Model Predictive Control

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1) Introduction & System Theory

Concept of MPC

$$U_k^*(x(k)) = \underset{U}{\text{argmin}} \sum l(x_{k+1}, u_{k+1}) \text{ where } U_k = \{u_k, \dots, u_{k+N-1}\}$$

- Problem defined by:
  - Objective that is minimized
  - Internal system model to predict system behavior
  - Constraints that have to be satisfied (Process and Control)
- At each sample time:
  - Measure / Estimate current state $x(k)$
  - Find optimal input sequence U_k^* for entire horizon N
  - implement only the ***first*** control action u_k^*
- Requirements for MPC:
  - Model of system and state estimator
  - Define optimal control problem and optimization problem
  - Solve optimization problem to get optimal control sequence and verify that closed-loop system performs as desired

- **Feasibility**: for the optimization problem, there may not exist a plan satisfying all constraints in future time steps
- **Stability**: Convergence of closed-loop is not automatically guaranteed
- **Robustness**: closed-loop system not robust against uncertainty / disturbance

System Theory - Models

Target models for MPC are mainly LTI state space models.

Nonlinear Time-Invariant Continuous-Time State Space

System I: $\dot{x} = g(x, u),\ y = h(x, u),\ g(x, u) = \text{dynamics},\ h(x, u) = \text{output}$

Transformation to standard form (n^{th} order ODE to n 1^{st} order ODE)

System Equation, n^{th} order ODE: $x^{(n)} + g(x, \dot{x}, \ddot{x}, \dots, x^{(n-1)}) = 0$

Define: $x_{j+1} = x^{(j)}, j = 0, \dots, n-1$, transform: $\dot{x}_1 = x_2, \dots, \dot{x}_n = -g(x_1, \dots, x_n)$

LTI Continuous-Time State Space

System II: $\dot{x} = A^c x + B^c u,\ y = Cx + Du$ with $x = x(t), u = u(t)$

Solution: $x(t) = e^{A^c(t-t_0)}x_0 + \int_{t_0}^{t} e^{A^c(t-\tau)}Bu(\tau)d\tau, \quad e^{A^c t} = \sum_{n=0}^{\infty} \frac{(A^c t)^n}{n!}$

Linearization around stationary operating points x_s, u_s

Taylor Expansion: $g(x) \approx g(\bar{x}) + \left.\frac{dg}{dx^T}\right|_{x=\bar{x}} \underbrace{(x - \bar{x})}_{\Delta x}, \quad \frac{dg}{dx^T} = \begin{bmatrix} \frac{dg_1}{dx_1} & \dots & \frac{dg_1}{dx_n} \\ \vdots & \ddots & \vdots \\ \frac{dg_n}{dx_1} & \dots & \frac{dg_n}{dx_n} \end{bmatrix}$

Linearization:

$$\dot{x} = \underbrace{g(x_s, u_s)}_{=0} + \underbrace{\left.\frac{\partial g}{\partial x^T}\right|_{\substack{x=x_s \\ u=u_s}}}_{=A^c} \underbrace{(x - x_s)}_{=\Delta x} + \underbrace{\left.\frac{\partial g}{\partial u^T}\right|_{\substack{x=x_s \\ u=u_s}}}_{=B^c} \underbrace{(u - u_s)}_{=\Delta u}$$

$$\Rightarrow \dot{x} - \underbrace{\dot{x}_s}_{=0} = \Delta\dot{x} = A^c \Delta x + B^c \Delta u$$

$$y = \underbrace{h(x_s, u_s)}_{y_s} + \underbrace{\left.\frac{\partial h}{\partial x^T}\right|_{\substack{x=x_s \\ u=u_s}}}_{=C} (x - x_s) + \underbrace{\left.\frac{\partial h}{\partial u^T}\right|_{\substack{x=x_s \\ u=u_s}}}_{=D} (u - u_s)$$

$$\Rightarrow \underbrace{\Delta y}_{y - y_s} = C\Delta x + D\Delta u$$

$\Rightarrow$ **Will use x instead of Δx from now on!**

Time-Invariant Discrete-Time State Space

System III: $x(k+1) = g(x(k), u(k)),\ y(k) = h(x(k), u(k))$

Finite computation time in control system: continuous time system has to be discretized with sampling time T_s: $t_{k+1} = t_k + T_s,\ u(t) = u(t_k), t \in [t_k, t_{k+1})$

Euler Discretization of Nonlinear Time-Invariant Models

Approximation from System I: $\dot{x^c}(t) = \frac{x^c(t+T_s) - x^c(t)}{T_s},\quad x(k) = x^c(t_0 + kT_s)$

System IV: $x(k+1) = x(k) + T_s * g^c(x(k), u(k)) = g(x(k), u(k))$

$y(k) = h^c(x(k), u(k)) = h(x(k), u(k))$

Euler Discretization of LTI Models

Approximation from System II: $A = I + T_s A_c, B = T_s B_c, C = C_c, D = D_c$

System V: $x(k+1) = Ax(k) + Bu(k) \mid y(k) = Cx(k) + Du(k)$

Exact Discretization of LTI Models

From Solution of System II: set $t_0 = t_k, x_0 = x(t_k), t = t_{k+1}$ and $u(t) = u(t_k)\ \forall\, t \in [t_k, t_{k+1})$, then under assumption of a **constant** $u(t)$ during a sampling interval, we find the **exact** discrete-time model predicting the state of the continuous-time system at time t_{k+1} given $x(t_k)$:

$$x(t_{k+1}) = e^{A^c T_s}x(t_k) + \int_{t_k}^{t_{k+1}} e^{A^c(t_{k+1}-\tau)}B^c d\tau\, u(t_k)$$

$$= \underbrace{e^{A^c T_s}}_{\triangleq A} x(t_k) + \underbrace{\int_0^{T_s} e^{A^c(T_s - \tau')}B^c d\tau'}_{\triangleq B} u(t_k)$$

$$= Ax(t_k) + Bu(t_k)$$

$\Rightarrow$ If A_c invertible: $B = (A^c)^{-1}(A - I)B^c$

Solution of Linear Discrete-Time Systems

$$x(k+N) = A^N x(k) + \sum_{i=0}^{N-1} A^i Bu(k + N - 1 - i)$$

System Theory– Analysis of LTI Systems

Coordinate Transformation

- Consider System V , Transformations can facilitate system analysis since many choices of state can yield same input-output behavior
- Consider $\tilde{x}(k) = Tx(k)$ with det $(T) \neq 0$ while $u(k)$ and $y(k)$ are unchanged:

$$\tilde{x}(k+1) = \underbrace{TAT^{-1}}_{\tilde{A}}\tilde{x}(k) + \underbrace{TB}_{\tilde{B}}u(k)$$

$$y(k) = \underbrace{CT^{-1}}_{\tilde{C}}\tilde{x}(k) + \underbrace{D}_{\tilde{D}}u(k)$$

Stability of LTI Systems

LTI System $\boldsymbol{x(k+1) = Ax(k)}$ is globally asymptotically stable, meaning $\lim_{k\to\infty} x(k) = 0\ \forall\, x(0) \in \mathbb{R}^n$, if and only if all eigenvalues $|\lambda_i| < 1$ of the matrix A

Controllability of LTI Systems

A discrete LTI system (System V) is controllable/reachable if for any pairs of states $x(0)$ & x^*, there exists a finite time N & input sequence U s.th $x^* = x(N)$:

$$x^* = x(N) = A^N x(0) + [B\ AB \dots A^{N-1}B] \begin{bmatrix} u(N-1) \\ \vdots \\ u(0) \end{bmatrix}$$

Cayley-Hamilton

Matrix A^k can be expressed as lin. comb. of $A^j, j = 0, \dots, n-1$ for $k \geq n$. So, for all $N \geq n$: range$(B\ AB \dots A^{N-1}B)$ = range$(B\ AB \dots A^{n-1}B)$

Controllability Matrix C and Solution

$C = [B\ AB \dots A^{n-1}B]$, Input Sequence $U = [u(n-1), u(n-2), \dots, u(0)]^T$

$\Rightarrow$ **system controllable if $C * U = x^* - A^N x(0)$ has solution for all right sides**

$\Rightarrow$ **Necessary and Sufficient condition**: rank$(C) = n \Rightarrow (A, B)$ controllable

- If system cannot be controlled in N steps to x^*, then it can't be controlled for any number of steps.
- System **stabilizable** if there is an input sequence to return the state to the origin asymptotically from an arbitrary initial state
- System stabilizable iff all of its uncontrollable modes are stable
- Controllability implies stabilizability
- **Stabilizability condition**: rank$(\lambda_i I - A \mid B) = n$ for all $|\lambda_i| \geq 1$

Observability of LTI Systems

A discrete LTI system $\boldsymbol{x(k+1) = Ax(k), y(k) = Cx(k)}$ is observable if there exists a finite N such that for every $x(0)$, the measurements $y(0), \dots, y(N-1)$ uniquely distinguish the initial state $x(0)$:

$$\begin{bmatrix} y(0) \\ \vdots \\ y(N-1) \end{bmatrix} = \begin{bmatrix} C \\ \vdots \\ CA^{N-1} \end{bmatrix} x(0)$$

Observability Matrix O and Solution

$O = [C\ CA \dots CA^{n-1}]^T$, above equation has solution if columns of O lin. independ

$\Rightarrow$ **Necessary and Sufficient condition**: rank$(O) = n \Rightarrow (A, C)$ observable

- System is **detectable** if it's possible to construct a sequence of state estimates from the measurements, which converges to the true state asymptotically.
- System is detectable if all of its unobservable modes are stable.
- Observability implies detectability
- **Detectability condition**: rank$(A^T - \lambda_i I \mid C^T) = n$ for all $|\lambda_i| \geq 1$

System Theory – Analysis of Nonlinear DT Systems

$\Rightarrow$ ***Note***: These are all sufficient conditions!

Stability of Nonlinear Systems

System IV & equilibrium $g(\bar{x}) = \bar{x}$: Stability defined in the sense of Lyapunov.

Lyapunov Stability

An equilibrium point $\bar{x}$ is Lyapunov **stable** if for every $\varepsilon > 0$ there is a $\delta(\varepsilon)$ s.th.

$$\|x(0) - \bar{x}\| < \delta(\varepsilon) \rightarrow \|x(k) - \bar{x}\| < \varepsilon \quad \forall\, k \geq 0$$

Lyapunov Asymptotic Stability

An equilibrium point $\bar{x} \in \Omega$ is **asymptotically stable** in Ω if it's Lyapunov stable and $\lim_{k\to\infty}\|x(k) - \bar{x}\| = 0$. If $\Omega = \mathbb{R}^n$, then it's **globally asymptotically stable**.

Lyapunov function

Consider the equilibrium point $\bar{x} = 0$ and $\Omega \subset \mathbb{R}^n$ a closed & bounded set containing the origin. Lyapunov function $V: \mathbb{R}^n \rightarrow \mathbb{R}$, continuous at the origin s.th: $V(0) = 0 \mid V(x) > 0,\ \forall\, x \in \Omega\backslash\{0\} \mid V(g(x)) - V(x) \leq -\alpha(x)\ \forall\, x \in \Omega, \alpha > 0$

Lyapunov Theorem – asymptotic stability

If system admits a Lyapunov function $V(x)$, then $x = 0$ is **asymptotically stable** for $\alpha(x)$ positive definite. For $\alpha(x)$ positive semidefinite, $x = 0$ is only stable.

Lyapunov Theorem – global asymptotic stability

If system admits a Lyapunov function $V(x)$ for $\Omega = \mathbb{R}^n$ and additionally $\|x\| \rightarrow \infty \Longrightarrow V(x) \rightarrow \infty$, then $x = 0$ is **globally asymptotically stable**.

Lyapunov indirect method

Let $A = \left.\frac{dg}{dx^T}\right|_{x=0}$ be the linearized matrix of the nonlinear system around $\bar{x} = 0$.

Then the origin is **locally asymptotically stable** if $|\lambda_i| < 1$ for all eigenvalues of A. If there is at least one $|\lambda_i| > 1$, the origin is **unstable**. If there is at least one $|\lambda_i| = 1$, we cannot say anything about stability → build Lyapunov function.

Stability of LTI Systems with Lyapunov

Consider an LTI system $\boldsymbol{x(k+1) = Ax(k)}$ and take $V(x) = x^T Px$ as candidate Lyapunov function satisfying properties of **Lyapunov Theorem**, with $P > 0$

$\Rightarrow$ From $V(Ax(k)) - V(x(k)) = x^T(k)(A^T PA - P)x(k) \leq -\alpha(x(k))$,

chose $\alpha(x(k)) = x^T(k)Qx(k),\ Q > 0$. P is found by Lyapunov Equation.

DT Lyapunov Equation & Existence

DT Lyapunov Equation $\boldsymbol{A^T PA - P = -Q}$, $Q > 0$ has a unique solution $P > 0$ **if and only if** A has all eigenvalues inside the unit circle $\Longleftrightarrow$ **system is stable**.

- Necessary and sufficient condition
- Stability is always global for linear systems!

Property of $\boldsymbol{P}$

P can be used to determine the infinite horizon cost-to-go for asympt. stable autonomous system $x(k+1) = Ax(k)$. Defining $\Psi(x(0)) = \sum_{k=0}^{\infty} x^T(k)Qx(k) = \sum_{k=0}^{\infty} x^T(0)(A^k)^T QA^k x(0) \Longrightarrow \Psi(x(0)) = x^T(0)Px(0) \Longrightarrow A^T PA = P - Q$

2) Unconstrained linear quadratic optimal control

Optimal Control - General

Finite horizon N optimal control for discrete-time systems and init. state x_0:

$J^*(x(0)) \coloneqq \min_U J(x(0), U) \mid$ subj. to $x_{i+1} = g(x_i, u_i), h(x_i, u_i) \leq 0, x_N \in \mathcal{X}_f$

Linear Quadratic Unconstrained Optimal Control

Linear System: $x(k+1) = Ax(k) + Bu(k)$

$$\text{Quadratic Cost:}\quad J^*(x(0)) = \min_U x_i^T Px_i + \sum_{i=0}^{N-1}(x_i^T Qx_i + u_i^T Ru_i)$$

subj. to: $x_{i+1} = Ax_i + Bu_i, \quad x_0 = x(0)$

weights: $P \succcurlyeq 0, P = P^T$terminal $\mid Q \succcurlyeq 0, Q = Q^T$state $\mid R \succ 0, R = R^T$ input

$\Rightarrow$ goal is to regulate state to the origin **without state or input constraints**.

Batch Approach

1. $X = S^x x(0) + S^u U = \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ \vdots \\ x_N \end{bmatrix} = \begin{bmatrix} I \\ A \\ \vdots \\ \vdots \\ A^N \end{bmatrix} x(0) + \begin{bmatrix} 0 & \dots & \dots & 0 \\ B & 0 & \dots & 0 \\ AB & B & 0 & 0 \\ \vdots & \ddots & \ddots & 0 \\ A^{N-1}B & \dots & AB & B \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ \vdots \\ u_{N-1} \end{bmatrix}$
2. $\bar{Q} \coloneqq$ blockdiag$(Q, \dots, Q, P)$ & $\bar{R} \coloneqq$ blockdiag$(R, \dots, R)$ & $H \coloneqq (S^u)^T \bar{Q} S^u + \bar{R}$ & $F \coloneqq (S^x)^T \bar{Q} S^u$ & $H > 0$:
   $J(x(0), U) = X^T \bar{Q} X + U^T \bar{R} U = U^T HU + 2x(0)^T FU + x(0)^T (S^x)^T \bar{Q} S^x x(0)$
3. Since problem is unconstrained and $J(x(0), U)$ is positive definite, can set $\nabla_U J(x(0), U) = 2HU + 2F^T x(0) = 0$
   $\rightarrow \boldsymbol{U^*(x(0)) = -((S^u)^T \bar{Q} S^u + \bar{R})^{-1}(S^u)^T \bar{Q} S^x x(0) = -H^{-1}F^T x(0)}$
4. $J^*(x(0), U) = x(0)^T[(S^x)^T \bar{Q} S^x - (S^x)^T \bar{Q}((S^u)^T \bar{Q} S^u + \bar{R})^{-1}(S^u)^T \bar{Q} S^x]x(0)$

Note: if there are state / input constraints, solving this problem by matrix inversion does not guarantee a feasible input sequence.

Recursive Approach / Finite Horizon LQR

1. Consider j-step problem at time $N - j$:

$$J_{N-j}^*(x_{N-j}) = \min_{U_{N-j}} x_N^T P_N x_N + \sum_{i=N-j}^{N-1}(x_i^T Qx_i + u_i^T Ru_i)$$

s. th.: $x_{i+1} = Ax_i + Bu_i, \quad i = [N-j, \dots, N-1], \quad \boldsymbol{P_N = P}$

1. Substituting equation for x_{N-j} into J_{N-j}^* and setting $\nabla_U J_{N-j}^* = 0$ leads to

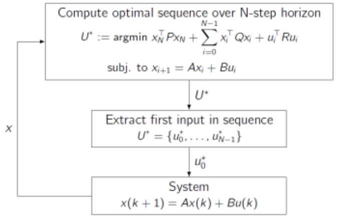
$$\boldsymbol{u_{N-j}^* = -(B^T P_{N-j+1} B + R)^{-1} B^T P_{N-j+1} A x_{N-j} = F_{N-j} x_{N-j}}$$

2. Optimal cost-to-go $J_{N-j}^*(x_{N-j}) = x_{N-j}^T P_{N-j} x_{N-j}$
3. Every P_j is related to P_{j+1} by the **Riccati Difference Equation (RDE)**

$$P_{N-j} = A^T P_{N-j+1} A + Q - A^T P_{N-j+1} B(B^T P_{N-j+1} B + R)^{-1} B^T P_{N-j+1} A$$

Receding Horizon Control

- For unconstrained system, will get a **constant linear controller**
- As Horizon Length grows, system gets stable and prediction more accurate



Infinite Horizon LQR

Optimal Solution

$J_\infty(x(0)) = \sum_{i=0}^{\infty}(x_i^T Qx_i + u_i^T Ru_i)$ with $x_{i+1} = Ax_i + Bu_i, x(0) = x_0$

- The optimal input is **time-invariant** (opposed to finite horizon)
  $\boldsymbol{u^*(k) = -(B^T P_\infty B + R)^{-1} B^T P_\infty A x(k) \coloneqq F_\infty x(k)}$
  and the infinite-horizon cost-to-go $\boldsymbol{J_\infty(x(k)) = x(k)^T P_\infty x(k)}$ are referred to the Recursive Dynamic Programming Approach.
- If RDE converges to a const. P_∞, we get **Algebraic Riccati Equation (ARE)**
  $P_\infty = A^T P_\infty A + Q - A^T P_\infty B(B^T P_\infty B + R)^{-1} B^T P_\infty A$
- In fact, if (A, B) stabilizable and $(Q^{\frac{1}{2}}, A)$ detectable, **initialized with $\boldsymbol{P_\infty = Q}$**, then the RDE converges to the unique positive definite solution P_∞ of the ARE.
- $\boldsymbol{F_\infty}$ = **LQR Controller → time-invariant**

Stability of Infinite Horizon LQR - Lemma

If the system is stabilizable and detectable, the optimal value function $J^*(x) = x^T P_\infty x$ is a valid Lyapunov function for the System $x_{i+1} = (A + BF_\infty)x_i$ where $F_\infty = -(B^T P_\infty B + R)^{-1} B^T P_\infty A$ and P_∞ solves the ARE for some $Q \succcurlyeq 0, R \succ 0$.

- If (A, B) stabilizable and $(Q^{\frac{1}{2}}, A)$ detectable, closed-loop system with $u(k) = F_\infty x(k)$ is **guaranteed to be asymptotically stable.**
- Asymptotic stability of $x_{i+1} = (A + BF_\infty)x_i$ can be proven by showing that $J^*(x)$ is a **Lyapunov Function** which implies that $\lim_{k\to\infty} x(k) = 0$
- $P_\infty \succ 0$ gives the requirements:
  1. $J^*(x(k)) = x(k)^T P_\infty x(k) = \sum_{i=k}^{\infty} x^T(i)(Q + F_\infty^T RF_\infty)x(i)$
  2. $J^*(x(k+1)) = \sum_{i=k+1}^{\infty} x^T(i)(Q + F_\infty^T RF_\infty)x(i) = J^*(x(k)) - x^T(k)(Q + F_\infty^T RF_\infty)x(k) < J^*(x(k))$

Distinct Choices for P in Finite Horizon Control

1. P can be chosen s.th. its solution matches the infinite horizon solution: make P equal to optimal cost from N to ∞ by computing the ARE for $P = P_\infty$.
2. If System asym. stable & assuming no control action after end of Horizon $[x(k+1) = Ax(k), k = N, \dots, \infty]$, determine P by Lyapunov: $P = A^T PA + Q$.
3. Assume we want state and input both to be zero after end of the finite Horizon. In this case, no P but an extra constraint is needed: $x_{i+N} = 0$.

3) Convex Optimization

Optimization in MPC

General form: $\min_{x \in dom(f)} f(x)$ sub. to: $g_i(x) \leq 0, \quad h_i(x) = 0$

- The i^{th} inequality constraint is **active** at $\bar{x}$ if $g_i(\bar{x}) = 0$, otherwise it's inactive.
- Equality constraints are always active.
- **Feasible Set**: $\mathcal{X} \coloneqq \{x \in dom(f) \mid \ g_i(x) \leq 0, i = 1, \dots, m, \quad h_i(x) = 0, i = 1, \dots, p\} \Longrightarrow$ Set of ***initial*** feasible states!
- **Optimizer**: set of feasible $x^*$with smallest cost $p^* = f(x^*) \triangleq \min f(x)$:
  $\text{argmin}_{x \in \mathcal{X}} f(x) \coloneqq \{x \in \mathcal{X} \mid f(x) = p^*\}$

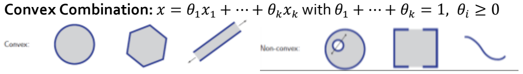
Optimality

- **Locally Optimal**: $x \in \mathcal{X}$ for some $R > 0$: $y \in \mathcal{X}. \|y - x\| \le R \Rightarrow f(y) \ge f(x)$
- **Globally Optimal**: $x \in \mathcal{X}$ satisfies: $y \in \mathcal{X} \Rightarrow f(y) \ge f(x)$
- **Unbounded**: if $p^* = -\infty$ **Infeasible**: $\mathcal{X}$ empty $\Leftrightarrow$ $p^* = \infty$ **Unconstr.**: $\mathcal{X} = \mathbb{R}^n$

Convex Sets

Definition: A set $\mathcal{X}$ is convex if and only if for any pair of points x and y in $\mathcal{X}$:
$$\lambda x + (1-\lambda) y \in \mathcal{X}, \quad \forall \lambda \in [0,1], \quad \forall x, y \in \mathcal{X}$$

- **Interpretation**: All line segments starting and ending in $\mathcal{X}$ stay within $\mathcal{X}$.
- **Convex Combination**: $x = \theta_1 x_1 + \cdots + \theta_k x_k$ with $\theta_1 + \cdots + \theta_k = 1$, $\theta_i \ge 0$

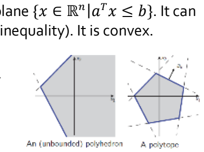


Hyperplanes and Halfspaces

- **Hyperplane** is defined by $\{x \in \mathbb{R}^n | a^T x = b\}$ for $a \ne 0$, where $a \in \mathbb{R}^n$ is the normal vector to the hyperplane. It is affine and convex.
- **Halfspace** is everything on one side of a hyperplane $\{x \in \mathbb{R}^n | a^T x \le b\}$. It can be **open** (strict inequality) or **closed** (non-strict inequality). It is convex.

Polyhedra and Polytopes (both convex)

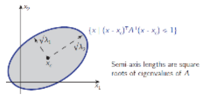
- **Polyhedron** is the intersection of finite number of closed halfspaces:
  $P := \{x | a_i^T x \le b_i, i = 1, \dots, n\} = \{x | Ax \le b\}$,
  $A := [a_1, \dots, a_m]^T, b := [b_1, \dots b_m]^T$



- **Polytope** is a bounded Polyhedron.

Ellipsoids

- An **Ellipsoid**, with x_c the centre of the ellipsoid and $A \succ 0$ pos. def. & symmetric, is:
  $$\{x | (x - x_c)^T A^{-1} (x - x_c) \le 1\}$$



Norm Balls

- The **Norm Ball** is defined by $\{x | \|x - x_c\| \le r\}$, x_c is the centre & r the radius.
- The Norm Ball is always convex for any norm. Most common $\ell_p - Norms$:

ℓ_1: $\|x\|_1 = \sum_i |x_i|$ $\quad$ ℓ_2: $\|x\|_2 = \sqrt{\sum_i x_i^2}$ $\quad$ ℓ_∞: $\|x\|_\infty = \max_i |x_i|$

Intersection

- Intersection of any convex sets / halfspaces is itself a convex set.
- **Proof**: for any a and b which lie both in both sets $\mathcal{X}$ & $\mathcal{Y}$, the point $\lambda a + (1-\lambda) b$ is $\in \mathcal{X}$ ***and*** $\in \mathcal{Y}$ $\Longrightarrow \in \mathcal{X} \cap \mathcal{Y}$.

Union of two sets ***not*** convex in general, even if the original sets are convex!

Convex Functions

Definition: a function $f: \mathrm{dom}(f) \to \mathbb{R}$ is (strictly) convex iff $\mathrm{dom(f)}$ is convex **and**
$$f(\lambda x + (1-\lambda) y) \le \lambda f(x) + (1-\lambda) f(y), \quad \forall \lambda \in (0,1), \quad \forall x, y \in \mathrm{dom}(f)$$

- The function is **concave** iff $\mathrm{dom}(f)$ is convex and $-f$ is convex.

First-Order Condition for Convexity

- A differentiable function f with a convex domain is convex iff

$$f(y) \ge f(x) + \nabla f(x)^T (y - x), \quad \forall x, y \in \mathrm{dom}(f), \nabla f(x) = \left[\frac{\partial f(x)}{\partial x_1}, \dots, \frac{\partial f(x)}{\partial x_n}\right]^T$$

- First-order approx. of f around any point x is a global underestimator of f

Second-Order Condition for Convexity

- A twice-differentiable function f with convex domain is convex iff
$$\nabla^2 f(x) \succeq 0, \quad \forall x \in \mathrm{dom}(f), \quad \nabla^2 f(x)_{ij} = \frac{\partial^2 f(x)}{\partial x_i \partial x_j}$$
- If $\mathrm{dom}(f)$ is convex **and** $\nabla^2 f(x) \succ 0$ $\forall x \in \mathrm{dom}(f)$, then f is strictly convex.

Level and Sublevel sets

- **Level Set L_α** of f for value α is the set of all $x \in \mathrm{dom}(f)$ for which $f(x) = \alpha$:
  $$L_\alpha := \{x | x \in \mathrm{dom}(f), f(x) = \alpha\}$$
- For $f: \mathbb{R}^2 \to \mathbb{R}$ these are **contour lines** of constant "height"
- **Sublevel Set C_α** of a function f for value α is defined by:
  $$C_\alpha := \{x | x \in \mathrm{dom}(f), f(x) \le \alpha\}$$
- Function f is convex $\Longrightarrow$ sublevel sets of f are convex for all α. But ***not*** $\Longleftarrow$!

Examples of Convex Functions: $\mathbb{R} \to \mathbb{R}$

	Convex	Concave
• Affine, $a, b \in \mathbb{R}$:	$ax + b$	$ax + b$
• Exponential, $a \in \mathbb{R}$:	e^{ax}	-
• Powers on $\mathbb{R}_{++}$:	x^α for $\alpha \le 0$ or $\alpha \ge 1$	x^α for $0 \le \alpha \le 1$
• Vector Norm on $\mathbb{R}^n$:	$\|x\|_p$, $p \ge 1$	-
• Logarithm / Entropy:	-	$\log(x)$ / $-x\log(x)$ on $\mathbb{R}_{++}$

Preserving Operations: non-negative weighted sum, composition with affine function, pointwise maximum and supremum, partial minimization.

Convex Optimization Problems

General form: $\quad \min_{x \in \mathrm{dom}(f)} f(x) \quad$ sub. to: $\quad g_i(x) \le 0, \quad a_i^T x = b_i \Leftrightarrow Ax = b$

- f, g_i are convex functions, $\mathrm{dom}(f)$ is conv. set, $h_i(x) = a_i^T x - b_i$ are all affine
- Feasible set of a convex optimization problem is convex

Local and Global Optimality

- For a convex optimization problem it holds: **locally = globally** optimal solution
- **Proof**: If x is loc. optimum, there are points y, z s.th. $f(y) < f(x)$ and $\|z - x\| \le R \Rightarrow f(z) \ge f(x) \longrightarrow$ problem can't be convex!

Linear Program (LP)

Affine cost and constraint functions: $\min_{x \in \mathbb{R}^n} c^T x \quad$ sub. to: $Gx \le h, \quad Ax = b$

- Solution Properties, where p^* is the optimal cost, X_{opt} is the set of optimizers
  Case 1: LP solution is unbounded, $p^* = -\infty$
  Case 2: LP solution is bounded, $p^* > -\infty$, unique optimizer, X_{opt} is a singleton
  Case 3: LP sol. is bounded & there are mult. optima, $X_{opt} \subseteq \mathbb{R}^S$ (un)bounded

Quadratic Program (QP)

$$\min_{x \in \mathbb{R}^n} \frac{1}{2} x^T H x + q^T x + r \quad \text{sub. to: } Gx \le h, \quad Ax = b \quad \to \text{convex if } H \succ 0$$

- If feasible set P not empty
  Case 1: the optimizer lies strictly inside the feasible polyhedron
  Case 2: The optimizer lies on the boundary of the feasible polyhedron

Optimality Conditions

The Lagrange Dual Problem

Lagrangian Function L

From General form with (primal) decision var. x: $\quad$ $\boldsymbol{L}: \mathrm{dom}(f) \times \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}$:

$$L(x, \lambda, \nu) = f(x) + \sum_{i=1}^m \lambda_i g_i(x) + \sum_{i=1}^p \nu_i h(x)$$

Dual Function d

$\boldsymbol{d}: \mathbb{R}^m \times \mathbb{R}^p$: $\quad$ $d(\lambda, \nu) = \inf_{x \in \mathrm{dom}(f)} L(x, \lambda, \nu)$

$\to d(\lambda, \nu)$ always **concave**, sets lower bounds for p^*: $d(\lambda, \nu) \le p^*$ $\forall (\lambda \ge 0, \nu \in \mathbb{R}^p)$
$\to$ If $d(\lambda, \nu) = -\infty$: $\quad \mathrm{dom}(d) := \{\lambda, \nu | d(\lambda, \nu) > -\infty\}$

Primal (P) Problem	Dual (D) Problem
$\min_x f(x)$	$\max_{\lambda, \nu} d(\lambda, \nu)$
sub. to: $g_i(x) \le 0, h_i(x) = 0$	sub. to: $\lambda \ge 0$

- Problem (D) is **convex** even if (P) is not and has optimal value $d^* \le p^*$
- Point (λ, ν) is **dual feasible** if $\lambda \ge 0$ & $(\lambda, \nu) \in \mathrm{dom}(d)$ (can be imposed in D)

Weak and Strong Duality

- **Weak**: it is always true that $d^* \le p^*$
- **Strong**: it is sometimes true that $d^* = p^*$. Strong duality doesn't hold for non-convex problems. Can impose conditions on convex problems to have $d^* = p^*$

Strog Duality for Convex Problems: Slater Condition

- Assume a Primal, convex (f, g_i) optimization Problem. If there is at least one **strictly feasible** point $\{x | Ax = b, g_i(x) < 0, \forall i\} \ne \emptyset$, then $\boldsymbol{p^* = d^*}$.

Optimality Conditions: Karush-Kuhn-Tucker (KKT)

Necessary conditions for optimality assuming all g_i and h_i are differentiable:

1) **Primal Feasibility**: $\quad g_i(x^*) \le 0, \quad i = 1, \dots, m \mid h_i(x^*) = 0, i = 1, \dots, p$
2) **Dual Feasibility**: $\quad \lambda^* \ge 0$
3) **Complementary Slackness**: $\lambda^* g_i(x^*) = 0, \quad i = 1, \dots, m$
   If Strong Duality: $f(x^*) = d(\lambda^*, \nu^*) = f(x^*) + \sum_{i=1}^m \lambda_i^* g_i(x^*) + \sum_{i=1}^p \nu_i^* h(x^*)$
   $\Longrightarrow \lambda_i^* = 0$ for $g_i(x^*) < 0$ & $g_i(x^*) = 0$ for $\lambda_i^* > 0$
4) **Stationarity** (gradient=0 at extremum):
   $\nabla_x L(x^*, \lambda^*, \nu^*) = \nabla f(x^*) + \sum_{i=1}^m \lambda_i \nabla g_i(x^*) + \sum_{i=1}^p \nu_i \nabla h(x^*) = 0$

General Optimization: Necessary Condition

If x^* & (λ^*, ν^*) primal & dual sol. (zero duality gap), then x^* & (λ^*, ν^*) satisfy KKT

Convex Optimization: Necessary & Sufficient Condition

If x^* and (λ^*, ν^*) satisfy KKT conditions, then x^* and (λ^*, ν^*) are primal & dual solutions ($\boldsymbol{p^* = d^*}$). If **Slater Condition** holds (strong duality), x^* and (λ^*, ν^*) are primal and dual solutions if and only if they satisfy KKT conditions.

- $p^* = f(x^*) = L(x^*, \lambda^*, \nu^*)$ due to Complementary Slackness
- $d^* = d(\lambda^*, \nu^*) = L(x^*, \lambda^*, \nu^*)$ due to Convexity of the functions & Stationarity

Sensitivity Analysis: Perturbed Problems

Primal (P) Problem	Dual (D) Problem	Assuming
$\min f(x)$	$\max_{\lambda, \nu} d(\lambda, \nu) - u^T \lambda - v^T \nu$	Strong duality for
sub. to: $g_i(x) \le u_i, h_i(x) = v_i$	sub. to: $\lambda \ge 0$	unperturbed problem

- Weak duality for pert. problem implies: $p^*(u, v) \ge d^*(\nu^*, \lambda^*) - u^T \lambda^* - v^T \nu^*$
  $= p^*(0,0) - u^T \lambda^* - v^T \nu^*$
- **Global Sensitivity**: change of $p^*(u, v)$ due to λ^*, ν^* large or small & sign of u, v
- **Local Sensitivity** of $p^*(u, v)$ if diff.able at (0,0): $\lambda_i^* = -\frac{\partial p^*(0,0)}{\partial u_i} \quad \nu_i^* = -\frac{\partial p^*(0,0)}{\partial v_i}$

4) Constrained Finite Time Optimal Control (CFTOC)

Constrained Linear Optimal Control

$$J^*(x_k) = \begin{cases} \min_U \underbrace{x_N^T P_N x_N}_{I_f(x_N)} + \sum_{i=0}^{N-1} \underbrace{(x_i^T Q x_i + u_i^T R u_i)}_{I(x_i, u_i)}, & \text{Euclidian: } P \succeq 0, Q \succeq 0, R \succ 0 \\ \min_U \underbrace{\|P x_N\|_p}_{I_f(x_N)} + \sum_{i=0}^{N-1} \underbrace{\|Q x_i\|_p + \|R u_i\|_p}_{I(x_i, u_i)}, & p = \{1, \infty\}: P, Q, R \text{ full column} \end{cases}$$

subj. to: $x_{i+1} = A x_i + B u_i, \quad x_i \in \mathcal{X}, x_N \in \mathcal{X}_f, u_i \in \mathcal{U}, x_0 = x(k), i = 0, \dots, N-1$

Feasible Sets

$\mathcal{X}_j$ is the set of states x_j at time $j = 0, \dots, N$ for which control problem is feasible:
$\mathcal{X}_j = \{x_j \in \mathbb{R}^n | \exists (u_j, \dots, u_{N-1}) \text{ s.th.: } x_i \in \mathcal{X}, u_i \in \mathcal{U}, x_N \in \mathcal{X}_f, x_{i+1} = A x_i + B u_i\}$
$\Longrightarrow$ Approximate Feasible Set by taking **convex hull** of initial feasible states.

Transformation: Quadratic Cost CFTOP $\Longrightarrow$ QP

Construction of QP with substitution (Batch Approach, $N{*}m$ optimizations)

1) **Cost**: $J(x_k, U) = U^T H U + 2 x_k^T F U + x_k^T Y x_k = [U^T \quad x_k^T] \begin{bmatrix} H & F^T \\ F & Y \end{bmatrix} [U^T \quad x_k^T]^T$
   subj. to: $GU \le w + E x_k$
2) **Constraints**: $\mathcal{X} = \{x | A_x x \le b_x\}, \mathcal{X}_f = \{x | A_f x \le b_f\}, \mathcal{U} = \{u | A_u u \le b_u\}$
   $\bar{A}_x = \mathrm{diag}(A_x, \dots, A_f), \bar{A}_u = \mathrm{diag}(A_u, \dots), \bar{b}_{x/u} = [b_{x/u}, \dots]^T \to \bar{A}_x X \le \bar{b}_x, \bar{A}_u U \le \bar{b}_u$
   Substitution: $X = S^X x(k) + S^U U \to \bar{A}_x S^U U \le \bar{b}_x - \bar{A}_x S^X x(k) \Longrightarrow GU \le w + Ex(k)$

$$G = \begin{bmatrix} A_u & 0 & \dots & 0 \\ 0 & A_u & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & A_u \\ 0 & 0 & \dots & 0 \\ A_x B & 0 & \dots & 0 \\ A_x A B & A_x B & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_f A^{N-1} B & A_f A^{N-2} B & \dots & A_f B \end{bmatrix}, E = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ -A_x \\ -A_x A \\ -A_x A^2 \\ \vdots \\ -A_f A^N \end{bmatrix}, w = \begin{bmatrix} b_u \\ b_u \\ \vdots \\ b_u \\ b_x \\ b_x \\ b_x \\ \vdots \\ b_f \end{bmatrix}$$

$\to$ For a given x_k, solution U^* can be found with a QP solver.

Construction of the QP without substitution ($N(m+n)$ optimizations)

1) **Cost**: $J^*(x_k) = \min_z [z^T \quad x_k^T] \begin{bmatrix} \bar{H} & 0 \\ 0 & Q \end{bmatrix} [z^T \quad x_k^T]^T$ subj. to: $G_{in} z \le w_{in} + E_{in} x_k$
   $G_{eq} z = E_{eq} x_k$ with $z = [x_1^T \dots x_N^T \; u_0^T \dots u_{N-1}^T]^T, \bar{H} = \mathrm{diag}(Q, \dots, Q, P, R, \dots, R)$
2) **Equalities** $x_{i+1} = A x_i + B u_i$: $G_{eq} = \begin{bmatrix} I & & & -B & & \\ -A & \ddots & & & \ddots & \\ & -A & I & & & -B \end{bmatrix}, E_{eq} = \begin{bmatrix} A \\ 0 \\ \vdots \end{bmatrix}$
3) **Inequalities**: with $\mathcal{X} = \{x | A_x x \le b_x\}, \mathcal{X}_f = \{x | A_f x \le b_f\}, \mathcal{U} = \{u | A_u u \le b_u\}$
   and $\bar{A}_x, \bar{A}_u, \bar{b}_x, \bar{b}_u$ from above: $G_{in} = \begin{bmatrix} 0 & \\ \bar{A}_x & \\ & \bar{A}_u \end{bmatrix}, w_{in} = \begin{bmatrix} \bar{b}_x \\ \bar{b}_u \end{bmatrix}, E_{in} = \begin{bmatrix} -A_x \\ 0 \end{bmatrix}$

Quadratic Cost State Feedback Solution

$J^*(x_k) = \min_U [U^T \quad x_k^T] \begin{bmatrix} H & F^T \\ F & Y \end{bmatrix} [U^T \quad x_k^T]^T$ subj. to: $GU \le w + E x_k$ is a multiparametric quadratic program (**mp-QP**) with the **solution properties:**

- First component of the solution: $u_0^* = \kappa(x_k)$ $\forall x_k \in \mathcal{X}_0$ where $\kappa: \mathbb{R}^n \to \mathbb{R}^m$ is continuous & piecewise affine on Polyhedra:
  $$\kappa(x) = F^j x + g^j \quad \text{if} \quad x \in CR^j, \quad j = 1, \dots, N^r$$
- Polyhedral sets $CR^j = \{x \in \mathbb{R}^n | H^j x \le K^j, j = 1, \dots, N^r\}$ are a partition of the feasible polyhedron $\mathcal{X}_0$
- The value function $J^*(x_k)$ is convex and piecewise quadratic on polyhedra

Transformation: P-Norm Cost CFTOP $\Longrightarrow$ LP

ℓ_∞ minimization

$\min_{x \in \mathbb{R}^n} \|x\|_\infty = \min_{x \in \mathbb{R}^n} [\max\{x_1, \dots, x_n, -x_1, \dots, -x_n\}] \quad$ subj. to: $Fx \le g$
$\Leftrightarrow \min_{x, t} t \quad$ subj. to: $x_i \le t, -x_i \le t \Leftrightarrow -\mathbf{1}t \le x \le \mathbf{1}t \quad$ & $Fx \le g$

ℓ_1 minimization

$\min_{x \in \mathbb{R}^n} \|x\|_1 = \min_{x \in \mathbb{R}^n} [\sum_{i=1}^m \max\{x_i, -x_i\}] \quad$ subj. to: $Fx \le g$
$\Leftrightarrow \min_{x, t} \sum_{i=1}^m t_i \quad$ subj. to: $x_i \le t, -x_i \le t \quad \Leftrightarrow \quad \min_{x, t} \mathbf{1}^T t \quad$ subj. to: $-t \le x \le t$ & $Fx \le g$

Construction of LP with substitution (∞ – Norm) ($s = (m+1)N + N + 1$)

1) **Cost**: $\min_z \varepsilon_N^x + \sum_{i=0}^{N-1} \varepsilon_i^x + \varepsilon_i^u$ subj. to: $\pm Q x_i \le \mathbf{1}\varepsilon_i^x, \pm P x_N \le \mathbf{1}\varepsilon_N^x, \pm R x_i \le \mathbf{1}\varepsilon_i^u$
2) **Substitution**: $x_i = S_i^x x(k) + S_i^u U \to -\mathbf{1}\varepsilon_i^x \le \pm Q x_i, -\mathbf{1}\varepsilon_N^x \le \pm P x_N, -\mathbf{1}\varepsilon_i^u \le \pm R u_i$
   $\Longrightarrow \min_z c^T z \quad$ subj. to: $\bar{G} z \le \bar{w} + \bar{S} x_k, \bar{G} = \begin{bmatrix} G_\varepsilon & G_U \\ 0 & G \end{bmatrix}, \bar{S} = \begin{bmatrix} S_\varepsilon \\ S \end{bmatrix}, \bar{w} = \begin{bmatrix} w_\varepsilon \\ w \end{bmatrix}$
$z := \{\varepsilon_0^x, \dots, \varepsilon_N^x, \varepsilon_0^u, \dots, \varepsilon_{N-1}^u, u_0^T, \dots, u_{N-1}^T\} \in \mathbb{R}^s \to$ for x_k, U^* obtained via LP solver

1-/∞-Norm State Feedback Solution

$\min_z c^T z \quad$ subj. to: $\bar{G} z \le \bar{w} + \bar{S} x_k$ is a multiparam. lin. program (**mp-LP**)

- First component of solution: $u_0^* = \kappa(x_0)$ $\forall x_0 \in \mathcal{X}_0$. ***Else, the same as mp-QP.***
- The value function $J^*(x_0)$ is convex and piecewise linear on polyhedra.

Quadratic vs. 1-/∞-Norm Cost

Let n = # optimization variables, FS= feasible set. Solution is either

Quadratic cost	Linear cost
• unique & in interior of FS	• unbounded
(no constraints active)	• unique at vertex of FS
• unique and on boundary of FS	(at least n active constraints)
(at least 1 active constraint)	• multi optima (a.l. 1 active constr.)

5) Invariance

Invariance: Region s.th. autonomous system satisfies constraints for all time
Controlled Invariance: Region, s.th. $\exists$ controller satisfying constraints $\forall$ t

Invariance

Constraint satisfaction for an autonomous system $x_{k+1} = g(x_k)$ or closed loop system $x_{k+1} = g(x_k, \kappa(x_k))$.

Positive Invariant Set: a set $\mathcal{O}$ is said to be a positive invariant set for the autonomous system if $x_k \in \mathcal{O} \Longrightarrow x_{k+1} \in \mathcal{O}$ $\forall k \in \{0, 1, \dots\}$

If the invariant set is within the constraints, it provides a set of initial states from which the trajectory will never violate the system constraints.

Maximal Positive Invariant Set: the set $\mathcal{O}_\infty \subset \mathcal{X}$ is the max. invariant set w.r.t. $\mathcal{X}$, $\mathcal{O}_\infty$ is invariant and $\mathcal{O}_\infty$ contains all invariant sets.

The maximal invariant set is the set of all states for which the system will remain feasible if it starts in $\mathcal{O}_\infty$.

Pre-Set: given a set $\mathcal{S}$ & the system $x_{k+1} = g(x_k)$, the pre-set of $\mathcal{S}$ is the set of states that evolve into the target set $\mathcal{S}$ in *one time* step: $\mathrm{pre}(\mathcal{S}) := \{x | g(x) \in \mathcal{S}\}$.
If $\mathcal{S} := \{x | Fx \le f\}$, then $\mathrm{pre}(\mathcal{S}) = \{x | FAx \le f\}$ for $x_{k+1} = A x_k$ (I)

Invariant Set Conditions

- A set $\mathcal{O}$ is a positive invariant set if and only if $\mathcal{O} \subseteq \mathrm{pre}(\mathcal{O})$
- **Necessary**: if $\mathcal{O} \nsubseteq \mathrm{pre}(\mathcal{O})$, then $\exists \bar{x} \in \mathcal{O}$ s.th. $\bar{x} \notin \mathrm{pre}(\mathcal{O})$. From the definition of $\mathrm{pre}(\mathcal{O})$, $g(\bar{x}) \notin \mathcal{O}$ and thus $\mathcal{O}$ is **not** a positive invariant set.
- **Sufficient**: if $\mathcal{O}$ is not a positive invariant set, then $\exists \bar{x} \in \mathcal{O}$ s.th. $g(\bar{x}) \notin \mathcal{O}$. This implies that $\bar{x} \in \mathcal{O}$ and $\bar{x} \notin \mathrm{pre}(\mathcal{O})$ and thus $\mathcal{O} \nsubseteq \mathrm{pre}(\mathcal{O})$.

Computing Invariant Sets – Algorithm

$\Omega_0 \leftarrow \mathcal{X}, \quad$ **loop**$\{\Omega_{i+1} \leftarrow \mathrm{pre}(\Omega_i) \cap \Omega_i, \quad$ ***if*** $\Omega_{i+1} = \Omega_i$: **return** $\Omega_\infty = \Omega_i\}$
Intersection for (I): $\mathrm{pre}(\Omega_i) \cap \Omega_i = \left\{x \middle| \begin{bmatrix} F \\ FA \end{bmatrix} x \le \begin{bmatrix} f \\ f \end{bmatrix}\right\}$
Algorithm generates $\Omega_i \to$ maximal positive invariant set $\mathcal{O}_\infty$ for $x_{k+1} = g(x_k)$.

Controlled Invariance

Controlled Invariant Set: a set $\mathcal{C} \subseteq \mathcal{X}$ is said to be a controlled invariant set if $x_k \in \mathcal{C} \Longrightarrow \exists u_k \in \mathcal{U}$ $s.th.$ $g(x_k, u_k) \in \mathcal{C}$ $\quad \forall k \in \mathbb{N}^+$

Defines the states for which there exists a controller that satisfies constraints $\forall t$

Maximal Control Invariant Set: the set $\mathcal{C}_\infty$ is said to be the max. control invariant set for $x_{k+1} = g(x_k, \kappa(x_k))$ subj. to the constraints $(x, u) \in \mathcal{X} \times \mathcal{U}$ if it is control invariant and contains all control invariant sets contained in $\mathcal{X}$.

For all states contained in the maximal control invariant set $\mathcal{C}_\infty$ there exists a control law, such that the system constraints are never violated.

Pre-Set: $\quad \mathrm{pre}(\mathcal{S}) := \{x | \exists u \in \mathcal{U} \text{ s.th. } g(x, u) \in \mathcal{S}\}$

Computing Control Invariant Sets – Algorithm

- A set $\mathcal{C}$ is a control invariant set if and only if $\mathcal{C} \subseteq \mathrm{pre}(\mathcal{C})$
- Since the ***same*** geometric condition hold as for Invariance, ***the same conceptual algorithm*** can be used to compute the control invariant sets.

Pre-Set Computation: Controlled System

Consider the system $x_{k+1} = A x_k + B u_k$ with constraints $u_k \in \mathcal{U} := \{u | Gu \le g\}$ and the set $\mathcal{S} := \{x | Fx \le f\}$. $\quad \mathrm{pre}(\mathcal{S}) = \{x | \exists u \in \mathcal{U}, Ax + Bu \in \mathcal{S}\} = \dots$
$\dots = \{x | \exists u \in \mathcal{U}, FAx + FBu \le f\} = \left\{x \middle| \exists u, \begin{bmatrix} FA & FB \\ 0 & G \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} \le \begin{bmatrix} f \\ g \end{bmatrix}\right\} \Rightarrow$ Projection

Control Law

Let $\mathcal{C}$ be the control invariant set for the system $x_{k+1} = g(x_k, u_k)$. A control law $\kappa(x_k)$ will guarantee that the system $x_{k+1} = g(x_k, \kappa(x_k))$ will satisfy the constraints for all time if: $g(x, \kappa(x)) \in \mathcal{C}$ $\forall x \in \mathcal{C}$.
With f as any function including $f(x, u) = 0$, we can find a control law:
$$\kappa(x) := \mathrm{argmin}\{f(x, u) | g(x, u) \in \mathcal{C}\}$$

- This doesn't ensure that system will converge, but **it will satisfy constraints**.
- By only ensuring to be in $\mathcal{C}$ for **one-time** step ahead, we get feasibility for infinite time since we can find optimal input satisfying constraints in the future.
- **MPC implicitly** describes a $\mathcal{C}$ that is easy to represent & compute. It turns an invariant set into a control invariant set with tractable computation.
- **Polyhedral** invar. sets represent maximum invar. set but can be complex, whereas **Ellipsoidal** invar. sets are smaller and easier but not the max. inv. set.

Computation of Ellipsoidal Invariant Set from Lyapunov

If $V: \mathbb{R}^n \to \mathbb{R}$ is a Lyapunov function for the system $x_{k+1} = g(x_k)$, then $Y := \{x | V(x) \le \alpha\}$ is an invariant set for all $\alpha \ge 0$.

- Invariance is ensured by the property **1)** $V(x) > 0 \; \forall x$, **2)** $V(g(x)) - V(x) < 0$ because once $V(x_k) \le \alpha \Rightarrow V(x_j) < \alpha$ for all $j \ge k$.
- Given constraints by $\mathcal{X}$ and a $P \succ 0$ s.th. $A^T P A - P \prec 0$, then with Lyapunov Function $V(x_k) = x_k^T P x_k$ maximize α s.th. invariant set Y_α is contained in $\mathcal{X}$:

$$Y_\alpha := \{x | x^T P x \le \alpha\} \subset \mathcal{X} := \{x | F x \le f\}$$
$$\Leftrightarrow \max_\alpha \; \alpha \quad \text{subj. to } h_{Y_\alpha}(F_i^T) \le f_i \quad \forall i \in \{1, \dots, n\}$$

Maximum Ellipsoidal Invariant Sets

- Support Function / Containment: $h_{Y_\alpha}(F_i^T) = \max_x F_i x$ subj. to $x^T P x \le \alpha$, direction F_i^T
- Transform to Ball: $y = P^{\frac{1}{2}} x \quad \Rightarrow \quad h_{Y_\alpha}(F_i^T) = \max_y F_i P^{-\frac{1}{2}} y$ subj. to $y^T y \le \alpha$, direction $P^{-\frac{1}{2}} F_i^T$
- Maximizer: $y^* = \frac{P^{-\frac{1}{2}} F_i^T}{\left\| P^{-\frac{1}{2}} F_i^T \right\|} \sqrt{\alpha} \quad \Rightarrow \quad h_{Y_\alpha}(F_i^T) = \left\| P^{-\frac{1}{2}} F_i^T \right\| \sqrt{\alpha}$
- **Largest Ellipse:** $\alpha^* = \max_\alpha \alpha$ subj. to $h_{Y_\alpha}(F_i^T)^2 \le f_i^2 \;\Leftrightarrow\; \alpha^* = \min_{i \in \{1, \dots, n\}} \frac{f_i^2}{F_i P^{-1} F_i^T}$

6) Feasibility and Stability

LQR	MPC
$J_\infty^*(x(k)) = \min \sum_{i=0}^{\infty} x_i^T Q x_i + u_i^T R u_i$	$J^*(x(k)) = \min_U \sum_{i=0}^{N-1} x_i^T Q x_i + u_i^T R u_i$
subj. to $x_{i+1} = A x_i + B u_i, x_0 = x(k)$	subj. to $x_{i+1} = A x_i + B u_i, x_0 = x(k)$
	$x_i \in \mathcal{X}, u_i \in \mathcal{U}$

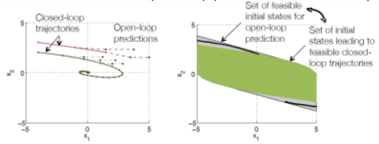
Assuming: $Q = Q^T \succcurlyeq 0, R = R^T \succ 0$

Loss of Feasibility & Stability in MPC

- **Decrease in prediction horizon N causes loss of stability!**
- Standard MPC does not guarantee feasibility (solution existence) nor stability
- Initial conditions determine if closed-loop trajectory leads to feasible states
- Investigate the stability properties for different horizons N and weights R

Unstable System: increasing $N \Rightarrow$ more constraints s.th. feasible set shrinks.
Stable System: increasing $N \Rightarrow$ feasible set grows.

- Finite N causes deviation btw. open-loop prediction & closed-loop system:



- Set of initial states for **open-loop is not invariant**: it may lead to infeasibility
- Set of initial states for **closed-loop is invariant** under MPC controller because there is always a feasible solution.
- **Green** set is a subset of open-loop set and already a controlled invariant set since there exists a controller for all states. This set is for a particular controller and is **not** the **max**. contr. inv. set.
- To get max contr. inv. set, would have to take $N \to \infty$ of open-loop set. This will shrink feasible set (e.g. the black lines in between for unstable system).

Infinite-Horizon

If we solve the RHC problem for $\boldsymbol{N = \infty}$ (as done for LQR), then open-loop trajectories are the **same** as the closed-loop trajectories. Hence:

- If problem is feasible, the closed-loop trajectories will be always feasible.
- If cost is finite, then states & inputs will converge asymptotically to the origin.

Finite-Horizon

"Short-sighted" approx. strategy of RHC for infinite horizon controller. But:

- **Feasibility**: After some steps the finite horizon optimal control problem may become infeasible (occurs without disturbances and model mismatch!)
- **Stability**: Generated control inputs may not lead to trajectories that converge to the origin.

Solution

Introduce terminal cost and constraints to explicitly ensure feasibility & stability:

$$J^*(x(k)) = \min_U I_f(x_N) + \sum_{i=0}^{N-1} I(x_i, u_i) \quad \Big| \quad x_i \in \mathcal{X}, u_i \in \mathcal{U}, i = 0, \dots, N-1$$
$$\text{subj. to } x_{i+1} = A x_i + B u_i, i = 0, \dots, N-1 \quad \Big| \quad x_N \in \mathcal{X}_f, \quad x_0 = x(k) \qquad \text{(II)}$$

I_f Terminal Cost, $\mathcal{X}_f$ Terminal Constraint: are chosen to mimic infinite horizon. It holds that $I(x_i, u_i) > 0$ for $x_i, u_i \ne 0$ and $I(0,0) = 0$.

$\Rightarrow$ **Note: Feasible set without terminal constraint is not invariant!**

Feasibility & Stability Guarantees in MPC

Main steps for proof: prove **recursive feasibility** by showing existence of feasible control sequence for all time when starting from a feasible initial point & prove **stability** by showing that optimal cost is a **Lyapunov function**.

Terminal Constraint at zero $x_N \in \mathcal{X}_f = 0$

Recursive Feasibility: Assume feasibility of $x(k)$ and $\mathcal{U}^* = \{u_0^*, \dots, u_{N-1}^*\}$ the optimal control sequence & corresponding state trajectory $\mathcal{X}^* = \{x(k), x_1^*, \dots, x_N^*\}$. Apply $u(k) = u_0^*$ and let system evolve to $x(k+1) = Ax(k) + Bu(k)$.

At $x(k+1) = x_1^*$, the control sequence $\tilde{\mathcal{U}} = \left\{ \underbrace{u_1^*, \dots, u_{N-1}^*}_{\in \mathcal{U}^*}, 0 \right\}$ is feasible:

apply 0 input: $A \underbrace{x_N^*}_{0} + B * 0 = 0 \Leftrightarrow \tilde{\mathcal{X}} = \{x_1^*, .., x_N^*, 0\} \Rightarrow$ not optimal, but **feasible**

Stability: take *shifted* candidate solution for $\tilde{J}$ which has *same* tail as optimal solution one time-step before for J^*. By optimizing, J^* can only be better or equal as $\tilde{J}$. We extend $\tilde{J}(x(k+1))$ with 0 step cost to compare with $J^*(x(k))$:

$$\tilde{J}(x(k+1)) = \sum_{i=1}^{N-1} I(x_i^*, u_i^*) + \underbrace{I(x_N^*, 0)}_{=0} + \{I(x_0^*, u_0^*) - I(x_0^*, u_0^*)\} = J^*(x(k)) - \underbrace{I(x_0^*, u_0^*)}_{>0}$$

$\Rightarrow J^*(x(k+1)) \le \tilde{J}(x(k+1)) < J^*(x(k)) \Rightarrow J^*(x)$ is a **Lyapunov Function** $\Rightarrow$ **Stability**

$\Rightarrow$ **Disadvantage of $\mathcal{X}_f = 0$: need large N to approximate max. cont. inv. set**

General Terminal Set

The Terminal Constraint $x_N = 0$ reduces the size of feasible set. Use convex set $\mathcal{X}_f$ to increase Region of Attraction. **Assumptions for Stability of MPC:** (III)

1) Stage cost is positive definite, i.e. it is strictly positive and only zero at the origin
2) Terminal set is invariant under the local control law $\kappa_f(x_i)$:
$x_{i+1} = Ax_i + B\kappa_f(x_i) \in \mathcal{X}_f \quad \forall x_i \in \mathcal{X}_f$ and all state & input constraints are satisfied in $\mathcal{X}_f$: $\mathcal{X}_f \subseteq \mathcal{X}, \kappa_f(x_i) \in \mathcal{U} \; \forall x_i \in \mathcal{X}_f$.
3) Terminal cost is a continuous Lyapunov function in the terminal set $\mathcal{X}_f$ and satisfies: $I_f(x_{i+1}) - I_f(x_i) \le -I(x_i, \kappa_f(x_i)) \quad \forall x_i \in \mathcal{X}_f$

Under these 3 Assumptions:

The closed-loop system under the MPC control law u_0^* is asymptotically stable and feas. set $\mathcal{X}_N$ is positive invariant for the system: $x(k+1) = Ax(k) + Bu_0^*(x(k))$.

Recursive Feasibility: Assume feasibility of $x(k)$ with optimal control $\mathcal{U}^*$. At $x(k+1) = x_1^*$, the control sequence $\tilde{\mathcal{U}} = \{u_1^*, \dots, u_{N-1}^*, \kappa_f(x_N^*)\}$ with trajectory $\tilde{\mathcal{X}} = \{x_1^*, \dots, x_N^*, Ax_N^* + B\kappa_f(x_N^*)\}$ is feasible: x_N^* is in $\mathcal{X}_f \to \kappa_f(x_N^*) \in \mathcal{U}$ is feasible (by Ass. 2) and $Ax_N^* + B\kappa_f(x_N^*)$ in $\mathcal{X}_f$ (by invariance in Ass. 2) $\Rightarrow \mathcal{X}_f$ **provides rec. feasibility**

Stability: use shift sequence: $\tilde{\mathcal{U}} = \{u_1^*, \dots, u_{N-1}^*, \kappa_f(x_N^*)\}$ & $\tilde{\mathcal{X}} = \{x_1^*, \dots, x_N^*, Ax_N^* + B\kappa_f(x_N^*)\}$
$J^*(x(k)) = \sum_{i=0}^{N-1} I(x_i^*, u_i^*) + I_f(x_N^*) \quad \Rightarrow \quad J^*(x(k+1)) \le \tilde{J}(x(k+1)) = \sum_{i=0}^{N-1} I(\tilde{x}_i, \tilde{u}_i) + I_f(\tilde{x}_{N+1}) = \sum_{i=1}^{N-1} I(x_i^*, u_i^*) + I(x_N^*, \kappa_f(x_N^*)) + I_f(Ax_N^* + B\kappa_f(x_N^*)) + \{I(x_0^*, u_0^*) - I(x_0^*, u_0^*)\} + I_f(x_N^*) - I_f(x_N^*)\} \le J^*(x(k)) - \underbrace{I(x_0^*, u_0^*)}_{>0}$ (by Ass. 3: $\sum I_{(\cdot)} \le 0$, leave out)

$\Rightarrow \; J^*(x(k+1)) \le \tilde{J}(x(k+1)) < J^*(x(k)) \;\;\&\;\; J^*(x(k+1)) - J^*(x(k)) \le -I(x_k, u_k)$

$\Rightarrow$ **$J^*(x)$ is Lyapunov, cl-system under MPC control law is asymptotically stable**

Proof: Bound of infinite horizon cost

From Ass. 3, take ∞-sum: $\sum_{j=i}^{\infty} I_f(x_{j+1}) - I_f(x_j) \le \sum_{j=i}^{\infty} -I(x_j, \kappa_f(x_j)) \Leftrightarrow$
$\underbrace{I_f(x_\infty)}_{\to 0} - I_f(x_i) \le \sum_{j=i}^{\infty} -I(x_j, \kappa_f(x_j)) \xrightarrow{I_f \text{ is Lyapunov}} \sum_{j=i}^{\infty} I(x_j, \kappa_f(x_j)) \le I_f(x_i)$

Choice of Terminal Set and Cost (LQR)

$$J^*(x(k)) = \min_U x_N^T P x_N + \sum_{i=0}^{N-1} x_i^T Q x_i + u_i^T R u_i \quad \text{subj. to (II)}$$

- Design unconstrained LQR control law $F_\infty = -(B^T P_\infty B + R)^{-1} B^T P_\infty A$
- P_∞ is solution of discrete ARE $\to$ choose $P = P_\infty$ as terminal weight
- Choose Terminal Set $\mathcal{X}_f$ to be the maximum invariant set for closed-loop system $x_{k+1} = (A + BF_\infty) x_k \in \mathcal{X}_f \quad \forall x_k \in \mathcal{X}_f \Rightarrow$ **Ellipsoidal** inv. set with Lyapunov
- all state & input constraints are satisfied in $\mathcal{X}_f$: $\mathcal{X}_f \subseteq \mathcal{X}, F_\infty x_k \in \mathcal{U} \quad \forall x_k \in \mathcal{X}_f$

This leads to:

1) Stage Cost is a positive definite function.
2) By construction, $\mathcal{X}_f$ is invariant under local control law $\kappa_f(x) = F_\infty x$
3) The Terminal Cost is a continuous Lyapunov Function in the Terminal Set $\mathcal{X}_f$ and satisfies: $x_{k+1}^T P x_{k+1} - x_k^T P x_k = \dots = -x_k^T(Q + F_\infty^T R F_\infty) x_N$

$\Longrightarrow$ **All Assumptions of Feasibility and Stability Theorem are verified.**

Notes

- Terminal constraint provides a ***sufficient condition*** for feasibility and stability
- Region of attraction without term. constraint may be larger than with ter. const
- In practice: enlarge horizon and check stability by sampling. With larger horizon N, region of attraction **approaches maximum control invariant set**.
- Closed-loop trajectories may not follow Assumptions made for open-loop pred.
- infinite horizon LQR controller is locally optimal so for quadratic cost it's the best choice we can take.

$\Longrightarrow$ **Infinite-horizon provides stability and invariance. Finite-horizon MPC may not be stable & may not satisfy constraints for all time!**

7) Practical Issues

Practical Issues

- **Tracking** of non-zero output set points $\to$ want to use MPC for tracking
- **Disturbance** / offset rejection s.th. system converges to desired set point
- **Constraints** restrict set of feasibility $\to$ want feasible set to be as large as possible

Reference Tracking

Consider linear model $x_{k+1} = Ax_k + Bu_k$, where $x \in \mathbb{R}^{n_x}, u \in \mathbb{R}^{n_u}$ and the Constraint set $\mathcal{X} = \{x | G_x x \le h_x\}, \mathcal{U} = \{x | G_u u \le h_u\}$.

$\Rightarrow$ **Goal: Track given reference $\boldsymbol{r}$ s.th. $\boldsymbol{z_k = Hx_k} \to \boldsymbol{r} \in \mathbb{R}^{n_r}$ as $\boldsymbol{k} \to \infty$.**

Steady-State Target Problem

Reference r is achieved by target state x_s if $z_s = Hx_s = r$, where x_s should be a steady-state s.th. there exists an input to keep system at target $x_s = Ax_s + Bu_s$

Target Condition: $\begin{matrix} x_s = Ax_s + Bu_s \\ Hx_s = r \end{matrix} \;\Leftrightarrow\; \underbrace{\begin{bmatrix} I - A & -B \\ H & 0 \end{bmatrix}}_{(n_x + n_r) \times (n_x + n_u)} \begin{bmatrix} x_s \\ u_s \end{bmatrix} = \begin{bmatrix} 0 \\ r \end{bmatrix}$ (IV)

- In case of multiple feasible u_s, compute cheapest steady-state (x_s, u_s) corresponding to r: $\quad \boldsymbol{\min u_s^T R_s u_s}$ subj. to **Target condition**, $x_s \in \mathcal{X}, u_s \in \mathcal{U}$
- If no solution exists, compute reachable set point closest to r:
$\quad \boldsymbol{\min (Hx_s - r)^T Q_s (Hx_s - r)}$ subj. to $\; x_s = Ax_s + Bu_s, \;\; x_s \in \mathcal{X}, u_s \in \mathcal{U}$

MPC for Reference Tracking

MPC Formulation

$$\min \|z_N - Hx_s\|_{P_z}^2 + \sum_{i=0}^{N-1} \underbrace{\|z_i - Hx_s\|_{Q_z}^2}_{\|Hx_i - Hx_s\|_{Q_z}^2 = \|x_i - x_s\|_{H^T Q H}^2} + \|u_i - u_s\|_R^2$$

$\Rightarrow$Used to set system to desired steady-state (x_s, u_s) yielding the output $z_k \to r$

Delta Formulation

- Set point tracking as regulation problem with same model equations:
$$\Delta x = x - x_s, G_x \Delta x \le h_x - G_x x_s, \qquad \Delta u = u - u_s, G_u \Delta u \le h_u - G_u u_s$$
- Obtain Target steady-state with Target cond. (IV), init. state: $\Delta x_k = x(k) - x_s$ apply regulation problem, find optimal ΔU^*, system input $u_0^* = \Delta u_0^* + u_s$:

$\boldsymbol{\min \sum_{i=0}^{N-1} \Delta x_i^T Q \Delta x_i + \Delta u_i^T R \Delta u_i + V_f(\Delta x_N)}$
s. th. $\Delta x_0 = \Delta x_k, \;\; \Delta x_{i+1} = A \Delta x_i + B \Delta u_i$
$G_x \Delta x \le h_x - G_x x_s, G_u \Delta u \le h_u - G_u u_s, \Delta x_N \in \mathcal{X}_f$

Convergence

- Assume target is feasible with $x_s \in \mathcal{X}, u_s \in \mathcal{U}$, chose terminal weight $V_f(x)$ and constraint $\mathcal{X}_f$ as in regulation case satisfying:
$\mathcal{X}_f \subseteq \mathcal{X}, Kx \in \mathcal{U} \; \forall x \in \mathcal{X}_f \quad \& \quad V_f(x_{k+1}) - V_f(x_k) \le -I(x_k, Kx_k) \; \forall x \in \mathcal{X}_f$
- If (x_s, u_s) satisfies $\boldsymbol{x_s \oplus \mathcal{X}_f \subseteq \mathcal{X}}$, $\boldsymbol{K\Delta x + u_s \in \mathcal{U} \;\forall\; \Delta x \in \mathcal{X}_f}$ then closed-loop system converges to target reference $x_k \to x_s$ and $z_k = Hx_k \to r$ for $k \to \infty$.

Proof: Invariance inherited by regulation case. Constraints: $x_s \oplus \mathcal{X}_f \subseteq \mathcal{X}$ ...
... $\to x \in \mathcal{X} \; \forall \; \Delta x = x - x_s \in \mathcal{X}_f$ and $K\Delta x + u_s \in \mathcal{U} \; \forall \; \Delta x \in \mathcal{X}_f \to u \in \mathcal{U}$.

Terminal Set

- Set of feasible targets may be significantly reduced. Enlarge set of feasible targets by scaling (shrinking) terminal set: $\boldsymbol{\mathcal{X}_f^{scaled} = \alpha \mathcal{X}_f = \{x | G_f x \le \alpha h_f\}}$.
- If $\mathcal{X}_f$ is invariant, then $\alpha \mathcal{X}_f$ is also. Chose α s.th. constraints are still satisfied.
- For targets at the **boundary of the constraints**: $\boldsymbol{x_N = x_s}$, which corresponds to a zero terminal set $\mathcal{X}_f = 0$ because $\Delta x = x - x_s, \; x_N = x_s \;\Rightarrow\; \Delta x = 0$.

$\Rightarrow$ If cl-loop system stable, set point achieved. Stability proof as for regulation.

MPC for Reference Tracking without Offset

- Constant disturbance acts on system causing system trajectory to deviate
- **Objective**: If system stabilized with disturbance, it converges without offset
- **Approach**: Model disturbance and estimate state to remove offset.

$\Rightarrow$ **Goal: Track reference $\boldsymbol{r}$ with measured output $\boldsymbol{y_k}$: $\boldsymbol{z_k = Hy_k \to r}$ as $\boldsymbol{k} \to \infty$**

Augmented Model (model the disturbance)

$$x_{k+1} = Ax_k + Bu_k + B_d d_k, \qquad d_{k+1} = d_k, \qquad y_k = Cx_k + C_d d_k$$

- Assume integral disturbance dynamics, $d \in \mathbb{R}^{n_d}$, Restriction on choice of B_d, C_d: Observability $\Rightarrow$ Aug. model observable if and only if (A, C) is observable and
$$\text{rank}\left(\begin{bmatrix} A - I & B_d \\ C & C_d \end{bmatrix} \right) = n_x + n_d = \textbf{full} \;\Longrightarrow\; \text{max. Dim. of disturbance: } n_d \le n_y$$
- **Intuition**: at st.-st. $\begin{bmatrix} A - I & B_d \\ C & C_d \end{bmatrix} \begin{bmatrix} x_s \\ d_s \end{bmatrix} = \begin{bmatrix} 0 \\ y_s \end{bmatrix}$ & given y_s: d_s must be uniq. defined

Linear State Estimation

- **State Observer** for augm. model, where $\hat{x}, \hat{d}$ are estimates, y measured output
$$\begin{bmatrix} \hat{x}_{k+1} \\ \hat{d}_{k+1} \end{bmatrix} = \begin{bmatrix} A & B_d \\ 0 & I \end{bmatrix} \begin{bmatrix} \hat{x}_k \\ \hat{d}_k \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u_k + \begin{bmatrix} L_x \\ L_d \end{bmatrix} (-y_k + C\hat{x}_k + C_d \hat{d}_k)$$
- **Error dynamics** where $L = \begin{bmatrix} L_x \\ L_d \end{bmatrix}$ to chose s.th. estimator is asympt. stable
$$\begin{bmatrix} x_{k+1} - \hat{x}_{k+1} \\ d_{k+1} - \hat{d}_{k+1} \end{bmatrix} = \left(\begin{bmatrix} A & B_d \\ 0 & I \end{bmatrix} + \begin{bmatrix} L_x \\ L_d \end{bmatrix} \begin{bmatrix} C & C_d \end{bmatrix} \right) \begin{bmatrix} x_k - \hat{x}_k \\ d_k - \hat{d}_k \end{bmatrix}$$

Steady-State Conditions

Steady-State for Observer

Suppose observer is asym. stable & number of outputs n_y = dimension of const. disturbance n_d. The Observer Steady-State satisfies:

Observer Steady-State: $\begin{bmatrix} A - I & B \\ C & 0 \end{bmatrix} \begin{bmatrix} \hat{x}_\infty \\ u_\infty \end{bmatrix} = \begin{bmatrix} -B_d \hat{d}_\infty \\ y_\infty - C_d \hat{d}_\infty \end{bmatrix}$

- y_∞ & u_∞ are the steady-state measured outputs and inputs.

$\Rightarrow$ Observer output $C\hat{x}_\infty + C_d \hat{d}_\infty$ tracks measurement y_∞ without offset.

Steady-State for Tracking

$$x_s = Ax_s + Bu_s + B_d \hat{d}_\infty, \qquad z_s = H(Cx_s + C_d \hat{d}_\infty) = r$$

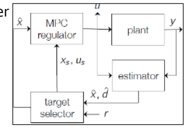
- Best forecast for steady-state disturbance is **current estimate**: $\hat{d}_\infty = \hat{d}$
- Target is modified to account for effect of disturbance on tracked variables
- Same procedure for regulation case with $r = 0$

Offset-Free Target Condition: $\begin{bmatrix} A - I & B \\ HC & 0 \end{bmatrix} \begin{bmatrix} x_s \\ u_s \end{bmatrix} = \begin{bmatrix} -B_d \hat{d} \\ r - HC_d \hat{d} \end{bmatrix}$ (V)

Offset-free Tracking Procedure

1) Estimate state & disturbance $\hat{x}, \hat{d}$ with State Observer
2) Obtain (x_s, u_s) from steady-state target (V)
3) Solve MPC problem for tracking with $\Delta x_i = x_i - x_s$:

$\boldsymbol{\min \sum_{i=0}^{N-1} \Delta x_i^T Q \Delta x_i + \Delta u_i^T R \Delta u_i + V_f(\Delta x_N)}$ (VI)
s. th. $x_0 = \hat{x}_k, d_0 = \hat{d}_k, x_{i+1} = Ax_i + Bu_i + B_d d_i$
$d_{i+1} = d_i, x_i \in \mathcal{X}, u_i \in \mathcal{U}, \Delta x_N \in \mathcal{X}_f$



Offset-free Tracking Main: Control law $u_0^* = \kappa(\hat{x}_k, \hat{d}_k, r)$. Consider the case $n_d = n_y$. Assume (VI) is feasible, no active constr. at st.-st. & cl-loop system

$$x(k+1) = Ax(k) + B\kappa(\hat{x}(k), \hat{d}(k), r) + B_d d$$
$$\hat{x}(k+1) = (A + L_x C)\hat{x}(k) + (B_d + L_x C_d)\hat{d}(k)$$
$$+ B\kappa(\hat{x}(k), \hat{d}(k), r) - L_x y(k)$$
$$\hat{d}(k+1) = L_d C\hat{x}(k) + (I + L_d C_d)\hat{d}(k) - L_d y(k)$$

converges, i.e. $\hat{x}_k \to \hat{x}_\infty, \hat{d}_k \to \hat{d}_\infty, y_k \to y_\infty$ as $k \to \infty$. **Then $\boldsymbol{z_k = Hy_k \to r}$**

Enlarging Feasible Set

MPC without Terminal Set

Considering MPC Stability Assumptions (III). **Terminal constraint reduces feasible set.** For unstable systems, it adds state constraints to problems with only input constraints. But feasible set without terminal constraint is not invariant.

$\Rightarrow$ **Goal:** MPC without terminal constraint but with guaranteed stability.

Terminal constraint can be removed while maintaining stability s.th. terminal state satisfies terminal constraint without enforcing it in optimization **IF**:
1) initial state lies in sufficiently small subset of feasible set and **2)** N is suffic. large.

$\Rightarrow$ **Solution of finite horizon MPC corresponds to infinite horizon solution.**

- **Advantage**: Controller defined in a larger feasible set
- **Attraction Region**: larger w/o term. set, approach max. con. inv. set w/ larger N

Soft Constrained MPC

$\Rightarrow$ **Goal:** Minimize duration / size of violation $\Longrightarrow$ can be conflicting

- **Multi-Objective Problem:** Approximation of Pareto Optimality.
Choosing type: If product must be discarded during constraint violation $\to$ **time.**
If large constraint violations lead to process shutdown or exceptions $\to$ **size.**

Problem Setup

$$\min \sum_{i=0}^{N-1} x_i^T Q x_i + u_i^T R u_i + I_\epsilon(\epsilon_i) + x_N^T P x_N + I_\epsilon(\epsilon_N)$$
s. th. $x_{i+1} = Ax_i + Bu_i, H_x x_i \le k_x + \epsilon_i, H_u u_i \le k_u, \epsilon_i \ge 0 \in \mathbb{R}^p$ (**Slack Variables**)

- **Penalty Function quadratic**: $I_\epsilon(\epsilon_i) = \epsilon_i^T S \epsilon_i, \;\; S \succ 0, \;\;$ possible init: $S = Q$
- **Penalty Function quadratic & linear:** $I_\epsilon(\epsilon_i) = \epsilon_i^T S \epsilon_i + v \|\epsilon_i\|_{1/\infty}$

$\Rightarrow$ **Requirement**: If original problem has feasible solution z^*, the softened problem should have the same solution z^* and $\epsilon = 0$. *Counter-Example*: $I_\epsilon(\epsilon) = s * \epsilon^2$

Exact Penalty Function: $I_\epsilon(\epsilon) = v * \epsilon$ satisfies the requirement for any $v > \lambda^* \ge 0$, with λ^* the optimal Lagrange multiplier of original problem.

- In practice, combined cost is used for exact penalty: $I_\epsilon(\epsilon) = v * \epsilon + s * \epsilon^2, s > 0$
- Multi constraints: $g_j(z) \le 0, j \in [1, r]$: $I_\epsilon(\epsilon) = \epsilon^T S \epsilon + v \|\epsilon_i\|_{1/\infty}$,
$\epsilon = [\epsilon_1, \dots, \epsilon_r]^T$ with $S \succ 0, v > \|\lambda^*\|_D \triangleq$ dual of chosen norm: $\|\cdot\|_1 \Leftrightarrow \|\cdot\|_\infty$

Tuning of Penalty Function

- **Quadratic**: Increase S: hardening of soft constr.: reduced size & longer duration.
- **Linear**: Increase v results in increasing peak violations & decreasing duration.
  - If weight v is chosen large enough, constraints are satisfied if possible
  - Large linear penalties make tuning difficult and cause numerical problems

Separation of Objectives

1) $\epsilon^{min} = \underset{U, \epsilon}{\text{argmin}} \; \epsilon_i^T S \epsilon_i + v^T \epsilon_i$
s. th. $\; x_i = Ax_i + Bu_i, \; H_x x_i \le K_x + \epsilon_i$
$\quad H_u u_i \le K_u, \;\; \epsilon_i \ge 0$

2) $\min \sum_{i=0}^{N-1} x_i^T Q x_i + u_i^T R u_i + x_N^T P x_N$
s. th. $\; x_i = Ax_i + Bu_i, \; H_x x_i \le k_x + \epsilon_i^{min}$
$\quad H_u u_i \le K_u$

$\Longrightarrow$ Simplifies tuning, constraints will be satisfied if possible.

8) Robust MPC

Uncertainty Models

- Random noise w changes evolution of system, model structure and impact of const. parameter θ unknown. Uncertain System: $x_{k+1} = g(x_k, u_k, w_k; \theta)$
- **Uncertainty Model**: $g(x_k, u_k, w_k; \theta) = Ax_k + Bu_k + w_k$, A, B known, w changes

$\Rightarrow$ **Goal**: **design control law $u(k) = \kappa(x(k))$ to**: 1) satisfy constraints for all disturbance realizations, 2) be stable (converge to neighborhood of origin), 3) optimize performance, maximizes the set satisfying 1) – 3)

Impact of Bounded Additive Noise

Cannot predict how state will evolve, but can compute a set of trajectories that the system *may* follow $\Rightarrow$ **Idea**: design control law satisfying constraints and stabilizing the system for all possible disturbances.

Uncertain State Evolution

Define $\phi_i(x_0, U, W)$ as the **state** that the system will be in at time i, starting from x_0, applying $U := \{u_0, \dots, u_{N-1}\}$ & observing the disturbance $W := \{w_0, \dots, w_{N-1}\}$

$$\phi_i = \underbrace{A^i x_0 + \sum_{j=0}^{i-1} A^j B u_{i-1-j}}_{\text{nominal system}} + \underbrace{\sum_{j=0}^{i-1} A^j w_{i-1-j}}_{\text{offset by disturbance}} = x_i + \sum_{j=0}^{i-1} A^j w_{i-1-j}$$

Cost to Minimize (3. condition)

Cost is a function of the acting disturbance $J(\phi_i) = \sum_{j=0}^{N-1} I(\phi_i, u_i) + I_f(\phi_N)$

$\Rightarrow$ Need to eliminate dependence of W. Options for this:

- Minimize expected Value: $J_N(x_0, U) := E[J(\phi_i)]$
- Take the worst-case: $J_N(x_0, U) := \max_W J(\phi_i)$
- Take the nominal case: $J_N(x_0, U) := J(x_0, U, 0)$ **(→ considered)**

Robust Constraint Satisfaction (1. condition)

Break up the MPC prediction into **two parts**:

$$\left.\begin{aligned} \phi_{i+1} &= A\phi_i + Bu_i + w_i \\ u_i &\in \mathcal{U} \\ \phi_i &\in \mathcal{X}\ \forall W \in \mathcal{W}^N \end{aligned}\right\}$$

- $i = 0, \dots, N-1$
- Optimize over control actions $\{u_0, \dots, u_{N-1}\}$
- Enforce constraints explicitly by imposing $\phi_i \in \mathcal{X}$ and $u_i \in \mathcal{U}$ for all sequences W

$$\left.\begin{aligned} \phi_N &\in \mathcal{X}_f \\ \phi_{i+1} &= (A + BK)\phi_i + w_i \end{aligned}\right\}$$

- $i = N, \dots$
- Assume control law to be linear $u_i = K\phi_i$
- Enforce constraints implicitly by constraining ϕ_N to be in an *robust invariant set* $\mathcal{X}_f \subseteq \mathcal{X}$ and $K\mathcal{X}_f \subseteq \mathcal{U}$ for the system $\phi_{i+1} = (A + BK)\phi_i + w_i$

Part 1) Robustly enforcing constraints of Linear System ($i = N, \dots$)

Robust Invariant Set: for an autonom. system $x_{k+1} = g(x_k, w_k)$ or closed loop system $x_{k+1} = g(x_k, \kappa(x_k), w_k)$. A set $\mathcal{O}^{\mathcal{W}}$ is said to be a robust positive invariant set for the autonom. system if $x_k \in \mathcal{O}^{\mathcal{W}} \implies x_{k+1} \in \mathcal{O}^{\mathcal{W}}\ \forall w \in \mathcal{W}$

Robust Pre-Set: given a set Ω & the system $x_{k+1} = g(x_k, w_k)$, the **pre-set** of Ω is the set of states that evolve into the **target set** Ω in *one time step for all values of the disturbance* $w \in \mathcal{W}$: $\operatorname{pre}^{\mathcal{W}}(\Omega) := \{x | g(x, w) \in \Omega\ \forall\ w \in \mathcal{W}\}$

Computing Robust Pre-Set for Linear System

System $g(x, w) = Ax_k + w_k$ & set $\Omega := \{x | Fx \le f\}$, where h_w = support function

$$\operatorname{pre}^{\mathcal{W}}(\Omega) := \left\{x | FAx \le f - \max_w Fw\right\} = \{x | FAx \le f - h_w(F)\}$$

- A set $\mathcal{O}^{\mathcal{W}}$ is a robust positive invariant set if and only if $\mathcal{O}^{\mathcal{W}} \subseteq \operatorname{pre}^{\mathcal{W}}(\mathcal{O}^{\mathcal{W}})$.

$\Rightarrow$ **Algorithm for computing Robust Invariant Set is the same as for nominal case: $\operatorname{pre}(\Omega) \to \operatorname{pre}^{w}(\Omega)$**

Part 2) Robustly ensuring constraints of sequence ϕ_i ($i = 1, \dots, N-1$)

Compute set of **tighter constraints** s.th. if nominal system meets these constraints, then the uncertain system will too. MPC is then done on nominal system

- Ensure $\phi_i(x_0, U, W) = \{x_i + \sum_{j=0}^{i-1} A^j w_{i-1-j} | W \in \mathcal{W}^i\} \subseteq \mathcal{X}$ (VII)
- Assume $\mathcal{X} = \{x | Fx \le f\} \iff Fx_i + F\sum_{j=0}^{i-1} A^j w_{i-1-j} \le f\ \forall\ W \in \mathcal{W}^i$

$$\Rightarrow Fx_i \le f - \max_w F\sum_{j=0}^{i-1} A^j w_{i-1-j} = f - h_{\mathcal{W}^i}\left(F\sum_{j=0}^{i-1} A^j\right)$$

- Need to ensure that $\phi_N(x_0, U, W) \subseteq \mathcal{X}_f \implies$ in the same way as above

Set Operations

- **Minkowski Sum**: A, B subsets of $\mathbb{R}^n$, then $A \oplus B := \{x + y | x \in A, y \in B\}$
- **Pontryagin Difference:** A, B subsets of $\mathbb{R}^n$: $A \ominus B = \{x | x + e \in A\ \ \forall e \in B\}$

$\Rightarrow$ ***Note***: **$A \ominus B \oplus B \subseteq A$ and not $= A$ depending on shapes of the sets!!**

$\Rightarrow$ So (VII) can be written as $\phi_i(x_0, U, W) \in x_i \oplus (\mathcal{W} \oplus A\mathcal{W} \oplus \dots A^{i-1}\mathcal{W}) \subseteq \mathcal{X}$, to enforce this condition, we require **the tightened constraints**:

$$x_i \in \mathcal{X} \ominus (\mathcal{W} \oplus A\mathcal{W} \oplus \dots A^{i-1}\mathcal{W}) \Leftrightarrow x_i \in \mathcal{X} \ominus \left(\bigoplus_{j=0}^{i-1} A^j\mathcal{W}\right)$$

Robust Open-Loop MPC

$\min_U \sum_{i=0}^{N-1} I(x_i, u_i) + I_f(x_N)$, subj. to $x_i \in \mathcal{X} \ominus (\mathcal{W} \oplus A\mathcal{W} \oplus \dots A^{i-1}\mathcal{W})$
$u_i \in \mathcal{U}$, $x_N \in \mathcal{X}_f \ominus (\mathcal{W} \oplus A\mathcal{W} \oplus \dots A^{N-1}\mathcal{W})$, $\mathcal{X}_f$ a robust invariant set for the system $x_{k+1} = (A + BK)x_k$ for stabilizing K.

$\Rightarrow$ We do nominal MPC, but with tighter constraints on the states. It is sure that if the nominal system satisfies the tighter constraints, then the uncertain system will satisfy the real constraints.

$\Rightarrow$ Robust open-loop MPC may have very small Region of Attraction, in particular for unstable systems → introduce Feedback to enlarge.

Robust Invariance

If U^* is the optimizer of the robust open-loop MPC problem for $x_k \in \mathcal{X}_0$, then it holds: $Ax_k + Bu_0^*(x_k) + w_k \in \mathcal{X}_0$ for all $w \in \mathcal{W}$

$\Rightarrow$ This follows because the computed trajectory at the current time is feasible for ***any*** disturbance and therefore also feasible for the actual observed one.

Closed-Loop Predictions

Want to optimize over a sequence of functions $\{u_0, \mu_1, \dots, \mu_{N-1}\}$, where $\mu_i(x_i): \mathbb{R}^n \to \mathbb{R}^m$ is a **control policy** and maps the state at time i to an input at time i.

$\Rightarrow$ Since we can't optimize over arbitrary functions, need some structure on μ:

- **Pre-Stabilization** $\mu_i = Kx + v_i$: $A + BK$ stable, fixed K, simple, conservative
- **Linear Feedback** $\mu_i = K_i x_i + v_i$: optimize over K_i & v_i, non-convex, difficult
- **Disturbance Feedback** $\mu_i = \sum_{j=0}^{i-1} M_{i,j} w_j + v_i$: optimize over $M_{i,j}$ & v_i, convex
- **Tube-MPC** $\mu_i = K(x - \bar{x}_i) + v_i$: $A + BK$ stable, fixed K, optimize over $\bar{x}_i, v_i$

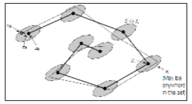
Tube-MPC

System $\boldsymbol{x_{k+1} = Ax_k + Bu_k + w_k}$. Separate available control authority in **2 parts:**
1) A portion that steers the noise-free system to the origin $z_{k+1} = Az_k + Bv_k$
2) A portion that compensates for deviations from this system: tracking controller to keep real trajectory close to the nominal $\boldsymbol{u_i = K(x_i - z_i) + v_i = Ke_i + v_i}$ for linear controller K which stabilizes nominal system.

$\Rightarrow$ fix controller K offline and optimize over nominal input trajectory $\{v_0, \dots, v_{N-1}\}$, which results in a convex problem.

Error Dynamics

- Define error $e_i = x_i - z_i$, resulting in:
  $e_{i+1} = (A + BK)e_i + w_i$ with Tube $\mu_i = u_i$
- Bound maximum error / how far real trajectory is from the nominal one:
  $e_{i+1} = (A + BK)e_i + w_i,\ \ w_i \in \mathcal{W}$



$\Rightarrow$ Dynamics $(A + BK)$ are stable, the set $\mathcal{W}$ is bounded, so there is some set $\mathcal{E}$ that e will stay inside for all time $\Rightarrow$ find **smallest $\mathcal{E}$** (minimal robust inv. set)

- Real trajectory stays near nominal one: $x_i \in z_i \oplus \mathcal{E}$. Must ensure that all possible state trajectories satisfy constraints: $z_i \oplus \mathcal{E} \subset \mathcal{X}$. Steps for ensuring this:

Tube-MPC Procedure

1) Computing set $\mathcal{E}$ that error will remain inside
2) Modify constraints on nominal trajectory $\{z_i\}$ s.th. $z_i \oplus \mathcal{E} \subset \mathcal{X}$ & $v_i \in \mathcal{U} \ominus K\mathcal{E}$
3) Formulate as convex optimization problem, prove constraints and stability

Part 1) Minimum Robust Invariant set $\mathcal{E}$ (mRPI)

- Robust Constraint Satisfaction: max. robust invariant set
- Now the **minimum robust invariant set (mRPI)** needed, in which states will remain inside despite the noise.
- For the system $x_{i+1} = Ax_i + w_i$ with $\boldsymbol{x_0 = 0}$, the state evolution is $x_i = \sum_{j=0}^{i-1} A^j w_{i-1-j}$

```
Input: A
Output: F_∞
  Ω_0 ← {0}
  loop
    Ω_{i+1} ← Ω_i ⊕ A^i W
    if Ω_{i+1} = Ω_i then
      return F_∞ = Ω_i
    end if
  end loop
```

- Set containing all possible states x_i is $F_i = \bigoplus_{j=0}^{i-1} A^j\mathcal{W}$, $F_0 = \{0\}$

$\Rightarrow$ **mRPI**: $\boldsymbol{F_\infty = \bigoplus_{j=0}^{\infty} A^j\mathcal{W}}$ & if $\exists\ n$ s.th. $F_n = F_{n+1}$ then $F_n = F_\infty$

Part 2) Modify Constraints: Tightening

Given nominal trajectory z_i, the error e_i will be in the set $\mathcal{E}$, so the noisy system trajectory $x_i = z_i + e_i$ can only be up to $\mathcal{E}$ far away from z_i:

$$x_i \in z_i \oplus \mathcal{E} = \{z_i + e | e \in \mathcal{E}\}$$

$\Rightarrow$ **Sufficient condition**: $z_i \in \mathcal{X} \ominus \mathcal{E} \implies x_i \in z_i \oplus \mathcal{E} \subseteq \mathcal{X}$
$\mathcal{E}$ is known offline and constraints $\mathcal{X} \ominus \mathcal{E}$ can also be computed offline

$\Rightarrow$ **Condition for input**: $v_i \in \mathcal{U} \ominus K\mathcal{E} \implies u_i \in K\mathcal{E} \oplus v_i \subset \mathcal{U}$

Tightening for Polytopic Constraints

- **State:** $Fx_i \le f \Leftrightarrow Fz_i + Fe_i \le f\ \forall\ e_i \in \mathcal{E} \Leftrightarrow Fz_i \le f - \max_{e_i \in \mathcal{E}} Fe_i$
- **Input:** $Hu_i \le h \Leftrightarrow Hv_i + HKe_i \le h\ \forall\ e_i \in \mathcal{E} \Leftrightarrow Hv_i \le h - \max_{e_i \in \mathcal{E}} HKe_i$

Part 3) Convex Optimization Formulation

Feasible set: $\mathcal{Z}(x_0) := \left\{ Z, V \left| \begin{array}{ll} z_{i+1} = Az_i + Bv_i & i \in [0,\ N-1] \\ z_i \in \mathcal{X} \ominus \mathcal{E} & i \in [0,\ N-1] \\ v_i \in \mathcal{U} \ominus K\mathcal{E} & i \in [0,\ N-1] \\ z_N \in \mathcal{X}_f & \\ x_0 \in z_0 \oplus \mathcal{E} & \end{array} \right. \right\}$

Cost function: $J(Z, V) := \sum_{i=0}^{N-1} l(z_i, v_i) + l_f(z_N)$

Optimization: $(V^*(x_0), Z^*(x_0)) = \operatorname{argmin}_{V,Z} \{J(Z, V) | (Z, V) \in \mathcal{Z}(x_0)\}$

Control law: $\mu_{\text{tube}}(x) := K(x - z_0^*(x)) + v_0^*(x)$

- Optimizing nominal system with tightened state & input constraints
- First tube centre is optimization variable → must be within $\mathcal{E}$ of x_0
- Cost w.r.t. tube centres (nom. system) & terminal set w.r.t. tightened constraint

$\Rightarrow$ **Note**: z_0 is also an optimization variable which allows to choose states within the tubes $\mathcal{E}$ as long as the real trajectory x_k is included. This gives a better cost and enlarges feasible set since only z_k is constrained to be within tightened constraints $\mathcal{X} \ominus \mathcal{E}$ s.th. x_k may also lie outside $\mathcal{X} \ominus \mathcal{E}$. Since we optimize z_0, $z_0^*(k)$ will not be linear anymore (will be replaned in every sampling instance).

Robust Constraint Satisfaction

Tube-MPC Assumptions

1) stage cost is positive definite function
2) Terminal set is invariant for the **nominal system** under local control law $\kappa_f(z)$:

$$Az + B\kappa_f(z) \in \mathcal{X}_f\ \forall\ z \in \mathcal{X}_f$$

All **tightened state & input constraints** are satisfied in $\mathcal{X}_f$:

$$\mathcal{X}_f \subseteq \mathcal{X} \ominus \mathcal{E},\ \kappa_f(z) \in \mathcal{U} \ominus K\mathcal{E}\ \forall\ z \in \mathcal{X}_f \iff K\mathcal{X}_f \subseteq \mathcal{U} \ominus K\mathcal{E}$$

4) Terminal cost is a continuous Lyapunov function in the terminal set $\mathcal{X}_f$:

$$I_f(Az + B\kappa_f(z)) - I_f(z) \le I(z, \kappa_f(z))\ \ \forall\ z \in \mathcal{X}_f \quad \textit{(as for General Terminal set)}$$

Proof Robust Invariance of Tube-MPC

Theorem: The set $\mathcal{Z} := \{x\ |\ \mathcal{Z}(x) \neq \emptyset\}$ is a robust invariant set of the system $x_k = Ax_k + B\mu_{tube}(x_k) + w_k$ subject to constraints $x, u\ \in \mathcal{X} \times \mathcal{U}$.

Proof: let $x_{k+1} = Ax_k + BK(x_k - z_0^*) + Bv_0^* + w_k$ the evolution after the first time-step. x_{k+1} may have many possible values but by construction, it is in the set $z_1 \oplus \mathcal{E}\ \forall\ \mathcal{W}$. Therefore the sequence $(\{v_1^*, \dots, v_{N-1}^*, \kappa_f(z_N^*)\}, \{z_1^*, \dots, z_N^*, Az_N^* + B\kappa_f(z_N^*)\})$ is feasible for all x_{k+1} *(as in nominal MPC)*.

Robust Stability of Closed-Loop System

Theorem: The state x_k of the system $x_{k+1} = Ax_k + B\mu_{tube}(x_k) + w_k$ converges in the limit to the set $\mathcal{E}$.

Poof: As in standard MPC it holds: $J^*(x(k)) = \sum_{i=0}^{N-1} I(z_i^*, v_i^*) + I_f(z_N^*)$

$$J^*(x(k+1)) \le \sum_{i=1}^{N} I(z_i^*, v_i^*) + I_f(z_{N+1}^*)$$

$$= \sum_{i=1}^{N-1} I(z_i^*, v_i^*) + I\left(z_N^*, \kappa_f(z_N^*)\right) + I_f\left(Az_N^* + B\kappa_f(z_N^*)\right)$$

$$= J^*(x(k)) - \underbrace{I(z_0^*, v_0^*)}_{\ge 0} \underbrace{- I_f(z_N^*) + I_f(z_{N+1}) + I\left(z_N^*, \kappa_f(z_N^*)\right)}_{\le 0\ (I_f \text{ is Lyapunov function in } \mathcal{X}_f)}$$

This shows that $\lim_{k\to\infty} I_f(z_0^*(x_k)) = 0$ and therefore $\lim_{k\to\infty} z_0^*(x_k) = 0$. BUT x_k does not tend to 0! It stays within robust inv. set centered at $z_0^*(x_k)$: $\lim_{k\to\infty} d(x_k, \mathcal{E}) = 0$.

Tube-MPC Implementation Procedure

OFFLINE

1) Choose stabilizing controller K s.th. $\|A + BK\| < 1$
2) Compute min. robust inv. set $\mathcal{E} = F_\infty$ for the system $x_{k+1} = (A + BK)x_k + w_k$
3) Compute tightened constraints $\tilde{\mathcal{X}} := \mathcal{X} \ominus \mathcal{E}$, $\tilde{\mathcal{U}} := \mathcal{U} \ominus K\mathcal{E}$
4) Chose terminal weight function I_f & constraint $\mathcal{X}_f$ satisfying Tube-assumptions
   $\Rightarrow$ Take **Optimal Cost** of LQR-control $J^*(z_0) = \sum_{i=0}^{\infty} z_i^T(Q + K^TRK)z_i = z_0^TPz_0$
   $\to$ $J^*(z) =$ Lyapunov Function for $z_{k+1} = (A + BK)z_k \to I_f(z) = z^TPz$
   $\Rightarrow$ Take **Terminal Control Law** to LQR control law $\kappa_f(x) = Kx$
   $\Rightarrow$ Take Maximum Invariant set for **Terminal Constraint** $\mathcal{X}_f$ as in nominal MPC.

ONLINE

1) Measure / Estimate state x
2) Solve the problem $(V^*(x), Z^*(x)) = \operatorname{argmin}_{V,Z}\{J(V, Z) | (V, Z) \in \mathcal{Z}(x)\}$ **(Part 3)**
3) Set the input to $u = K(x - z_0^*(x)) + v_0^*(x) = Ke + v$

Tube-MPC Pros	vs. Cons
∘ less conservative than open-loop robust MPC (actively compensating noise)	∘ sub-optimal MPC
∘ works for unstable systems, optimization simple	∘ reduced feasible set
	∘ $\mathcal{W}$ not known exactly

Robust-MPC Pros	vs. Cons
∘ feasible set is invariant & we know Exactly when controller will work	∘ complex, conservativ
∘ easier to tune: robustness vs. performance	∘ must know largest $\mathcal{W}$
	∘ feasible set small

9) Robustness of Nominal MPC

Want to control the noisy system $x_{k+1} = Ax_k + Bu_k + w_k$ by ignoring the noise and using standard MPC procedure → we will reach convergence to neighborhood of origin (*for linear systems!!*)

Convergence Proof

- Optimal cost $J^*(x)$ is Lyapunov function for nominal system:
  $J^*(Ax + Bu^*(x)) - J^*(x) \le -I(x, u^*(x))$
  and in next time-step the state is: $x_{k+1} = Ax_k + Bu^*(x_k) + w_k$
- Assuming optimal cost J^* is continuous (true for linear systems, convex constraints, continuous stage cost):
  $|J^*(Ax + Bu^*(x) + w) - J^*(Ax + Bu^*(x))|$
  $\le \gamma\|Ax + Bu^*(x) + w - (Ax + Bu^*(x))\| = \gamma\|w\|$
- **Lyapunov decrease is bounded** as: $J^*(Ax + Bu^*(x) + w) - J^*(x) =$
  $J^*(Ax + Bu^*(x) + w) - J^*(x) - J^*(Ax + Bu^*(x)) + J^*(Ax + Bu^*(x))$
  $\le J^*(Ax + Bu^*(x)) - J^*(x) + \gamma\|w\| \le -I(x, u^*(x)) + \gamma\|w\|$
- Amount of decrease grows with $\|x\|$
- Amount of increase upper bounded by $\max\{\|w\|\ |\ w \in \mathcal{W}\}$

$\Rightarrow$ We will move towards origin until there is a trade-off between size of x & w. This is called an **Input-To-State Stable** system.

Nominal-MPC for uncertain Systems: Pros	vs. Cons
∘ simple, $\mathcal{W}$ not needed to be known	∘ difficult to determine reg. of attract.
∘ effective in practice	∘ hard tuning (robustness vs. performance)
∘ feasible set is large	∘ works for lin. systems, non-lin. systems need continuity assumptions
∘ region of attraction may be larger than other approaches	

Further Examples

- $[a, b] \oplus [c, d] = [a + c, b + d]$ & $[a, b] \ominus [c, d] = [a - c, b - d]$