C4.1 : 24, 26, 42, 66
C4.2: 6, 12, 16, 22, 31, 63
C4.3: 5, 39

C4.1

Absolute Extrema on Finite Closed Intervals

In Exercises 21–36, find the absolute maximum and minimum values of each function on the given interval. Then graph the function. Identify the points on the graph where the absolute extrema occur, and include their coordinates.

24. $f(x) = 4 - x^3, \quad -2 \le x \le 1$

25. $F(x) = -\frac{1}{x^2}, \quad 0.5 \le x \le 2$

26. $F(x) = -\frac{1}{x}, \quad -2 \le x \le -1$

1. Absolute max. & min. ;
2. Graph the function
3. Graph where the absolute extrema occur & coordinates

24. $f(x) = 4 - x^3, \ -2 \le x \le 1$

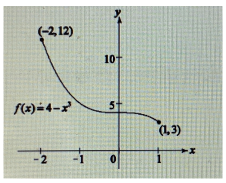
$f'(x) = -3x^2 \le 0$

$\therefore$ f(x) is a decreasing function in interval $[-2, 1]$.

$\therefore f(x) \in [3, 12]$

$\therefore$ the max. absolute value is 12, the min. absolute value is 3.

$f(-2) = 12, \ f(1) = 3 \qquad (-2, 12) \quad (1, 3)$



26. $F(x) = -\frac{1}{x}, \ -2 \le x \le -1$

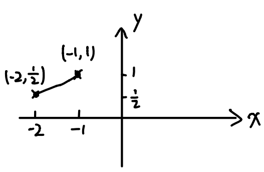
$F'(x) = \frac{1}{x^2} \ge 0$

$\therefore$ F(x) is an increasing function in the interval $[-2, -1]$

$\therefore F(x) \in [\frac{1}{2}, 1]$

$\therefore$ F(x) will reach absolute max. value 1 at $x = -1$;

F(x) will reach absolute min. value $\frac{1}{2}$ at $x = -2$.



In Exercises 41–44, find the function's absolute maximum and minimum values and say where they occur.

41. $f(x) = x^{4/3}, \quad -1 \le x \le 8$

42. $f(x) = x^{5/3}, \quad -1 \le x \le 8$

$f(x) = x^{5/3} \qquad f'(x) = \frac{5}{3} \cdot x^{\frac{2}{3}} \ge 0$

$\therefore$ f(x) is an increasing function in the interval $[-1, 8]$

$f(-1) = (-1)^{\frac{5}{3}} = \sqrt[3]{(-1)^5} = \sqrt[3]{-1} = -1$; $f(8) = 8^{\frac{5}{3}} = (2^3)^{\frac{5}{3}} = 2^5 = 32$.

$\therefore f(x) \in [-1, 32]$

$f(-1) = -1, \ f(8) = 32$

$\therefore$ f(x) will reach absolute max. value 32 at $x = 8$;

f(x) will reach absolute min. value -1 at $x = -1$.

66. No critical points or endpoints exist We know how to find the extreme values of a continuous function $f(x)$ by investigating its values at critical points and endpoints. But what if there *are* no critical points or endpoints? What happens then? Do such functions really exist? Give reasons for your answers.

Then, the function will have no extreme values in its domain, if there are no boundary points or critical points.

Such functions do exist.

For example. $f(x) = x \ , \ x \in (-\infty, +\infty)$

C4.2

Checking the Mean Value Theorem

Find the value or values of c that satisfy the equation

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

in the conclusion of the Mean Value Theorem for the functions and intervals in Exercises 1–8.

1. $f(x) = x^2 + 2x - 1, \quad [0, 1]$

2. $f(x) = x^{2/3}, \quad [0, 1]$

3. $f(x) = x + \frac{1}{x}, \quad \left[\frac{1}{2}, 2\right]$

4. $f(x) = \sqrt{x - 1}, \quad [1, 3]$

5. $f(x) = \sin^{-1} x, \quad [-1, 1]$

6. $f(x) = \ln(x - 1), \quad [2, 4]$

Recall:

THEOREM 4—The Mean Value Theorem
Suppose $y = f(x)$ is continuous over a closed interval $[a, b]$ and differentiable on the interval's interior (a, b). Then there is at least one point c in (a, b) at which

$$\frac{f(b) - f(a)}{b - a} = f'(c). \tag{1}$$

$f(x) = \ln(x-1) \qquad x \in [2, 4] \qquad a = 2, \ b = 4$

$f(2) = \ln 1 = 0, \ f(4) = \ln 3$

$\frac{f(b) - f(a)}{b - a} = \frac{\ln 3 - 0}{4 - 2} = \frac{1}{2} \cdot \ln 3$

$f'(x) = \frac{1}{x-1} \cdot 1 = \frac{1}{x-1}$

$f'(c) = \frac{1}{c-1} = \frac{1}{2} \ln 3$

$\Rightarrow c - 1 = \frac{2}{\ln 3}$

$c = 1 + \frac{2}{\ln 3} \approx 2.82 \in [2, 4]$

Which of the functions in Exercises 9–14 satisfy the hypotheses of the Mean Value Theorem on the given interval, and which do not? Give reasons for your answers.

9. $f(x) = x^{2/3}$, $[-1, 8]$

10. $f(x) = x^{4/5}$, $[0, 1]$

11. $f(x) = \sqrt{x(1-x)}$, $[0, 1]$

12. $f(x) = \begin{cases} \frac{\sin x}{x}, & -\pi \le x < 0 \\ 0, & x = 0 \end{cases}$

Does not. Because $f(x)$ is not continuous at $x=0$.

$$f(x) = \begin{cases} \frac{\sin x}{x}, & -\pi \le x < 0 \\ 0, & x = 0 \end{cases}$$

$\lim_{x\to 0^-} \frac{\sin x}{x} = 1$ but $f(0) = 0$

$\therefore \lim_{x\to 0^-} \frac{\sin x}{x} \ne f(0)$

$\therefore f(x)$ is not continuous at $x=0$

16. For what values of a, m, and b does the function

$$f(x) = \begin{cases} 3, & x = 0 \\ -x^2 + 3x + a, & 0 < x < 1 \\ mx + b, & 1 \le x \le 2 \end{cases}$$

satisfy the hypotheses of the Mean Value Theorem on the interval $[0, 2]$?

$f(x)$ should be continuous at every point in interval $[0,2]$

$$\begin{cases} \lim_{x\to 0^+} -x^2+3x+a = f(0) \Rightarrow 0+0+a=3 \Rightarrow a=3 \\ \lim_{x\to 1^-} -x^2+3x+a = f(1) \Rightarrow -1+3+3 = m+b \\ \qquad m+b=5 \end{cases}$$

$f(x)$ should be differentiable in interval $(0,2)$.

$\therefore \frac{d}{dx}(-x^2+3x+3)\Big|_{x=1} = \frac{d}{dx}(mx+b)\Big|_{x=1}$

$\Rightarrow -2x+3\Big|_{x=1} = m$

$-2+3 = m$

$\Rightarrow m=1$

$\therefore b = 5-m = 5-1 = 4$

$\therefore a=3,\ m=1,\ b=4.$

Show that the functions in Exercises 21–28 have exactly one zero in the given interval.

22. $f(x) = x^3 + \frac{4}{x^2} + 7$, $(-\infty, 0)$

$f(x) = x^3 + 4x^{-2} + 7$, $x \in (-\infty, 0)$

$f'(x) = 3x^2 + 4\cdot(-2)\cdot x^{-3} = 3x^2 - \frac{8}{x^3}$

$\because x \in (-\infty, 0)$

$\therefore -x^3 > 0$

$\therefore -\frac{8}{x^3} > 0$

$\therefore 3x^2 - \frac{8}{x^3} > 0$

or if you can find $f(-2)=0$

$\therefore f(x) < 0$ if $x < -2$;

$f(x) > 0$ if $x > -2$

$\therefore f(x)$ only have one zero.

$\therefore f(x)$ is an increasing function in interval $(-\infty, 0)$

$f(-1) = (-1)^3 + \frac{4}{(-1)^2} + 7 = -1+4+7 = 10$, $f(-10) = (-10)^3 + \frac{4}{(-10)^2} + 7 < 0$

$\therefore$ There must be one zero in interval $(-10, -1)$

$\therefore f(x)$ has exactly one zero in $(-\infty, 0)$

31. Suppose that $f'(x) = 2x$ for all x. Find $f(2)$ if

a. $f(0) = 0$ **b.** $f(1) = 0$ **c.** $f(-2) = 3$.

$\because f'(x) = 2x$

$\therefore f(x) = x^2 + C$

(a) $f(0) = 0$

$\Rightarrow 0^2 + C = 0$

$C = 0$

$\therefore f(x) = x^2$

$\therefore f(2) = 4$

(b) $f(1) = 0$

$1^2 + C = 0$

$C = -1$

$\therefore f(x) = x^2 - 1$

$\therefore f(2) = 3$

(c) $f(-2) = 3$

$(-2)^2 + C = 3$

$C = 3-4$

$C = -1$

$\therefore f(x) = x^2 - 1$

$\therefore f(2) = 3$

63. Suppose that $f'(x) \le 1$ for $1 \le x \le 4$. Show that $f(4) - f(1) \le 3$.

Recall:

> **THEOREM 4—The Mean Value Theorem**
> Suppose $y = f(x)$ is continuous over a closed interval $[a, b]$ and differentiable on the interval's interior (a, b). Then there is at least one point c in (a, b) at which
> $$\frac{f(b) - f(a)}{b - a} = f'(c). \qquad (1)$$

$\because f'(x) \le 1$ for $1 \le x \le 4$

$\therefore f(x)$ is differentiable on $1 \le x \le 4$

$\therefore f(x)$ is continuous on $1 \le x \le 4$.

$\therefore f(x)$ satisfies the conditions of the Mean Value Theorem.

$\therefore \exists\, c \in (1,4)$, $\frac{f(4)-f(1)}{4-1} = f'(c)$

$\therefore \frac{f(4)-f(1)}{3} = f'(c) \le 1$

$\therefore f(4) - f(1) \le 3$

C4.3

Analyzing Functions from Derivatives

Answer the following questions about the functions whose derivatives are given in Exercises 1–14:

a. What are the critical points of f?

b. On what open intervals is f increasing or decreasing?

c. At what points, if any, does f assume local maximum or minimum values?

5. $f'(x) = (x-1)e^{-x}$ **6.** $f'(x) = (x-7)(x+1)(x+5)$

(a)

$f'(x) = (x-1)\cdot e^{-x}$

$f'(x) = 0$

$\Rightarrow (x-1)\cdot e^{-x} = 0$

$x - 1 = 0$

$x = 1$

$\therefore$ The critical point at $x=1$

(b) $f'(x) = (x-1)\cdot e^{-x}$

$\because e^{-x} > 0$

$\therefore f'(x) > 0$ if $x > 1$;

$f'(x) < 0$ if $x < 1$.

$\therefore f(x)$ is increasing on $(1, +\infty)$,

$f(x)$ is decreasing on $(-\infty, 1)$.

(c) $\because f(x)$ is decreasing on $(-\infty, 1)$ and increasing on $(1, +\infty)$.

$\therefore f(x)$ will reach local (also absolute) min. at $x = 1$.

In Exercises 19–46:

a. Find the open intervals on which the function is increasing and those on which it is decreasing.

b. Identify the function's local extreme values, if any, saying where they occur.

39. $h(x) = x^{1/3}(x^2 - 4)$

(a) $h(x) = x^{1/3}(x^2-4) = x^{7/3} - 4\cdot x^{1/3}$

$h'(x) = \frac{7}{3}\cdot x^{\frac{4}{3}} - \frac{4}{3}\cdot x^{-\frac{2}{3}}$

$h'(x) = 0$

$\Rightarrow \frac{7}{3} x^{\frac{4}{3}} = \frac{4}{3}\cdot x^{-\frac{2}{3}}$

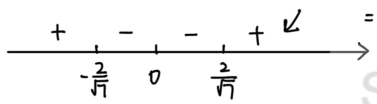
$7\cdot x^2 = 4\cdot x^0$ or $x_1 = 0$

$x^2 = \frac{4}{7}$

$\Rightarrow x_1 = 0,\ x_2 = \frac{2}{\sqrt{7}},\ x_3 = -\frac{2}{\sqrt{7}}$

$\frac{7}{3}\cdot x^{\frac{4}{3}} - \frac{4}{3}\cdot x^{-\frac{2}{3}}$

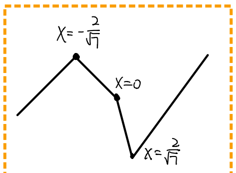
$= \frac{1}{3} x^{-\frac{2}{3}}(7\cdot x^2 - 4)$



$\therefore$ $h(x)$ is increasing on $(\frac{2}{\sqrt{7}}, +\infty)$ and $(-\infty, -\frac{2}{\sqrt{7}})$.

decreasing on $(-\frac{2}{\sqrt{7}}, 0)$ and $(0, \frac{2}{\sqrt{7}})$.

(b)



just see it's decreasing or increasing.

Local maximum:

$h(-\frac{2}{\sqrt{7}}) = (-\frac{2}{\sqrt{7}})^{\frac{1}{3}}\cdot(\frac{4}{7}-4) = \frac{24}{7}\cdot(\frac{2}{\sqrt{7}})^{\frac{1}{3}} = \frac{24\cdot\sqrt[3]{2}}{7^{\frac{7}{6}}}$ at $x = -\frac{2}{\sqrt{7}}$.

Local minimum:

$h(\frac{2}{\sqrt{7}}) = (\frac{2}{\sqrt{7}})^{\frac{1}{3}}\cdot(\frac{4}{7}-4) = -\frac{24}{7}\cdot(\frac{2}{\sqrt{7}})^{\frac{1}{3}} = -\frac{24\cdot\sqrt[3]{2}}{7^{\frac{7}{6}}}$ at $x = \frac{2}{\sqrt{7}}$.