Recursive Estimation

Lecture 9
The Extended Kalman Filter (EKF)

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Learning Objectives

Topic: The Extended Kalman Filter (EKF)

Objectives

- You can derive the *discrete-time Extended Kalman Filter* (EKF) for a nonlinear discrete-time system.
- You can explain the derivation of the *hybrid EKF* for a nonlinear system with continuous-time process model and discrete-time measurement model.
- You can explain the *underlying assumptions and approximations* in the derivation of the EKF and their implications on the EKF results.
- You can assess for *what types of problems* an EKF is (likely) a suitable solution.
- You can *design and implement* an EKF.

The Extended Kalman Filter

We discuss the *Extended Kalman Filter* (EKF) as an extension of the KF to nonlinear systems. The EKF is derived by linearizing the nonlinear system equations about the latest state estimate. First, we introduce the *discrete-time EKF* as an estimator for discrete-time nonlinear process and measurement equations. We then discuss the case where the process is governed by nonlinear continuous-time dynamics, which yields the *hybrid EKF*.

Outline

The Extended Kalman Filter

Discrete-Time EKF

Hybrid EKF

Discrete-Time EKF

We consider the nonlinear discrete-time system for $k = 1, 2, \ldots$:

$$
\begin{aligned}
x(k) &= q_{k-1}\big(x(k{-}1), v(k{-}1)\big) && \mathrm{E}\left[x(0)\right] = x_0, \mathrm{Var}\left[x(0)\right] = P_0 \\
& && \mathrm{E}\left[v(k{-}1)\right] = 0, \mathrm{Var}\left[v(k{-}1)\right] = Q(k{-}1) \\
z(k) &= h_k\big(x(k), w(k)\big) && \mathrm{E}\left[w(k)\right] = 0, \mathrm{Var}\left[w(k)\right] = R(k)
\end{aligned}
$$

Assumptions:

- $x(0)$, $\{v(\cdot)\}$, and $\{w(\cdot)\}$ are mutually independent
- q_{k-1} is continuously differentiable with respect to $x(k{-}1)$ and $v(k{-}1)$
- h_k is continuously differentiable with respect to $x(k)$ and $w(k)$

We introduce the EKF for a system without input $u(k-1)$, but the discussion directly extends to a system with input (the known input $u(k-1)$ can be absorbed in the explicit time dependency of $q_{k-1}(\cdot)$).

Derivation

The key idea in the derivation of the EKF is simple: in order to obtain a state estimate for the nonlinear system above, we linearize the system equations about the current state estimate, and we then apply the (standard) KF prior and measurement update equations to the linearized equations.

Process Update (1/2)

Assume we have computed $\hat{x}_{\mathrm{m}}(k-1)$ and $P_{\mathrm{m}}(k-1)$ as (approximations of) the conditional mean and variance of the state $x(k-1)$ given the measurements $z(1{:}k-1)$. Linearizing $x(k) = q_{k-1}\big(x(k-1), v(k-1)\big)$ about $\hat{x}_{\mathrm{m}}(k-1)$ and $\mathrm{E}\left[v(k-1)\right] = 0$ yields

$$
\begin{aligned}
x(k) &\approx q_{k-1}(\hat{x}_{\mathrm{m}}(k-1), 0) \\
&\quad + \underbrace{\frac{\partial q_{k-1}(\hat{x}_{\mathrm{m}}(k-1), 0)}{\partial x}}_{=:A(k-1)} \cdot (x(k-1) - \hat{x}_{\mathrm{m}}(k-1)) + \underbrace{\frac{\partial q_{k-1}(\hat{x}_{\mathrm{m}}(k-1), 0)}{\partial v}}_{=:L(k-1)} \cdot v(k-1) \\
&= A(k-1)\, x(k-1) + \underbrace{L(k-1)\, v(k-1)}_{=:\tilde{v}(k-1)} + \underbrace{q_{k-1}(\hat{x}_{\mathrm{m}}(k-1), 0) - A(k-1)\, \hat{x}_{\mathrm{m}}(k-1)}_{=:\xi(k-1)} \\
&= \boxed{A(k-1)}\, x(k-1) + \tilde{v}(k-1) + \xi(k-1),
\end{aligned}
$$

where $\xi(k-1)$ is treated as a known input, and the process noise $\tilde{v}(k-1)$ has zero-mean and variance $\mathrm{Var}\left[\tilde{v}(k-1)\right] = L(k-1)\, Q(k-1)\, L^T(k-1)$.

Process Update (2/2)

We can now apply the KF prior update equations to the linearized process equation:

$$\begin{aligned}\hat{x}_{\mathrm{p}}(k) &= A(k-1)\hat{x}_{\mathrm{m}}(k-1) + \xi(k-1) \\ &= q_{k-1}(\hat{x}_{\mathrm{m}}(k-1), 0) \qquad \text{(by substituting } \xi(k-1)\text{)} \\ P_{\mathrm{p}}(k) &= \boxed{A(k-1)}P_{\mathrm{m}}(k-1)\boxed{A^T(k-1)} + L(k-1)Q(k-1)L^T(k-1).\end{aligned}$$

Intuition: predict the mean state estimate forward using the nonlinear process model and update the variance according to the linearized equations.

Measurement Update (1/2)

We linearize $z(k) = h_k\big(x(k), w(k)\big)$ about $\hat{x}_\mathrm{p}(k)$ and $\mathrm{E}\left[w(k)\right] = 0$:

$$
\begin{aligned}
z(k) &\approx h_k(\hat{x}_\mathrm{p}(k), 0) + \underbrace{\frac{\partial h_k(\hat{x}_\mathrm{p}(k), 0)}{\partial x}}_{=:H(k)} \cdot (x(k) - \hat{x}_\mathrm{p}(k)) + \underbrace{\frac{\partial h_k(\hat{x}_\mathrm{p}(k), 0)}{\partial w}}_{=:M(k)} \cdot w(k) \\
&= H(k)x(k) + \underbrace{M(k)w(k)}_{=:\tilde{w}(k)} + \underbrace{h_k(\hat{x}_\mathrm{p}(k), 0) - H(k)\hat{x}_\mathrm{p}(k)}_{=:\zeta(k)} \\
&= H(k)x(k) + \tilde{w}(k) + \zeta(k),
\end{aligned}
$$

where $\tilde{w}(k)$ has zero mean and variance $\mathrm{Var}\left[\tilde{w}(k)\right] = M(k)R(k)M^T(k)$.

Compared to the measurement equation that we used in the derivation of the KF, there is the additional term $\zeta(k)$, which is known. It is straightforward to extend the KF measurement update to this case (for example, by introducing the auxiliary measurement $z(k) - \zeta(k)$).

Measurement Update (2/2)

Applying the KF measurement update to the linearized measurement equation yields:

$$\begin{aligned}
K(k) &= \underline{P_\mathrm{p}(k) H^T(k) \big(H(k) P_\mathrm{p}(k) H^T(k) + M(k) R(k) M^T(k)\big)}^{-1} \\
\hat{x}_\mathrm{m}(k) &= \hat{x}_\mathrm{p}(k) + K(k) \left(\bar{z}(k) - H(k)\hat{x}_\mathrm{p}(k) - \zeta(k)\right) \\
&= \hat{x}_\mathrm{p}(k) + K(k) \left(\bar{z}(k) - h_k(\hat{x}_\mathrm{p}(k), 0)\right) \qquad \text{(by substituting } \zeta(k)\text{)} \\
P_\mathrm{m}(k) &= \big(I - K(k) H(k)\big) P_\mathrm{p}(k).
\end{aligned}$$

Intuition: correct for the mismatch between the actual measurement $\bar{z}(k)$ and its nonlinear prediction $h_k(\hat{x}_\mathrm{p}(k), 0)$, and correct the variance according to the linearized equations.

Summary of the Discrete-Time EKF Equations

Initialization: $\hat{x}_\mathrm{m}(0) = x_0$, $P_\mathrm{m}(0) = P_0$.

Step 1 (S1): Prior update/Prediction step

$$\hat{x}_\mathrm{p}(k) = q_{k-1}(\hat{x}_\mathrm{m}(k{-}1), 0)$$

$$P_\mathrm{p}(k) = A(k{-}1)P_\mathrm{m}(k{-}1)A^T(k{-}1) + L(k{-}1)Q(k{-}1)L^T(k{-}1)$$

where

$$A(k{-}1) := \frac{\partial q_{k-1}(\hat{x}_\mathrm{m}(k{-}1), 0)}{\partial x} \quad \text{and} \quad L(k{-}1) := \frac{\partial q_{k-1}(\hat{x}_\mathrm{m}(k{-}1), 0)}{\partial v}$$

Step 2 (S2): A posteriori update/Measurement update step

$$K(k) = P_\mathrm{p}(k)H^T(k)\left(H(k)P_\mathrm{p}(k)H^T(k) + M(k)R(k)M^T(k)\right)^{-1}$$

$$\hat{x}_\mathrm{m}(k) = \hat{x}_\mathrm{p}(k) + K(k)\left(\bar{z}(k) - h_k(\hat{x}_\mathrm{p}(k), 0)\right)$$

$$P_\mathrm{m}(k) = \left(I - K(k)H(k)\right)P_\mathrm{p}(k)$$

where

$$H(k) := \frac{\partial h_k(\hat{x}_\mathrm{p}(k), 0)}{\partial x} \quad \text{and} \quad M(k) := \frac{\partial h_k(\hat{x}_\mathrm{p}(k), 0)}{\partial w}$$

Remarks (1/3)

- The process and measurement noise are assumed to be zero-mean.

- The matrices $A(k-1)$, $L(k-1)$, $H(k)$, and $M(k)$ are obtained from linearizing the system equations about the current state estimate (which depends on real-time measurements). Hence, the EKF gains cannot be computed off-line, even if the model and noise distributions are known for all k.

- The EKF variables $\hat{x}_\mathrm{p}(k)$, $\hat{x}_\mathrm{m}(k)$, $P_\mathrm{p}(k)$, and $P_\mathrm{m}(k)$ are only approximations of mean and variance. For example, in the prior update, $\hat{x}_\mathrm{p}(k)$ would only accurately capture the mean update if the expected value operator $\mathrm{E}\left[\cdot\right]$ and q_{k-1} commuted; that is, if

$$\mathrm{E}\left[q_{k-1}\big(x(k-1), v(k-1)\big)\right] = q_{k-1}\big(\mathrm{E}\left[x(k-1)\right], \mathrm{E}\left[v(k-1)\right]\big).$$

  This is not true for a general nonlinear function q_{k-1}, and may be a really bad assumption in the case of strong nonlinearities (it holds, however, for linear q_{k-1}).

Remarks (2/3)

- Even though the EKF variables do not capture the true conditional mean and variance, they are often still referred to as the prior/posterior mean and variance.

- If the actual state and noise values are close to the values that we linearize about (i.e. if $x(k{-}1) - \hat{x}_{\mathrm{m}}(k{-}1)$, $v(k{-}1)$, $x(k) - \hat{x}_{\mathrm{p}}(k)$, and $w(k)$ are all close to zero), then the linearization is a good approximation of the actual nonlinear dynamics. This assumption may, however, be bad. In the case of Gaussian noise, for example, the above quantities are not guaranteed to be small since the noise is actually unbounded.

Remarks (3/3)

- Despite the fact that the EKF is a (possibly crude) approximation of the Bayesian state estimator and general convergence guarantees cannot be given, the EKF has proven to work well in many practical applications. As a rule of thumb, an EKF often works well for (mildly) nonlinear systems with unimodal distributions.

- Solving the Bayesian state estimation problem for a general nonlinear system is often computationally intractable. Hence, the EKF may be seen as a computationally tractable approximation (trade-off: tractability vs. accuracy).

Outline

The Extended Kalman Filter

Discrete-Time EKF

Hybrid EKF

Hybrid EKF

In practice, one often encounters problems where the process is governed by continuous-time dynamics, while the measurements are taken at discrete time instants. Hence, the system is described by

$$\boxed{\begin{aligned} \dot{x}(t) &= q\big(x(t), v(t), t\big) \\ z[k] &= h_k\big(x[k], w[k]\big) \qquad\qquad \mathrm{E}\left[w[k]\right] = 0, \mathrm{Var}\left[w[k]\right] = R. \end{aligned}}$$

- We consider constant measurement noise variance R for simplicity.
- We use parentheses $(\cdot)$ to denote continuous-time signals and square brackets $[\cdot]$ to denote discrete-time signals; that is,

$$x[k] := x(kT), \qquad T = \text{constant sampling time.}$$

One way to deal with this problem is to discretize the process dynamics and then implement the discrete-time EKF. Here we present a different solution, where we work directly with the continuous-time dynamics.

Continuous-Time Process Noise

For discrete-time noise $v_\mathrm{d}[k]$ that is zero mean and independent in time, it holds that

$$\mathrm{E}\left[v_\mathrm{d}[k]\right] = 0 \quad \text{and} \quad \mathrm{E}\left[v_\mathrm{d}[k] v_\mathrm{d}^T[k+n]\right] = Q\delta_\mathrm{d}[n], \tag{1}$$

$$E[v_d(k) v_d^T(k+n)] = \begin{cases} Q & n=0 \\ 0 & n \neq 0 \end{cases}$$

where n is an integer.

The function $\delta_\mathrm{d}[n]$ takes the value 1 for $n = 0$ and zero otherwise, that is $\delta_\mathrm{d}[0] = 1$, $\delta_\mathrm{d}[n] = 0$, $n \neq 0$.

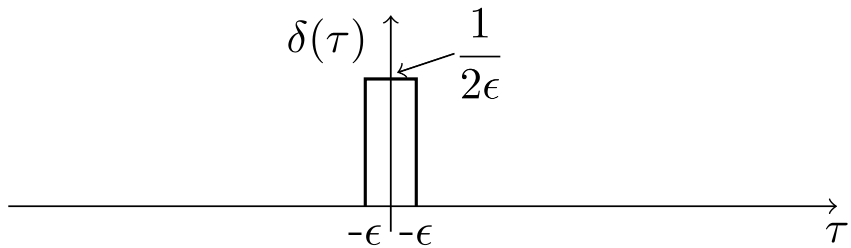
A signal for which (1) holds is called "white noise". We extend this to continuous-time white noise $v(t)$:

$$\mathrm{E}\left[v(t)\right] = 0 \quad \text{and} \quad \mathrm{E}\left[v(t) v^T(t+\tau)\right] = Q_\mathrm{c}\delta(\tau),$$

where $\delta(\tau)$ is the continuous-time Dirac pulse.

Continuous-Time Dirac Pulse

The continuous-time Dirac pulse $\delta(\tau)$ may be defined informally as the limit, as ϵ tends to zero, of the following function:



The Dirac pulse has the property

$$\int_a^b \xi(\tau)\delta(\tau)\,d\tau = \xi(0), \quad \text{for all } a < 0, \text{ and } b > 0,$$

where $\xi(\tau)$ is a real-valued function that is continuous at 0.

Note that a rigorous treatment of this is beyond the scope of this class; truly continuous-time white noise as defined above cannot exist since it has infinite power. Nonetheless, the concept of continuous-time white noise is useful for theoretical analysis, and many continuous-time noise signals can be approximated by white noise.

Continuous-time process update

Process equation with noise characteristics:

$$\boxed{\begin{aligned} \dot{x}(t) = q(x(t), v(t), t), \qquad & \mathrm{E}\left[x(0)\right] = x_0,\ \mathrm{Var}\left[x(0)\right] = P_0 \\ & \mathrm{E}\left[v(t)\right] = 0,\ \mathrm{E}\left[v(t)v^T(t+\tau)\right] = Q_\mathrm{c}\delta(\tau) \end{aligned}}$$

and $v(\cdot)$, $x(0)$ are independent.

We consider $0 \leq t \leq T$ for now and generalize this later.

Assume $\hat{x}_\mathrm{m}[0] = \mathrm{E}\left[x(0)\right]$ and $P_\mathrm{m}[0] = \mathrm{Var}\left[x(0)\right]$. Analogously to the discrete-time case, we predict the estimate forward using the process model. This time, we integrate the continuous-time dynamics instead of performing a discrete update step.

Mean

$\hat{x}(t)$是 $\begin{cases} \dot{a} = q(a, 0, t) \\ a(0) = \hat{x}_m[0] = E[x(0)] \end{cases}$ 的解

Let $\hat{x}(t)$ solve

$$\dot{\hat{x}}(t) = q(\hat{x}(t), 0, t), \quad \text{for } 0 \leq t \leq T \text{ and } \hat{x}(0) = \hat{x}_{\mathrm{m}}[0].$$

We *assume* that $\hat{x}(t) \approx \mathrm{E}[x(t)]$, and we set $\hat{x}_{\mathrm{p}}[1] = \hat{x}(T)$.

$\triangle$ def

$$\hat{x}_m[0] = E[x(0)] \qquad P_m[0] = Var(x(0))$$

$$\Rightarrow \hat{x}(t) \Rightarrow \hat{x}_p[1] \overset{def}{=} \hat{x}(T)$$

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let $F(x, v) = q(x, v, t) - q(\hat{x}, 0, t)$ on $(\hat{x}, 0)$

$$\dot{\tilde{x}} = x - \hat{x} = q(x, v, t) - q(\hat{x}, 0, t)$$

$$= F(x, v) \approx F(\hat{x}, 0) + \frac{\partial F}{\partial x}\Big|_{(\hat{x}, 0)}(x - \hat{x}) + \frac{\partial F}{\partial v}\Big|_{(\hat{x}, 0)}(v - 0)$$

$$= 0 + \frac{\partial q(\hat{x}, 0, t)}{\partial x}(x - \hat{x}) + \frac{\partial q(\hat{x}, 0, t)}{\partial v} \cdot v$$

Variance (1/3)

$$= \frac{\partial f(\hat{x},0,t)}{\partial x}\tilde{x}(t) + \frac{\partial f(\hat{x},0,t)}{\partial v}v(t)$$

$A(t)$ $L(t)$.

Let $\tilde{x}(t) = x(t) - \hat{x}(t)$. *Assuming* that $\tilde{x}(t)$ and $v(t)$ are small (which is not necessarily a good assumption, since $v(t)$ may be arbitrarily large.), and that all quantities except $v(t)$ are sufficiently smooth,

$$\dot{\tilde{x}}(t) \approx A(t)\tilde{x}(t) + L(t)v(t),$$

$$A(t) := \frac{\partial q(\hat{x}(t), 0, t)}{\partial x}, \quad L(t) := \frac{\partial q(\hat{x}(t), 0, t)}{\partial v}.$$

We then have

$$\Rightarrow \quad \tilde{x}(t+\tau) \approx \tilde{x}(t) + \int_t^{t+\tau} A(\xi)\tilde{x}(\xi) + L(\xi)v(\xi)\, d\xi.$$

It can be shown that linearization about $\tau = 0$ yields

$$\begin{aligned}\tilde{x}(t+\tau) &\approx \tilde{x}(t) + \int_t^{t+\tau} A(\xi)\tilde{x}(\xi) + L(\xi)v(\xi)\, d\xi \\ &\approx \tilde{x}(t) + \tau A(t)\tilde{x}(t) + L(t)\int_t^{t+\tau} v(\xi)\, d\xi + O(\tau^2),\end{aligned}$$

where $O(\tau^2)$ denotes second and higher-order terms. Note that the integral of $v(\xi)$ cannot be approximated, since $v(\xi)$ is not continuous.

Variance (2/3)

We define $P(t) := \mathrm{Var}\left[x(t)\right] \approx \mathrm{E}\left[\tilde{x}(t)\tilde{x}^T(t)\right]$, since we assume $\hat{x}(t) \approx \mathrm{E}\left[x(t)\right]$. We therefore have:

$$\begin{aligned} P(t+\tau) \approx & P(t) + \tau A(t)P(t) + \tau P(t)A^T(t) \\ & + L(t)\int_t^{t+\tau}\int_t^{t+\tau} \mathrm{E}\left[v(\xi)v^T(s)\right]\, d\xi\, ds\, L^T(t) + O(\tau^2). \end{aligned}$$

Using the fact that

$$\int_t^{t+\tau} \mathrm{E}\left[v(\xi)v^T(s)\right] d\xi = \int_t^{t+\tau} Q_\mathrm{c}\delta(\xi - s)d\xi = \begin{cases} Q_\mathrm{c} & s \in (t, t+\tau) \\ 0 & \text{otherwise,} \end{cases}$$

simplifies the previous equation to

$$\begin{aligned} P(t+\tau) &\approx P(t) + \tau A(t)P(t) + \tau P(t)A^T(t) + L(t)\int_t^{t+\tau} Q_\mathrm{c}\, ds\, L^T(t) + O(\tau^2) \\ &\approx P(t) + \tau A(t)P(t) + \tau P(t)A^T(t) + \tau L(t)Q_\mathrm{c}\, L^T(t) + O(\tau^2). \end{aligned}$$

Variance (3/3)

Taking the limit as $\tau \to 0$:

$$\dot{P}(t) = \lim_{\tau \to 0} \frac{P(t+\tau) - P(t)}{\tau} \approx A(t)P(t) + P(t)A^T(t) + L(t)Q_c L^T(t).$$

We can thus approximate $P(t)$ by solving the above matrix differential equation for $0 \leq t \leq T$ with initial condition $P(0) = P_\mathrm{m}[0]$. We then set $P_\mathrm{p}[1] = P(T)$.

Note that the approximation of $P(t)$ is due to the nonlinearity of the function $q(x, v, t)$. If q were affine in x and v, and under some mild assumptions, the derivation shown would be exact.

Summary of the Hybrid EKF equations (1/2)

Initialization: $\hat{x}_{\mathrm{m}}[0] = x_0$, $P_{\mathrm{m}}[0] = P_0$.

Step 1 (S1): Prior update/Prediction step

Solve

$$\dot{\hat{x}}(t) = q(\hat{x}(t), 0, t), \quad \text{for } (k-1)T \leq t \leq kT \text{ and } \hat{x}((k-1)T) = \hat{x}_{\mathrm{m}}[k-1].$$

Then $\hat{x}_{\mathrm{p}}[k] := \hat{x}(kT)$.

Solve

$$\dot{P}(t) = A(t)P(t) + P(t)A^T(t) + L(t)Q_{\mathrm{c}}L^T(t), \text{ for } (k-1)T \leq t \leq kT \\ \text{and } P((k-1)T) = P_{\mathrm{m}}[k-1],$$

where

$$A(t) = \frac{\partial q(\hat{x}(t), 0, t)}{\partial x} \quad \text{and} \quad L(t) = \frac{\partial q(\hat{x}(t), 0, t)}{\partial v}.$$

Then $P_{\mathrm{p}}[k] := P(kT)$.

Summary of the Hybrid EKF equations (2/2)

Step 2 (S2): A posteriori update/Measurement update step

The measurement update step is identical to the one for the discrete-time EKF since the measurement model is still discrete-time.

$$K[k] = P_{\mathrm{p}}[k] H^T[k] \left(H[k] P_{\mathrm{p}}[k] H^T[k] + M[k] R M^T[k] \right)^{-1}$$

$$\hat{x}_{\mathrm{m}}[k] = \hat{x}_{\mathrm{p}}[k] + K[k] \left(\bar{z}[k] - h_k(\hat{x}_{\mathrm{p}}[k], 0) \right)$$

$$P_{\mathrm{m}}[k] = \left(I - K[k] H[k] \right) P_{\mathrm{p}}[k]$$

where

$$H[k] := \frac{\partial h_k(\hat{x}_{\mathrm{p}}[k], 0)}{\partial x} \quad \text{and} \quad M[k] := \frac{\partial h_k(\hat{x}_{\mathrm{p}}[k], 0)}{\partial w}.$$

Remarks

- The process and measurement noise are assumed to be zero-mean.

- The hybrid EKF requires the solution of a vector and a matrix ordinary differential equation (ODE), which is typically done using numerical ODE solvers (such as Runge-Kutta methods as, for example, implemented in Matlab's `ode45`). The accuracy of the numerical integration largely depends on the order of the solver. Generally, numerical accuracy is at the cost of increased computation.

- We have discussed continuous-time dynamics within the setting of the EKF (i.e. for nonlinear systems). The same discussion applies to linear systems (which are just a special case of nonlinear systems). Furthermore, one can extend this discussion to the case where also the measurement model is a continuous-time model (linear or nonlinear) and derive the continuous-time KF (for a continuous-time linear system) and the continuous-time EKF (for a continuous-time nonlinear system).