C8.2

Recall:

Case 1 If m **is odd**, we write m as $2k+1$ and use the identity $\sin^2 x = 1 - \cos^2 x$ to obtain

$$\sin^m x = \sin^{2k+1} x = (\sin^2 x)^k \sin x = (1-\cos^2 x)^k \sin x. \quad (1)$$

Then we substitute $u = \cos x$ and $du = -\sin x\, dx$.

Case 2 If n **is odd** in $\int \sin^m x \cos^n x\, dx$, we write n as $2k+1$ and use the identity $\cos^2 x = 1 - \sin^2 x$ to obtain

$$\cos^n x = \cos^{2k+1} x = (\cos^2 x)^k \cos x = (1-\sin^2 x)^k \cos x.$$

We then substitute $u = \sin x$ and $du = \cos x\, dx$.

Case 3 If **both** m **and** n **are even** in $\int \sin^m x \cos^n x\, dx$, we substitute

$$\sin^2 x = \frac{1-\cos 2x}{2}, \qquad \cos^2 x = \frac{1+\cos 2x}{2} \quad (2)$$

to reduce the integrand to one in lower powers of $\cos 2x$.

Powers of Sines and Cosines

Evaluate the integrals in Exercises 1–22.

1. $\int \cos 2x\, dx$ 2. $\int_0^{\pi} 3\,\mathrm{si}$

3. $\int \cos^3 x \sin x\, dx$ 4. $\int \sin^4$

5. $\int \sin^3 x\, dx$ 6. $\int \cos^3$

5. $\int \sin^3 x \cdot dx = \int \sin x \cdot \sin^2 x \cdot dx = \int \sin x\,(1-\cos^2 x)\cdot dx$

let $u = \cos x$, $du = -\sin x \cdot dx$

$= -\int (1-\cos^2 x)(-\sin x)\cdot dx = -\int (1-u^2)\cdot du$

$= \int u^2 - 1\, du = \frac{1}{3}u^3 - u + C = \frac{1}{3}\cdot\cos^3 x - \cos x + C$

8. $\int_0^{\pi} \sin^5 \frac{x}{2}\, dx$

$\int_0^{\pi} \sin^5 \frac{x}{2}\cdot dx$ $\qquad d\left(\frac{x}{2}\right) = \frac{1}{2}\cdot dx$

$= 2\int_0^{\pi} \sin^5 \frac{x}{2}\cdot \frac{1}{2}dx$

$= 2\int_0^{\pi} \sin^5 \frac{x}{2}\cdot d\left(\frac{x}{2}\right)$ $\qquad$ let $\frac{x}{2} = u$

$= 2\int_0^{\frac{\pi}{2}} \sin^5 u \cdot du = 2\int_0^{\frac{\pi}{2}} \sin^4 u \cdot \sin u \cdot du = -2\int_0^{\frac{\pi}{2}} \sin^4 u \cdot (-\sin u)\cdot du$

$= -2\int_0^{\frac{\pi}{2}} (1-\cos^2 u)^2 \cdot d(\cos u)$ $\qquad$ let $t = \cos u$

$= -2\int_1^0 (1-t^2)^2 \cdot dt = -2\int_1^0 1 - 2t^2 + t^4 \cdot dt$

$= -2\left[t - \frac{2}{3}t^3 + \frac{1}{5}t^5\right]_1^0 = -2\left[0 - \left(1 - \frac{2}{3} + \frac{1}{5}\right)\right]$

$= 2\left(1 - \frac{2}{3} + \frac{1}{5}\right) = 2\left(\frac{1}{3} + \frac{1}{5}\right) = 2\times\frac{8}{15} = \frac{16}{15}$

18. $\int 8\cos^4 2\pi x\, dx$

$\int 8\cdot\cos^4(2\pi x)\cdot dx$

$= \int 8\left[\frac{1+\cos(4\pi x)}{2}\right]^2 \cdot dx = \int 8\cdot\frac{1+2\cos(4\pi x)+\cos^2 4\pi x}{4}\cdot dx$

$= 2\int 1 + 2\cos(4\pi x) + \cos^2(4\pi x)\cdot dx$

$= 2\int 1 + 2\cos(4\pi x) + \frac{1+\cos 8\pi x}{2}\, dx$

$= \int 2 + 4\cos(4\pi x) + 1 + \cos(8\pi x)\cdot dx = \int 3 + 4\cos(4\pi x) + \cos(8\pi x)\, dx$

$= 3x + 4\times\frac{1}{4\pi}\sin 4\pi x + \frac{1}{8\pi}\cdot\sin(8\pi x) + C$

$= 3x + \frac{1}{\pi}\cdot\sin 4\pi x + \frac{1}{8\pi}\cdot\sin(8\pi x) + C$

C8.3

Using Trigonometric Substitutions

Evaluate the integrals in Exercises 1–14.

8. $\int \sqrt{1-9t^2}\, dt$

Recall: $\sin^2\theta + \cos^2\theta = 1$ $\qquad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$\int \sqrt{1-9t^2}\cdot dt$ $\qquad$ let $t = \frac{1}{3}\cdot\sin\theta \Rightarrow 3t = \sin\theta \Rightarrow \theta = \sin^{-1}3t$

$\left(t^2 = \frac{1}{9}\sin^2\theta\right)$

$= \int \sqrt{1-\sin^2\theta}\cdot d\left(\frac{1}{3}\cdot\sin\theta\right)$

$= \int \sqrt{\cos^2\theta}\cdot\frac{1}{3}\cdot\cos\theta\cdot d\theta$

$= \frac{1}{3}\cdot\int |\cos\theta|\cdot\cos\theta\cdot d\theta$

$= \frac{1}{3}\cdot\int \cos^2\theta\cdot d\theta = \frac{1}{3}\int \frac{1+\cos 2\theta}{2}\cdot d\theta = \frac{1}{6}\int(1+\cos 2\theta)\cdot d\theta$

$= \frac{1}{6}\left(\theta + \frac{1}{2}\sin 2\theta\right) + C$ $\qquad \sin\theta = 3t \qquad \theta = \sin^{-1}3t$

$= \frac{1}{6}\theta + \frac{1}{6}\cdot\sin\theta\cdot\cos\theta + C$ $\qquad \cos\theta = \sqrt{1-\sin^2\theta} = \sqrt{1-9t^2}$

$= \frac{1}{6}\cdot\sin^{-1}3t + \frac{1}{6}\cdot 3t\cdot\sqrt{1-9t^2} + C$

$= \frac{1}{6}\cdot\sin^{-1}3t + \frac{1}{2}\cdot t\sqrt{1-9t^2} + C$

C8.4

Nonrepeated Linear Factors

In Exercises 9–16, express the integrand as a sum of partial fractions and evaluate the integrals.

13. $\int_4^8 \frac{y\, dy}{y^2 - 2y - 3}$

$\begin{matrix} 1 & & 1 \\ & \times & \\ 1 & & -3 \end{matrix}$

$\frac{y}{y^2-2y-3} = \frac{y}{(y+1)(y-3)} = \frac{A}{y+1} + \frac{B}{y-3} = \frac{A(y-3)+B(y+1)}{(y+1)(y-3)}$

$\therefore y = A(y-3) + B(y+1) = y(A+B) - 3A + B$

$\therefore \begin{cases} A+B=1 \\ -3A+B=0 \end{cases} \Rightarrow \begin{cases} A = \frac{1}{4} \\ B = \frac{3}{4} \end{cases}$

$\therefore \frac{y}{y^2-2y-3} = \frac{1}{4}\cdot\frac{1}{y+1} + \frac{3}{4}\cdot\frac{1}{y-3}$

$\int_4^8 \frac{y}{y^2-2y-3}\cdot dy$

$= \int_4^8 \frac{1}{4}\,\frac{1}{y+1} + \frac{3}{4}\,\frac{1}{y-3}\, dy$

$= \frac{1}{4}\int_4^8 \frac{1}{y+1}\cdot dy + \frac{3}{4}\int_4^8 \frac{1}{y-3}\cdot dy$

$= \frac{1}{4}\cdot\left[\ln(y+1)\right]_4^8 + \frac{3}{4}\left[\ln(y-3)\right]_4^8$

$= \frac{1}{4}\left(\ln 9 - \ln 5\right) + \frac{3}{4}\left(\ln 5 - \ln 1\right)$

$= \frac{1}{4}\ln 9 - \frac{1}{4}\ln 5 + \frac{3}{4}\ln 5$

$= \frac{1}{4}\ln 9 + \frac{1}{2}\ln 5$

$= \frac{1}{2}\ln 3 + \frac{1}{2}\cdot\ln 5 = \frac{1}{2}\ln 15$

C8.7

Evaluating Improper Integrals

The integrals in Exercises 1–34 converge. Evaluate the integrals without using tables.

12. $\displaystyle\int_2^{\infty} \frac{2\,dt}{t^2-1}$

$$\int_2^{\infty} \frac{2}{t^2-1}\cdot dt = \lim_{b\to+\infty} \int_2^b \frac{2}{t^2-1}\cdot dt$$

$$\frac{2}{t^2-1} = \frac{2}{(t+1)(t-1)} = \frac{A}{t+1} + \frac{B}{t-1} = \frac{A(t-1)+B(t+1)}{(t+1)(t-1)}$$

$$\Rightarrow A(t-1)+B(t+1) = t(A+B) - A + B = 2$$

$$\therefore \begin{cases} A+B=0 \\ -A+B=2 \end{cases} \Rightarrow \begin{cases} A=-1 \\ B=1 \end{cases}$$

$$\therefore \frac{2}{t^2-1} = -\frac{1}{t+1} + \frac{1}{t-1}$$

$$= \lim_{b\to+\infty} \int_2^b \frac{1}{t-1} - \frac{1}{t+1}\, dt$$

$$= \lim_{b\to+\infty} \left[\ln(t-1) - \ln(t+1)\right]_2^b$$

$$= \lim_{b\to+\infty} \left[\ln\left(\frac{t-1}{t+1}\right)\right]_2^b$$

$$= \lim_{b\to+\infty} \ln\left(\frac{b-1}{b+1}\right) - \ln\left(\frac{2-1}{2+1}\right)$$

$$= \ln 1 - \ln\left(\frac{1}{3}\right)$$

$$= 0 - \ln 3^{-1} = \ln 3$$

$$\lim_{b\to+\infty} \ln\left(\frac{b-1}{b+1}\right) = \lim_{b\to+\infty} \ln\left(\frac{1-\frac{1}{b}}{1+\frac{1}{b}}\right) = \ln 1$$

70. $\int_{-\infty}^{\infty} f(x)\,dx$ **may not equal** $\lim_{b\to\infty} \int_{-b}^{b} f(x)\,dx$. Show that

$$\int_0^{\infty} \frac{2x\,dx}{x^2+1}$$

diverges and hence that

$$\int_{-\infty}^{\infty} \frac{2x\,dx}{x^2+1}$$

diverges. Then show that

$$\lim_{b\to\infty} \int_{-b}^{b} \frac{2x\,dx}{x^2+1} = 0.$$

1) Show $\int_0^{\infty} \frac{2x}{x^2+1}\cdot dx$ diverges.

$$\int_0^{+\infty} \frac{2x}{x^2+1}\cdot dx = \lim_{b\to+\infty} \int_0^b \frac{2x}{x^2+1}\cdot dx$$

$$\int_0^b \frac{2x}{x^2+1}\cdot dx = \int_0^b \frac{1}{x^2+1}\cdot d(x^2) = \int_0^b \frac{1}{x^2+1}\cdot d(x^2+1)$$

$$= \left[\ln(x^2+1)\right]_0^b = \ln(b^2+1) - \ln 1 = \ln(b^2+1)$$

$$\because \lim_{b\to+\infty} \ln(b^2+1) = +\infty$$

$\therefore \int_0^{\infty} \frac{2x}{x^2+1}\cdot dx$ diverges. $\therefore \int_{-\infty}^{+\infty} \frac{2x}{x^2+1}\cdot dx$ diverges

2) Show $\int_{-\infty}^{+\infty} \frac{2x}{x^2+1}\cdot dx = 0$

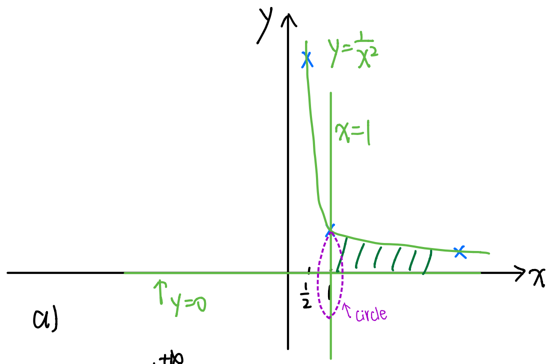
$$\int_{-\infty}^{+\infty} \frac{2x}{x^2+1}\cdot dx = \lim_{b\to+\infty} \int_{-b}^{b} \frac{2x}{x^2+1}\cdot dx$$

$$\int_{-b}^{b} \frac{2x}{x^2+1}\cdot dx = \left[\ln(x^2+1)\right]_{-b}^{b} = \ln(b^2+1) - \ln(b^2+1) = 0$$

77. Consider the infinite region in the first quadrant bounded by the graphs of $y = \frac{1}{x^2}$, $y = 0$, and $x = 1$.

a. Find the area of the region.

b. Find the volume of the solid formed by revolving the region (i) about the x-axis; (ii) about the y-axis.



a)

$$\text{Area} = \int_1^{+\infty} \frac{1}{x^2}\cdot dx$$

$$= \lim_{b\to+\infty} \int_1^b \frac{1}{x^2}\cdot dx = \lim_{b\to+\infty} \left[-\frac{1}{x}\right]_1^b$$

$$= \lim_{b\to+\infty} \left(-\frac{1}{b}\right) - (-1) = \lim_{b\to+\infty} -\frac{1}{b} + 1 = 0 + 1 = 1$$

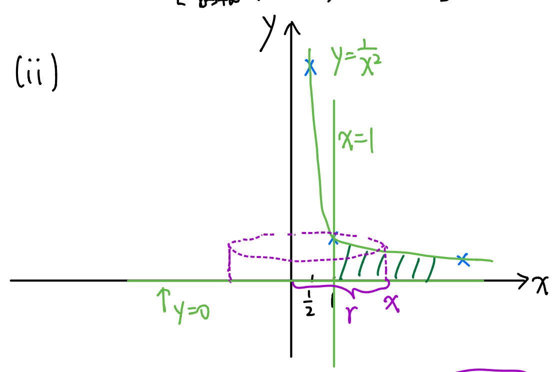
b) $r(x) = \frac{1}{x^2}$

(i)
$$V = \int_1^{+\infty} \pi\cdot r^2(x)\cdot dx = \int_1^{+\infty} \pi\cdot \frac{1}{x^4}\cdot dx = \pi\int_1^{+\infty} x^{-4}\cdot dx$$

$$= \pi\cdot \lim_{b\to+\infty} \int_1^b x^{-4}\cdot dx = \pi\cdot \lim_{b\to+\infty} \left[\frac{1}{-3} x^{-3}\right]_1^b$$

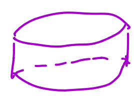
$$= \pi\left[\lim_{b\to+\infty} \left(-\frac{1}{3}\cdot b^{-3}\right) - \left(-\frac{1}{3}\right)\right] = \pi\left(0 + \frac{1}{3}\right) = \frac{1}{3}\pi.$$

(ii)



$r(x) = x$

Side Area of



$$V = \int_1^{+\infty} 2\pi r\cdot \frac{1}{x^2}\cdot dx$$

$$= \lim_{b\to+\infty} \int_1^b 2\pi x\cdot \frac{1}{x^2}\cdot dx$$

$$= \lim_{b\to+\infty} \int_1^b 2\pi\cdot \frac{1}{x}\cdot dx$$

$$= 2\pi\cdot \lim_{b\to+\infty} \int_1^b \frac{1}{x}\cdot dx$$

$$= 2\pi\cdot \lim_{b\to+\infty} \left[\ln x\right]_1^b$$

$$= 2\pi\cdot \lim_{b\to+\infty} (\ln b - \ln 1)$$

$$= 2\pi\cdot \lim_{b\to+\infty} \ln b = +\infty$$