Exam level questions :

1. Given $f(x), g(x)$ are differentiable on R

$f(0)=0,\ f'(0)=3,\ f'(-1)=2,\ g(0)=-1,\ g'(0)=2$

Goal : find $\frac{d}{dx}f(x+g(x))\big|_{x=0}$

$$\frac{d}{dx}f(x+g(x)) = f'[x+g(x)]\cdot\frac{d}{dx}[x+g(x)]$$ chain rule .

$$= f'[x+g(x)]\cdot(1+g'(x))$$

$$\frac{d}{dx}f(x+g(x))\big|_{x=0} = f'(0+g(0))\cdot(1+g'(0))$$
$$= f'(g(0))\cdot(1+g'(0))$$
$$= f'(-1)\cdot(1+2)$$
$$= 2\times 3 = 6$$

2、 Given $f(x)=\begin{cases} a\cdot e^{bx} & \text{for } x>0 \\ 2x-1 & \text{for } x\le 0\end{cases}$

Goal : Find a, b that f is continuous & differentiable at $x=0$

① Continuous :

$f(0)=\lim_{x\to 0^+} f(x)$

$f(0)=-1 \qquad \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} a\cdot e^{bx} = a\cdot e^0 = a$

$\therefore\ a=-1$

② differentiable at $x=0 \iff \lim_{x\to 0^+} f'(x) = \lim_{x\to 0^-} f'(x)$

$\lim_{x\to 0^+} f'(x) = \lim_{x\to 0^+}\frac{d}{dx}-e^{bx} = \lim_{x\to 0^+} -b\cdot e^{bx} = -b\cdot e^0 = -b$

$\lim_{x\to 0^-} f'(x) = \lim_{x\to 0^-}\frac{d}{dx}(2x-1) = \lim_{x\to 0^-} 2 = 2$

$\therefore\ b=-2$

$\therefore \begin{cases} a=-1 \\ b=-2.\end{cases}$

2.1.

Do the graphs of the functions f in Exercises 13–17 have tangent lines at the given points? If yes, what is the tangent line?

13. $f(x)=\sqrt{|x|}$ at $x=0$ **14.** $f(x)=(x-1)^{4/3}$ at $x=1$
15. $f(x)=(x+2)^{3/5}$ at $x=-2$
16. $f(x)=|x^2-1|$ at $x=1$

Before starting : have tangent line at x_0

$\iff \lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h} = \text{constant}/\pm\infty$

$+$ continuous at $x=x_0$.

$\iff$ continuous at $x=x_0$

$\lim_{x\to x_0^-}\frac{f(x)-f(x_0)}{x-x_0} = \lim_{x\to x_0^+}\frac{f(x)-f(x_0)}{x-x_0}$

16. $f(x)=|x^2-1|$ at $x=1$

Way 1 : $\lim_{h\to 0}\frac{f(1+h)-f(1)}{h} = \lim_{h\to 0}\frac{|(1+h)^2-1|-0}{h}$

$= \lim_{h\to 0}\frac{|h^2+2h|}{h} = \lim_{h\to 0}\frac{|h(h+2)|}{h} = \begin{cases}\lim_{h\to 0^+}\left|\frac{\cancel{h}(h+2)}{\cancel{h}}\right| = 2, & h>0 \\ \lim_{h\to 0^-} -\left|\frac{\cancel{h}(h+2)}{\cancel{h}}\right| = -2, & h<0\end{cases}$

Thus, limitation does not exist.

No tangent at $x=1$.

Way 2 : $\lim_{x\to 1^+} f'(x) = \lim_{x\to 1^+}(x^2-1)' = \lim_{x\to 1^+} 2x = 2$

$\lim_{x\to 1^-} f'(x) = \lim_{x\to 1^-}(1-x^2)' = \lim_{x\to 1^-} -2x = -2$

$\because \lim_{x\to 1^+} f'(x) \ne \lim_{x\to 1^-} f'(x)$

$\therefore$ no tangent at $x=1$.

19. (a) Find the slope of $y=x^3$ at the point $x=a$.
(b) Find the equations of the straight lines having slope 3 that are tangent to $y=x^3$.

19. (a)

$y'=3x^2 \qquad \frac{dy}{dx}\big|_{x=a} = 3a^2$

but here, we are supposed to use limitation :

$\lim_{h\to 0}\frac{(a+h)^3-a^3}{h} = \lim_{h\to 0}\frac{\cancel{a^3}+3a^2h+3ah^2+h^3-\cancel{a^3}}{h}$

$=\lim_{h\to 0} 3a^2+3ah+h^2 = 3a^2+0+0 = 3a^2.$

(b) tangent to $y=x^3$

$\Rightarrow$ slope $=\frac{d}{dx}x^3 = 3x^2$

$\Rightarrow 3x^2=3 \Rightarrow x^2=1$

$\Rightarrow x=\pm 1$

$\therefore$ This line will cross $(1,1)$ or $(-1,-1)$

$\therefore\ y=3(x-1)+1$ or $y=3(x+1)-1$
$=3x-2 \qquad\qquad = 3x+2$

23. For what value of the constant k is the line $x+y=k$ normal to the curve $y=x^2$?

$y=-x+k \Rightarrow$ slope $=-1$

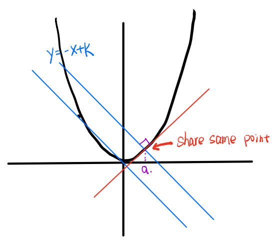
$\frac{dx^2}{dx}\big|_{x=a} = 2x\big|_{x=a} = 2a \xleftrightarrow{\text{normal to}} -\frac{1}{2a}$

when $-\frac{1}{2a}=-1 \Rightarrow a=\frac{1}{2}$

$(\frac{1}{2},\frac{1}{4})$ is on the curve $y=x^2$

$x+y=k$ should cross $(\frac{1}{2},\frac{1}{4})$

$\Rightarrow k=\frac{3}{4}$



2.2.

EXERCISES 2.2

Make rough sketches of the graphs of the derivatives of the functions in Exercises 1–4.

1. The function f graphed in Figure 2.18(a).
2. The function g graphed in Figure 2.18(b).
3. The function h graphed in Figure 2.18(c).
4. The function k graphed in Figure 2.18(d).
5. Where is the function f graphed in Figure 2.18(a) differentiable?
6. Where is the function g graphed in Figure 2.18(b) differentiable?

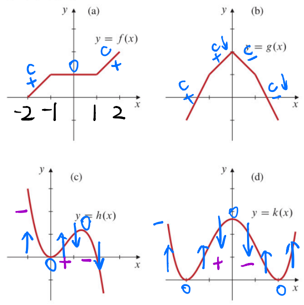
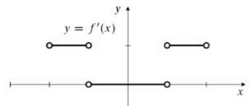


Figure 2.18

Use a graphics utility with differentiation capabilities to plot the graphs of the following functions and their derivatives. Observe the relationships between the graph of y and that of y' in each case. What features of the graph of y can you infer from the graph of y'?

Section 2.2 The Derivative (page 107)

1.



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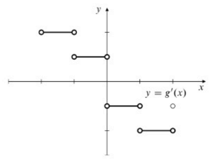

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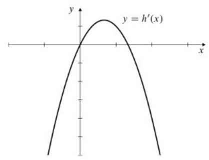
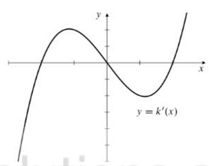


4.



5. on the intervals:
$(-2,-1), (-1,1), (1,2)$

6. $(-2,-1), (-1,0), (0,1), (1,2)$

26. How should the function $g(x) = x^2 \operatorname{sgn} x$ be defined at $x = 0$ so that it is continuous there? Is it then differentiable there?

27. Where does $h(x) = |x^2 + 3x + 2|$ fail to be differentiable?

26. $\operatorname{sgn} x = \begin{cases} 1, & \text{if } x>0 \\ -1, & \text{if } x<0 \end{cases}$

$g(x) = x^2 \cdot \operatorname{sgn} x = \begin{cases} x^2, & \text{if } x>0 \\ -x^2, & \text{if } x<0 \end{cases}$

If $g(x)$ is continuous at $x=0$, then

$\lim_{x\to 0^+} g(x) = \lim_{x\to 0^-} g(x) = g(0)$

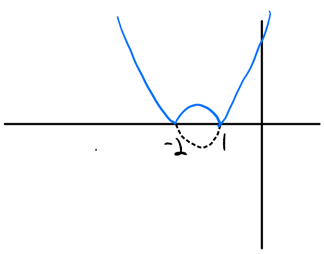
$\therefore g(0) = 0$

Differentiable?

$\lim_{x\to 0^+} g'(x) = \lim_{x\to 0^+} 2x = 0$ $\quad \because \lim_{x\to 0^+} g'(x) = \lim_{x\to 0^-} g'(x)$

$\lim_{x\to 0^-} g'(x) = \lim_{x\to 0^-} -2x = 0$ $\quad \therefore$ Yes.

27. $h(x) = |x^2+3x+2| = |(x+1)(x+2)|$



when $x=-1$ or $x=-2$.

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58. Find values of a and b that make

$$f(x) = \begin{cases} ax+b, & x<0 \\ 2\sin x + 3\cos x, & x \ge 0 \end{cases}$$

differentiable at $x = 0$.

1) $f(x)$ should be continuous at $x=0$.

$\Leftrightarrow \lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x)$

$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} ax+b = b$

$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} 2\sin x + 3\cos x = 2\sin 0 + 3\cos 0 = 3$

$\therefore b = 3$.

2) $\lim_{x\to 0^-} f'(x) = \lim_{x\to 0^+} f'(x)$ OR $\lim_{h\to 0} \dfrac{f(0+h)-f(0)}{h} = m$.

$\lim_{x\to 0^-} f'(x) = \lim_{x\to 0^-} (ax+b)' = \lim_{x\to 0^-} a = a$

$\lim_{x\to 0^+} f'(x) = \lim_{x\to 0^+} (2\sin x + 3\cos x)' = \lim_{x\to 0^+} 2\cos x - 3\sin x = 2$

$\therefore a = 2$

2.9

EXERCISES 2.9

In Exercises 1–8, find dy/dx in terms of x and y.

1. $xy - x + 2y = 1$ **2.** $x^3 + y^3 = 1$

3. $x^2 + xy = y^3$ **4.** $x^3y + xy^5 = 2$

1. $x'y + xy' - x' + 2y' = 0$

$\Rightarrow y + xy' - 1 + 2y' = 0$

$y'(x+2) = 1-y$

$y' = \dfrac{1-y}{x+2}$

2. $x^3+y^3 = 1$

$3x^2 + 3y^2 \cdot y' = 0$

$y' = -\dfrac{x^2}{y^2}$

3. $x^2+xy = y^3$

$2x + (y+xy') = 3y^2 \cdot y'$

$y'(3y^2-x) = 2x+y$

$y' = \dfrac{2x+y}{3y^2-x}$

4. $x^3y + xy^5 = 2$

$3x^2 \cdot y + x^3 \cdot y' + y^5 + x \cdot 5y^4 \cdot y' = 0$

$y'(x^3+5xy^4) = -(3x^2y+y^5)$

$\Rightarrow y' = -\dfrac{3x^2y+y^5}{x^3+5xy^4}$

3.3

In Exercises 19–48, differentiate the given functions. If possible, simplify your answers.

19. $y = e^{5x}$ **20.** $y = xe^x - x$

21. $y = \dfrac{x}{e^{2x}}$ **22.** $y = x^2e^{x/2}$

23. $y = \ln(3x-2)$ **24.** $y = \ln|3x-2|$

19. $y = e^{5x}$

$y' = e^{5x} \cdot 5$

24. $y = \ln|3x-2|$

$y' = \dfrac{3}{3x-2}$

$(x \ne \frac{2}{3})$

Recall: $(\ln x)' = \frac{1}{x}$ with $x>0$

when $3x-2>0 \Rightarrow x>\frac{2}{3}$

$y = \ln(3x-2) \Rightarrow y' = \dfrac{1}{3x-2} \times 3 = \dfrac{3}{3x-2}$

when $3x-2<0 \Rightarrow x<\frac{2}{3}$

$y = \ln(2-3x) \Rightarrow y' = \dfrac{1}{2-3x} \cdot (-3) = \dfrac{3}{3x-2}$