optimex!

T1

Problem 1. (Knapsack Problem) The following 8 objects with the given weights and profits are available for delivery.

Object	1	2	3	4	5	6	7	8	
Weight (kg)	153	54	191	66	239	137	148	249	$:= w$ $(1,8)$
Profit (€)	232	73	201	50	141	79	48	38	$:= p$ $(1,8)$
Profit density (€/kg)	1.52	1.35	1.05	0.76	0.59	0.58	0.32	0.15	

The "Knapsack" (for example a trailer, vehicle, container or a rocket), in which the objects should be transported, has a weight limitation of **645 kg**. There is no space limitation. Formulate the optimization problem which maximizes the profit of the delivery without exceeding the weight limitation. Note that objects can't be partially transported and each object is available as often as needed.

$x = [x_1 \cdots x_8]$ $(1,8)$: Number of objects.

Problem: $\min \; -p \cdot x^T$

s.t. $w \cdot x^T \leq 645$, where $x_1 \dots x_8 \in \mathbb{Z}_+$

Problem 2 $\frac{t}{m}$ $r\frac{2}{t}$

(Cargo Optimization) An airfreight has three compartments for storing cargo: front, centre and rear. These compartments have the following limits on both weight and space:

Compartment	Weight capacity (tonnes) $(3,1)$ b	Space capacity (m^3) $(3,1)$ d
Front	10	6800
Centre	16	8700
Rear	8	5300

Furthermore, the weight of the cargo in the respective compartments must satisfy the proportion **10:16:8** as in their weight limits to maintain the balance of the plane.

The following four cargoes are available for shipment on the next flight.

Cargo	Weight (tonnes) $(4,1)$ a	Volume (m^3/ton) $(4,1)$ v	Profit (€/ton) $(4,1)$ r
C1	18	480	310
C2	15	650	380
C3	23	580	350
C4	12	390	285

Any proportion of these cargoes can be accepted. The objective is to determine how much (if any) of each cargo C1, C2, C3 and C4 should be accepted and how to distribute each among the compartments such that the total profit for the flight is maximized. It is assumed that cargoes can be split into any fractions. Formulate the optimization problem.

$$\begin{array}{c|ccc|l} & F & C & R & \\ \hline X := & x_{11} & \cdots & x_{13} & C_1 \\ (4,3) & \vdots & & & C_2 \\ & & & & C_3 \\ & x_{41} & \cdots & x_{43} & C_4 \end{array}$$

(tons)

$p := \left(\sum \; \sum \; \sum\right)^T$ $(3,1)$ ← position, $\quad c := \begin{pmatrix} \sum \\ \sum \\ \sum \\ \sum \end{pmatrix}$ $(4,1)$ ← Cargo

Problem: $\min \; -r^T \cdot c$

s.t. $c \leq a$

$p \leq b$

$v^T X \leq d^T$

$16p_1 - 10p_2 = 0$

$8p_2 - 16p_3 = 0$

} proportion

T2

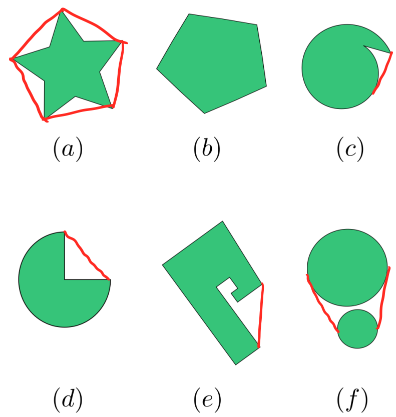
Problem 1. (Semidefinite matrices and cones)

a) Show that the eigenvalues of a positive semidefinite matrix are nonnegative.

b) Prove the following equivalence for the positive semidefinite cone in $\mathcal{S}^2$.

$$\boldsymbol{X} = \begin{pmatrix} x & y \\ y & z \end{pmatrix} \in \mathcal{S}^2_{\geq 0} \Longleftrightarrow x \geq 0, z \geq 0, xz \geq y^2.$$

Problem 2. (Convex hull) Draw the *convex hull* for all the sets given in the figure.



a) Def. positive semidefinite Matrix:

$x^T \cdot M x \geq 0 \quad \forall x \in \mathbb{R}^n \setminus 0$

Eigenvalues: $M \cdot v = a \cdot v \Rightarrow v^T \cdot M \cdot v \geq 0$

$\Rightarrow v^T M v = a \cdot v^T v = a \, \|v\|^2 \geq 0 \Rightarrow a \geq 0$

b) "$\Rightarrow$": Eigenvalue: $\begin{vmatrix} x-\lambda & y \\ y & z-\lambda \end{vmatrix} = (x-\lambda)(z-\lambda) - y^2 \stackrel{!}{=} 0$

$\Rightarrow \lambda^2 + (-z-x)\lambda + xz - y^2 = 0$

$\Rightarrow \lambda = \dfrac{z+x \pm \sqrt{(z+x)^2 - 4(xz-y^2)}}{2} \geq 0$

$= \dfrac{z+x \pm \sqrt{z^2+x^2-2xz+4y^2}}{2} = \dfrac{z+x \pm \sqrt{(z-x)^2+(2y)^2}}{2}$

$$\begin{cases} z+x+\sqrt{(z-x)^2+(2y)^2} \geq 0 \\ z+x-\sqrt{(z-x)^2+(2y)^2} \geq 0 \Rightarrow z+x \geq \sqrt{(z-x)^2+(2y)^2} \end{cases}$$

① $z+x \geq 0$

② $xz \geq 0$

$\Rightarrow \begin{cases} x \geq 0 \\ z \geq 0 \end{cases}$

$\cancel{z^2+x^2}+2zx \geq \cancel{z^2+x^2}-2zx+4y^2$

$xz \geq y^2$

"$\Leftarrow$" back to Eigenvalue: nonnegative! $\square$

T3

Problem 1. (Convex sets) Fill the following blank spaces with T or F (for true and false, respectively).

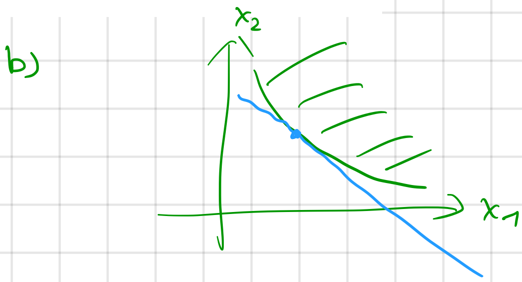
- **a)** T Any affine set is convex.
- **b)** F Any convex set is affine.
- **c)** F Any open set is affine. → "open" ≠ unlimited ($\mathbb{R}$)
- **d)** F $\mathbb{R}^n$ is the only affine set in $\mathbb{R}^n$.
- **e)** T If $n \geq 1$, there exist infinitely many affine sets in $\mathbb{R}^n$.

} Line Example!

Problem 2. (Supporting hyperplanes) Represent each of the following closed, convex sets $\mathcal{C} \subseteq \mathbb{R}^2$ as an intersection of halfspaces.

- **a)** $\mathcal{C} = \{\boldsymbol{x} \in \mathbb{R}^2 \,|\, x_2 \geq e^{x_1}\}$.
- **b)** $\mathcal{C} = \{\boldsymbol{x} \in \mathbb{R}^2_{>0} \,|\, x_1 x_2 \geq 1\}$.

HS: $\{x \in \mathbb{R}^n \mid a^T x \leq b\}$, $a \neq 0$



a)

$x_2 = e^{\tilde{x}_1} \cdot x_1 + b$

s.t. $e^{\tilde{x}_1} = e^{\tilde{x}_1} \cdot \tilde{x}_1 + b \Rightarrow b = e^{\tilde{x}_1} - e^{\tilde{x}_1} \tilde{x}_1$

$= e^{\tilde{x}_1}(1 - \tilde{x}_1)$

$\Rightarrow$ HS: $\bigcap_{t \in \mathbb{R}} \{x_2 \geq e^t \cdot x_1 + e^t(1-t)\}$

Matrix $\rightsquigarrow$ $\bigcap_{t \in \mathbb{R}} \left\{x \in \mathbb{R}^2 \,\middle|\, (e^t, -1)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \leq e^t(t-1)\right\}$

b)

$x_2 = -\frac{1}{\tilde{x}_1^2} x_1 + b$

s.t. $\frac{1}{\tilde{x}_1} = -\frac{1}{\tilde{x}_1} + b$

$\Rightarrow b = \frac{2}{\tilde{x}_1}$

$\Rightarrow$ HS: $\bigcap_{t \in \mathbb{R}^+} \{x_2 \geq -\frac{x_1}{t^2} + \frac{2}{t}\}$

Matrix $\rightarrow$ $\bigcap_{t \in \mathbb{R}^+} \left\{x \in \mathbb{R}^2 \,\middle|\, (-\frac{1}{t^2}, -1)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \leq -\frac{2}{t}\right\}$

$\boxed{T4}$

Problem 1. (Convex and concave functions) Decide which of the following functions are convex or concave and give reasons.

a) $f(x) = |x|$, $x \in \mathbb{R}$ Def.

b) $f(\boldsymbol{x}) = \|\boldsymbol{x}\|^p$, $\boldsymbol{x} \in \mathbb{R}^n$ and $p \geq 1$ thg

c) $f(x) = e^x - 1$, $x \in \mathbb{R}$ "

d) $f(\boldsymbol{x}) = x_1 x_2$, $\boldsymbol{x} \in \mathbb{R}^2_{>0}$ "

e) $f(\boldsymbol{x}) = \frac{1}{x_1 x_2}$, $\boldsymbol{x} \in \mathbb{R}^2_{>0}$ "

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y)$$

$$\lambda \in [0,1]$$

a) $f(\lambda x + (1-\lambda)y) = |\lambda x + (1-\lambda)y|$ $\leq$

Right side: $= \lambda|x| + (1-\lambda)|y| = |\lambda x| + |(1-\lambda)y|$ $\boxed{\text{convex}}$

b) $\|\lambda x + (1-\lambda)y\|^p \overset{!}{\leq} \lambda\|x\|^p + (1-\lambda)\|y\|^p$ $p \geq 1$.

triangle inequality of norm: $\|\lambda x + (1-\lambda)y\|^p \leq \|\lambda x\|^p + \|(1-\lambda)y\|^p$

$\leq \lambda \|x\|^p \ldots$ $\boxed{\text{convex}}$

c) $f''(x) = e^x > 0 \Rightarrow$ $\boxed{\text{strictly convex}}$

d) $\nabla^2 f(x) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ check EV: $\begin{vmatrix} 0-\lambda & 1 \\ 1 & 0-\lambda \end{vmatrix} = (-\lambda)^2 - 1 \overset{!}{=} 0$

$\lambda^2 = 1$, $\lambda = \pm 1$ $\Rightarrow$ f either convex nor concave

e) $\nabla^2 f(x) = \begin{pmatrix} \frac{\partial}{\partial x_1}\left(\frac{1}{x_2}\cdot\left(-\frac{1}{x_1^2}\right)\right) & \frac{\partial}{\partial x_1}\left(\frac{1}{x_1}\cdot\left(-\frac{1}{x_2^2}\right)\right) \\ \frac{\partial}{\partial x_2}\left(\frac{1}{x_2}\cdot\left(-\frac{1}{x_1^2}\right)\right) & \frac{\partial}{\partial x_2}\left(\frac{1}{x_1}\cdot\left(-\frac{1}{x_2^2}\right)\right) \end{pmatrix} = \begin{pmatrix} \frac{1}{x_2}\cdot\frac{2}{x_1^3} & +\frac{1}{x_2^2}\cdot\frac{1}{x_1^2} \\ \frac{1}{x_1^2 x_2^2} & \frac{2}{x_1 x_2^3} \end{pmatrix}$

$\frac{2}{x_2 x_1^3} > 0$, $|\nabla^2 f(x)| = \frac{4}{x_1^4 x_2^4} - \frac{1}{x_1^4 x_2^4} = \frac{3}{x_1^4 x_2^4} > 0 \Rightarrow \nabla^2 f(x)$ p.d.

b). $f := h(g(x))$ convex $\Leftarrow$ $g(x)$ convex, $h(x)$ convex $\nearrow$

$g(x)$ concave, $h(x)$ convex $\searrow$

$f(x) = \|x\|^p$, $p \geq 1$.

Norm: convex

$(\,)^p$: convex $\nearrow$

$\boxed{\text{T5}}$

精选

Problem 1. (Running average) Suppose $f : \mathbb{R} \to \mathbb{R}$ is a differentiable convex function. Show that its *running average* F, defined as

$$F(x) = \frac{1}{x}\int_0^x f(t)\,\mathrm{d}t, \quad x > 0,$$

is convex.

$$F'(x) = -\frac{1}{x^2}\int_0^x f(t)\,dt + \frac{1}{x} f(x)$$

$$\frac{d}{dx}\left(\int_a^x g(t)\,dt\right) = \frac{d}{dx}\left(G(x) - G(a)\right) = \frac{d}{dx}G(x) - \frac{d}{dx}\underbrace{G(a)}_{\text{const.}} = \frac{d}{dx}G(x)$$

$$F''(x) = \frac{2}{x^3}\int_0^x f(t)\,dt - \frac{1}{x^2}f(x) - \frac{1}{x^2}f(x) + \frac{1}{x}f'(x)$$

$$= \frac{2}{x^3}\int_0^x f(t)\,dt - \frac{2}{x^2}f(x) + \frac{1}{x}f'(x) \quad \cdots \quad (*)$$

$\boxed{1}$: f convex $\Leftrightarrow$ $f(t) \geq f(x) + f'(x)(t-x) \quad \forall t, x \in \mathbb{R}$.

$$\Rightarrow \int_0^x f(t)\,dt \geq \int_0^x f(x) + f'(x)(t-x)\,dt$$

$$= x\cdot f(x) + \frac{1}{2}f'(x)(x^2) - x^2 f'(x)$$

$$= x\cdot f(x) - \frac{1}{2}x^2 f'(x) \quad \forall x > 0$$

$$(*): \quad F''(x) = \frac{2}{x^3}\left(\int_0^x f(t)\,dt - x f(x) + \frac{1}{2}x^2 f'(x)\right) \geq 0 \quad \square$$

思路：1. 题目：已知 f convex，求 $\int f$ convex.

2. 过程：用 second order 列式子

用 first order 为条件，套 $\int$

靠近目标式子

T5

Problem 2. (Definition of convexity) Suppose $f : \mathbb{R} \to \mathbb{R}$ is convex $a, b \in \textbf{dom}\, f$ with $a < b$.

a) Show that

$$f(x) \le \frac{b-x}{b-a} f(a) + \frac{x-a}{b-a} f(b)$$

for all $x \in [a, b]$.

b) Show that

$$\frac{f(x)-f(a)}{x-a} \le \frac{f(b)-f(a)}{b-a} \le \frac{f(b)-f(x)}{b-x}$$

for all $x \in (a, b)$. Draw a sketch that illustrates this inequality.

c) Suppose that f is differentiable. Use the result in (b) to show that

$$f'(a) \le \frac{f(b)-f(a)}{b-a} \le f'(b).$$

d) Suppose that f is twice differentiable. Use the result in (c) to show that $f''(a) \ge 0$ and $f''(b) \ge 0$.

a) "Def": $f(\lambda x + (1-\lambda) y) \le \lambda f(x) + (1-\lambda) f(y)$

$: f(x) = f(\lambda a + (1-\lambda) b) \le \lambda f(a) + (1-\lambda) f(b) \quad (*)$

$\frac{b-x}{b-a} + \frac{x-a}{b-a} = 1 \Rightarrow \lambda := \frac{b-x}{b-a}, \quad (1-\lambda) = \frac{x-a}{b-a} \quad \square$

b) $(*)$: $f(x) - f(a) \le (\lambda - 1) f(a) + (1-\lambda) f(b)$

$= (1-\lambda)(f(b) - f(a))$

with $(1-\lambda) := \frac{x-a}{b-a}$ $\quad = \frac{x-a}{b-a}(f(b) - f(a)) \Rightarrow$ "1. $\le$"

$(*)$: $f(b) - f(x) \ge -\lambda f(a) - f(b) + \lambda f(b) + f(b)$

$= \lambda (f(b) - f(a))$

with $\lambda := \frac{b-x}{b-a}$: $\quad = \frac{b-x}{b-a}(f(b) - f(a)) \Rightarrow$ "2. $\le$"

c) $x \to a$ or $x \to b$ $\Rightarrow \frac{f(x \to a) - f(a)}{x-a} = f'(a)$

d) $a \to b$ or $b \to a$

$\Rightarrow \lim_{a \to b} \frac{f'(b) - f'(a)}{b-a} \ge 0$, $\quad f'(b) - f'(a) \ge 0$, from c

T6

Problem 1. (Inverse of an increasing convex function) Suppose $f : \mathbb{R} \to \mathbb{R}$ is strictly increasing and convex on its domain (a, b). Let g denote its inverse, i.e., the function with domain $(f(a), f(b))$ and $g(f(x)) = x$ for $a < x < b$. What can you say about convexity or concavity of g?

$f : \mathbb{R} \to \mathbb{R}$: $\nearrow$ convex in (a,b)

$g : (f(a), f(b)) \to (a,b)$, $g(f(x)) = x$, $\nearrow$

w.l.o.g : choose $x_1 < x_2$. $\lambda \in [0,1]$

(f) "Def" : $f(\lambda x_1 + (1-\lambda)x_2) < \lambda f(x_1) + (1-\lambda) f(x_2)$

$g \nearrow$ $\iff$ $g(f(\lambda x_1 + (1-\lambda)x_2)) < g(\lambda f(x_1) + (1-\lambda) f(x_2))$

g def $\iff$ $\lambda x_1 + (1-\lambda) x_2 < g(\lambda f(x_1) + (1-\lambda) f(x_2))$

g def $\iff$ $\lambda g(f(x_1)) + (1-\lambda) g(f(x_2)) < g(\lambda f(x_1) + (1-\lambda) f(x_2))$

$y_1 := f(x_1)$
$y_2 := f(x_2)$ $\iff$ $\lambda g(y_1) + (1-\lambda) g(y_2) < g(\lambda y_1 + (1-\lambda) y_2)$

$\iff$ g is strictly concave.

Alternative : $g(f(x)) = x$. $\Rightarrow \frac{\partial}{\partial x} g(f(x)) = \frac{\partial}{\partial f(x)} g(f(x)) \cdot \frac{\partial}{\partial x} f(x)$

$\iff \frac{\partial}{\partial f(x)} g(f(x)) = \frac{1}{\frac{\partial}{\partial x} f(x)} > 0$

$$\frac{\partial}{\partial x} \frac{\partial}{\partial f(x)} g(f(x)) = \frac{\partial}{\partial x} \cdot \left(\frac{\partial}{\partial x} f(x)\right)^{-1} = - \frac{\frac{\partial^2}{\partial x^2} f(x)}{\left(\frac{\partial}{\partial x} f(x)\right)^2}$$

also $= \frac{\partial \frac{\partial g(f(x))}{\partial f(x)}}{\partial f(x)} \cdot \frac{\partial f(x)}{\partial x} = \frac{\partial}{\partial x} f(x) \cdot \boxed{\frac{\partial^2}{\partial f(x)^2} g(f(x))}$

$$= - \frac{\frac{\partial^2}{\partial x^2} f(x)}{\left(\frac{\partial}{\partial x} f(x)\right)^3} < 0 \Rightarrow \text{concave!}$$

T6

Problem 2. (Norm reformulation) Linear optimization problems are convex optimization problems for which the objective function and all constraint functions are affine.

Recall that for $\boldsymbol{x} \in \mathbb{R}^n$,

$$||\boldsymbol{x}||_\infty = \max_{i=1,\dots,n} |x_i| \quad \text{and} \quad ||\boldsymbol{x}||_1 = \sum_{i=1}^{n} |x_i|$$

holds. Find an equivalent linear formulation for the following optimization problems.

a) minimize $||\boldsymbol{A}\boldsymbol{x} - \boldsymbol{b}||_\infty$

b) minimize $||\boldsymbol{A}\boldsymbol{x} - \boldsymbol{b}||_1$

c) minimize $||\boldsymbol{A}\boldsymbol{x} - \boldsymbol{b}||_1$ subject to $||\boldsymbol{x}||_\infty \leq 1$

d) minimize $||\boldsymbol{x}||_1$ subject to $||\boldsymbol{A}\boldsymbol{x} - \boldsymbol{b}||_\infty \leq 1$

e) minimize $||\boldsymbol{A}\boldsymbol{x} - \boldsymbol{b}||_1 + ||\boldsymbol{x}||_\infty$

LP:

convex + Affine

SFLP: $\min \ c^T x + d$

s.t. $Gx \leq h$

$Ax = b$

can have several constraints

a) $||Ax-b||_\infty = \max_{i=1,\dots,n} |(Ax-b)_i| = \max_{i=1\dots n} |a_i^T x - b_i|$

Let $t \geq |a_i^T x - b_i|$, $\Leftrightarrow$ $-t \leq a_i^T x - b_i \leq t$

$$\boxed{\begin{aligned} &\min \quad t \\ &\text{s.t.} \quad Ax - b \leq t \cdot \vec{1} \\ &\qquad\ -Ax + b \leq t \cdot \vec{1} \end{aligned}}$$

b) $||Ax-b||_1 = \sum_{i=1}^{n} |(Ax-b)_i| = 1^T |Ax-b|$

$\min \quad u^T \cdot 1$

s.t. $\quad Ax - b \leq u$

$\qquad -Ax + b \leq u$

Let $u \in \mathbb{R}^m$: $u_k \geq |Ax-b|_k$

☆ 不等式的传递性

c) $||x||_\infty = \max_{i=1\dots n} |x_i| \leq 1 \Leftrightarrow$ $x \leq 1$

$-x \leq 1$

$\Rightarrow \quad \min \quad u^T 1$

s.t. $\quad Ax - b \leq u$

$\qquad -Ax + b \leq u$

$\qquad x \leq 1$

$\qquad -x \leq 1$

T6

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a) minimize $||\boldsymbol{Ax} - \boldsymbol{b}||_\infty$

b) minimize $||\boldsymbol{Ax} - \boldsymbol{b}||_1$

c) minimize $||\boldsymbol{Ax} - \boldsymbol{b}||_1$ subject to $||\boldsymbol{x}||_\infty \le 1$

d) minimize $||\boldsymbol{x}||_1$ subject to $||\boldsymbol{Ax} - \boldsymbol{b}||_\infty \le 1$

e) minimize $||\boldsymbol{Ax} - \boldsymbol{b}||_1 + ||\boldsymbol{x}||_\infty$

d) $||x||_1 = \sum_{i=1}^{n} |x_i|$ Let $u \in \mathbb{R}^n$. $u_k \geq |x_k| \ \forall k = 1 \cdots n$

$\min ||x||_1 \Leftrightarrow \min u^T \cdot \mathbb{1}$, s.t. $\begin{cases} x_k \le u_k \\ -x_k \le u_k \end{cases}$

$||Ax-b||_\infty = \max_{i=1\cdots n} |(Ax-b)_i| \le 1$

$\Leftrightarrow \begin{cases} Ax-b \le 1 \\ -Ax+b \le 1 \end{cases}$ $\Rightarrow$

$$\boxed{\begin{aligned} \min \quad & u^T \cdot \mathbb{1} \\ \text{s.t.} \quad & x \le u \\ & -x \le u \\ & Ax-b \le 1 \\ & -Ax+b \le 1 \end{aligned}}$$

e) $||Ax-b||_1 + ||x||_\infty$

$= \sum_{i=1}^{n} |(Ax-b)_i| + \max_{i=1\cdots n} |x_i|$

Let $u, v \in \mathbb{R}^n$, and $\begin{cases} u_i \geq |(Ax-b)_i| \\ v_j \geq |x_j| \end{cases} \quad \forall i,j \in \{1 \cdots n\}$.

So, $\Rightarrow$ $\min \ u^T \cdot \mathbb{1} + v$

s.t. $Ax - b \le u$

$-Ax+b \le u$

$x \le v$

$-x \le v$

T7

11

Problem 1. (Products and quotients of convex functions) Let $f, g : \mathbb{R} \to \mathbb{R}_{>0}$. Prove the following statements.

a) If f and g are convex and not decreasing (or not increasing), then the product $p(x) = f(x)g(x)$ is convex.

b) If f and g are concave, f is not decreasing and g is not increasing (or vice versa), then the product $p(x) = f(x)g(x)$ is concave.

c) If f is convex and not decreasing as well as g is concave and not increasing, then the quotient $q(x) = \frac{f(x)}{g(x)}$ is convex.

a) $p(\lambda x + (1-\lambda)y) \overset{?}{\leq} \lambda p(x) + (1-\lambda)p(y)$

$\Leftrightarrow$ $f(\lambda x + (1-\lambda)y)\, g(\lambda x + (1-\lambda)y) \overset{?}{\leq} \lambda f(x)g(x) + (1-\lambda) f(y)g(y)$

$\leq (\lambda f(x) + (1-\lambda) f(y))(\lambda g(x) + (1-\lambda) g(y))$

$= \lambda^2 f(x)g(x) + \lambda(1-\lambda)(f(x)g(y) + f(y)g(x)) + (1-\lambda)^2 f(y)g(y)$

$= \lambda^2 f(x)g(x) + \lambda(1-\lambda) f(x)g(y) + \lambda(1-\lambda) f(y)g(x) + (1-\lambda)^2 f(y)g(y)$

$= -\lambda(1-\lambda) f(x)g(x) + \lambda f(x)g(x) + \dots + \dots - \lambda(1-\lambda) f(y)g(y) + (1-\lambda) f(y)g(y)$

$= \lambda f(x)g(x) + (1-\lambda) f(y)g(y) + \lambda(1-\lambda)\big(f(x)g(y) + f(y)g(x) - f(x)g(x) - f(y)g(y)\big)$

$= \lambda f(x)g(x) + (1-\lambda) f(y)g(y) + \lambda(1-\lambda)\big(f(y) - f(x)\big)\big(g(x) - g(y)\big)$

$\leq \lambda f(x)g(x) + (1-\lambda) f(y)g(y)$

!! b) Alternative:

$p'(x) = f'(x)g(x) + f(x)g'(x)$

$p''(x) = \underset{\leq 0}{f''(x)}\,\underset{\geq 0}{g(x)} + \underbrace{2f'(x)g'(x)}_{\text{negative}} + \underset{\geq 0}{f(x)}\,\underset{\leq 0}{g''(x)} \leq 0$

c) 1 Use a): $\frac{1}{g(x)}$ convex and not decreasing

2 $q'(x) = \frac{f'(x)}{g(x)} - \frac{f(x)}{g^2(x)} \cdot g'(x)$

$q''(x) = \frac{f''(x)}{g(x)} - \frac{f'(x)g'(x)}{g^2(x)} - \frac{f'(x)g'(x)}{g^2(x)} - \frac{f(x)g''(x)}{g^2(x)} + 2\frac{f(x)g'(x)g'(x)}{g^3(x)}$

$= \underset{\geq 0}{\frac{f''(x)}{g(x)}} - \underset{\geq 0}{\frac{2f'(x)g'(x)}{g^2(x)}} - \underset{\geq 0}{\frac{f(x)g''(x)}{g^2(x)}} + \underset{\geq 0}{\frac{2f(x)(g'(x))^2}{g^3(x)}} \geq 0$

T7

Problem 2. (Second-order condition for convexity) Let $f : \mathcal{C} \to \mathbb{R}$ be a twice differentiable function on a convex set $\mathcal{C} \subset \mathbb{R}^n$. Prove the following statements.

a) Let $n = 1$, then f is convex, iff $f''(x) \geq 0$, $\forall x \in \mathcal{C}$.

b) f is convex, iff $\nabla^2 f(\boldsymbol{x}) \geq 0$, $\forall \boldsymbol{x} \in \mathcal{C}$.

"Restriction of a convex function to a line"

a) " f convex $\iff f''(x) \geq 0 \quad \forall x \in C$ "

" $\Rightarrow$ " First-order: $\begin{cases} f(y) \geq f(x) + f'(x)(y-x) \\ f(x) \geq f(y) + f'(y)(x-y) \end{cases}$

$$\iff f'(x)(y-x) \leq f(y) - f(x) \leq f'(y)(y-x)$$

$$\iff f'(y)(y-x) \geq f'(x)(y-x)$$

w.l.o.g. $y \geq x \iff f'(y) - f'(x) \geq 0 \iff \frac{f'(y)-f'(x)}{y-x} \geq 0$

" $\Leftarrow$ " $f''(z) \geq 0 \quad \forall z \in C$.

Integration by parts

$$\boxed{0 \leq \int_x^y f''(z)(y-z)\,dz} = f'(z)(y-z)\Big|_x^y - \int_x^y f'(z)(-1)\,dz$$

$$= 0 - f'(x)(y-x) + f(y) - f(x) \qquad \text{"First-Order"}$$

b) $\underset{C \to \mathbb{R}}{f}$ convex $\iff g : g(t) = f(x+tv)$ convex, $x + tv \in C$

$\frac{t}{m} \gamma \frac{2}{t}$

$g'(t) = \nabla f(x+tv) \cdot v$, $\boxed{g''(t) = v^T \nabla^2 f(x+tv)\, v}$

" $\Rightarrow$ " f convex $\Rightarrow g''(t) = v^T \nabla^2 f(x+tv)\, v \geq 0 \Rightarrow \nabla^2 f(x+tv) \geq 0$

" $\Leftarrow$ " $\nabla^2 f(x+tv) \geq 0 \Rightarrow g''(t) \geq 0 \Rightarrow f$ convex.

$$\begin{cases} \vec{s} \mapsto f(\vec{s}) = t \\ t \mapsto \vec{h}(t) := \vec{x} + t\vec{v} \\ t \mapsto g(t) := f(\vec{h}(t)) \end{cases}$$

$g'(t) = \frac{\partial f(\vec{s})}{\partial \vec{s}} \cdot \frac{\partial \vec{h}(t)}{\partial t} = \nabla f(\vec{x} + t\vec{v}) \cdot \vec{v}$

$\frac{\partial}{\partial t} \nabla f(\vec{x} + t\vec{v}) \cdot \vec{v}$

$\nabla^2 f(\vec{x} + t\vec{v})\, \vec{v} \cdot \vec{v}$

T8

Problem 1. (Optimality criterion for convex problems) Let $f : \mathcal{C} \to \mathbb{R}$ with $\mathcal{C} \subseteq \mathbb{R}^n$ be a differentiable objective function of a convex optimization problem in standard form and $M[h,g] \subseteq \mathcal{C}$ the corresponding feasible set. $\Longrightarrow$ M is convex!

a) Show that $\boldsymbol{x}^* \in M[h,g]$ is optimal, iff

"$\Rightarrow$"

$$\nabla f\left(\boldsymbol{x}^*\right)^T \left(\boldsymbol{y} - \boldsymbol{x}^*\right) \geq 0$$

for all $\boldsymbol{y} \in M[h,g]$.

b) Let $f(\boldsymbol{x}) = \frac{1}{2}\boldsymbol{x}^T \boldsymbol{P}\boldsymbol{x} + \boldsymbol{q}^T\boldsymbol{x} + r$, $\boldsymbol{x} \in \mathbb{R}^3$, with

$$\boldsymbol{P} = \begin{pmatrix} 13 & 12 & -2 \\ 12 & 17 & 6 \\ -2 & 6 & 12 \end{pmatrix}, \qquad \boldsymbol{q} = \begin{pmatrix} -22 \\ -14.5 \\ 13.0 \end{pmatrix}, \qquad r = 1$$

be the objective function with constraints

$$-1 \leq x_i \leq 1 , \quad i = 1, 2, 3 .$$

Show that the point $\boldsymbol{x}^* = (1, 0.5, -1)^T$ is an optimal solution.

Hint: Use the first order condition for part (a).

a) "$\Rightarrow$" $\quad z := \lambda y + (1-\lambda) x^*$, with $y \in M[h,g]$, $\lambda \in [0,1]$

$\Rightarrow z \in M[h,g]$, as M is convex set!

Goal: $\boxed{\frac{\partial z(\lambda)}{\partial \lambda} = y - x^*}$

$\frac{\partial}{\partial r} f(z(r)) = \nabla f \cdot \frac{\partial z(r)}{\partial \lambda}$

$\nabla f := \frac{f(h+\Delta t) - f(h)}{\Delta h}$

$$\frac{f(\lambda y + (1-\lambda) x^*) - f(x^*)}{\lambda} \geq 0$$

$$\Leftrightarrow \lim_{\lambda \to 0} \frac{f(\lambda \vec{y} + (1-\lambda)\vec{x}^*) - f(\vec{x}^*)}{\lambda} \geq 0$$

$$\Leftrightarrow \lim_{\lambda \to 0} \frac{\partial}{\partial \lambda} f(z(r)) = \nabla f(\vec{x}^*)^T (\vec{y} - \vec{x}^*)$$

"$\Leftarrow$": $\nabla f(\vec{x}^*)^T (\vec{y} - \vec{x}^*) \geq 0 \quad \forall \vec{y} \in M[h,g]$

$$f(\vec{y}) \geq f(\vec{x}^*) + \underbrace{\nabla f(\vec{x}^*)^T (\vec{y} - \vec{x}^*)}_{\geq 0}$$

$\Leftrightarrow f(\vec{y}) \geq f(\vec{x}^*) \quad \forall \vec{y} \in M[h,g]$

$\Leftrightarrow \vec{x}^*$ is optimal minimum.

Problem 1. (Optimality criterion for convex problems) Let $f : \mathcal{C} \to \mathbb{R}$ with $\mathcal{C} \subseteq \mathbb{R}^n$ be a differentiable objective function of a convex optimization problem in standard form and $M[h, g] \subseteq \mathcal{C}$ the corresponding feasible set.

T8

a) Show that $\boldsymbol{x}^* \in M[h, g]$ is optimal, iff

$$\nabla f\left(\boldsymbol{x}^*\right)^T\left(\boldsymbol{y}-\boldsymbol{x}^*\right) \geq 0$$

for all $\boldsymbol{y} \in M[h, g]$.

b) Let $f(\boldsymbol{x}) = \frac{1}{2}\boldsymbol{x}^T\boldsymbol{P}\boldsymbol{x} + \boldsymbol{q}^T\boldsymbol{x} + r\ ,\ \boldsymbol{x} \in \mathbb{R}^3$, with

$$\boldsymbol{P} = \begin{pmatrix} 13 & 12 & -2 \\ 12 & 17 & 6 \\ -2 & 6 & 12 \end{pmatrix}, \qquad \boldsymbol{q} = \begin{pmatrix} -22 \\ -14.5 \\ 13.0 \end{pmatrix}, \qquad r = 1$$

be the objective function with constraints

$$-1 \leq x_i \leq 1\,, \quad i = 1, 2, 3\,.$$

Show that the point $\boldsymbol{x}^* = (1, 0.5, -1)^T$ is an optimal solution.

Hint: Use the first order condition for part (a).

b) $\nabla f(\vec{x}^*)^T(\vec{y} - \vec{x}^*) \overset{?}{\geq} 0 \quad \forall\ y_1, y_2, y_3 \in [-1, 1]$

$= (P\vec{x}^* + \vec{q})^T (\vec{y} - \vec{x}^*)$

$= \left(\begin{pmatrix} 13 & 12 & -2 \\ 12 & 17 & 6 \\ -2 & 6 & 12 \end{pmatrix}\begin{pmatrix} 1 \\ 0{,}5 \\ -1 \end{pmatrix} + \begin{pmatrix} -22 \\ -14{,}5 \\ 13 \end{pmatrix}\right)^T \cdot \begin{pmatrix} y_1 - 1 \\ y_2 - 0{,}5 \\ y_3 + 1 \end{pmatrix}$

$= \begin{pmatrix} 13+6+2-22 \\ 12+8{,}5-6-14{,}5 \\ -2+3-12+13 \end{pmatrix}^T \cdot = (-1, 0, 2) \begin{pmatrix} y_1 - 1 \\ y_2 - 0{,}5 \\ y_3 + 1 \end{pmatrix}$

$= -y_1 + 1 + 2y_3 + 2 = -y_1 + 2y_3 + 3 \overset{!}{\geq} 0$

T9

Problem 1. (Dual problem bounds) For the following optimization problems with optimization variable $\boldsymbol{x} \in \mathbb{R}^2$, compute the dual problem and the maximum lower bound d^* for the optimal value p^*.

1. L : Lagrangian
2. $L_D = \inf L$
   $\hookrightarrow x^* = \ldots$
3. DP $\to d^* \le p^*$

a)

$$\begin{aligned} \text{minimize} \quad & 2x_1^2 + 8x_2^2 \\ \text{subject to} \quad & 3x_1 + 6x_2 = 10 \end{aligned}$$

b) $\frac{t}{u_1} \sqrt{e}$

$$\begin{aligned} \text{maximize} \quad & 2x_1x_2 \\ \text{subject to} \quad & x_1^2 + x_2^2 = 1 \end{aligned}$$

Remark: Convert problem (b) into a minimization problem first.

a) $L(\vec{x}, \lambda) = 2x_1^2 + 8x_2^2 + \lambda(3x_1 + 6x_2 - 10)$

$L_D(\; = \inf_{\vec{x} \in \mathbb{R}^2} L(\vec{x}, \lambda)$: $\nabla L(\vec{x}, \lambda) = \begin{pmatrix} 4x_1 + 3\lambda \\ 16x_2 + 6\lambda \end{pmatrix} \overset{!}{=} 0$

$\Rightarrow \begin{cases} x_1^* = -\frac{3}{4}\lambda \\ x_2^* = -\frac{3}{8}\lambda \end{cases}$

$\Rightarrow L_D(\lambda) = 2 \cdot \; \cdot \lambda^2 + 8 \cdot \frac{9}{64} \cdot \lambda^2 + \lambda\left(-\frac{9}{4}\lambda - \frac{9}{4}\lambda - 10\right)$

$= -\frac{\;}{4}\lambda^2 - 10\lambda = \lambda\left(-\frac{9}{4}\lambda - 10\right)$

DP: $\max L_D(\lambda)$, $-\frac{9}{4}\lambda - 10 \overset{!}{=} 0$, $\lambda = -\frac{10 \cdot 4}{9}$, $\lambda^* = -\frac{20}{9}$

$\Rightarrow \begin{cases} x_1^* = \frac{5}{3} \\ x_2^* = \frac{5}{6} \end{cases}$, feasible!

b) $\min \; -2x_1x_2$

s.t. $x_1^2 + x_2^2 = 1$

$L(\vec{x}, \mu) = -2x_1x_2 + \mu(x_1^2 + x_2^2 - 1)$

$L_D(\mu) = \inf_{\vec{x} \in \mathbb{R}^2} L(\vec{x}, \mu) = \begin{cases} \mu < 1: & -\infty \\ \mu \geq 1: & -\mu \end{cases}$

$\Rightarrow$ DP: $\max \; -\mu$

s.t. $\mu \geq 1$.

$d^* = -1$.

$x_{1,2}^* = \pm\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

T9

Problem 2. (Sum rate maximization in OFDM) A single-user OFDM system provides n orthogonal subcarriers for downlink data transmission. For subcarrier $i = 1, \dots, n$, the data rate is computed from the signal-to-noise ratio $u_i \in \mathbb{R}_{>0}$ as well as the transmit power $p_i \in \mathbb{R}_{\geq 0}$ on this subcarrier as

$$r_i = \log(1 + p_i u_i)\,.$$

The goal of *sum rate maximization* is to find a transmit power vector $\boldsymbol{p} = (p_1, \dots, p_n)$ such that the overall rate R, defined as the sum of the rates r_i of all subcarriers, is maximal. A global power budget P limits the combined transmit power which can be spent over all subcarriers.

a) Formulate the sum rate maximization problem as a convex optimization problem in standard form.

b) State the KKT conditions for this problem. Derive an expression describing the relationship between a primal optimal $\boldsymbol{p}^*$ and a dual optimal $\boldsymbol{\lambda}^*$, respectively.

$$\boxed{\begin{aligned} \min \quad & -\sum_{i=1}^{n} \log(1+p_i u_i) \\ \text{s.t.} \quad & \sum_{i=1}^{n} p_i - P \leq 0 \\ & -p_i \leq 0 \qquad \forall i = 1 \dots n \end{aligned}}$$

b) KKT:

1. $\sum_{i=1}^{n} p_i - P \leq 0, \quad -p_i \leq 0, \quad \forall i = 1 \dots n$
2. $\lambda_P \geq 0, \quad \lambda_i \geq 0, \quad \forall i = 1 \dots n$
3. $\lambda_P \cdot \left(\sum_{i=1}^{n} p_i - P\right) = 0, \quad \lambda_i \cdot p_i = 0 \quad \forall i = 1 \dots n$
4. $\frac{\partial}{\partial p_i}\left[-\sum_{i=1}^{n} \log(1+p_i \underbrace{u_i}_{\text{const.}}) + \lambda_P \cdot \sum_{i=1}^{n} p_i - P - \lambda_i p_i\right] = 0$

(4): $= -\frac{u_i}{1+p_i u_i} + \lambda_P - \lambda_i = 0 \iff \lambda_P - \lambda_i = \frac{u_i}{1+p_i u_i} > 0$

$\Rightarrow \lambda_P > 0$ (*)

$\iff p_i = \frac{1}{\lambda_P - \lambda_i} - \frac{1}{u_i}$

(3): $\boxed{3.1.a}$ $p_i > 0 \Rightarrow \lambda_i = 0$

$\Rightarrow p_i = \frac{1}{\lambda_P} - \frac{1}{u_i}, \quad i = 1 \dots n$

$\boxed{3.1.b}$ $p_i = 0$

$p_i^* = \left(\frac{1}{\lambda_P} - \frac{1}{u_i}\right)^+$

$\boxed{3.2}$

$\sum_{i=1}^{n} p_i^* = P$

$\sum_{i=1}^{n} \left(\frac{1}{\lambda_P} - \frac{1}{u_i}\right)^+ = P$

10

Problem 1. (Weighted sum rate maximization in OFDMA) In a multi-user OFDM system, k users compete for the available n subcarriers on the downlink. The parameters are the signal-to-noise ratios $u_{i,j} \in \mathbb{R}_{>0}$ for subcarrier i and user j, as well as a normalized weight vector $\boldsymbol{w} = (w_1, \dots, w_k) \in \mathbb{R}^k_{>0}$ with $\sum_{j=1}^k w_j = 1$.

The goal of *weighted sum rate maximization* is to find transmit powers $p_{i,j} \in \mathbb{R}_{geq0}$ for subcarrier i and user j such that the weighted sum rate $\sum_{j=1}^k w_j R_j$ is maximized. A global power budget P once again limits the combined transmit power, and each subcarrier can only be used by a single user.

a) Formulate the weighted sum rate maximization problem analogously to the single-user problem. Give a reason why this problem is not convex.

b) Assume that an allocating function $a\colon \{1, \dots, n\} \to \{1, \dots, k\}$ assigns each subcarrier i to a user $a(i)$. On the basis of this allocation, state the KKT conditions for the problem. Derive an expression describing the relationship between a primal optimal $\boldsymbol{p}^*$ and a dual optimal $\boldsymbol{\lambda}^*$, respectively.

c) Given a fixed allocation a, how does the solution to the multi-user problem differ from the solution to the single-user case?

a) $\min \; -\sum_{j=1}^{k} w_j \cdot \sum_{i=1}^{n} \log(1 + p_{i,j} u_{i,j})$

$$\text{s.t.} \quad \sum_{j=1}^{k} \sum_{i=1}^{n} p_{i,j} - P \leq 0$$

$$-p_{i,j} \leq 0 , \quad i = 1 \dots n , \; j = 1 \dots k$$

$$p_{i,j} \cdot p_{i,l} = 0 , \quad i = 1 \dots n , \; j \neq l$$

Not affine !

b) $\min \; -\sum_{i=1}^{n} w_{a(i)} \log(1 + p_{i,a(i)} u_{i,a(i)})$

$$\text{s.t.} \quad \sum_{i=1}^{n} p_{i,a(i)} - P \leq 0$$

$$-p_{i,a(i)} \leq 0 , \quad i = 1 \dots n$$

Analogeously : $p_i^* = \left(\frac{w_{a(i)}}{\lambda_P} - \frac{1}{u_i} \right)^+ , \quad \sum_{i=1}^{n} \left(\frac{w_{a(i)}}{\lambda_P} - \frac{1}{u_i} \right)^+ = P$

c) *The only difference between the single- and the multi-user problem is the waterlevel which is used to compute p_i^*.*

In the single-user case, there is a single waterlevel given by $1/\lambda_P$.

In the multi-user case, on the other hand, each user k has an individual waterlevel. These waterlevels are given by w_j/λ_P, which means that the waterlevels of each user are weighted according to the weight vector w that appears in the objective function.

By increasing the weight wj, user j gets a larger part of the power budget, and vice versa. A good choice of w ensures that users under bad channel conditions are still able to transmit data and improves the overall fairness in the system.

$\boxed{10}$

Problem 2. (One-dimensional trust region problem) Consider the one-dimensional, real-valued trust region problem.

$$\begin{array}{ll} \text{minimize} & ax^2 + 2bx \\ \text{subject to} & x^2 \le 1. \end{array}$$

a) Determine all pairs (a, b) for which the problem is non-convex.

In the following the problem shall be non-convex.

b) Calculate the dual function $L_D(\lambda)$

c) Give the optimal parameter λ^* which maximizes L_D and the corresponding value d^*.

d) Show that the optimal value of the primal problem p^* equals d^*.

a) $ax^2 + 2bx$ non-convex: $'' = 2a \iff 2a < 0$

$x^2 - 1$ non-convex: never,

So: $\{(a,b) \mid a < 0,\ a, b \in \mathbb{R}\}$

b) $L(x,\lambda) = ax^2 + 2bx + \lambda x^2 - \lambda = (a+\lambda)x^2 + 2bx - \lambda$, $\lambda \geq 0$

$$L_D(\lambda) = \inf_{x \in \mathbb{R}} L(x,\lambda) = \begin{cases} a+\lambda > 0: & \cancel{(a+\lambda)}\dfrac{\cancel{4}b^2}{\cancel{4}(a+\lambda)^{\cancel{2}}} - \dfrac{{}^2\cancel{4}b^2}{\cancel{2}(a+\lambda)} - \lambda = \dfrac{-b^2}{a+\lambda} - \lambda \\ a+\lambda = 0 \wedge b = 0: & -\lambda \\ \text{else}: & -\infty \end{cases}$$

c) $\max \; -\dfrac{b^2}{a+\lambda} - \lambda$

s.t. $-\lambda \leq 0$

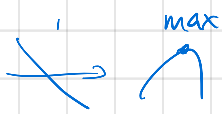
$-a-\lambda < 0$

$L_D' = -1 + b^2 \cdot \dfrac{1}{(a+\lambda)^2} = \dfrac{b^2}{(a+\lambda)^2} - 1$

$L_D'' = b^2 \cdot \left(-\dfrac{2}{(a+\lambda)^3}\right) \leq 0$

"shall be non-convex"

$L_D' \overset{!}{=} 0$: $(a+\lambda^*)^2 = b^2$

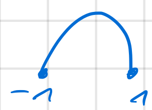


$\lambda^* = |b| - a$, $d^* = -\dfrac{b^2}{|b|} - |b| + a$

$= a - 2|b|$

d) Feasible set: $[-1, 1]$

Objective function: non-convex:



So local minima are achievable at borders:

$p^* = \min(f(\pm 1)) = \min(a \pm 2b) = a - 2|b| = d^*$

T11

Problem 1. (Backtracking line search) Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a strict convex function with $\nabla^2 f(\boldsymbol{x}) \preceq M\boldsymbol{I}_n$ for $M > 0$ and $\Delta\boldsymbol{x}$ the descent direction at $\boldsymbol{x} \in \mathbb{R}^n$.

a) Show that the backtracking line search stopping criterion holds for

$$0 < t \leq -\frac{\nabla f(\boldsymbol{x})^T \Delta\boldsymbol{x}}{M\,\|\Delta\boldsymbol{x}\|_2^2} . \qquad (*)$$

b) Use the above result to derive an upper bound on the number of backtracking iterations.

Backtracking line search. $\alpha \in (0, \frac{1}{2})$, $\beta \in (0,1)$

START $t=1$, REPEAT $t := \beta t$, UNTIL $f(x + t\Delta x) < f(x) + \alpha t \nabla f(x)^T \Delta x$

a) Taylor's theorem: $f(x) = P_k(x) + R_k(x)$, k times diff'

$$P_k(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(k)}(a)}{k!}(x-a)^k$$

$$R_k(x) = \frac{f^{(k+1)}(\xi_L)}{(k+1)!}(x-a)^{k+1}, \quad a \leq \xi_L \leq x.$$

$k=1$: $\quad f(y) = f(a) + \nabla f(a)^T (y-a) + (y-a)^T \cdot \frac{\nabla^2 f(z)^T}{2} \cdot (y-a)$

$$\Rightarrow \quad f(x + t\Delta x) = f(x) + \nabla f(x)^T (t\Delta x) + \frac{1}{2} \cdot \underbrace{(t\Delta x)^T \cdot \nabla^2 f(z)^T (t\Delta x)}_{\leq \frac{t^2}{2} \|\Delta x\|_2^2 \cdot M \cdot \cancel{I_n}}$$

To prove: $\quad \nabla f(x)^T \cdot \cancel{t}\Delta x + \frac{t^{\cancel{2}}}{2} \|\Delta x\|_2^2 \cdot M < \alpha \cancel{t} \nabla f(x)^T \Delta x$

$$\Leftrightarrow \quad (1-\alpha)\, \nabla f(x)^T \Delta x + \frac{tM}{2} \|\Delta x\|_2^2 < 0$$

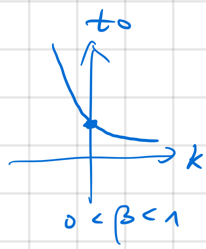
$$\Leftrightarrow \quad t < -2(1-\alpha) \frac{\nabla f(x)^T \Delta x}{M \|\Delta x\|_2^2}$$

$\Leftrightarrow$ Compared to $(*)$, to prove $\quad -2(1-\alpha) < -1$

$$2(1-\alpha) > 1 \qquad \square$$

b) $\quad t_0 := -\dfrac{\nabla f(x)^T \Delta x}{M \|\Delta x\|_2^2}$

$\beta^k \cdot t \leq t_0$ with $t = 1$. $\qquad \beta^k \leq t_0$



$$k \geq \frac{\log t_0}{\log \beta}$$

$\boxed{11}$

Problem 2. (Pure Newton method) Consider the minimization of the following functions. Plot f, g and their derivatives. Apply the pure Newton method for fixed step size $t = 1$ and calculate the values for the first few iterations. Calculate the difference to the minimum in each iteration.

a) The function $f : \mathbb{R} \longrightarrow \mathbb{R}$ with $f(x) = \log(e^x + e^{-x})$ has a unique minimizer $x^* = 0$. Use the starting values $x^{(0)} = 1$ and in a second run $x^{(0)} = 1.1$.

b) The function $g : \mathbb{R}_{>0} \longrightarrow \mathbb{R}$ with $g(x) = -\log(x) + x$ has a unique minimizer $x^* = 1$. Use the starting value $x^{(0)} = 3$.

Hint: Note that $\tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$ for $x \in \mathbb{R}$.

a) $f(x) = \log(e^x + e^{-x})$, $x^{(0)} = 1$, / $x^{(0)} = 1,1$

$$\nabla f(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \tanh(x) \quad , \quad \nabla^2 f(x) = 1 - \tanh(x)^2$$

$$\Delta x_{nt} = -\nabla^2 f(x)^{-1} \nabla f(x) = -\frac{\tanh(x)}{1 - \tanh(x)^2} \quad , \quad \lambda^2 = \frac{\tanh(x)^2}{1 - \tanh(x)^2}$$

$$x^{(k+1)} = x^{(k)} - \frac{\tanh(x)}{1 - \tanh(x)^2}$$

	$x^{(k)}$, $x^{(0)} = 1$	$x^{(k)}$, $x^{(0)} = 1,1$
0	1	1,1
1	−0,81343	−1,1286
2	0,40934	1,234
3	−0,0473	−1,695
4	$7,058 \cdot 10^{-5}$	5,7154
5		−23021 !

b) $g(x) = -\log(x) + x$. $x^{(0)} = 3$

$$\nabla g(x) = -\frac{1}{x} + 1 \quad , \quad \nabla^2 g(x) = \frac{1}{x^2}$$

$$\Delta x_{nt} = -\frac{-\frac{1}{x} + 1}{\frac{1}{x^2}} = -\left(-\frac{x^2}{x} + x^2\right) = x - x^2$$

$$\lambda^2 = \frac{1 + \frac{1}{x^2} - \frac{2}{x}}{\frac{1}{x^2}} = x^2 - 2x + 1$$

$$x^{(k+1)} = x^{(k)} + (x^{(k)} - x^{(k)2})$$

$x^{(1)} = 3 + 3 - 9 = -3$! out of domain ! Failed!

12

Problem 1. (Barrier method) Let

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & 2 \le x \le 4 \end{array}$$

be an optimization problem with $f(x) = x + 1$, $x \in \mathbb{R}$. The feasible set is $[2, 4]$ and the optimal solution $x^* = 2$. Formulate the logarithmic barrier function $\Phi(x)$ and calculate the optimal solution $x^*(t)$ of the problem

$$\text{minimize} \quad tf(x) + \Phi(x)$$

with $x \in \mathbb{R}$ and constant $t > 0$. Illustrate the development of $x^*(t)$ and $f(x^*(t))$ for increasing t. What happens for $t \to \infty$?

$\Phi(x) = -\sum_{i=1}^{s} \log(-g_i(x))$

no equality constraints

1. Centering step: $x^*(t)$: $\min \; tx + t + \Phi(x)$

with $\Phi(x) = -\log(-x+4) - \log(x-2)$.

Compute x^*: $0 \overset{!}{=} \nabla(tx + t + \Phi(x))$

$$= t + \frac{1}{-x+4} - \frac{1}{x-2} \Rightarrow t = \frac{1}{x-2} + \frac{1}{4-x}$$

$$(x-2)(4-x) = \frac{2}{t}$$

$$\Rightarrow -x^2 + 6x - 8 - \frac{2}{t} = 0. \quad x^* = \frac{-6 \pm \sqrt{36 - (32 + \frac{8}{t})}}{-2}$$

$$(\text{positive}) = \frac{6 - \sqrt{4 - \frac{8}{t}}}{2} = 3 - \sqrt{1 - \frac{2}{t}}$$

$t \to \infty$: $x^* = 3 - 1 = 2$: optimum achieved!

12

Problem 2. (Branch-and-bound algorithm for a 0-1 linear program)

a) A network operator can offer $n \in \mathbb{N}$ different services to its customers with revenues $c_1, \dots, c_n \in \mathbb{R}$ corresponding to each service. Each service requires a certain bandwidth $v_1, \dots, v_n \in \mathbb{R}$ within the frequency band available to the network operator, whose width is given as $B \in \mathbb{R}$. A service can at most be offered to one customer. Formulate the optimization problem which maximizes the revenue as an integer linear programming problem.

ILP

b) Solve the knapsack problem by using branch-and-bound algorithm for $n = 3$, and $c_i = v_i$ for $1 \leq i \leq 3$, where $c_1 = c_2 = 2$, $c_3 = 3$ and $B = 6$.

a) Offering vector : $\vec{x} = (x_1 \cdots x_n)^T$, where $x_i = 1$ means offering this service, $x_i = 0$ means not.

$$\boxed{\begin{aligned} \max \quad & c^T \cdot x \\ \text{s.t.} \quad & v^T \cdot x - B \leq 0 \\ & x \in \{0,1\}^n \end{aligned}}$$

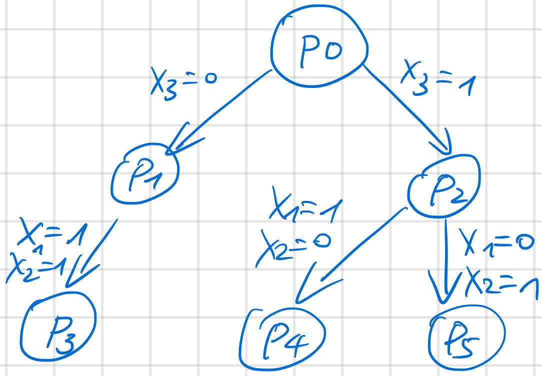
b)

$$\begin{aligned} \max \quad & 2x_1 + 2x_2 + 3x_3 \\ \text{s.t.} \quad & 2x_1 + 2x_2 + 3x_3 - 6 \leq 0 \\ & x_1, x_2, x_3 \in \{0,1\} \end{aligned}$$

1 Keep x_3 fixed: $\begin{cases} x_3 = 0 : & x_1 + x_2 \leq 3 \quad (1.1) \\ x_3 = 1 : & x_1 + x_2 \leq \frac{3}{2} \quad (1.2) \end{cases}$

(1.1) : $x_1 = x_2 = 1$: $p^* = 4$

(1.2) : $\begin{cases} x_1 = 0, \; x_2 = 1 \\ x_1 = 1, \; x_2 = 0 \end{cases}$



	x^*	p^*	
P_0	$\{x_1, x_2, x_3\}$	/	
P_1	$\{x_1, x_2, 0\}$	/	
P_2	$\{x_1, x_2, 1\}$	/	
P_3	$\{1, 1, 0\}$	4	
$P_{4,5}$	$\{1, 0, 1\}, \{0, 1, 1\}$	5	*