5.6

5. $\int \frac{x\,dx}{(4x^2+1)^5}$

let $t=4x^2+1$

$dt = 8x\cdot dx \Rightarrow x\cdot dx = \frac{1}{8}\cdot dt$

$\int \frac{x\cdot dx}{(4x^2+1)^5} \overset{t=4x^2+1}{=\!=\!=} \int \frac{\frac{1}{8}}{t^5}\cdot dt$

$= \frac{1}{8}\int t^{-5}\cdot dt$ $\qquad (-\frac{1}{4}t^{-4})' = t^{-5}$

$= \frac{1}{8}\cdot\left(-\frac{1}{4}\cdot t^{-4}\right)+C$

$= -\frac{1}{32}t^{-4}+C$

$= -\frac{1}{32(4x^2+1)^4}+C$

6. $\int \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx$

let $t=\sqrt{x}$ $\qquad (\sqrt{x})' = \frac{1}{2\sqrt{x}}$

$dt = \frac{1}{2\sqrt{x}}\cdot dx$

$\Rightarrow 2dt = \frac{1}{\sqrt{x}}\cdot dx$

$\int \frac{\sin\sqrt{x}}{\sqrt{x}}\cdot dx = \int \sin t\cdot 2dt$

$= -2\cdot\cos t + C$

$= -2\cos\sqrt{x}+C$

12. $\int \frac{\ln t}{t}\,dt$

let $u=\ln t$

$du = \frac{1}{t}\cdot dt$

$\int \frac{\ln t}{t}\cdot dt$

$= \int u\cdot du = \frac{1}{2}u^2+C$

$= \frac{1}{2}(\ln t)^2 + C$

17. $\int \frac{dx}{e^x+1}$ $\qquad d(e^x+1) = de^x$

let $t=e^x+1$ $\qquad$ or let $t=e^{-x}$

$dt = e^x\cdot dx \Rightarrow dt = (t-1)\cdot dx$

$\Rightarrow dx = \frac{1}{t-1}\cdot dt$

$\int \frac{1}{t(t-1)}\cdot dt$

$= \int\left(\frac{1}{t-1}-\frac{1}{t}\right)\cdot dt$

$= \ln(t-1) - \ln t + C$

$= \ln\left(1-\frac{1}{t}\right)+C$

$= \ln\left(1-\frac{1}{e^x+1}\right)+C$

or $= \ln\left[(1+e^{-x})^{-1}\right]+C$

$= -\ln(1+e^{-x})+C$

$1-\frac{1}{e^x+1}$
$= \frac{e^x}{(e^x+1)}\cdot\frac{e^{-x}}{e^{-x}}$
$= \frac{1}{1+e^{-x}}$

18. $\int \frac{dx}{e^x+e^{-x}}$

$\int \frac{1}{e^x+e^{-x}}\cdot dx = \int \frac{e^x}{e^x(e^x+e^{-x})}\cdot dx$

$= \int \frac{e^x}{e^{2x}+1}\cdot dx$ $\qquad$ let $t=e^x$ (then $e^{2x}=t^2$)

$dt = e^x\cdot dx$

$= \int \frac{1}{t^2+1}\cdot dt$

$= \tan^{-1}t+C$

$= \tan^{-1}e^x+C$

16. $\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}+C$

35. $\int \frac{\sin^2 x}{\cos^4 x}\,dx$

$\int \frac{\sin^2 x}{\cos^4 x}\cdot dx$

$= \int \tan^2 x\cdot\sec^2 x\cdot dx$ $\qquad$ recall: $(\tan x)' = \sec^2 x$

let $u=\tan x$

$du = \sec^2 x\cdot dx$

$= \int u^2\cdot du = \frac{1}{3}u^3+C = \frac{1}{3}\tan^3 x + C$

$$\int U\,dV = UV - \int V\,dU.$$

6.1

1. $\int x\cos x\,dx$ $\qquad$ **2.** $\int (x+3)e^{2x}\,dx$

1. $\int x\cdot\cos x\cdot dx$

$= \int x\cdot d(\sin x) = x\cdot\sin x - \int \sin x\cdot dx = x\cdot\sin x + \cos x + C$

2. $\int (x+3)\cdot e^{2x}\cdot dx$ $\qquad d(\frac{1}{2}\cdot e^{2x}) = \frac{1}{2}\cdot e^{2x}\cdot 2 = e^{2x}$

$= \int (x+3)\cdot d\left(\frac{1}{2}e^{2x}\right)$

$= \frac{1}{2}(x+3)\cdot e^{2x} - \int \frac{1}{2}\cdot e^{2x}\cdot d(x+3)$

$= \frac{1}{2}(x+3)\cdot e^{2x} - \int \frac{1}{2}\cdot e^{2x}\cdot dx$

$= \frac{1}{2}(x+3)\cdot e^{2x} - \frac{1}{4}e^{2x}+C$

3. $\int x^2\cos\pi x\,dx$ $\qquad$ **4.** $\int (x^2-2x)e^{kx}\,dx$

$\int x^2\cdot\cos\pi x\cdot dx$

$= \int x^2\cdot d\left(\frac{1}{\pi}\cdot\sin\pi x\right)$

$= \frac{1}{\pi}\cdot\sin\pi x\cdot x^2 - \int \frac{1}{\pi}\cdot\sin\pi x\cdot dx^2$

$= \frac{x^2\cdot\sin\pi x}{\pi} - \int \frac{1}{\pi}\cdot\sin\pi x\cdot 2x\cdot dx$

$= \frac{x^2\cdot\sin\pi x}{\pi} + \int \frac{2}{\pi^2}\cdot x\, d(\cos\pi x)$

$= \frac{x^2\cdot\sin\pi x}{\pi} + \frac{2}{\pi^2}\cdot x\cdot\cos\pi x - \int \frac{2}{\pi^2}\cdot\cos\pi x\cdot dx$

$= \frac{x^2\cdot\sin\pi x}{\pi} + \frac{2x\cdot\cos\pi x}{\pi^2} - \frac{2}{\pi^3}\cdot\sin\pi x + C$

4. $\int (x^2-2x)e^{kx}\,dx$

$\int (x^2-2x)\cdot e^{kx}\cdot dx = \int (x^2-2x)\cdot d\left(\frac{1}{k}\cdot e^{kx}\right)$

$= \frac{1}{k}\cdot e^{kx}(x^2-2x) - \int \frac{1}{k}\cdot e^{kx}\cdot d(x^2-2x)$

$= \frac{1}{k}\cdot e^{kx}(x^2-2x) - \int \frac{1}{k}\cdot e^{kx}\cdot(2x-2)\cdot dx$

$= \frac{1}{k}\cdot e^{kx}(x^2-2x) - \int \frac{2}{k}\cdot(x-1)\cdot d\left(\frac{1}{k}\cdot e^{kx}\right)$

$= \frac{1}{k}\cdot e^{kx}(x^2-2x) - \frac{2}{k}\cdot(x-1)\cdot\frac{1}{k}\cdot e^{kx} + \int \frac{2}{k}\cdot\frac{1}{k}\cdot e^{kx}\cdot d(x-1)$

$= \frac{1}{k}\cdot e^{kx}(x^2-2x) - \frac{2}{k}\cdot(x-1)\cdot\frac{1}{k}\cdot e^{kx} + \int \frac{2}{k^2}\cdot e^{kx}\cdot dx$

$= \frac{1}{k}\cdot e^{kx}(x^2-2x) - \frac{2}{k^2}(x-1)\cdot e^{kx} + \frac{2}{k^3}\cdot e^{kx}+C$

5. $\int x^3 \ln x \, dx$

$$\int x^3 \cdot \ln x \cdot dx$$
$$= \int \ln x \cdot d\left(\tfrac{1}{4}x^4\right)$$
$$= \tfrac{1}{4}\cdot \ln x \cdot x^4 - \int \tfrac{1}{4}\cdot x^4 \cdot d(\ln x)$$
$$= \tfrac{1}{4}\cdot \ln x \cdot x^4 - \int \tfrac{1}{4}\cdot x^4 \cdot \tfrac{1}{x}\cdot dx$$
$$= \tfrac{1}{4}\cdot \ln x \cdot x^4 - \int \tfrac{1}{4}\cdot x^3 \cdot dx$$
$$= \tfrac{1}{4}\cdot \ln x \cdot x^4 - \tfrac{1}{4}\cdot\tfrac{1}{4}\cdot x^4 + C$$
$$= \tfrac{1}{4}\cdot \ln x \cdot x^4 - \tfrac{1}{16}\cdot x^4 + C$$

6. $\int x(\ln x)^3 \, dx$

$$\int x\cdot(\ln x)^3 \cdot dx \qquad \text{(way2)}$$
$$= \int (\ln x)^3 \cdot d\left(\tfrac{1}{2}x^2\right)$$
$$= (\ln x)^3\cdot\tfrac{1}{2}x^2 - \int \tfrac{1}{2}x^2 \cdot d\left((\ln x)^3\right)$$
$$= (\ln x)^3\cdot\tfrac{1}{2}x^2 - \int \tfrac{1}{2}\cdot x^2\cdot 3\cdot \ln^2 x\cdot\tfrac{1}{x}\cdot dx$$
$$= (\ln x)^3\cdot\tfrac{1}{2}x^2 - \int \tfrac{3}{2}\, x\cdot \ln^2 x\cdot dx$$
$$= \tfrac{1}{2}x^2\cdot(\ln x)^3 - \int \tfrac{3}{2}\ln^2 x\cdot d\left(\tfrac{1}{2}x^2\right)$$
$$= \tfrac{1}{2}x^2\cdot(\ln x)^3 - \tfrac{3}{2}\cdot\ln^2 x\cdot\tfrac{1}{2}\cdot x^2 + \int \tfrac{3}{2}\cdot\tfrac{1}{2}x^2\cdot d(\ln^2 x)$$
$$= \tfrac{1}{2}x^2\cdot(\ln x)^3 - \tfrac{3}{4}\ln^2 x\cdot x^2 + \int \tfrac{3}{2}\cdot\cancel{\tfrac{1}{2}}\cdot x^{\cancel{2}}\cdot\cancel{2}\ln x\cdot\cancel{\tfrac{1}{x}}\cdot dx$$
$$= \tfrac{1}{2}x^2\cdot(\ln x)^3 - \tfrac{3}{4}\ln^2 x\cdot x^2 + \int \tfrac{3}{2}\, x\cdot\ln x\cdot dx$$
$$= \tfrac{1}{2}x^2\cdot(\ln x)^3 - \tfrac{3}{4}\ln^2 x\cdot x^2 + \int \tfrac{3}{2}\ln x\cdot d\left(\tfrac{1}{2}x^2\right)$$
$$= \tfrac{1}{2}x^2\cdot(\ln x)^3 - \tfrac{3}{4}\ln^2 x\cdot x^2 + \tfrac{3}{2}\cdot\ln x\cdot\tfrac{1}{2}x^2 - \int \tfrac{3}{2}\cdot\tfrac{1}{2}x^2\cdot d(\ln x)$$

$$\int \tfrac{3}{4}x^2\cdot\tfrac{1}{x}\cdot dx = \int \tfrac{3}{4}\cdot x\cdot dx$$

$$= \tfrac{1}{2}x^2\cdot(\ln x)^3 - \tfrac{3}{4}\ln^2 x\cdot x^2 + \tfrac{3}{4}\cdot x^2\cdot\ln x - \tfrac{3}{8}x^2 + C$$
$$= \tfrac{x^2}{2}\left[(\ln x)^3 - \tfrac{3}{2}(\ln x)^2 + \tfrac{3}{2}\cdot\ln x - \tfrac{3}{4}\right] + C$$

$$\int x\cdot(\ln x)^n\cdot dx = I_n$$
$$= \int (\ln x)^n\cdot d\left(\tfrac{1}{2}x^2\right)$$
$$= (\ln x)^n\cdot\tfrac{1}{2}x^2 - \int \tfrac{1}{2}x^2\cdot d\left((\ln x)^n\right)$$
$$= \tfrac{1}{2}x^2\cdot(\ln x)^n - \tfrac{1}{2}\int x^2\cdot n(\ln x)^{n-1}\cdot\tfrac{1}{x}\cdot dx$$
$$= \tfrac{1}{2}x^2\cdot(\ln x)^n - \tfrac{n}{2}\int x\cdot(\ln x)^{n-1}\cdot dx$$
$$= \tfrac{1}{2}x^2\cdot(\ln x)^n - \tfrac{n}{2}\cdot I_{n-1}$$

$$\Rightarrow I_n = \tfrac{1}{2}x^2\cdot(\ln x)^n - \tfrac{n}{2}\cdot I_{n-1}$$
$$\therefore \int x\cdot(\ln x)^3\cdot dx = I_3$$
$$= \tfrac{1}{2}x^2\cdot(\ln x)^3 - \tfrac{3}{2}\cdot I_2$$
$$= \tfrac{1}{2}x^2\cdot(\ln x)^3 - \tfrac{3}{2}\left[\tfrac{x^2}{2}\left((\ln x)^2 - \ln x + \tfrac{1}{2}\right)\right] + C$$
$$= \tfrac{x^2}{2}\left[(\ln x)^3 - \tfrac{3}{2}(\ln x)^2 + \tfrac{3}{2}\ln x - \tfrac{3}{4}\right] + C$$

$$I_2 = \tfrac{1}{2}\cdot x^2\cdot(\ln x)^2 - I_1$$
$$I_1 = \tfrac{1}{2}\cdot x^2\cdot\ln x - \tfrac{1}{2}\cdot I_0$$
$$I_0 = \int x\cdot dx = \tfrac{1}{2}x^2 + C$$
$$I_1 = \tfrac{1}{2}x^2\cdot\ln x - \tfrac{1}{2}\cdot\tfrac{1}{2}x^2 + C$$
$$= \tfrac{x^2}{2}\left(\ln x - \tfrac{1}{2}\right) + C$$
$$I_2 = \tfrac{1}{2}x^2\cdot(\ln x)^2 - \tfrac{x^2}{2}\left(\ln x - \tfrac{1}{2}\right) + C$$
$$= \tfrac{x^2}{2}\left[(\ln x)^2 - \ln x + \tfrac{1}{2}\right] + C$$

6.2

4. $\int \frac{x^2}{x-4} \, dx$

$$\begin{array}{r} x+4 \\ x-4\,\overline{)\,x^2 \qquad} \\ x^2-4x \\ \hline 4x \\ 4x-16 \\ \hline 16 \end{array}$$

$$\int \frac{x^2}{x-4}\cdot dx = \int \left(x+4+\frac{16}{x-4}\right)\cdot dx$$
$$= \tfrac{1}{2}x^2 + 4x + 16\cdot\ln|x-4| + C$$

11. $\int \frac{x-2}{x^2+x} \, dx$

$$\frac{x-2}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1} = \frac{A(x+1)+Bx}{x(x+1)} = \frac{x(A+B)+A}{x(x+1)}$$

$$\begin{cases} A+B=1 \\ A=-2 \end{cases} \Rightarrow \begin{cases} A=-2 \\ B=3 \end{cases} \qquad \therefore \frac{x-2}{x(x+1)} = -\frac{2}{x} + \frac{3}{x+1}$$

$$\int \frac{x-2}{x^2+x}\cdot dx = \int\left(-\frac{2}{x} + \frac{3}{x+1}\right)\cdot dx$$
$$= -2\cdot\ln|x| + 3\cdot\ln|x+1| + C$$

13. $\int \frac{dx}{1-6x+9x^2}$

$$\frac{1}{1-6x+9x^2} = \frac{1}{(1-3x)^2}$$

$$\int \frac{dx}{1-6x+9x^2} = \int \frac{dx}{(1-3x)^2} = \frac{1}{3}\int \frac{-1}{(1-3x)^2}\cdot d(1-3x)$$
$$= \frac{1}{3}\cdot\frac{1}{1-3x} + C$$
$$= \frac{1}{3(1-3x)} + C$$

$$\left(\tfrac{1}{x}\right)' = -\tfrac{1}{x^2}$$

or let $t = 1-3x$

$dt = -3\cdot dx$

$$\int \frac{dx}{(1-3x)^2} = \int \frac{1}{t^2}\cdot\frac{1}{-3}\,dt$$
$$= \frac{1}{3t} + C$$
$$= \frac{1}{3(1-3x)} + C$$