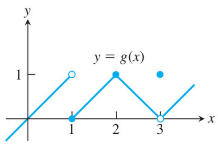
C2.2 : 1,5,24,35,39,63

Limits from Graphs

1. For the function $g(x)$ graphed here, find the following limits or explain why they do not exist.

a. $\lim_{x\to1} g(x)$ **b.** $\lim_{x\to2} g(x)$ **c.** $\lim_{x\to3} g(x)$ **d.** $\lim_{x\to2.5} g(x)$



DEFINITION Let $f(x)$ be defined on an open interval about c, except possibly at c itself. We say that the **limit of $f(x)$ as x approaches c is the number L**, and write

$$\lim_{x\to c} f(x) = L,$$

if, for every number $\varepsilon > 0$, there exists a corresponding number $\delta > 0$ such that

$$|f(x) - L| < \varepsilon \quad \text{whenever} \quad 0 < |x - c| < \delta.$$

a. $\lim_{x\to1^+} g(x) = 0$, $\lim_{x\to1^-} g(x) = 1$, $\lim_{x\to1^+} g(x) \neq \lim_{x\to1^-} g(x)$

Does not exist. As x approaches 1 from the right, $g(x)$ approaches 0. As x approaches 1 from the left, $g(x)$ approaches 1. There is no single number L that all the values $g(x)$ get arbitrarily close to as $x \to 1$.

b. $\lim_{x\to2^+} g(x) = 1$, $\lim_{x\to2^-} g(x) = 1$, $\lim_{x\to2^+} g(x) = \lim_{x\to2^-} g(x) = 1$

$\therefore \lim_{x\to2} g(x) = 1.$

c. $\lim_{x\to3^-} g(x) = 0$, $\lim_{x\to3^+} g(x) = 0$, $\lim_{x\to3^+} g(x) = \lim_{x\to3^-} g(x) = 0$.

$\therefore \lim_{x\to3} g(x) = 0$

d. $\lim_{x\to2.5^-} g(x) = 0.5$, $\lim_{x\to2.5^+} g(x) = 0.5$, $\lim_{x\to2.5^-} g(x) = \lim_{x\to2.5^+} g(x) = 0.5$

$\therefore \lim_{x\to2.5} g(x) = 0.5$.

Existence of Limits

In Exercises 5 and 6, explain why the limits do not exist.

5. $\lim_{x\to0} \frac{x}{|x|}$ **6.** $\lim_{x\to1} \frac{1}{x-1}$

5. $\lim_{x\to0} \frac{x}{|x|}$

approach from left approach from right

$\lim_{x\to0^-} \frac{x}{|x|} = \lim_{x\to0^-} \frac{x}{-x} = -1$; $\lim_{x\to0^+} \frac{x}{|x|} = \lim_{x\to0^+} \frac{x}{x} = 1$

$\therefore \lim_{x\to0^-} \frac{x}{|x|} \neq \lim_{x\to0^+} \frac{x}{|x|}$ $\therefore$ the limit does not exist.

Limits of quotients Find the limits in Exercises 23–42.

23. $\lim_{x\to5} \frac{x-5}{x^2-25}$ **24.** $\lim_{x\to-3} \frac{x+3}{x^2+4x+3}$

24. $\lim_{x\to-3} \frac{x+3}{x^2+4x+3}$

$= \lim_{x\to-3} \frac{x+3}{(x+3)(x+1)} = \lim_{x\to-3} \frac{1}{x+1} = -\frac{1}{2}$

35. $\lim_{x\to9} \frac{\sqrt{x}-3}{x-9}$

35. $\lim_{x\to9} \frac{\sqrt{x}-3}{x-9} = \lim_{x\to9} \frac{\cancel{\sqrt{x}-3}}{(\cancel{\sqrt{x}-3})(\sqrt{x}+3)} = \lim_{x\to9} \frac{1}{\sqrt{x}+3} = \frac{1}{3+3} = \frac{1}{6}$

39. $\lim_{x\to2} \frac{\sqrt{x^2+12}-4}{x-2}$

39. $\lim_{x\to2} \frac{\sqrt{x^2+12}-4}{x-2} = \lim_{x\to2} \frac{(\sqrt{x^2+12}-4)(\sqrt{x^2+12}+4)}{(x-2)(\sqrt{x^2+12}+4)}$

$= \lim_{x\to2} \frac{x^2+12-16}{(x-2)(\sqrt{x^2+12}+4)} = \lim_{x\to2} \frac{x^2-4}{(x-2)(\sqrt{x^2+12}+4)}$

$= \lim_{x\to2} \frac{\cancel{(x-2)}(x+2)}{\cancel{(x-2)}(\sqrt{x^2+12}+4)} = \frac{2+2}{\sqrt{4+12}+4} = \frac{4}{8} = \frac{1}{2}$

Using the Sandwich Theorem

63. If $\sqrt{5-2x^2} \le f(x) \le \sqrt{5-x^2}$ for $-1 \le x \le 1$, find $\lim_{x\to0} f(x)$.

THEOREM 4—The Sandwich Theorem Recall:

Suppose that $g(x) \le f(x) \le h(x)$ for all x in some open interval containing c, except possibly at $x = c$ itself. Suppose also that

$$\lim_{x\to c} g(x) = \lim_{x\to c} h(x) = L.$$

Then $\lim_{x\to c} f(x) = L$.

$\lim_{x\to0} \sqrt{5-2x^2} = \sqrt{5}$, $\lim_{x\to0} \sqrt{5-x^2} = \sqrt{5}$ $\therefore \lim_{x\to0} \sqrt{5-2x^2} = \lim_{x\to0} \sqrt{5-x^2} = \sqrt{5}$

$\because \sqrt{5-2x^2} \le f(x) \le \sqrt{5-x^2}$, for $-1 \le x \le 1$

According to the sandwich Theorem, $\lim_{x\to0} f(x) = \sqrt{5}$

C2.3 : 17, 25, 33, 34, 41, 49

Finding Deltas Algebraically

Each of Exercises 15–30 gives a function $f(x)$ and numbers L, c, and $\varepsilon > 0$. In each case, find the largest open interval about c on which the inequality $|f(x) - L| < \varepsilon$ holds. Then give a value for $\delta > 0$ such that for all x satisfying $0 < |x - c| < \delta$, the inequality $|f(x) - L| < \varepsilon$ holds.

15. $f(x) = x + 1, \quad L = 5, \quad c = 4, \quad \varepsilon = 0.01$

16. $f(x) = 2x - 2, \quad L = -6, \quad c = -2, \quad \varepsilon = 0.02$

17. $f(x) = \sqrt{x + 1}, \quad L = 1, \quad c = 0, \quad \varepsilon = 0.1$

18. $f(x) = \sqrt{x}, \quad L = 1/2, \quad c = 1/4, \quad \varepsilon = 0.1$

19. $f(x) = \sqrt{19 - x}, \quad L = 3, \quad c = 10, \quad \varepsilon = 1$

20. $f(x) = \sqrt{x - 7}, \quad L = 4, \quad c = 23, \quad \varepsilon = 1$

21. $f(x) = 1/x, \quad L = 1/4, \quad c = 4, \quad \varepsilon = 0.05$

22. $f(x) = x^2, \quad L = 3, \quad c = \sqrt{3}, \quad \varepsilon = 0.1$

23. $f(x) = x^2, \quad L = 4, \quad c = -2, \quad \varepsilon = 0.5$

24. $f(x) = 1/x, \quad L = -1, \quad c = -1, \quad \varepsilon = 0.1$

25. $f(x) = x^2 - 5, \quad L = 11, \quad c = 4, \quad \varepsilon = 1$

Recall:

$|f(x) - L| < \varepsilon$ holds for all x that statisfies $0 < |x - c| < \delta$

17. $f(x) = \sqrt{x+1} \quad L = 1, \ c = 0, \ \varepsilon = 0.1 \qquad x+1 \geq 0 \Rightarrow x \geq -1$

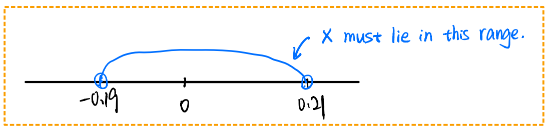
$|\sqrt{x+1} - 1| < 0.1 \qquad 0 < |x - 0| < \delta$

$-0.1 < \sqrt{x+1} - 1 < 0.1 \qquad 0 < |x| < \delta$

$0.9 < \sqrt{x+1} < 1.1 \qquad -\delta < x < \delta$

$0.81 < x+1 < 1.21 \qquad \therefore \delta_{max} = 0.19. \ \langle \delta \in (0, 0.19] \rangle$

$-0.19 < x < 0.21$



25. $f(x) = x^2 - 5, \quad L = 11, \quad c = 4, \quad \varepsilon = 1$

$|x^2 - 5 - 11| < 1 \qquad 0 < |x - 4| < \delta$

$\Rightarrow |x^2 - 16| < 1 \qquad -\delta < x - 4 < \delta$

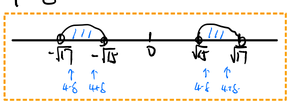
$\Rightarrow -1 < x^2 - 16 < 1 \qquad 4 - \delta < x < 4 + \delta$

$15 < x^2 < 17$

$\therefore x \in (\sqrt{15}, \sqrt{17}) \cup (-\sqrt{17}, -\sqrt{15})$

no need to consider, because $c = 4$.

we also could discard this condition later.



$\therefore \begin{cases} 4 - \delta > \sqrt{15} \\ 4 + \delta < \sqrt{17} \end{cases}$ or $\begin{cases} 4 - \delta > -\sqrt{17} \\ 4 + \delta < -\sqrt{15} \end{cases} \Rightarrow \begin{cases} \delta < 4 + \sqrt{17} \\ \delta < -4 - \sqrt{15} \end{cases}$ no solution in this case!

$\Rightarrow \begin{cases} \delta < 4 - \sqrt{15} \approx 0.127 \\ \delta < \sqrt{17} - 4 \approx 0.123 \end{cases}$

$\therefore \delta_{max} = \sqrt{17} - 4 \approx 0.123. \ \langle \delta \in (0, \sqrt{17} - 4] \rangle$

Using the Formal Definition

Each of Exercises 31–36 gives a function $f(x)$, a point c, and a positive number ε. Find $L = \lim_{x \to c} f(x)$. Then find a number $\delta > 0$ such that

$$|f(x) - L| < \varepsilon \quad \text{whenever} \quad 0 < |x - c| < \delta.$$

31. $f(x) = 3 - 2x, \quad c = 3, \quad \varepsilon = 0.02$

32. $f(x) = -3x - 2, \quad c = -1, \quad \varepsilon = 0.03$

33. $f(x) = \dfrac{x^2 - 4}{x - 2}, \quad c = 2, \quad \varepsilon = 0.05$

33. $f(x) = \dfrac{x^2 - 4}{x - 2} \quad (x \neq 2)$

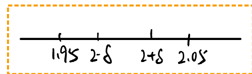
$\lim_{x \to 2} \dfrac{x^2 - 4}{x - 2} = \lim_{x \to 2} \dfrac{(x-2)(x+2)}{x - 2} = \lim_{x \to 2} x + 2 = 4 \quad \therefore L = 4$

$|f(x) - 4| < 0.05 \qquad 0 < |x - 2| < \delta$

$\Rightarrow \left| \dfrac{x^2 - 4}{x - 2} - 4 \right| < 0.05 \qquad -\delta < x - 2 < \delta$

$\Rightarrow -0.05 < x + 2 - 4 < 0.05 \qquad 2 - \delta < x < 2 + \delta$

$1.95 < x < 2.05 \ (x \neq 2)$



$\therefore \begin{cases} 2 - \delta > 1.95 \Rightarrow \delta < 0.05 \\ 2 + \delta < 2.05 \Rightarrow \delta < 0.05 \end{cases} \Rightarrow \delta_{max} = 0.05 \quad \langle \delta \in (0, 0.05] \rangle$

34. $f(x) = \dfrac{x^2 + 6x + 5}{x + 5}, \quad c = -5, \quad \varepsilon = 0.05$

$f(x) = \dfrac{x^2 + 6x + 5}{x + 5} = \dfrac{(x+5)(x+1)}{x + 5} = x + 1 \quad (x \neq -5)$

$\lim_{x \to -5} f(x) = \lim_{x \to -5} x + 1 = -5 + 1 = -4 \quad \therefore L = -4$

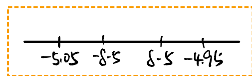
$|f(x) - L| < \varepsilon \qquad 0 < |x - (-5)| < \delta$

$-\delta < x + 5 < \delta$

$-0.05 < (x+1) - (-4) < 0.05 \qquad -\delta - 5 < x < \delta - 5$

$-0.05 < x + 5 < 0.05$

$-5.05 < x < -4.95 \ (x \neq -5)$



$\therefore \begin{cases} -5.05 < -\delta - 5 \\ \delta - 5 < -4.95 \end{cases} \Rightarrow \begin{cases} \delta < 0.05 \\ \delta < 0.05 \end{cases} \quad \therefore \delta_{max} = 0.05 \quad \langle \delta \in (0, 0.05] \rangle$

Prove the limit statements in Exercises 37–50.

37. $\lim_{x \to 4} (9 - x) = 5$ **38.** $\lim_{x \to 3} (3x - 7) = 2$

39. $\lim_{x \to 9} \sqrt{x - 5} = 2$ **40.** $\lim_{x \to 0} \sqrt{4 - x} = 2$

41. $\lim_{x \to 1} f(x) = 1$ if $f(x) = \begin{cases} x^2, & x \neq 1 \\ 2, & x = 1 \end{cases}$

Recall:

DEFINITION Let $f(x)$ be defined on an open interval about c, except possibly at c itself. We say that the **limit of $f(x)$ as x approaches c is the number L**, and write

$$\lim_{x \to c} f(x) = L,$$

if, for every number $\varepsilon > 0$, there exists a corresponding number $\delta > 0$ such that

$$|f(x) - L| < \varepsilon \quad \text{whenever} \quad 0 < |x - c| < \delta.$$

$|f(x) - L| < \varepsilon$

$|x^2 - 1| < \varepsilon$ since x won't exactly equal to 1 for limitation.

$-\varepsilon < x^2 - 1 < \varepsilon$

$1 - \varepsilon < x^2 < 1 + \varepsilon$ since ε is always small enough.

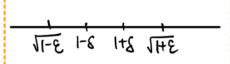
$\sqrt{1-\varepsilon} < x < \sqrt{1+\varepsilon}$

Then, figure out whether there exists $\delta > 0$ could make it whenever $0 < |x - c| < \delta$.

$0 < |x - 1| < \delta$

$-\delta < x - 1 < \delta$

$1 - \delta < x < 1 + \delta$



If exists:

$\begin{cases} \sqrt{1-\varepsilon} < 1 - \delta \\ 1 + \delta < \sqrt{1+\varepsilon} \end{cases} \Rightarrow \begin{matrix} \delta < 1 - \sqrt{1-\varepsilon} \\ \delta < \sqrt{1+\varepsilon} - 1 \end{matrix} \Rightarrow \delta < \sqrt{1+\varepsilon} - 1$

About $1 - \sqrt{1-\varepsilon} \ ? \ \sqrt{1+\varepsilon} - 1$

Assume $1 - \sqrt{1-\varepsilon} < \sqrt{1+\varepsilon} - 1$

$2 < \sqrt{1+\varepsilon} + \sqrt{1-\varepsilon}$

$4 < 1 + \cancel{\varepsilon} + 1 - \cancel{\varepsilon} + 2\sqrt{1+\varepsilon} \cdot \sqrt{1-\varepsilon}$

$2 < 2\sqrt{1+\varepsilon} \cdot \sqrt{1-\varepsilon}$

$1 < \sqrt{1+\varepsilon} \cdot \sqrt{1-\varepsilon}$

$1^2 < (1+\varepsilon)(1-\varepsilon)$

$1 < 1 - \varepsilon^2$

$\varepsilon^2 < 0 \rightarrow$ Assumption is wrong!

$\therefore 1 - \sqrt{1-\varepsilon} > \sqrt{1+\varepsilon} - 1$

$\therefore$ For every number $\varepsilon > 0$, there exists a corresponding number $\delta = \sqrt{1+\varepsilon} - 1$ or $\delta = \min\{\sqrt{1+\varepsilon} - 1, 1 - \sqrt{1-\varepsilon}\}$ (or smaller positive number such that $|f(x) - 1| < \varepsilon$ whenever $0 < |x - 1| < \sqrt{1+\varepsilon} - 1$ or $0 < |x - 1| < \min\{\sqrt{1+\varepsilon} - 1, 1 - \sqrt{1-\varepsilon}\}$.

49. $\lim_{x \to 0} x \sin \frac{1}{x} = 0$

Solution 1: Use Sandwich theorem

when $x > 0$. $-x \leq x \cdot \sin \frac{1}{x} \leq x$

$\lim_{x \to 0^+} -x = 0$, $\lim_{x \to 0^+} x = 0$

$\therefore$ by sandwich theorem, $\lim_{x \to 0^+} x \cdot \sin \frac{1}{x} = 0$

when $x < 0$, $x \leq x \cdot \sin \frac{1}{x} \leq -x$

$\lim_{x \to 0^-} x = 0$, $\lim_{x \to 0^-} -x = 0$.

$\therefore$ by sandwich theorem, $\lim_{x \to 0^-} x \cdot \sin \frac{1}{x} = 0$

$\therefore \lim_{x \to 0} x \cdot \sin \frac{1}{x} = 0$.

Solution 2: Use definition.

$L = 0$, $c = 0$, $f(x) = x \cdot \sin \frac{1}{x}$

$|f(x) - L| < \varepsilon$ $\qquad 0 < |x - c| < \delta$

$\left| x \cdot \sin \frac{1}{x} - 0 \right| < \varepsilon$ $\qquad 0 < |x - 0| < \delta$

$\left| x \cdot \sin \frac{1}{x} \right| < \varepsilon$ $\qquad 0 < |x| < \delta$

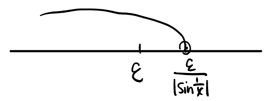
$\Rightarrow |x| \cdot \left|\sin \frac{1}{x}\right| < \varepsilon$

$|x| < \dfrac{\varepsilon}{\left|\sin \frac{1}{x}\right|}$

$0 \leq \left|\sin \frac{1}{x}\right| \leq 1$

$\Rightarrow \dfrac{1}{\left|\sin \frac{1}{x}\right|} \geq 1$

$\Rightarrow \dfrac{\varepsilon}{\left|\sin \frac{1}{x}\right|} \geq \varepsilon$



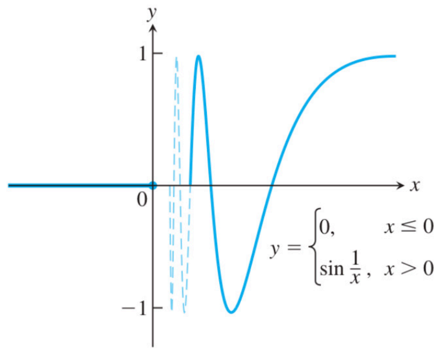
So, $\delta = \varepsilon$ can statisfy the definition.

$\therefore$ For every number $\varepsilon > 0$, there exists a corresponding number $\delta = \varepsilon$ (or smaller positive number) such that $|f(x) - 2| < \varepsilon$ whenever $0 < |x - c| < \delta$

$\therefore \lim_{x \to 0} x \cdot \sin \frac{1}{x} = 0$.

C2.4 : 5, 18, 23, 27, 53

5. Let $f(x) = \begin{cases} 0, & x \le 0 \\ \sin\frac{1}{x}, & x > 0. \end{cases}$

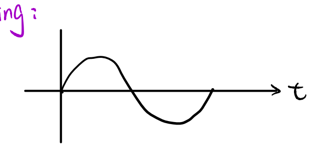


a. Does $\lim_{x\to 0^+} f(x)$ exist? If so, what is it? If not, why not?
b. Does $\lim_{x\to 0^-} f(x)$ exist? If so, what is it? If not, why not?
c. Does $\lim_{x\to 0} f(x)$ exist? If so, what is it? If not, why not?

a. No.
Because when x approaches to 0, $\sin\frac{1}{x}$ won't approach to any single value.

Understanding:

$\sin t$:



$\sin\frac{1}{x}$: $x \to 0^+$, $\frac{1}{x} \to +\infty$.
Let $t = \frac{1}{x}$. So, when $t \to +\infty$, $\sin t$ will fluctuate between $[-1, 1]$, rather than approach to a fixed value.

b. $\lim_{x\to 0^-} f(x) = 0$.

C. Not exist. Because $\lim_{x\to 0^+} f(x)$ does Not exist.

18. a. $\lim_{x\to 1^+} \frac{\sqrt{2x}\,(x-1)}{|x-1|}$ **b.** $\lim_{x\to 1^-} \frac{\sqrt{2x}\,(x-1)}{|x-1|}$

a. $\lim_{x\to 1^+} \frac{\sqrt{2x}\,(x-1)}{|x-1|} = \lim_{x\to 1^+} \frac{\sqrt{2x}\,(x-1)}{x-1} = \lim_{x\to 1^+} \sqrt{2x} = \sqrt{2}$

b. $\lim_{x\to 1^-} \frac{\sqrt{2x}\,(x-1)}{|x-1|} = \lim_{x\to 1^-} \frac{\sqrt{2x}\,(x-1)}{1-x} = \lim_{x\to 1^-} -\sqrt{2x} = -\sqrt{2}$

Using $\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$

Find the limits in Exercises 23–46.

23. $\lim_{\theta\to 0} \frac{\sin\sqrt{2\theta}}{\sqrt{2\theta}}$ 2

25. $\lim_{y\to 0} \frac{\sin 3y}{4y}$ 2

27. $\lim_{x\to 0} \frac{\tan 2x}{x}$ 2

23. $\lim_{\theta\to 0} \frac{\sin\sqrt{2}\theta}{\sqrt{2}\theta} = \lim_{\sqrt{2}\theta\to 0} \frac{\sin\sqrt{2}\theta}{\sqrt{2}\theta} = \lim_{t\to 0} \frac{\sin t}{t} = 1$

27. $\lim_{x\to 0} \frac{\tan 2x}{x} = \lim_{x\to 0} \frac{\sin 2x}{x\cdot\cos 2x}$

$= \lim_{x\to 0} \frac{2\cdot\sin x\cdot\cos x}{x\cdot\cos 2x}$

$= \lim_{x\to 0} \frac{2\sin x\cdot\cos x}{x\cdot\cos 2x}$

$= \lim_{x\to 0} 2\times\frac{\sin x}{x}\times\frac{\cos x}{\cos 2x}$ $\to 1$ $\frac{\cos(0)}{\cos(0)} = 1$

$= \left(\lim_{x\to 0} \frac{\sin x}{x}\right)\cdot\left(\lim_{x\to 0} \frac{2\cdot\cos x}{\cos 2x}\right)$

$= 1\times 2 = 2$

Use the definitions of right-hand and left-hand limits to prove the limit statements in Exercises 53 and 54.

53. $\lim_{x\to 0^-} \frac{x}{|x|} = -1$ **54.** $\lim_{x\to 2^+} \frac{x-2}{|x-2|} = 1$

DEFINITIONS (a) Assume the domain of f contains an interval (c, d) to the right of c. We say that $f(x)$ has **right-hand limit L at c**, and write

$$\lim_{x\to c^+} f(x) = L$$

if for every number $\varepsilon > 0$ there exists a corresponding number $\delta > 0$ such that

$$|f(x) - L| < \varepsilon \quad \text{whenever} \quad c < x < c + \delta.$$

(b) Assume the domain of f contains an interval (b, c) to the left of c. We say that f has **left-hand limit L at c**, and write

$$\lim_{x\to c^-} f(x) = L$$

if for every number $\varepsilon > 0$ there exists a corresponding number $\delta > 0$ such that

$$|f(x) - L| < \varepsilon \quad \text{whenever} \quad c - \delta < x < c.$$

$x \to 0^-$, therefore $x < 0$. $L = -1$, $c = 0$

$|f(x) - L| < \varepsilon$

$\left|\frac{x}{-x} - (-1)\right| < \varepsilon$

$|-1+1| < \varepsilon$

$\Rightarrow 0 < \varepsilon$

$\therefore$ $|f(x) - L| < \varepsilon$ is always true independent of x.

$\therefore$ For every $\varepsilon > 0$, any $\delta > 0$ could make $|f(x) - (-1)| < \varepsilon$ whenever $-\delta < x < 0$.

$\therefore \lim_{x\to 0^-} \frac{x}{|x|} = -1$.