Recursive Estimation

Lecture 12
Observer-Based Control and the Separation Principle

ETH Zurich

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Learning Objectives

Topic: Observer-Based Control and the Separation Principle

Objectives

- You can explain the *structure of an observer-based controller*.
- You can design an *LTI state observer/estimator* and a *static-gain state-feedback controller* for an LTI system.
- You can derive the *separation principle* for LTI systems.
- You can explain the *separation theorem* for LTI systems and quadratic cost and its implications for controller design.
- You are aware of *potential problems* when combining a state estimator with a state-feedback controller *for general systems*.

Observer-Based Control and the Separation Principle

An important application of state estimation is feedback control: an estimate of the state (typically the mean estimate) is used as input to a state-feedback controller. This scheme is called observer-based control, and it is a common way of designing an output-feedback controller (i.e. a controller that does not have access to perfect state measurements).

Outline

Observer-Based Control and the Separation Principle

A Common Strategy for Control

LTI Observer

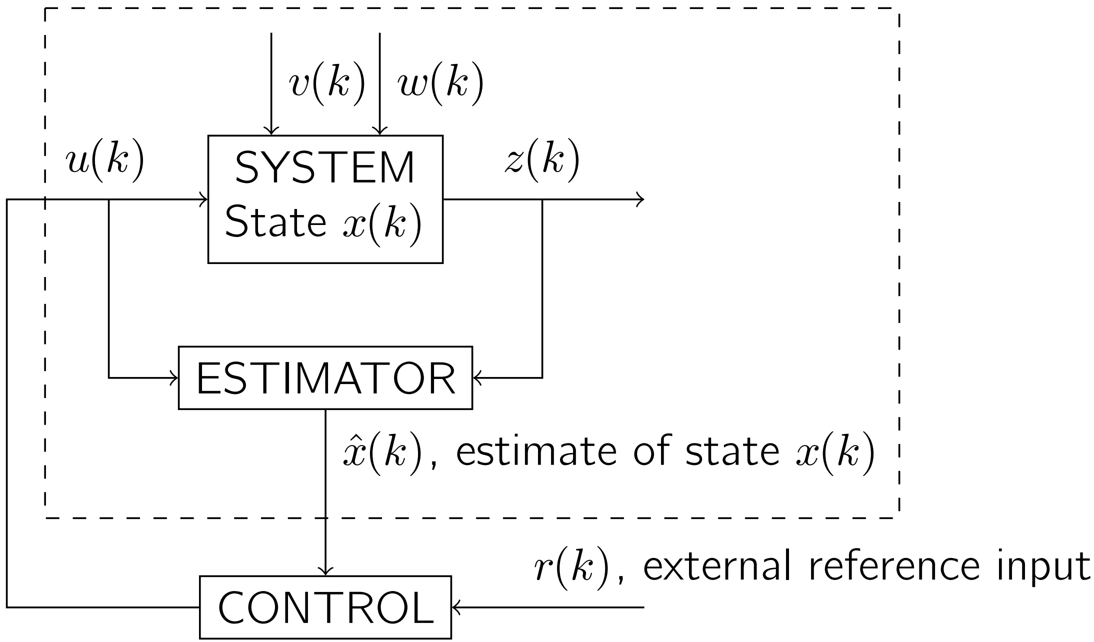
Static State-Feedback Control

Separation Principle

Separation Theorem

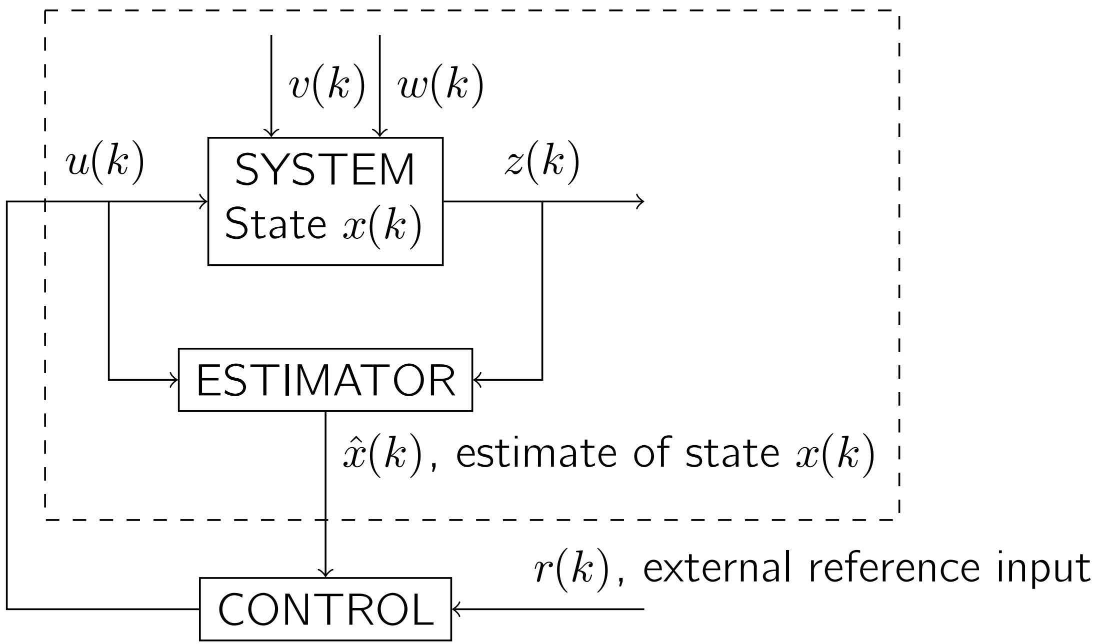
A Common Strategy for Control (1/3)

Many modern methods for control design assume knowledge of the state $x(k)$; that is, these methods can be used to design *state-feedback* controllers. If, however, perfect state measurements are not available (i.e. if $z(k) \neq x(k)$), one can use an estimate $\hat{x}(k)$ of the state instead as the input to the state-feedback controller, as shown in the diagram below.



A Common Strategy for Control (2/3)

In this class, you learned how to design state estimators/observers under the assumption that $u(k)$ is given (the dashed block below). In other classes (for example, *Dynamic Programming and Optimal Control*), you may have learned how to design controllers (i.e. to find the input $u(k)$) under the assumption that the system state is known. The below scheme is the combination of the two designs.



A Common Strategy for Control (3/3)

Important question: *Does it make sense to separate the two steps of state estimator and state-feedback controller design?*

- Often yes, sometimes in a provable way.
- We discuss the case of LTI systems in detail, and we prove that the feedback control system resulting from the combination of a stable LTI observer with a stable static-gain controller is stable. Even more, if both designs are optimal, the combination of the two is optimal as well (in the sense of minimizing a quadratic cost).
- In practice, the separation of the two design problems allows you to manage the design complexity by focusing on the design of a state estimator without caring about the controller, and vice versa.
- In the case of nonlinear systems, there are counterexamples, where a stable state estimator and a stable state-feedback controller together yield an unstable closed-loop system (see examples in the problem set).

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LTI Observer (1/2)

We consider a linear time-invariant (LTI) system,

$$\begin{aligned} x(k) &= A\,x(k-1) + Bu(k-1) + v(k-1) \\ z(k) &= Hx(k) + w(k). \end{aligned}$$

- $v(k-1)$ and $w(k)$ are zero-mean CRVs representing noise.
- We are interested in constructing an estimate $\hat{x}(k)$ of the state $x(k)$.
- In the absence of noise, we require that $\hat{x}(k)$ converges to $x(k)$ as $k \to \infty$.
- For the case with noise, this corresponds to the mean of $\hat{x}(k)$ converging to the mean of $x(k)$ and the variance of $\hat{x}(k)$ being bounded.

LTI Observer (2/2)

We consider an LTI observer of the following form (also called a *Luenberger observer*),

$$\begin{aligned}\hat{x}(k) &= A\,\hat{x}(k-1) + Bu(k-1) + K\left(\bar{z}(k) - \hat{z}(k)\right)\\ \hat{z}(k) &= H\left(A\,\hat{x}(k-1) + Bu(k-1)\right).\end{aligned}$$

- K is a static correction matrix that is to be designed.
- $A\hat{x}(k-1) + Bu(k-1)$ is what we predict the state should be according to the process model and the current state estimate.

Connection to Steady-State Kalman Filter

The LTI observer has the same structure as the steady-state Kalman Filter (KF) derived in previous lectures,

$$\hat{x}(k) = \big(I - K_\infty H\big) A\, \hat{x}(k{-}1) + \big(I - K_\infty H\big) B\, u(k{-}1) + K_\infty\, \bar{z}(k).$$

In fact, the steady-state KF is one way to design the observer gain matrix K (note the difference to the time varying Kalman Filter, where $K(k)$ is time dependent).

Remarks

- We analyze the dynamics of the observer error $e(k) = x(k) - \hat{x}(k)$ in the absence of noise ($v(k{-}1) = 0,\ w(k) = 0$, $x(0)$ deterministic), therefore $\bar{z}(k) = z(k)$. We get

$$\begin{aligned} e(k) &= A\,x(k{-}1) + Bu(k{-}1) - A\,\hat{x}(k{-}1) - Bu(k{-}1) - K\left(z(k) - \hat{z}(k)\right) \\ &= A\,e(k{-}1) - K\left(HA\,x(k{-}1) + HBu(k{-}1) - HA\,\hat{x}(k{-}1) - HBu(k{-}1)\right) \\ &= (I - KH)A\,e(k{-}1). \end{aligned}$$

  That is, $e(k) \to 0$ as $k \to \infty$ for all initial errors $e(0)$ *if and only if* $(I - KH)A$ is stable (i.e. all eigenvalues have magnitude strictly less than one).

- Theorem from Linear Systems Theory:

  There exists such a K if and only if (A, HA) is detectable.

- Furthermore, one can show that: (A, HA) is detectable if and only if (A, H) is detectable. (**PSET 6: P1**)

Summary

If (A, H) is detectable, we can construct a matrix K such that $(I - KH)A$ is stable.

- Pole placement design: you can use the `place()` command in Matlab to find a K that places the eigenvalues of the error dynamics corresponding to the observable modes at desired locations (unobservable modes remain unchanged). (**PSET 6: P2**)

- Steady-state KF design: yields the *optimal* K (in the sense of minimizing the steady-state mean squared error), given the noise statistics of $v(k)$ and $w(k)$. Here we also need the assumption that (A, G) is stabilizable, where $\mathrm{Var}\left[v(k)\right] = GG^T$ – see Lecture 8.

Alternative Formulation

An alternative formulation, which also appears often in the literature, uses the previous measurement in its update equation,

$$\begin{aligned}\hat{x}(k{+}1) &= A\,\hat{x}(k) + Bu(k) + K\left(\bar{z}(k) - \hat{z}(k)\right)\\ \hat{z}(k) &= H\hat{x}(k).\end{aligned}$$

In the deterministic case, the error dynamics for this observer are given by

$$e(k{+}1) = (A - KH)\,e(k).$$

- One can find a K that yields a stable $A - KH$ if and only if (A, H) is detectable.
- This observer essentially involves a delay since the estimate $\hat{x}(k)$ depends on the measurement $\bar{z}(k{-}1)$ rather than $\bar{z}(k)$.

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Static State-Feedback Control

We now consider the design of a controller assuming that we have perfect state information $(z(k) = x(k))$. Furthermore, we assume that we have no process noise and consider the deterministic system

$$\begin{aligned} x(k) &= Ax(k{-}1) + Bu(k{-}1) \\ z(k) &= x(k), \end{aligned}$$

for which we seek to design a linear, static feedback law

$$u(k) = Fx(k) = Fz(k)$$

by choosing the static gain matrix F. The closed-loop dynamics are:

$$x(k) = (A + BF)\, x(k{-}1).$$

Hence, the system is stable if and only if $A + BF$ is stable.

From Linear Systems Theory: Such an F exists $\iff$ (A, B) is *stabilizable*.

Control Design

- Pole placement design: place (controllable) poles at desired closed-loop pole locations.

- *Linear Quadratic Regulator* (LQR) design: find F that minimizes the quadratic cost

$$J_{\mathsf{LQR}} = \sum_{k=0}^{\infty} x^T(k)\bar{Q}\,x(k) + u^T(k)\,\bar{R}u\,(k),$$

where $\bar{Q} = \bar{Q}^T \geq 0$ and $\bar{R} = \bar{R}^T > 0$ are weighting matrices.

LQR and Steady-State Kalman Filter

Interestingly, the optimal F for the LQR design can be found using the same tools that we used to obtain the steady-state KF gain K_∞, namely by solving the *Discrete Algebraic Riccati Equation* (DARE). In particular, under the assumptions that (A, B) is stabilizable and (A, G) is detectable with $\bar{Q} = GG^T$, the optimal stabilizing controller is given by

$$F = -(B^T P B + \bar{R})^{-1} B^T P A,$$

where $P = P^T \geq 0$ is the unique positive semidefinite solution to the DARE

$$P = A^T P A + \bar{Q} - A^T P B \big(B^T P B + \bar{R} \big)^{-1} B^T P A.$$

Notice the substitutions $A \to A^T$, $H \to B^T$, $Q \to \bar{Q}$, and $R \to \bar{R}$ when comparing this equation to the DARE we obtained for the steady-state KF. Because of this relationship, the steady-state KF design and the LQR design are often referred to as *dual* problems.

Static State-Feedback Control

- In the static state-feedback control design shown above, we assumed that we have perfect state information ($z(k) = x(k)$).

- Now, what happens if we do not have perfect state measurements ($z(k) \neq x(k)$), but we use $\hat{x}(k)$ instead of $x(k)$ for feedback? Will the system still be stable? This question is answered by the separation principle.

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Separation Principle (1/2)

We consider the deterministic system (the case with noise is analyzed in (PSET 6: P3))

$$
\begin{aligned}
x(k) &= Ax(k\text{–}1) + Bu(k\text{–}1) \\
z(k) &= Hx(k)
\end{aligned}
$$

with observer and controller given by

$$
\begin{aligned}
\hat{x}(k) &= A\hat{x}(k\text{–}1) + Bu(k\text{–}1) + K\left(z(k) - \hat{z}(k)\right) \\
\hat{z}(k) &= H\left(A\hat{x}(k\text{–}1) + Bu(k\text{–}1)\right) \\
u(k) &= F\hat{x}(k).
\end{aligned}
$$

We assume that $(I - KH)A$ and $(A + BF)$ are stable; that is, both the estimator dynamics and the state-feedback dynamics are stable.

Question: is the overall system stable?

Separation Principle (2/2)

We derive the closed-loop dynamics. As before, let $e(k) = x(k) - \hat{x}(k)$. When deriving the estimation error dynamics, we did not make *any* assumptions on $u(k\text{–}1)$, so the derivation certainly holds for $u(k\text{–}1) = F\hat{x}(k\text{–}1)$. Consequently,

$$e(k) = (I - KH)Ae(k\text{–}1).$$

For the state equation, we get

$$\begin{aligned} x(k) &= Ax(k\text{–}1) + BF\hat{x}(k\text{–}1) \\ &= Ax(k\text{–}1) + BF\left(\hat{x}(k\text{–}1) + x(k\text{–}1) - x(k\text{–}1)\right) \\ &= (A + BF)x(k\text{–}1) - BFe(k\text{–}1), \end{aligned}$$

which, together with the above, yields the closed-loop dynamics

$$\begin{bmatrix} x(k) \\ e(k) \end{bmatrix} = \begin{bmatrix} A + BF & -BF \\ 0 & (I - KH)A \end{bmatrix} \begin{bmatrix} x(k\text{–}1) \\ e(k\text{–}1) \end{bmatrix}.$$

Remarks

- The eigenvalues of the closed-loop dynamics are given by the eigenvalues of $(I - KH)A$ and $(A + BF)$. Therefore, the overall system is stable. This is called the *separation principle* for LTI systems.

- Including noise in the analysis does not affect stability, there will simply be $v(k–1)$ and $w(k)$ terms driving the system. (**PSET 6: P3**)

- Under some mild assumptions, the above analysis generalizes to the time-varying case.

- In general, the separation principle does *not* hold for nonlinear systems. (**PSET 6: P5**)

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Separation Theorem (1/2)

It turns out that we can say something even stronger for LTI systems. Consider the case with noise:

$$
\begin{aligned}
x(k) &= Ax(k-1) + Bu(k-1) + v(k-1) \qquad && v(k-1) \sim \mathcal{N}(0, Q) \\
z(k) &= Hx(k) + w(k) && w(k) \sim \mathcal{N}(0, R).
\end{aligned}
$$

The control objective is to find the control policy that minimizes

$$
J_{\mathsf{LQG}} = \lim_{N \to \infty} \mathrm{E}\left[\frac{1}{N} \sum_{k=0}^{N-1} \left(x^T(k) \bar{Q}\, x(k) + u^T(k) \bar{R}\, u(k) \right) \right]
$$

where $u(k)$ can depend on current and past measurements $z(1{:}k)$ (causal strategy).

Separation Theorem (2/2)

The *optimal* strategy to solve the control problem given above is:

1. Design a steady-state KF. (The filter *does not* depend on $\bar{Q}$ and $\bar{R}$.) The filter provides an estimate $\hat{x}(k)$ of $x(k)$.

2. Design an optimal state-feedback strategy $u(k) = Fx(k)$ for the deterministic LQR problem

$$x(k) = Ax(k-1) + Bu(k-1)$$

   that minimizes

$$J_{\mathsf{LQR}} = \sum_{k=0}^{\infty} x^T(k)\,\bar{Q}\,x(k) + u^T(k)\,\bar{R}\,u(k).$$

   (The feedback gain F *does not* depend on the noise statistics Q and R.)

3. Put both together.

Remarks

- This control design is called *Linear Quadratic Gaussian* (LQG) control.

- The fact that the combination of the optimal state estimator (steady-state KF) and the optimal state-feedback controller (LQR) is the globally optimal control strategy is often referred to as the *separation theorem* for LTI systems and quadratic cost.

- The *separation principle* described in the previous section denotes the fact that the combination of a stable state estimator with a stable state-feedback controller yields a *stable* closed-loop system. The *separation theorem* refers to the *optimality* of this scheme.