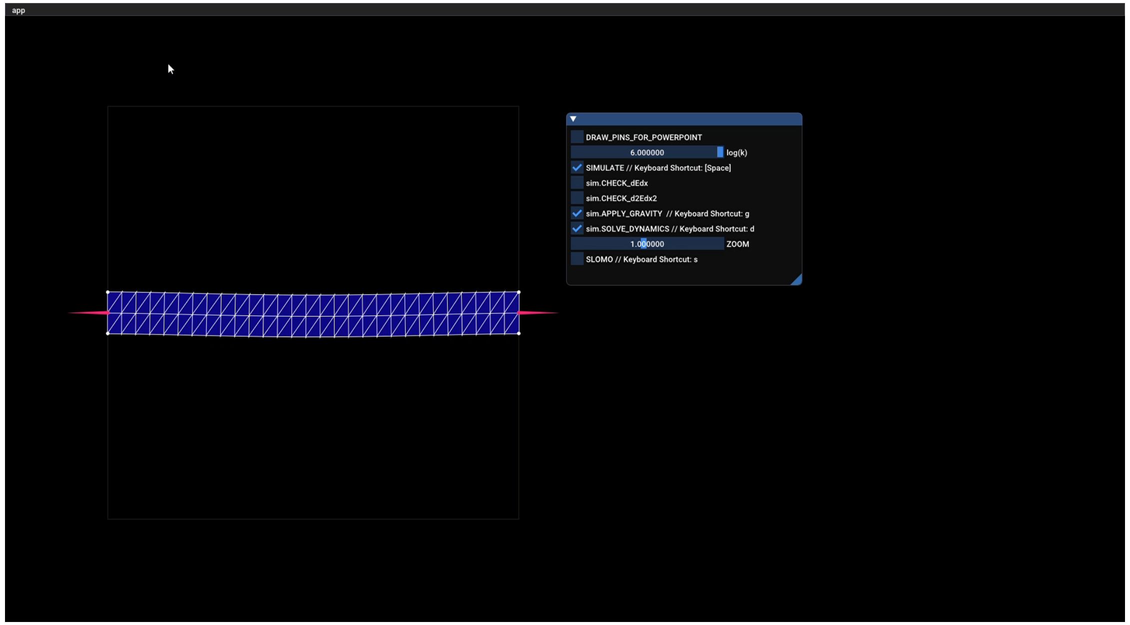
Assignment 4

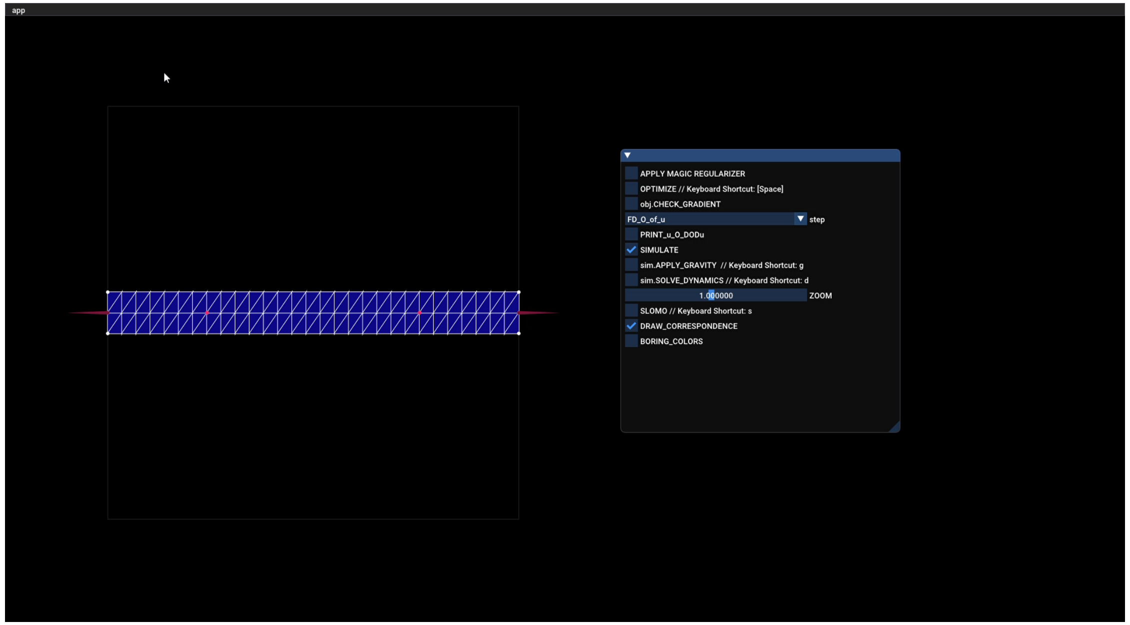
Assignment credits to James Bern

His tutorial http://crl.ethz.ch/teaching/computational-motion-20/videos/tutorial-a3.mp4

Tutorial slides http://crl.ethz.ch/teaching/computational-motion-20/slides/tutorial-a3.pdf

Assignment 4





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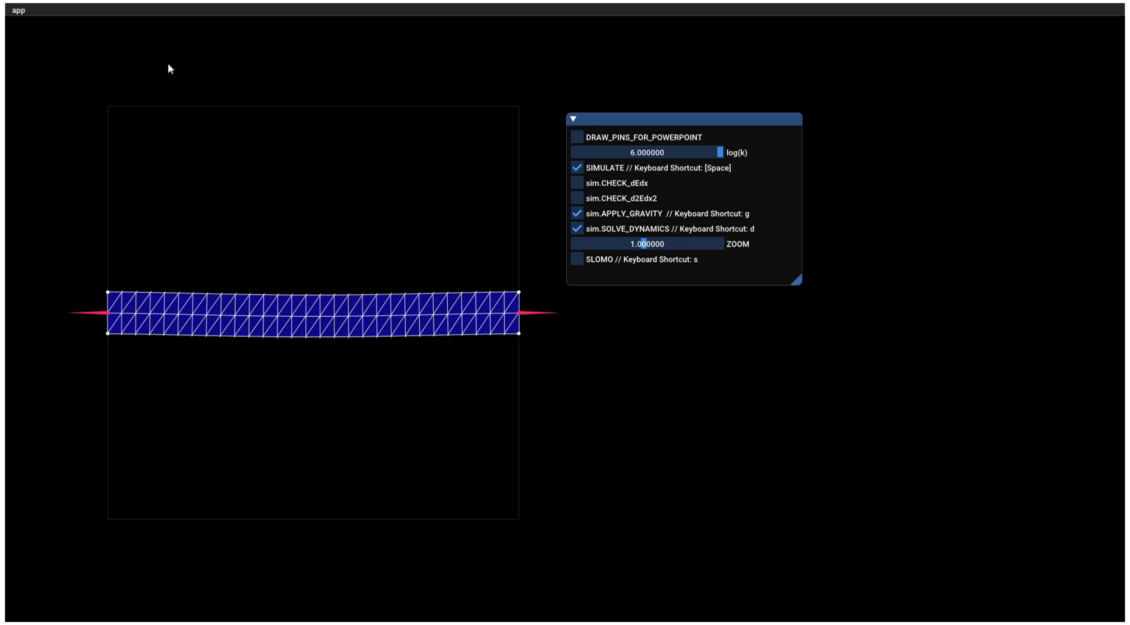
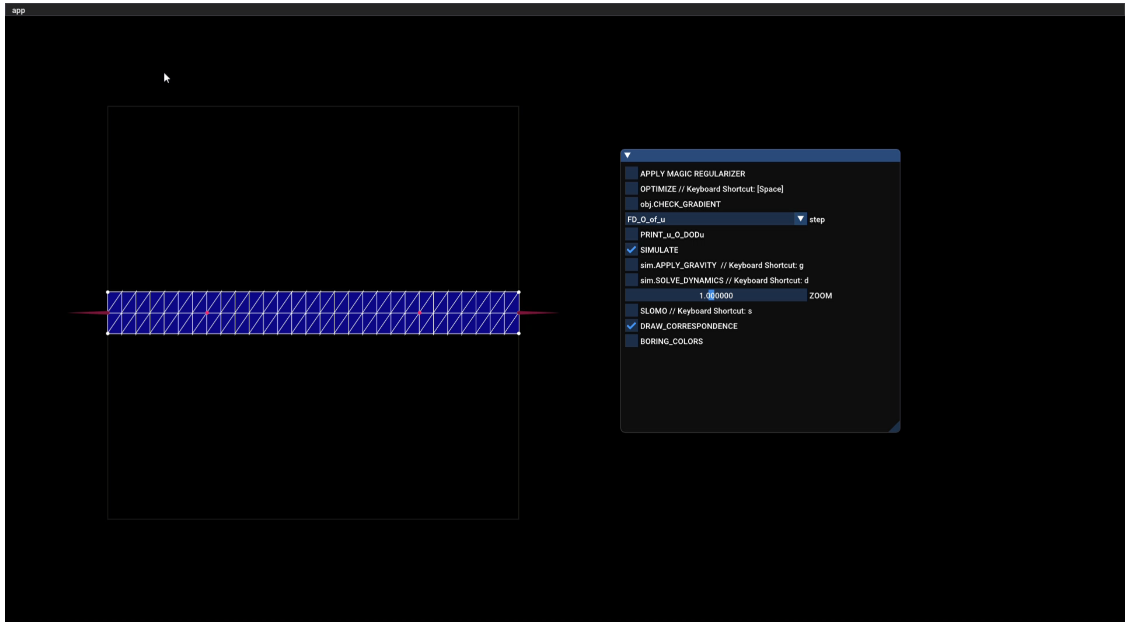
Assignment 4



Forward Problem



Inverse Problem

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Forward Problem – finding dynamic or static force equilibrium

- Dynamic problem $f_{int} + f_{ext} = ma$
- Static problem $f_{int} + f_{ext} = 0$
  - No notion of time or inertia effects

Forward Problem – update rule, residual vector

- Dynamic problem $\mathrm{f}_{int} + \mathrm{f}_{\mathrm{ext}} = ma$
  - $\frac{M(v^{n+1} - \mathrm{v}^n)}{\Delta t} = f_{int}(x^{n+1}) + f_{ext}$
  - $M(v^{n+1} - \mathrm{v}^n) = \Delta t f_{int}(x^{n+1}) + \Delta t f_{ext}$
  - $M(x^{n+1} - x^n - v^n \Delta t) = \Delta t^2 (f_{int}(x^{n+1}) + f_{ext})$
  - $r(x^{n+1}) = M(x^{n+1} - x^n - v^n \Delta t) - \Delta t^2 (f_{int}(x^{n+1}) + f_{ext})$

- Static problem $f_{int} + f_{ext} = 0$
  - $f_{int}(x') + f_{ext} = 0$

Forward Problem – Casting as optimization problems

- Dynamic problem $\mathrm{f}_{int} + \mathrm{f}_{ext} = ma$
  - $\frac{M(v^{n+1} - \mathrm{v}^n)}{\Delta t} = f_{int}(x^{n+1}) + f_{ext}$
  - $M(v^{n+1} - \mathrm{v}^n) = \Delta t f_{int}(x^{n+1}) + \Delta t f_{ext}$
  - $M(x^{n+1} - x^n - v^n \Delta t) = \Delta t^2 (f_{int}(x^{n+1}) + f_{ext})$
  - $r(x^{n+1}) = M(x^{n+1} - x^n - v^n \Delta t) - \Delta t^2 (f_{int}(x^{n+1}) + f_{ext})$
  - $\min_x \frac{1}{2} x^T M x + \mathrm{x}^\mathrm{T} \mathrm{M}(\mathrm{x}^\mathrm{n} - \mathrm{v}^\mathrm{n} \Delta t) + \Delta t^2 (E_{int} - f_{ext} \cdot x)$
- Static problem $f_{int} + f_{ext} = 0$
  - $f_{int}(x') + f_{ext} = 0$
  - $\min_x E_{int} + f_{ext} \cdot x$

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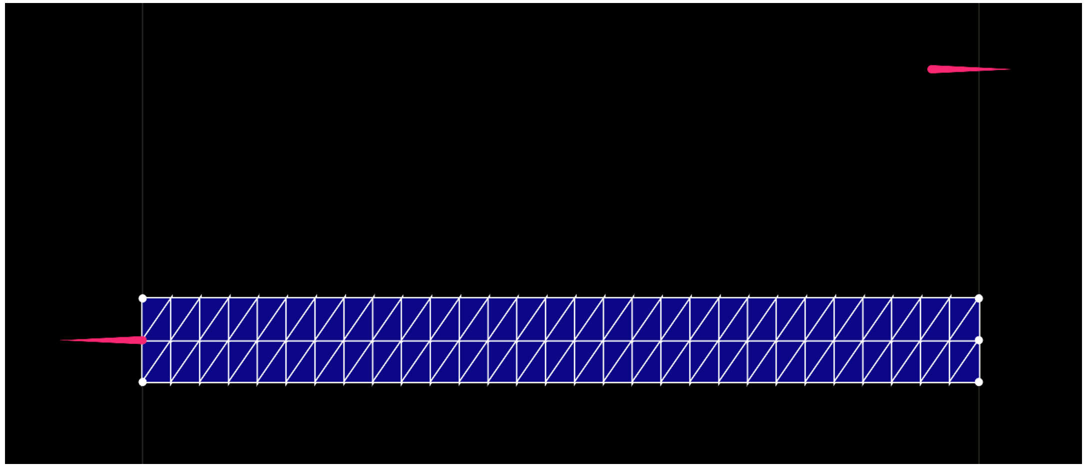
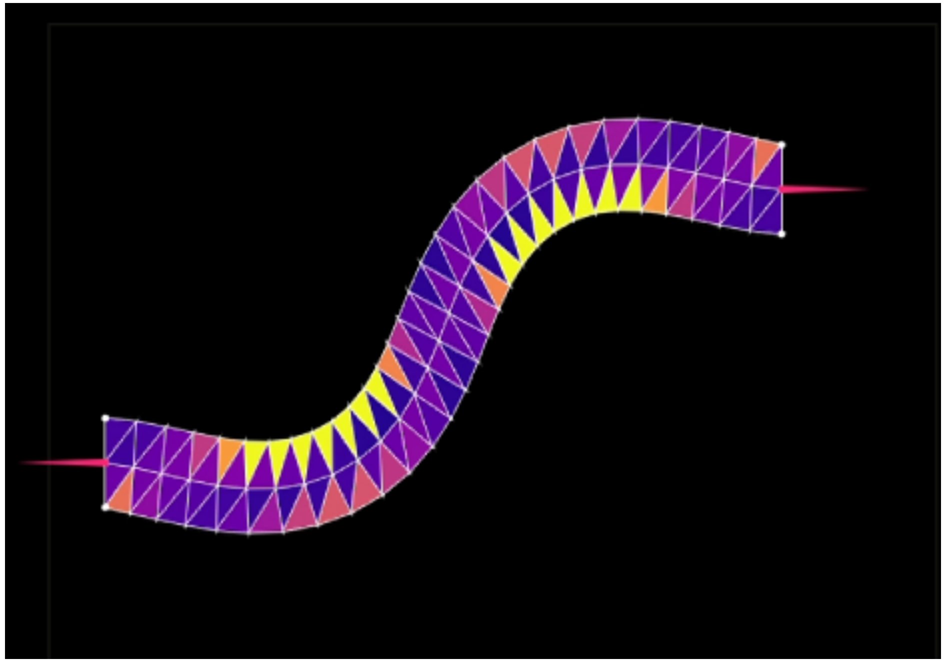
Forward Problem – Our case

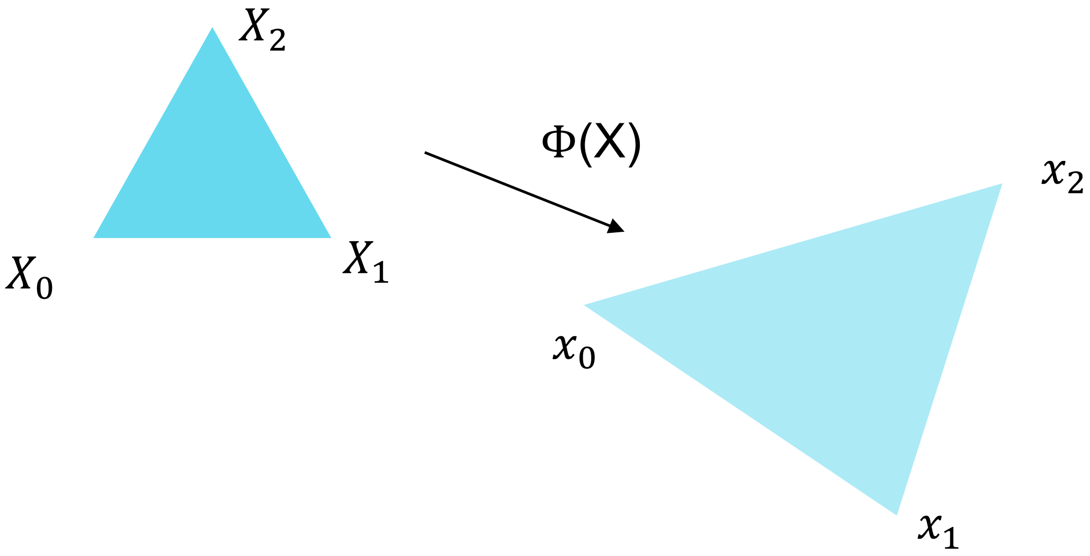
- $\min_{x} E_{elastic} + E_{pins}$
- $E_{pins} = \frac{k_p}{2}||x - p||^2$
  - $f_{pins} = -\frac{\partial E_{pins}}{\partial x}$





FEM Discretization

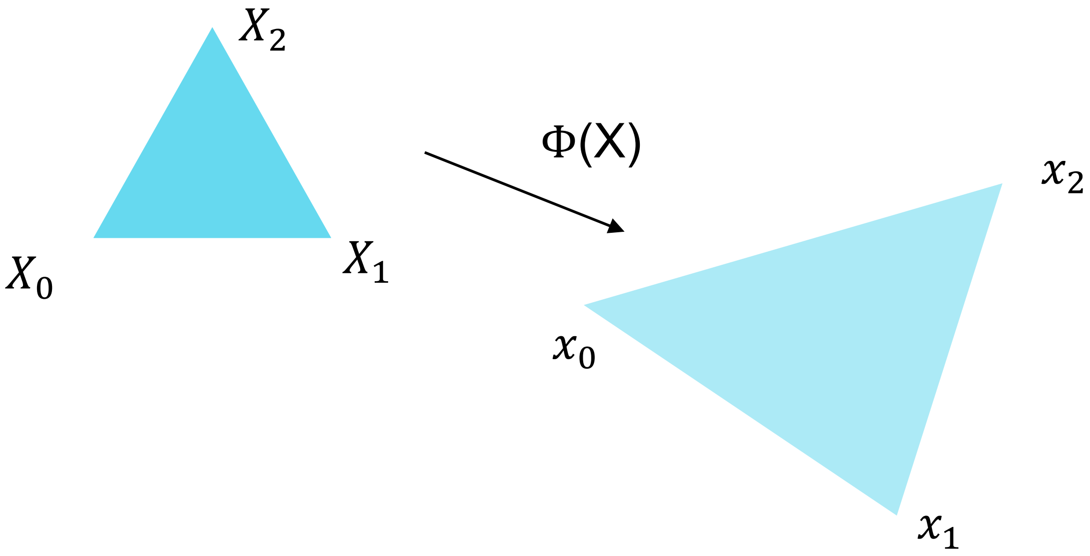
- Linear triangle elements



- $F = \frac{\partial \mathrm{x}}{\partial X}$
- $||\mathrm{F}||_F^{\ 2}$?

FEM Discretization

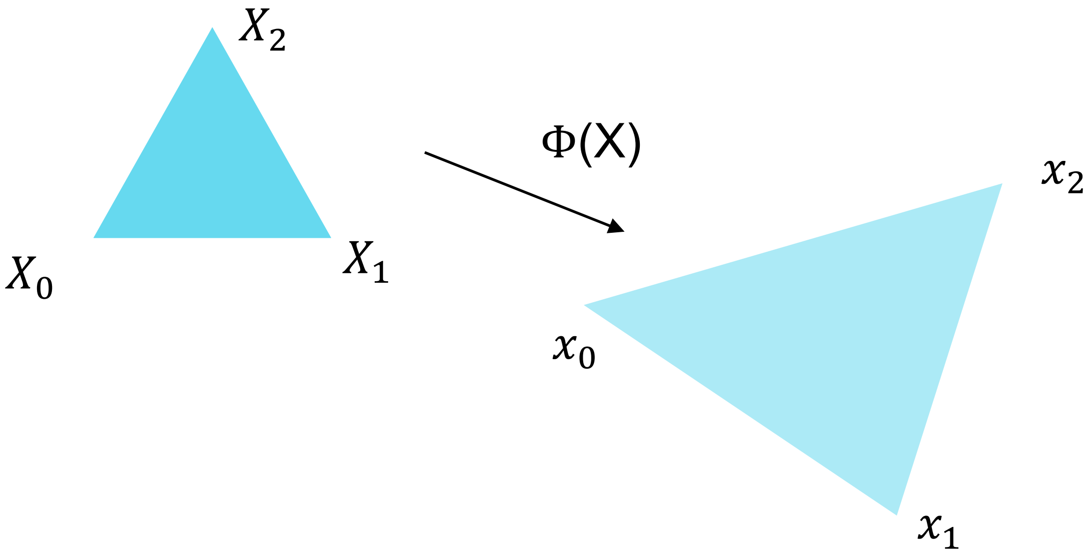
- Linear triangle elements



- $F = \frac{\partial x}{\partial X}$
- $||F||_F{}^2$?
- $||F - I||_F{}^2$?

FEM Discretization

- Linear triangle elements



- $F = \frac{\partial x}{\partial X}$
- $||F||_F^{\ 2}$?
- $||F - I||_F^{\ 2}$?
- $\frac{1}{2}||F^T F - I||_F^{\ 2}$?

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NeoHookean

- $\Psi(\mathrm{F}) = \frac{\mu}{2}\, tr(\boldsymbol{\mathcal{F}}^T\boldsymbol{\mathcal{F}} - \boldsymbol{I}) - \mu \log(J) + \frac{\lambda}{2}(\log(J))^2$

NeoHookean

- $\Psi(\mathrm{F}) = \frac{\mu}{2}\, tr(\boldsymbol{\mathcal{F}}^T\boldsymbol{\mathcal{F}} - \boldsymbol{I}) - \mu\log(J) + \frac{\lambda}{2}(\log(J))^2$
- $J = Det(F)$
- Log barrier to prevent inversion

NeoHookean

- $\Psi(\mathrm{F}) = \frac{\mu}{2}\, tr(\boldsymbol{\mathcal{F}}^T\boldsymbol{\mathcal{F}} - \boldsymbol{I}) - \mu\log(J) \ + \frac{\lambda}{2}(\log(J))^2$
- $J = Det(F)$
- Log barrier to prevent inversion

- $E_{elastic} = \int \Psi(F)d\omega = \sum_{i=0}^{n_{tri}} \Psi(F)$
  - $f_{elastic} = \frac{\partial E_{elastic}}{\partial F}\frac{\partial F}{\partial \mathrm{x}}$

Forward problem - Optimization

$$\min_{x} E_{elastic}(x) + E_{pin}(x, u)$$

Forward problem - Optimization

$$\min_{x} \; O(x) = E_{elastic}(x) + E_{pin}(x, u)$$

$$\frac{dO}{dx} = \frac{\partial E_{elastic}}{\partial x} + \frac{\partial E_{pin}}{\partial x} = 0$$

Forward problem - Optimization

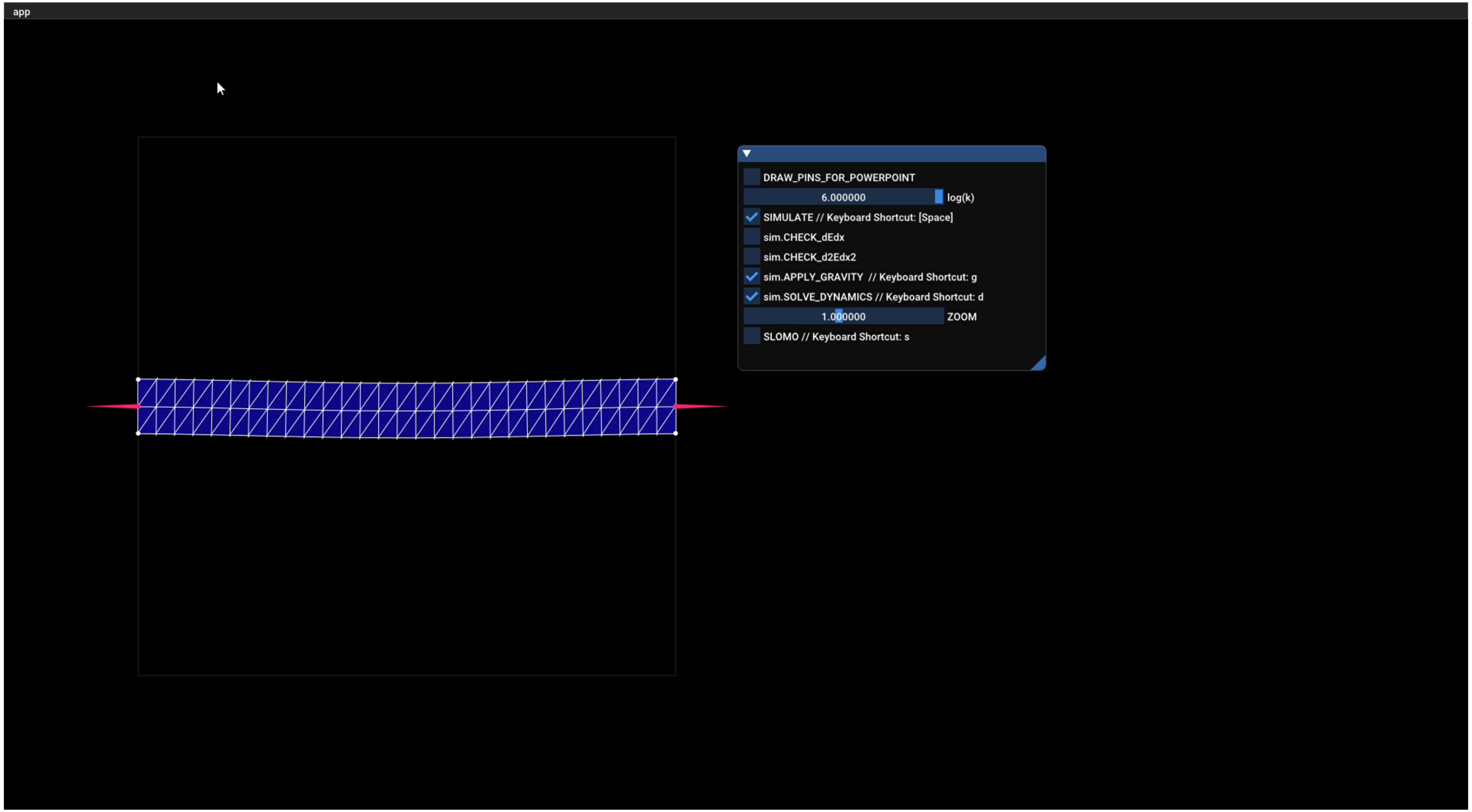
$$\min_{x} \; O(x) = E_{elastic}(x) + E_{pin}(x, u)$$

$$\frac{dO}{dx} = \frac{\partial E_{elastic}}{\partial x} + \frac{\partial E_{pin}}{\partial x} = 0$$

Newton's method

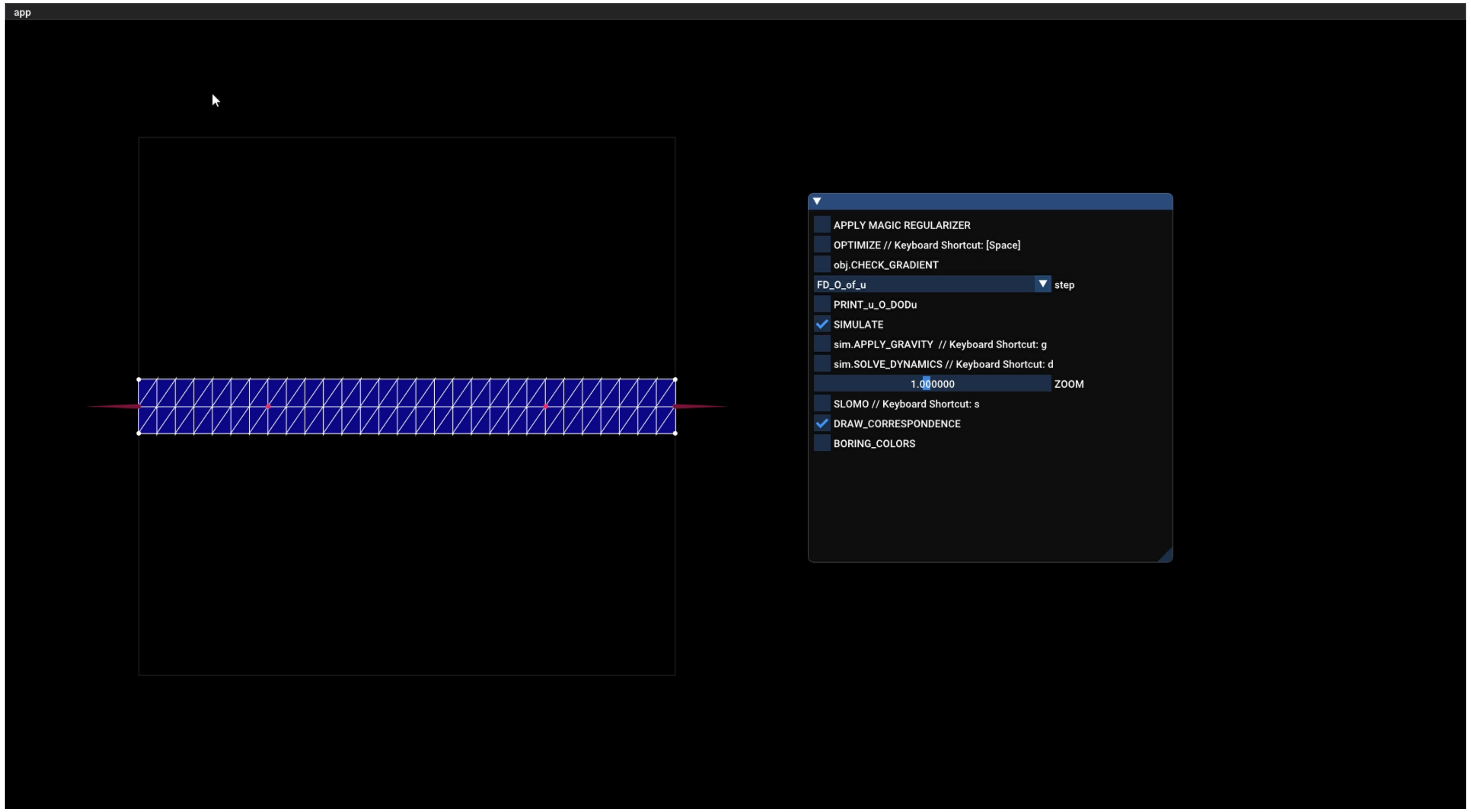
$$\left(\frac{\partial^2 E_{elastic}}{\partial x^2} + \frac{\partial E_{pin}}{\partial x}\right)\Delta x = -\frac{dO}{dx}$$

Demo revisit



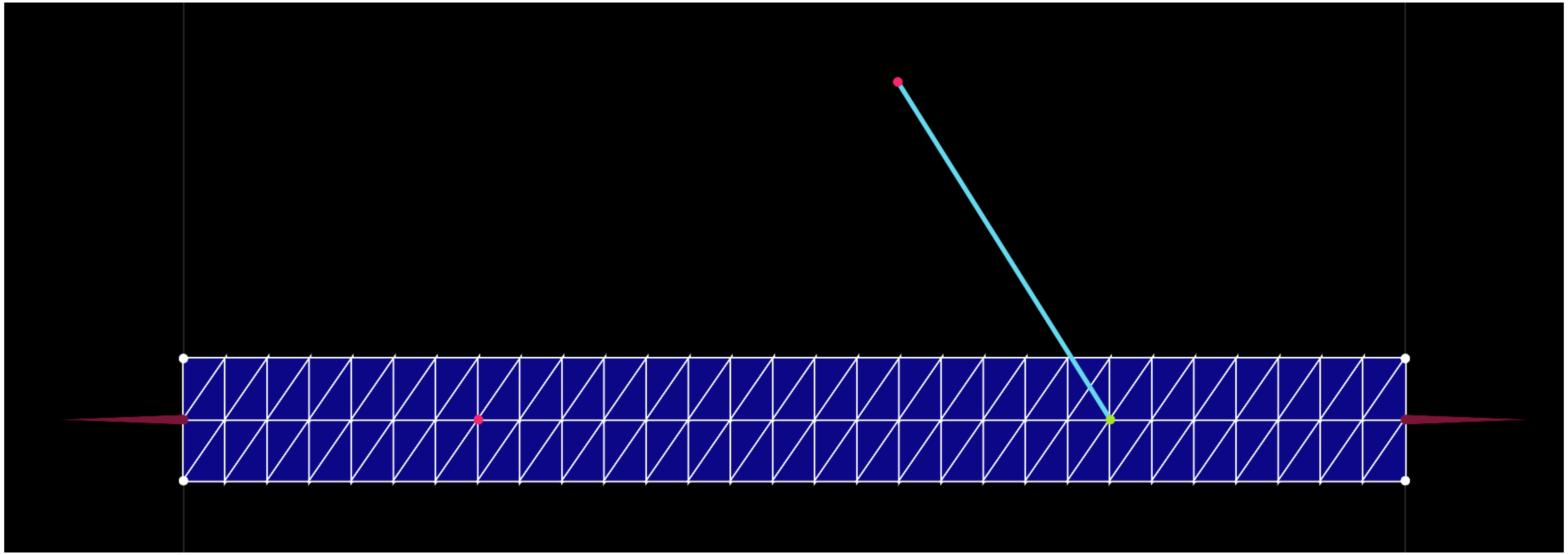
Inverse Problem – finding control/design parameters

- What are the control parameters for the handles to reach target position?



Inverse Problem – Constrained Optimization

$$\min \quad \frac{1}{2}\left|\left|x - x'\right|\right|^2$$



Inverse Problem – Constrained Optimization

$$\min \quad \frac{1}{2}||x - x'||^2$$

What are the design variables?

Inverse Problem – Constrained Optimization / Sensitivity Analysis

$$\min \quad \frac{1}{2}||x - x'||^2$$

What are the design variables? => Control parameters u

$$\min_{u} O(x(u), u) = \frac{1}{2}||x(u) - x'||^2$$
$$s.t. \frac{\partial E_{total}}{\partial x} = 0$$

We are interested in gradient-based methods, hence the requirement for computing

$$\frac{dO}{du}$$

Inverse Problem – Sensitivity Analysis

$$\min_{u} O(x(u), u) = \frac{1}{2} ||x(u) - x'||^2$$

$$\frac{dO}{du} = \frac{\partial O}{\partial u} + \frac{\partial O}{\partial x}\frac{dx}{du}$$

Inverse Problem – Sensitivity Analysis

$$\min_{u} O(x(u), u) = \frac{1}{2}\left|\left|x(u) - x'\right|\right|^2$$

$$\frac{dO}{du} = \frac{\partial O}{\partial u} + \frac{\partial O}{\partial x}\frac{dx}{du}$$

What is $\frac{\partial O}{\partial u}$

Inverse Problem – Sensitivity Analysis

$$\min_{u} O(x(u), u) = \frac{1}{2} \left|\left|x(u) - x'\right|\right|^2$$

$$\frac{dO}{du} = \frac{\partial O}{\partial u} + \frac{\partial O}{\partial x}\frac{dx}{du}$$

What is $\frac{\partial O}{\partial u}$ =>0

What is $\frac{\partial O}{\partial x}$?

Inverse Problem – Sensitivity Analysis

$$\min_{u} O(x(u), u) = \frac{1}{2} ||x(u) - x'||^2$$

$$\frac{dO}{du} = \frac{\partial O}{\partial u} + \frac{\partial O}{\partial x} \frac{\mathrm{d}x}{du}$$

What is $\frac{\partial O}{\partial u}$ =>0

What is $\frac{\partial O}{\partial x}$ =>$x(u) - x'$

Inverse Problem – Sensitivity Analysis

$$\min_{u} O(x(u), u) = \frac{1}{2}\left|\left|x(u) - x'\right|\right|^2$$

$$\frac{dO}{du} = \frac{\partial O}{\partial u} + \frac{\partial O}{\partial x}\frac{dx}{du}$$

What is $\frac{\partial O}{\partial u}$ =>0

What is $\frac{\partial O}{\partial x}$ =>$x(u) - x'$

What is $\frac{dx}{du}$?

Recall that x is a function of u, once u is updated we have to solve the forward problem to find a new x

Inverse Problem – Sensitivity Matrix $\frac{dx}{du}$

- We know that the simulation must at equilibrium, (constraint)

$$\frac{\partial E_{total}}{\partial x} = 0$$

Inverse Problem – Sensitivity Matrix $\frac{\partial x}{\partial u}$

- We know that the simulation must at equilibrium

$$\frac{\partial E}{\partial x} = 0$$

Differentiate both sides w.r.t our design parameters, using chain-rule we have

$$\frac{\partial^2 E}{\partial x \partial u} + \frac{\partial^2 E}{\partial x^2}\frac{dx}{du} = 0$$

Inverse Problem – Sensitivity Matrix $\frac{\partial x}{\partial u}$

- We know that the simulation must at equilibrium

$$\frac{\partial E}{\partial x} = 0$$

Differentiate both sides w.r.t our design parameters, using chain-rule we have

$$\frac{\partial^2 E}{\partial x \partial u} + \frac{\partial^2 E}{\partial x^2}\frac{dx}{du} = 0$$

Re-arrange the equation

$$\frac{dx}{du} = -\left(\frac{\partial^2 E}{\partial x^2}\right)^{-1} \frac{\partial^2 E}{\partial x \partial u}$$

How to compute with code

$$\left(\frac{\partial^2 E}{\partial x^2}\right) \frac{dx}{du} = -\frac{\partial^2 E}{\partial x \partial u}$$

Inverse Problem – Sensitivity Matrix $\frac{\partial x}{\partial u}$

- Putting things together

$$\frac{\partial E}{\partial x} = 0$$

$$\frac{\partial^2 E}{\partial x \partial u} + \frac{\partial^2 E}{\partial x^2} \frac{dx}{du} = 0$$

$$\frac{dx}{du} = -\left(\frac{\partial^2 E}{\partial x^2}\right)^{-1} \frac{\partial^2 E}{\partial x \partial u}$$

Inverse Problem – Sensitivity Matrix $\frac{\partial x}{\partial u}$

- Putting things together

$$\frac{\partial E}{\partial x} = 0$$

$$\frac{\partial^2 E}{\partial x \partial u} + \frac{\partial^2 E}{\partial x^2}\frac{dx}{du} = 0$$

$$\frac{dx}{du} = -\left(\frac{\partial^2 E}{\partial x^2}\right)^{-1} \frac{\partial^2 E}{\partial x \partial u}$$

$$\frac{dO}{du} = \frac{\partial O}{\partial u} + \frac{\partial O}{\partial x}\frac{\partial x}{\partial u}$$

Inverse Problem – Sensitivity Matrix $\frac{\partial x}{\partial u}$

- Putting things together

$$\frac{\partial E}{\partial x} = 0$$

$$\frac{\partial^2 E}{\partial x \partial u} + \frac{\partial^2 E}{\partial x^2}\frac{dx}{du} = 0$$

$$\frac{dx}{du} = -\left(\frac{\partial^2 E}{\partial x^2}\right)^{-1} \frac{\partial^2 E}{\partial x \partial u}$$

$$\frac{dO}{du} = \frac{\partial O}{\partial u} + \frac{\partial O}{\partial x}\frac{\partial x}{\partial u}$$

$$\frac{dO}{du} = \frac{\partial O}{\partial u} - \frac{\partial O}{\partial x}\left(\frac{\partial^2 E}{\partial x^2}\right)^{-1} \frac{\partial^2 E}{\partial x \partial u}$$

Inverse Problem – Sensitivity Matrix $\frac{\partial x}{\partial u}$

- Putting things together

$$\frac{\partial E}{\partial x} = 0$$

$$\frac{\partial^2 E}{\partial x \partial u} + \frac{\partial^2 E}{\partial x^2}\frac{dx}{du} = 0$$

$$\frac{dx}{du} = -\left(\frac{\partial^2 E}{\partial x^2}\right)^{-1} \frac{\partial^2 E}{\partial x \partial u}$$

$$\frac{dO}{du} = \frac{\partial O}{\partial u} - \frac{\partial O}{\partial x}\left(\frac{\partial^2 E}{\partial x^2}\right)^{-1} \frac{\partial^2 E}{\partial x \partial u}$$

- Here we go!

Inverse Problem – Sensitivity Matrix $\frac{\partial x}{\partial u}$

- We know that the simulation must at equilibrium

$$\frac{\partial E}{\partial x} = 0$$

$$\frac{\partial^2 E}{\partial x \partial u} + \frac{\partial^2 E}{\partial x^2}\frac{dx}{du} = 0$$

$$\frac{dx}{du} = -\left(\frac{\partial^2 E}{\partial x^2}\right)^{-1} \frac{\partial^2 E}{\partial x \partial u}$$

$$\frac{dO}{du} = \frac{\partial O}{\partial u} - \frac{\partial O}{\partial x}\left(\frac{\partial^2 E}{\partial x^2}\right)^{-1} \frac{\partial^2 E}{\partial x \partial u}$$

- Well, let break down things a bit further
- What is $\left(\frac{\partial^2 E}{\partial x^2}\right)$? => Elasticity Hessian => You already have in the forward pass

Inverse Problem – Sensitivity Matrix $\frac{\partial x}{\partial u}$

- We know that the simulation must at equilibrium

$$\frac{\partial E}{\partial x} = 0$$

$$\frac{\partial^2 E}{\partial x \partial u} + \frac{\partial^2 E}{\partial x^2}\frac{dx}{du} = 0$$

$$\frac{dx}{du} = -\left(\frac{\partial^2 E}{\partial x^2}\right)^{-1} \frac{\partial^2 E}{\partial x \partial u}$$

$$\frac{dO}{du} = \frac{\partial O}{\partial u} - \frac{\partial O}{\partial x}\left(\frac{\partial^2 E}{\partial x^2}\right)^{-1} \frac{\partial^2 E}{\partial x \partial u}$$

- Well, let break down things a bit further
- What is $\left(\frac{\partial^2 E}{\partial x^2}\right)$? What is $\frac{\partial^2 E}{\partial x \partial u}$? => $-\frac{\partial f}{\partial u}$ => Your job!

Demo Re-visit

Inverse Problem – Adjoint method (equivalent)

- Formulate the Lagrangian

$$L(x(u),\ \lambda,\ u) = O(x(u), u) + \lambda^T \left(\frac{\partial E}{\partial x}\right)$$

We know that $\frac{dO}{du} = \frac{\partial L}{\partial u}$, since the constraint should be 0

Inverse Problem – Adjoint method (equivalent) $\frac{\partial L}{\partial u}$

- Formulate the Lagrangian $L(x(u),\ \lambda, u) = O(x(u), u) + \lambda^T\ (\frac{\partial E}{\partial x})$

Inverse Problem – Adjoint method (equivalent) $\frac{\partial L}{\partial u}$

- Formulate the Lagrangian $L(x(u), \lambda, u) = O(x(u), u) + \lambda^T \left(\frac{\partial E}{\partial x}\right)$

$$\frac{\partial L}{\partial u} = \frac{\partial O}{\partial u} + \frac{\partial O}{\partial x}\frac{dx}{du} + \lambda^T\left(\frac{\partial^2 E}{\partial x^2}\frac{dx}{du} + \frac{\partial^2 E}{\partial x \partial u}\right)$$

Inverse Problem – Adjoint method (equivalent) $\frac{\partial L}{\partial u}$

- Formulate the Lagrangian $L(x(u),\ \lambda, u) = O(x(u), u) + \lambda^T\ (\frac{\partial E}{\partial x})$

$$\frac{\partial L}{\partial u} = \frac{\partial O}{\partial u} + \frac{\partial O}{\partial x}\frac{dx}{du} + \lambda^T(\frac{\partial^2 E}{\partial x^2}\frac{dx}{du} + \frac{\partial^2 E}{\partial x \partial u})$$

$$\frac{\partial L}{\partial u} = \frac{\partial O}{\partial u} + (\lambda^T\frac{\partial^2 E}{\partial x^2} + \frac{\partial O}{\partial x})\frac{dx}{du} + \lambda^T\frac{\partial^2 E}{\partial x \partial u}$$

Inverse Problem – Adjoint method (equivalent) $\frac{\partial L}{\partial u}$

- Formulate the Lagrangian $L(x(u),\ \lambda, u) =\ O(x(u), u) + \lambda^T\ (\frac{\partial E}{\partial x})$

$$\frac{\partial L}{\partial u} = \frac{\partial O}{\partial u} + \frac{\partial O}{\partial x}\frac{dx}{du} + \lambda^T(\frac{\partial^2 E}{\partial x^2}\frac{dx}{du} + \frac{\partial^2 E}{\partial x \partial u})$$

$$\frac{\partial L}{\partial u} = \frac{\partial O}{\partial u} + (\lambda^T \frac{\partial^2 E}{\partial x^2} + \frac{\partial O}{\partial x})\frac{dx}{du} + \lambda^T \frac{\partial^2 E}{\partial x \partial u}$$

- What do we want here?

Inverse Problem – Adjoint method (equivalent) $\frac{\partial L}{\partial u}$

- Formulate the Lagrangian $L(x(u),\ \lambda, u) = \ O(x(u), u) + \lambda^T\ (\frac{\partial E}{\partial x})$

$$\frac{\partial L}{\partial u} = \frac{\partial O}{\partial u} + \ \frac{\partial O}{\partial x}\frac{dx}{du} + \lambda^T(\frac{\partial^2 E}{\partial x^2}\frac{dx}{du} + \frac{\partial^2 E}{\partial x \partial u})$$

$$\frac{\partial L}{\partial u} = \frac{\partial O}{\partial u} + (\lambda^T\frac{\partial^2 E}{\partial x^2} + \frac{\partial O}{\partial x})\frac{dx}{du} + \lambda^T\frac{\partial^2 E}{\partial x \partial u}$$

- Want the $\frac{dx}{du}$ to go away, to do so we find λ such that

$$\lambda^T\frac{\partial^2 E}{\partial x^2} + \frac{\partial O}{\partial x} = 0$$

Inverse Problem – Adjoint method (equivalent) $\frac{\partial L}{\partial u}$

- Formulate the Lagrangian $L(x(u),\ \lambda, u) = \ O(x(u), u) + \lambda^T\ (\frac{\partial E}{\partial x})$

$$\frac{\partial L}{\partial u} = \frac{\partial O}{\partial u} + \ \frac{\partial O}{\partial x}\frac{dx}{du} + \lambda^T(\frac{\partial^2 E}{\partial x^2}\frac{dx}{du} + \frac{\partial^2 E}{\partial x \partial u})$$

$$\frac{\partial L}{\partial u} = \frac{\partial O}{\partial u} + (\lambda^T\frac{\partial^2 E}{\partial x^2} + \frac{\partial O}{\partial x})\frac{dx}{du} + \lambda^T\frac{\partial^2 E}{\partial x \partial u}$$

- Want the $\frac{dx}{du}$ to go away, to do so we find λ such that

$$\lambda^T\frac{\partial^2 E}{\partial x^2} + \frac{\partial O}{\partial x} = 0$$

$$\lambda^{\mathrm{T}} = -\left(\frac{\partial O}{\partial x}\right)\left(\frac{\partial^2 E}{\partial x^2}\right)^{-1}$$

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Inverse Problem – Adjoint method (equivalent) $\frac{\partial L}{\partial u}$

- Formulate the Lagrangian $L(x(u),\ \lambda, u) = O(x(u), u) + \lambda^T\left(\frac{\partial E}{\partial x}\right)$

$$\frac{\partial L}{\partial u} = \frac{\partial O}{\partial u} + \frac{\partial O}{\partial x}\frac{dx}{du} + \lambda^T\left(\frac{\partial^2 E}{\partial x^2}\frac{dx}{du} + \frac{\partial^2 E}{\partial x \partial u}\right)$$

$$\frac{\partial L}{\partial u} = \frac{\partial O}{\partial u} + \left(\lambda^T\frac{\partial^2 E}{\partial x^2} + \frac{\partial O}{\partial x}\right)\frac{dx}{du} + \lambda^T\frac{\partial^2 E}{\partial x \partial u}$$

- Want the $\frac{dx}{du}$ to go away, to do so we find λ such that

$$\lambda^T\frac{\partial^2 E}{\partial x^2} + \frac{\partial O}{\partial x} = 0$$

$$\lambda^{\mathrm{T}} = -\left(\frac{\partial O}{\partial x}\right)\left(\frac{\partial^2 E}{\partial x^2}\right)^{-1}$$

$$\frac{dO}{du} = \frac{\partial L}{\partial u} = \frac{\partial O}{\partial u} - \left(\frac{\partial O}{\partial x}\right)\left(\frac{\partial^2 E}{\partial x^2}\right)^{-1}\frac{\partial^2 E}{\partial x \partial u}$$

Inverse Problem – Adjoint method (equivalent) $\frac{\partial L}{\partial u}$

- Formulate the Lagrangian $L(x(u), \lambda, u) = O(x(u), u) + \lambda^T \left(\frac{\partial E}{\partial x}\right)$

$$\frac{\partial L}{\partial u} = \frac{\partial O}{\partial u} + \frac{\partial O}{\partial x}\frac{dx}{du} + \lambda^T\left(\frac{\partial^2 E}{\partial x^2}\frac{dx}{du} + \frac{\partial^2 E}{\partial x \partial u}\right)$$

$$\frac{\partial L}{\partial u} = \frac{\partial O}{\partial u} + \left(\lambda^T \frac{\partial^2 E}{\partial x^2} + \frac{\partial O}{\partial x}\right)\frac{dx}{du} + \lambda^T \frac{\partial^2 E}{\partial x \partial u}$$

- Want the $\frac{dx}{du}$ to go away, to do so we find λ such that

$$\lambda^T \frac{\partial^2 E}{\partial x^2} + \frac{\partial O}{\partial x} = 0$$

$$\lambda^{\mathrm{T}} = -\left(\frac{\partial O}{\partial x}\right)\left(\frac{\partial^2 E}{\partial x^2}\right)^{-1}$$

$$\frac{dO}{du} = \frac{\partial L}{\partial u} = \frac{\partial O}{\partial u} - \left(\frac{\partial O}{\partial x}\right)\left(\frac{\partial^2 E}{\partial x^2}\right)^{-1}\frac{\partial^2 E}{\partial x \partial u}$$

Same! Yay!

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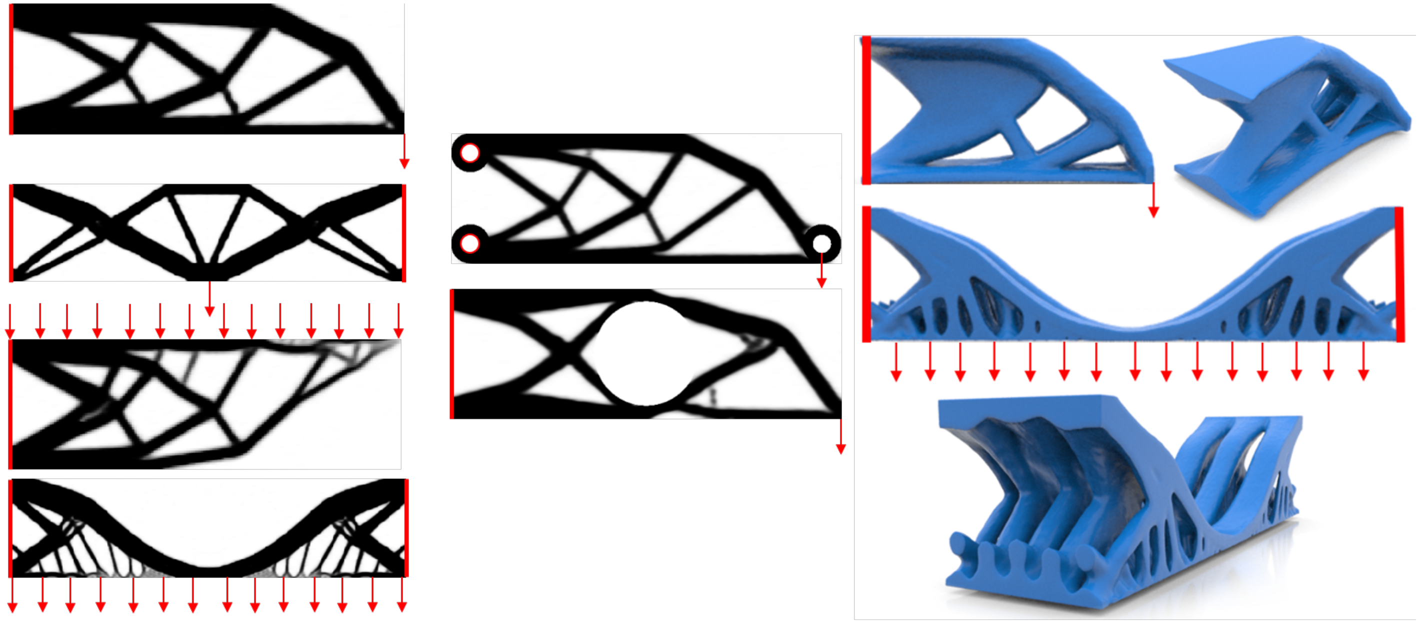
What else can we do with Sensitivity Analysis?

- Topology Optimization!!!!



2D Standard Examples — Curved Boundary Conditions — 3D Designs

Zehnder et al. https://arxiv.org/abs/2102.10782

Demo & Q&A

- Thanks