# Classification

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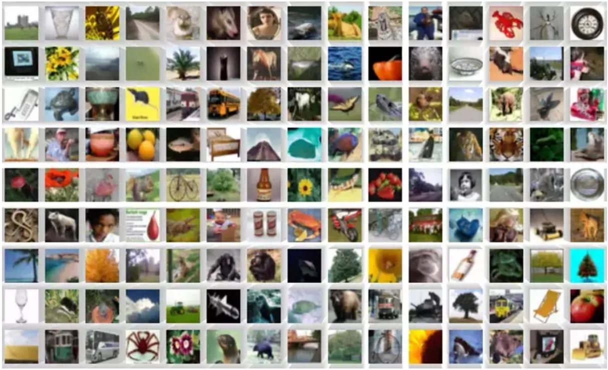
# Classification problem

- Classification: assign a label (or category, class) to an observation based on its features
- $\mathcal{X}$: input space (e.g. $\mathbb{R}^d$); $\mathcal{Y}$: output space (e.g. $\{1, 2, \ldots, K\}$)
- $x \in \mathcal{X}$: feature vector, input, data point...
- $y \in \mathcal{Y}$: label, category, class...
- Classifier: a mapping $f : \mathcal{X} \to \mathcal{Y}$
- Goal: construct a classifier $f$ that accurately predicts the label $y$ given the features $x$

# MNIST dataset

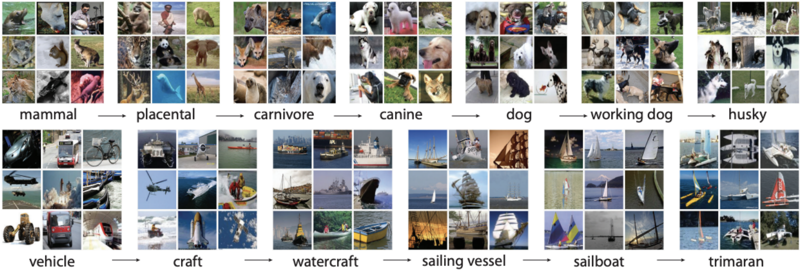
- Input: 28x28 gray scale (1 channel) images, i.e., $\mathcal{X} = \mathbb{R}^{28\times 28}$ or $\mathbb{R}^{784}$
- Output: digits 0 through 9 (i.e., $\mathcal{Y} = \{0, 1, \ldots, 9\}$)

# CIFAR datasets



- Input: $32 \times 32$ RGB color (3 channels) images, i.e., $\mathcal{X} = \mathbb{R}^{32\times 32\times 3}$ or $\mathbb{R}^{3072}$
- Output: 10 classes (airplanes, cars, birds, cats, deer, dogs, frogs, horses, ships, and trucks) or 100 classes

# ImageNet dataset




- Input: varies, often high-resolution (often $224 \times 224 \times 3$)
- Output: 1000 different categories

# Mathematical set-up

- Modeling assumption: the data (input-output pairs) come from an underlying data distribution $\rho$ over $\mathcal{X} \times \mathcal{Y}$
- Training data: $(x_1, y_1), \ldots, (x_n, y_n) \overset{\text{i.i.d.}}{\sim} \rho$
- Error metric: for any given classifier $f$, its risk, defined as the average (expected) classification error on a new data is

$$R(f) := \mathbb{P}_{(X,Y)\sim\rho}(f(X) \neq Y)$$

- Supervised learning: build a classifier $f$ based on training data, that makes the average classification error as small as possible

# Questions

- Does there exists a "best" classifier?

  *— this lecture*

- Can we construct this "best" classifier with the information of $\rho$?

  *— this lecture*

- What can we do when we only have a finite number of training data?

  *— next few weeks*

# Bayes optimal classifier: binary case

- Consider the binary case: $\mathcal{Y} = \{0, 1\}$
- Define the Bayes classifier: for any $x \in \mathcal{X}$,

$$f^\star(x) := \begin{cases} 1, & \text{if } \mathbb{P}(Y = 1 \mid X = x) \geq \mathbb{P}(Y = 0 \mid X = x), \\ 0, & \text{otherwise.} \end{cases}$$

**Theorem 2.1 (Bayes optimal classifier: binary case)**

*The Bayes classifier $f^\star$ minimizes the misclassification error, i.e.,*

$$f^\star \in \operatorname*{arg\,min}_{f : \mathcal{X} \to \mathcal{Y}} \mathbb{P}_{(X,Y)\sim\rho}(f(X) \neq Y).$$

# Proof of Theorem 2.1

We need to show that, for any classifier $f : \mathcal{X} \to \mathcal{Y}$,

$$R(f) = \mathbb{P}(f(X) \neq Y) \geq \mathbb{P}(f^\star(X) \neq Y) = R(f^\star)$$

By tower property,

$$\begin{aligned}
\mathbb{P}(f(X) \neq Y) &= \mathbb{E}\left[\mathbb{1}_{f(X)\neq Y}\right] \\
&= \mathbb{E}_X\left[\mathbb{E}\left[\mathbb{1}_{f(X)\neq Y} \mid X\right]\right] && \text{(tower property)} \\
&= \mathbb{E}_X\left[\mathbb{P}\left(f(X) \neq Y \mid X\right)\right] \\
&\geq \mathbb{E}_X\left[\mathbb{P}\left(f^\star(X) \neq Y \mid X\right)\right] && \text{(why?)} \\
&= \mathbb{E}_X\left[\mathbb{E}\left[\mathbb{1}_{f^\star(X)\neq Y} \mid X\right]\right] \\
&= \mathbb{E}\left[\mathbb{1}_{f^\star(X)\neq Y}\right] && \text{(tower property)} \\
&= \mathbb{P}(f^\star(X) \neq Y).
\end{aligned}$$

It suffices to check

$$\mathbb{P}\left(f(X) \neq Y \mid X\right) \geq \mathbb{P}\left(f^\star(X) \neq Y \mid X\right).$$

# Proof of Theorem 2.1 (cont.)

Observe that

$$\mathbb{P}(f^\star(X) \neq Y \mid X) = \begin{cases} \mathbb{P}(Y = 0 \mid X) & \text{if } \mathbb{P}(Y = 1 \mid X) \geq \mathbb{P}(Y = 0 \mid X) \\ \mathbb{P}(Y = 1 \mid X) & \text{if } \mathbb{P}(Y = 1 \mid X) \geq \mathbb{P}(Y = 0 \mid X) \end{cases}$$

$$= \min\left\{\mathbb{P}(Y = 1 \mid X), \mathbb{P}(Y = 0 \mid X)\right\}$$

and

$$\mathbb{P}(f(X) \neq Y \mid X) = \begin{cases} \mathbb{P}(Y = 0 \mid X) & \text{if } f(X) = 1 \\ \mathbb{P}(Y = 1 \mid X) & \text{if } f(X) = 0 \end{cases}$$

$$\geq \min\left\{\mathbb{P}(Y = 1 \mid X), \mathbb{P}(Y = 0 \mid X)\right\}.$$

Therefore

$$\mathbb{P}(f^\star(X) \neq Y \mid X) \geq \mathbb{P}(f(X) \neq Y \mid X).$$

# A few remarks

Bayes optimal classifier

$$f^\star(x) \coloneqq \begin{cases} 1, & \text{if } \mathbb{P}(Y = 1 \mid X = x) \geq \mathbb{P}(Y = 0 \mid X = x), \\ 0, & \text{otherwise.} \end{cases}$$

- Depends on the true underlying data distribution $\rho$
- The optimal classifier might not be unique
- When $\mathcal{X}$ is discrete, it is equivalent to

$$f^\star(x) \coloneqq \begin{cases} 1, & \text{if } \mathbb{P}(X = x, Y = 1) \geq \mathbb{P}(X = x, Y = 0), \\ 0, & \text{otherwise.} \end{cases}$$

# Bayes risk: binary case

- Bayes risk:

$$R^\star := \mathbb{P}_{(X,Y)\sim\rho}(f^\star(X) \neq Y)$$

- The Bayes risk serves as a lower bound for the classification error that any practical classifier can achieve:

$$R^\star = \min_{f:\mathcal{X}\to\mathcal{Y}} \mathbb{P}_{(X,Y)\sim\rho}(f(X) \neq Y).$$

- It represents the inherent uncertainty in the classification problem due to overlapping distributions of the classes.
- Excess risk: $R(f) - R^\star$

# Bayes optimal classifier: multiclass setting

- Consider the multiclass case: $\mathcal{Y} = \{1, \dots, K\}$
- Define the Bayes classifier: for any $x \in \mathcal{X}$,

$$f^\star(x) \coloneqq \arg\max_{y \in \mathcal{Y}} \mathbb{P}(Y = y \mid X = x)$$

**Theorem 2.2 (Bayes optimal classifier: multiclass case)**

*The Bayes classifier $f^\star$ minimizes the misclassification error, i.e.,*

$$f^\star \in \arg\min_{f : \mathcal{X} \to \mathcal{Y}} \mathbb{P}_{(X,Y)\sim\rho}(f(X) \neq Y).$$

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**Theorem 2.2 (Bayes optimal classifier: multiclass case)**

*The Bayes classifier $f^\star$ minimizes the misclassification error, i.e.,*

$$f^\star \in \arg\min_{f : \mathcal{X} \to \mathcal{Y}} \mathbb{P}_{(X,Y)\sim\rho}(f(X) \neq Y).$$

**Proof:** similar to Theorem 2.1, it suffices to check for any classifier $f$

$$\mathbb{P}\left(f(X) \neq Y \mid X\right) \geq \mathbb{P}\left(f^\star(X) \neq Y \mid X\right).$$

# More general loss function?

- Consider more general loss function $\ell : \mathcal{Y} \times \mathcal{Y} \to \mathbb{R}$
- Define the risk for a classifier $f : \mathcal{X} \to \mathcal{Y}$ as

$$R_\ell(f) := \mathbb{E}_{(X,Y)\sim\rho}[\ell(f(X), Y)]$$

- Example: with 0-1 loss $\ell(y, y') = \mathbb{1}\{y \neq y'\}$, we recover the average classification error

$$R(f) = \mathbb{P}_{(X,Y)\sim\rho}(f(X) \neq Y)$$

- Goal: find $f$ that minimizes the risk $R_\ell(f)$ (the Bayes classifier might not be optimal...)

Question: Can you think of settings where other types of loss functions are more appropriate than the 0-1 loss?

# Example: traffic signs


- $\mathcal{Y} = \{\text{stop sign}, 50 \text{ mph}, 40 \text{ mph}\}$.
- Predicting 50 mph when it is actually a stop sign is worse than predicting 40 mph when it is actually 50mph.
- 0-1 loss is not suitable here...

# Example: traffic signs


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- Predicting 50 mph when it is actually a stop sign is worse than predicting 40 mph when it is actually 50mph.
- 0-1 loss is not suitable here...

We will discuss classification with general loss later if time permits

# Supervised learning

- Go back to 0-1 loss
- In practice, we don't know $\rho$. It is in general impossible to compute the Bayes classifier $f^\star$
- Goal: build a classifier $f : \mathcal{X} \to \mathcal{Y}$ based on training data $(x_1, y_1), \ldots, (x_n, y_n) \overset{\text{i.i.d.}}{\sim} \rho$
- Hope: achieve small excess risk $R(f) - R^\star$
- High-level framework:
    - Make some modeling assumptions on $\rho$
    - Design a good classifier $f$ under this setup
    - For example, a good classifier may satisfy

$$R(f) - R^\star \leq h(n)$$

      where $h(n)$ is a function of the sample size $n$ describing the rate of convergence, e.g., $h(n) = O(1/n)$.

*Linear Methods for Classification*

# Linear classifiers

- Linear classifiers: decision boundaries are linear hyperplanes
  - Hyperplane $\mathcal{H}_{\boldsymbol{\beta},\beta_0} = \{\boldsymbol{x} \in \mathbb{R}^d : \langle \boldsymbol{\beta}, \boldsymbol{x} \rangle + \beta_0 = 0\}$
  - Half planes cut by $\mathcal{H}_{\boldsymbol{\beta},\beta_0}$:

$$\mathcal{H}^+_{\boldsymbol{\beta},\beta_0} = \{\boldsymbol{x} \in \mathbb{R}^d : \langle \boldsymbol{\beta}, \boldsymbol{x} \rangle + \beta_0 \geq 0\},$$
$$\mathcal{H}^-_{\boldsymbol{\beta},\beta_0} = \{\boldsymbol{x} \in \mathbb{R}^d : \langle \boldsymbol{\beta}, \boldsymbol{x} \rangle + \beta_0 < 0\}.$$

  - Example: in the binary case, the linear classifier has the form

$$f(\boldsymbol{x}) = \mathbb{1}\{\boldsymbol{x} \in \mathcal{H}^+_{\boldsymbol{\beta},\beta_0}\}$$

- Three approaches to learn a linear classifier from the data:
  - Linear discriminant analysis (LDA)
  - Logistic regression
  - Perceptrons and Support vector machines (SVMs)

# Linear discriminant analysis (LDA)

- Model set-up: $\mathcal{X} = \mathbb{R}^d$, $\mathcal{Y} = \{1, \dots, K\}$. For $k = 1, \dots, K$,

$$\mathbb{P}(Y = k) = \omega_k, \qquad X \mid Y = k \sim \mathcal{N}(\boldsymbol{\mu}_k, \boldsymbol{\Sigma})$$

  where $\omega_k \geq 0$, $\sum_{k=1}^K \omega_k = 1$, $\boldsymbol{\mu}_k \in \mathbb{R}^d$, $\boldsymbol{\Sigma} \in \mathbb{S}_+^d$

- The Bayes classifier under this setup: for any $\boldsymbol{x}$, compute

$$\delta_k(\boldsymbol{x}) := \underbrace{\boldsymbol{x}^\top \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}_k - \frac{1}{2} \boldsymbol{\mu}_k^\top \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}_k + \log \omega_k}_{\propto \mathbb{P}(Y=k \mid X=\boldsymbol{x}) + \text{constant}}.$$

  Let $f^\star(\boldsymbol{x}) = \arg\max_{1 \leq k \leq K} \delta_k(\boldsymbol{x})$.

- Issue: model parameters are unknown...

# Plug-in approach

- Plug-in approach: replace the unknown parameters with reliable estimates
- Suppose we have i.i.d. data $(\boldsymbol{x}_1, y_1), \ldots, (\boldsymbol{x}_n, y_n) \overset{\text{i.i.d.}}{\sim} \rho$
- For each $1 \le k \le K$, let $n_k = \sum_{i=1}^{n} \mathbb{1}\{y_i = k\}$ and

$$\widehat{\boldsymbol{\mu}}_k = \frac{1}{n_k} \sum_{i: y_i = k} \boldsymbol{x}_i, \qquad \widehat{\omega}_k = \frac{n_k}{n}$$

- Estimate the covariance matrix

$$\widehat{\boldsymbol{\Sigma}} = \frac{1}{N - K} \sum_{k=1}^{K} \sum_{i: y_i = k} \big(\boldsymbol{x}_i - \widehat{\boldsymbol{\mu}}_k\big)\big(\boldsymbol{x}_i - \widehat{\boldsymbol{\mu}}_k\big)^\top$$

- Replace $\boldsymbol{\mu}_k$, $\omega_k$, $\boldsymbol{\Sigma}$ with $\widehat{\boldsymbol{\mu}}_k$, $\widehat{\omega}_k$, $\widehat{\boldsymbol{\Sigma}}$

$$\widehat{\delta}_k(\boldsymbol{x}) := \underbrace{\boldsymbol{x}^\top \widehat{\boldsymbol{\Sigma}}^{-1} \widehat{\boldsymbol{\mu}}_k - \frac{1}{2} \widehat{\boldsymbol{\mu}}_k^\top \widehat{\boldsymbol{\Sigma}}^{-1} \widehat{\boldsymbol{\mu}}_k + \log \widehat{\omega}_k}_{\text{linear in } \boldsymbol{x}} .$$

# Generalization

- Consider a more general set-up: for $k = 1, \ldots, K$, assume

$$\mathbb{P}(Y = k) = \omega_k, \qquad X \mid Y = k \sim \mathcal{N}(\boldsymbol{\mu}_k, \boldsymbol{\Sigma}_{k})$$

  where $\omega_k \geq 0$, $\sum_{k=1}^K \omega_k = 1$, $\mu_k \in \mathbb{R}^d$, $\boldsymbol{\Sigma}_k \in \mathbb{S}_+^d$
- This setup will lead to the so-called quadratic discriminant analysis (QDA)
- Homework: derive QDA
  - What is the Bayes classifier under this setup?
  - How to derive a practical (data-driven) classifier?
  - Is this still a linear classifier?

# Logistic regression

- Model set-up: $\mathcal{X} = \mathbb{R}^d$, $\mathcal{Y} = \{0, 1, \ldots, K\}$. Let

$$\mathbb{P}(Y = k \mid \boldsymbol{x}) = \frac{\exp(\boldsymbol{\beta}_k^\top \boldsymbol{x} + \beta_{0,k})}{1 + \sum_{k'=1}^{K} \exp(\boldsymbol{\beta}_{k'}^\top \boldsymbol{x} + \beta_{0,k'})}, \quad (1 \le k \le K),$$
$$\mathbb{P}(Y = 0 \mid \boldsymbol{x}) = \frac{1}{1 + \sum_{k'=1}^{K} \exp(\boldsymbol{\beta}_{k'}^\top \boldsymbol{x} + \beta_{0,k})},$$

where the parameters $\boldsymbol{\beta}_k \in \mathbb{R}^d$, $\beta_{0,k} \in \mathbb{R}$ for $k = 1, \ldots, K$

# Logistic regression

- Model set-up: $\mathcal{X} = \mathbb{R}^d \times \{1\}$, $\mathcal{Y} = \{0, 1, \ldots, K\}$. Let

$$\mathbb{P}(Y = k \mid \boldsymbol{x}) = \frac{\exp(\boldsymbol{\beta}_k^\top \boldsymbol{x})}{1 + \sum_{k'=1}^K \exp(\boldsymbol{\beta}_{k'}^\top \boldsymbol{x})}, \qquad (k = 1, \ldots, K),$$

$$\mathbb{P}(Y = 0 \mid \boldsymbol{x}) = \frac{1}{1 + \sum_{k'=1}^K \exp(\boldsymbol{\beta}_{k'}^\top \boldsymbol{x})},$$

  where the parameters $\boldsymbol{\beta}_k \in \mathbb{R}^{d+1}$ for $k = 1, \ldots, K$

- Bayes classifier:

$$f(\boldsymbol{x}) = \begin{cases} \text{argmax}_{1 \le k \le K} \boldsymbol{\beta}_k^\top \boldsymbol{x}, & \text{if } \max_{1 \le k \le K} \boldsymbol{\beta}_k^\top \boldsymbol{x} > 0, \\ 0, & \text{otherwise.} \end{cases}$$

- Estimate $\boldsymbol{\beta}_k$'s: maximum likelihood estimation (MLE)

# Maximum likelihood estimation

- Suppose we have i.i.d. data $(\boldsymbol{x}_1, y_1), \ldots, (\boldsymbol{x}_n, y_n)$
- The negative log-likelihood function

$$\ell(\boldsymbol{\beta}) = -\frac{1}{n} \sum_{k=1}^{K} \sum_{i: y_i = k} \boldsymbol{x}_i^\top \boldsymbol{\beta}_k + \frac{1}{n} \sum_{i=1}^{n} \log \left[ 1 + \sum_{k'=1}^{K} \exp(\boldsymbol{x}_i^\top \boldsymbol{\beta}_{k'}) \right]$$

- Maximum likelihood estimation (MLE)

$$\widehat{\boldsymbol{\beta}} := \arg\min_{\boldsymbol{\beta}} \ell(\boldsymbol{\beta})$$

- Convex optimization: solve by e.g., gradient descent

$$\boldsymbol{\beta}^{t+1} = \boldsymbol{\beta}^t - \eta \nabla \ell(\boldsymbol{\beta}^t) \qquad (t = 0, 1, \ldots)$$

# A brief introduction to gradient descent

Gradient descent (GD) for solving $\min_{\boldsymbol{\beta}\in\mathbb{R}^d} L(\boldsymbol{\beta})$:

$$\boldsymbol{\beta}^{t+1} = \boldsymbol{\beta}^t - \eta\nabla L(\boldsymbol{\beta}^t) \qquad (t = 0, 1, \ldots)$$

When $\eta$ is properly small, GD satisfy the following properties:

- For smooth function $L$, GD is a descent algorithm: $L(\boldsymbol{\beta}^{t+1}) \le L(\boldsymbol{\beta}^t)$
- For convex + smooth function $L$, GD satisfies

  $$L(\boldsymbol{\beta}^t) - L(\boldsymbol{\beta}^\star) \le O\left(\frac{\|\boldsymbol{\beta}^0 - \boldsymbol{\beta}^\star\|_2^2}{t}\right) \qquad (t = 0, 1, \ldots)$$

  for any minimizer $\boldsymbol{\beta}^\star$
- For strongly convex + smooth function $L$, GD satisfies

  $$\|\boldsymbol{\beta}^{t+1} - \boldsymbol{\beta}^\star\|_2 \le (1-\kappa)^t\|\boldsymbol{\beta}^0 - \boldsymbol{\beta}^\star\|_2 \qquad (t = 0, 1, \ldots)$$

  for some $\kappa \in (0, 1)$, where $\boldsymbol{\beta}^\star$ is the unique minimizer

# Stochastic gradient descent

Consider the following empirical risk minimization problem

$$\min_{\boldsymbol{\beta} \in \mathbb{R}^d} L(\boldsymbol{\beta}) := \frac{1}{n} \sum_{i=1}^{n} g(\boldsymbol{\beta}; \boldsymbol{x}_i),$$

where $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$ are training data points.

- **Stochastic gradient descent:** for $t = 0, 1, \ldots,$

$$\boldsymbol{\beta}^{t+1} = \boldsymbol{\beta}^t - \eta \nabla g(\boldsymbol{\beta}^t; \boldsymbol{x}_{i_t}) \quad \text{where} \quad \boldsymbol{x}_{i_t} \overset{\text{ind.}}{\sim} \text{Unif}\{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n\}$$

- **Gradient descent:** for $t = 0, 1, \ldots,$

$$\boldsymbol{\beta}^{t+1} = \boldsymbol{\beta}^t - \eta \nabla L(\boldsymbol{\beta}^t) = \boldsymbol{\beta}^t - \eta \frac{1}{n} \sum_{i=1}^{n} \nabla g(\boldsymbol{\beta}; \boldsymbol{x}_i)$$

- **Advantage of SGD:** much faster updates, especially for large datasets, but still enjoys nice properties (sometimes even better than GD!)

# Gradient descent methods

**Example:** GD / SGD for logistic regresion

**Take-away:** (stochastic) gradient descent is the default method for solving unconstrained optimization problem

— simple and effective!

**Recommended reading materials:** Lecture 1 and 10 of the course

*Large-Scale Optimization for Data Science*

by Prof. Yuxin Chen (UPenn); Lecture on GD and SGD

*Perceptrons and SVMs*

# Linearly separable data

- Consider binary classification: $\mathcal{X} = \mathbb{R}^d$ and $\mathcal{Y} = \{1, -1\}$
- Training data: $(\boldsymbol{x}_1, y_1), \ldots, (\boldsymbol{x}_n, y_n)$
- **Linearly separable data:** $\exists$ a separating hyperplane $\mathcal{H}_{\boldsymbol{\beta},\beta_0}$ s.t.

$$y_i \cdot (\boldsymbol{x}_i^\top \boldsymbol{\beta} + \beta_0) > 0 \qquad (i = 1, \ldots, n)$$

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- by merging $\beta_0$ into $\boldsymbol{\beta}$ and adding 1 to $\boldsymbol{x}_i$'s, this assumption becomes: $\exists\, \boldsymbol{\beta}_{\mathsf{sep}} \in \mathbb{R}^{d+1}$

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$$y_i \cdot \boldsymbol{x}_i^\top \boldsymbol{\beta}_{\mathsf{sep}} > 0 \qquad (i = 1, \ldots, n)$$

- **Goal:** search a separating hyperplane indexed by $\widehat{\boldsymbol{\beta}}$

$$y_i \cdot \boldsymbol{x}_i^\top \widehat{\boldsymbol{\beta}} > 0 \qquad (i = 1, \ldots, n)$$

(note that $\boldsymbol{\beta}_{\mathsf{sep}}$ is not known a priori)

# Perceptron Learning Algorithm

- For every $\boldsymbol{\beta} \in \mathbb{R}^{d+1}$, define the set $\mathcal{M}_{\boldsymbol{\beta}} := \underbrace{\{i : y_i \cdot \boldsymbol{x}_i^\top \boldsymbol{\beta} \le 0\}}_{\text{misclassified points}}$
- Target: minimize the perceptron loss

$$\sigma(\boldsymbol{\beta}) := -\sum_{i \in \mathcal{M}_{\boldsymbol{\beta}}} y_i \cdot \boldsymbol{x}_i^\top \boldsymbol{\beta} \propto \sum_{i \in \mathcal{M}_{\boldsymbol{\beta}}} \mathrm{dist}(\boldsymbol{x}_i, \mathcal{H}_{\boldsymbol{\beta}})$$

  where $\mathcal{H}_{\boldsymbol{\beta}} = \{\boldsymbol{x} : \boldsymbol{x}^\top \boldsymbol{\beta} = 0\}$

- Algorithm: initialize with $\boldsymbol{\beta}^0 \in \mathbb{R}^{d+1}$, for $t = 0, 1, \ldots,$ update

$$\boldsymbol{\beta}^{t+1} = \boldsymbol{\beta}^t + \eta y_i \boldsymbol{x}_i, \quad \text{for a random } i \in \mathcal{M}_{\boldsymbol{\beta}^t}$$

  where $\eta > 0$ is the step size; in fact, we can take $\eta = 1$ here...

# Perceptron Learning Algorithm

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$$\boldsymbol{\beta}^{t+1} = \boldsymbol{\beta}^t + y_i \boldsymbol{x}_i, \quad \text{for a random } i \in \mathcal{M}_{\boldsymbol{\beta}^t}$$

- Interpretation: SGD with step size 1 (kind of...)

# Convergence theory

**Theorem 2.3**

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- only works linearly separable data. If the classes cannot be separated by a hyperplane, the algorithm will not converge

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**Limitations:**

- solutions not unique: might converge to an unstable hyperplane
- only works linearly separable data. If the classes cannot be separated by a hyperplane, the algorithm will not converge
- the “finite” number of steps can be very large