costs. The two problems of design and testing are very often considered separately in the literature and therefore optimizing each problem individually cannot lead to overall optimization of the design and the validation of products.

In the present work, a new formulation is proposed by combining the design, the failure and the testing costs. The optimal number of tests is not specified only on the basis of confidence criterion but also by including the testing cost and delay. In this way, both the product design and the validation testing are optimized by considering the initial cost, the failure risk and the testing effort to demonstrate the reliability. In the following sections, the reliability demonstration tests are presented, then the optimization formulation is developed and applied to three numerical examples.

2. Demonstration test

When considering a limited number of tests, the estimates of resistance mean and standard deviation cannot be precisely defined, as the epistemic statistical uncertainties are high. For this reason, the resistance distribution parameters should be considered as random [7], in order to better estimate the in-service reliability.

2.1. Product reliability

For any product or structure, the state of safety can be described by a performance function [18,19], traditionally defined by the difference between the resistance R and the applied stress S; i.e. $G(R,S)=R-S$. Each of R and S is described by its probability density function, noted as $f_R(r)$ and $f_S(s)$, respectively. According to the stress–resistance inference model shown in Fig. 1, the probability of failure is given by

$$P_f = P[G(R,S) \leq 0] = \int_{R \leq S} f_{R,S}(r,s)dr\, ds \quad (1)$$

where $f_{R,S}(r,s)$ is the joint probability density function for resistance and stress. Due to numerical difficulties in evaluating the above integral, an efficient and practical approximation can be obtained by First and Second order Reliability Methods FORM/SORM [19], which are based on the calculation of a reliability measure, known as the reliability index β [20], by solving the following optimization problem:

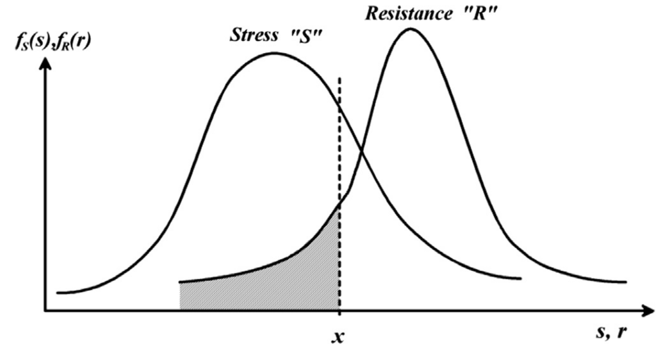

Fig. 1. Stress and resistance inference.

$$\beta = \underset{u_i}{\operatorname{argmin}} \sqrt{\sum_{\mathrm{i}} u_i^2} \text{ subject to } G(r,s)=0 \quad (2)$$

where u_i are the uncorrelated normalized random variables, obtained by probabilistic transformation of the physical variables $u_i = T_i(x_j)$. There are many algorithms available to solve the reliability problem (2), such as standard optimization schemes and HL-RF algorithms [19,20]. The solution of this problem gives the reliability index β and the failure probability can be deduced by FORM approximation $P_f = \Phi(-\beta)$; where $\Phi(.)$ is the Gaussian cumulated probability function.

In the special case of independent normal distributions of R and S, with respective mean values and standard deviations: μ_R, μ_S, σ_R and σ_S, the solution of problem (2) leads to the reliability index written as

$$\beta = \frac{\mu_R - \mu_S}{\sqrt{\sigma_R^2 + \sigma_S^2}} \quad (3)$$

2.2. Reliability demonstration

As the number of product tests cannot cover all the population, the real values of resistance parameters – mean value μ_R and standard deviation σ_R – are not precisely known, as large amount of uncertainties comes from the use of small sample size. The reliability level can be demonstrated by considering the test sample estimates as epistemic random variables within the reliability analysis model [7], rather than just deterministic distribution parameters. This leads to system modeling by compound variables (i.e. random variables whose distribution parameters are random). Having these compound variables, the probability distribution becomes conditional on the parameter estimates $f_{R|m_R,s_R}(r|m_R,s_R)$, where m_R and s_R are respectively the mean and the standard deviation estimates of resistance, which are approximating the unknown real mean and standard deviation μ_R and σ_R. In other words, the computed reliability becomes also random, as it depends on the randomness of the sample mean and standard deviation. Without loss of generality, it is assumed in this paper that the coefficient of variation of the resistance $v_R = \sigma_R/\mu_R$ is known from previous engineering experience on similar production processes.

Consider a number of tests N_{test} to be performed on identical units, where the measured resistances are r_1, r_2,…,$r_{N_{test}}$. The arithmetic mean $\bar{r} = (r_1 + r_2 + \cdots + r_{N_{test}})/N_{test}$ can be considered as a realization of the estimate $\overline{R}$. Under this observation, the product design should satisfy the admissible failure probability in operation $P_{f_{adm}}$

$$P_f(\bar{r}) = \Pr[G(R,S) \leq 0\ \mu_R = \bar{r}] \leq P_{f_{adm}}\ \forall \bar{r} \in \overline{R} \quad (4)$$

where $\Pr[\cdot]$ is the probability operator, $P_f(\bar{r})$ is the failure probability of the product, $P_{f_{adm}}$ is the admissible failure probability (corresponding to the target reliability level) and $\bar{r}$ is the observed mean of the sample. Eq. (4) should be satisfied whatever the value of $\bar{r}$ drawn from the distribution of the estimate random variable $\overline{R}$, with probability density function $f_{\overline{R}}(\overline{r})$, mean value $m_{\overline{R}}$ and standard deviation $\sigma_{\overline{R}}$. The failure probability can thus be computed by integration over the possible domain of the estimate variable

$$P_f = \int_0^{\infty} P_f(\overline{r}) f_{\overline{R}}(\overline{r}) d\overline{r} \tag{5}$$

Due to randomness in the sampling procedure, the admissible failure probability cannot be guaranteed without uncertainties, and therefore confidence intervals should be considered. When the tests are performed at the stress level S_{test}, the decision-maker has to confirm that the probability of having appropriate product reliability is higher than a given confidence level $(1-\alpha)$; this condition can be written as

$$\Pr[\Pr[G(R,S) \le 0 \,|\, \mu_R = \overline{R},\ S_{test} = m_{\overline{R}}] \le P_{f_{adm}}] \ge 1-\alpha \tag{6}$$

where α is the acceptable proportion that the product does not fit the desired reliability. This expression allows us to specify the test load that ensures the in-service target reliability, with a confidence probability of $1-\alpha$. It can be directly solved, in order to define the test load $S_{test} = m_{\overline{R}}$, where $m_{\overline{R}}$ is the mean resistance of the tested sample (Fig. 2).

By introducing the reliability index concept, Eq. (6) can be equivalently written in the following form:

$$\Pr\left[\beta(\overline{R}) - \beta_{adm} \le 0\right] \le \alpha \tag{7}$$

where $\beta(\overline{R})$ is the reliability index of the product, given as a function of the mean resistance estimate $\overline{R}$ and β_{adm} is the target reliability index, related to the admissible failure probability by the following relationship: $P_{f_{adm}} = \Phi(-\beta_{adm})$. In the case of normal distributions of R and S, Eq. (7) takes the following form:

$$\Pr\left[\frac{\overline{R}-\mu_S}{\sqrt{(v_R\overline{R})^2 + (v_S\mu_S)^2}} + \Phi^{-1}(P_{f_{adm}}) \le 0\right] \le \alpha \tag{8}$$

where v_R and v_S are respectively the coefficients of variation of resistance and stress. The above equation can be considered as a new reliability problem with respect to the following performance function:

$$G_{test}(\overline{R}) = \overline{R} - \mu_S + \Phi^{-1}(P_{f_{adm}})\sqrt{(v_R\overline{R})^2 + (v_S\mu_S)^2} \tag{9}$$

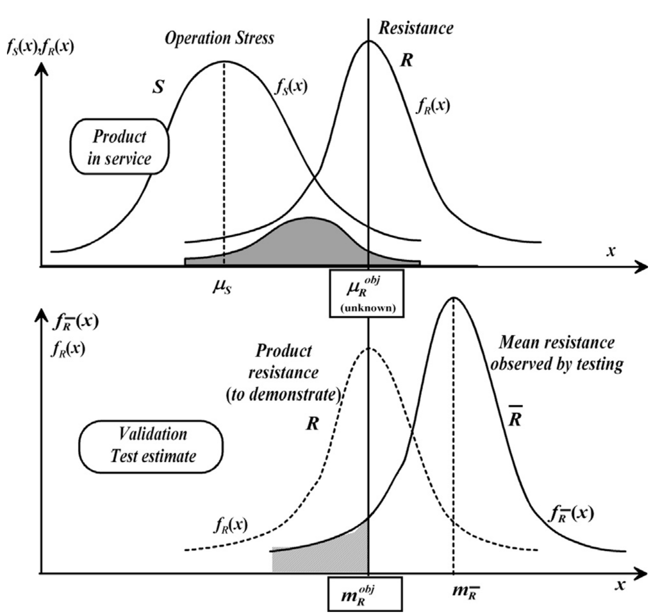

Fig. 2. Product and test strength distributions.

This performance function contains only one random variable which is the estimate of the mean resistance. Under normality assumption for the estimate $\overline{R}$, the test reliability index is therefore defined as:

$$\beta_{test}(m_{\overline{R}}) = \frac{1}{v_{\overline{R}} m_{\overline{R}}}\left[m_{\overline{R}} - \mu_S - \beta_{adm}\sqrt{(v_R m_{\overline{R}})^2 + (v_S\mu_S)^2}\right] \tag{10}$$

where $v_{\overline{R}}$ is the coefficient of variation of the mean estimate $\overline{R}$, which can be related to the coefficient of variation of the resistance by $v_{\overline{R}} = v_R/\sqrt{N_{test}}$; the error in this relationship vanishes when the normality assumption holds for the mean estimate (i.e. either the number of tests is significant; e.g. $N_{test} \ge 5$, or the resistance distribution is normal whatever the number of tests). By setting the test load to $S_{test} = m_{\overline{R}}$, the test design should satisfy the following condition:

$$\beta_{test}(S_{test}) \ge \beta_\alpha = -\Phi^{-1}(\alpha) \tag{11}$$

Combining Eqs. (10) and (11), the test load and number can thus be computed by solving the following equation:

$$\left(1 - \beta_\alpha \frac{v_R}{\sqrt{N_{test}}}\right) S_{test} - \mu_S - \beta_{adm}\sqrt{(v_R S_{test})^2 + (v_S\mu_S)^2} \ge 0 \tag{12}$$

with $\beta_\alpha = -\Phi^{-1}(\alpha)$ and $\beta_{adm} = -\Phi^{-1}(P_{f_{adm}})$. It is clear from this equation that the test parameters (i.e. load S_{test} and number N_{test}) must be specified in terms of the required reliability level of the product and the confidence level of the test, in addition to the dispersion of stress and resistance. This formulation has the advantage of considering both operating and testing uncertainties, where the coupled effect plays an important role in reliability demonstration. In other words, the influence of the random strength on the product reliability should be correlated with the precision of parameter estimates. The high sensitivity of reliability with respect to resistance requires high accuracy of test estimates. On the opposite, when resistance does not have much influence on the product reliability (for example, due to large dispersion of the applied loading), it becomes useless to get high precision on the resistance parameters, and consequently the testing cost and time can be saved without loss of product reliability estimation.

3. Design and validation cost optimization

3.1. Cost model

The Life Cycle Cost (LCC) analysis is an efficient tool for choosing the most cost-effective approaches from a series of alternatives. Studies reported in [21] showed that the design of product influences between 70% and 85% the total cost of a product. Therefore, the designers can substantially reduce the LCC of products by optimizing the product parameters according to the operating requirements, under the available information. Many factors can have significant influence on the design, such as the manufacturing costs and tools, the performance and quality requirements, the number of products to be manufactured, the cost and the total number of tests, etc. The general expression of the LCC should also include the costs of design, validation, manufacturing, warranty and overhead [22], in addition to operating costs of maintenance, inspection, repair and failure. This kind of problems is considered as a decision making problem according to the hierarchy of structural reliability measures [23].

The present work is focused on the coupling of design, failure and validation costs, in order to define the best product parameters and validation strategy. In other words, we assume herein that manufacturing, warranty and overhead costs are independent from the selected values of the design parameters. The proposed model aims at addressing the question about the optimal number of tests which satisfies the reliability target and gives the demonstration necessary to validate the product under the specified confidence level. In this scope, the total cost of the product is given by:

$$C_T(d_i, x_j, y_k) = C_D(d_i) + C_F(d_i, x_j) + C_V(y_k) \tag{13}$$

where d_i are the design parameters, x_j are the distribution parameters of the random variables, y_k are the validation test parameters, $C_T(d_i, x_j, y_k)$ is the total cost, $C_D(d_i)$ is the initial cost of design and manufacturing, depending on design parameters, $C_F(d_i, x_j)$ is the expected failure cost, depending on design and uncertainty parameters, and $C_V(y_k)$ is the cost of the demonstration tests, depending on test parameters. Eq. (13) represents the two major quantifiable characteristics in product life cycle – reliability and quality. While the reliability is related to initial and failure costs, the quality is related to validation tests demonstrating that the product is conformal to design specifications.

It is easy to see that the number of tests to be optimized concerns two conflicting needs – the product cost related to the required reliability level and the test cost related to the number of tested specimens and configurations. While increasing the number of tests improves the reliability demonstration, and therefore reduces the product cost, it increases the testing cost itself. The optimization problem can therefore be formulated in order to achieve an optimal specification of the number of tests, regarding the product reliability.

Without loss of generality, a simple model can be defined by considering that the design cost of the product is proportional to its resistance according to a power law; it is to note that the term "design cost" corresponds to the initial cost including manufacturing. The design cost can be written as:

$$C_D(d_i) = n_p \quad C_{D0} = n_p c_P m_R^w \tag{14}$$

where n_P is the total number of manufactured product units, C_{D0} is the cost of one unit, c_P is the specific cost of each product unit,m_R is the mean resistance defined to guarantee the target reliability, and w is a model parameter to specify the type of cost-resistance relationship. The expected failure cost can be computed by the equation:

$$C_F(d_i, x_j) = C_f P_f(d_i, x_j) \tag{15}$$

where C_f is the cost of failure consequences and $P_f(d_i, x_j)$ is the failure probability as a function of the design variables and the statistical parameters; it is assumed herein that the failure consequences C_f are independent from the design parameters d_i.

Concerning the cost of demonstration tests, it is proposed to consider the configuration where the tests are carried out sequentially on a number of benches in parallel. The testing costs are divided into setup and specimen-related costs. The proposed cost formulation is as following:

$$C_V(y_k) = c_B n_B + n_B n_V (c_V + C_{D0}) + n_V T_V B_V \tag{16}$$

where n_B is the number of test benches, n_V is the number of reliability demonstration tests to be carried out on each bench, c_B is the setup cost of each bench, c_V is the setup cost of each tested specimen, $C_{D0} = c_P m_R^w$ is the cost of tested specimen until failure, T_V is the time span for each test and B_V is the benefit loss due to production stop when waiting for test results.

3.2. Optimization problem

Fig. 3 illustrates the evolution of various costs as a function of the number of validation tests. The increase of test number and time will directly increase the testing cost. On the opposite, the expected failure cost decreases with testing due to the reduction of statistical uncertainties. In parallel, the design cost tends to decrease with the number of tests because the uncertainty

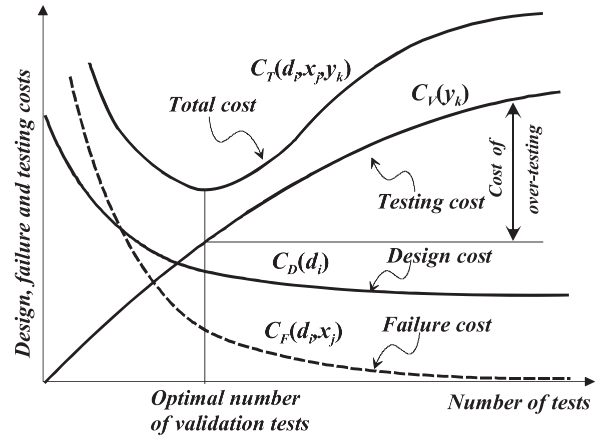

Fig. 3. Design and validation cost.

reduction leads to the reduction of the safety margins and therefore the design becomes smaller. As can be seen in Fig. 3, the sum of the above three curves leads to the total product cost taking the form of a U-shaped curve with a minimum point corresponding to the optimal test number and the associated design parameters.

In this work, three optimization models are proposed in order to determine the best set of design and testing parameters; these models are described in the following:

Model 1: Double-constraint design and test cost optimization. This formulation considers the total initial cost, given by the sum of design and validation costs, under the constraints of satisfying the reliability target and the test confidence requirements. The optimization parameters are the target mean resistance $m_{R_{obj}}$, on one hand, and the test stress S_{test} and number N_{test}, allowing to guarantee the objective resistance with sufficient confidence, on the other hand.

$$\begin{aligned} &\min_{d_i, y_k} C_D(d_i) + C_V(y_k) \\ &\text{under :} \\ &\frac{m_{R_{obj}} - \mu_S}{\sqrt{(v_R m_{R_{obj}})^2 + (v_S \mu_S)^2}} \geq \beta_{adm} \\ &\frac{S_{test} - m_{R_{obj}}}{\frac{v_R S_{test}}{\sqrt{N_{test}}}} \geq \beta_\alpha \end{aligned} \tag{17}$$

Model 2: Single-constraint total cost.
This formulation considers the total cost, including design, failure and validation costs, under the embedded constraint condition containing product reliability and test confidence. In this formulation, the cost of failure consequences plays a significant role in setting the reliability level of the product.

$$\begin{aligned} &\min_{d_i, y_k} C_T(d_i, x_j, y_k) = C_D(d_i) + C_F(d_i, x_j) + C_V(y_k) \\ &\text{under :} \\ &\left(1 - \beta_\alpha \frac{v_R}{\sqrt{N_{test}}}\right) S_{test} - \mu_S - \beta_{adm} \sqrt{(v_R S_{test})^2 + (v_S \mu_S)^2} \geq 0 \end{aligned} \tag{18}$$

Model 3: Double-constraint total cost.
In order to ensure a robust solution of Model 2, an additional constraint on the reliability target can be included to avoid the case of under-estimating the failure cost. The reliability constraint is only active when the failure consequences are small compared to the design and testing costs, and hence the

optimal design can lead to low product reliability, which may not be allowed by the designer.

$$\begin{aligned}&\min_{d_i, y_k} C_T(d_i, x_j, y_k) = C_D(d_i) + C_F(d_i, x_j) + C_V(y_k)\\&\text{under :}\\&\frac{m_{R_{obj}} - \mu_S}{\sqrt{(v_R m_{R_{obj}})^2 + (v_S \mu_S)^2}} \geq \beta_{adm}\\&\left(1 - \beta_\alpha \frac{v_R}{\sqrt{N_{test}}}\right) S_{test} - \mu_S - \beta_{adm}\sqrt{(v_R S_{test})^2 + (v_S \mu_S)^2} \geq 0\end{aligned} \tag{19}$$

The solution of each of these problems leads to the optimal set of product and validation parameters, by satisfying the constraints related to the target reliability β_{adm} and the test confidence β_α.

4. Numerical examples

Three numerical examples are considered to show the interest of the proposed formulation in various testing situations. While the first example deals with a generic case of stress–resistance inference, the second example deals with the configuration of testing equipment and the third example considers the acceleration time uncertainties.

4.1. Stress–resistance inference model

The stress–resistance inference model is described by the performance function $G(R,S) = R - S$, where the resistance R is considered as normally distributed with coefficient of variation of 0.1 and the stress S is deterministic equal to 0.4. The target reliability index is $\beta_{adm} = 3.8$ and the test confidence is 0.95; i.e. $\beta_\alpha = \beta_{0.05} = 1.645$. The cost of each product is $C_p = 10 m_R$ currency units (m_R being the resistance guaranteed by testing at the specified confidence level), and the number of manufactured products is equal to 5000 units. For illustration purpose, only one test bench is considered, the production is not stopped during testing and the tested specimen cost is neglected; the cost of each test is $c_V = 100$ currency units. Under the constraint of satisfying the target probability, Fig. 4 plots the product cost and the total expected cost in terms of the number of tests. Each point of these curves is a solution of the optimal design under reliability constraint, with a given number of validation tests. Although the product cost decreases monotonically by performing more tests, the total cost increases beyond the optimum point due to the increase of testing cost without significant decrease of design cost. The solution of the optimization Model 1 Eq. (17) gives the optimal number of tests which is equal to 10 tests with a total cost of 3.5×10^7 currency units.

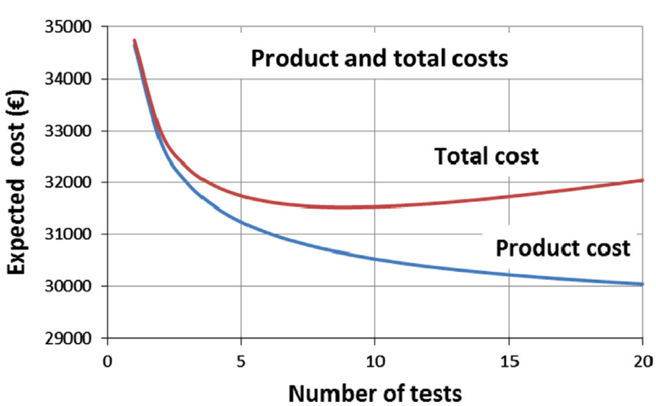

Fig. 4. Design and validation cost.

Table 1
Optimal number of tests and production corresponding to one test.

Test-to-product cost ratio	Optimal number of tests for 5000 products	Minimum number of products for the first validation test
10	10	135
15	7	200
20	6	265

Table 2
Optimal total costs and test number versus reliability level.

Failure probability	Optimal number of tests	Total cost (currency unit)
10^{-1}	8	2.51×10^7
10^{-3}	9	3.15×10^7
10^{-5}	10	3.78×10^7

For different test-to-product cost ratios, the second column in Table 1 provides the optimal number of validation tests to be performed when the production size is equal to 5000 units. The third column in the same table indicates the minimum number of manufactured products to justify the setting of one test; i.e. the validation testing is not economical when the production size is lower than the given product numbers. It can be seen that when the test cost is twenty times the product cost, it is required to perform 6 validation tests, while at least 10 tests are required when the test cost equals to ten times the product cost. In the same table, it can be seen that low production size does not justify validation testing and this size increases with the increase of testing cost. This means that below a given number of products, with the numerical data in this example, it is more interesting to overdesign the product than to spend money on validation testing. Naturally, this minimum number depends on the considered data, especially costs and probabilities.

Table 2 indicates the optimal number of tests and the minimum total cost, in terms of the admissible failure probability. As can be seen, increasing the reliability target requires additional costs, due to larger number of tests and larger required resistance. When the target goes from 10^{-1} to 10^{-5}, the total cost increases by 50% (i.e. from 2.51×10^7 to 3.78×10^7) and the testing cost increases by 25% (i.e. from 8 to 10 tests).

These results offer various alternatives to designers and suppliers, allowing them to make decisions according to their preferences. It becomes possible to optimally set the design parameters and the testing capacity to ensure the reliability of the products in operation.

4.2. Optimal pipe testing

Consider the manufacturing of pipes, with length L, radius r and thickness t, aimed at conveying fluid under pressure p (Fig. 5a). The pipe testing is carried out by applying hydrostatic pressure until failure. Several test benches are to be installed to allow for testing in parallel, and thus reducing the total test duration; it is to note that, the specimens are tested sequentially on each bench (i.e. in series). In operation, the applied pressure p and the material strength f_Y are considered as random, and the performance function is given by:

$$G(f_y, p) = f_Y t - pr \tag{20}$$

By assuming normal distributions for simplicity, without loss of generality, the reliability index can be explicitly obtained as a

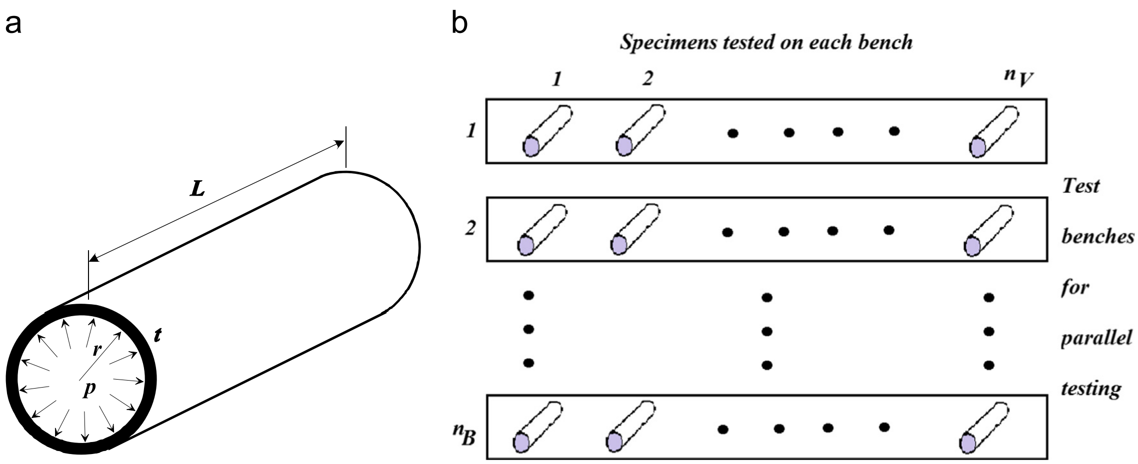

Fig. 5. Pipe and testing configurations. (a) Pipe configuration and (b) Testing configuration.

Table 3
Tube parameters and cost data.

Model parameters	Symbol	Value
Pipe radius (mm)	r	220
Pipe length (mm)	L	6000
Production rate (pipes)	n_P	8500
Test duration (day)	T_V	0.5
Daily factory benefit (€/day)	B_V	7380
Cost per unit weight (€/kg)	C_S	1.8
Bench setup cost (€)	c_B	$3900\left(\frac{f_{test}}{500}\right)^2$
Test setup cost (€)	c_V	241
Pipeline failure cost (€)	C_f	8 million
Design variables	**Symbol**	**Interval**
Pipe wall thickness (mm)	t	$1 \le t \le 20$
Test stress (MPa)	f_{test}	$500 \le f_{test} \le 1500$
Number of benches	n_B	$n_B \ge 1$
Number of test/bench	n_V	$n_V \ge 1$
Random variables	**Symbol**	**Statistical parameters**
Internal pressure	p	$m_p = 4$ MPa, $v_p = 0.10$
Material strength	f_Y	$m_{f_Y} = 500$ MPa, $v_{f_Y} = 0.08$

function of the design variables, as following:

$$\beta(t, r) = \frac{m_{f_Y} t - m_p r}{\sqrt{(v_{f_Y} m_{f_Y} t)^2 + (v_p m_p r)^2}} \tag{21}$$

where m_{f_Y}, m_p, v_{f_Y} and v_p are respectively the mean values and the standard deviations of the yield stress and the applied pressure, r is the pipe radius and t is the pipe wall thickness. The concerned production contains $n_p = 8500$ pipes of this type and the manufacturing benefits is equal to 7380€/day. The product cost is given by

$$C_D(t) = n_p\ C_{D0}(t) = n_p c_P (2\pi \rho_S r t L) \tag{22}$$

where $\rho_S = 7850$ kg/m^3 is the steel density, $C_{D0}(t)$ is the cost of a single pipe, and $C_D(t)$ is the cost of the whole production. The pipe thickness t is the only design variable, as all the other parameters of the pipe are fixed by the client. The testing cost is defined by

$$C_V(n_B, n_V, T_V, S_{test}, t) = c_B n_B + n_B n_V (c_V + C_{D0}(t)) + n_V T_V B_V \tag{23}$$

where n_B, n_V, T_V and S_{test} are respectively the number of test benches, the number of tested specimens on each bench, the test duration and the test validation stress, c_B and c_V are respectively the setup costs for benches and specimens, and B_V is the benefit

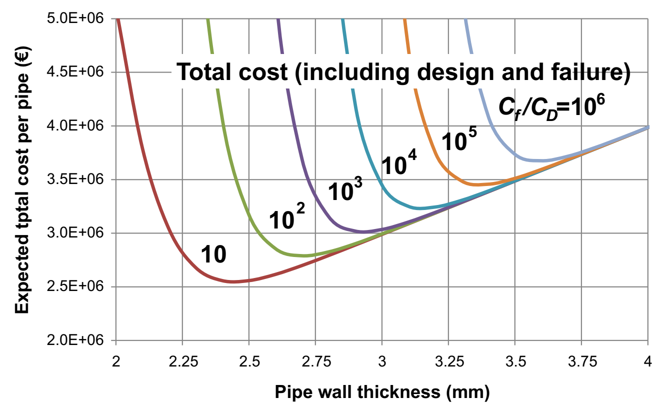

Fig. 6. Total cost/pipe, in terms of failure-to-design cost ratio.

from pipe production (this benefit is lost due to testing, as the production is stopped, waiting for test results). Table 3 provides the numerical data of the problem; the random variables are considered as normally distributed and the coefficient of variation are supposed to include various sources of uncertainties. The expected failure cost is given by

$$C_F(t) = C_f P_f(t) \tag{24}$$

where C_f is the cost of failure consequences. In the following, two cases are considered, corresponding to perfect and uncertain resistance estimates, respectively.

4.2.1. Design under perfect resistance estimate

When considering a very large number of tests, the estimate of the mean strength can be considered as free from epistemic uncertainties, and therefore the product mean strength is exactly given by the point estimate. By considering two levels of reliability target, namely $P_{f_{adm}} = 10^{-4}$ and 10^{-6}, the minimization of the initial cost $C_D(t)$ gives an optimal wall thickness $t = 2.83$ and $t = 3.25$ mm, respectively, with corresponding minimum design cost of 331.1 and 381.3€/pipe. In this case, the increase of the target reliability increases the design cost by 15.1%.

The optimal design is now considered by minimizing the total design cost $C_T(t) = C_D(t) + C_F(t)$ instead of the initial cost only. In this case, the constraint on the admissible failure probability can be removed as this probability is already included in the total cost, through the expected failure cost. Fig. 6 depicts the expected total cost as a function of the pipe wall thickness, for various ratios of failure-to-design costs. It is easy to see that the optimum wall thickness increases significantly with the increase of the failure cost. The optimal thickness goes from 2.45 mm when the failure cost is 10 times the design cost, up to 3.59 mm in case of extremely high consequences (i.e. failure cost equals to 10^6 times the design cost). As indicated in Table 4, the optimal setting of the target reliability increases with the increase of the failure cost. The comparison of the design and total costs shows that, in all cases, the expected failure cost $C_f P_f$ is very small at the optimal design (i.e. less than 0.1% of the total cost).

4.2.2. Imperfect test outcome

Let us now consider the epistemic uncertainties due to the limited number of tests. The problem (18) has to be solved in order to determine the optimal design and testing arrangement; i.e. pipe wall thickness t, test stress $S_{test} = f_{Y,test}$ (i.e. yield stress at test), number of test benches n_B and number of specimens to be tested on each bench n_V. The optimization problem takes the following form:

$$\min_{t, f_{Y,test}, n_B, n_V} C_T(t, f_{Y,test}, n_B, n_V) = C_D(t) + C_F(t, f_{Y,test}) + C_V(n_B, n_V, f_{Y,test})$$
under :
$$\left(1 - \beta_\alpha \frac{v_{f_Y}}{\sqrt{n_B n_V}}\right) f_{Y,test} t - m_p r - \beta_{adm}\sqrt{(v_{f_Y} f_{Y,test} t)^2 + (v_p m_p r)^2} \geq 0$$
with :
$$C_D(t, f_{Y,test}) = n_p C_S (2\pi \rho_S r t L)\left(\frac{f_{Y,test}}{500}\right)^{0.2}$$
$$C_F(t) = C_f P_f(t)$$
$$C_V(n_B, n_V, f_{Y,test}) = c_B n_B \left(\frac{f_{Y,test}}{500}\right)^{1.5} + n_B n_V (c_V + C_S\ (2\pi\ \rho_S\ r\ t\ L)) +\ n_V T_V B_V \quad (25)$$

In these expressions, the term $\left(\frac{f_{Y,test}}{500}\right)$ is introduced to relate the design cost to the selected material strength, on one hand, and to take account for testing bench equipment to be used with higher stress levels, on the other hand. By setting the target reliability of the pipeline to 10^{-4} and the probability of test acceptance to 10^{-3}, the solution of the above problem provides the optimal parameters as $t = 2.6$ mm, $f_{Y,test} = 584.8$ MPa, $n_B = 3$ and $n_V = 4$, leading to the minimum cost of 328.8€/pipe. In order to understand the role of testing costs on the optimal setting of the parameters, the graphs in Fig. 7 depict the evolution of the optimal number of benches and tested specimens on each bench as functions of their setup costs. As can be seen, the increase of bench setup cost leads

Table 4
Optimal total costs of each pipe and test number versus reliability level.

Failure cost C_f	Optimal wall thickness t (mm)	Minimum design unit cost (€/pipe)	Minimum total unit cost (€/pipe)	Optimal failure probability
$10C_D$	2.45	286.7	299.8	4.56×10^{-3}
10^2C_D	2.70	316.4	328.2	3.73×10^{-4}
10^3C_D	2.93	343.4	354.6	3.29×10^{-5}
10^4C_D	3.15	369.3	380.5	3.02×10^{-6}
10^5C_D	3.37	394.9	406.2	2.84×10^{-7}
10^6C_D	3.59	420.9	432.4	2.73×10^{-8}

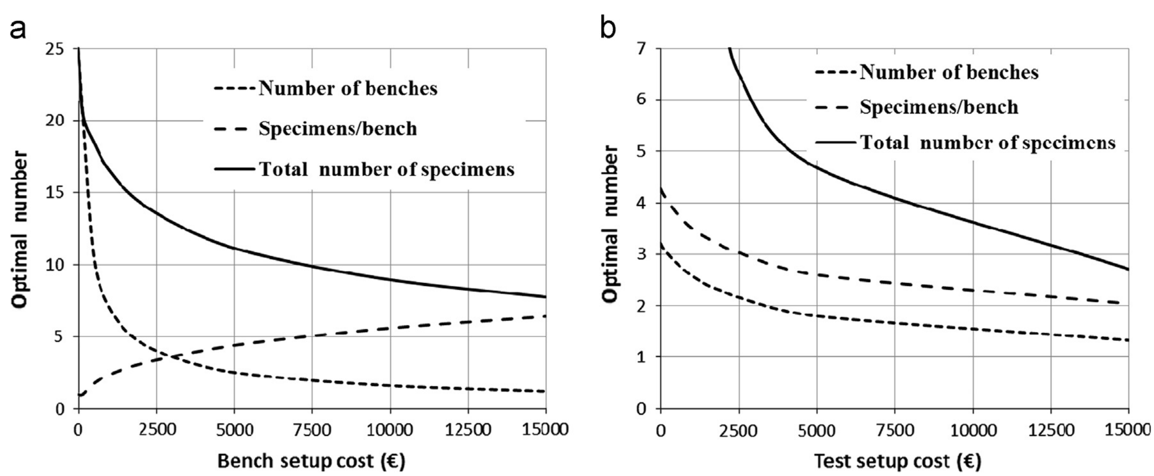

Fig. 7. Influence of bench and test costs on the optimal testing arrangement.

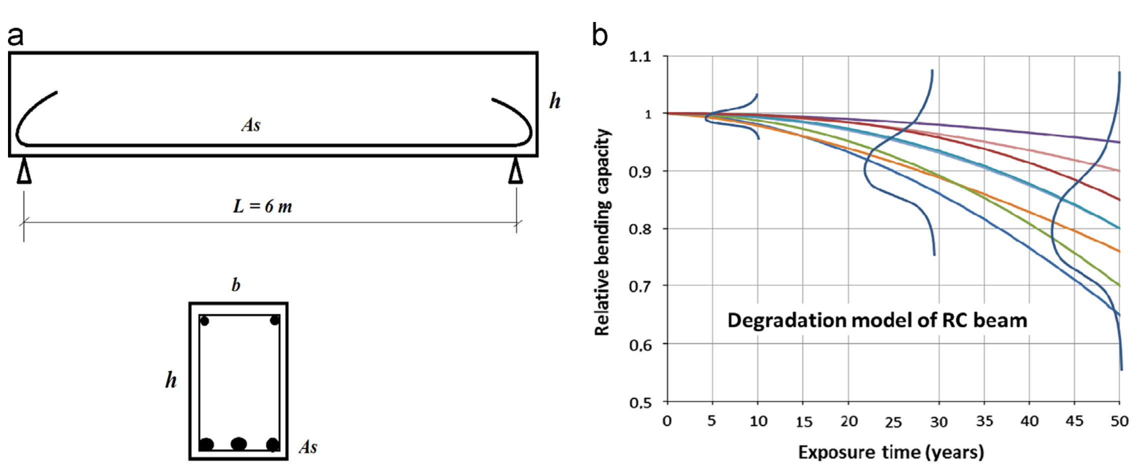

Fig. 8. RC beam layout and time-dependent degradation. (a) RC beam configuration and (b) Degradation with service time.

to a reduction of the number of benches to be used, which is compensated by the increase of the number of tests on each bench; the total number of tested specimens is also reduced as the testing cost becomes higher. When the test setup increases, both bench and test numbers decrease due to the increase of the overall testing cost. It is also to note that the test number affects the applied test stress and the pipe thickness, but the variation is small and remains below 5%. These parameters are much more sensitive to the failure cost, as its increase implies more wall thickness and additional validation tests.

4.3. Precast reinforced concrete beam

The mass production of precast reinforced concrete beams has to be validated for the prescribed design lifespan. Under constant cross-section breadth (Fig. 8), the beam design consists in defining the height h and the steel area A_S to resist the applied ultimate bending moment M_{Ed}. During the service life, the precast beam in the construction has to fit the objective bending moment at 50 years of time-span, noted M_R^{obj}, which is computed by

$$M_R^{obj} = A_{S,50} f_Y \left(h - c - \frac{A_{S,50} f_Y}{2 b f_C} \right) \tag{26}$$

where b and h are respectively the cross-section breadth and height, c is the concrete cover, $A_{S,50}$ is the steel area at 50 years of service life (for illustrative purpose, the initial steel area is reduced by 15% to take into account the corrosion), f_C is the concrete strength and f_Y is the steel yield stress. To validate the product reliability against corrosion, several RC beams have to be tested under accelerated environmental conditions. After an exposure time T_V, the beam is loaded until failure, in order to determine the residual resisting bending moment $M_R(T_V)$. By varying the test environment aggressiveness, the accelerated test can predict the resisting moment at 50 years of service life. However, the accuracy of test predictions depends on two parameters – the number of samples under testing N_{test} and the test duration T_V (which is related to the acceleration process). For this reason, an empirical relationship is proposed herein for the coefficient of variation of the mean bending capacity, as following:

$$v_{\overline{M}_R} = \frac{c_{M_R}}{\sqrt{N_{test}}} \times \frac{1}{2 T_{test}^{\,0.72}} \tag{27}$$

According to this expression, the test uncertainties can be reduced by increasing the number of tested specimens and the test exposure time (i.e. through the decrease of test environment

Table 5
Data for precast RC beam design and testing.

Model parameters	Symbol	Value
Beam breadth (m)	b	0.20
Beam length (m)	L	6
Reinforcement steel length (m)	L_S	7
Steel mass (kg/m^3)	ρ_S	7850
Steel cost per unit weight (€/kg)	C_U	1.8
Concrete cost per unit volume (€/m^3)	C_C	150
Number of beam products	n_P	1500
Bench setup cost (€)	c_B	3000
Test setup cost (€)	c_V	50
Testing delay losses (€/month)	C_L	16,500
Failure cost (€)	C_f	5,000,000
Cost of non-conformity (€)	C_{NC}	800,000
Design variables	**Symbol**	**Interval**
Beam height (mm)	h	$0.2 \le h \le 1.5$
Steel area (mm)	A_S	$2 \le A_S \le 50$
Number of tested beams	n_B	$1 \le n_B \le 20$
Testing exposure time (years)	T_V	$0 \le T_V \le 3$
Random variables	**Symbol**	**Statistical parameters**
Applied bending moment	M_{Ed}	$m_{M_{Ed}} = 4$ MPa, $v_{M_{Ed}} = 0.20$
Yield stress	f_Y	$m_{f_Y} = 500$ MPa, $v_{f_Y} = 0.08$

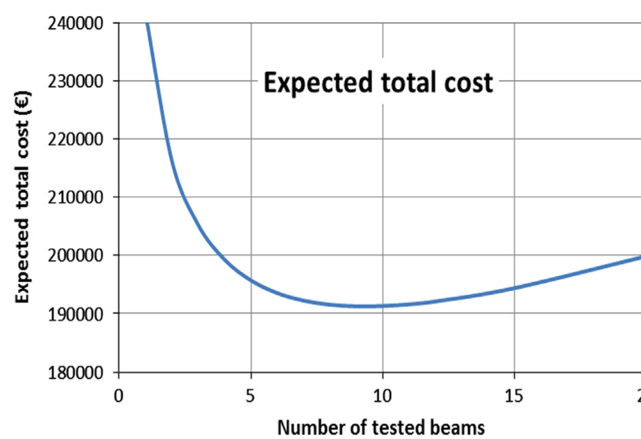

Fig. 10. Design and validation cost.

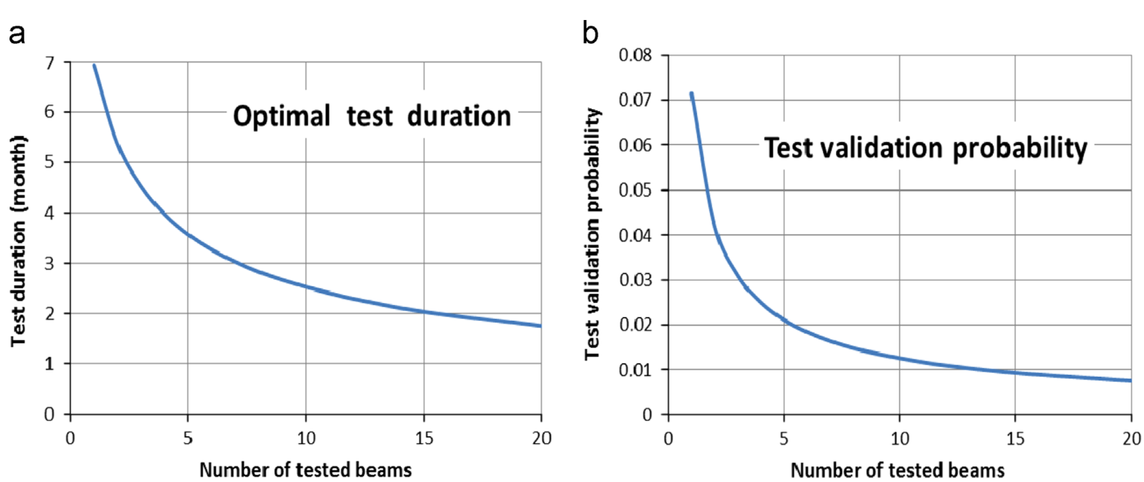

Fig. 9. Design and validation cost.

aggressiveness). The cost model is defined as following:

$$
\begin{aligned}
&C_T(h, A_S, n_B, T_V) = C_D(h, A_S) + C_F(h, A_S, T_V) + C_V(n_B, T_V)\\
&\text{with :}\\
&C_D(h, A_S) = n_p\left(C_C bhL + C_S \rho_S A_S L_S\right)\\
&C_F(h, A_S, T_V) = C_f \Pr\left[M_R^{obj} \le M_{Ed} | M_R \ge M_R^{obj}\right] \Pr\left[M_R \ge M_R^{obj}\right] + C_{NC} \Pr\left[M_R < M_R^{obj}\right]\\
&C_V(n_B, T_V) = n_B(c_B + c_V + C_{D0}) + C_L T_V
\end{aligned}
\tag{28}
$$

where C_C and C_S are respectively the concrete and the steel unit costs, C_f is the failure cost, C_{NC} is the cost of product non-conformity (i.e. destruction, reduction of product rating), c_B and c_V are respectively the bench and the test setup costs, C_{D0} is the cost of each tested beam, C_L are the losses due to testing delay and ρ_S is the steel density. Table 5 provides the deterministic, random and cost parameters for the beam production. In the reliability analysis, the coefficient of variation of the resisting moment is taken as 1.5 times the coefficient of variation of steel yield stress in order to take account for other kinds of uncertainties (i.e. dimensions, concrete cover and strength, etc.).

The graphs in Fig. 9 show the evolution of the optimal test exposure time and the corresponding test validation probability, in terms of the number of tested beams. As can be seen, the increase of the number of tested beams reduces not only the test duration, but also the threshold probability for product acceptance. In both graphs, when the number of tests is greater than 15, the decrease rate becomes slow and quasi-linear. Fig. 10 plots the expected total cost as a function of the number of tests (the other design parameters are set to their optimal values). Strong decrease of the total cost is observed for test numbers less than 5, indicating that this parameter plays a strong role in this range. The optimal solution is found with 9 tests and the optimal design is $h^* = 0.54$ m and $A_S{}^* = 8.63$ cm^2, leading to an optimal expected total cost of 191,252€ for the production, a failure probability of 8.96×10^{-4}, a test duration of 2.67 months and a test validation probability of 1.4×10^{-2}. Compared to the case of only one tested beam ($C_T{}^* = 241,638$€ and $T_V{}^* = 6.94$ months), the above optimal solution leads to a cost reduction of 20.9%, with a reduction of testing duration by 61.5%.

5. Conclusion

Reliability demonstration testing is an important tool for product design and safety. In fact, the most demanding problem in the industrial world is how to achieve the best products in terms of reliability and quality, with the lowest cost. In this sense, the reliability demonstration tests play an important role in life cycle cost of products.

The present paper couples the design and testing problems in order to set the optimal number of product validation tests. The proposed formulation aims at setting both design and testing parameters, by fitting the target reliability with the minimum total cost. The compound uncertainties approach is applied to include the testing uncertainties in the reliability-based design optimization. Compared to classical formulations where design and testing are considered separately, the proposed model meets cost-effective requirements as the coupled optimization problem is considered. The numerical examples show clearly the interest of such a formulation, by balancing design and validation costs. The proposed formulation provides the designers with comprehensive decision-making tools for improved design and validation through optimal testing strategies.

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