

# Geometric reconstruction from point-normal data

Eleanor G. Rieffel  
FXPAL  
rieffel@fxpal.com

Don Kimber  
FXPAL  
kimber@fxpal.com

Jim Vaughan  
FXPAL  
jimv@fxpal.com

## Abstract

Creating virtual models of real spaces and objects is cumbersome and time consuming. This paper focuses on the problem of geometric reconstruction from sparse data obtained from certain image-based modeling approaches. A number of elegant and simple-to-state problems arise concerning when the geometry can be reconstructed. We describe results and counterexamples, and list open problems.

## 1 Introduction.

While three-dimensional virtual models have long been used in industry for design, the increased speed and graphics capabilities of today's computers, higher bandwidth, and the popularity of virtual environments mean that virtual models are becoming ever easier to view, manipulate, and distribute. This improved ease of use has spawned an increasing desire for better methods to create models, including models of real objects and spaces.

At FXPAL, we are particularly interested in the use of virtual models in a factory setting [11] and in surveillance [15, 26]. Common applications include training, immersive telepresence, military exercises, and design and testing of emergency response plans. Other applications range from virtual tourism [28, 27] and psychiatric treatment for post-traumatic stress disorder [14], phobias [21], and autism [30]. Real estate offices are beginning to use three-dimensional models to support the creation of virtual tours [4]. Not only are marketing departments beginning to make models of their products available to potential purchasers, but applications are springing up around these models. For example, MyDeco [5] enables users to create models of a three-dimensional space, place models of real furniture and other home accessories that are available for purchase in the virtual space, and then buy any of these products directly from the site. Virtual worlds such as Second Life [7] are filled with more or less realistic models of real places and objects. Google Earth [3] now includes three-dimensional models of various buildings.

Unfortunately, creating virtual models of real objects and spaces remains cumbersome and time con-



Figure 1: Model of an IKEA Bookcase cabinet generated by the Pantheia system.

suming. Current state of the art modeling is done by artists using interactive modeling tools, often supported by measurement and photographs of the real space. An ambitious long term research goal is to automatically construct such models from collected images; fully automatic approaches are not yet possible. FXPAL's Pantheia system [17, 25] enables users to create models by *marking up* the real world with pre-printed uniquely identifiable markers. Predefined meanings associated with the markers guide the system in creating models. The position and outward pointing normal at each marker can be estimated from user-captured images or video of the marked-up space. Point-normal data, consisting of the position and outward pointing normal, can be obtained using other technologies such as range scanners.

This paper focuses on the problem of reconstructing the geometry from the marker information. Our initial attempts at reconstruction used *ad hoc* reconstruction algorithms and markup placement strategies. When we tried to model a new space, we often needed to place additional markers, add meanings to the markup language, or revise the reconstruction algorithm to make it more powerful. This paper is the result of our work to place the geometric reconstruction aspect of our system on a firmer formal footing.

## 2 Related Work.

This section discusses two types of related work. First, we discuss related work in the area of image-based modeling. Then we survey previous work in polyhedral reconstruction from simple geometric data.

Researchers such as [23, 24, 13] work on non-marker-based methods for constructing models from images. Their work advances progress on the hard problem of deducing geometric structure from image features. Instead, we make the problem simpler by placing markers that are easily detected and have meanings that greatly simplify the geometric deduction. Furthermore, a marker-based approach enables users to specify which parts of the scene are important. In this way, Pantheia handles clutter removal and certain occlusion issues easily, since it renders what the markers indicate rather than what is seen.

From a large number of photographs of a place or object, visual features, such as SIFT features [19], can be extracted and rendered as point cloud models [28, 29]. More generally, the area of ‘point based graphics’ provides methods for representing surfaces by point data, without requiring other graphics primitives such as meshes [16]. These methods have been used as primitives for modeling tools [9]. Amenta *et al.* [10] describe the point based notion of ‘surfels’ which are points and normals. Our markers specify one dimension over a surfel: the orientation of the marker within its plane. Generally, point based methods use large numbers of surface points, and aim to produce smooth surfaces. Our system aims to produce polyhedral models with low polygon count from sparse geometric data.

There is a rich history of work on reconstructing polyhedra from partial descriptions. See Lucier [20] for a survey. There is little work, however, on reconstructing polyhedra from sparse point-plane or point-normal data, let alone more complex metadata. Biedl *et al.* discuss several polygon reconstruction problems based on point-normal data and related data [12]. Their reconstruction results are limited to two dimensions.

## 3 Overview of the Pantheia System.

For markers, the Pantheia system uses the two-dimensional two dimensional barcode style markers of ARToolKitPlus [1] (see Figure 2). Pantheia [25, 17] takes as input a set of images, identifies the markers in each image, and determines the relative pose of the marker to camera for each marker in each image. If the pose of a marker in the world is known then the pose of the camera for each image can be determined. Conversely, if the pose of a camera is known then the pose of any marker identified in that image can be cal-

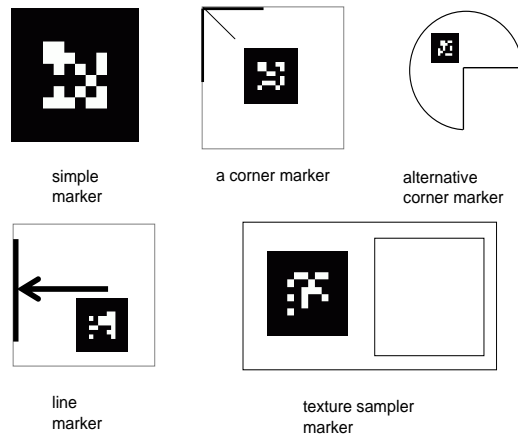


Figure 2: Examples of visual markup markers

culated. From a set of images that meets a few simple conditions, the pose of every marker and the pose of the camera for every image can be estimated. Pantheia obtains good estimates using ARToolKitPlus [1] together with the sparse bundle adjustment package SBA [18] that globally optimizes the pose estimates for all markers and images.

Pantheia creates models using a markup language [17] that includes elements specifying appearance characteristics, interactive elements, and geometric properties of the scene. This paper concerns only the geometric reconstruction aspect of the system supported by the following marker language elements:

- planar (plane, wall, ceiling, floor, door),
- shape (parametric shape, extrusion),
- modifier (edge, corner).

This paper reports on results of our ongoing effort to define a simple yet powerful markup language and to develop robust reconstruction algorithms that together support the creation of a large class of virtual models. The output of our system can be thought of as the description of a virtual model and its dynamic capabilities in terms of a language such as COLLADA [2], which supports the expression of physics, or of a language such as VRML [8] together with physics specifications for a physics engine such as ODE [6] in order to support interactive elements. Pantheia currently saves models as VRML together with metadata files that specify relations between named parts of the VRML scene graph.

The user is encouraged, when placing markers on the same plane, to use markers that have the same plane ID. Furthermore, planes that are close to aligned are forced to align by averaging their normal vectors.

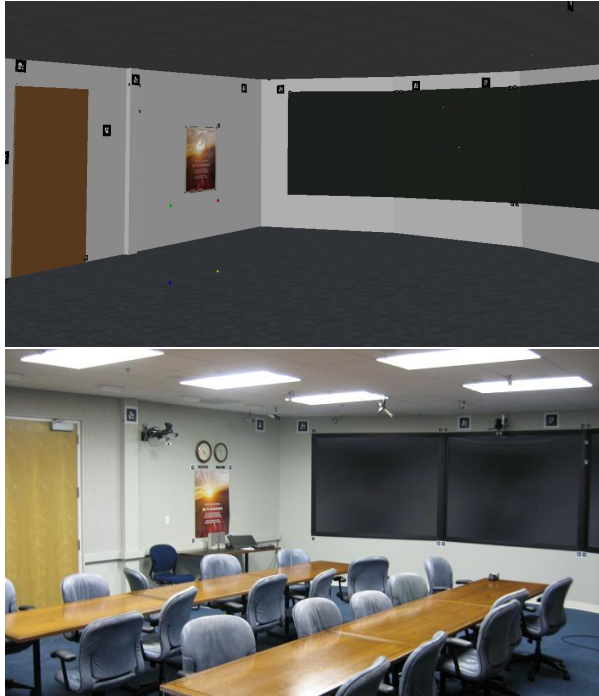


Figure 3: Pantheia generated model of a room

Finally, there is an option that takes all planes close to being aligned with one of the coordinate planes and snaps them to being precisely parallel to the coordinate plane. These features all help with the robustness of the estimation. Accuracy results for an early version of the system are reported in [17]. More details on the Pantheia system can be found in both [17, 25].

As Pantheia’s designers, we can choose which markup strategies to suggest to users, which marker meanings to make available, and which reconstruction algorithm to use. Our aim is to design markup strategies that, for a broad class of models, specify a unique model, are easy for users to understand and carry out, and have an efficient reconstruction algorithm.

#### 4 Formal Framework.

This section contains definitions used through the paper, and sets up the formal framework in which we discuss markup strategies and reconstruction algorithms. The framework is somewhat abstract in that we assume that the position and orientation of all markers is known. When marker poses are obtained using vision techniques, some markers may not be visible to a camera. For those markers, a pose cannot be obtained. We do not handle these cases, nor do we discuss robustness questions. Both are interesting areas for future work.

Section 4.1 begins with some basic geometric defini-

tions. Section 4.2 formally defines a *markup description* and related concepts. Section 4.3 defines various types of marker data considered in this paper. Section 4.4 discusses the markup strategies considered in this paper. Section 4.5 covers elaboration techniques for constructing more complex polyhedra from simpler ones.

**4.1 Basic geometric definitions.** We begin with some basic definitions. This paper is primarily concerned with constructing polyhedra from marker data. Surprisingly, there are a number of variations as to how the terms “polygon” and “polyhedron” are defined. For clarity, we state the definitions we use in this paper.

**DEFINITION 4.1.** A *polygon* is a closed, connected, two-dimensional region of a plane whose boundary consists of a finite set of segments,  $e_1, e_2, \dots, e_n$ , called *edges* with endpoints,  $v_1, v_2, \dots, v_n$ , called *vertices*. Furthermore, every vertex  $v_i$  is the endpoint of two edges, each edge ends in two vertices, and two edges can intersect only in a vertex.

**DEFINITION 4.2.** A *polyhedron*, *pl. polyhedra*, is a closed, connected, three-dimensional volume of three-dimensional Euclidean space whose boundary consists of a finite set of polygons called *faces* such that every edge of every polygon is shared with exactly one other polygon, two faces may intersect only at edges or vertices shared by the two polygons, and the faces that share a vertex can be ordered so that face  $f_i$  shares an edge with  $f_{i+1} \pmod q$ , the number of such faces).

This definition allows polyhedra to have holes. Our theorems hold for polyhedra of arbitrary genus.

**DEFINITION 4.3.** A region is *convex* if, for every pair of points in the region, the line segment connecting those points is entirely contained in the region.

**DEFINITION 4.4.** A *polyhedron* is *orthogonal* if all of its faces are parallel to one of the three coordinate planes. A set of polyhedra is *orthogonal* if all of the polyhedra in the set are orthogonal.

**4.2 Markup definitions.** We model a marker as a numeric identifier together with metadata. Minimally the metadata includes the position of each marker. The metadata may include other information about the marker or its placement, such as the outward pointing normal, or meanings from the markup language associated with sets of markers. We assume that all marker information is known – that all markers have been seen and robustly estimated.

**DEFINITION 4.5.** A *markup description* of a model is a set  $I_M$  of marker indices, together with the metadata associated with those markers.

Let  $\mathcal{X}$  be the class of models under consideration.

**DEFINITION 4.6.** A *markup placement strategy*  $f_P$  is a mapping  $f_P : \mathcal{X} \rightarrow \mathcal{D}$  from the class of models  $\mathcal{X}$  to the space of markup descriptions  $\mathcal{D}$ .

A markup strategy may or may not be deterministic. For example, a deterministic strategy might be, “for each face of the model, place a marker on the face at the centroid of its vertices”. A nondeterministic strategy might be, “For each face of the model, place a marker somewhere on the face”.

**DEFINITION 4.7.** A *markup reconstruction rule*  $f_R$  is a mapping from the class of markup descriptions  $\mathcal{D}$  to the class of models  $\mathcal{X}$ . A *markup reconstruction algorithm* is a procedure which implements a reconstruction rule.

**DEFINITION 4.8.** A *markup system* is a markup placement strategy  $f_P$  together with a reconstruction rule  $f_R$ . A markup system is complete and faithful if for every  $m \in \mathcal{X}$ ,  $f_R(f_P(m)) = m$ .

**4.3 Marker data and metadata types.** This paper considers three main types of marker data.

**DEFINITION 4.9.** *Point marker data* consists of the position of the center of the marker, or the position of a point specified by the metadata relative to the center of the marker. The corner marker shown in Figure 2 is an example of a marker that specifies a position that is not the center of the marker.

**DEFINITION 4.10.** *Point-plane metadata* consists of the position of the marker and the plane in which it lies.

**DEFINITION 4.11.** *Point-normal metadata* consists of the position and outward pointing normal at each marker.

To each of these basic types, various levels of additional metadata can be added. Useful types of additional metadata include IDs that indicate that all markers with that ID share a property such as being on the same face or defining the same polyhedron, orderings of a set of markers that indicate, for example, the order in which to traverse the vertices of a face, and relationships, such as two faces share an edge.

#### 4.4 Some Markup Strategies.

**DEFINITION 4.12.** The *marker-per-face* markup strategy is any placement of at least one marker on each face.

Most of this paper will discuss reconstruction from a marker-per-face strategy with point-plane or point-normal metadata and possibly additional metadata. As mentioned in Section 2, much more common in the literature are discussions of reconstruction from a vertex markup strategy in which every vertex is marked. We are less interested in such strategies than marker-per-face strategies because they require more precision in placement on the part of a user and require markers to be placed in places that may be out of reach or even hidden. In Section 5.3, we discuss a few results related to vertex or edge markup strategies.

**4.5 Elaboration.** Elaboration is a way to create a more complex polyhedron from a base polyhedron by gluing a polyhedron to a face or removing a polyhedron aligned with the face from the base polyhedron. *Extrusions* and *intrusions* are special cases of elaboration. The reconstruction results related to elaboration focus on reconstruction of polyhedra that can be obtained by elaborating a convex polyhedra with extrusions and intrusions of convex polygons.

**DEFINITION 4.13.** *Orthogonal Polygonal Extrusion* or just *Extrusion*: A polygon in the interior of a face of the base polyhedron is extruded outward, perpendicular to the face, a constant amount.

**DEFINITION 4.14.** *Orthogonal Polygonal Intrusion* or just *Intrusion*: A polygon in the interior of a face is pushed inward, perpendicular to the face, a constant amount.

We will also consider polyhedra obtained by taking the union of separately defined but intersecting polyhedra, including ones obtained by *gluing* along a shared face.

### 5 Reconstruction Results.

Our goal is to understand under what circumstances unambiguous reconstruction is possible from complete knowledge of all marker poses and metadata. At the same time we would like the user’s task to be as simple as possible. We have two starting points, one easy for the user but supporting reconstruction of only a small class of polyhedra, the other general but burdensome on the user.

**THEOREM 5.1. (CONVEX POLYHEDRON)** *A marker-per-face strategy with point-plane metadata is sufficient to uniquely specify an arbitrary convex polyhedron.*

**THEOREM 5.2. (DENSE MARKUP)** *For any scene containing finitely many polyhedra, if markers are placed*

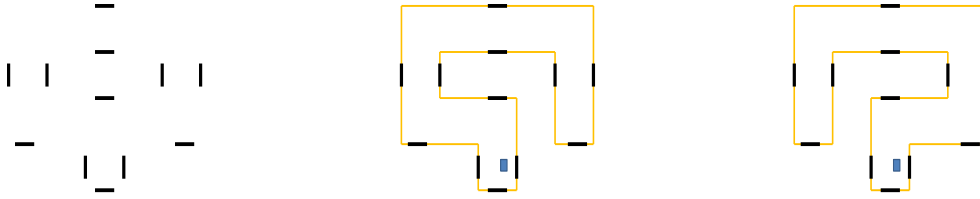


Figure 5: A configuration of markers that supports two different, but topologically equivalent, interpretations under the “single marker per face” markup system.

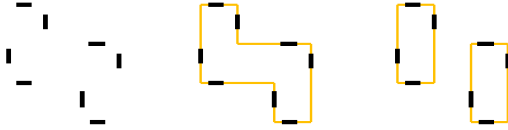


Figure 4: A configuration of markers ambiguous under the “single marker per face” markup strategy. Two polyhedral interpretations of this marker set are shown.

sufficiently densely, then the scene can be reconstructed. More precisely, for any scene containing a finite number of polyhedra, there is an  $\epsilon$  such that if every point is within  $\epsilon$  of a marker, then the scene can be unambiguously reconstructed.

The problem with the marker-per-face approach is that Theorem 5.3 and Theorem 5.4 show that both convexity, and the requirement that the scene contains only one convex polyhedron, are necessary to guarantee unambiguous reconstruction. The two main concerns with respect to dense placement are that while there is a constructive means to determine  $\epsilon$ , as the proof shows, it is not easy for a user to visually determine what density suffices, and it requires users to place many more markers than are necessary, including in places that may be hard to reach or even hidden.

The rest of this section pursues ways of enabling users to specify broader classes of models while keeping the burden on the user low in terms of the number

of markers a user must place, the complexity of the instructions, and the precision with which the user must place the markers.

Short proofs follow or preface the statement of the results. Longer proofs are contained in Appendix A. For example, the proofs of both Theorem 5.1 and Theorem 5.2 are contained in Appendix A.

**5.1 Point-plane marker-per-face results.** Theorem 5.1 states that a marker-per-face strategy with point-plane metadata suffices to uniquely reconstruct a convex polyhedron. Convexity is required, however, to avoid ambiguity.

**THEOREM 5.3.** *A marker-per-face strategy with point-plane metadata is not sufficient to uniquely specify a non-convex polyhedron.*

*Proof.* Figure 4 provides a simple counterexample. This example forms the basis for a more complex arrangement of markers that are consistent with two different polyhedra, shown in Figure 5, that have the same topology. Both examples can be extended to three dimensions by placing two markers on both an upper and lower face in the positions marked by the gray squares.

Similarly, that there is only one polyhedron in the scene is required to avoid ambiguity.

**THEOREM 5.4.** *A marker-per-face strategy with point-plane metadata is not sufficient to uniquely specify multiple convex polyhedra.*

*Proof.* Figure 6.

Theorem 5.3 and Theorem 5.4 hold for point-normal metadata as well.

The ambiguities in the case of multiple convex polyhedra is easily overcome by adding a polyhedron ID

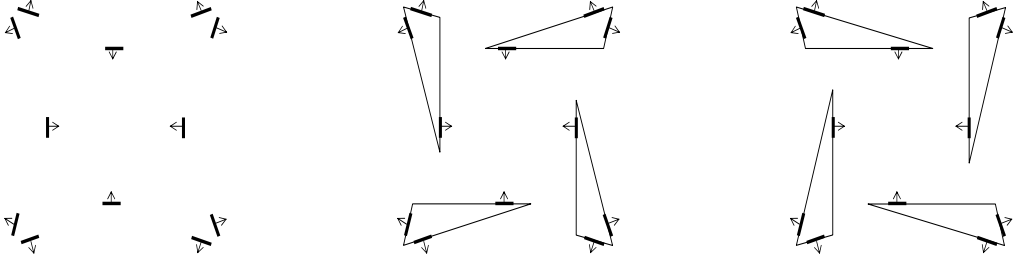


Figure 6: A configuration of markers that is ambiguous for the ‘collection of non-intersecting convex polygons’ model class with the ‘single marker per edge’ markup rule.

to the metadata so that all markers placed on the same polyhedron have the same polyhedron ID and markers placed on different polyhedra have different polyhedron IDs. The user’s task is now slightly more burdensome in that the user must keep track of different sets of markers for different polyhedra in the scene, but that is a relatively light additional burden.

**THEOREM 5.5.** *A marker-per-face strategy with point-plane and polyhedron ID metadata suffices to uniquely specify multiple convex polyhedra.*

We obtain more general results by considering elaborations of simple polyhedra.

**5.2 Markup Containing Elaboration Information.** We begin by stating the most general elaboration result we have and then discuss situations in which less elaborate markup suffices.

**THEOREM 5.6. (HIERARCHICAL ELABORATION)** *Any polyhedral scene that can be constructed by starting with a finite set of convex polyhedra and iteratively elaborating faces with orthogonal convex polygonal extrusions and intrusions can be unambiguously reconstructed from a marker-per-face strategy with point-plane metadata together with the following additional metadata that precisely specifies an elaboration hierarchy: each of the polyhedra from which the scene will be built must have its own polyhedron ID, and each of its faces identified as faces of the initial set of polyhedra from which the construction is made, and each elaboration has its own ID and the markers for an elaboration indicate on which base face the elaboration is made.*

By including additional ordering information in the metadata, extrusions and intrusions from nonconvex polygons can also be handled.

**THEOREM 5.7.** *Theorem 5.6 can be extended to the case of orthogonal nonconvex polygonal intrusions and extrusions if, for each elaboration, the markers on the faces perpendicular to the base of the elaboration include additional metadata providing an ordering, say clockwise, of these faces.*

A few remarks are in order.

- Separate IDs are required for each elaboration in order to handle cases in which the elaboration and markup resemble that shown in Figure 6.
- A given scene can often be constructed by different series of elaborations. This nonuniqueness of the hierarchical elaboration supports different users’ conceptualization of how a scene can be built, but is also potentially a source of confusion when a user considers how best to mark up a scene.
- Because the different polyhedra each have their own ID, this proof applies to scenes in which the different polyhedra intersect. The model can be rendered as intersecting polyhedra, but for efficiency reasons it may be advisable to simplify the resulting model by removing all regions internal to the final collection of polyhedra. While this capability extends the class of scenes we can reconstruct, it can add to the confusion of a user as to how best to markup a scene. One subcase of intersecting polyhedra is polyhedra that can be obtained from simpler polyhedra by gluing along their boundaries.

Many real world objects and spaces are orthogonal elaborations of convex polyhedra. An even larger class of objects is well approximated by such polyhedra. When the set of extrusions and intrusions in each face are derived from a set of non-intersecting rectangles that are axis aligned for some choice of two perpendicular



axes, the separate elaborations do not need different elaboration IDs.

**THEOREM 5.8.** *Let  $\mathcal{P}$  be a set of polyhedra obtained from a set of convex polyhedron by intrusions and extrusions, where all of the intrusions and extrusions in each face are non-intersecting, orthogonal rectangular intrusions. This set of polyhedra can be uniquely reconstructed from a marker-per-face strategy with point-normal metadata in which*

- *all markers defining a single polyhedron have an associated polyhedron ID,*
- *the markers specify the base faces of each polyhedron, and*
- *all markers indicating an elaboration specify the base face in which the elaboration is done.*

Even the class of real world objects and spaces that can be obtained simply by orthogonal intrusions of a box is reasonably large. In this case, a simple marker-per-face strategy with point-normal metadata suffices without the need for additional metadata.

**THEOREM 5.9.** *Let  $P$  be a polyhedron obtained from a box by a set of non-intersecting rectangular intrusions made in the interior of one face and aligned with the sides of this face. The polyhedron  $P$  can be uniquely reconstructed from a marker-per-face strategy with point-normal metadata.*

Theorem 5.9 can be strengthened to only need point-plane metadata, though the proof is more involved.

**THEOREM 5.10.** *Let  $P$  be a polyhedron obtained from a box by a set of non-intersecting rectangular intrusions made in the interior of one face and aligned with the sides of this face. The polyhedron  $P$  can be uniquely reconstructed from a marker-per-face strategy with point-plane metadata.*

**5.3 Other sorts of markup.** As mentioned in Section 2, researchers have looked at polyhedral reconstruction from vertex data. We are less interested in such reconstruction results because they require placement of markers in hard to reach, or even hidden, places. Nevertheless we state a few results.

O’Rourke gave an elegantly simple algorithm [22] that reconstructs orthogonal polygons from vertex data provided there are no degenerate vertices connecting two collinear segments. This condition together with orthogonality means that from every vertex extends

exactly one horizontal segment and exactly one vertical segment. The reconstruction is then obtained using a straightforward *connect-the-dots* algorithm: Consider each row of vertices. In each row, the first vertex must connect to the second, the third to the fourth, and so on. Connecting the vertices of each column in pairs in the analogous way, completes the argument. O’Rourke’s algorithm can be extended to three-dimensions, again with a prohibition against no  $180^\circ$  vertices. This prohibition is more stringent in the three-dimensional case in that it rules out shapes we might like to cover such as one brick lying perpendicularly across another.

**THEOREM 5.11.** *An orthogonal polyhedron in which all vertices do not connect any two collinear segments can be uniquely and quickly reconstructed via a connect-the-dots algorithm.*

Reconstruction of a much more general class of polyhedra is possible from vertex data together with some additional metadata.

**THEOREM 5.12.** *Any polyhedral scene in which every face is convex can be reconstructed from a vertex markup with metadata that includes the face ID of all faces meeting at that vertex.*

*Proof.* Each face can be reconstructed by taking the convex hull of all of the vertices that are marked as associated with that face.

General polyhedra can be reconstructed if the metadata also includes ordering information.

**THEOREM 5.13.** *Any polyhedral scene can be reconstructed from vertex markup if the metadata includes not only the face ID but also ordering information that specifies the clockwise order in which the vertices would be traversed when walking around the boundary of the face.*

*Proof.* The boundary of each face can be reconstructed by placing a segment between each vertex and the next one in the ordering.

## 6 Future Work.

We have explored a number of markup strategies with various types of metadata in search of markup systems that are easy for users to carry out and that unambiguously reconstruct an attractive class of models. We are in the process of improving our system on the basis of these results. We look forward to working with artists to test our system, and to learn which strategies are most intuitive and effective for such users.

A number of open research questions remain. We did not consider robustness issues. When the marker pose is obtained from camera estimates, there are interesting questions as to the best markup strategies in the face of inaccurate estimates or failing to detect one or more markers altogether. Robustness in light of user errors in entering metadata is also interesting. A number of our results are not sharp, such as Theorem 5.6, in that polyhedra not covered by the theorem can be reconstructed from the marker strategy and metadata. The question is whether the statement can be extended to a well-defined class of polyhedra. Similarly, in some cases, polyhedral scenes could be reconstructed with less metadata. With care, some classes of polyhedra obtained by extensions from polygons with boundaries intersecting the boundary of the face could be covered. Consideration of non-orthogonal extrusions may be fruitful. We reconstructed a few non-polyhedral shapes such as cylinders. Extending these results to more general parametrized shapes would yield a much richer class of scenes that can be reconstructed. We primarily explored a marker-per-face strategy leaving open the question of whether a few markers placed more strategically would enable unambiguous reconstruction of a more general class of polyhedra. In general, we continue to search for practical markup strategies that yield unambiguous reconstruction of scenes from a sparse set of markers.

## References

- [1] ARToolkit. <http://www.hitl.washington.edu/artoolkit/>.
- [2] COLLADA. <http://www.collada.org>.
- [3] Google earth. <http://earth.google.com/>.
- [4] Heartwood studios. <http://www.hwd3d.com/>.
- [5] Mydeco. <http://mydeco.com/rooms/planner/>.
- [6] Open Dynamics Engine. <http://www.ode.org/>.
- [7] Second life. <http://www.secondlife.com/>.
- [8] VRML. <http://www.w3.org/MarkUp/VRML/>.
- [9] Pointset3D. <http://graphics.ethz.ch/pointshop3d/>.
- [10] N. Amenta and Y. Kil. Defining point-set surfaces. In *SIGGRAPH 2004*, pages 264–270, 2004.
- [11] M. Back and T. Childs. High-tech chocolate: exploring 3D and mobile applications for factories. In *SIGGRAPH '09*, pages 1–1, New York, NY, USA, 2009. ACM.
- [12] T. Biedl, S. Durocher, and J. Snoeyink. Reconstructing polygons from scanner data. In *18th Fall Workshop on Computational Geometry*, 2008.
- [13] F. Dellaert, S. Seitz, C. Thorpe, and S. Thrun. Structure from motion without correspondence. In *Proceedings of CVPR'00*, 2000.
- [14] J. Difede and H. G. Hoffman. Virtual reality exposure therapy for world trade center post-traumatic stress disorder: A case report. *CyberPsychology and Behavior*, 5(6):529–535, 2002.
- [15] A. Girgensohn, D. Kimber, J. Vaughan, T. Yang, F. Shipman, T. Turner, E. G. Rieffel, L. Wilcox, F. Chen, and T. Dunnigan. DOTS: Support for effective video surveillance. In *ACM Multimedia 2007*, pages 423 – 432, 2007.
- [16] M. Gross and H. Pfister. *Point-based Graphics*. Elsevier, 2007.
- [17] D. Kimber, C. Chen, E. Rieffel, J. Shingu, and J. Vaughan. Marking up a world: Visual markup for creating and manipulating virtual models. In *Proceedings of Immerscom09*, 2009.
- [18] M. Lourakis and A. Argyros. sba: A generic sparse bundle adjustment C/C++ package based on the Levenberg-Marquardt algorithm. <http://www.ics.forth.gr/lourakis/sba/>.
- [19] D. G. Lowe. Object recognition from local scale-invariant features. In *ICCV '99: Proceedings of the International Conference on Computer Vision-Volume 2*, page 1150, 1999.
- [20] B. Lucier. Unfolding and reconstructing polyhedra. <http://uwspace.uwaterloo.ca/handle/10012/1037>, 2006.
- [21] K. Moore, B. Wiederhold, M. Wiederhold, and G. Riva. Panic and agoraphobia in a virtual world. *Cyberpsychology and Behavior*, 5(3):197–202, 2006.
- [22] J. O'Rourke. Uniqueness of orthogonal connect-the-dots. *Computational Morphology*, pages 99–104, 1988.
- [23] M. Pollefeys. Visual 3D modeling from images. <http://www.cs.unc.edu/marc/tutorial>.
- [24] M. Pollefeys and L. V. Gool. Visual modeling: from images to images. *Journal of Visualization and Computer Animation*, 13:199–209, 2002.
- [25] E. Rieffel, D. Kimber, J. Vaughan, S. Gattepally, and J. Shingu. Interactive models from images of a static scene. In *CGVR 2009*, 2009.
- [26] E. G. Rieffel, A. Girgensohn, D. Kimber, T. Chen, and Q. Liu. Geometric tools for multicamera surveillance systems. In *ICDSC 2007*, pages 132 – 139, 2004.
- [27] W. Shao and D. Terzopoulos. Autonomous pedestrians. *Graph. Models*, 69(5-6):246–274, 2007.
- [28] N. Snavely, S. M. Seitz, and R. Szeliski. Photo tourism: exploring photo collections in 3D. In *SIGGRAPH '06: ACM SIGGRAPH 2006 Papers*, pages 835–846, 2006.
- [29] N. Snavely, S. M. Seitz, and R. Szeliski. Modeling the world from internet photo collections. *International Journal of Computer Vision*, 80(2):189–210, 2008.
- [30] D. Strickland. Virtual reality for the treatment of autism. *Studies in Health Technology and Informatics*, 44:81 – 86, 1997.

## A Proofs for results in Section 5.

This appendix contains proofs of results in Section 5. It begins with proofs of Theorem 5.1 and Theorem 5.2.

To prove Theorem 5.1, we make use of the following Lemma and Corollary.



LEMMA A.1. *A convex polyhedron  $P$  lies entirely on one side of the plane defined by any of its faces.*

*Proof.* By contradiction. Suppose there are points of the polyhedron  $P$  on both sides of the plane  $R$  defined by a face  $F$ . We show that such a polyhedron cannot be convex. Let  $p$  be a point on face  $F$ . The face bounds the polyhedron on one side, so there exists an  $\epsilon > 0$  such that there are no points within  $\epsilon$  of  $p$  in one of the open half-spaces  $H$  defined by  $R$ . Since by hypothesis the polyhedron contains points on both sides of the plane  $R$ , let  $p_1$  be a point in the polyhedron contained in the half-space  $H$ . Since the polyhedron is convex by hypothesis, the entire line segment between  $p$  and  $p_1$  must be contained in the polyhedron. This line segment contains points within  $\epsilon$  of  $p$ . We have reached a contradiction, thus our supposition that there are points of  $P$  on both sides of  $F$  must be false.

THEOREM A.1. (COROLLARY TO LEMMA A.1) *In a convex polyhedron, the intersection of any plane  $R$  defined by a face  $F$  of a convex polyhedron with any other face  $F'$  of the polyhedron must be contained entirely within an edge of  $F'$ .*

*Proof.* If the plane  $R$  intersected the face  $F'$  anywhere other than in an edge of  $F'$ , the face  $F'$  would contain points on both sides of the plane  $R$ , but that is impossible by Lemma A.1.

*Proof.* [Proof of Theorem 5.1, Convex Polyhedron] For each marker, determine the half-space which contains all the other markers, as per Lemma A.1. Take the intersection of all these half-spaces. The intersection of convex sets is convex, and half-spaces are convex, so this intersection is convex.

We want to prove that this convex polyhedron  $P_I$  is the same as the original one,  $P_0$ , that was marked. To do so, it suffices to show that no other convex polyhedron has faces with the same set of bounding planes. By Lemma A.1, any convex polyhedron lies entirely on one side of the plane defined by any of its faces. By this Lemma, any other convex polyhedron defined by these faces must be contained in  $P_I$ . To prove the converse, suppose  $P_0$  were strictly contained in  $P_I$ . In this case its face set must be different. Because the face planes of  $P_I$  and  $P_0$  are identical, and  $P_0 \subset P_I$ , a face  $F_0$  of  $P_0$  must be strictly contained a face  $F_I$  of  $P_I$ . Let  $E_0$  be an edge of  $F_0$  that is not an edge of  $F_I$ ; in particular,  $E_0$  is a line segment in the interior of  $F_I$ . The edge  $E_0$  must arise as the intersection of two faces of  $P_0$ , so must be contained in the intersection of two face planes  $R_I$  and  $R_J$  of  $P_0$ , one for the face  $F_I$  and one for another face,  $F_J$ , of  $P_0$ . The two planes  $R_I$  and  $R_J$  are also face

planes for  $P_I$ . By construction  $R_J$  intersects  $F_I$  in its interior, but that contradicts Corollary A.1. Thus  $P_0$  and  $P_I$  must be the same polyhedron.

We now turn to proving Theorem 5.2.

*Proof.* [Proof of Theorem 5.2, Dense Markup.] The markers define a finite number of planes. Partition each plane into regions along lines of intersection with all other planes. Remove from consideration all infinite regions. Each of the remaining regions contains a disk of positive radius for some radius  $\epsilon$ . Take the minimum  $\epsilon_{min}$  of these positive radii over all of the regions. Since the set of regions is a finite set, this minimum exists and is positive. Unambiguous reconstruction of the scene can be done from any marker strategy that places at least one marker within distance  $d < \epsilon_{min}$  of every point on every face in the polyhedral scene. The reconstruction algorithm simply keeps every region in which there is a marker and discards the rest.

**A.0.1 Proofs for Elaborations on a Convex Polyhedron.** This section contains proofs for the results stated in Section 5.2.

*Proof.* [Proof of Theorem 5.6.] The marker metadata specified in Theorem 5.6 specified the entire hierarchy of elaborations by specifying to which face each elaboration is made. The marker metadata tells us which markers lie on faces belonging to the set of initial polyhedra on which the hierarchical construction is made. Start by identifying these markers. Because each initial polyhedron has its own ID specified by the marker metadata, we can reconstruct the initial set of polyhedra by Theorem 5.1.

For each face on this initial set of polyhedra, identify any elaborations made to this face as indicated by the marker metadata. Each elaboration has its own ID, so to construct the elaboration, identify all markers with that ID that are perpendicular to the face. Projecting the markers onto the face defines a polygon. If there are any markers with this elaboration ID parallel to the face, they determine the depth of the extrusion or intrusion. If no such marker exists, the elaboration must be an intrusion and must go all the way through the polyhedron. Repeat this process for all faces in the initial set of polyhedra. The order in which each face is considered is irrelevant. This process is then repeated for the next level of the hierarchy: each face just created is considered in turn and any elaborations to those faces are determined. This process is continued for all levels in the elaboration hierarchy until there are all elaborations have been incorporated.

We now prove some two-dimensional results to support the proofs of Theorem 5.9 and Theorem 5.10. We first prove a uniqueness results and then give an algorithm that performs the reconstruction. The proofs and algorithms make use of the following concepts.

**DEFINITION A.1.** A set of polygons is *consistent* with a set of markers if every marker is on (and aligned with) an edge of one of the polygons.

**DEFINITION A.2.** A consistent set of polygons is *fully consistent* with a set of markers if in addition every edge in the set of polygons has at least one marker on it (and aligned with it).

**DEFINITION A.3.** A marker is considered *in the interior* of a polygon if its center is in the interior of the polygon, or it is on an edge of the polygon but not aligned with it. Markers on an edge, and aligned with it, are not considered in the interior.

Consider the case of orthogonal non-intersecting convex polygons; in other words, we are considering a set of axis-aligned rectangles. Suppose this collection of rectangles is marked up using a marker-per-face strategy with point normal data. The markers can be divided into four classes depending on the direction of their outward facing normal,  $\mathcal{L}$ ,  $\mathcal{R}$ ,  $\mathcal{T}$ , and  $\mathcal{B}$ , where the normals for the markers in  $\mathcal{L}$  point to the left, to the right in  $\mathcal{R}$ , to the top in  $\mathcal{T}$ , to the bottom in  $\mathcal{B}$ .

The rest of this section shows that a set of non-intersecting orthogonal rectangles can be uniquely reconstructed from any marker-per-face strategy with point-normal data. The idea behind the proof is that, in such a markup of an arbitrary collection of axis aligned rectangles, if we can find one rectangle  $P$  that belongs to a set of rectangles that is fully consistent with the marker set, and if we can show that every fully consistent set of rectangles contains this rectangle, by induction there is only one set of rectangles fully consistent with the markers. We begin with a lemma that finds such a rectangle  $P$ .

**LEMMA A.2.** *Let  $\mathcal{M}$  be an arbitrary set of markers placed according to a marker-per-face strategy with associated point-normal data on an arbitrary set  $S$  of non-intersecting, orthogonal rectangles. There is a unique rectangle  $P$  consistent with the left-most (and top-most among these if there is a tie) marker  $M_L \in \mathcal{L}$  that is contained in all sets of rectangles that are fully consistent with this set of markers. Moreover, the rectangle  $P$  is the only rectangle consistent with marker  $M_L$  that has markers on and aligned with each of its edges and no markers in its interior.*

*Proof.* Suppose  $M_L$  lies on a rectangle  $P$  that is a member of a fully consistent set of non-intersecting rectangles  $S$  with respect to  $\mathcal{M}$ , and that  $M_L$  also lies on a rectangle  $P'$  that is a member of another fully consistent set of non-intersecting rectangles  $S'$ . Let  $M_R$  be the topmost marker in  $\mathcal{R}$  on the right side of  $P$ , and  $M_T$  and  $M_B$  be the left-most markers in  $\mathcal{T}$  and  $\mathcal{B}$  on the top and bottom, respectively, of  $P$ . (See Figure 7.) Similarly, let  $M'_R$  be the topmost marker on the right side of  $P'$ , and  $M'_T$  and  $M'_B$  be the left-most markers on the top and bottom, respectively, of  $P'$ . Given the way we chose  $M_L$ , it must be the top-most marker on the left side of both  $P$  and  $P'$ . Furthermore, without loss of generality, we may assume that marker  $M'_T$  is either to the right of  $M_T$ , or  $M'_T$  and  $M_T$  have the same horizontal component. (Otherwise, switch  $P$  and  $P'$ .) The marker  $M'_T$  must not be higher than  $M_T$  since otherwise  $M_T$  would be in  $P'$ , which is impossible since  $P'$  is part of a consistent set of non-intersecting rectangles. Unless  $M_T$  and  $M'_T$  are the same marker, they cannot be equal in height, because then  $M_T$  would be a top marker on  $P'$  to the left of  $M'_T$ . We are left with two cases to consider: either  $M'_T$  is to the right and lower than  $M_T$ , or  $M_T = M'_T$ .

Consider the case in which  $M'_T$  is to the right and lower than  $M_T$ . The marker  $M'_T$  must be to the right of  $M_R$ , since  $M'_T$  cannot be in  $P$ . And, by definition,  $M'_T$  is above  $M_L$ . There are two cases: either  $M_R$  is above  $M_L$  or  $M_R$  is below  $M_L$ , as illustrated in Figure 7. (If  $M_L$  and  $M_R$  were equal in height, the  $P'$  would contain  $M_R$ .) Suppose  $M_R$  is above  $M_L$ . Then  $P'$  must be below  $M_R$ . But now  $M_R$  cannot be part of any rectangle in a set consistent with  $P'$ ; there are no markers that could form the bottom of such a rectangle because there are no markers in  $\mathcal{B}$  below and to the left of  $M_R$  and above  $P'$  since  $P$  contains no markers in its interior. Similarly, suppose  $M_R$  is below  $M_L$ . Then  $P'$  must be above  $M_R$ . But then  $M_R$  cannot be part of a rectangle in a set consistent with  $P'$  because there are no markers in  $\mathcal{T}$  below  $P'$  and above and to the left of  $M_R$ . So the only alternative left open to us is that  $M'_T$  and  $M_T$  are the same marker. Similar arguments show that  $M_B$  and  $M'_B$  must be the same marker, and that  $M_R$  and  $M'_R$  must be the same marker. Thus  $P$  is unique.

**THEOREM A.2.** *A set  $\mathcal{R}$  of non-intersecting axis-aligned rectangles can be uniquely reconstructed from any set of markers  $\mathcal{M}$  placed according to a marker-per-face strategy with point-normal data.*

*Proof.* We proceed by induction on the number of rectangles (which we do not assume is known).

Let  $O$  be the bounding orthogonal rectangle, the smallest orthogonal rectangle that contains all the

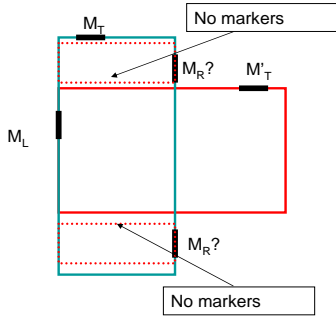


Figure 7: Diagram supporting the proof of Lemma A.2.

markers. Each edge of  $O$  has at least one marker on it, and aligned with it, because by definition of  $O$  each edge of  $O$  must intersect the set of rectangles, and orthogonality guarantees that this intersection must be in an edge, not just a point; we can find  $O$  from the markers by finding all markers that define a line which is a boundary of the set - a line is on the boundary if all markers are in only one of the half-planes defined by the line - and letting  $O$  be the intersection of all of these half planes.

**Case  $n = 1$ :** If there is only one rectangle,  $O$  will be that rectangle. We know we are in this case if all of the markers are on and aligned with an edge of  $O$ .

**Case  $n = k$ :** By Lemma A.2, any fully consistent set of rectangles must contain the rectangle  $P$ . Now exclude from  $\mathcal{M}$  any markers that are consistent with  $P$ . The remaining markers must be consistent with a set of  $k - 1$  rectangles. By induction, there is only one such set.

Thus there is only one set of rectangles fully consistent with the set of markers  $\mathcal{M}$ .

The orthogonality assumption in Theorem A.2 is necessary.

**THEOREM A.3.** *A set of non-intersecting convex polygons cannot always be reconstructed from a set of markers that includes at least one marker per edge.*

*Proof.* A counterexample is shown in Figure 6.

**A.0.2 A reconstruction algorithm for Theorem A.2.** This section describes a constructive algorithm

for finding the unique set of rectangles fully consistent with a marker set. The algorithm proceeds by finding a rectangle  $P$  that is consistent with the top left-most marker  $M_L$ . The rest of this subsection describes how that is done. Once it is done, all markers consistent with  $P$  are ignored, the new top left-most marker is considered, and a new rectangle is found. The algorithm continues until all markers have been taken care of.

We set out to find a rectangle consistent with  $M_L$  that has markers on and aligned with each edge and no markers in its interior. Consider all markers in  $\mathcal{T}$  that are to the right and above  $M_L$ ; one of these must be able to form a rectangle with  $M_L$  that is part of a fully consistent set of rectangles for the marker set. Consider first the left-most of these markers and, if there are multiple left-most markers, take the top-most one of these. Call this marker  $M$ .

Partition  $O$  into an irregular grid with lines defined by the markers in  $\mathcal{M}$ . Consider the smallest rectangle consistent with markers  $M_L$  and  $M$  that is made up of grid segments. If this rectangle contains any markers in its interior,  $M$  cannot form a rectangle with  $M_L$  that is part of a fully consistent set, so go on to consider the next left-most marker. Eventually a marker  $M_{Tcand}$  is found such that the smallest grid-aligned rectangle  $R$  containing  $M_L$  and  $M_{Tcand}$  does not have any markers in its interior. (There must be at least one, since by assumption, the markers mark the original set of non-intersecting rectangles, the set we are trying to find.)

Check if there are markers on, and aligned with all sides of  $R$ . If there are, we take this rectangle to be  $P$ . Otherwise, expand the rectangle until this property holds as follows. Let  $S$  be the set of markers in  $\mathcal{B}$  that are below and to the right of  $M_L$ . The set  $S$  cannot be empty since that would mean  $M_L$  is not on a rectangle that is part of a fully consistent set. Among the markers in  $S$ , consider the subset  $S_M$  of markers that do not have any markers in  $S$  directly above them or above them to their left. The left-most marker in  $S$  (and top-most, if there is more than one left-most) satisfies this criterion. Consider first the left-most (and top-most if there is a tie) of these markers,  $M_{Bcand}$ . If the smallest rectangle consistent with  $M_{Bcand}$ ,  $M_{Tcand}$ , and  $M_L$  has markers in its interior, go on to the next left-most marker in  $S_M$ . If there are no markers in the interior, find the left-most marker or markers in  $\mathcal{R}$  and in the strip bounded on the top by  $M_{Tcand}$ , on the bottom by  $M_{Bcand}$ , and to the right of  $M_L$ . If there is no such marker, then  $M_{Tcand}$ ,  $M_L$ , and  $M_{Bcand}$ , cannot form a marker consistent rectangle, so go on to consider the next candidate for  $M_{Bcand}$ . If none of the possible candidates for  $M_{Bcand}$  work, then try the next possible  $M_{Tcand}$ . If there is

no marker in the interior of the rectangle defined by  $M_{Tcand}$ ,  $M_L$ , and  $M_{Bcand}$ , find the left-most marker in  $\mathcal{R}$  that is in the strip bounded on the top by  $M_{Tcand}$ , on the bottom by  $M_{Bcand}$ , and on the left by  $M_L$ . If there is no such marker, then  $M_{Tcand}$ ,  $M_L$ , and  $M_{Bcand}$ , cannot form a marker consistent rectangle, so go on to consider the next candidate for  $M_{Bcand}$ . Again, if none of the candidates  $M_{Bcand}$  work, then consider the next candidate  $M_{Tcand}$ . Eventually some set works, since we know the markers form a fully consistent set for some set of rectangles.

**DEFINITION A.4.** An infinite polygonal *cylinder* consists of a 3D region obtained by infinitely extending in both directions the edges of a polygon (that lies in 3D space) perpendicular to the face of the polygon.

**DEFINITION A.5.** A polygonal *half-cylinder* consists of a 3D region obtained by infinitely extending in the edges of a polygon in one of the directions perpendicular to the face of the polygon.

*Proof.* [Proof of Corollary 5.8] For each set of markers with the same polyhedron ID, construct the faces of the base polyhedron from the markers with metadata indicating that they are on one of the faces of the base polyhedron. For each base face  $F$ , consider all markers  $S_F$  that indicate that they are on faces made by elaboration of this face. Find all such markers that are perpendicular to this face, and project them onto the plane defined by this face. By hypothesis, the result defines a set of rectangles on the plane. Apply Theorem A.2 to obtain the uniquely determined set of rectangles. Each rectangle corresponds to an intrusion or extrusion. Consider the infinite rectangular cylinder defined by each of the rectangles. If there is no marker contained in  $S_F$  in the interior of this cylinder, this rectangle defines an intrusion that goes all the way through the polyhedron. If there are markers in  $S_F$  inside that cylinder, they must all lie on the same plane, since rectangles defining the extrusions and intrusions do not intersect by hypothesis. This plane determines the extent of the extrusion or intrusion. The same construction can now be applied to all elaborations of faces that resulted from this level of elaboration, and can continue to be applied at each level until all elaborations have been handled.

*Proof.* [Proof of Theorem 5.9] Consider all  $xy$ -aligned markers. All of these markers lie on faces that are either boundaries of the original box or are faces obtained from making interior orthogonal rectangular intrusions in the same face. Because the intrusions are in the interior, all of the markers obtained by intrusion have

$z$  coordinates strictly between those of the boundary regions. So the  $xy$  boundary planes can be obtained by taking the extremal  $z$  values from among the  $xy$ -aligned markers. The same process can be repeated for  $xz$ - and  $yz$ -aligned markers to obtain the original box.

Now consider all markers in the interior of the box. The markers fall into six classes determined by the direction of their outward facing normal,  $\mathcal{L}$ ,  $\mathcal{R}$ ,  $\mathcal{T}$ ,  $\mathcal{B}$ ,  $\mathcal{F}$ ,  $\mathcal{G}$ , where the normals for the markers in  $\mathcal{L}$  point to the left, to the right in  $\mathcal{R}$ , to the top in  $\mathcal{T}$ , to the bottom in  $\mathcal{B}$ , to the front in  $\mathcal{F}$ , to the back in  $\mathcal{G}$ . Since all interior markers come from intrusions, and these intrusions were all made in the same face, exactly one of these classes is empty. The boundary face with that class's normal is the face in which the intrusions were made.

To finish the construction, we project all of the non-boundary markers that are perpendicular to the face onto the face. The result defines a set of non-intersecting axis-aligned rectangles. To this set, apply Theorem A.2 to obtain the unique fully consistent set of rectangles. Now, just as in the proof of Theorem 5.8, for each rectangle, consider the infinite rectangular cylinder it defines. If there are no non-boundary markers inside that cylinder, then the intrusion goes all the way through the box. If there are non-boundary markers inside the cylinder, they must all be in the same plane since the set of rectangles does not intersect. These markers define the depth of the intrusion.

*Proof.* [Proof of Theorem 5.10] The proof finds the faces of the bounding box in the same way as the proof for Theorem 5.9. Because we no longer have access to the normal direction, the method to find the face in which the intrusion was made is more involved.

By hypothesis all of the interior markers must be obtained by intrusion from one face. Consider each face in turn. Among the interior markers parallel to a face, find the marker closest to the face. If there are no markers parallel to a face, all intrusions must be from this face and must go all the way through the box. If there is a marker, check to see if there are perpendicular markers closer to the face. If not, this is not the face in which the intrusions were made. If so, this is the face, because if the closest marker were on the side, rather than the bottom, of an intrusion, there would be no markers closer to the face than it.

In this way we have determined the face in which all the intrusions have been made. The proof finishes in exactly the same way as the proof for Theorem 5.9.