On Siegel's formula in the theory of classical group

André Weil
Translated by Yu LUO

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Abstract

This is a translation of André Weil's 1965 paper "Sur La Formule de Siegel Dans La Théorie des Groupes Classiques", published in Acta Mathematica. The translation was primarily accomplished with the assistance of AI. Any errors or inaccuracies are my responsibility. For typos, corrections, or suggestions, please contact yluo237@wisc.edu.

In the end of the previous memoir ([14, n^o 52]), I have claimed a formula which (a slightly more general form which will be given here) contains most of the results obtained by Siegel in his work on quadratic forms. Essentially, it's about an identity whose second term is an Eisenstein series, while the first is obtained by writing a series (which generalizes the theta series) depending on certain parameters, and integrating the series over the region related to the fundamental domain of certain discrete group. As will be seen, we can deduce, among other applications, a determination of "Tamagawa number" of most classical groups, as well as a partial demonstration of the "Hasse principal" and an "appriximation theorem" for these groups.

We will try to simultaneously treat different types of classical groups, following Siegel's idea, which contains first of all, a semisimple algebra A with involution and a module X over A. One can always, as we know, reduce to either in cases (I) where A is simple, or to the case (II) A is sum of two simple components exchange by the given involution; the case (I) itself can be subdivided into several types (involutions of first and the second kind, etc.). It is true that in many situations, I cannot avoid separating into many cases, and the reader will not be stuck if they put off the reading of those complicated results. In many senses, the case (I) is the most interesting; it would have been tempting to limit ourselves to this case if the extension of the base field (and in particular, the "localization", i.e., passing from the base field to one of its completions) did not necessarily lead from (I) to (II) when we start from an involution of the second kind. As a result, we cannot exclude the case (II) without excluding these involutions. Similarly, we will see that the most interesting applications of our results presents when we take X a simple modules on A. But, when we extend the base field, a simple A-module will generally not to be simple, unless A is a matrix algebra of the base field. If from the beginning we restrict us to these algebras, and simple modules on these algebras, we could have present the theory much shorter, and more enjoyable for reading, and which nevertheless still brought out the essential feature of the theory. Also for the reader who are unfamiliar with this type of question might find it helpful, during his first read, to only consider this case in mind (even further assuming that the involution in A is given by the symmetric matrix, such that correspond to the theory of quadratic forms and orthogonal groups) and neglect all the cases that are not visible unless it is trivial. However, it is a truth which has been seem from the evolution of the contemporary mathematics that specialization can not simplify a complicated theory but generalization can, as long as the theory is valuable; Of course, here I am not referring to those superficial or artificial generations, which is sometimes suggested by fashion, but to those generations which put the theory in a natural framework and be free from the adventitious circumstances which is encombrant and disfigured. There is no double that what we present here is only a fragment of the future arithmetic theory of semi-simple groups. But it can be advantageous, with a view to progress in this direction, to have the most extensive comparison material possible; it is in this spirit that I have sought to give at least to the method here (directly inspired by that of Siegel in [12]) followed all the generality it involves, rather than to present a mere sampling.

We will reply on the results in the theory of semi-simple groups obtained in recent years by Borel and Harish-Chandra, by Borel, and by Godement. The validity of these results is provisionally acquired in the case of characteristic 0. As a consequence, we will limit ourselves exclusively to this case from Chapter III. Most of the results in Chapter I and II are true in any characteristic, or at least when the characteristic is not 2. The main results and notations of [14] are supposed to be known to the readers.

Table of notations

(Cf. also the table of notations tables of [14])

Chapter I

n°1 : F_Φ^*.

n°2 : $F_\Phi, (A), (B), (B_0), (B_1)$.

n°3 : $|\omega|_v, X^o$.

n°4 : $U_v^o, |\lambda\omega|_A, \Delta_G$.

n°5 : $(\omega/f_*\eta)_y$.

n°8 : (W).

n°9 : $T, a_\tau, \Theta(T)$.

n°10 : T_A^+, Θ^+.

Chapter II

n°14 : $\mathcal{A}, \iota, \tau, X, X^*$.

n°15 : $\otimes_\iota, I(X), I(X)^*, Q(X/\mathcal{A})$.

n°16 : $\mathcal{A}_v, A_v, A, \mathfrak{k}_v, \mathcal{B}, m_v, X, X_v, n_v$.

n°17 : B.

n°18 : (I), (II), $\mathfrak{k}, \xi \to \xi', h, \eta, \mathfrak{k}', \mathfrak{z}, \mathfrak{z}_0, \mathfrak{r}_m$.

n°19 : $h[x], \delta, \varepsilon$.

n°21 : δ.

n°22 : $G, U(i)$.

n°24 : $\mathrm{Aut}(X), \mathrm{Aut}(B), \bar{v}, \mathrm{Aut}(X,i)$.

n°26 : $\mathfrak{R}, m_K, n_K, p_K, q_K$.

n°27 : $(\mathrm{I}_0), (\mathrm{I}_1), (\mathrm{I}_2), (\mathrm{I}_3), (\mathrm{I}_4)$.

n°28 : $G_1, Ps'(X, \mathcal{A})$.

Chapter III

n°31 : $(\mathbf{A})$.

n°34 : $(\mathbf{A}')$.

n°36 : $\bar{U}(i)$.

n°37 : θ_i.

n°38 : $(\mathbf{B}), (\mathbf{B}')$.

Chapter IV

n°39 : $E(\Phi)$.

n°41 : $E_Z(\Phi), E_X(\Phi), E_X, E_Z, E$.

n°43 : $U(i), U(i)_K, U(i)_v, U(i)_A, |\theta_i|_v, |\theta_i|_A$.

Chapter VI

n°51 : $\rho, \rho_1, \widetilde{G}, \widetilde{\rho}, I(\Phi), I_1(\Phi), \widetilde{I}(\Phi)$.

I Analytic Preliminaries

1. In the following, we will denote a locally compact abelian group as G. As in [14], we denote G^* the dual of G, and $\langle g, g^* \rangle$ the value in g of the character of G, which correspond to $g^* \in G^*$. We will also denote the Haar measure dg for G, and $dg*$ for G^*, respectively. Recall that the Fourier transformation defines an isomorphism, between a part of $L^2(G)$ and $L^2(G^*)$, and also between "Bruhat-Schwartz" space $S(G)$, $S(G^*)$ respectively associated to G and G^* (cf. [14], n^o 11). By definition, a "tempered distribution" on G is a continuous linear form on $S(G)$, that is to say, the space of tempered distributions on G is the dual of $S(G)$. It follows that the the Fourier transformation determines an isomorphism between the space of tempered distributions on G and on G^*. In particular, if φ is a bounded locally integrable function on G, $\varphi(g)dg$ can be considered as a tempered distribution, and therefore its Fourier transform is a tempered distribution on G^*. Recall again that a function $\Phi \in \mathcal{S}(G)$ has compact support are everywhere dense in the space of compact support continuous functions on G; we conclude that, if a tempered distribution T on G is positive (that's to say, $T(\Phi) \geq 0$ whenever $\Phi \geq 0$), this defines a measure (called "tempered") on G, and (all the measures) can be identified with this one.

In the following content, we will give a locally compact space with a measure dx, and a continuous map f from X to G; from the §2, X will be an abelian group, and dx an Haar measure of X. To all functions $\Phi \in L^1(X)$, we will associate to a function F^*_Φ on G^*, defined by the formula

$$F^*_\Phi(g^*) = \int_X \Phi(x) \cdot \langle f(x), g^* \rangle dx, \tag{1}$$

one can immediately verify that this is continuous and bounded. With these notations, we have the following lemma

Lemma 1. *Let τ a bounded locally integrable function on G^*; let T a tempered distribution of G, being the Fourier transform τ. For any compact neighborhood W of 0 in G, let φ_W a continuous function ≥ 0 on G with support contained in W, such that $\int \varphi_W dg = 1$; and let $t_W = \widetilde{\varphi}_W * T * \varphi_W$. Also let t_W be a continuous and bounded function on G; and, for any $\Phi \in L^1(X)$ such that the integral*

$$S(\Phi) = \int_{G^*} F^*_\Phi(-g^*)\tau(g^*)dg^* \tag{2}$$

is absolute convergent, we also have

$$S(\Phi) = \lim_W \int_X \Phi(x) t_W(f(x))dx, \tag{3}$$

with the limit being taken in the filtered order of neighborhood of 0 *in* G*. Moreover, if the absolute convergence of* (2) *is uniform in a bounded subset* B *of* $L^1(X)$*, then the second part of* (3) *convergent uniformally on* B*.*

Let φ^*_W be the Fourier transform of φ_W; we have

$$|\varphi^*_W(g^*)| \leq \int \varphi_W dg = 1, \quad \int |\varphi^*_W(g^*)|^2 dg^* = \int \varphi_W^2 dg < +\infty.$$

As we all know that, t_W (which is a "regularization" of T) is the Fourier transform of $|\varphi^*_W|^2\tau$, that is to say, we have

$$t_W(g) = \int |\varphi^*_W(g^*)|^2 \tau(g^*) \cdot \langle g, -g^* \rangle dg^*;$$

therefore, it is a bounded continuous function on G. Using Lebesgue-Fubini, we have, for all $\Phi \in L^1(X)$:

$$\int_X \Phi(x) t_W(f(x))dx = \int_{G^*} \left(\int_X \Phi(x) \cdot \langle f(x), -g^* \rangle dx \right) |\varphi_W^*(g^*)|^2 \tau(g^*) dg^*,$$

and as a consequence, if (2) is absolute convergent:

$$S(\Phi) - \int_X \Phi(x) t_W(f(X))dx = \int_{G^*} (1 - |\varphi_W^*(g^*)|^2) F_\Phi^*(-g^*) \cdot \tau(g^*) dg^*. \tag{4}$$

Let K a compact set of G^*, and let $\varepsilon > 0$; there is a neighborhood W_0 of 0 in G such that for any $g \in W_0$ and $g^* \in K$, $|\langle g, g^* \rangle - 1| \leq \varepsilon$ by the following:

$$|\varphi_W^*(g^*) - 1| = \left| \int \varphi_W(g) \cdot (\langle g, g^* \rangle - 1) dg \right| \leq \varepsilon$$

which holds for any $W \subset W_0$ and $g^* \in K$; in other words, we have $\lim_W \varphi_W^* = 1$, uniformly over all compact subsets of G^*, the limit is understood as it was said in in the statement of lemma 1. The first assertion of the lemma then follows immediately from (4). As for the second one, the hypothesis means that for all $\varepsilon > 0$ we give a correspondence for a compact subset K of G^* such that we have

$$\int_{G^* - K} |F_\Phi^*(-g^*) \tau(g^*)| dg^* \leq \varepsilon$$

for any $\Phi \in B$; this is being said, the conclusion also follows from (4).

2. Let's assume now that X is a locally compact abelian group and that dx an Haar measure on X. We will have to take the following remark, which for more convenience we write into the following lemma:

Lemma 2. *Let $(S_\alpha)_{\alpha \in A}$ be a family of tempered distribution on X, and let $\mathcal{F}$ a filter on A. Suppose that $(S_\alpha(\Phi))_{\alpha \in A}$ convergent on $\mathcal{F}$ for all $\Phi \in \mathcal{S}(X)$, the convergence is uniform on all compact subset of $\mathcal{S}(X)$. Then $S(\Phi) = \lim_{\mathcal{F}} S_\alpha(\Phi)$ is a tempered distribution.*

It is in fact clear that it is a linear form on $\mathcal{S}(X)$, continuous on all compact subset of $\mathcal{S}(X)$. As $\mathcal{S}(X)$ is the inductive limit of measurable spaces $\mathcal{S}(H, H')$ (cf. [14], n^{o}11), the conclusion follows immediately.

Lemma 3. *With the same hypothesis and notation as Lemma 1 in* n^o*1, and suppose moreover that X is a locally compact abelian group, with dx a Haar measure on X, and that* (2) *is absolute convergent for all $\Phi \in \mathcal{S}(X)$ and uniformally for all compact subset of $\mathcal{S}(X)$. Then* (2) *defines a tempered distribution S on X, whose support contained in $f^{-1}(\operatorname{supp} T)$; if T is of positive measure, then so is of S; and if T coincides with a measure $\rho(g)dg$ in a open subset U of G, ρ is a continuous function on U, S coincide with the measure $\rho(f(x))dx$ in $f^{-1}(U)$.*

According to Lemma 1, we can also consider S defined by means of (3); the lemma 1 and 2 also show that S is a tempered distribution. The assertion of the support of S result due to the fact that the support of t_W is contained in $\operatorname{supp}(T) + W + (-W)$. If T is a positive measure, we have $t_W \geq 0$ for any W, so $S(\Phi) \geq 0$ for all $\Phi \geq 0$. Finally, if T coincide with $\rho(g)dg$ in U, we haev $\lim_W t_W = \rho$ uniformly in all compact subset of U, so $\lim_W t_W \circ f = \rho \circ f$ uniformly in all compact subset of $f^{-1}(U)$, hence the last assertion of the lemma follows, by the virtue of (3).

Proposition 1. *Suppose X and G are locally compact abelian groups, let f be a continuous map from X to G, satisfies the following condition:*

(A) for any $\Phi \in \mathcal{S}(X)$, the function F^*_Φ defined by (1) is integrable on G^*; and the integration $\int |F^*_\Phi| dg^*$ convergent uniformly on all compact subset of $\mathcal{S}(X)$.

Then we can find, one and only one way, to match every $g \in G$ with a positive measure μ_g on X, whose support contained in $f^{-1}(\{g\})$, so that, for all continuous functions Φ on X with compact support, the function F_Φ on G defined by $F_\Phi(g) = \int \Phi d\mu_g$ is continuous and satisfies that $\int F_\Phi dg = \int \Phi dx$. Moreover, the μ_g are tempered measure; and for all $\Phi \in \mathcal{S}(X)$, F_Φ is continuous, belongs to $L^1(G)$, satisfies that $\int F_\Phi dg = \int \Phi dx$ and is the Fourier transform of F^*_Φ.

If (A) is satisfied, we can, for any g, apply the lemma 3 with f and a "Dirac distribution" $T = \delta_g$, that's it to say, a mass 1 concentrated on g. Hence exist a tempered positive measure μ_g, with support contained in $f^{-1}(\{g\})$, such that for all $\Phi \in \mathcal{S}(X)$, $F_\Phi(g) = \int \Phi d\mu_g$ is the value at g of the Fourier transform of F^*_Φ; since $F^*_\Phi \in L^1(G^*)$ for $\Phi \in \mathcal{S}(X)$, F_Φ is therefore bounded and continuous. For $\Phi \in \mathcal{S}(X)$, with $\Phi \geq 0$, we have $F_\Phi \geq 0$; so F^*_Φ is therefore a continuous function of positive type, and as a result is a Fourier transform of a positive bounded measure; this cannot be other than $F_\Phi(g)dg$, which shows that F_Φ belongs to $L^1(G)$. But (for example, by virtue of lemma 5 of [14], n^{o}14) all function $\Phi \in \mathcal{S}(X)$ is written, if it's real valued, as the difference of two positive functions of $\mathcal{S}(X)$, so for all case as the finite linear combinations of such functions. So we have $F_\Phi \in L^1(G)$ for any $\Phi \in \mathcal{S}(X)$. Therefore, it follows that F^*_Φ is the Fourier transform of F_Φ, in particular, we have $F^*_\Phi(0) = \int F_\Phi dg$; as we have $F^*_\Phi(0) = \int \Phi dx$ by the definition of F^*_Φ, this give, for any $\Phi \in \mathcal{S}(X)$, with the announced equalities. Next, let Φ a continuous function on X, with the support C compact; let U a compact neighborhood of 0 in X, and $C' = C + U$. It is easy to see that there exists a function $\Psi \in \mathcal{S}(X)$ which is ≥ 0 on X and ≥ 1 on C', and, for any $\varepsilon > 0$, a function $\Phi' \in \mathcal{S}(X)$, of support contained in C', such that $|\Phi' - \Phi| \leq \varepsilon$. Therefore, the support of F_Φ, $F_{\Phi'}$ is contained in $f(C')$, and we have $|F_{\Phi'} - F_\Phi| \leq \varepsilon F_\Psi$. We can immediately conclude that F_Φ is continuous and that has $\int F_\Phi dg = \int \Phi dx$. It remains to prove the uniqueness of μ_g. Let (μ'_g) a family of measures having the properties stated in proposition 1, and let $\mu''_g = \mu'_g - \mu_g$. We will have, for all continuous function Φ with compact support:

$$\int \left(\int \Phi(x) d\mu''_g(x) \right) dg = 0.$$

By replacing $\Phi(x)$ to $\Phi(x)\varphi(f(x))$, where φ is a continuous function on G, we obtain

$$\int \left(\int \Phi(x) d\mu''_g(x) \right) \varphi(g) dg = 0.$$

This relation must be satisfied for all φ; on the other hand, the hypotheses we made implies that $\int \Phi d\mu''_g$ is a continuous function of $g \in G$, it follows that it must vanishes for any Φ, so that $\mu''_g = 0$.

Proposition 2. *Let G a locally compact abelian group, Γ a discrete subgroup of G such that G/Γ is compact, and G_* a discrete subgroup of G^* that corresponds to Γ by duality. Let X a locally compact abelian group and f a continuous map from X to G satisfies the following condition:*

(B) for any $\Phi \in \mathcal{S}(X)$ and $g^* \in G^*$, the series

$$\sum_{\gamma^* \in \Gamma_*} |F^*_\Phi(g^* + \gamma^*)| \tag{5}$$

is convergent, which is uniform for any compact subset of $\mathcal{S}(X) \times G^*$.

Then f satisfies the condition (A) in proposition 1. Furthermore, if F_Φ once again designates the Fourier transform of F^*_Φ defined by (1), we have, for any function $\Phi \in \mathcal{S}(X)$:

$$\sum_{\gamma \in \Gamma} F_\Phi(\gamma) = \sum_{\gamma^* \in \Gamma_*} F^*_\Phi(\gamma^*),$$

the series of the two sides are absolutely convergent.

The previous assertion means that, if $\varepsilon > 0$, and if C is a compact subset of $\mathcal{S}(X)$, it exists a compact subset K of G^* such that the intagration of $|F^*_\Phi|$ on $G^* - K$ is $\leq \varepsilon$ for any $\Phi \in C$. Now, as Γ is discrete, G^*/Γ_* is compact; hence there exists a compact subset C^* of G^* such that $G^* = C^* + \Gamma_*$. Therefore, according to (B), there exists, for any $\delta > 0$, a finite subset F of Γ_* such that we have

$$\sum_{\gamma^* \in \Gamma_* - F} |F^*_\Phi(g^* + \gamma^*)| \leq \delta$$

whenever $\Phi \in C$ and $g^* \in C^*$. Our condition will satisfies if we take $\delta \leq m(C^*)^{-1}\varepsilon$, where $m(C^*)$ is the measure of C^* on dx, and $K = C^* + F$.

We can hence apply the f in proposition 1; as a result, there exists a measure μ_g with the properties stated in there. On the other hand, let be π a canonical homomorphism from G to G/Γ, and let $\bar{f} = \pi \circ f$. As Γ_* is the dual of G/Γ, (B) implies that the map $\bar{f}$ from X to G/Γ satisfies the condition (A) of the proposition 1. There exists hence for all g a positive tempered measure $\nu_{\pi(g)}$, with support contained in the intersection of $f^{-1}(\{g + \gamma\})$ for $\gamma \in \Gamma$, such that we have, for all $\Phi \in \mathcal{S}(X)$:

$$\int \Phi d\nu_{\pi(g)} = \sum_{\gamma^* \in \Gamma_*} F^*_\Phi(\gamma^*) \cdot \langle g, -\gamma^* \rangle, \tag{6}$$

the first term is a continuous function. In particular, let's take Φ a function with compact support C; as $F_\Phi(g) = \int \Phi d\mu_g$ by the proposition 1, the support of F_Φ is contained in $f(C)$. Also the function $\bar{F}$ defined by

$$\bar{F}(g) = \sum_{\gamma \in \Gamma} F_\Phi(g + \gamma) = \int \Phi \cdot (\sum_{\gamma \in \Gamma} d\mu_{g+\gamma}),$$

is continuous and determines, by pass to the quotient, a continuous function on G/Γ, so the Fourier coefficient are given by

$$c(\gamma^*) = \int_{G/\Gamma} \bar{F}(g) \cdot \langle g, \gamma^* \rangle d\dot{g} = \int_G F_\Phi(g) \cdot \langle g, \gamma^* \rangle dg = F^*_\Phi(\gamma^*);$$

in the second term, we have pose $\dot{g} = \pi(g)$. As $\nu_{\pi(g)}$ is a tempered measure, it follows that the series $\sum F_\Phi(g + \gamma)$ is absolute convergent for all positive function $\Phi \in \mathcal{S}(x)$, and also for all $\Phi \in \mathcal{S}(X)$. By applying (2) for $g = 0$, and taking into account what we have just demonstrated, we obtain the second part of the proposition.

We can express the second part of the proposition 2 by saying that the Possion formula is true for the functions F_Φ, F^*_Φ; it would be trivial if these functions belongs to $\mathcal{S}(G)$ and $\mathcal{S}(G^*)$, respectively, but the simple examples show that this is not the case in general.

On the other hand, we know, for $\Phi \in \mathcal{S}(X)$, we denote Φ_{g^*} the function defined by $\Phi_{g^*}(x) = \Phi(x)\langle f(x), g^* \rangle$. We will apply the proposition 2 to those function f satisfies the following conditions:

(B_0) $(\Phi, g^*) \to \Phi_{g^*}$ is a continuous map from $\mathcal{S}(X) \times G^*$ to $\mathcal{S}(X)$;

(B_1) the series $\sum_{\gamma^* \in \Gamma_*} |F^*_\Phi(\gamma^*)|$ is uniformly convergent for all compact subset of $\mathcal{S}(X)$. If we replace Φ by $\Phi_{g_0^*}$, $F^*_\Phi(g^*)$ clearly change to $F^*_\Phi(g^* + g_0^*)$; we can conclude that (B_0) and (B_1) implies (B), hence that the proposition 2 can apply to all functions satisfies these two conditions.

3. We will now complete the part in [14] concerns about the "adelic" space (cf. [14]. n^{os} 29-30), and first of all fix the choice of measures in these spaces.

We will reply on the following general observation. Let G be a locally compact abelian group, and $\chi(x, y)$ a non-degenerated bicharacter of $G \times G$, that is to say, a function of the form $\langle x, y\beta \rangle$, where β is a isomorphism from G to G^*. We can then identify G with its dual by means of the χ by setting $\langle x, y \rangle = \chi(x, y)$, which amounts to identify them by means of the isomorphism β; this is what we have done essentially for the local fields in n^{o}24 of [14]. and for adelic ring in n^{o}29 of [14]. When this is the case, we see immediately that there exists one and only one choice of Haar measure for G for which it is its own dual under the identification; we say that this measure is *autodual* with respect to χ.

Keeping our nations in (see [14] n^{o}29), we will agree once for all a choice of the completion of k_ν of the base field k of the autodual Haar measure with respect to the bicharacter $\chi_v(xy)$, and denote it by $|dx|_\nu$. Also, for almost all ν, the measure $\mathfrak{o}_\nu$ of the ring of integers of k_ν has the value 1.

Let U now be an analytic variety of dimension n on k_ν; the choice of measure $|dx|_\nu$ on k_ν allows, as we know (cf. p.ex. [13], Chap. II), to deduce from any holomorphic differential forms ω of degree n on U a positive measure on U, denote by $|\omega|_\nu$. Let's recall that, if U is an open subset of $(k_\nu)^n$ and that ω is given by $\omega = f(x) dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n$, $|\omega|_\nu$ will be defined by

$$|\omega|_\nu = |f(x)|_\nu \prod_{i=1}^{n} |dx_i|_\nu,$$

where $|a|_\nu$ naturally denoted, for all $a \in k_\nu$, the module of the automorphism $x \to ax$ of k_ν if $a \neq 0$, and 0 if $a = 0$. The definition of $|\omega|_\nu$ in the general case follows immediately from this, by transport of the structure over the local charts.

In particular, let X_k be a vector space of dimension n over k; let X_k be its dual, X° a base of X_k over k, and $(X^*)^\circ$ the base of X^*_k dual to X°. Denote X, as usual, the extension of X_k to the universal domain; let $x_1, \cdots, x_n$ be the coordinates in X^* relative to $(X^*)^\circ$. Let $dx = dx_1 \wedge \cdots \wedge dx_n$, $dx^* = dx_1^* \wedge \cdots \wedge dx_n^*$; then, if we consider the identification in nst24 and 29 of [14], the measure $|dx|_\nu$ in X_ν and $|dx^*|_\nu$ in X^*_ν that are dual to each other. This is always how we choose the measures in the spaces of the form X_ν, X^*_ν in which follows, it being understood that his choice is relate to the choice of the character χ of A_k, assumed once for all, and to that of the dual bases X°, $(X^*)^\circ$.

In the adele ring A_k of k, the "Tamagawa measure" (the one for which A_k/k is of measure 1) is autodual with respect to $\chi(xy)$, since k is associated to itself by the duality when we identify A_k to its dual by means of $\chi(xy)$. It follows that this measure is always of the form $\prod |dx|_\nu$ whenever $|dx|_\nu$ are chosen as has just been said; that is to say, on each open subgroups

$$\prod_{\nu \in S} k_\nu \times \prod_{\nu \notin S} \mathfrak{o}_\nu$$

of A_k, the measure in question is equal to the product of the measures $|dx|_\nu$ in the usual sense; we denote it by $|dx|_A$. We also conclude that, if X_k is as above, and if the measure $|dx|_\nu$ are defined as above on the spaces X_ν (by means of a base X° of Xk), the measure $|dx|_A = \prod |dx|_\nu$ on X_A (cf. [14], n^{o}29, (29)) is nothing but the Tamagawa measure on X_A, for which X_A/X_k is of measure 1; $|dx|_A$ is thus independent of the choice of X°, and even of also of χ.

4. More generally, let U be the algebraic variety without multiple points[1], of dimension n, rational over k, or even a finite disjoint union of such varieties. By a *gauge* on U, we mean a differential form ω of degree n on U, rational over k, everywhere finite and $\neq 0$; according to § 3, a gauge ω on U determines for all ν a measure $|\omega|_\nu$ on U_ν.

Each time that U is a affine variety and k_ν a discrete valuated fields, we denote by U_ν° the set of points on U with coordinates in $\mathfrak{o}_\nu$; this is a compact analytic variety on k_ν, possible empty. Similarly, each time U is represented by a finite union of biregular [2] $\varphi_i(U_i)$ of affine varieties U_i, we denote U_ν° the union of sets $\varphi_i((U_i)_\nu^\circ)$; of course, this depends on U_i and on φ_i, but we know that the other choice of U_i and of φ_i only modify U_ν° for a finite number of places ν of k (so that the choice of U_i and φ_i has no influence on the definition cf. [13, Chap. I]. The adelic variety U_A attach to U is then defined as the union, or more precisely the inductive limit, of the sets

$$U_S^\circ = \prod_{\nu\in S} U_\nu \times \prod_{\nu\notin S} U_\nu^\circ.$$

We know that (cf. [13, Chap. II]), for almost all ν, we can define a variety $\overline{U}_\nu$ of dimension n, without multiple point, on the finite field $\mathfrak{o}_\nu/\mathfrak{p}_\nu$, the *reduction of* U_ν° *modulo* $\mathfrak{p}_\nu$, such that we have

$$\int_{U_\nu^\circ} |\omega|_\nu = N(\mathfrak{p}_\nu)^{-n} \cdot N_\nu$$

if N_ν is the number of rational points of $\overline{U}_\nu$ over $\mathfrak{o}_\nu/\mathfrak{p}_\nu$. Moreover, if we denote by m_ν the common value of two members of the formula, and if v is the number of connected components of U, it results from an estimation of Langweil ([10, Theorem 1]) that $\lim_\nu m_\nu = v$ following a filtered completion of the finite parts of the set of places of k. We conclude in particular that $U_\nu^\circ \neq \emptyset$ for almost all ν. This is the case, if, for all places ν of k, we give ourselves a number $\lambda_\nu > 0$, we say that (λ_ν) is a *system of convergence factors* of U if the product $\prod(\lambda_\nu m_\nu)$ is absolute convergent [3]. When this is the case, we can consider, on each of the sets U_S°, the product measure of the measures $\lambda_\nu|\omega|_\nu$ and on U_A, the measure coincide with this on U_S° such that S is large enough; this measure is called the *Tamagawa measure* determined on U_A by the gauge ω and the system of factors $\lambda = (\lambda_\nu)$ and will be denoted by $|\lambda\omega|_A$; it does not change if we replace ω by $c\omega$ for any $c \in k, c \neq 0$.

Consider in particular a connected algebraic group G defined over k; on G, there always exist gauges, and every gauge is "relatively invariant" (that is, invariant up to a scalar factor) under all left or right translations. With an obvious symbol, we can write, if ω is an gauge of G:

$$\omega(axb) = \chi(a)\chi'(b)\omega(x)$$

[1] LUO: this is defined in Weil's algebraic geometry book, I think it means singular point

[2] LUO:LUO: according to [13], I think it means isomorphism

[3] We will often consider the restricted products $\prod a_\nu$, either the set of places of k, or over the complement of a finite part of this set; some of the a_ν may be zero. We say that $\prod a_\nu$ is absolute convergent if there exists a finite subset S of the set of places of k, such that a_ν is defined and $\neq 0$ for all $\nu \notin S$ and that the partial product $\prod a_\nu$ restrict to $\nu \in S$ is absolute convergent in the usual sense.

for any $a \in G, b \in G$l χ, χ' are characters of G, rational over k. Then the gauges $\chi^{-1}\omega$, $\chi'^{-1}\omega$ are left and right invariant resp., and these are, up to constant factor, the only ones having these properties. We have in particular

$$\omega(a^{-1}xa) = \Delta_G(a)\omega(x)$$

with $\Delta_G = \chi^{-1}\chi'$; the character Δ_G is called the *algebraic modulus* (or sometimes simply modulus) of G; it is independent of choice of ω. If $\Delta_G = 1$, G is is said to be *unimodular*; this is the case for any reductive group and for any unipotent group. If ω is a gauge of G, and if λ is a system of convergence factors for G, the measures $|\omega|_\nu$ and $|\lambda\omega|_A$ are relatively invariant, on the groups G_ν and G_A respectively; these are Haar measures if ω is left invariant. Consequently, for G_A to be unimodular (as a locally compact group), it is necessary and sufficient that G be so (as an algebraic group). We will denote by 1 the system of factors (λ_ν) for which $\lambda_\nu = 1$ for all ν; according to a general result due to Serre, this is a system of convergence factors for any semisimple group; for classical groups, which will be the only ones considered in the rest of this work, this result is demonstrated in [13].

If G is a non-connected algebraic group, with all connected components rational over k, we can no longer assert that every gauge is relatively invariant on G; but it remains true that every gauge relatively invariant on the left is also invariant on the right, and such gauges exist; if we restrict ourselves to considering these cases, the facts stated above remain valid. We will say that G is unimodular if this is the case for the connected component G_0 of the neutral element in G. If G is unimodular, Δ_G is a character of G/G_0, and its values are roots of unity.

5. Let X, Y be algebraic varieties and f a map from X to Y; let x a simple point of X where f is defined, and suppose that the point $y = f(x)$ is simple[4] on Y; let T_x, T_y be the vector spaces of X in x and Y in y respectively. We say then, as is known, that f is *submersive* (or "**of maximum rank**") at x if the linear map from T_x to T_y tangent to f at x is surjective.

Suppose X, Y and f are rational over k; suppose f is "**generically submersive**", that is to say, submersive at a generic point of X over k. Then the set X' of points of X where f is submersive is k-open in X, and f determines a submersive morphism f' from X' to Y. Moreover, the map f' is k-open, so that in particular the set-theoretic image $Y' = f'(X')$ of X' by f' is k-open in Y; and, for all $y \in Y'$, the set-theoretic inverse image $F_y = f'^{-1}(\{y\})$ of y in X' by f' is a variety without multiple point [5], or a union of such varieties, pairwise disjoint.

The hypotheses and notations being as above, let ω, η be gauges on X' and on Y'; denote by $f'_*\eta$ the inverse image of η by f' on X' (which we will sometimes also denote, with an obvious symbolism, $\eta(f'(x))$ if ω, η, are denoted $\omega(x), \eta(y)$). We verify immediately that there exists a covering $(X_\lambda)_{\lambda\in L}$ of X' by finite number of k-open subsets X_λ of X', and, in each X_λ, a differential form θ_λ everywhere finite, rational over k, such that ω coincide with $\theta_\lambda \wedge f'_*\eta$ in X_λ for all λ. Moreover, under these conditions, the forms θ_λ, θ_μ induce the same differential form on $F_y \cap X_\lambda \cap X_\mu$ for all $y \in Y'$, λ and μ; there is thus on each **fiber** F_y a differential form θ_y and coincides on $F_y \cap X_\lambda$ for all λ, with that induced by θ_λ on $F_u \cap X_\lambda$; finally, we verify immediately that θ_y is a gauge on F_y, rational over the field $k(y)$. We will write, symbolically, $\theta_y = (\omega/f'_*\eta)_y$, or again:

$$\theta_y(x) = \left(\frac{\omega(x)}{\eta(f'(x))}\right)_y.$$

[4] LUO: this is defined in Weil's algebraic geometry book

[5] LUO: this is defined in Weil's algebraic geometry book, I think it means reduced

In particular, let us take for X, Y and f an algebraic group G, the homogeneous space $H = G/g$ defined by G and a subgroup g of G, and the canonical map π of G onto H; this is everywhere submersive; the **fibers** $\pi^{-1}(\{y\})$ are the classes xg according to g in G; in particular, if $y_0 = \pi(e)$, we have $\pi^{-1}(\{y_0\}) = g$. Suppose there exsits on H a relatively invariant gauge η, that is such that $\eta(ay) = \psi(a)\eta(y)$ for any $a \in G$ (cf. §4), thus such that $\omega(axb) = \chi(a)\chi'(b)\omega(x)$. We can then, as above, define $\theta_y = (\omega/\pi^*\eta)_y$; in particular, we verity immediately that θ_{y_0} is a relative invariant gauge on g, and more precisely that we have

$$\theta_{y_0}(a\xi\beta) = \chi(\alpha)\psi(\alpha)^{-1}\chi'(\beta)\theta_{y_0}(\xi)$$

for any $\alpha \in g, \beta \in g$. It follows that, on g, ψ coincides with $\Delta_g\Delta_G^{-1}$. This result admits a reciprocal (cf. [13, Theorem 2.4.1]). We will note in particular that $\psi = 1$ implies that Δ_G coincides with Δ_g on g; this will be the case whenever g is an invariant subgroup of G, since then we can take for η a left-invariant gauge on the quotient group G/g.

6. The above results interests us in the case when k is a local field k_ν; then, as we saw in §4, the gauges ω, η determine measure $|\omega|_\nu$, $\eta|_\nu$ on X'_ν, Y'_ν, and similarly $|\theta_y|$ determines a measure $|\theta_y|_\nu$ on $(F_u)_\nu$ for every $y \in Y'_\nu$. The application of the Lebesgue-Fubini theorem to the neighborhood of each points of X'_ν shows that for any continuous function Φ with compact support on X'_ν:

$$\int_{X'_\nu} \Phi(x)|\omega(x)|_\nu = \int_{Y'_\nu} \left(\int_{(F_u)_\nu} \Phi(z)|\theta_y(z)|_\nu \right) |\eta(y)|_\nu, \tag{7}$$

the function to be integrated on Y'_ν in the second term of the equality being then continuous with compact support it is also immediate (as at the end of the proof of Proposition 1, §2) that the family of measures $|\theta_y|_\nu$, with respective supports $(F_y)_\nu$, is the only one possessing these properties. As usual in integration theory, formula (7) remains valid for any function Φ integrable on X'_ν, or even for any function $\Phi \geq 0$ locally integrable on X_ν, with respect to the measure $|\omega|_\nu$. If the variety X itself is without multiple point[6] , and ω is a gauge on X, we do not modify the integral of the first member of (7) by taking it on X_ν instead of X'_ν, since $X_\nu - X'_\nu$ is a union of subvarieties of X_ν of codimension ≥ 1, in finite number, and is therefore of zero measure; an analogous remark applies to Y_ν and Y'_ν.

Suppose in particular that X and Y are vector spaces; then X_ν, Y_ν are vector spaces over k_ν; suppose that the map f is generically submersive, defined everywhere on X, and that it satisfies condition (A) of Proposition 1 of §2; this proposition therefore allows us to define, for any $y \in Y_\nu$, a positive tempered measure μ_y, with support contained in $f^{-1}(\{y\})$, having the properties stated in that Proposition. In particular, we will have, for any continuous function Φ with compact support on X_ν:

$$\int_{X_\nu} \Phi(x)|dx|_\nu = \int_{Y_\nu} \left(\int \Phi d\mu_u \right) |dy|_\nu,$$

the function to be integrated over Y_ν in the right hand side is continuous. On the other hand, f being generically submersive, we can, using the same notations as above, define a gauge $\theta_y = (dx/f'_*dy)_y$ on F_u for any $y \in Y'$, then a measure $|\theta_y|$ on $(F_y)_\nu$ for any $y \in Y_{\nu'}$, and this satisfies (7). The comparision between these formulas then shows that μ_y coincides with $|\theta_y|_\nu$ on $(F_u)_\nu$, or, which amounts to the same thing, on $f^{-1}(\{y\})_\nu$, in the neighborhood of any point of this latter set where f is submersive.

[6]LUO: I think it means singular point

7. We will now introduce a type of integral that plays a grand role, not only in the rest of te work, but also in the other questions of number theory (for example, the **Selberg formula**). Let G be an algebraic group, X a variety on which G operates by a law denoted by $(g, x) \to gx$; let us suppose that G, X and this law defined over the field k. Let μ a positive measure on $G_{\mathbb{A}}/G_k$, or equivalently, a positive measure on $G_{\mathbb{A}}$, invariant under right multiplication by G_k. Suppose that X_k is discrete in $X_{\mathbb{A}}$, which will be the case in particular for any X is an affine variety. For any continuous function Φ with compact support on $X_{\mathbb{A}}$, the function

$$g \to \sum_{\xi \in X_k} \Phi(g\xi)$$

will be continuous on $G_{\mathbb{A}}$, and invariant under right multiplication by G_k. Let us define

$$I_\mu(\Phi) = \int_{G_{\mathbb{A}}/G_k} \sum_{\xi \in X_k} \Phi(g\xi) \cdot d\mu(g), \tag{8}$$

and suppose that this integral is absolute convergent for all continuous function Φ with compact support on $X_{\mathbb{A}}$; this will be the case, for example, for any μ is compact support on $G_{\mathbb{A}}/G_k$. Then I_μ is a positive measure on $X_{\mathbb{A}}$, whose support is contained in the closure of the union of orbits of points of X_k under $G_{\mathbb{A}}$.

We can further **decompose I_μ according to the orbits** of G_k in X_k. Let Ω be the set of these latter orbits; for each $\omega \in \Omega$, let us choose a representative ξ_ω of X_k for the equivalence relation determined by G_k in X_k. For each $\omega \in \Omega$, let $g(\omega)$ be the stabilizer of ξ_ω in G. Let $I_{\mu,\omega}(\Phi)$ be the integral analogous to (8), with the summation restricted to $\xi \in \omega$; we then have

$$I_\mu = \sum_{\omega \in \Omega} I_{\mu,\omega},$$

and we can also write

$$I_{\mu,\omega}(\Phi) = \int_{G_{\mathbb{A}}/g(\omega)_k} \Phi(g\xi_\omega) d\mu(g). \tag{9}$$

Let $H(\omega) = G/g(\omega)$, and, for ω given in Ω, let φ be the canonical morphism of G onto $H(\omega)$, as algebraic varieties (i.e., over the universal domain) and f a map from G in X given by $f(g) = g\xi_\omega$. As f is constant on the right cosets modulo $g(\omega)$, we can write $f = j \circ \varphi$ where j is injective morphism from $H(\omega)$ into X; more precisely, j determines a k-isomorphism of $H(\omega)$ onto the orbit of ξ_ω by G in X; moreover, if we denote by $X(\omega)$ the closure of this orbit in X (in the sense of Zariski topology), the orbit in question of a k-open subset of $X(\omega)$, and $X(\omega)$ is a k-closed subset of X, and even a subvariety of X if G is connected. From this, we deduce that the map $\varphi_{\mathbb{A}}$, $j_{\mathbb{A}}$ from $G_{\mathbb{A}}$ into $H(\omega)_{\mathbb{A}}$ and $H(\omega)_{\mathbb{A}}$ in $X_{\mathbb{A}}$, respectively; $X(\omega)_{\mathbb{A}}$ is identified with the closed subset of $X_{\mathbb{A}}$, and $j_{\mathbb{A}}$ maps $H(\omega)_{\mathbb{A}}$ into $X(\omega)_{\mathbb{A}}$. Furthermore, $\varphi_{\mathbb{A}}$ is constant on the cosets modulo $gomega)_{\mathbb{A}}$ in $G_{\mathbb{A}}$ and can thus be written as $\varphi_{\mathbb{A}} = \psi \circ \varphi_1$, where φ_1 is the canonical map from $G_{\mathbb{A}}$ onto $G_{\mathbb{A}}/g(\omega)_{\mathbb{A}}$ and ψ an injective map from $G_{\mathbb{A}}/g(\omega)_{\mathbb{A}}$ into $H(\omega)_{\mathbb{A}}$. Finally, we can write $\varphi_1 = \varphi' \circ \varphi''$, where φ'' is the canonical map from $G_{\mathbb{A}}$ onto $G_{\mathbb{A}}/g(\omega)_k$ and φ' a map from $G_{\mathbb{A}}/g(\omega)_k$ onto $G_{\mathbb{A}}/g(\omega)_{\mathbb{A}}$.

By hypothesism (8) is absolutely convergent for all function Φ with compact support; a fortiori, the same is true for (9), and consequently $I_{\mu,\omega}$ is a positive measure on $X_{\mathbb{A}}$, with support obviously contained in $X(\omega)_{\mathbb{A}}$. If we denote by μ' the measure on $G_{\mathbb{A}}/g(\omega)_k$ deduced from the measure μ on $G_{\mathbb{A}}$ by passing to the quotient, $I_{\mu,\omega}$ is the image of μ' by the map $j_{\mathbb{A}} \circ \psi \circ \varphi'$ of $G_{\mathbb{A}}/g(\omega)_k$ in $X_{\mathbb{A}}$ (Cf.

Bourbaki, *Integration*, Chap. V, §6). It follows, as is known, that μ' has an image μ_ω in $H(\omega)_{\mathbb{A}}$ by $\psi \circ \varphi'$ and that we have

$$I_{\mu,\omega}(\Phi) = \int_{H(\omega)_{\mathbb{A}}} \Phi(j_{\mathbb{A}}(h))d\mu_\omega(h). \tag{10}$$

8. To go further, let us assume the following condition (which generalizes Witt's theorem):

(W) For any orbit $\omega \in \Omega$ and field $K \supset k$, the map φ_K from G_K into $H(\omega)_K$ determined by the canonical map φ from g to $H(\omega)$ is surjective.

It is the same to say that, for each $L \supset K \supset k$ and $\xi \in X_k$, all points of X_K that is in the orbit of ξ under G_L is in the orbit of ξ under G_K. It follows from this, in particular, that $H(\omega) = H(\omega')$ implies $\omega = \omega'$.

We will also assume that G_k is dense in G in the sense of the Zariski topology; according to a result of Rosenlicht ([11], p.44), this is always the case if G is a connected linear algebraic group and if k is of characteristic 0. We know (cf. [13], pp. 3-4) that the map ψ, defined above, from $G_{\mathbb{A}}/g(\omega)_{\mathbb{A}}$ into $H(\omega)_{\mathbb{A}}$ is an isomorphism, so that we can identify these spaces by means of ψ.

Finally, let suppose that we have taken for μ a relatively invariant measure on $G_{\mathbb{A}}$, invariant on the right by G_k, and that convergence of (8) is assured under these conditions, for any continuous Φ with compact support; that of (9) and (10) will therefore also be. According to known results on homogeneous spaces (cf. [4], §2,n^{os}5-8), we can calculate the right hand side of (9) by integrating first over the classes followed $g(\omega)_{\mathbb{A}}/g(\omega)_k$, by means of a suitable relatively invariant measure ν, then over $G_{\mathbb{A}}/g(\omega)_{\mathbb{A}}$ by means of suitable measure. The function to be integrated being constant on the classes following $g(\omega)_{\mathbb{A}}$, it follows that $g(\omega)_{\mathbb{A}}/g(\omega)_k$ must be of finite measure for ν, therefore that ν is bi-invariant and $g(\omega)$ is unimodular (loc. cit. n^{o}6, corollaire 3). The same results then show that μ must be invariant on the right by $g(\omega)_{\mathbb{A}}$ and that the measure μ_ω defined by (10) is relatively invariant by $G_{\mathbb{A}}$; if ν is taken such that $g(\omega)_{\mathbb{A}}/g(\omega)_k$ is of measure 1, we have $\mu_\omega = \mu/\nu$; if μ is invariant on the left by $G_{\mathbb{A}}$, μ_ω will also be so.

9. We will now give convergence criterion for certain integrals of the form (8). Relying on reduction theory, *we will assume for the rest of this chapter that the field k is of characteristic* 0. We will first recall the main results of the theory in question (cf. [3], [2], [6]).

A **trivial torus** over k is an algebraic group T, direct product of isomorphic factors (over k) to the multiplicative group with one variable $\mathbb{G}_m = \mathrm{GL}_1$. Since $(\mathbb{G}_m)_{\mathbb{A}}$ is nothing but the group I_k of ideles of k, it follows that, if T is a trivial torus of dimension d, $T_{\mathbb{A}}$ is isomorphic to $(I_k)^d$. For $\tau \in \mathbb{R}_+^*$ (i.e., $\tau \in \mathbb{R}$, $\tau > 0$), denote a_τ the idele given by $a_\tau = (a_v)$ with $a_v = \tau$ for all infinite places v of k (i.e., for any k_v is isomorphic to $\mathbb{R}$ or $\mathbb{C}$) and $a_v = 1$ for all other places; let $\Theta(\mathbb{G}_m)$ denote the subgroup I_k formed by the a_τ for $\tau \in \mathbb{R}_+^*$; for $T = (\mathbb{G}_m)^d$, denote $\Theta(T)$ the subgroup of $\Theta(\mathbb{G}_m)^d$ of $T_{\mathbb{A}} = (I_k)^d$. Then, for any trivial torus T, we will denote by $\Theta(T)$ the image of $\Theta((\mathbb{G}_m)^d)$ for an isomorphism from $(\mathbb{G}_m)^d$ to T, d being the dimension of T; it is easy to see that the image is independent of the choice of the isomorphism in question. One could also give as follows an invariant definition of $\Theta(T)$, by means of the question $R_{k/\mathbb{Q}}$ (the **restriction of the base field**: cf. [13], Chap. I, n^{o}1.3); this operation applies to T gives a torus $T' = R_{k/\mathbb{Q}}(T)$ defined over $\mathbb{Q}$, non-trivial if $k \neq \mathbb{Q}$, and one can identify $T_{\mathbb{A}}$ (i.e., $T_{\mathbb{A}_k}$) with $T'_{\mathbb{A}_\mathbb{Q}}$. Then, if T'' is a maximal trivial torus over $\mathbb{Q}$ in T', $\Theta(T)$ is the connected component of 1 in T''_∞ considered as a subgroup of $T''_{\mathbb{A}_\mathbb{Q}}$, thus of $T'_{\mathbb{A}_\mathbb{Q}} = T_{\mathbb{A}}$.

10. Let G be a reductive algebraic group defined over k; if G is not connected, we mean by this that the connected component G_0 of the neutral element e in G is reductive. Let T be a maximal split torus (over k) in G. If ρ is a representation of G, that is to say, a homomorphism, rational over k, from G to the group $\mathrm{Aut}(X)$ of automorphism of a vector space X, we can choose for X a basis formed of **eigenvectors** of T; this means that, for each vector a of this basis, we will have, for $t \in T$, $\rho(t)a = \lambda(t)a$, where λ is a character of T (i.e., a homomorphism from T to $\mathbb{G}_m = \mathrm{GL}_1$, rational over k); the charaters λ that appear in this way are called the **weight** of ρ with respect to T. In particular, the weight of the adjoint representation of G in its Lie algebra are called the *roots*; among these, we can choose a certain number, $\alpha_1, \ldots, \alpha_r$ in such a way that any root α can be put in one and only one way in the form $\alpha = \prod \alpha_i^{n_i}$ with integer exponents n_i, and that, for all roots α, the n_i are all ≥ 0 or all ≤ 0; we will write $\alpha \succ 1$in the right case, $\alpha \prec 1$ in the second. The eigenvectors relative to the roots $\alpha \succ 1$ form the lie algebra of a connected subgroup P of G (**minimal parabolic subgroup** relative to k) which contains T; Δ_P being as usual the (algebraic) module of P, the restriction of Δ_P^{-1} to T is the product of all roots $\alpha \succ 1$ The homogeneous space G_0/P is isomorphic to a projective variety; as on the other hand we show that $(G_0)_{\mathbb{A}}/P_{\mathbb{A}}$ is isomorphic to $(G_0/P)_{\mathbb{A}}$, it follows that it is a compact space; since $G_{\mathbb{A}}/(G_0)_{\mathbb{A}}$ is compact (cf. [2],n^{o}1.9), the same is therefore true of $G_{\mathbb{A}}/P_{\mathbb{A}}$, so that there exists a compact subset C_1 of $G_{\mathbb{A}}$ such that $G_{\mathbb{A}} = C_1 \cdot P_{\mathbb{A}}$.

We will denote by $P_{\mathbb{A}}^{\circ}$ the subgroup pf $P_{\mathbb{A}}$, intersection of the kernel of homomorphism $p \to |\chi(p)|_{\mathbb{A}}$ of $P_{\mathbb{A}}$ in $\mathbb{R}_+^*$ when we take for χ all the characters of P (rational over k); of course, $|a|_{\mathbb{A}}$, for all $a \in I_k$, dentoes the module of the automorphism $x \to ax$ of $\mathbb{A}_k$. Then $P_{\mathbb{A}}^{\circ}/P_k$ is compact, so that there exists a compact subset C_2 of $P_{\mathbb{A}}^{\circ}$ such that $P_{\mathbb{A}}^{\circ} = C_2 \cdot O_k$. Moreover, the group of characters of P in induces on T a subgroup of finite index of the group of characters of T; we immediately conclude that the canonical homomorphism of $P_{\mathbb{A}}$ to $P_{\mathbb{A}}/P_{\mathbb{A}}^{\circ}$ induces on $\Theta(T)$ an isomorphism of $\Theta(T)$ to $P_{\mathbb{A}}/P_k^{\circ}$; $P_{\mathbb{A}}$ is therefore a semi-direct product of $\Theta(T)$ and of $P_{\mathbb{A}}^{\circ}$; we can write $P_{\mathbb{A}} = \Theta(T) \cdot P_{\mathbb{A}}^{\circ}$, and the map $(\theta, p_0) \to \theta p_0$ is a homomorphism of $\Theta(T) \times P_{\mathbb{A}}^{\circ}$ to $P_{\mathbb{A}}$.

As the module $|\Delta_P(p)|_{\mathbb{A}}$ of $P_{\mathbb{A}}$ (as a locally compact group) takes the value 1 on the invariant subgroup $P_{\mathbb{A}}^{\circ}$ of $P_{\mathbb{A}}$, the latter is unimodular. Let $d\theta$, dp_0 be Haar measures on $\Theta(T)$ and on $P_{\mathbb{A}}^{\circ}$, respectively; the image of the product measure $d\theta dp_0$ by the homoemorphism $\theta, p_0) \to \theta p_0$ from $\Theta(T) \times P_{\mathbb{A}}^{\circ}$ to $P_{\mathbb{A}}$ will then be a Haar measure on $P_{\mathbb{A}}$, which we will also denote by $d\theta dp_0$. Then the measure $d'p = |\Delta_P(\theta)|_{\mathbb{A}}^{-1} d\theta dp_0$ will be right invariant on $P_{\mathbb{A}}$.

Finally, we will denote by $T_{\mathbb{A}}^{+}$ the set of $t \in T_{\mathbb{A}}$ such that $|\alpha(t)|_{\mathbb{A}} \leq 1$ for all roots $\alpha \prec 1$, and we will set $\Theta^+ = \Theta(T) \cap T_{\mathbb{A}}^{+}$ and $P_{\mathbb{A}}^{+} = \Theta^+ \cdot P_{\mathbb{A}}^{\circ} = T_{\mathbb{A}}^{+} \cdot P_{\mathbb{A}}^{\circ}$. We know ([6], Lemma 1, p.17) that, if C is a compact subset of $P_{\mathbb{A}}$, the union of $\theta C \theta^{-1}$, for $\theta \in \Theta^+$, is relatively compact in $P_{\mathbb{A}}$; in particular, if C_2 is chosen as stated, that is to say compact and such that $P_{\mathbb{A}}^{\circ} = C_2 \cdot P_k$, we will denote by C_3 the closure of the union of $\theta C_2 \theta^{-1}$ for $\theta \in \Theta^+$; it is a compact subset of $P_{\mathbb{A}}^{\circ}$.

With these notations, the main result of the reduction theory is satsted as follows: *there exists a compact subset C OF $G_{\mathbb{A}}$ such that $G_{\mathbb{A}} = C \cdot P_{\mathbb{A}}^{+} \cdot G_k$.* If we set $N = C \cdot P_{\mathbb{A}}^{+}$, this implies that the image of N/P_k in the canonical map of $G_{\mathbb{A}}/P_k$ to $G_{\mathbb{A}}/G_k$ is $G_{\mathbb{A}}/G_k$.

11. Now let dg be a Haar measure on $G_{\mathbb{A}}$. To study the convergence of integrals on $G_{\mathbb{A}}/G_k$, we will rely on the following lemma:

Lemma 4. *Let G, Θ^+ be as stated in* n^{o}10. *Then there exists a compact subset C_0 of $G_{\mathbb{A}}$ and a* *constant* $\gamma > 0$ *such that*

$$\int_{(} G_{\mathbb{A}}/G_k)|F(g)|dg \leq \gamma \int_{\Theta^+} F_0(\theta) \cdot |\Delta_P(\theta)|_A^{-1} d\theta \tag{11}$$

whenever F, F_0 *are functions, locally integrable on* $G_{\mathbb{A}}/G_k$ *and on* Θ^+ *respectively, such that* $|F(c\theta)| \leq F_0(\theta)$ *for all* $c \in C_0$ *and* $\theta \in \Theta^+$.

Let I denote the left-hand side of (11) and φ_N the characteristic function of the set $N = C \cdot P_{\mathbb{A}}^+$ introduced at the end of n^{o}10. We will have

$$I \leq \int_{(} N/P_k)|F(g)|dg = \int_{(} G_{\mathbb{A}}/P_k)|F(g)|\varphi_N(g)dg. \tag{12}$$

We will transform this last integral using the theory of quasi-invariant measures in homogeneous spaces (see [4, §2,n^{os}5-8]. According to this, we can construct on $G_{\mathbb{A}}$ a continuous function h, everywhere > 0, such that $h(gp) = h(g)|\Delta_P(p)|_{\mathbb{A}}$ for all $g \in G_{\mathbb{A}}$, $p \in P_{\mathbb{A}}$, then on $G_{\mathbb{A}}/P_{\mathbb{A}}$ a positive measure λ such that, for any function $f \geq 0$ locally integrable on $G_{\mathbb{A}}/P_k$:

$$\int_{(} G_{\mathbb{A}}/P_k)f(g)dg = \int_{(} G_{\mathbb{A}}/P_{\mathbb{A}}) \left(h(g) \int_{(} P_{\mathbb{A}}/P_k)f(gp)d'p \right) d\lambda(\dot{g})$$

where $\dot{g}$ is the canonical image of $g \in G_{\mathbb{A}}$ in $G_{\mathbb{A}}/P_{\mathbb{A}}$, and where $d'p$ is the right-invariant measure on $P_{\mathbb{A}}$ introduced in n^{o}10; the condition imposed on h implies that the function to be integrated over $G_{\mathbb{A}}/P_B A$, which is written as a function of $g \in G_{\mathbb{A}}$, is constant on the classes modulo $P_{\mathbb{A}}$ and can therefore be considered as a function on $G_{\mathbb{A}}/P_{\mathbb{A}}$. Applying this formula to the last member of (12), we obtain

$$I \leq \int_{(} G_{\mathbb{A}}/P_{\mathbb{A}})\psi(\dot{g})d\lambda(\dot{g}),$$

where ψ is the function defined by the formula

$$\psi(\dot{g}) = h(g) \int_{(} P_{\mathbb{A}}/P_k)|F(gp)|\varphi_N(gp)d'p.$$

In the second member of this, we can take for g any representative of the class $\dot{g}$ modulo $P_{\mathbb{A}}$ in $G_{\mathbb{A}}$; since $G_{\mathbb{A}} = C_1 \cdot P_{\mathbb{A}}$, we can therefore assume that we have taken $g \in C_1$. But then we have $\varphi_N(gp) = 0$ when $p \notin C_1^{-1}N$. Let's define:

$$Q = C_1^{-1}N \cap P_{\mathbb{A}} = (C_1^{-1}C \cap P_{\mathbb{A}}) \cdot P_{\mathbb{A}}^+ = (C_1^{-1}C \cap P_{\mathbb{A}}) \cdot \Theta^+ \cdot P_{\mathbb{A}}^\circ;$$

let γ_1 also be the upper bound of h on C_1, and let $F_1(p)$, for all $p \in P_{\mathbb{A}}$, be the upper bound of $|F(gp)|$ for $g \in C_1$. We will therefore have:

$$\psi(\dot{g}) \leq \gamma_1 \int_{(} Q/P_k)|F(gp)|d'p \leq \gamma_1 \int_{(} Q/P_k)F_1(p)d'p,$$

and consequently, since $G_{\mathbb{A}}/P_{\mathbb{A}}$ is compact:

$$I \leq \gamma_2 \int_{(} Q/P_k)F_1(p)d'p$$

provided that the constant γ_2 is suitably chosen.

As in n^{o}10, let's identify $P_{\mathbb{A}}/P^\circ_{\mathbb{A}}$ with $\Theta(T)$; it is immediate moreover that any compact subset of $\Theta(T)$ is contained in a set of the form $\theta_0\Theta^+$, with $\theta_0 \in \Theta(T)$; applying this remark to the image of $C_1^{-1}C \cap P_{\mathbb{A}}$ in $P_{\mathbb{A}}/P^\circ_{\mathbb{A}} = \Theta(T)$, we conclude that there exists $\theta_0 \in \Theta(T)$ such that Q is contained in $\theta_0\Theta^+ \cdot P^\circ_{\mathbb{A}}$. Since on the other hand we have $P^\circ_{\mathbb{A}} = C_2 \cdot P_k$, we obtain $Q \subset \theta_0\Theta^+ \cdot C_2 \cdot P_k$, and consequently

$$I \leq \gamma_2 \int_(\theta_0\Theta^+ \cdot C_2) F_1(p) d'p.$$

As $d'p$ was defined in n^{o}10 by means of $d'p = |\Delta_P(\theta)|^{-1}_{\mathbb{A}} d\theta dp_0$, this can also be written as

$$I \leq \gamma_2 \int_{C_2} \left(\int_{\Theta^+} F_1(\theta_0\theta o_0) \cdot |\Delta_P(\theta_0\theta)|^{-1}_{\mathbb{A}} d\theta \right) dp_0.$$

But, as we saw in n^{o}10, there is a compact subset C_3 of $P^\circ_{\mathbb{A}}$ such that $\theta C_2 \theta^{-1} \subset C_3$, therefore also $\theta_0\theta C_2 = \theta_0 C_3 \theta$, whatever $\theta \in \Theta^+$. So if we denote by $F_2(\theta)$, for all $\theta \in \Theta^+$, the upper bound of $F_1(p\theta)$ for $p \in \theta_0 C_3$, we obtain

$$I \leq \gamma \int_(\theta^+) F_2(\theta) \cdot |\Delta_P(\theta)|^{-1}_{\mathbb{A}} d\theta$$

provided that the constant γ is suitably chosen. It follows that the assertion of the lemma is verified if we take $C = C_1\theta_0 C_3$.

In the statement of Lemma 4, we could, without changing anything in substance, substitute $T^+_{\mathbb{A}}/T_k$ for Θ^+; we would thus obtain an equivalent statement, less convenient to use directly, but which would have the advantage of not giving a privileged role to the places at infinity (and probably, consequently, of remaining valid in all characteristics).

12. We will now apply lemma 4 to integrals of the form (8) considered in n^{os}7-8, limiting ourselves to the case where μ is a Haar measure on $G_{\mathbb{A}}$ and X is an affine space on which G operates by means of a representation ρ of G in $\mathrm{Aut}(X)$. Keeping the same notations as in n^{os}10-11, we will denote by Λ the group of characters of T, and, for any $\lambda \in \Lambda$, we will denote by m_λ the dimension over k of the space of vectors $a \in X_k$ such that $\rho(t)a = \lambda(t)a$ for all $t \in T$. The characters λ of T for which $m_\lambda > 0$ are the weights of the representation ρ; m_λ is the multiplicity of the weight λ.

Lemma 5. *Let G, T, Θ^+ be as stated in* n^o*10; let ρ be a representation of G in the group* $\mathrm{Aut}(X)$ *of automorphisms of an affine space X; for each character λ of T, let m_λ be its multiplicity as a weight of ρ. Then the integral*

$$I(\Phi) = \int_{G_{\mathbb{A}}/\Theta_k} \sum_{\xi \in X_k} \Phi(\rho(g)\xi) \cdot dg \tag{13}$$

is absolutely convergent for any function $\Phi \in \mathcal{S}(X_{\mathbb{A}})$ whenever the integral

$$\int_{\Theta^+} \prod_{\lambda} \sup(1, |\lambda(\theta)|^{m_\lambda}_{\mathbb{A}}) \cdot |\Delta_\rho(\theta)|^{-1}_{\mathbb{A}} d\theta$$

is convergent; and, when this is the case, (13) *defines a tempered positive measure I.*

If $I(\Phi)$ is absolutely convergent for all functions $\Phi \in \mathcal{S}(X_{\mathbb{A}})$, Lemma 5 of [14], n^{o}41, shows that it is uniformly convergent on any compact subset of $\mathcal{S}(X_{\mathbb{A}})$, from which it follows, according to Lemma 2 of n^{o}2, that I is a tempered distribution, hence a tempered positive measure. Let C_0 now be a compact subset of $G_{\mathbb{A}}$ having the property stated in Lemma 4 of n^{o}11 above. For Φ given in $\mathcal{S}(X_{\mathbb{A}})$, there exists, according to lemma 5 of [14], n^{o}41, a function $\Phi_1 \in \mathcal{S}(X_{\mathbb{A}})$ such that we have

$$|\Phi(\rho(c)x)| \leq \Phi_1(x)$$

for any $c \in C_0$ and $x \in X_{\mathbb{A}}$. The application of lemma 4 to (13) then shows that $I(\Phi)$ is absolutely convergent provided that the integral

$$I_1 = \int_{\Theta^+} \sum_{\xi \in X_k} \Phi_1(\rho(\theta)\xi) \cdot |\Delta_\rho(\theta)|_A^{-1} d\theta$$

is also convergent.

Let X_∞ denote the product $\prod X_v$ extended to the places at infinity of k (or, equivalently, let $X_\infty = X_k \otimes \mathbb{R}$, the tensor product being taken over $\mathbb{Q}$; cf. [14, n^{o}39]); we can write $X_{\mathbb{A}} = X_\infty \times X'$, where X' is defined like $X_{\mathbb{A}}$ but using the places v of k for which k_v has discrete valuation. Taking into account the definition of $\mathcal{S}(X_{\mathbb{A}})$ (cf. [14, n^{os}29 and 39][14]), we can assume that Φ_1 has been chosen of the form

$$\Phi_1(x) = \Phi_\infty(x_\infty)\Phi'(x'),$$

where x_∞, x' are the projections of $x \in X_{\mathbb{A}}$ on X_∞ and X', with $\Phi_\infty \in \mathcal{S}(X_\infty)$, Φ' being the characteristic function of an open compact subgroup of X'. The set of $\xi \in X_k$ whose projection on X' belongs to the support of Φ' is then a lattice $L \subset X_k$, that is, a finitely generated abelian group such that $\mathbb{Q}L = X_k$, and I_1 can be written:

$$I_1 = \int_{\Theta^+} \sum_{\xi \in L} \Phi_\infty(\rho(\theta)\xi) \cdot |\Delta_\rho(\theta)|_A^{-1} d\theta.$$

For any weight λ of ρ, let X_λ be the subspace of X_k, of dimension m_λ over k, formed by the eigenvectors of weight λ, i.e., the vectors a such that $\rho(t)a = \lambda(t)a$ for $t \in T$; X_k is the direct sum of the X_λ. Let $d = [k : \mathbb{Q}]$; let

$$(a_{\lambda_i})_{1 \leq i \leq m_{\lambda_d}}$$

be a basis of X_λ over $\mathbb{Q}$; by replacing if necessary the a_{λ_i} by $N^{-1}a_{\lambda_i}$, where N is a suitable integer, we can assume that the lattice L is contained in the subgroup of X_k generated by the set of a_{λ_i}. The a_{λ_i} also form a basis of X_∞ over $\mathbb{R}$; for $x_\infty \in X_\infty$, we can thus write

$$x_\infty = \sum_{\lambda,i} x_{\lambda_i} a_{\lambda_i}$$

with $x_{\lambda_i} \in \mathbb{R}$; then, if $\alpha > 1$, there will be a constant C such that

$$\Phi_\infty(x_\infty) \leq C \prod_{\lambda,i} (1 + |x_{\lambda_i}|^\alpha)^{-1}.$$

On the other hand, under these conditions

$$\rho(\theta)x_\infty = \sum_{\lambda,i} \lambda(\theta) x_{\lambda_i} a_{\lambda_i},$$

and, if x_∞ is the projection on X_∞ of an element ξ of the lattice L, all the x_{λ_i} are integers by virtue of the choice of bases (a_{λ_i}). We therefore have

$$\sum_{\xi\in L}\Phi_\infty(\rho(\theta)\xi) \leq C\prod_\lambda\left(\sum_{n=-\infty}^{+\infty}\frac{1}{1+\lambda(\theta)^\alpha|n|^\alpha}\right)^{m_\lambda d}$$
$$\leq C'\prod_\lambda \sup(1,\lambda(\theta)^{-1})^{m_\lambda d}$$

where C' is a suitable constant. If we observe that we have $|\lambda(\theta)|_{\mathbb{A}} = \lambda(\theta)^d$ for any character λ of T and for any $\theta \in \Theta$, we see that this indeed gives the announced conclusion.

If we apply Lemma 5 to the case where $X = 0$, and where consequently ρ is the trivial representation of G, we recover the result of Borel and Harish-Chandra, according to which $G_{\mathbb{A}}/G_k$ is of finite measure when the center of G contains no trivial torus other than 1. It would naturally be of interest to examine whether the condition for convergence of (13), given by Lemma 5, is necessary as well as sufficient.

13. Finally, in applying the above results, we will need to use the following lemma, where we have again denoted by a_τ, for $\tau \in \mathbb{R}_+^*$, the idele (a_v) given by $a_v = \tau$ for every place at infinity v of k, and $a_v = 1$ for every other place v:

Lemma 6. *Let $(X_k^\alpha)_{1\leq\alpha\leq n}$ and Y_k be vector spaces over k; let $X_k = \prod_\alpha X_k^{(\alpha)}$, and let p be a morphism from X to Y, rational over k and such that $p(0,x^{(2)},...,x^{(n)}) = 0$ for any $x^{(2)},...,x^{(n)}$. Let C_0 be a compact subset of $\mathcal{S}(X_{\mathbb{A}})$, and let $N \geq 0$. Then there exists a function $\Phi_0 \in \mathcal{S}(X_{\mathbb{A}})$ such that we have*

$$|\tau_1^N\Phi(a_{\tau_1}x^{(1)},...,a_{\tau_n}x^{(n)})| \leq \Phi_0(x)$$

whenever $\Phi \in C_0$, $\tau_1 \geq 1$,...,$\tau_n \geq 1$, $x = (x^{(1)},...,x^{(n)}) \in X_{\mathbb{A}}$, $p(x) \in Y_k$ and $p(x) \neq 0$.

Let us recall that a "morphism" from one affine space to another is nothing other than a polynomial application; the hypothesis made on p therefore amounts to saying that, if we choose bases for X_k and Y_k over k, the coordinates of $p(x)$ are expressed as polynomials with coefficients in k in terms of those of x. We will denote by D the largest of the degrees of these polynomials. On the other hand, let us write $X_{\mathbb{A}} = X_\infty \times X'$, where X_∞ and X' are defined as in n^{o}12 in the proof of Lemma 5, and similarly $X_{\mathbb{A}}^{(\alpha)} = X_\infty^{(\alpha)} \times X'^{(\alpha)}$ and $Y_{\mathbb{A}} = Y_\infty \times Y'$; p evidently determines applications from X_∞ to Y_∞ and from X' to Y'. Let us choose bases for $X_\infty^{(\alpha)}$ and Y_∞ over $\mathbb{R}$, and, for $x_\infty^{(\alpha)} \in X_\infty^{(\alpha)}$ (resp. $y_\infty \in Y_\infty$), let us denote by $r_\alpha(x_\infty^{(\alpha)})$ (resp. $s(y_\infty)$) the sum of the squares of the coordinates of $x_\infty^{(\alpha)}$ (resp. of y_∞) with respect to these bases. Let us also set, for $x_\infty = (x_\infty^{(1)},\cdots,x_\infty^{(n)}) \in X_\infty$:

$$r'(x_\infty) = \sum_{\alpha\geq 2} r_\alpha(x_\infty^{(\alpha)}), \quad r(x_\infty) = r_1(x_\infty^{(1)}) + r'(x_\infty).$$

Since $p(x)$ vanishes whenever $x^{(1)} = 0$, there is a constant $C > 0$ such that, for any $x_\infty \in X_\infty$:

$$s(p(x_\infty)) \leq C \cdot r_1(x_\infty^{(1)}) \cdot r(x_\infty)^{(D-1)},$$

and consequently, for $\tau_1 \geq 1$:

$$s(p(x_\infty)) \leq C\tau_1^{-2}(\tau_1^2 r_1(x_\infty^{(1)}) + r'(x_\infty))^D.$$

Let us agree to set, for $\tau = (\tau_1, ..., \tau_n)$, $\tau_1 > 0$, ..., $\tau_n > 0$, and $x_\tau \in X_\infty$:

$$\tau x_\infty = (\tau_1 x_\infty^{(1)}, ..., \tau_n x_\infty^{(n)});$$

the inequality we have just obtained shows that we will have, with even stronger reason, whenever $\tau_1 \geq 1, ..., \tau_n \geq 1$:

$$s(p(\tau x_\infty)) \leq C\tau_1^{-2} r(\tau x_\infty)^D.$$

Let us now apply, as in n^{o}12, lemma 5 of [14, n^{o}41]; it shows that we can choose $\Phi_1 \in \mathcal{S}(X_\mathbb{A})$ such that $|\Phi(x)| \leq \Phi_1(x)$ for any $\Phi \in C_0$ and $x = (x_\infty, x') \in X_\mathbb{A}$, and even that we can assume Φ_1 of the form

$$\Phi_1(x) = \Phi_\infty(x_\infty)\Phi'(x'),$$

where $\Phi_\infty \in \mathcal{S}(X_\infty)$ and where Φ' is the characteristic function of an open compact subgroup of X'. Let E be the set of points $x = (x_\infty, x')$ of $X_\mathbb{A}$ such that $p(x) \in Y_k, p(x) \neq 0$ and $\Phi'(x') \neq 0$; we will show that, on E, $s(p(x_\infty))$ has a lower bound $\varepsilon > 0$. Indeed, if this were not the case, there would be a sequence of points $x_v = (x_{v\infty}, x_{v'})$ of E such that the sequence $p(x_{v\infty})$ tends towards 0 in Y_∞. As the support of Φ' is compact, we can assume at the same time that the sequence x'_v tends towards a limit $\overline{x}'$, thus that $p(x'_v)$ tends towards $p(\overline{x}')$. But then the sequence of points $y_v = p(x_v)$ tends towards a limit $\overline{y}$ in $Y_\mathbb{A}$, for which we have $\overline{y}_\infty = 0$, that is to say $\overline{y}_v = 0$ for every place at infinity v of k. As the y_v belong to $Y_k - 0$, which is discrete in $Y_\mathbb{A}$, we have $\overline{y} \in Y_k$, $\overline{y} \neq 0$, thus $\overline{y}_v \neq 0$ whatever v, hence a contradiction. Taking into account the inequality demonstrated above, we therefore have, for $x \in E$, $\tau_1 \geq 1, ..., \tau_n \geq 1$:

$$\tau_1 \leq C' r(\tau x_\infty)^(D/2)$$

with $C' = (C/\varepsilon)^{1/2}$.

Let us now set, for all $i \geq 0$:

$$a_i = \sup_{x_\infty \in X_\infty} (r(x_\infty)^i \Phi_\infty(x_\infty)).$$

Let M be an integer $\geq ND/2$. According to lemma 4 of [14, n^{o}41], there is a $\varphi \in \mathcal{S}(\mathbb{R})$ such that, for all $r \in \mathbb{R}$:

$$\varphi(r) \geq \inf_{i \geq 0}(a_{M+i}|r|^{-i}).$$

For x and τ as above, we will have, for any $i \geq 0$:

$$\begin{aligned}\tau_1^{2M/D}\Phi_\infty(\tau x_\infty) &\leq C'^{2M/D} r(\tau x_\infty)^M \Phi_\infty(\tau x_\infty) \\ &\leq C'^{2M/D} a_{M+1} r(\tau x_\infty)^{-i} \leq C'^{2M/D} a_{M+i} r(x_\infty)^{-i},\end{aligned}$$

and consequently, with even stronger reason

$$\tau_1^N \Phi_\infty(\tau x_\infty) \leq C'^{2M/D}\varphi(r(x_\infty)).$$

We will therefore satisfy the conditions of the lemma by setting

$$\Phi_0(x) = C'^{2M/D}\varphi(r(x_\infty))\Phi'(x').$$

II Algebraic Preliminary

In this Chapter, we will assume once and for all that the base field k is not of characteristic 2. Whenever there is a need to make additional assumptions about k, this will be explicitly mentioned.

14. Taking up the hypotheses and notations of [14, n^{o}49], we consider, on the base field k, an algebra $\mathcal{A}$ equipped with an involution ι and a trace function τ, as well as left modules over $\mathcal{A}$. Eventually, we may need to "extend" these data to a field K containing k; this means we take the tensor product $\mathcal{A}_K$ of $\mathcal{A}$ and K over k, endowing it with the obvious algebra structure over K and the involution and trace function that are "naturally" derived from ι and τ; by abuse of notation, we will still denote these as ι and τ (instead of ι_K, τ_K). Similarly, the extension by K of a left $\mathcal{A}$-module X will give a left $\mathcal{A}_K$-module X_K, etc.

Due to the fact that k is not of characteristic 2, certain results from [14, Chap. V], can be clarified or stated more simply. For example, it has already been observed in n^o. 31 of [14] that $Ps(X)$ is then isomorphic to the symplectic group. More generally, let's denote by $\mathrm{Sp}(X/\mathcal{A})$ the group of automorphisms σ of the left $\mathcal{A}$-module $X \oplus X*$ that satisfy $\sigma\sigma^I = 1$, σ^I being defined as usual by

$$\sigma = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \quad \sigma^I = \begin{pmatrix} \delta^* & -\beta^* \\ -\gamma^* & \alpha^* \end{pmatrix}$$

Then, if $\sigma \in \mathrm{Sp}(X/\mathcal{A})$, there is an element f and only one from $Q(X/\mathcal{A})$ such that (σ, f) belongs to $Ps(X/\mathcal{A})$; an easy calculation shows that f is given by the formulas

$$\begin{aligned} f(x, x^*) &= \frac{1}{2}\tau(F(x, x^*; x, x^*)), \\ F(x, x^*; y, y^*) &= \{x\alpha + x^*\gamma, y\beta\} + \{y^*\gamma, x\beta + x^*\delta\}^\iota; \end{aligned}$$

due to the fact that $\sigma\sigma^I = 1$, F is here a Hermitian form on $X \oplus X^*$. It follows that $Ps(X/\mathcal{A})$ is isomorphic to $\mathrm{Sp}(X/\mathcal{A})$. Moreover, according to n^{o}31 of [14], the symplecticity condition of σ is equivalent to saying that σ leaves invariant the alternating k-bilinear form

$$[x_1, x_2^*] - [x_2, x_1^*] = \tau(\{x_1, x_2^*\} - \{x_2, x_1^*\}^\iota).$$

By replacing (x_1, x_1*) with (tx_1, tx_1^*) where $t \in A$, we see that this amounts to saying that σ leaves invariant the form $\{x_1, x_2^*\} - \{x_2, x_1^*\}^\iota$, which is sesquilinear and anti-Hermitian on $(X \oplus X^*) \times (X \oplus X^*)$.

15. As is well known, we can form the tensor product over $\mathcal{A}$ of a right $\mathcal{A}$-module X and a left $\mathcal{A}$-module Y; it is the vector space over k, quotient of the tensor product $X \otimes_k Y$ of X and Y (or rather of the vector spaces over k, respectively underlying X and Y), taken over k, by the subspace of this product generated by elements $xt \otimes y - x \otimes ty$ for $x \in X, y \in Y, t \in \mathcal{A}$. But, by means of the involution ι, any left $\mathcal{A}$-module X can also be considered as a right $\mathcal{A}$-module, by the formula $xt = t^\iota x$ $(x \in X, t \in \mathcal{A})$. This leads to defining a tensor product, which we will denote by $\otimes_\iota$, between left $\mathcal{A}$-modules X, Y; by definition, $X \otimes_\iota Y$ will be the vector space over k, quotient of $X \otimes_k Y$ by the subspace generated by elements $t^\iota x \otimes y - x \otimes ty$ $(x \in X, y \in Y, t \in \mathcal{A})$; the image of $x \otimes y$ in this quotient will be denoted $x \otimes_\iota y$.

As the dual of $X \otimes_k Y$ is nothing other than the space of k-bilinear forms on $X \times Y$, the dual of $X \otimes_\iota Y$ is identified with the space of k-bilinear forms f on $X \times Y$ that satisfy $f(t^\iota x, y) = f(x, ty)$

for all $x \in X, y \in Y, t \in \mathcal{A}$. But, if $x \in X, y \in Y$, the mapping $t \to f(tx, y)$ is a k-linear form on $\mathcal{A}$, which can thus, in one and only one way, be written in the form $t \to \tau(tF(x, y))$, where F is a mapping from $X \times Y$ to $\mathcal{A}$; we then immediately verify that, under these conditions, F is sesquilinear. Conversely, these formulas make any sesquilinear form on $X \times Y$ correspond to an element of the dual of $X \otimes_\iota Y$. It amounts to the same to say that this dual is identified with the space of sesquilinear forms F on $X \times Y$ by means of the formula

$$[x \otimes_\iota y, F] = \tau(F(x, y)). \tag{14}$$

We will have to make almost exclusive use of these notions in the case where $X = Y$. In this case, let s be the automorphism of order 2 of $X \otimes_\iota X$ defined by $s(x \otimes_\iota y) = y \otimes_\iota x$. As the characteristic of k is not 2, $X \otimes_\iota X$ is the direct sum of the space $I(X)$ of elements invariant under s and the space $I^-(X)$ of elements u such that $su = -u$; $I(X)$ and $I^-(X)$ are the subspaces of $X \otimes_\iota X$ respectively generated by elements of the form $x \otimes_\iota x$ (for $x \in X$) and by elements of the form $x \otimes_\iota y - y \otimes_\iota x$ (for $x \in X, y \in X$). Moreover, we can, in an obvious way, identify $I(X)$ with $(X \otimes_\iota X)/I^-(X)$.

Once and for all, we will agree to denote by i_X the mapping $x \to x \otimes_\iota x$ from X to $I(X)$, that is to say, we will set $i_X(x) = x \otimes_\iota x$. This mapping is evidently quadratic, meaning that, for any choice of bases of X and $I(X)$ over k, the coordinates of $i_X(x)$ are expressed as homogeneous polynomials of the second degree in terms of those of x.

Let F then be a sesquilinear form on $X \times X$, which we identify with an element of the dual of $X \otimes_\iota X$ by means of (14). If F is Hermitian, we will have, for all x, y:

$$[x \otimes_\iota y, F] = [y \otimes_\iota x, F],$$

thus $[u, F] = 0$ for $u \in I^-(X)$; in other words, F belongs to the dual of $(X \otimes_\iota X)/I^-(X)$, that is to say, according to the conventions above, to the dual $I(X)*$ of $I(X)$, considered as a subspace of the dual of $X \otimes_\iota X$. Conversely, if this is the case, we will have, for all $x \in X, y \in X, t \in A$:

$$\tau(F(tx, y)) = \tau(F(y, tx)),$$

which can also be written, since F is sesquilinear and τ is invariant under ι:

$$\tau(tF(x, y)) = \tau(F(y, x)t^\iota) = \tau(t \cdot F(y, x)^\iota);$$

this relation holding for any t, it follows that F is Hermitian. For $x = y$, the formulas above give

$$[i_X(x), F] = \tau(F(x, x)),$$

so that we can identify $I(X)^*$ with the space of Hermitian forms on $X \times X$ by means of this formula. We have seen elsewhere in n^{o}49 of [14] that this latter space is identified with the space $Q(X/\mathcal{A})$ of $\mathcal{A}$-quadratic forms f on X by means of $f(x) = \tau(F(x, x))$; it follows that the formula $f(x) = [i_X(x), f]$ allows us to identify $Q(X/\mathcal{A})$ with $I(X)^*$.

Let Z be a submodule of X, and suppose that Z admits a complement Z' in X (which will always be the case if A is semisimple); we can therefore write $X = Z \oplus Z'$. It is immediate that then $I(X)$ is canonically identified with the direct sum of $I(Z)$, $Z \otimes_\iota Z'$ and $I(Z')$, and that the subspace of $I(X)$ generated by the elements $i_X(z)$ for $z \in Z$ is $I(Z)$. In other words, we have the right, under these conditions, to identify $I(Z)$ with the subspace of $I(X)$ generated by the image of Z by i_X, and to identify i_Z with i_X.

We will also note that, if K is a field containing k, $I(X_K)$ is identified with $I(X)_K$, and consequently $Q(X_K/\mathcal{A}_K)$ with $Q(X/\mathcal{A})_K$. We will always denote by i_X the application from X_K to $I(X)_K$ derived in the obvious manner from i_X.

The following property of i_X will play an important role later on:

Lemma 7. *For the map i_X from X to $I(X)$ to be submersive at a point x of X, it is necessary and sufficient that $I(X/\mathcal{A}x) = \{0\}$, or in other words, that the space $Q((X/\mathcal{A}x)/\mathcal{A})$ be reduced to $\{0\}$.*

The differential of i_X at x is indeed given by

$$di_X(x) = x \otimes_\iota dx + dx \otimes_\iota x,$$

so that the linear application from X to $I(X)$, tangent to i_X at x, is $y \to x \otimes_\iota y + y \otimes_\iota x$. For this not to be surjective, it is necessary and sufficient that there exists $f \neq 0$ in $I(X)^*$ such that we have

$$[x \otimes_\iota y + y \otimes_\iota x, f] = 0$$

for all y. Identifying $I(X)^*$ with the space of Hermitian forms on $X \times X$ as stated above, we see that this condition is equivalent to the following: there exists on $X \times X$ a Hermitian form $F \neq 0$ such that $\tau(F(x, y)) = 0$ for all $y \in X$. Replacing y by ty, thus $F(x, y)$ by $F(x, y)t^\tau$, in this condition, we see that it amounts to saying that $F(x, y) = 0$ for all y; then replacing x by tx, we see that this amounts to saying that $F(z, y) = 0$ for all $z \in \mathcal{A}x$ and $y \in X$. Let $Z = \mathcal{A}x$; Z is the submodule of X generated by x. By passing to the quotient, the space of Hermitian forms on $X \times X$ having the property we just stated is identified with the space of Hermitian forms on $(X/Z) \times (X/Z)$, that is to say, with the dual of $I(X/Z)$, or equivalently with $Q((X/Z)/\mathcal{A})$, which completes the demonstration.

16. *From now on, we will assume once and for all that the algebra $\mathcal{A}$ is semisimple.* We will first state, in the language that will be convenient for us, some of the classical structure theorems concerning these algebras, and establish some notations.

Such an algebra $\mathcal{A}$, as we know, is a direct sum of a finite family of simple algebras $\mathcal{A}_\nu (\nu \in \mathbb{N})$; for each ν, we will choose once and for all a simple left $\mathcal{A}_\nu$-module A_ν, which we can also obviously consider as an $\mathcal{A}$-module (the annihilator of $\mathcal{A}_\nu$ in $\mathcal{A}$ being the sum of $\mathcal{A}_\mu$ for $\mu \neq \nu$). We will denote by A the direct sum of the A_ν, considered as a left $\mathcal{A}$-module. The ring $\mathrm{End}(\mathcal{A}_\nu)$ of endomorphisms of $\mathcal{A}_\nu$, considered as a left module, either over $\mathcal{A}_\nu$ or over $\mathcal{A}$, is a division algebra over k (thus a field, commutative or not) which we will denote by $\mathfrak{k}_\nu$; $\mathcal{A}_\nu$ is then a right vector space over $\mathfrak{k}_\nu$, whose dimension will be denoted by m_ν, and $\mathcal{A}_\nu$ is nothing other than the ring of endomorphisms of this vector space. Consequently, $\mathcal{A}_\nu$ is isomorphic to the ring $M_{m_\nu}(\mathfrak{k}_\nu)$ of matrices of order m_ν over $\mathfrak{k}_\nu$. We will denote by B the direct sum of the $\mathfrak{k}_\nu$; then the A_ν, and consequently A, can, in an obvious way, be considered as right $\mathcal{B}$-modules; $\mathcal{B}$, operating thus on A, is nothing other than the ring $\mathrm{End}_{\mathcal{A}}(A)$ of endomorphisms of A considered as a left $\mathcal{A}$-module. Reciprocally, $\mathcal{A}$, operating on the left on A, is the ring $\mathrm{End}_{\mathcal{B}}(A)$ of endomorphisms of A considered as a right $\mathcal{B}$-module. As the operations of $\mathcal{A}$ and $\mathcal{B}$ on A are permutable with each other, we can consider A as a bimodule over $\mathcal{A}$ and over $\mathcal{B}$.

Let X be a left $\mathcal{A}$-module; it is a direct sum of simple modules, each of which is isomorphic to one of the A_ν; we will denote by X_ν the sum of those simple components of X which are isomorphic to A_ν, and by n_ν the number of these components. The family of integers $n = (n_\nu)_{\nu\in\mathbb{N}}$ will be called the *rank* of X; X is uniquely determined by its rank, up to an isomorphism. If $x \in X$, the rank of the submodule $\mathcal{A}x$ of X generated by x will be called the *rank* of x; if this is equal to the rank of X, that is to say if x generates X, we will agree to say that x is of *maximal rank* in X; for such elements to exist in X, it is necessary and sufficient that X be **monogenic**.

17. With the same notations as above, let $B = \mathrm{Hom}_{\mathcal{A}}(X, A)$; this is the set of $\mathcal{A}$-linear applications from X to A, these being considered as left $\mathcal{A}$-modules; we will consider B as operating on the right on X, and we will give it its "natural" structure of right $\mathcal{B}$-module. Similarly, let B_ν be the set of A_ν-linear maps from X_ν to A_ν; B_ν is **naturally** a right vector space over $\mathfrak{k}_\nu$, and, as such, it has dimension n_ν; B is then the direct sum of the B_ν. The map of X to A defined by an element b of B being denoted $x \to xb$, it follows that, for all $x \in X$, $b \to xb$ is a $\mathcal{B}$-linear map from B to A, these being considered as right $\mathcal{B}$-modules; and every $\mathcal{B}$-linear map from B to A is of this form. In other words, we can identify X with the space $\mathrm{Hom}_{\mathcal{B}}(B, A)$ of these maps. This identification allows us to state the following lemma:

Lemma 8. *Let $\mathcal{A}$, A, X, B be as above; let x_0, x_1 be two elements of X, and let N_0, N_1 be their kernels when considered as maps from B to A. Then, for $\mathcal{A}x_0 supset \mathcal{A}x_1$, it is necessary and sufficient that $N_0 \subset N_1$; for $\mathcal{A}x_0 = X$, it is necessary and sufficient that $N_0 = \{0\}$.*

It is clear that $\mathcal{A}x_0 \supset \mathcal{A}x_1$ is equivalent to $x_1 \in \mathcal{A}x_0$ and implies $N_0 \subset N_1$. To prove the converse, we can limit ourselves to the case where $\mathcal{A}$ is a simple algebra, the general case resulting trivially from this case. We can therefore assume that A and B are right vector spaces over a field $\mathfrak{k}$ and that $\mathcal{A} = End_{\mathfrak{k}}(A)$ and $X = \mathrm{Hom}_{\mathfrak{k}}(B, A)$. Let C_0, C_1 be the images of B in A by x_0, x_1; by passing to the quotient, x_0 determines an isomorphism u_0 of B/N_0 onto C_0, and, if $N_1 \supset N_0$, x_1 determines an $\mathfrak{k}$-linear application u_1 of B/N_0 onto C_1. There then exists an endomorphism a of A which coincides with $u_1 \circ u_0^{-1}$ on C_0; for example, we can complete its determination by imposing that it vanishes on an arbitrarily chosen supplement of C_0 in A. We then have $x_1 = ax_0$, which proves the first part of the lemma. The second part follows if we observe that to every $b \in B$ corresponds at least one $x \in X$ such that $xb \neq 0$.

It follows in particular from Lemma 8 that, if $\mathcal{A}$ is a simple algebra, the rank of an element x of X, as defined above, is none other than the rank of x as an $\mathfrak{k}$-linear application from B to A.

18. If we have also given an involution ι on $\mathcal{A}$, it determines on the set N of simple components $\mathcal{A}_\nu$ of $\mathcal{A}$ a permutation $\nu \to \nu'$ of order 2. The determination of $\mathcal{A}$, as an algebra with involution, then reduces to that of "simple algebras with involution" $\mathcal{A}_\nu$ (for $\nu = \nu'$) and $\mathcal{A}_\nu \oplus \mathcal{A}_\mu$ (for $\nu' = \mu \neq \nu$). As we know (cf. for ex. [1], Chap. X), these are necessarily, up to an isomorphism, of one of the following types:

(I) $\nu = \nu^\iota$; $\mathcal{A}_\nu$ is the algebra of matrices $M_m(\mathfrak{k})$ over a division algebra $\mathfrak{k}$ equipped with an involution $\xi \to \xi'$, and ι is the involution $x \to h^{-1} \cdot {}^t x' \cdot h$, where h is an invertible matrix satisfying ${}^t h' = \eta h$, with $\eta = \pm 1$;

(II) $\nu^\iota = \mu \neq \nu$; $\mathcal{A}_\nu$ is the algebra of matrices $M_m(\mathfrak{k})$ over a division algebra $\mathfrak{k}$; $\mathcal{A}_\mu$ is the algebra $M_m(\mathfrak{k}')$, where $\mathfrak{k}'$ is such that there exists an anti-isomorphism α of $\mathfrak{k}$ onto $\mathfrak{k}'$; and ι is the involution $(x, y) \to ({}^t y^{\alpha^{-1}}, {}^t x^\alpha)$ of $M_m(\mathfrak{k}) \oplus M_m(\mathfrak{k}')$.

For obvious reasons, most of the problems that can be posed about semisimple algebras with involution reduce to the corresponding problems in the **simple** cases (I) and (II). We will now finish setting the notations concerning these and demonstrate various auxiliary results.

In case (I), we are given a division algebra $\mathfrak{k}$ over k, equipped with an involution $\xi \to \xi'$. We will denote by $\mathfrak{z}$ the center of $\mathfrak{k}$, and $\mathfrak{z}_0$ the subfield of $\mathfrak{z}$ formed by the elements of $\mathfrak{z}$ invariant under the automorphism induced on $\mathfrak{z}$ by $\xi \to \xi'$; depending on whether the latter is or is not the identical automorphism, we have $\mathfrak{z} = \mathfrak{z}_0$, or $\mathfrak{z}$ is a quadratic extension of $\mathfrak{z}_0$.

If V is a vector space over k, we will extend $\xi \to \xi'$ to a k-linear application of $V \otimes_k \mathfrak{k}$ onto itself by setting $(v \otimes \xi)' = v \otimes \xi'$ for $v \in V$, $\xi \in \mathfrak{k}$. In particular, if V is the space $M_{m,n}(k)$ of matrices with m rows and n columns over k, $V \otimes_k \mathfrak{k}$ will be the analogous space $M_{m,n}(\mathfrak{k})$ over $\mathfrak{k}$, and, if $x \in M_{m,n}(\mathfrak{k})$, x' will be the matrix obtained by applying to each element of the matrix x the involution $\xi \to \xi'$. We will write $M_m(k)$, $M_m(\mathfrak{k})$ instead of $M_{m,m}(k)$, $M_{m,m}(\mathfrak{k})$. For all m, $u \to {}^t u'$ is an involution of $M_m(\mathfrak{k})$. We will denote by τ_m the reduced trace taken in $M_m(\mathfrak{k})$ over $\mathfrak{z}_0$ *considered as the base field.*

Let h be an invertible element of $M_m(\mathfrak{k})$ such that ${}^t h' = \eta h$ with $\eta = \pm 1$. We then take for $\mathcal{A}$ the algebra $M_m(\mathfrak{k})$ equipped with the involution ι defined by $u^\iota = h^{-1} \cdot {}^t u' \cdot h$. We easily verify that, if λ is a k-linear form on $\mathfrak{z}_0$, $\tau = \lambda \circ \tau_m$ is a trace function on $\mathcal{A}$ provided that $\lambda \neq 0$, and that conversely every trace function on $\mathcal{A}$ is of this form.

In accordance with the conventions of n^{o}16, we will set here $A = M_{m,1}(\mathfrak{k})$; $\mathcal{A}$ operating on A by matrix multiplication, A is indeed a simple left $\mathcal{A}$-module. Similarly, for all n, $M_{m,n}(\mathfrak{k})$ is, for $\mathcal{A}$ operating on the left by matrix multiplication, an $\mathcal{A}$-module of rank n. Whatever n and p may be, every $\mathcal{A}$-linear application from $M_{m,n}(\mathfrak{k})$ to $M_{m,p}(\mathfrak{k})$ is of the form $x \to x\alpha$ with $\alpha \in M_{n,p}(\mathfrak{k})$. In particular, if we set $X = M_{m,n}(\mathfrak{k})$ and, as in n^{o}17, $B = \mathrm{Hom}_{\mathcal{A}}(X, A)$, we can identify B with $M_{n,1}(\mathfrak{k})$, then X with $\mathrm{Hom}_{\mathfrak{k}}(B, A)$, that is, with the space of $\mathfrak{k}$-linear applications from B to A when these are considered as right vector spaces over $\mathfrak{k}$. As we have already observed, the rank of an element x of X in the sense of n^{o}16 coincides with its rank as a matrix with m rows and n columns over $\mathfrak{k}$. We will note that the ring $\mathrm{End}_{\mathcal{A}}(X)$ of endomorphisms of X considered as a left $\mathcal{A}$-module, and the ring $\mathrm{End}_{\mathfrak{k}}(B)$ of endomorphisms of B considered as a right vector space over $\mathfrak{k}$, both identify with $M_n(\mathfrak{k})$ and therefore identify with each other.

19. If $X = M_{m,n}(\mathfrak{k})$, $Y = M_{m,p}(\mathfrak{k})$, we can easily see that any sesquilinear form F on $X \times Y$ can be written as $F(x, y) = x \cdot w \cdot {}^t(hy)'$, with $w \in M_{n,p}(\mathfrak{k})$; the space of these forms, that is, according to n^{o}15, the dual of $X \otimes_\iota Y$, can therefore be identified with $M_{n,p}(\mathfrak{k})$ by means of this formula; if we make this identification, (14) is written as

$$[x \otimes_\iota y, w] = \tau(xw \cdot {}^t(hy)' = \lambda(\tau_n(w \cdot {}^t(hy)' \cdot x)).$$

But $(w_1, w_2) \to \lambda(\tau_n(w_1 \cdot {}^t w_2'))$ is a non-degenerate bilinear form on $M_{n,p}(\mathfrak{k}) \times M_{n,p}(\mathfrak{k})$; if we identify $M_{n,p}(\mathfrak{k})$ with its dual by means of this form, we see that $X \otimes_\iota Y$ is also identified with $M_{n,p}(\mathfrak{k})$, by means of the formula $x \otimes_\iota y = {}^t x' \cdot h \cdot y$. In particular, for $X = Y$, that is for $n = p$, we see that the automorphism s of $X \otimes_\iota X$ defined by $s(x \otimes_\iota y) = y \otimes_\iota x$ becomes the automorphism $w \to \eta\, {}^t w'$ of $M_n(\mathfrak{k})$; $I(X)$ is the space of $w \in M_n(\mathfrak{k})$ such that $w = \eta\, {}^t w'$, and i_X is the map $x \to {}^t x' \cdot h \cdot x$ from X to $I(X)$.

Following the usage, if V and W are right vector spaces over $\mathfrak{k}$, φ an $\mathfrak{k}$-linear application from V to W, and $\mathfrak{k}$ a sesquilinear form (for example η-hermitian) on $W \times W$, we will denote by $f[\varphi]$ the sesquilinear form $f \circ (\varphi, \varphi)$ on $V \times V$, where (φ, φ) denotes the map $(v_1, v_2) \to (\varphi(v_1), \varphi(v_2))$ from $V \times V$ to $W \times W$. The spaces A, X, B being as above, let us then consider on $A \times A$ the η-hermitian form defined by $(a_1, a_2) \to {}^t a_1' \cdot h \cdot a_2$, and agree to also denote it by h; similarly, to each $w \in M_n(\mathfrak{k})$ such that $w = \eta\, {}^t w'$, let us correspond the η-hermitian form $(b_1, b_2) \to {}^t b_1' \cdot w \cdot b_2$ on $B \times B$, and identify $I(X)$ with the space of η-hermitian forms on $B \times B$ by means of this correspondence. It then follows from what precedes that, if we identify X with $\mathrm{Hom}_{\mathfrak{k}}(B, A)$ as above, i_X is none other than the application $x \to h[x]$ from this latter space to the space of η-hermitian forms on $B \times B$.

Lemma 9. *Let δ and δ' be respectively the dimensions, over k, of $\mathfrak{k}$ and of the space of elements ξ of $\mathfrak{k}$ such that $\xi' = \eta\xi$, and let $\varepsilon = \delta'/\delta$. Then, if X is an $\mathcal{A}$-module of rank n, $I(X)$ has the dimension $\delta n(n + 2\varepsilon - 1)/2$ over k. Moreover, for i_X to be submersive at a point x of X, it is necessary and sufficient that x be of rank n or that $\varepsilon = 0$ and x be of rank $n - 1$.*

The first assertion results from what precedes; the second follows from this, from Lemma 7 of n^{o}15, and from the fact that, if x is of rank r, $X/\mathcal{A}x$ is of rank $n - r$.

20. Still placing ourselves in case (I), we will, with the above notations, specify the structure of the group $Ps(X/\mathcal{A})$ when X is an $\mathcal{A}$-module of rank n.

Let φ be an isomorphism of X onto $M_{m,n}(\mathfrak{k})$. As X^* has the same dimension as X over k, namely δmn, X^* is also of rank n. Using the results of n^{o}19, we see immediately that we can choose, in one and only one way, an isomorphism ψ of X^* onto $M_{m,n}(\mathfrak{k})$ so as to have

$$\{x, x^*\} = (x\varphi) \cdot {}^t(x^*\psi)' \cdot h.$$

Let us denote by ω the isomorphism $(x, x^*) \to (x\varphi, x^*\psi)$ of $X \oplus X^*$ onto $M_{m,n}(\mathfrak{k}) \oplus M_{m,n}(\mathfrak{k})$, or, which amounts to the same thing, onto $M_{m,2n}(\mathfrak{k})$. According to n^{o}14, for an automorphism σ of $X \oplus X^*$ to be symplectic, it is necessary and sufficient that it leaves invariant the sesquilinear form

$$((x_1, x_1^*), (x_2, x_2^*)) \to \{x_1, x_2^*\} - \{x_2, x_1^*\}^\iota$$

on $(X \oplus X^*) \times (X \oplus X^*)$. As shown by an easy calculation, it amounts to the same to say that the automorphism $s = \omega\sigma\omega^{-1}$ of $M_{m,2n}(\mathfrak{k})$ leaves invariant the form

$$(u_1, u_2) \to u_1 \cdot e \cdot {}^t u_2' \cdot h$$

on $M_{m,2n}(\mathfrak{k}) \times M_{m,2n}(\mathfrak{k})$, where e denotes the $(-\eta)$-hermitian matrix given by

$$e = \begin{pmatrix} 0 & 1_n \\ -\eta \cdot 1_n & 0 \end{pmatrix}. \tag{15}$$

But this condition is evidently equivalent to $s \cdot e \cdot {}^t s' = e$. In other words, the application $(\sigma, f) \to \omega\sigma\omega^{-1}$ is an isomorphism of $Ps(X/\mathcal{A})$ onto the group of elements s of $M_2n(\mathfrak{k})$ that satisfy $s \cdot e \cdot {}^t s' = e$. Moreover, it is clear that the image of $P(X/\mathcal{A})$ under this isomorphism is the set of elements of this latter group that are of the form

$$s = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$$

with a, b, d belonging to $M_n(\mathfrak{k})$.

21. *In case (II)*, we will proceed similarly by first considering a division algebra $\mathfrak{k}$ over k, whose center will be denoted by $\mathfrak{z}_0$. We will denote by $\mathfrak{k}'$ the **opposite** algebra to k, and by $\xi \to \xi'$ the **canonical** anti-automorphism of $\mathfrak{k}$ onto $\mathfrak{k}'$, as well as its inverse, such that $(\xi')' = \xi$ and $\xi \to \xi'$ induces the identity on $\mathfrak{z}_0$ (which is therefore the common center of $\mathfrak{k}$ and $\mathfrak{k}'$).

To simplify notation, in this n^o, we will agree to write $M_{m,n}$, $M'_{m,n}$ instead of $M_{m,n}(\mathfrak{k})$, $M_{m,n}(\mathfrak{k}')$, and similarly M_m, M'_m instead of $M_m(\mathfrak{k})$, $M_m(\mathfrak{k}')$. We will take $\mathcal{A} = M_m \oplus M'_m$, with the involution ι on $\mathcal{A}$ given by $(u,v)^\iota = ({}^t v', {}^t u')$. We will denote by τ_m the reduced trace taken either in M_m or in M'_m, on $\mathfrak{z}_0$ considered as the base field. The trace function τ on $\mathcal{A}$ will then be given by

$$\tau(u,v) = \lambda(\tau_m(u) + \tau_m(v)),$$

where λ is a k-linear form on $\mathfrak{z}_0$, and $\lambda \neq 0$.

We will set $A = A_0 \oplus A'_0$, with $A_0 = M_{m,1}$ and $A'_0 = M'_{m,1}$; we note that A'_0 can be identified in an obvious way with the dual of A_0 over $\mathfrak{k}$. Any $\mathcal{A}$-module of rank (p,q) is isomorphic to the module $M_{m,p} \oplus M'_{m,q}$, $\mathcal{A}$ operating on it by matrix multiplication in each component. Any $\mathcal{A}$-linear application from $M_{m,p} \oplus M'_{m,q}$ to $M_{m,r} \oplus M'_{m,s}$ is of the form $(x,y) \to (x\alpha, y\beta)$, with $\alpha \in M_{p,r}, \beta \in M'_{q,s}$. In particular, if we set again $B = \mathrm{Hom}_{\mathcal{A}}(X,A)$, and take $X = X_0 \oplus X'_0$ with $X_0 = M_{m,p}$, $X'_0 = M'_{m,q}$, we will have $B = B_0 \oplus B'_0$ with $B_0 = M_{p,1}$, $B'_0 = M'_{q,1}$, then $X_0 = Hom_{\mathfrak{k}}(B_0, A_0)$ and $X'_0 = \mathrm{Hom}_{\mathfrak{k}'}(B'_0, A'_0)$.

If X is as above, and if $Z = M - m, r \oplus M'_{m,s}$, any sesquilinear form on $X \times Z$ is written:

$$((x,y),(z,t)) \to (x \cdot v \cdot {}^t t', y \cdot w \cdot {}^t z')$$

with $v \in M_{p,s}$ and $w \in M'_{q,r}$; this allows to identify the dual of $X \otimes_\iota Y$ with $M_{p,s} \oplus M'_{q,r}$. Reasoning as in n^{o}19, we conclude that we can identify $X \otimes_\iota X$ with $M_{q,p} \oplus M'_{p,q}$, then that we can identify $I(X)$ with $M_{q,p}$ in such a way as to have $i_X(x,y) = {}^t y' \cdot x$. This gives in particular:

Lemma 10. *Let δ be the dimension of $\mathfrak{k}$ over k. Then, if X is an $\mathcal{A}$-module of rank (p,q), $I(X)$ has dimension δpq over k; and, for i_X to be submersive at a point x of rank (p',q') in X, it is necessary and sufficient that $p = p'$ or $q = q'$.*

Still using the same notations, we can identify the dual X^* of X with $M_{m,q} \oplus M'_{m,p}$, in such a way that

$$\{(x,y),(x^*,y^*)\} = (x \cdot {}^t(y^*)', y \cdot {}^t(x^*)').$$

The dual of a module of rank (p,q) is therefore of rank (q,p). By setting $n = p+q$, $X \oplus X^*$ is obviously identified with $M_{m,n} \oplus M'_{m,n}$; according to what we have seen, the ring of endomorphisms of this latter module is identified with $M_n \oplus M'_n$. For an automorphism $\sigma = (\lambda, \mu)$ of $X \oplus X^*$ to be symplectic, it is necessary and sufficient that it leaves invariant the sesquilinear form

$$\{(x_1,y_1),(x_2^*,y_2^*)\} - \{(x_2,y_2),(x_1^*,y_1^*)\}^\iota$$

An easy calculation shows that, if we set

$$e = \begin{pmatrix} 0 & 1_p \\ -1_q & 0 \end{pmatrix},$$

this condition is equivalent to $\mu = e^{-1} \cdot {}^t\lambda'^{-1} \cdot e$. It follows that with these notations, the map $(\sigma, f) \to \lambda$ is an isomorphism of $Ps(X/\mathcal{A})$ onto $\mathrm{GL}(n, \mathfrak{k})$. We then easily verify that the image of $P(X/\mathcal{A})$ by this isomorphism is the subgroup of $\mathrm{GL}(n, \mathfrak{k})$ formed by elements of the form

$$\lambda = \begin{pmatrix} \lambda_1 & \lambda_2 \\ 0 & \lambda_3 \end{pmatrix}$$

with $\lambda_1 \in \mathrm{GL}(p, \mathfrak{k}), \lambda_2 \in M_{p,q}(\mathfrak{k}), \lambda_3 \in \mathrm{GL}(q, \mathfrak{k})$.

22. Returning to the general case of n^{os}16-17, we will denote by G the group of elements u of $\mathcal{A}$ such that $u \cdot u^\iota = 1$. It is clear that G is the direct product of analogous groups relative to the **simple** components of $\mathcal{A}$, these being of type (I) or (II). In case (I), using the notations from n^{o}18, G is the group of elements u of $M_m(\mathfrak{k})$ that satisfy ${}^tu' \cdot h \cdot u = h$, that is to say $h[u] = h$; in other words, if $\mathrm{Aut}(A) = \mathrm{GL}(m, \mathfrak{k})$ denotes the group of automorphisms of the right vector space $A = M_{m,1}(\mathfrak{k})$ over $\mathfrak{k}$, G is the subgroup $\mathrm{Aut}(A, h)$ of $\mathrm{Aut}(A)$ formed by the elements of $\mathrm{Aut}(A)$ that leave invariant the η-Hermitian form h. In case (II), using the notations from n^{o}21, G is the group of elements of $\mathcal{A}$ of the form $(u, {}^tu'^{-1})$ with $u \in \mathrm{Aut}(A_0) = \mathrm{GL}(m, \mathfrak{k})$, and is therefore isomorphic to this latter group. In the general case, G is thus a product of **classical groups**.

The definition of i_X shows that we have, for any $x \in X$, $t \in \mathcal{A}$:

$$i_X(tx) = (tx) \otimes_\iota (tx) = x \otimes_\iota t^\iota(tx).$$

It follows that, for $u \in G$, we have $i_X(ux) = i_X(x)$ for any $x \in X$. In other words, i_X is invariant under G operating on the left on X.

In the following chapters, we will need various results on the orbits of G, and of certain groups related to G, in X; these results will be established in n^{os}22-25. In the case where $\mathcal{A}$ is of type (I), and where X is of rank 1 over $\mathcal{A}$, they are essentially contained in Witt's theorem; the reader who only has this case in mind (cf. Introduction) can therefore dispense with reading the greater part of the demonstrations that will follow.

Proposition 3. *For any $i \in I(X)$, let $U(i)$ be the set of elements of $i_X^{-1}(\{i\})$ which are of maximal rank in X; then, if $U(i)$ is not empty, it is an orbit of G in X.*

Let $x_0 \in U(i)$, which means that we have $i_X(x_0) = i$ and $\mathcal{A}x_0 = X$. It is clear that, if $x_1 = ux_0$ with $u \in G$, we have $x_1 \in U(i)$. To prove the converse, it suffices evidently to treat cases (I) and (II). In case (I), with the notations of n^{os}. 18-19, the hypothesis means that $h[x_0] = h[x_1] = i$ and that x_0, x_1 are of maximal rank, that is to say (according to Lemma 8 of n^{o}17) that they are of kernel $\{0\}$ if we consider them as maps of B in A. Then x_0, x_1 determine isomorphisms $\overline{x}_0$, $\overline{x}_1$ of B onto subspaces L_0, L_1 of A, and $z = \overline{x}_1\overline{x}_0^{-1}$ is an isomorphism of L_0 onto L_1. Let h_0, h_1 be the η-Hermitian forms induced by h on $L_0 \times L_0$ and on $L_1 \times L_1$, respectively. The hypothesis $h[x_0] = h[x_1]$ is equivalent to $h_0 = h_1[z]$, that is to say that z, in an obvious sense, is an isomorphism of (L_0, h_0) onto (L_1, h_1). Witt's theorem then says that z can be extended to an automorphism u of (A, h), that is to say to an element of G; and we indeed have $x_1 = ux_0$.

In case (II), let's take up the notations of n^{o}21, and consider two elements (x_0, y_0), (x_1, y_1) of $U(i)$. We see, as above, that y_0, y_1 determine isomorphisms $\overline{y}_0, \overline{y}_1$ of B_0' onto subspaces L_0', L_1' of A_0', and $\overline{y}_1\overline{y}_0^{-1}$ is an isomorphism of L_0' onto L_1', which can be extended to an automorphism v of A_0', so that we have $y_1 = vy_0$. By replacing (x_0, y_0) with $({}^tv'^{-1} \cdot x_0, vy_0)$, we are thus brought back to treating the case where $y_0 = y_1$. Let then $z_0 = {}^t y_0'$, so that we have $z_0 x_0 = z_0 x_1$. As $z_0 \in M_{q,m}(\mathfrak{k})$, we can consider z_0 as a $\mathfrak{k}$-linear map of $A_0 = M_{m,1}(\mathfrak{k})$ into $M_{q,1}(\mathfrak{k})$; let N_0 be the kernel of this map. On the other hand, x_0, x_1 determine isomorphisms $\overline{x}_0$, $\overline{x}_1$ of B_0 onto subspaces L_0, L_1 of A_0, so that $\overline{u} = \overline{x}_1 \overline{x}_0^{-1}$ is an isomorphism of L_0 onto L_1. As we have $z_0(x_1 - x_0) = 0$, we have $\overline{u}a - a \in N_0$ for any $a \in L_0$, and similarly $\overline{u}^{-1}a - a \in N_0$ for any $a \in L_1$; we immediately conclude that $L_0 + N_0 = L_1 + N_0$, and also that $\overline{u}$ induces on $L_0 \cap N_0$ an isomorphism of $L_0 \cap N_0$ onto $L_1 \cap N_0$; we can then extend this last isomorphism to an automorphism v of N_0. As $\overline{u}$ and v coincide on $L_0 \cap N_0$, there exists an endomorphism u_0 of $L_0 + N_0$ and only one which coincides with $\overline{u}$ on L_0 and with v on N_0; similarly, there exists an endomorphism u_1 of $L_0 + N_0$ and only one which coincides with $\overline{u}^{-1}$ on L_1 and with v^{-1} on N_0. We thus have $u_0 u_1 = 1$, so that u_0 is an automorphism of $L_0 + N_0$. Let finally L' be a complement of $L_0 + N_0$ in A_0, and u the automorphism of A_0 which coincides with u_0 on $L_0 + N_0$ and with the identity on L'. We then have $x_1 = u x_0$; and $u - 1$ maps A_0 into N_0, so that we have $z_0 u = z_0$, or equivalently $y_0 = {}^t u'^{-1} \cdot y_0$. This completes the demonstration of proposition 3.

Corollary. *For two points x_0, x_1 of X to belong to the same orbit for G, it is necessary and sufficient that we have $i_X(x_0) = i_X(x_1)$ and $\mathcal{A}x_0 = \mathcal{A}x_1$.*

These conditions are obviously necessary. Suppose them satisfied, and let $X' = \mathcal{A}x_0$. As $\mathcal{A}$ is semisimple, X' has a supplement in X; consequently, according to n^o. 15, we can identify $I(X')$ with the subspace of $I(X)$ generated by the elements $i_X(x')$ for $x' \in X'$, and $i_{X'}$ with the application of X' in this subspace of $I(X)$ which is induced by i_X. Then x_0 and x_1 are elements of X' of maximal rank in X', and we have $i_{X'}(x_0) = i_{X'}(x_1)$; according to Proposition 3, they therefore belong to the same orbit of G in X'.

23. Proposition 3 shows that, if $U(i)$ is not empty, it is a homogeneous space with respect to the group G. To completely determine its structure, we need to know the stability group g_0, in G, of an element x_0 of $U(i)$.

Here again, we can limit ourselves to cases (I) and (II). In case (I), let's consider i as an η-Hermitian form on $B \times B$; let B' be the subspace of B which, with respect to i, is orthogonal to all vectors of B, and let B" be a supplement of B' in B; then the η-Hermitian form induced by i on B" $\times$ B" is non-degenerate. Let r be the rank of i; then B', B" are of dimensions $n - r$ and r, respectively. Let $x_0 \in U(i)$; as x_0 is of maximal rank, it defines an isomorphism $\overline{x}_0$ of B onto a subspace L_0 of A; as we have $i = h[x_0]$, the form induced by h on $L_0 \times L_0$ is that which is deduced from i by $\overline{x}_0$. Consequently, if L', L" are the images of B', B" by x_0, the form induced by h on L" $\times$ L" is non-degenerate, and L' is the intersection of L_0 with the orthogonal of L_0 with respect to h in A. Let M be the orthogonal of L" with respect to h in A; the form induced by h on $M \times M$ is non-degenerate; as L' is a **totally isotropic** subspace of M, we know ([5], p. 21) that there exists a totally isotropic subspace L_1' of M, supplementary in M to the orthogonal of L' in M. Let finally L_1 be the orthogonal of L"$+ L' + L_1'$ in A. It is easy to verify that A is the direct sum of L", L', L_1, L_1'; thus L_1 is of dimension $m - 2n + r$, which implies $m \geq 2n - r$. Moreover, the form h_1 induced by h on $L_1 \times L_1$ is non-degenerate. Then, if we extend any element of $\mathrm{Aut}(L_1, h_1)$ to an automorphism of A by imposing that it induces the identity on $L_0 + L_1'$, we obtain a subgroup g_1 of the stability group g_0 of x_0 in G, which is evidently isomorphic to $\mathrm{Aut}(L_1, h_1)$; moreover, we easily see that g_0 is the semidirect product of g_1 and a group g_0' which induces the identity on L" and on L', which leaves invariant the spaces $L' + L_1$ and $L' + L_1 + L_1' = M$, and which induces the identity on $(L' + L_1)/L'$ and on $M/(L' + L_1)$; g_0' is thus unipotent, so that ultimately g_0 is isomorphic to a semidirect product of $\mathrm{Aut}(L_1, h_1)$ and a unipotent group. In particular, if i is non-degenerate, thus of rank n, L' and L_1' reduce to $\{0\}$, L_1 is of dimension $m - n$, and g_0 is isomorphic to $\mathrm{Aut}(L_1, h_1)$.

In case (II), let $i \in M_{q,p}(\mathfrak{k})$, and let r be the rank of i; $U(i)$ is the set of couples $(x, {}^t z')$ with x of rank p in $M_{m,p}(\mathfrak{k})$, z of rank q in $M_{q,m}(\mathfrak{k})$, and $zx = i$. Let $(x_0, {}^t z_0')$ be such a couple, and let g_0 be its stability group in G; when we identify G with $\mathrm{GL}(m, \mathfrak{k})$ as in n^{o}22, g_0 is identified with the group of $u \in GL(m, \mathfrak{k})$ such that $ux_0 = x_0$ and $z_0 u = z_0$. Let L_0 be the image of $B_0 = M_{p,1}(\mathfrak{k})$ by x_0 in $A_0 = M_{m,1}(\mathfrak{k})$; L_0 is of dimension p. Let N_0 be the kernel of z_0 considered as an application of A_0 in $M_{q,1}(\mathfrak{k})$; N_0 is of dimension $m - q$. As i is of rank r, z_0 induces on L_0 an map of rank r from L_0 to $M_{q,1}(\mathfrak{k})$, so that $L_0 \cap N_0$ has dimension $p - r$. Let $L' = L_0 \cap N_0$; let L", L_1 be supplements of L' in L_0 and in N_0, respectively; L_1 is of dimension $m - p - q + r$, which implies $m \geq p + q - r$. Let L_1' be a supplement of $L_0 + N_0$ in A_0, and let $M = L' + L_1 + L_1'$. If we extend any automorphism of L_1 to an automorphism of A_0 by imposing that it induces the identity on $L_0 + L_1'$, we obtain a subgroup g_1 of g_0, isomorphic to $\mathrm{Aut}(L_1)$. Let g_0' be the subgroup of G which induces the identity on L" and on L', which leaves invariant the spaces $L' + L_1$ and M, and which induces the identity on $(L' + L_1)/L'$ and on $M/(L' + L_1)$. Then g_0 is the semidirect product of g_1 and the unipotent group g_0'. If $p = q = r$, L' and L_1' reduce to $\{0\}$, L_1 is of dimension $m - p$, and g_0 is isomorphic to $\mathrm{Aut}(L_1)$. Finally, we note that, in case (II), the condition $m \geq p + q - r$ is not only necessary, but also sufficient for $U(i)$ to be non-empty.

24. At the same time as G, we will operate on X, but on the right, the ring $\mathrm{End}_{\mathcal{A}}(X)$ of endomorphisms of X considered as a left $\mathcal{A}$-module, and in particular the group of invertible elements of this ring, that is to say the automorphisms of X, a group which we will simply denote by $\mathrm{Aut}(X)$. If B is defined as in n^o. 17, it is immediate that $\mathrm{End}_{\mathcal{A}}(X)$ is identified with the ring $\mathrm{End}_{\mathcal{B}}(B)$ of endomorphisms of B considered as a $\mathcal{B}$-module on the right; consequently, $\mathrm{Aut}(X)$ is identified with the group $\mathrm{Aut}(B)$ of automorphisms of B.

By transport of structure, any automorphism v of X determines an automorphism of $I(X)$ which we will denote by $\overline{v}$, so that we will have $i_X(xv) = i_X(x)\overline{v}$. For any $i \in I(X)$, we will denote by $\mathrm{Aut}(X, i)$ the subgroup of $\mathrm{Aut}(X)$ which leaves i invariant, that is to say the group of $v \in \mathrm{Aut}(X)$ such that $i\overline{v} = i$. We will agree on the other hand to make the group $G \times \mathrm{Aut}(X)$ operate on X by means of the formula

$$(u, v) \cdot x = uxv^{-1} \quad (u \in G, x \in X, v \in \mathrm{Aut}(X)).$$

Under these conditions, it is clear that the set $i_X^{-1}(\{i\})$ is invariant under the group $G \times \mathrm{Aut}(X, i)$. We will now demonstrate some auxiliary results, which will be used in Chapter III, concerning the orbits of this latter group and some of its subgroups in $i_X^{-1}(\{i\})$; in the case where $\mathcal{A}$ is of type (I) and X of rank 1, these results are trivial or are trivial consequences of what precedes.

Taking up the notations of n^o. 17, let's first consider the space $B^* = \mathrm{Hom}_{\mathcal{A}}(X^*, A)$ attached to X^* as B is to X; supposing each **simple** component of $\mathcal{A}$ is put in the form described in n^{o}18, and the corresponding components of X put in the form described in n^{os}19 and 21, we will define an isomorphism of $X \otimes_\iota X$ onto the space $\mathrm{Hom}_{\mathcal{B}}(B, B*)$ of morphisms from B to B^* for their structures of right $\mathcal{B}$-modules; for this, it will evidently suffice to describe this isomorphism separately in cases (I) and (II); for brevity, we will not concern ourselves with knowing to what extent it is **canonical**. In case (I), we identify X with $M_{m,n}(\mathfrak{k})$ and $X \otimes_\iota X$ with $M_n(\mathfrak{k})$ as in n^o. 19, this latter identification being made by means of the formula $x \otimes_\iota y = {}^t x' \cdot h \cdot y$. As in n^o. 20, let's identify X^* with $M_{m,n}(\mathfrak{k})$ by means of the formula $\{x, x^*\} = x \cdot {}^t(x^*)' \cdot h$; then identify B and B^* with $M_{n,1}(\mathfrak{k})$ as in n^o. 18. Under these conditions, to any element w of $X \otimes_\iota X$, we will make correspond the morphism λ_w from B to B^* given by $\lambda_w(b) = w \cdot b$ for all $b \in B$. In particular, for $i \in I(X)$, λ_i will thus be the morphism $b \to i \cdot b$; on the other hand, according to n^o. 19, i then satisfies ${}^t i' = \eta i$, and is the matrix of the η-Hermitian form $(b_1, b_2) \to {}^t b_1' \cdot i \cdot b_2$ on $B \times B$; it follows that the kernel of λ_i in B is the subspace B' of B which is orthogonal, with respect to i, to all vectors of B. Moreover, for $x \in X$ and $i = i_X(x)$, we have $i = {}^t x' \cdot h \cdot x$, so that the kernel of x in B is contained in that of λ_i.

Similarly, in case (II), we will again identify X with $M_{m,p} \oplus M'_{m,q}$ and $X \otimes_\iota X$ with $M_{q,p} \oplus M'_{p,q}$ as in n^o. 21, this latter identification being made by means of the formula

$$(x, y) \otimes_\iota (z, t) = ({}^t t' \cdot x, {}^t z' \cdot y)$$

for x and z in $M_{m,p}$ and y and t in $M'_{m,q}$. We will identify X^* with $M_{m,q} \oplus M'_{m,p}$ and B with $M_{p,1} \oplus M'_{q,1}$ as in n^o. 21, and similarly B^* with $M_{q,1} \oplus M'_{p,1}$. Then, for any $(u, v) \in X \otimes_\iota X$, we will define the morphism $\lambda_{(u,v)}$ from B to B^* by setting, for any element (b, c) of B:

$$\lambda_{(u,v)}(b, c) = (ub, vc).$$

In particular, for $(x, y) \in X$ and $i = i_X(x, y)$, λ_i will be given, according to n^o. 21, by the formula

$$\lambda_i(b, c) = ({}^t y' \cdot x \cdot b, {}^t x' \cdot y \cdot c),$$

from which it follows that the kernel of (x, y) in B is contained in that of λ_i, as in case (I).

25. These definitions being extended in an obvious manner, component by component, to the general case where $\mathcal{A}$ is only assumed to be semisimple, we see that we thus make correspond to each $i \in I(X)$ a morphism λ_i from B to B^* having the properties that result from what precedes. In particular, for $i = i_X(x)$, the kernel of λ_i always contains that of x when x is considered as an application from B to A.

We will say that an element i of $I(X)$ is *non-degenerate* if the kernel of λ_i reduces to $\{0\}$; the formulas of n^o. 24 show immediately in cases (I) and (II) that then λ_i is an isomorphism from B onto B^*, and consequently that then X is isomorphic to its dual X^*; it follows that this conclusion remains valid in the general case. For non-degenerate elements to exist in $I(X)$, it is therefore necessary that X be isomorphic to X^*; but this is not sufficient. If $x \in X$ and $i_X(x)$ is non-degenerate, the kernel of x in B must be $\{0\}$; according to Lemma 8 of n^o. 17, this amounts to saying that x is of maximal rank. In other words, if i is non-degenerate, we have $U(i) = i_X^{-1}(\{i\})$; consequently, according to Proposition 3 of n^o. 22, the group G then operates transitively on $i_X^{-1}(\{i\})$.

On the other hand, let $i \in I(X)$; let B' be the kernel of λ_i; let B" be a supplement of B' in B. Let g' denote the subgroup of $\mathrm{Aut}(B)$ formed by the automorphisms of B which leave B' and B" invariant and induce the identity on B". We verify immediately, in case (I) and in case (II), that g' leaves i invariant; identifying $\mathrm{Aut}(B)$ with $\mathrm{Aut}(X)$ as in n^o. 24, we can therefore consider g' as a subgroup of $\mathrm{Aut}(X, i)$; and this conclusion, once verified in cases (I) and (II), evidently remains valid in the general case. With these notations, we then have the following result:

Proposition 4. *For two elements of $i_X^{-1}(\{i\})$ to belong to the same orbit of $G \times g'$, it is necessary and sufficient that they have the same rank.*

The condition is evidently necessary. Conversely, let x_0 and x_1 be two elements of $i_X^{-1}(\{i\})$, and let N_0 and N_1 be their kernels in B; according to what precedes, N_0 and N_1 are sub-modules of the kernel B' of λ_i for the structure of right $\mathcal{B}$-module of B. But $\mathcal{B}$ is the direct sum of the fields $\mathfrak{k}_v$; if we decompose B, B', N_0 and N_1 into their components relative to the $\mathfrak{k}_v$, these components are right vector spaces over the $\mathfrak{k}_v$, and the hypothesis we have made on the ranks of x_0 and x_1 amounts to saying that, for each v, the components of N_0 and of N_1 relative to $\mathfrak{k}_v$ are spaces of the same dimension over $\mathfrak{k}_v$, both contained in the corresponding component of B'. There is therefore an automorphism v' of B' which transforms N_1 into N_0. Then the automorphism v of B which coincides with v' on B' and with the identity on B" belongs to g' and therefore leaves i invariant, so that x_0v still belongs to $i_X^{-1}(\{i\})$; and x_0v has the same kernel N_1 as x_1, which implies, according to Lemma 8 of n^o. 17 and Proposition 3 of n^o. 22, that x_0v and x_1 belong to the same orbit of G.

Corollary. *For two elements of $i_X^{-1}(\{i\})$ to belong to the same orbit of $G \times \mathrm{Aut}(X, i)$, it is necessary and sufficient that they have the same rank.*

The condition is evidently necessary, and proposition 4 shows that it is sufficient.

Proposition 4 and its corollary show in particular that the groups $G \times g'$ and $G \times \mathrm{Aut}(X, i)$ have only a finite number of orbits in $i_X^{-1}(\{i\})$. We will observe that the condition involved here is generally weaker than that of the corollary of proposition 3 of n^o. 22, but is equivalent to it in certain cases. Let indeed $n = (n_v)$ and $n' = (n'_v)$ be the ranks of X and of a point x_0 of X, and suppose that, for all v, we have either $n'_v = n_v$, or $n'_v = 0$ (which will occur for example if x_0 is of maximal rank in X, but also whenever $n_v \leq 1$ for all v). It is clear that then we will have $\mathcal{A}x_0 = \mathcal{A}x_1$ whenever x1 has the same rank as x_0.

26. Suppose $\mathcal{A}$ absolutely semisimple, which amounts to saying that, for all ν, the center of $\mathfrak{k}_\nu$ is separable over k; it will always be so when k is of characteristic 0. Then, if we extend $\mathcal{A}$ by a field $K \supset k$, the extended algebra $\mathcal{A}_K$ is still semisimple; if at the same time, as was said in n^o. 14, we extend by K the involution ι, the trace τ, and the modules over $\mathcal{A}$, we can apply to them all that has been said above. We will now describe the changes of structure that such an extension can entail; it will suffice for this to place ourselves successively in cases (I) and (II). To simplify, we will assume in both cases that $\mathfrak{z}_0 = k$, the general case reducing to this one for well-known reasons; recall that $\mathfrak{z}_0$ was defined, in case (I), as being the set of elements of the center of $\mathfrak{k}$ which are invariant under ι, and, in case (II), as being the common center of $\mathfrak{k}$ and $\mathfrak{k}'$. It follows, in case (I), that the center $\mathfrak{z}$ of $\mathfrak{k}$ is either k, or of degree 2 over k.

Let's first place ourselves in case (I). Suppose first that we have $\mathfrak{z} \neq k$ and $K \supset z$; then $\mathfrak{z}_K$ is the direct sum of two fields isomorphic to K, and $\mathcal{A}_K$ is the direct sum of two simple algebras with center K, exchanged by the involution ι; the extension by K thus makes us pass from type (I) to type (II). Moreover, $\mathfrak{k}_K$ is then the direct sum of two simple algebras, anti-isomorphic to each other, which we can write as $M_\nu(\mathcal{K})$ and $M_\nu(\mathcal{K}')$, where $\mathcal{K}$ and $\mathcal{K}'$ are two division algebras of center K, anti-isomorphic to each other, so that $\mathcal{A}_K$ will be isomorphic to the direct sum of $M_{m_K}(\mathcal{K})$ and $M_{m_K}(\mathcal{K}')$, with $m_K = m\nu$. If δ denotes as before the dimension of $\mathfrak{k}$ over k, and if $\mathfrak{d}_K$ denotes that of $\mathcal{K}$ over K, we have $\delta = 2\delta_K\nu^2$. If X is a left $\mathcal{A}$-module of rank n, X_K will be an $\mathcal{A}_K$-module of rank (n_K, n_K) with $n_K = n\nu$.

Suppose now that we are in case (I) with $\mathfrak{z} = k$, which gives $\mathfrak{z}_K = K$, or with $\mathfrak{z} \neq k$ and $K \not\supset \mathfrak{z}$, which implies that $\mathfrak{z}_K$ is a quadratic extension of K. Then $\mathcal{A}_K$ is a simple algebra; $\mathfrak{k}_K$ is of the form $M_\nu(\mathcal{K})$, where $\mathcal{K}$ is a division algebra with center $\mathfrak{z}_K$, and $\mathcal{A}_K$ is then isomorphic to $M_{m_K}(\mathcal{K})$

with $m_K = m\nu$. If δ and δ_K are the dimensions of $\mathfrak{k}$ over k and of $\mathcal{K}$ over K, respectively, we have $\mathfrak{d} = \mathfrak{d}_K \nu^2$. If X is a left $\mathcal{A}$-module of rank n, X_K is a left $\mathcal{A}_K$-module of rank $n_K = n\nu$. Finally, it has already been noted in n^o. 15 that $I(X_K)$ is identified with $I(X)_K$, so that the dimension of $I(X_K)$ over K is equal to that of $I(X)$ over k; applying Lemma 9 of n^o. 19 to these dimensions, we conclude that we have

$$2\varepsilon_K - 1 = (2\varepsilon - 1)\nu \tag{16}$$

if ε is defined as in n^o. 19, and if ε_K is the number defined similarly for the involution algebra $\mathcal{A}_K$ over K.

If $\mathcal{A}$ is of type (II), $\mathcal{A}_K$ will also be of type (II); $\mathfrak{k}_K$ will be of the form $M_\nu(\mathcal{K})$, $\mathcal{K}$ being a division algebra with center K; if $\mathcal{K}'$ is anti-isomorphic to $\mathcal{K}$, $\mathcal{A}_K$ will be the direct sum of $M_{m_K}(\mathcal{K})$ and $M_{m_K}(\mathcal{K}')$, with $m_K = m\nu$; we will have $\delta = \delta_K \nu^2$ if δ and δ_K are the dimensions of $\mathfrak{k}$ over k and of $\mathcal{K}$ over K. Finally, if X is a left $\mathcal{A}$-module of rank (p, q), X_K will be a left $\mathcal{A}_K$-module of rank (p_K, q_K) with $p_K = p\nu$, $q_K = q\nu$.

Note that, if we are in case (I) with $\mathfrak{z} \neq k$, we necessarily have $\varepsilon = \frac{1}{2}$; if we therefore agree to define ε as having the value $\frac{1}{2}$ each time $\mathcal{A}$ is of type (II), formula (16) will remain valid in all cases.

Finally, it is immediate that the dual $(X_K)^*$ of X_K can always be identified with $(X^*)_K$, and that any element of X of maximal rank in X is still of maximal rank in X_K. We can also verify that any element of $I(X)$ which is non-degenerate in $I(X)$ is still so in $I(X_K)$.

27. In all cases that interest us, k is either a local field, or a field of algebraic numbers, or a field of algebraic functions of dimension 1 over a finite field. If k is one of these fields, it is well known that any division algebra $\mathfrak{k}$ over k on which there exists an involution leaving invariant all elements of the center of $\mathfrak{k}$ is commutative or is an algebra of quaternions over its center. We easily conclude that the involution algebras of type (I) over k are distributed as follows into subtypes, each characterized by the corresponding value of the number ε defined in Lemma 9 of n^o. 19:

(I_0) $\varepsilon = 0$; this is the case where we have, with the notations of n^o. 18, $\mathfrak{k} = \mathfrak{z} = \mathfrak{z}_0$ and $\eta = -1$, the involution ι being thus defined by an alternating form h on a commutative field $\mathfrak{z}_0$; the group G is the symplectic group with m variables, $G = \mathrm{Sp}(m, \mathfrak{z}_0)$. As h must be non-degenerate, m is even.

(I_1) $\varepsilon = \frac{1}{4}$; we have $\mathfrak{z} = \mathfrak{z}_0$, and $\mathfrak{k}$ is a quaternion algebra over $\mathfrak{z}_0$; $\xi \to \xi'$ is the **usual** involution on $\mathfrak{k}$, defined by the fact that $\xi + \xi'$ is the reduced trace of ξ over $\mathfrak{z}_0$; and $\eta = 1$. The form h is Hermitian (or, if one wants to be precise, Hermitian-quaternionic) on $\mathfrak{k}$, with respect to the usual involution.

(I_2) $\varepsilon = \frac{1}{2}$; this is the case where $\mathfrak{z}$ is a quadratic extension of $\mathfrak{z}_0$, that is, where ι induces a non-trivial automorphism on the center of $\mathcal{A}$; it is customary to say then that ι is an involution **of the 2nd kind**. We can assume that $\eta = 1$, that is, that h is a Hermitian form.

(I_3) $\varepsilon = \frac{3}{4}$; we have $\mathfrak{z} = \mathfrak{z}_0$, $\mathfrak{k}$ is a quaternion algebra over $\mathfrak{z}_0$, $\xi \to \xi'$ is the usual involution on $\mathfrak{k}$, and $\eta = -1$, that is, h is anti-Hermitian, or, if one wants to be precise, anti-Hermitian-quaternionic, with respect to the usual involution.

(I_4) $\varepsilon = 1$; we have $\mathfrak{k} = \mathfrak{z} = \mathfrak{z}_0$ and $\eta = 1$; h is a quadratic form, and G is the orthogonal group of h.

The effect on these subtypes of an extension by a field $K \supset k$ is immediately deduced from formula (16) of n^o. 26. As in n^o. 26, let's assume for simplicity that we have $\mathfrak{z}_0 = k$. Then the extension by K transforms an algebra of type (I_0) or (I_4) into an algebra of the same type; it transforms an algebra of type (I_1) or (I_3) into an algebra of the same type, if it does not "decompose" the corresponding quaternion algebra $\mathfrak{k}$ (that is, if $\mathfrak{k}_K$ is without zero divisors), and transforms type (I_1) into (I_0), and (I_3) into (I_4), in the contrary case. Finally, as we saw in n^o. 26, it transforms an algebra of type (I_2) into an algebra of the same type, or of type (II), depending on whether K does not contain the center $\mathfrak{k}$ of the algebra in question or contains $\mathfrak{k}$. In particular, by extension to the universal domain (or to any algebraically closed field), the cases (I_1), (I_2), (I_3) reduce respectively to (I_0), (II), (I_4), while these subsist as they are.

28. We defined the group G in n^o. 22 as the group of elements u of $\mathcal{A}$ such that $u \cdot u^\iota = 1$; the above results allow in each case to determine the structure of G as an algebraic group (that is, after extension to the universal domain).

For this, let's agree to denote by $\mathcal{A}_\nu^{(1)}$, for each ν, the group of elements of $\mathcal{A}_\nu$ whose reduced norm relative to the center of $\mathcal{A}_\nu$ is equal to 1; let $\mathcal{A}^{(1)} = \prod \mathcal{A}_\nu^{(1)}$, and let $G_1 = G \cap \mathcal{A}^{(1)}$. In cases (I_0) and (I_1), we have $G = G_1$; G is connected, simple and simply connected for all $m > 0$. In cases (I_2) and (II), G is connected, reductive, and product (non-direct) of its center and G_1; G_1 reduces to $\{1\}$ for $m = 1$, $\mathfrak{k} = \mathfrak{z}$ in case (I_2), for $m = 1$, $\mathfrak{k} = \mathfrak{k}' = \mathfrak{z}_0$ in case (II), and is connected, simple and simply connected in any other case. Finally, in cases (I_3) and (I_4), G, as an algebraic group, has two connected components, one of which is G_1; it turns out that, in case (I_3), any point of G, rational over the base field, belongs to G_1; as for G_1, it is reduced to $\{1\}$ for $m = 1$ in case (I_4); it's a connected torus of dimension 1, for $m = 2$ in case (I_4) and for $m = 1$ in case (I_3); it is connected and semisimple in any other case; it is even simple for $m = 3$ and for $m \geq 5$ in case (I_4), and for $m \geq 3$ in case (I_3). These statements are to be modified in an obvious way if we don't assume $\mathfrak{z}_0 = k$.

From this, we easily deduce, in each case, the structure of the group $Ps(X/\mathcal{A})$. In case (II), this structure is given by the results of n^o. 21, where we defined an isomorphism of $Ps(X/\mathcal{A})$ onto $\mathrm{GL}(p+q, \mathfrak{k})$; in this case, the subgroup of $Ps(X/\mathcal{A})$ which corresponds to $\mathrm{SL}(p+q, \mathfrak{k})$ by this isomorphism will be denoted by $Ps'(X/\mathcal{A})$. In case (I), it follows from n^o. 20 that $Ps(X/\mathcal{A})$ is the group analogous to G determined by the algebra $M_{2n}(\mathfrak{k})$ equipped with the involution $u \to e \cdot {}^t u' \cdot e^{-1}$, e being the element of $M_{2n}(\mathfrak{k})$ given by (15) and thus satisfying ${}^t e' = -\eta e$; if $\mathcal{A}$ is of type (I_ν) with $0 \leq r \leq 4$, this latter algebra is of type $(I_{4-\nu})$. If $\mathcal{A}$ is of type (I_0) and we have $n = 1$, we will set $Ps'(X/\mathcal{A}) = \{1\}$; otherwise, we will denote by $Ps'(X/\mathcal{A})$ the group (analogous to G_1) formed by the elements of $Ps(X/\mathcal{A})$ whose reduced norm, taken in the algebra $M_{2n}(\mathfrak{k})$ relative to its center, has the value 1. By virtue of this definition, $Ps'(X/\mathcal{A})$ reduces to $\{1\}$ if $\mathcal{A}$ is of type (I_0) and if $n = 1$, or if $\mathcal{A}$ is of type (II) with $\mathfrak{k} = \mathfrak{z}_0$ and if $p + q = 1$, and is connected and semisimple in any other case. If $\mathcal{A}$ is only assumed semisimple, we will denote by $Ps'(X/\mathcal{A})$ the direct product of the analogous groups relative to the simple components of $\mathcal{A}$; this group is therefore always either connected and semisimple, or reduced to $\{1\}$.

29. Here finally are two auxiliary results concerning the parabolic subgroup $P(X/\mathcal{A})$ of $Ps(X/\mathcal{A})$ which was defined in n^o. 51 of [14] and whose structure in cases (I) and (II) was given in n^{os}. 20 and 21 above.

Lemma 11. *$P(X/\mathcal{A}) \cdot Ps'(X/\mathcal{A})$ is a subgroup of finite index of $Ps(X/\mathcal{A})$.*

It suffices naturally to verify this when $\mathcal{A}$ is of type (I) or (II). In case (II), according to n^o. 21, the first group is even equal to the second. When $\mathcal{A}$ is of type (I_0) with $n > 1$, or of type (I_1), (I_3) or (I_4), $Ps'(X/\mathcal{A})$ coincides with $Ps(X/\mathcal{A})$ or is of index 2 in it. When $\mathcal{A}$ is of type (I_0) with

$n = 1$, $P(X/\mathcal{A})$ is of index 2 in $Ps(X/\mathcal{A})$. In case (I_2), let's denote by N the reduced norm taken in $M_{2n}(\mathfrak{k})$ relative to $\mathfrak{z}$, so that $Ps'(X/\mathcal{A})$ is the kernel of the homomorphism of $Ps(X/\mathcal{A})$ into the multiplicative group of $\mathfrak{z}$ which is induced by N; if ν is the index of the image of $P(X/\mathcal{A})$ in that of $Ps(X/\mathcal{A})$ by this homomorphism, ν will also be the index of $P(X/\mathcal{A}) \cdot Ps'(X/\mathcal{A})$ in $Ps(X/\mathcal{A})$. But, according to proposition 8 of [14], n^o. 51, any element of $Ps(X/\mathcal{A})$ belongs to the bilateral class with respect to $P(X/\mathcal{A})$ defined by an element of the form $s_1 = d'(\gamma_1) \otimes e_2$, where $\otimes$ refers to a decomposition into direct sum $X = X_1 \oplus X_2$ and where γ_1 is an isomorphism of X_1^* onto X_1 that can be chosen at will. It is then easy to see that, if X_1 and X_2 are given, we can choose γ_1 in such a way that $N(s_1) = 1$; consequently, in this case, the two groups in question coincide.

Lemma 12. *Let Δ_P be the algebraic module of the group $P = P(X/\mathcal{A})$. Let $g \in Q(X/\mathcal{A})$, $\lambda \in \mathrm{Aut}(X)$ and $p = t(g)d(\lambda)$; and let $\Delta(\lambda)$ be the determinant, with respect to any basis of $Q(X/\mathcal{A})$ over k, of the automorphism $f \to f \circ \lambda$ of $Q(X/\mathcal{A})$. Then we have $\Delta_P(p) = \Delta(\lambda)^{-1}$.*

Indeed, if $p = t(g)d(\lambda)$ is a generic element of P, and if dg, $d\lambda$ are invariant measures on the vector space $Q(X/\mathcal{A})$ and on $\mathrm{Aut}(X)$, respectively, $dp = dg \wedge d\lambda$ is a right-invariant measure on P. The conclusion follows immediately from this and the results of n^o. 4. Of course, $f \circ \lambda$ here denotes the quadratic form $x \to f(x\lambda)$, which was denoted $f^{\lambda^{-1}}$ in [14].

III The Local Problem

30. Before addressing the main subject of this Chapter, we will establish another auxiliary result. Let G be a unimodular algebraic group, defined over k; let P and U be algebraic subgroups of G, also defined over k, such that $P \cap U = \{e\}$ and the Lie algebra of G is the direct sum of P and U. Then $(p, u) \to pu$ is a k-isomorphism of $P \times U$ onto a k-open subset of G (in the sense of Zariski topology); for $g = pu \in P \cdot U$ we can therefore write $p = \varpi(g), u = \sigma(g)$ where ϖ and σ are morphisms of $P \cdot U$ into P and into U, respectively. Let dh, dp, du be left-invariant gauges in G, P and U, respectively. In $P \cdot U$, we can write $dg = f(p, u)dp \wedge du$, where f is a finite and non-zero function everywhere in $P \times U$; we will then have $f(p'p, uu') = f(p, u)\Delta_U(u')^{-1}$, and consequently f differs from Δ_U^{-1} only by a constant factor. If we assume that U is also unimodular, we will have $dg = dp \wedge du$ up to a constant factor which we can assume to be equal to 1.

For any $g_0 \in G$, the set $U_0 = U \cap P \cdot Ug_0$ is a k-open subset of U. For $u \in U_0$, let $p' = \varpi(ug_0^{-1}), u' = \sigma(ug_0^{-1})$; we will have $ug_0^{-1} = p'u'$, from which $p'^{-1}u = u'g_0$ and consequently $p' = \varpi(u'g_0)^{-1}, u = \sigma(u'g_0$ and $u' \in U_0' = U \cap P \cdot Ug_0^{-1}$; it follows that $u \to u'$ is an isomorphism of U_0 onto U_0'. We then have, in $P \cdot U_0$, $d(pug_0^{-1}) = d(pp'u')$, that is,

$$dp \wedge du = d(pp') \wedge du' = \Delta_P(p')dp \wedge du'$$

and consequently

$$du = \Delta_P(p')du'. \tag{17}$$

Suppose now that k is a local field, and also that G is a reductive group and P a **parabolic** subgroup (not necessarily minimal) of G, which means that G/P is isomorphic to a projective variety. To simplify the notation, we will write G, P, U instead of G_k, P_k, U_k; we will denote by $|dg|$, $|dp|$, $|du|$ the Haar measures on these groups derived from the gauges dg, dp, du as stated in n^{o}3 of Chapter I. As usual, for $x \in k$, we will denote by $|x|$ the module of x in k. With these notations, G, P and U are k-analytic varieties, and G/P is a compact k-analytic variety; the same is true for $V = P\backslash G$ of left cosets of P in G.

Lemma 13. *Let ψ be a representation of P in the multiplicative group $\mathbb{R}_+^*$; let $\psi' = |\Delta_P| \cdot \psi$; let f_0 be a continuous function, everywhere > 0 on G, such that $f_0(pg) = \psi(p) f_0(g)$ for any $p \in P$, $g \in G$. Then, for $\int_U f_0(u)|du| < +\infty$, it is necessary that, for any $g_0 \in G$, there exists a neighborhood U' of e in U such that $\psi'(\varpi(ug_0)^{-1})$ is integrable in $U' \cap P \cdot U g_0^{-1}$, and it is sufficient that for all left coset of P in G contains an element g_0 that possesses this property.*

Since $V = P \backslash G$ is compact, it is necessary and sufficient, for f_0 to be integrable on U, that every point of V (i.e., every left coset Pg_0 following P) has a neighborhood V' such that f_0 is integrable in $U \cap V'$; since Ug_0 is transversal to Pg_0 at g_0, we can assume that V' is of the form $V' = P \cdot U' g_0$, where U' is a neighborhood of e in U. The condition we just stated then means that $f_0(u)|du|$ is integrable in $U \cap P \cdot U' g_0$. By making the change of variable $u = \sigma(u' g_0)$, this can also be written, according to (17):

$$\int_{U_1'} f_0(p' u' g_0) |\Delta_P(p')| \cdot |du'| < +\infty$$

with $p' = \varpi(u' g_0)^{-1}$ and $U_1' = U' \cap P \cdot U g_0^{-1}$. If we take into account the hypotheses made on f_0, we obtain the announced result.

It is well know that, if ψ is given, there always exists a function f_0 having the properties stated in Lemma 13 (cf. [4, n^{o}4, prop. 7]). Consequently, if the condition of Lemma 13 is satisfied, and if C is a compact subset of G such that $G = P \cdot C$, the integral $\int_U f(u)|du|$ will be absolutely convergent, uniformly on any set of continuous functions f on G, uniformly based on C and satisfying $f(pg) = \psi(p) f(g)$ for any $p \in P$ and $g \in G$. Indeed. if $|f| \leq M$ on C, and if $m = \inf_C(f_0)$, we will have $|f_0^{-1} f| \leq m^{-1} M$ on C, and thus also at all points of G since $f_0^{-1} f$ is constant on the left cosets of P in G.

31. *Throughout the rest of this chapter, k will denote a local field with characteristic other than 2, A a semisimple algebra over k equipped with an involution, and X a left A-module.*

We will retain the hypothesis and notations from [14, Chap. II, no. 24], from [14, Chap. V, nos. 49-51], and from Chapter II above. For brevity, we will refer Q, Ps, P instead of $Q(\mathcal{X}/\mathcal{A})$, $Ps(\mathcal{X}/\mathcal{A})$, $P(\mathcal{X}/\mathcal{A})$.

In this Chapter, we propose the main application to Proposition 1 of Chapter I, n^{o}2, to the case when X is described as above, where $G = i(x)$, and where we take for f the map i_X of X into $I(X)$. For this, we must first determine in which case i_X satisfies the condition (A) of this proposition.

We will thus set, for $\Phi \in \mathcal{S}(X)$ and $i^* \in I(X)^*$:

$$F_\Phi^*(i^*) = \int_X \Phi(x) \chi([i_X(x), i^*]) |dx|.$$

Identifying $I(X)^*$ with $Q = Q(\mathcal{X}/\mathcal{A})$ as stated in n^{o}15, we can further write, for $q \in Q$:

$$F_\Phi^*(q) = \int_X \Phi(x) \chi(q(x)) |dx|.$$

Saying that i_X satisfies condition (A) is then equivalent to saying that the function F_Φ^* defined is integrable on Q, and is uniformly with respect to Φ on any comapct subset of $\mathcal{S}(X)$; when this is the case, we will say that $\mathcal{A}$ *and* $\mathcal{X}$ *have property (A)* or *satisfying condition (A)*.

For brevity, we will limit ourselves to the case where X is isomorphic to its dual X^*. We will proof among other things the following result:

Proposition 5. *If X is isomorphic to its dual X^*, it is necessary and sufficient, for $\mathcal{A}$ and $\mathcal{X}$ to have property (A), that this be true for each simple component of $\mathcal{A}$ and the corresponding component of $\mathcal{X}$. For $\mathcal{A}$ and $\mathcal{X}$ to have property (A), it suffices that $m \geq 2n + 4\varepsilon - 2$ if $\mathcal{A}$ is of type (I) and $\mathcal{X}$ of rank n, or $m \geq 2p$ if $\mathcal{A}$ is of type (II) and X of rank (p, p).*

Let γ an isomorphism from X^* to X. With the notations from [14, Chap. III], we will have

$$F^*_\Phi(q) = |\gamma|^{\frac{1}{2}} d'(\gamma) t(q) \Phi(0).$$

However, by virture of the definition of the group $\mathrm{Mp}(\mathcal{X}/\mathcal{A})$ and its canonical projection π to $Ps = Ps(\mathcal{X}/\mathcal{A})$ (cf. [14, Chap. III, and Chap. V, n^{o}51]), we can define on Ps a continuous function f_Φ by setting, for all $S \in \mathrm{Mp}(\mathcal{X}/\mathcal{A})$:

$$f_\Phi(\pi(S)) = |Sd'(\gamma)\Phi(0)|.$$

For any element $p = t(q)d(\lambda)$ of P, with $q \in Q$, $\lambda \in \mathrm{Aut}(\mathcal{X})$, let $\psi(p) = |\lambda|^{\frac{1}{2}}_{\mathcal{X}}$, where $|\lambda|_{\mathcal{X}}$ denotes as usual the modulus of the automrphism λ of $\mathcal{X}$. Let us also set $\Phi' = Sd'(\gamma)\Phi$, and $s = \pi(S)$; according to the definition of $t(q)$ and of $d(\lambda)$ in n^{os}13 and 34 of [14], we will have

$$t(q)d(\lambda)\Phi'(x) = |\lambda|^{\frac{1}{2}}_{\mathcal{X}} \Phi(x\lambda)\chi(q(x))$$

and consequently

$$f_\Phi(ps) = |t(q)d(\lambda)\Phi'(0)| = |\lambda|^{\frac{1}{2}}_{\mathcal{X}} \cdot |\Phi'(0)| = \psi(p) f_\Phi(s).$$

Let U be the subgroup of Ps formed by the elements $t'(q')$ with $q' \in Q' = Q(\mathcal{X}^*/\mathcal{A})$; for $G = pS$, the subgroups P and U indeed have the properties stated above in n^{o}30. Moreover, $q \mapsto -q \circ \gamma$ is an isomorphism from Q to Q', so that $q \mapsto t'(-q \circ \gamma)$ is an isomorphism from Q to U. We can easily verify that, for any $q \in Q$:

$$t'(-q \circ \gamma) = d'(\gamma) t(q) d'(\gamma)^{-1}.$$

It follows that, under these conditions:

$$|F^*_\Phi(q)| = |\gamma|^{\frac{1}{2}} f_\Phi(t'(-q \circ \gamma)),$$

which shows that the integrability of F^*_Φ on Q is equivalent to that of f_Φ on U. Taking into account the final remark of n^{o}30, we see that the condition (A) for $\mathcal{A}$ and $\mathcal{X}$ is equivalent to the condition of Lemma 13 of n^{o}30 for the group $G = Ps$, P and U and the representation ψ of P as we have just defined them.

32. The module Δ_P of P, which appears in the condition of Lemma 13, is given by Lemma 12 of n^{o}29. On the other hand, in this condition, we can take as representation of the left classes following P of those given by Proposition 8 of [14, n^{o}51]. According to this, to obtain such representatives, we must choose in all possible ways a submodule X_1 of X which is isomorphic to its dual X_1^*; for each X_1, we choose an isomorphism γ_1 from X_1^* to X_1 and a complement X_2 of X_1 in X; finally we choose, in all possible ways, an element q_1 of $Q(\mathcal{X}_1/\mathcal{A})$, and, for each q_1, we write

$$s_0 = d'(\gamma_1)t(q_1) \otimes e_2. \tag{18}$$

The s_0 form the system of representatives in the question.

With the notations of n^{o}30, it remains to calculate $\psi'(p')$ for $p' = \varpi(us_0)^{-1}, u \in U \cap P \cdot Us_0^{-1}$, s_0 being given by (18). Since $u \in U$, we can write it in the form $u = t'(q')$ with $q' \in Q'$. By definition of ϖ, there will then be $q'' \in Q'$ such that we have

$$t'(q')s_0 = p'^{-1}t'(q''). \tag{19}$$

Let us write $p' = t(q)d(\lambda)$ with $q \in Q$, $\lambda \in \mathrm{Aut}(X)$, and let us explain the elements of the group $\mathrm{Sp}(\mathcal{X}/\mathcal{A})$ that correspond the two terms in (19). If we denote

$$\sigma_0 = \begin{pmatrix} \alpha_0 & \beta_0 \\ \gamma_0 & \delta_0 \end{pmatrix}$$

for the one corresponding to s_0, and let ρ' denote the symmetric morphism from X^* to X associated with the quadratic form q' on X^*, we thsu find that (19) implies in particular the relation $\lambda^* = \rho'\beta_0 + \delta_0$.

Let us also denote by ρ_1 the symmetric morphism from X_1 to X_1^* associated with q_1; for the decompositions $X = X_1 \oplus X_2$, $X^* = X_1^* \oplus X_2^*$ of X and X^* in direct sums, β_0 and δ_0 are written in matrix form as follows:

$$\beta_0 = \begin{pmatrix} -\gamma_1^{*-1} & 0 \\ 0 & 0 \end{pmatrix}, \quad \delta_0 = \begin{pmatrix} \gamma_1\rho_1 & 9 \\ 0 & 1 \end{pmatrix}.$$

If we write ρ' in matrix form for the same decomposition of X and X^*, we will have, since ρ' is symmetric:

$$\rho' = \begin{pmatrix} \rho_1' & \tau \\ \tau^* & \rho_2' \end{pmatrix} \tag{20}$$

where ρ_1', ρ_2' are symmetric morphisms of X_1* to X_1 and of X_2^* to X_2, respectively, and where τ is an morphism from X_1^* to X_2. We will then have

$$\lambda^* = \begin{pmatrix} \gamma_1\rho_1 - \rho_1'\gamma_1^{*-1} & 0 \\ -\tau^*\gamma_1^{*-1} & 1 \end{pmatrix},$$

or, which amounts to the same, and by further setting $\overline{\rho}_1 = \gamma_1\rho_1\gamma_1^*$:

$$\lambda = \begin{pmatrix} \gamma_1^{-1}(\overline{\rho}_1 - \rho_1') & -\gamma_1^{-1}\tau \\ 0 & 1 \end{pmatrix} \tag{21}$$

Here $\overline{\rho}_1$ is the symmetric morphism from X_1^* to X_1 associated with the quadratic form $\overline{q}_1 = q_1 \circ \gamma_1$ on X_1^*, so that $\overline{\rho}_1 - \rho_1'$ is a symmetric morphism from X_1^* to X_1; if this is an isomorphism, (21) defines an automorphism λ of X_1. Moreover, when this last condition is satisfied, it is easy to see, by applying propositio 1 of [14, n^{o}7], to $s = d(\lambda)t'(q')s_0d'(\gamma)$, that we can determine q and q'' in a way to satisfy (19), which shows that then $u = t'(q')$ belongs to $P \cdot Us_0^{-1}$.

Still with the same notations, we then have, accodring to what precedes:

$$\psi'(p') = \psi'(\varpi(us_0)^{-1}) = |\lambda|_X^{\frac{1}{2}}|\Delta(\lambda)|^{-1}. \tag{22}$$

Therefore, for (A) to be satisfied, it is necessary and sufficient that this expression, considered as a function of q', is integrable in the neighborhood of 0 in Q', and this for all s_0 of the form (18).

33. This question obviously relates to analogous questions concerning the simple components of $\mathcal{A}$; this proves the first assertion of Proposition 5. Let us now suppose that $\mathcal{A}$ is of type (I); then, with our usual notation, $\mathrm{Aut}(\mathcal{X})$ is identified with $\mathrm{GL}(n, \mathfrak{k})$, that is to say, the group of invertible elements of $M_n(\mathfrak{k})$. We know that any character of this group, rational over k (in the sense of the theory of algebraic groups) is of the form $\lambda \mapsto v_n(\lambda)^N$, where N is an integer and v_N is the reduced norm taken in $M_n(\mathfrak{k})$ relative to k. This will therefore be the case, in particular, for $\Delta(\lambda)$, and also for the determinant $D(\lambda)$ of the automorphism λ of X, taken with respect to any basis of X over k. To complete the determination of these, it suffices to calculate them, as well as $v_n(\lambda)$, in the case where λ is of the form $t \cdot 1$, with $t \in k, t \neq 0$. If δ denotes as always the dimension of $\mathfrak{k}$ over k, and α^2 that of $\mathfrak{k}$ over its center $\mathfrak{z}$, we will have, for this choice of λ:

$$v_n(\lambda) = t^{\delta n/\alpha}, \quad D(\lambda) = t^{\delta mn},$$

and, according to Lemma 9 of n^{o}19:

$$\Delta(\lambda) = t^{\delta n(n+2\varepsilon-1)}.$$

We therefore have, for any $\lambda \in \mathrm{Aut}(\mathcal{X})$:

$$D(\lambda) = v_n(\lambda)^{\alpha m}, \quad \Delta(\lambda) = v_n(\lambda)^{\alpha(n+2\varepsilon-1)}.$$

As we have elsewhere $|\lambda|_{\mathcal{X}} = |D(\lambda)|$, (22) can therefore be written:

$$\psi'(p) = |v_n(\lambda)|^{\alpha(m-2n-4\varepsilon+2)/2}.$$

But λ is given by (21) each time (21) defines an automorphism of $\mathcal{X}$. Let $\rho = \widetilde{\rho}_1 - \rho_1'$; if n_1 is the rank of the module $\mathcal{X}_1$, ρ is an η-Hermitian matrix with n_1 rows and n_1 columns, and we have:

$$v_n(\lambda) = v_{n_1}(\gamma_1)^{-1} \cdot v_{n_1}(\rho).$$

We have to express that $\psi'(p')$, considered as a function of q', or, which amounts to the same thing, of ρ', or, which again amounts to the same thing by (20), of (ρ_1', ρ_2', τ), is integrable in the neighborhood of 0 in the complement of the set of points where $v_{n_1}(\rho) = 0$. This is equivalent to saying that $\psi'(p')$, considered as a function of ρ, is integrable in this complement in the neighborhood of $\widetilde{\rho}_1$. In other words, we have proof the following:

Lemma 14. *Let $\mathcal{A}$ be of type (I), and $\mathcal{X}$ of rank n. For $\mathcal{A}$ and $\mathcal{X}$ to have property (A), it is necessary and sufficient that, for all $n_1 \leq n$, the function*

$$\varphi(\rho) = |v_{n_1}(\rho)|^{\alpha(,-2n-4\varepsilon+2)/2}$$

be locally integrable in the neighborhood of all points in the space of η-hermitian matrices with n_1 rows and n_1 columns over $\mathfrak{k}$.

As this function is everywhere continuous for $m \geq 2n+4\varepsilon-2$, the second assertion of Proposition 5 is contained in this result.

Now let $\mathcal{A}$ be of type (II), and X of rank (p,p). With our usual notations, $\mathcal{X}$ is identified with the direct sum of $M_{m,p}(\mathfrak{k})$ and of $M_{m,p}(\mathfrak{k})$, and consequently $\mathrm{Aut}(\mathcal{X})$ with

$$\mathrm{GL}(p, \mathfrak{k}) \times \mathrm{GL}(p, \mathfrak{k}').$$

Let v_p, v'_p be the reduced norms taken relative to k in $M_p(\mathfrak{k})$ and $M_p(\mathfrak{k})$, respectively. For the same reason as above, we see easily that, if $\lambda = (v, w)$ belongs to $\mathrm{Aut}(X)$, we have:

$$|\lambda|_X = |v_p(v)v'_p(w)|^{\alpha m}, \quad |\Delta(\lambda)| = |v_p(v)v'_p(w)|^{\alpha n},$$

where we have again denoted by α^2 the dimension of $\mathfrak{k}$ over its center. Proceeding then as in the case (I), we obtain the following result:

Lemma 15. *Let $\mathcal{A}$ be of type (II), and $\mathcal{X}$ of rank (p, p). For $\mathcal{A}$ and $\mathcal{X}$ to have property (A), it is necessary and sufficient that, for all $p_1 \leq p$, the function*

$$\varphi(r) = |v_{p_1}(r)|^{\alpha(m-2p)}$$

be locally integrable in the neighborhood of every point in the space $M_{p_1}(\mathfrak{k})$.

The last assertion of Proposition 5 is a particular case of this lemma.

34. In fact, a slightly more detailed analytis would show that, under the condition of Lemma 14, the function $|v_n(\rho)|^{\alpha s}$ is locally integrable in the neighborhood of any point, in the space of η-hermitian elements of $M_n(\mathfrak{k})$, whenever $s > -\varepsilon$ if $\varepsilon \neq 0$, whenever $s? - \frac{1}{2}$ if $\varepsilon = 0$, and that it ceases to be so in the neighborhood of 0 if s does not satisfy this condition. Similarly, under the condition of Lemma 15, the condition for local integrability of $|v_p(r)|^{\alpha s}$ is $s > -1$. Finally, if we cease to assume that X is isomorphic to its dual, we find that the first asseration of Proposition 5 remains truel and, in case (II), we find that (A) is equivalent to $m \geq p + q$ if $\mathcal{X}$ is of rank (p, q).

Let us agree to say that $\mathcal{A}$ and $\mathcal{X}$ satisfy condition (A') if $\varepsilon = 0$ and $m > 2n - 3$ or if $\varepsilon \neq 0$ and $m > 2n + 2\varepsilon - 2$ in case (I), if $m \geq p + 1$ in case (II), and, in the general case, if these last conditions are satisfied by each simple component of $\mathcal{A}$ and by the corresponding component of $\mathcal{X}$. It follows from what we have just said that in fact (A) is always equivalent to (A'). Moreover, if $\mathcal{A}$ is of type (I), (A') differs from the sufficient condition provided by Proposition 5 only in cases $(I_3), (I_4)$; (A') gives $m \geq 2n$ in case (I_3) and $, \geq 2n + 1$ in the case (I_4). while Proposition 5 would only give $m \geq 2n + 1$ and $m \geq 2n + 2$, respectively.

35. In the rest of this Chapter, we will assume once and for all that condition (A) is satisfied, that is to say, we can apply Proposition 1 of n^{o}2 to i_X. We can therefore, in one and only one way, define for all $i \in I(\mathcal{X})$ a measure μ_i, with support contained in $i_X^{-1}(\{i\})$, such that, for any continuous function Φ with compact support on $\mathcal{X}$, the function $F_\Phi(i) = \int \Phi d\mu_i$ is continuous on $I(X)$ and satisfies $\int \Phi |dx| = \int F_\Phi |di|$.

With the notatinos of Chapter II, n^{o}24, let $u \in G$, $v \in \mathrm{Aut}(\mathcal{X})$. We have $i_X(uxv) = i_X(x)\overline{v}$, where $\overline{v}$ is the automorphism of $I(X)$ deduced from v by transport of structure. In the formula

$$\int_X \Phi(x)|dx| = \int_{I(X)} \left(\int \Phi(x) d\mu_i(x) \right) |di|,$$

let's replace Φ with the function Ψ given by

$$\Psi = \Phi(u^{-1}xv^{-1});$$

then replace x by uxv in both members, and i by $i\overline{v}$ in the right hand side. This gives

$$|u|_X \cdot |v|_X \int_X \Phi(x)|dx| = |\overline{v}| \int_{I(X)} \left(\int \Phi(x) d\mu_{i\overline{v}}(uxv) \right) |di|,$$

where $|u|_X$, $|v|_X$, $|\overline{v}|_{I(X)}$ denote of course the modules of $u, v, \overline{v}$ in X, X and $I(X)$, respectively. By the virtue of the uniqueness of the μ_i recalled above, we conclude:

$$d\mu_{i\overline{v}}(uxv) = |u|_X \cdot |v|_X \cdot |\overline{v}|_{I(X)}^{-1} \cdot d\mu_i(x). \tag{23}$$

In particular, for all $i \in I(X)$, μ_i is relatively invariant under $G \times \mathrm{Aut}(X, i)$, and we have, for $\mu = \mu_i, u \in G, v \in \mathrm{Aut}(X, i)$:

$$d\mu(uxv) = |u|_X \cdot |v|_X \cdot |\overline{v}|_{I(X)}^{-1} \cdot d\mu(x). \tag{24}$$

But the Corollary of Proposition 4 of Chapter II, n^{o}25, shows taht $G \times \mathrm{Aut}(X, i)$ has only finite number of orbits in $i_X^{-1}(\{i\})$; this allows us to apply the following elementary remark, which we will formulate as a lemma:

Lemma 16. *Let G a locally compact group, countable at infinity, operating on a locally compact space E; suppose that G has only a finite number of distinct orbits E_α in E. Then each orbit E_α is relatively open in its closure and is isomorphic to the quotient G/g_α of G by the stabilizer g_α of one of its points; and any measure μ on E can be written in one and only one way as a sum of measures μ_α respectively supported by the orbits E_α.*

We will proveed by recurrence on the number N of orbits E_α. Let a_α, for $1 \leq \alpha \leq N$, be representatives of theses orbits. Let V be a compact neighborhood of e in G; ny hypothesis, G admits a covering by a countable family of sets of the form $g_i V$; the sets $g_i V a_\alpha$ thus form a covering of E. As all locally compact space has the **Baire property**, it follows from that at least one of the $g_i V a_\alpha$ has an interior point; for such value α, the orbit $E_\alpha = G a_\alpha$ is thus open in E, and even so in its closure in E. The same reason then shows that, if V' is a compact open neighborhood of e in V, $V' a\alpha$ has a interior point; thus $x \to x a_\alpha$ is a open map of G onto E_α, and, if g_α is the stabilizer of a_α, E_α is isomorphic to G/g_α. To obtain the first part of the lemma, one only needs to apply the recurrence hypothesis to $E' = E - E_\alpha$, which is closed in E and is the union of $N-1$ orbits of G. Finally, let μ be a measure on E, μ_α its restriction to the open orbit E_α, and $\mu' = \mu - \mu_\alpha$; the support of μ' being contained in E', we can apply the recurrence hypothesis to it, hence the last conclusion.

Applying Lemma 16 to the measure μ_i and to the orbits of the group $G \times \mathrm{Aut}(X, i)$ in $i_X^{-1}(\{i\})$, we see that we can write μ_i as a sum of measures supported by these orbits respectively; it then follows from the uniqueness of this decomposition that each of these latter measure satisfies, like μ_i, the condition (24).

36. We will now finish the determing μ_i by showing that at most one of these orbits can carry such measure. For this, let us introduce another definition. Suppose that $\mathcal{A}$ is of type (I) or (II); we will agree to say that an element x of X is *of quasimaximal rank* if it is of maximal rank, or if A is of type (I_0), if $m = 2n - 2$, if $i_X(x) = 0$ and if x is of rank $n - 1$. In generael, we will say that an element x of X is *of quasimaximal rank* if, for each of the simple components of $\mathcal{A}$, the corresponding component if x is of quasimaximal rank. For any $i \in I(X)$, we will denote by $\overline{U}(i)$ the set of elements of $i_X^{-1}(\{i\})$ that are of maximal rank in X. We observe that, if $\mathcal{A}$ is of type (I) and if element x of X is of maximal rank n and satisfies $i_X(x)$, we have $m \geq 2n$; this is a special case of the results in n^{o}23. We immediately conclude that, first in the cases (I) and (II), then in the general case, that all elements of $\overline{U}(i)$ have the same rank for a given i; according to the corollary of Proposition 4 of Chapter II, n^{o}25, $\overline{U}(i)$ is therefore either empty or an orbit of $G \times \mathrm{Aut}(X, i)$.

Lemma 17. *Suppose that $\mathcal{A}$ and X satisfy the condition (A') of n^{o}34; let $i \in I(X)$, and let U an orbit of $G \times \mathrm{Aut}(X, i)$ in $i_X^{-1}(\{i\})$ on which there exists a measure $\mu \neq 0$ satisfying (24). Then $U = \overline{U}(i)$.*

Let x_0 a point of U, such that $i = u_X(x_0)$; let $n = (n_v)$ and $n' = (n'_v)$ be the ranks of X and of x_0, respectively. Resuming the notations of Chapter II, n^{o}25, let us denote by N_0 the kernel of x_0 in B, by B' that of $\lambda - i$, by B'' a complement of B' in B, and by g' the group of automorphism of B that keeps B' and B'' invariant and induce the identity on B''. Then N_0 is a submodule of B', and, according to Proposition 4 of n^{o}5 and its corollary, U is the orbit of x_0 under $G \times g'$. Let G_0 be the stabilizer of x_0 in $G \times g'$. As G and g' are unimodular, so is $G \times g'$; according to known theorem on relatively invariant measures in homogeneous spaces (cf. [4],§2,n^{o}5, th3), all measures on U, relatively invariant under $G \times g'$, must satisfy, for all $(u, v) \in G_0$, the formula

$$d\mu(uxv^{-1}) = |\Delta_{G_0}(u, v)| d\mu(x). \tag{25}$$

Let g'' the group of elements v of g' such that $(1, v) \in G_0$, or in other words, such that $x_0 = x_0 v$; if B'_0 is a complement of N_0 in B', we can also define g'' as formed of the automorphisms v of B which, for the direct sum decomposition $B = N_0 \oplus B'_0 \oplus B''$ of B, can be written in matrix form as follows:

$$v = \begin{pmatrix} v'' & w & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{26}$$

with $v'' \in \mathrm{Aut}(N_0)$, $w \in \mathrm{Hom}(B'_0, N_0)$. Let G'' be the subgroup of G_0 formed by the elements of the form $(1, v)$, with $v \in g''$; for v given by (26), let $U''(v'', w)$ denote the element $(1, v)$ of G''.

On the other hand, let G' be the group of elements (u, v) of G_0 such that v induces the identity on N_0 and leaves B'_0 invariant. If $U_0 = (u, v)$ belongs to G_0, we have $ux_0v^{-1} = x_0$, which implies that v leaves N_0 invariant; we then easily see that we can, in one and only one way, determine $v_1 \in g''$ such that $v \cdot v_1^{-1}$ induces identity on N_0 and leaves B'_0 invariant; it amounts to the same to say that U_0 can be written in one and only one way in the form $U_0 = U'U''$, with $U' \in G'$, $U'' \in G''$. We thus have $G_0 = G' \cdot G''$, and G_0 is the semidirect product of G' and G''. Let dU' be a left-invariant gauge on G', and dv'', dw be invariant gauges on $\mathrm{Aut}(N_0)$ and on the additive group $\mathrm{Hom}(B'_0, N_0)$, respectively. If we write $U_0 = U' \cdot U''(v'', w)$ and set

$$\omega(U_0) = dU' \wedge dv'' \wedge dw, \tag{27}$$

we immediately see that $\omega(U_0)$ is a left-invariant gauge, and thus also right-invariant, on G_0, by means of which we can calculate Δ_{G_0} as stated in n^{o}4 of Chapter I.

Suppose now that we have $n'_v < n_v$ for a certain value of v. Let's use the notations from n^{os}16,17 and 25; let's also denote by N_v, B_v, B'_v, B''_v the components relative to $\mathfrak{k}_v$ of N_0, B, B'_0 and B'', respectively; these are right vector space over $\mathfrak{k}_v$; B_v is the direct sum of N_v, B'_v and B''_v; B_v is of dimension n_v, and N_v of dimension $n_v - n'_v$, over $\mathfrak{k}_v$. For $t \in k$, $t \neq 0$, let v''_t be the automorphism of N_0 that induces $t \cdot 1$ on N_v and identity on all other components of N_0; let v_t the automorphism of B that induces v''_t on N_0 and identity on B_0 and on B''; $(1, v_t)$ belongs to G''. We will apply (24) with $u = 1$, $v = v_t^{-1}$, and (25) with $u = 1$, $v = v_t$, and compare the results. As $(1, v_t)$ is permutable with any elements of G'' and of dv'' is invariant on the right and left in $\mathrm{Aut}(N_0)$, we have, with the notation of (27):

$$\omega((1, v_t)^{-1} \cdot U_0 \cdot (1, v_t)) = dU' \wedge dv'' \wedge d(v_t^{-1} w).$$

Taking into accouont the results of n^{o}4 of Chapter I, we will have, by setting $r = n_v - n'_v$ and denoting by s the dimensino of B'_v over $\mathfrak{k}_v$:

$$\Delta_{G_0}(1, v_t) = t^{-\delta ts},$$

where δ is the dimension of $\mathfrak{k}_v$ over k.

Let's first consider the case where $\mathcal{A}_v$ is of type (I); using our usual notations for this type, let's write m, b instead of m_v, n_v, and b' instead of n'_v, hence $r = n - n'$. It is immediate that we have:

$$|v_t|_X = |t|^{\delta rm}, |\overline{v}_t|_{I(X)} = |t|^{\delta r(n+2\epsilon-1)}.$$

The comparison of (24) and of (25) then gives

$$m = n + 2\varepsilon - 1 + s.$$

As $s \leq n' \leq n01$, we conclude $m \leq 2n + 2\varepsilon - 2$, which contradicts hypothesis (A') if $\varepsilon \neq 0$. If $\varepsilon = 0$, (A') gives $m \geq 2n - 2$; we must therefore have $m = 2n - 2$, $s = n' = n - 1$, $r = 1$, so that $N_v \oplus B'_v$ is of dimension n and coincide with B_v. As $N_v \oplus B'_v$ is the space of vectors of B_v orthogonal to B_v with respect to the η-hermitian form determined by the component i_v of i relative to $\mathcal{A}_v$, we conclude that $i_v = 0$, which complete the proof that the component of x_0 relative to $\mathcal{A}_v$ is of quasimaximal rank in the sense of the definition given above.

Finally, suppose that the involution ι transform $\mathcal{A}_v$ into $\mathcal{A}_{v'}$, with $v' \neq v$, so that $\mathcal{A}_v \oplus \mathcal{A}_{v'}$ is of type (II); using our usual notations for this type, let's write m, p, q instead of m_v, n_v, $n_{v'}$, and p' instead of n'_v, hence $r = p - p'$. We then have

$$|v_t|_X = |t|^{\delta rm}, \quad |\overline{v}_t|_{I(X)} = |t|^{\delta rq}.$$

The comparison of (24) and of (25) implies that $m = q + s$. As we have $s \leq p' \leq p - 1$, this contradicts (A'), this finish the proof.

37. If (A) and (A') are both satisfied, it follows from n^{o}35 and Lemma 17 that, for all $i \in I(X)$, the measure μ_i is carried by the orbit $\overline{U}(i)$. The Lemma 9 of n^{o}19 and 10 of n^{o}21 show moreover that i_X is submersive at every point of $\overline{U}(i)$; and it is immediate, first in cases (I) and (II), then in general, that $\overline{U}(i)$ is alwayas an open subset of $i_X^{-1}(\{i\})$. According to n^{o}6 of Chapter I, μ_i coincides with the measure $|\theta_i|$ determined on $\overline{U}(i)$ by the gauge

$$\theta_i(x) = \left(\frac{dx}{di_X(x)}\right)_i, \tag{28}$$

this formula having the meaning explained in n^{o}6 of Chapter I; in other words, for any $x_0 \in \overline{U}(i)$, $\theta_i(x)$ coincides in the neighborhood of x_0 with the form induced on $\overline{U}(i)$ by any differential form $\eta(x)$ in X satisfying in the neighborhood of x_0 the relation $dx = \eta(x) \wedge di_X(x)$.

We can summarize the above results as follows:

Proposition 6. *Suppose that $\mathcal{A}$ and X satisfies (A) and (A'); let $i \in I(\mathcal{X})$; let $\overline{U}(i)$ be the set of points of $i_X^{-1}(\{i\})$ of quasimaximal rank in X, and let θ_i be the gauge on $\overline{U}(i)$ defined by (28). Then the measure $|\theta_i|$ on $\overline{U}(i)$ determines a temper measure in $\mathcal{X}$. Moreover, for all $\Phi \in \mathcal{S}(\mathcal{X})$, the function F_Φ, F^*_Φ respectively defined on $I(\mathcal{X})$ and on $I(\mathcal{X})^*$ by the formulas*

$$F_\Phi(i) = \int_{\overline{U}(i)} \Phi(x)|\theta_i(x)|, \quad F^*_\Phi(i^*) = \int_X \Phi(x)\chi([i_X(x), i^*])|dx|$$

are continuous, integrable, and are the Fourier transform of each other; in particular, we have $\int \Phi|dx| = \int F_\Phi|di|$.

In reality, as announced in n^{o}34, (A) is equivalent to (A'); but, in the following chapters, it will suffices for us to know that one and the other of these conditions are consequences of the sufficient conditions stated in Proposition 5 of n^{o}31.

38. When the hypotheses of Proposition 6 are verified, it shows in particular that we can only have $\mu_i = 0$, for i given in $I(\mathcal{X})$, if $\overline{U}(i)$ is empty; it is often important to know in which cases this circumstance can happen; in this regard, we will content ourselves with stating the following results, which in substance are well known. Let us first agree to say that $\mathcal{A}$ and X *satisfy condition* (B) if $m > 2n + 4\varepsilon - 2$ in the case (I), if $m > p + q$ in the case (II), and in general cse, if these conditions are satisfied by each simple component of $\mathcal{A}$ and by the corresponding component of $\mathcal{X}$; let us also agree to say that $\mathcal{A}$ and X *satisfy condition* (B') if $, \geq 2n + 4\varepsilon - 2$ in the case (I), if $p = q$ and $m \geq p + 1$ in the case (II), and, in the general case, if these conditions are satisfied by each simple component of $\mathcal{A}$ and by the corresponding component of X; (B') is nothing other than the sufficient condition of Proposition 5 of n^{o}31, which, according to this proposition, implies (A), and which obviously also implies (A').

This being established, it has already been observed in n^{o}23 of Chapter II that, in the case (II), the condition $, \geq p + q$, i.e., (B'), is necessary and sufficient for $U(0)$ to be non-empty, and sufficient for $U(i)$ to be non-empty whatever $i \in I(\mathcal{X})$. In the case (I_0), it is shown elementary that $, \geq 2n - 2$, i.e., again (B'), is necessary and sufficient for $U(0)$ to be non-empty, and that then $U(i)$ is non-empty whatever $i \neq 0$; it would be so even if we did not assume that k is a local field. In the case (I) with $\varepsilon \neq 0$, we know, for $n = 1$ (cf. for example [9]), and we easily verify from there by recurrence on n, in the general case, that $U(i)$ is never empty when condition (B) is satisfied, provided that k is a local field with *discrete valuation.* Consequently, if k is a local field with discrete valuation, (B) always implies $U(i) \neq$ for any $i \in I(\mathcal{X})$.

Let us finally recall that, if $\mathcal{A}$ is of type (I) on the local field k and X is of rank $n = 1$, the condition $U(0) \neq \emptyset$ is equivalent to the compactness of the group G; G is never compact in case (II).

IV The Siegel Eisenstein Series

39. *From now on, we will assume once and for all that the base field k is a field of algebraic numbers*, and we will adopt the hypotheses and notations of [14], Chapter II and V, as well as Chapter I and II above; in particular, $\mathcal{A}_k$ will always denote a semisimple algebra over k, and X_k a left $\mathcal{A}_k$-module.

Let $\Phi \in \mathcal{S}(X_\mathcal{A})$; the group $\mathrm{Mp}(X/\mathcal{A})_\mathcal{A}$ operates on $\mathcal{S}(X_\mathcal{A})$ as explained in [14], Chapter III and V, so that $S\Phi$ is defined for all $S \in \mathrm{Mp}(X/\mathcal{A})_\mathcal{A}$; in particular, for $s \in P_s(X_k/\mathcal{A}_k)$, $\mathbf{r}_k(s)\Phi$ is defined, and is even given explicitly by formula (38) of [14], Chapter III, n^{o}40.

By means of this formula, or directly, we immediately see that $\mathbf{r}_k(p)\Phi(0) = \Phi(0)$ for any $p \in P(X_k/\mathcal{A}_k)$; it follows that, for a given Φ, $\mathbf{r}_k(p)\Phi(0)$ is constant on the left classes according to $P(X_k/\mathcal{A}_k)$ in $P_s(X_k/\mathcal{A}_k)$. If R is a system of representatives of these classes, the series

$$E(\Phi) = \sum_{s \in R} \mathbf{r}_k(s)\Phi(0), \tag{29}$$

which we will call the *Eisenstein-Siegel series* relative to $\mathcal{A}_k$ and X_k, is therefore independent of the choice of R. Its study will form the main object of this chapter.

The primary task will be to provide a sufficient condition for the convergence of the series $E(\Phi)$. Let us first observe that the question does not arise when $\mathcal{A}$ is of type (II) and when $p = 0$ or $q = 0$, since then, according to n^{o}21, $P(X_k/\mathcal{A}_k)$ coincide with $P_s(X_k/\mathcal{A}_k)$, so that $E(\Phi)$ reduces to $\Phi(0)$, nor when $\mathcal{A}$ is of type (I_0) with $n = 1$, since then, as noted in n^{o}29, $P(X_k/\mathcal{A}_k)$ is of index 2 in $P_s(X_k/\mathcal{A}_k)$.

To address the question in the general case, we will follow a method analogous to taht used in the treat the **local problem** in n^{o}31. For brevity, let us write again P_s, P, Q instead of $P_s(X/\mathcal{A})$, $P(X/\mathcal{A})$, $Q(X/\mathcal{A})$, so that the series $E(\Phi)$ is extended to P_k/P_{s_k}; π denoting this time the canonical projection of $\mathrm{Mp}(X/\mathcal{A})_\mathcal{A}$ to $P_{s_\mathcal{A}}$, we will define a continuous function f_Φ on $P_{s_\mathcal{A}}$ by setting, for all $S \in \mathrm{Mp}(X/\mathcal{A})_\mathcal{A}$:

$$f_\Phi(\pi(S)) = |S\Phi(0)|.$$

For any element $p = t(q)d(\lambda)$ of $P_\mathcal{A}$, with $q \in Q_\mathcal{A}$, $\lambda \in \mathrm{Aut}(X)_\mathcal{A}$, let $\psi(p) = |\lambda|_X^{\frac{1}{2}}$, where $|\lambda|_X$ denotes the module of the automorphism λ of $X_\mathcal{A}$. For exactly the same reason as we did in n^{o}31 in the localcase, we see that $f_\Phi(ps) = \psi(p)f_\Phi(s)$ for any $p \in P_\mathcal{A}$ and $s \in P_{s_\mathcal{A}}$. For $\lambda \in \mathrm{Aut}(X)_k$, λ leaves X_k invariant, which is a discrete subgroup of $X_\mathcal{A}$ with compact quotient, so that $|\lambda|_X = 1$; we therefore have $\psi(p) = 1$ for $p \in P_k$.

Lemma 18. *Let f_0 be a continuous function, everywhere > 0 on $P_{s_\mathcal{A}}$, such that $f_0(ps) = \psi(p)f_0 9s)$ for any $p \in P_\mathcal{A}$ and $s \in P_{s_\mathcal{A}}$. Also, for the series $E(\Phi)$ to be absolutely convergent for any $\Phi \in \mathcal{S}(X_\mathcal{A})$, it is necessary and sufficient that the series $\sum_R f_0(s)$ be convergent; and, when this is the case, $E(\Phi)$ is absolutely convergent, uniformly with respect to Φ on any compact subset of $\mathcal{S}(X_\mathcal{A})$.*

As P is a parabolic subgroup of P_s, the space $P_A \backslash P_{s_\mathcal{A}}$ of left classes according to P_A in $P_{s_\mathcal{A}}$ is compact, and isomorphic to $(P\backslash P_s)_A$; let C_0 be a compact subset of P_{s_A} such that $P_{s_A} = P_A \cdot C_0$. For the same reason as we did in the local case at the end of n^{o}30, we immediately see that we can, for any compact subset C of $\mathcal{S}(X_A)$, there exists a corresponding $c > 0$ such that $f_0^{-1}f_\Phi \leq c$ on P_{s_A} for any $\Phi \in C$; therefore, if $\sum_R f_0(s)$ is convergent, $E(\Phi)$ is absolute and uniformly convergent on C. As for the converse, let Φ be such that $\Phi(0) \neq 0$, then $f_\Phi(e) \neq 0$; by virtue of the compactness of C_0, we can choose $s_i \in P_{s_A}$ and $c_i > 0$, finite in number, such that $f_0(s) \leq \sum c_i f_\Phi(ss_i)$ for any $s \in C_0$, and consequently for any $s \in P_{s_A}$. For each i, let $S_i \in \mathrm{Mp}(X/\mathcal{A})_A$ such that $\pi(S_i) = s_i$, and $\Phi_i = S_i\Phi$; we will have $f_\Phi(ss_i) = f_{\Phi_i}(s)$. The absolute convergence of the series $E(\Phi_i)$ therefore implies that of $\sum_R f_0(s)$.

Similar in the local case, there always exists functions f_0 having the properties stated in Lemma 18. We can immediately conclude from Lemma 18, in particular, that the absolute convergence of $E(\Phi)$ for any $\Phi \in \mathcal{S}(X_{\mathcal{A}})$, for given $\mathcal{A}$ and X, is equivalent to the conjunction of analogous conditions relative to each of the simple components of $\mathcal{A}$ and the corresponding component of X. In other words, it suffices to discuss the convergence of $E(\Phi)$ when $\mathcal{A}$ is of type (I) or (II). Moreover, by means of the **restriction of base field** (cf. [13], Chap. I), we see that, in both of these cases, we can always $\mathfrak{z}_0 = k$; for the analogous reasons, we will almost always make this hypothesis in the rest of this work.

40. We will now establish the following theorem:

Theorem 1. *For the series $E(\Phi)$, defined by (29), of n^{o}39, to be absolutely convergent regardless of $\Phi \in \mathcal{S}(X_A)$, and uniformly so on any compact subset of $\mathcal{S}(X_A)$, it suffices that $\mathcal{A}_k$ and X_k satisfy condition (B) of n^{o}38; in particular, it suffices for this that $m > 2n + 4\varepsilon - 2$ if $\mathcal{A}_k$ is of type (I), and that $m > p + q$ if $\mathcal{A}_k$ is of type (II).*

By what precedes, we can limit ourselves to considering cases (I) and (II), assuming $\mathfrak{z}_0 = k$; we will further exclude the trivial case where $\mathcal{A}_k$ is of type (II) and where $p = 0$ or $q = 0$, since then $E(\Phi)$ reduces to $\Phi(0)$, and also the case (I_0) with $n = 1$, where $E(\Phi)$ comprises only two terms. In all other cases, let us consider again the subgroup $P_{s'} = P_{s'}(X/\mathcal{A})$ of $P_s(X/\mathcal{A})$ that we introduce in n^{o}28; let $P' = P_{s'} \cap P$, and let R' be a system of representatives of the left classes of P'_k in $P_{s'_k}$. Then $P_{s'}$ is is connected and semisimple, P' is a parabolic subgroup of $P_{s'}$, and the Lemma 11 of Chapter II, n^{o}29, shows that we can take for R the union of a finite number of sets of the form $R'r$; it turns out that this number is equal to 2 in the case (I_0) and to 1 in all other cases, but this is of little importance here. However, if r is given in P_{s_k}, and if f_0 is the function considered in Lemma 18 of n^{o}39, $f_0(s)^{-1} f_0(sr)$ is bounded on P_{s_A}; therefore, for the series $\sum_R f_0(s)$ to be convergent, it is necessary and sufficient that the analogous series restricted to R' be so.

We will then apply to this series a convergence criterion due to Godement (cf. [7]), according to which it suffices, for the series in question to be convergent. that we have, for all $p \in P'_A$:

$$\psi(p) = |\Delta_{P'}(p)|_A^{-\tau} \tag{30}$$

with $\tau > 1$; naturally, ψ is here the function introduced in n^{o}39. As P' is an invariant subgroup of P, $\Delta_{P'}$ coincide with Δ_P on P' (cf. the final remark of n^{o}5 of Chapter I), and Δ_P is given by Lemma 12 of Chapter II, n^{o}29, that is to say, by $\Delta_P(p) = \Delta(\lambda)^{-1}$ for $p = t(g)d(\lambda)$; moreover, we then have $\psi(p) = |D(\lambda)|_A^{\frac{1}{2}}$ if $D(\lambda)$ denotes, as in n^{o}33, the determinant of λ with respect to a basis of X_k over k. In case (I), the calculation of $D(\lambda)$, $\Delta(\lambda)$ has already been carried out in n^{o}33 on the occasion of the local problem; it shows that ψ is indeed of the form (30), with $\tau = m/(2n+4\varepsilon-2)$, which completes the proof in this case. In case (II), we have a completely analogous calculation, based on the results of n^{o}21 of Chapter II; as it presents no difficulty, we will omit it here; we find that $d(\lambda)$ and $\Delta(\lambda)$ are in general (or, more precisely, each time that $p \neq q$) independent characters of $\mathrm{Aut}(X)$, but that, on P', we again have a relation of the form (30), with $\tau = m/(p+q)$, hence the announced result.

Various examples suggest that the sufficient condition of Theorem 1 is also necessary, excluding of course the case where $\mathcal{A}_k$ has a component of type (II) with $pq = 0$. It would be interesting to settle this question in the general case[7].

[7] LUO: settled now?

41. In the remaining of the Chapter, we suppose that $\mathcal{A}_k$ and X_k satisfy condition (B) of n^{o}38. We will explicate $E(\Phi)$ by taking as the system of representatives R of P_k/P_{s_k} the one given by Proposition 8 of [14], n^{o}51, which we have already used in the local case in n^{o}32 of Chapter III. In other words, R consists of all elements of P_{s_k} of the form $s = d'(\gamma_1)t(q_1) \otimes e_2$ when we choose in all possible ways a submodule X_1 of X_j isomorphic to its dual X_1^*, moreover, for each X_1, an isomorphism γ_1 from X_1^* to X_1 and an complement X_2 of X_1 in x_k, and we take for q_1 all elements of $Q(X_1/\mathcal{A}_k)$. For such an element s, we can calculate $\mathbf{r}_k(s)\Phi$ using the formula (38) of [14], n^{o}40, or directly using the formula of [14], n^{o}13. By writting Z_k instead of X_1, we obtain

$$\mathbf{r}_k(s)\Phi(0) = \int_{Z_{\mathbb{A}}} \Phi(z)\chi(q_1(z))|dz|_{\mathbb{A}}.$$

This leads to grouping in the series $E(\Phi)$ all terms related to the same submodule Z_k of X_k. We will therefore agree to write, whenever Z_k is a submodule of X_k. We will therefore agree to write, whenever Z_k is a submodule of X_k isomorphic to its dual Z_k^*:

$$E_Z(\Phi) = \sum_{q \in Q_Z} \int_{Z_{\mathbb{A}}} \Phi(z)\chi(q(z))|dz|_{\mathbb{A}} \tag{31}$$

where we have set $Q_z = Q(Z_k/\mathbb{A}_k)$. We have hence:

$$E(\Phi) = \sum_{Z_k} E_Z(\Phi), \tag{32}$$

the summation being extended to all submodules Z_k of X_k such taht Z_k is isomorphic to Z_k^*; and, by Theorem 1, (B) implies that the series (31) and (32) are absolutely convergent, uniformly in Φ on any compact subset of $\mathcal{S}(X_{\mathbb{A}})$.

To study the series $E_Z(\Phi)$, we can obviously limit ourselves to the case where X_k itself is isomorphic to its dual X_k^*, and where $Z_k = X_k$. Suppose this is the case, and identify $Q(X_k/\mathcal{A}_k)$ with the dual $I(X_k)^* = I(X)_k^*$ of the space $I(X_k) = I(X)_k$ as stated in n^{o}15 of Chapter II. Let us also set, for all $i^* \in I(X)_{\mathbb{A}}^*$:

$$F_\Phi^*(i^*) = \int_{X_{\mathbb{A}}} \Phi(x)\chi([i_X 9x), i^*])|dx|_{\mathbb{A}}.$$

We will then have:

$$E_X(\Phi) = \sum_{i^* \in I(X)_k} F_\Phi^*(i^*),$$

and the absolute convergence of this series, uniformly in Φ on any compact subset of $\mathcal{S}(X_{\mathbb{A}})$, follows from what precedes. In other words, if we substitute $X_{\mathbb{A}}$, $I(X)_{\mathbb{A}}$, $I(X)_k$ and i_X by for X, G, Γ and f, respectively, in Proposition 2 of n^{o}2, Chapter I, the condition (B_1) at the end of this n^{o}is satisfied. As, by the virtue of the result of [14], condition (B_0) is obviously also satisfied, it follows that the condition (B) of Proposition 2 is satisfied. Therefore, by the virtue of this proposition:

$$E_X(\Phi) = \sum_{i \in I(X)_k} F_\Phi(i) \tag{33}$$

where F_Φ is the Fourier transform of F^*_Φ; moreover, this Fourier transform is given, for all $i \in I(X)_\mathbb{A}$, by the formula

$$F_\Phi(i) = \int \Phi(x) d\mu_i(x), \tag{34}$$

where μ_i is a positive tempered measure on $X_\mathbb{A}$, of support contained in $i_X^{-1}(\{i\})$; and F_Φ and F^*_Φ are continuous and integrable functions, on $I(X)_\mathbb{A}$ and $I(X)^*_\mathbb{A}$ respectively. Finally, the Proposition 2 of n^{o}2 shows that the right hand side of (33) is absolutely convergent; as the μ_i are all positive measures, we conclude form this, by means of Lemma 5 of [14], n^{o}41, that the right hand side converges uniformly on any compact subset of $\mathcal{S}(X_\mathbb{A})$. According to Lemma 2 of n^{o}2, this shows that E_X is a positive tempered measure, given by

$$E_X = \sum_{i \in I(X)_k} \mu_i.$$

Finally, we conclude similarly by means of (32) that E is a positive tempered measure, sum of the measures E_Z.

42. Suppose again, until further notice, that $\mathcal{A}_k$ and X_k asatisfy (B) and that X_k is isomorphic to its dual X^*_k. According to the result of n^{o}26 of Chapter II, and in particular according to formula (16) of this n^o, it follows that, for any field $K \supset k$, $\mathcal{A}_K$ and X_K have these same properties, and therefore they furthermore satisfy the stronger condition (B') of n^{o}38 of Chapter III. Consequencely, for all v, $\mathcal{A}_v$ and X_v satisfy properties (A) and (A') of Chapter III, so that we can apply all the results of this Chapter to them, and in particular the Proposition 6 of n^{o}37. As (B) implies $m > 2n-2$ in the case (I_0), there will be no distinction to make here between the varieties defined by $U(i)$ and $\overline{U}(i)$ in Chapter III.

Let us then take for Φ, in the formula of n^{o}41, a function of the form

$$\Phi(x) = \prod_v \Phi_v(x_v) \quad (x = (x_v) \in X_\mathbb{A}),$$

where the product extends to all places v of k, where Φ_v belongs to $\mathcal{S}(X_v)$ for any v, and where, for almost all v, Φ_v is the characteristic function of X_v°. With the choice of the function Φ, denote by F_v and F^*_v for all v, the functions respectively defined on $I(X)_v$ and on $I(X)^*_v$ by the formulas

$$F_v(i) = \int_{U_v(i)} \Phi_v(x)|\theta_i(x)|_v, \quad F^*_v(i^*) = \int_{X_v} \Phi_v(x)\chi_v([i_X(x), i^*])|dx|_v;$$

here we have denote by $U_v(i)$ the variety formed by the points of maximal rank of $i_X^{-1}(\{i\})$ in X_v, and θ_i the gauge defined on this variety by formula (28) of n^{o}37. According to Proposition 6 of n^{o}37, F_v and F^*_v are continuous, integrable, and are Fourier transform of each other. According to the hypothesis made on Φ, we immediately see that F^*_v takes the constant value 1 on $(I(X)^*)^\circ_v$ for almost all v; here, in accordance with our general notations (cf. Chapter I, n^{o}3), $(I(X)^*)^\circ_v$ denotes the lattice in $I(X)^*_v$ generated by an arbitrary chosen basis $(I(X)^*)^\circ$ of $I(X)^*_k$ over k.

It is then immediate that we have, for any $i^* = (i^*_v) \in I(X)^*_\mathbb{A}$:

$$F^*_\Phi(i^*) = \prod_v F^*_v(i^*_v);$$

for all i^*, according to what precedes, almost all factors of the second member have the value 1. We deduce from this

$$\int |F_\Phi^*(i^*)| \cdot |di^*|_{\mathbb{A}} = \prod_v \int |F_v^*(i^*)| \cdot |di^*|_v.$$

In this relation, the left hand side is $< +\infty$; it is $\neq 0$ unless $F_\Phi^* = 0$; moreover, we can always modify a finite number of the functions $\Phi_v \geq 0$ and $\Phi_v \neq 0$ such that to have $F_\Phi \neq 0$ and consequently $F_\Phi^* \neq 0$. As almost all factors of the right hand side are ≥ 1 according to what precedes, it follows that the right hand side is absolutely convergent (in the sense defined in note (^{1}) of page 8). We easily conclude that the Fourier transform F_Φ of F_Φ^* is the product of the Fourier transform F_v of F_v^*, that is to say, we have, for any $i = (i_v) \in I(X)_{\mathbb{A}}$:

$$F_\Phi(i) = \prod_v F_v(i_v),$$

the product of the right hand side is absolutely convergent.

If we denote by μ_v the tempered measure on $I(X)_v$ determined by the measure $|\theta_{i_v}|_v$ on $U_v(i_v)$, $F_v(i_v)$ is nothing other than $\mu_v(X_v^\circ)$ each time Φ_V is the characteristic function of X_v°. The formula above therefore shows that the product of $\mu_v(X_v^\circ)$ is absolutely convergent, and that the measure μ_i that appears in the expression (34) is nothing but $\prod \mu_v$.

43. When i belongs to $I(X)_k$, the set $i_X^{-1}(\{i\})$, on the universal domain, is a k-closed subset of X. We will denote by $U(i)$ the set of points of maximal rank of this set; it is a k-open subset of $i_X^{-1}(\{i\})$; by Proposition 3 of Chapter II, n^{o}17, when $U(i)$ is nonempty, it is an orbit of the group G, also taken on the universal domain. We conclude easily from Lemma 8 of Chapter II, n^{o}17, that, if $K \supset k$, the set $U(i)_K$ of K-rational points of $U(i)$ is the set of points if $i_X^{-1}(\{i\})$ in X_K that are of maximal rank in X_K. In particular, for $K = k_v$, we see that $U(i)_v$ is the set that was denoted by $U_v(i)$ of n^{o}42.

Lemma 19. *For all $i \in I(X)_k$, $\mathfrak{1}$ is a system of convergence factors for $U(i)$, and we have $\mu_i = |\theta_i|_{\mathbb{A}}$.*

If Φ is chosen as in n^{o}42, we have, according to what precedes:

$$\int_{X_{\mathbb{A}}} \Phi d\mu_i = \prod_v \int_{U(i)_v} \Phi_v |\theta_i|_v.$$

Let $\lambda = (\lambda_v)$ be a system of convergence factors for $U(i)$. As the canonical injection of $U(i)$ into X is a morphism, we have, for almost all v, $U(i)_v \subset X_v^\circ$. If we have that $\Phi_v \geq 0$ for all v, we will have, according to the definitions of n^{o}4 of Chapter I:

$$\int_{U(i)_{\mathbb{A}}} \Phi \cdot |\lambda \theta_i|_{\mathbb{A}} = \prod_v \lambda_v \int_{U(i)_v} \Phi_v \cdot |\theta_i|_v.$$

The comparison of these formulas shows that, if we have the right to take $\lambda = 1$, Φ induces on $U(i)_{\mathbb{A}}$ an integrable function for $|\theta_i|_{\mathbb{A}}$, and that we then have $\mu_i = |\theta_i|_{\mathbb{A}}$.

It remains to proof the first assertion of the lemma; for this it is evidently to consider the case where $\mathcal{A}_k$ is of type (I) or (II) with $\mathfrak{z}_0 = k$. If $U(i)$ contains a k-rational point x_0, let g_0 be the stabilizer of x_0 in G. Then $U(i)$ is isomorphic to G/g_0, and everything comes down (according to [13], Theorem 2.4.2 and 2.4.3) to verifying that there is a system of convergence factors common to G and g_0; but this is an immediate consequence of the results of n^{o}23 of Chapter II, combined with the known results on classical groups (cf. [13]).

Suppose that $U(i)_k$ is empty; according to the remarks of n^{o}38 of Chapter III, this implies that $\mathcal{A}_k$ is neither of type (II), nor of type (I_0). By the virtue of the **Hasse principle** (which will be discussed in n^{o}53 of Chapter V), this also implies that $U(i)_{\mathbb{A}}$ is empty; but, as we do not wish to make use of the principle here, we will reduce this case to the preceding one as follows. By hypothesis (B), we have $m \geq 2n$; as in Chapter II, for the space $M_{m,1}(\mathfrak{k})$, it is easy to deduce from there that we can define on $\mathbb{A} \times \mathbb{A}$ a non-degenerate η-hermitian form h_1 such that $h_1[x_1] = i$ has a solution x_1 of maximal rank in X_k, or in other words that $U_1(i)_k$ is not empty, $U_1(i)$ being defined from h_1 as $U(i)$ is from h. Let S be a set of places of k, including in particular all places at infinite, all **even** places (those for which $\mathfrak{p}_v$ divide 2) and the places v, finite in number according to n^{o}4 of Chapter I, for which $U(i)_v^\circ$ of $U_1(i)_v^\circ$ is empty; for $v \notin S$, let N_1, N_v^1 be the number of points of $U(i)_v^\circ$ and of $U_1(i)_v^\circ$, respectively, modulo $\mathfrak{p}_v$; we are reduced to proof that $\prod(N_v/N_v^1)$ is absolutely convergent, since then, according to the results recalled in n^{o}4 of Chapter I, any system of convergence factors for $U_1(i)$ will also be one for $U(i)$. We then easily verify that, for almost all v, N_v is the number of elements of a non-empty set, which is a homogeneous space analogous to $U(i)_k$ over the finite field $\mathfrak{o}_k/\mathfrak{p}_v$; consequently, it is the quotient of the number of elements of a group analogous to G that of a group analogous to g_0; the same is true, of course, for N_v^1. The known formula on the number of elements of classical groups over finite fields then immediately shows that $N_v = N_v^1$ for almost all v if $\mathcal{A}_k$ is of type (I_1) or (I_2); this shows that $N_v/N_v^1 = 1 + O(N\mathfrak{p}_v^{-2})$ if $\mathcal{A}_k$ is of type (I_4), and also if $\mathcal{A}_k$ is of type (I_3) since then $\mathcal{A}_v$ is of type (I_4) for almost all v. It can be observed that the complete development of the indications that precede contains the proof of the results recalled above on the system of factors of classical groups and would therefore make unnecessary the distinction of the two cases above and the introduction of the orbit $U_1(i)$.

44. We can now summarize the results obtained in this Chapter.

Theorem 2. *Suppose that $\mathcal{A}_k$ and X_k satisfy the condition (B), and that X_k is isomorphic to its dual X_k^*. Let us pose, for $\Phi \in \mathcal{S}(X_{\mathbb{A}})$:*

$$E_X(\Phi) = \sum_{q \in Q_X} \int_{X_{\mathbb{A}}} \Phi(x)\chi(q(x))|dx|_{\mathbb{A}},$$

where the summation is extended to $Q_X = Q(X_k/\mathcal{A}_k)$. Also the series of the right hand side is absolutely convergent, and E_X is a positive tempered measure. Moreover, for all $i \in I(X)_k$, $\mathfrak{l}$ is a system of convergence factors for the variety $U(i)$ of points of $i_X^{-1}(\{i\})$ of maximal rank; and, if θ_i denoted the gauge on this variety defined by the formula

$$\theta_i(x) = \left(\frac{dx}{di_X(x)}\right)_i,$$

the measure $|\theta_i|_{\mathbb{A}}$ on $U(i)_{\mathbb{A}}$ defined a positive tempered measure μ_i in $X_{\mathbb{A}}$, and we have

$$E_X = \sum_{i \in I(X)_k} \mu_i.$$

Theorem 3. *Suppose that $\mathcal{A}_k$ and X_k satisfy the condition (B). Let us pose, for $\Phi \in \mathcal{S}(X_{\mathbb{A}})$:*

$$E(\Phi) = \sum_{s \in P_k \backslash P_{s_k}} \mathbf{r}_k(s)\Phi(0),$$

where $P_{s_k} = P_s(X_k/\mathcal{A}_k)$, $P_k = P(X_k/\mathcal{A}_k)$. Also, the series of right hand side is absolutely convergent; E is a positive tempered measure; and we have:

$$E = \sum_{Z_k} E_Z.$$

Where the summation is extended to all submodules Z_k of X_k such that each Z_k is isomorphic to its dual Z_k^*, and where E_Z, for all Z_k, is defined by the Theorem 2.

Formally, Theorem 3 remains true, but becomes trivial, in the case where $\mathcal{A}_k$ is of type (II) and where $p = 0$ or $q = 0$, similarly if (B) is not satisfied; $E(\Phi)$ then reduce to $\Phi(0)$, and the only submodule of X_k isomorphic to its dual is $Z_k = \{0\}$. In general, we will denote E_0, instead of $E_{\{0\}}$, the measure E_Z corresponding to $Z_k = \{0\}$; it is evidently nothing other than the Dirac measure δ_0 relative to point 0.

If $\mathcal{A}_k$ is of type (I) and X_k of rank 1, or if $\mathcal{A}_k$ is of type (II) and X_k of rank $(1,1)$, we have $E = E_0 + E_X$ because then there is no submodule of X_k, isomorphic to its dual, other than $\{0\}$ and X_k. In particular, if $\mathcal{A}_k$ is of type (I_0) and X_k of rank 1, we have already observed in n^{o}39 that the Eisenstein-Siegel series which defines E reduces to two terms; moreover, as we then have $Q_X = \{0\}$, we have $E_X(\Phi) = \int \Phi |dx|_{\mathbb{A}}$; most of the above results are trivial in this case.

45. The measures introduced above have important invariance properties that we will state now.

For this, we will transpose by duality, to the space of tempered distributions on $X_{\mathbb{A}}$, the operation of the metaplectic group $\mathrm{Mp}(X/\mathcal{A})_{\mathbb{A}}$ on $\mathcal{S}(X_{\mathbb{A}})$; in other words, for any element S of this group, and all tempered distribution T on $X_{\mathbb{A}}$, we define T^S by means of $T^S(\Phi) = T(S\Phi)$ for all $\Phi \in \mathcal{S}(X_{\mathbb{A}})$. The definition of the Eisenstein-Siegel series $E(\Phi)$ then shows that E is invariant by $\mathbf{r}_k(s)$ for all $s \in Ps(X_k/\mathcal{A}_k)$, which we will express more briefly by saying that E is invariant by $Ps(X_k/\mathcal{A}_k)$. By the virtue of the Proposition 8 of [14], n^{o}51, it is the same to say that E is invariant by the following:

(a) $\mathbf{t}(q)$, for all $q \in Q(X_k/\mathcal{A}_k)$; in other words, wehave $E(\Phi) = E(\Phi')$ *each time Φ' is defined by $\Phi'(x) = \Phi(x)\chi(q(x))$, with $q \in Q(X_k/\mathcal{A}_k)$.*

(b) $\mathbf{d}(\lambda)$, for all $\lambda \in \mathrm{Aut}(X_k)$; in other words, *$E$ is invariant by the automorphism $x \to x\lambda$ of $X_{\mathbb{A}}$ for all $\lambda \in \mathrm{Aut}(X_k/\mathcal{A}_k)$;*

(c) , $\mathbf{d}'(\gamma_1) \otimes e_2$ each time we have written X_k as direct sum of two submodules X_1, X_2 where the first is isomorphic to its dual, and that γ_1 is an isomorphism from X_1^* to X_1; in other words, each time it is so and we have identified $X_1^* \oplus X_2$ with $X_1 \oplus X_2$ by means of γ, *E is invariant by the* ***partial Fourier transform*** *relative to* $(X_1)_{\mathbb{A}}$.

When X_k is isomorphic to its dual, (c) implies in particular the following condition:

(c') *E is invariant by the Fourier transform when we identify $X_{\mathbb{A}}^*$ with $X_{\mathbb{A}}$ by means of an isomorphism γ from X_k^* to X_k.*

Conversely, (c'), joint with (a) and (b), implies (c) each time $Ps(X_k/\mathcal{A}_k)$ is generated by $d'(\gamma)$ and $P(X_k/\mathcal{A}_k)$. This is the case for example each time X_k is isomorphic to its dual and that $\mathcal{A}_k$ has no component of type (I_0), as we see by repeating the proof of Corollary 3 of the Proposition 6, [14], n^{o}47.

On the other hand, the corollary of the Proposition 9, [14], n^{o}51, immediately shows that E possesses the following property:

(d) *E is invariant under the automorphism $x \to ux$ of $X_{\mathbb{A}}$ for each $u \in G_{\mathbb{A}}$*, or in other words for all $u \in \mathcal{A}_{\mathbb{A}}$ such that $u \cdot u^t = 1$.

In fact, this corollary shows that each term of the series which defines E possesses this property; it is therefore the same for the measure E_Z, because thses have been introduced in n^{o}41 as partial series of the Eisenstein-Siegel series. As each of the measures E_Z, by Theorem 2, has its support contained in the union of the sets $i_X^{-1}(\{i\})$ for $i \in I(X)_k$, and that we have $\chi(q(x)) = 1$ on each of these sets for $q \in Q(X_k/\mathcal{A}_k)$, the E_z also have property (a). It is also evident that, if an isomorphism λ of X_k transform Z_k to Z'_k, it transforms E_Z into $E_{Z'}$; in particular, E_Z is invariant by $\mathrm{Aut}(Z_k/\mathcal{A}_k)$.

V The Uniqueness Theorem

46. When $\mathcal{A}_k$ is of type (I), we can characterize by intrinsic properties of the measure E and the measure E_X that we studied in Chapter V; this is the object of the present Chapter. The method we will follow extends without difficulty to the case where $\mathcal{A}_k$ is of type (II) and X_k of rank $(1,1)$, provided we replace everywhere the group $Ps(X/\mathcal{A})$ by the subgroup $Ps'(X/\mathcal{A})$ which was defined in Chapter II, n^{o}28. No could also happen that the same result extends to the case where $\mathcal{A}_k$ is of type (II) and X_k is of rank (p,p) for $p > 1$, but we would need for this a modification of Lemma 6 of n^{o}13 which I don't know if it is valid[8]. The case (II), which moreover does not seem for the moment to lend itself to interesting arithmetic applications, will be completely left aside in the following contents.

In this Chapter, we will assume once and for all that $\mathcal{A}_k$ *is of type* (I) and that $\mathcal{A}_k$ *and* X_k *satisfy condition* (B), that is to say $m > 2n + 4\varepsilon - 2$; we will also assume that $\mathfrak{z}_0 = k$. As before, we will most often write Ps instead of $Ps(X/\mathcal{A})$, also Ps_k instead of $Ps(X_k/\mathcal{A}_k)$ and $Ps_{\mathbb{A}}$ instead of $Ps(X/\mathcal{A})_{\mathbb{A}}$, and similarly $\mathrm{Mp}_{\mathbb{A}}$ instead of $\mathrm{Mp}(X/\mathcal{A})_{\mathbb{A}}$. We will identify Ps_l with its image in $\mathrm{Mp}_{\mathbb{A}}$ by means of $\mathbf{r}_k$ each time that this will not lead to confusion; as in n^{o}45, this makes it possible to say of a tempered measure on $X_{\mathbb{A}}$ that is invariant by Ps_k when it is acted by $\mathbf{r}_k(s)$ for any $s \in Ps_k$ in the sense that was explained in n^{o}45. On the other hand, as in n^{o}45, we will say that a (tempered or not) measure in $X_{\mathbb{A}}$ is invariant by an element u of $G_{\mathbb{A}}$ if it is by the map $x \to ux$ of $X_{\mathbb{A}}$ onto itself. Recall that, by the Corollary of Proposition 9, [14], n^{o}51, the automorphisms $\Phi \to S\Phi$, and $\Phi(x) \to \Phi(ux)$ of $\mathcal{S}(X_{\mathbb{A}})$, for $S \in \mathrm{Mp}_{\mathbb{A}}$ and $u \in G_{\mathbb{A}}$, are permutable; it is thus the same for the corresponding automorphisms of the space of tempered distributions on $X_{\mathbb{A}}$.

Let $\widetilde{E}$ be a tempered measure on $X_{\mathbb{A}}$, invariant by Ps_k, and let $\Phi \in \mathcal{S}(X_{\mathbb{A}})$; then $S \to \widetilde{E}(S\Phi)$ is a continuous function on $\mathrm{Mp}_{\mathbb{A}}$, invariant by the left action of Ps_k. We will start by giving conditions for this functions to be bounded on $\mathrm{Mp}_{\mathbb{A}}$, uniformly in Φ on any compact subset of $\mathcal{S}(X_{\mathbb{A}})$; for this, we will apply to the group ps the results of the *reduction theory* that we recalled in n^{os}9 and 10 of Chapter I.

[8] LUO:Does it valid now?

As in n^{os}18 − 19 of Chapter II, let us identify X_k with $M_{m,n}(\mathfrak{k})$, and consequently $\mathrm{Aut}(X_k)$ with $\mathrm{GL}(n,\mathfrak{k})$. For $x \in M_{m,n}(\mathfrak{k})$, we will denote by $x_1,\cdots,x_n$ the columns of the matrix x, so that we have $x_\alpha \in M_{m,1}(\mathfrak{k})$ for $1 \le \alpha \le n$, and we will write $x = (x_1,\cdots,x_n)$. Let $t = (t_1,\cdots,t_n)$ be an element of $(\mathbb{G}_m)^n$, where $\mathbb{G}_m$ denotes as usual the multiplicative group in one variable; we will denote by λ_t the automorphism of X defined by the diagonal matrix whose diagonal elements are $t_1,\cdots,t_n$; it is also written as

$$x = (x_1,\cdots,x_n) \to x\lambda_t = (x_1t_1,\cdots,x_nt_n).$$

As in n^{o}19 of Chapter II, let us identify $I(X)_k$ with the space of η-hermitian matrices $i = (i_{\alpha\beta})_{1\le\alpha,\beta\le n}$ on $\mathfrak{k}$. For $t \in (\mathbb{G}_m)^n$, we will denote by $\overline{\lambda}_t$ the automorphism of $I(X)$ determined by the automorphism λ_t of X; it is also written as

$$i = (i_{\alpha\beta}) \to i\overline{\lambda}_t = (i_{\alpha\beta}t_\alpha t_\beta).$$

If δ denotes as always the dimension of $\mathfrak{k}$ over k, the determinants of λ_t and of $\overline{\lambda}_t$, with respect to bases of X_k and of $I(X)_k$ over k, will be respectively

$$D(\lambda_t) = (t_1\cdots t_n)^{m\delta}, \quad D(\overline{\lambda}_t) = (t_1\cdots t_n)^{(n+2\varepsilon-1)\delta}.$$

We conclude that the gauge $\theta_i(x)$ on $U(i)$, defined in Theorem 2 of Chapter V, n^{o}44, is transformed by λ_t into the gauge

$$\theta_i(x\lambda_t^{-1}) = (t_1\cdots t_n)^{(-m+n+2\varepsilon-1)\delta}\theta_{i'}(x) \tag{35}$$

on $U(i')$, with $i' = i\overline{\lambda}_t$.

In particular, for $t \in (I_k)^n$, λ_t and $\overline{\lambda}_t$ be automorphisms of $x_\mathbb{A}$ and of $I(X)_\mathbb{A}$, respectively. If we set, for brevity, $|t|_\mathbb{A} = |t_1\cdots t_n|_\mathbb{A}$, the modulus of λ_t in $X_\mathbb{A}$ will be $|\lambda_t|_X = |t|_\mathbb{A}^{m\delta}$, so that we will have, for $\Phi \in \mathcal{S}(X_\mathbb{A})$, $x = (x_1,\cdots,x_n)$:

$$\mathbf{d}(\lambda_t)\Phi(x) = |t|_\mathbb{A}^{m\delta/2}\Phi(x_1t_1,\cdots,x_nt_n). \tag{36}$$

47. As we saw in n^{o}28 of Chapter II, Ps is a reductive algebraic group. We will denote by T the image of $(\mathbb{G}_m)^n$ in Ps by $t \to d(\lambda_t)$; it is well known that T is a trivial maximal torus of Ps. The determination of the roots of T in Ps is done without difficulty; we find that we can order them so that the strictly positive roots (that is to say $\neq 1$ and *succ* 1) are the $t_\alpha t_\beta^{-1}$ and the $t_\alpha t_\beta$ for $1 \le \alpha < \beta \le n$, as well as the t_α^2 for $1 \le \alpha \le n$ in the case of $\varepsilon > 0$. Let $T_\mathbb{A}^+$ be the subset of $T_\mathbb{A}$ defined by means of these roots as it was said in n^{o}10 of Chapter I; according to the results of this n^o, there is a comapct subset c_1 of $Ps_\mathbb{A}$ such that we have $Ps_\mathbb{A} = C_1 \cdot T_\mathbb{A}^+ \cdot Ps_k$. Let $T'_\mathbb{A}$ be a subset of $T_\mathbb{A}$ defined by the elements $d(\lambda - t)$ of $T_\mathbb{A}$ for which we have

$$|t_1|_\mathbb{A} \ge \cdots \ge |t_n|_\mathbb{A} \ge 1.$$

For $\varepsilon > 0$, we verify immediately that we have $T_\mathbb{A}^+ = T_\mathbb{A}'^{-1}$, so taht by setting $C = C_1^{-1}$, we will have $Ps_\mathbb{A} = Ps_k \cdot T'_\mathbb{A} \cdot C$. If $\varepsilon = 0$, let us apply the results obtained about Ps in n^{o}20 of Chapter II, where we must take $\mathfrak{k} = k$ and $\eta = -1$; we see that the matrix

$$s_1 = \begin{pmatrix} 1_{n-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1_{n-1} & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

belongs to Ps_k; it is moreover the element of Ps_k which, with our usual notations, is written as $e' \otimes d'(\gamma'')$ if we write X_k as the direct sum of the modules X'_k, X''_k respectively whose elements of the form $(x_1, \cdots, x_{n-1}, 0)$ and $(0, \cdots, 0, x_n)$ and we take for e' the neutral element of $Ps(X'_k/\mathcal{A}_k)$ and for γ'' the obvious isomorphism of $(X''_k)^*$ onto X''_k when we identify X''_k and $(X''_k)^*$ with $M_{m,1}(\mathfrak{k})$ as it was said in n^{o}20 of Chapter II. This being set, we verify immediately that $T^+_{\mathbb{A}}$ is the union of ${T'}^{-1}_{\mathbb{A}}$ and of $s_1^{-1}{T'}^{-1}_{\mathbb{A}}s_1$. It follows that we will again have $Ps_{\mathbb{A}} = Ps_k \cdot T'_{\mathbb{A}} \cdot C$ provided we take $C = C_1^{-1} \cup s_1 C_1^{-1}$.

By abuse of language, we will agree in what follows to identify $(I_k)^n$ with $T_{\mathbb{A}}$ by means of the isomorphism $t \to d(\lambda_t)$, and also with its image in $\mathrm{Mp}_{\mathbb{A}}$ by means of the isomorphism $t \to d(\lambda_t)$, and also with its image in $\mathrm{Mp}_{\mathbb{A}}$ by means of the isomorphism $t \to \mathbf{d}(\lambda_t)$, whenever there is no risk of confusion. With this convention, we can therefore write $\mathrm{Mp}_{\mathbb{A}} = Ps_k \cdot T'_{\mathbb{A}} \cdot \pi^{-1}(C)$, where π is as usual the canonical projection of $\mathrm{Mp}_{\mathbb{A}}$ onto $Ps_{\mathbb{A}}$; $\pi^{-1}(C)$ is then a compact subset of $\mathrm{Mp}_{\mathbb{A}}$. This entails the following lemma:

Lemma 20. *Let $\widetilde{E}$ be a tempered measure on $X_{\mathbb{A}}$, invariant by Ps_k; let $T''_{\mathbb{A}}$ a subset of $T'_{\mathbb{A}}$ such that we have $T'_{\mathbb{A}} \subset T_k \cdot T''_{\mathbb{A}} \cdot C'$, C' being a compact subset of $T_{\mathbb{A}}$. Then, for the function $S \to \widetilde{E}(S\Phi)$ to be bounded on $\mathrm{Mp}_{\mathbb{A}}$, uniformly in Φ on any compact subset of $\mathcal{S}(X_{\mathbb{A}})$, it is necessary and sufficient that it being so on $T''_{\mathbb{A}}$.*

48. Let μ' be a measure (tempered or not) on $X_{\mathbb{A}}$. For it to satisfy condition (a) of n^{o}45, Chapter V, or in other words for it to be invariant by $\mathbf{t}(q)$ each time $q \in Q(X_k/\mathcal{A}_k)$, it is obviously necessary and sufficient that its support be contained in the set of points x of $X_{\mathbb{A}}$ such that $\chi(q(x)) = 1$ whatever $q \in Q(X_k/\mathcal{A}_k)$. If we identify $Q(X_k/\mathcal{A}_k)$ with $I(X)^*_k$ as it in n^{o}15 of Chapter II, this last condition amounts to saying that we have $\chi([i_X(x), i^*]) = 1$ for any $i^* \in I(X)^*_k$, or again $i_X(x) \in I(X)_k$. In other words, condition (a) is equivalent to the following:

(a') *the support of μ' is contained in the union of sets $i_X^{-1}(\{i\})$ for $i \in I(X)_k$.*

Suppose that at the same time μ' satisfies condition (b) of n^{o}45, Chapter V, that is to say is invariant by $x \to x\lambda$ for each $\lambda \in \mathrm{Aut}(X_k)$; then, if $\overline{\lambda}$ is the automorphism of $I(X)_k$ determined by λ, λ determines the permutation $\overline{\lambda}$ on the set of measures μ'_i, that is to say we have, for $i \in I(X)k$, $i' = i\overline{\lambda}$:

$$d\mu'_i(x\lambda^{-1}) = d\mu'_{i'}(x).$$

On the other hand, identifying as above $I(X)_k$ with a subspace of $M_n(\mathfrak{k})$, let us agree, for any $i \in I(X)_k$, to denote by $i_1, \cdots, i_n$ in the columns of the matrix i, and to write $i = (i_1, \cdots, i_n)$. We will thus have $i_\alpha \in M_{n,1}(\mathfrak{k})$ for $1 \leq \alpha \leq n$; if $i = i_X(x)$, we have $i_\alpha = {}^t x' \cdot h \cdot x_\alpha$. For $0 \leq \alpha \leq n$, we will denote by $I_k^{(\alpha)}$ the set of elements $i = (i_1, \cdots, i_n)$ of $I(X)_k$ such that $i_1 = \ldots = i_\alpha = 0$ and $i_{\alpha+1} \neq 0$; $I(X)_k$ is thus the disjoint union of $I_k^{(\alpha)}$ for $0 \leq \alpha \leq n$.

Lemma 21. *Let $\widetilde{E}$ a positive tempered measure, invariant by T_k, whose support is contained in the union of $i_X^{-1}(\{i\})$ for $i \in I_k^{(0)}$. Then the function $S \to \widetilde{E}(S\Phi)$ is bounded on $T'_{\mathbb{A}}$, uniformly in Φ on any compact subset of $\mathcal{S}(X_{\mathbb{A}})$.*

As in n^{o}9 of Chapter I, let us denote by $\Theta(T)$ the set of elements of $T_{\mathbb{A}}$ of the form $(a_{\tau_1}, \cdots, a_{\tau_n})$, with $\tau_\alpha \in \mathbb{R}^*_+$ for $1 \leq \alpha \leq n$; let $\Theta' = \Theta(T) \cap T'_{\mathbb{A}}$; Θ' is the set of elements of the form above for which $\tau_1 \geq \cdots \geq \tau_n \geq 1$. There is obviously a compact subset C' of $T'_{\mathbb{A}}$ such that $T'_{\mathbb{A}} = T_k \cdot \Theta' \cdot C'$.

Let C_0 a compact subset of $\mathcal{S}(X_{\mathbb{A}})$; let C_0' be the set of $S\Phi$ for $S \in C'$, $\Phi \in C_0$. Applying Lemma 6 of Chapter I, n^{o}13, to the space $X_k = M_{m,n}(\mathfrak{k})$, $X_k^{(\alpha)} = M_{m,1}(\mathfrak{k})$ for $1 \leq \alpha \leq n$, $Y_k = M_{n,1}(\mathfrak{k})$, and to the morphism $x \to p(x) = {}^t x' \cdot h \cdot x_1$ of X to Y; we conclude that there exists $\Phi_0 \in \mathcal{S}(X_{\mathbb{A}})$ such that we have $|S\Phi| \leq \Phi_0$ on the support of $\widetilde{E}$ whenever $S \in \Theta'$, $\Phi \in C_0'$. The conclusion of the lemma follows immediately.

Lemma 21 is particularly useful in teh case $n = 1$, since then $I(X)_k = I_k^{(0)} \cup \{0\}$. We thus obtain for example the following result:

Proposition 7. *Let $\mathcal{A}_k$ be of type (I) and X_k of rank 1, satisfying (B), that is to say $m > 4\varepsilon$. Let ν a positive measure with compact support on $G_{\mathbb{A}}/G_k$, such that $\nu(G_{\mathbb{A}}/G_k) = 1$. Then the formula*

$$E'(\Phi) = \int_{G_{\mathbb{A}}/G_k} \sum_{\xi \in X_k} \Phi(u\xi) \cdot d\nu(u)$$

defines a positive tempered emeasure E' on $X_{\mathbb{A}}$, invariant on Ps_k; and the function

$$S \to E'(S\Phi) - E(S\Phi)$$

is bounded on $\mathrm{Mp}_{\mathbb{A}}$, uniformly on Φ for all comapct subsets of $\mathcal{S}(X_{\mathbb{A}})$.

Let C be a compact subset of $G_{\mathbb{A}}$ whose image in $G_{\mathbb{A}}/G_k$ contains the support of ν; let c_0 a compact subset of $\mathcal{S}(X_{\mathbb{A}})$. According to Lemma 5 of [14], n^{o}41, there is $\Phi_0 \in \mathcal{S}(X_{\mathbb{A}})$ such that $|\Phi(ux)| \leq \Phi_0(x)$ such that $|\Phi(ux)| \leq \Phi_0(x)$ for any $\Phi \in C_0$, $u \in C$, $x \in X_{\mathbb{A}}$. The integral defining $E'(\Phi_0)$ is obviously majorized [9] by the analogous integral extended to C, which Φ replaced by $|\Phi|$, and is thus in priority majorized by $v(C) \sum \Phi_0(\xi)$ for $\Xi \in C_0$. By Lemma 2 of Chapter I, n^{o}2, it thus defines a tempered distribution E', which is obviously a positive measure. The invariance of E' by Ps_j is an immediate consequence of Theoremf 6 of [14], n^{o}41, joint with Proposition 9 of [14], n^{o}51. Let us now write

$$\widetilde{E}(\Phi) = \int_{G_{\mathbb{A}}/G_k} \sum_{\xi \in X_k - \{0\}} \Phi(u\xi) \cdot d\nu(u),$$

which gives $E' = \widetilde{E} + \delta_0$ by virtue of the hypothesis made on ν. Let us apply Lemma 6 of Chapter I, n^{o}13, taking $n-1$, $X_k = Y_k$, p being the identical map of X onto X. Using the same argument on $\widetilde{E}(\Phi)$ as we just did on $E'(\Phi)$, we conclude that there exists $\Phi_0 \in \mathcal{S}(X_{\mathbb{A}})$ such that $\widetilde{E}(S\Phi)$ is majoried by $v(C) \sum \Phi_0(\xi)$ whenever $S \in \Theta'$ and $\Phi \in C_0$. On the other hand, let us apply to E the Theorem 2 and 3 of Chapter V, n^{o}44; with the notations of these theorems, we first have $E = E_X + \delta_0$, and $E_X = \mu_0 + \widetilde{E}_X$, where $\widetilde{E}_X$ has support in the union of sets $i_X^{-1}(\{i\})$ for $i \in I(X)_k$, $i \neq 0$. We can thus apply Lemma 21 to $\widetilde{E}_X$, which shows that $\widetilde{E}_X(S\Phi)$ remains bounded for $S \in \Theta'$, $\Phi \in C_0$. Finally, as μ_0 is the measure determined by the gauge θ_0 on $U(0)_{\mathbb{A}}$, we have, according to formula (35) of n^{o}46:

$$d\mu_0(x\lambda_t^{-1}) = |t|_{\mathbb{A}}^{(-m+2\varepsilon)\delta} d\mu_0(x)$$

for $t \in I_k$, and consequently, according to formula (36) of n^{o}46:

$$\int \mathbf{d}(\lambda_t)\Phi \cdot d\mu_0 = |t|_{\mathbb{A}}^{-(m-4\varepsilon)\delta/2} \int \Phi \cdot d\mu_0.$$

[9] LUO: I think it means less equals.

Because by hypothesis we have $m > 4\varepsilon$, this expression remains bounded for $|t|_{\mathbb{A}} \geq 1$, or in other words on $T'_{\mathbb{A}}$, uniformly in Φ on any compact subset of $\mathcal{S}(X_{\mathbb{A}})$. As we have $E' - E = \widehat{E} - \mu_0 - \widehat{E}_X$, it follows that $E' - E$ satisfies the conditions of Lemma 20 of n^{o}47, which completes the proof.

Proposition 7 applies for example to the case where we have $\nu = \delta_1$, that is to say $E'(\Phi) = \sum \Phi(\xi)$. The result obtained then includes as particular cases the classical theorems about the principle part of modular formed defined by a theta series which is given by an Eisenstein series (or, which amounts to the same thing, the results on the **singular series** which gives the principal part of the number of representatives of an integer by a quadratic form).

49. As shown by the example of the measure δ_0, we do not as the right to replace $I_k^{(0)}$ by $I(X)_k$, nor by $I_k^{(\alpha)}$ for $\alpha > 0$, in the Lemma 21, unless we introduce additional hypotheses on $\widetilde{E}$, this will be done in Lemma 23

To express ourselves more conveniently, we will make the following convention. Let $i \in I(X)_l$; let j be the canonical injection of $U(i)$ into X, which is a morphism of algebraic variety; j also determine an injective map $j_{\mathbb{A}}$ from $U(i)_{\mathbb{A}}$ to $X_{\mathbb{A}}$, and more precisely into $i_X^{-1}(\{i\})$. We will say taht a measure in $X_{\mathbb{A}}$ is *carried by* $U(i)_{\mathbb{A}}$ if it is the image of a measure on $U(i)_{\mathbb{A}}$ by $j_{\mathbb{A}}$. For example, this is the case, by definition, of the measure μ_i, image of $|\theta_i|_{\mathbb{A}}$ by $j_{\mathbb{A}}$, which appears in Theorem 2 of Chapter V, n^{o}44. In the case where i is a non-degenerate element of $I(X)_k$, it follows from the remarks of Chapter II, n^{o}25, that $j_{\mathbb{A}}$ is an isomorphism of $U(i)_{\mathbb{A}}$ to $i_X^{-1}(\{i\})$; in this case, all the measures with supported contained in $i_X^{-1}(\{i\})$ is carried by $U(i)_{\mathbb{A}}$.

On the other hand, in what follows, we will once and for all choose a place v of k, and we will write $X_{\mathbb{A}} = X_v \times X'$, where X' is defined from X_w and of X_w°, for $w \neq v$, just as $X_{\mathbb{A}}$ is defined from X_v and X_v°; for $x \in X_{\mathbb{A}}$, we will write $x = (x_v, x')$, where x_v and x' are the projections of x to X'. We will similarly write $U(i)_{\mathbb{A}} = U(i)_v \times U(i)'$.

Lemma 22. *Let $i \in I(X)_k$, and let G'_v a subgroup of G_v which acts transitively on $U(i)_v$. Let μ a positive tempered measure on $U(i)_{\mathbb{A}}$ and invariant under G'_v. Then, for all functions $\Phi' \in \mathcal{S}(X')$, we can corresponds to a measure $c(\Phi')$ in such a way that, for any $\Phi_v \in \mathcal{S}(X_v)$:*

$$\int \Phi_v(x_v)\Phi'(x')d\mu((x_v, x')) = c(\Phi') \int_{U(i)_v} \Phi_v \cdot |\theta_i|_v. \tag{37}$$

By hypothesis, μ is the image of a measure ν on $U(i)_{\mathbb{A}}$, that is to say that the left hand side of (37) is the integral of $\Phi_v(x_v)\Phi'(x')$ on $U(i)_v \times U(i)'$ with respect to ν. Suppose first that $\Phi' \geq 0$. By hypothesie, the integral in the question is finite whenever Φ_v is ≥ 0 and belongs to $\mathcal{S}(X_v)$, thus also whenever Φ_v is continuous with compact support on X_v, and for stronger reason whenever Φ_v is compact support on $U(i)_v$. It can therefore be written as $\int \Phi_v d\nu_v$, where ν_v is a positive measure on $U(i)_v$. According to the hypothesis made on μ, ν_v is invariant on G'_v. We can moreover assume that G'_v is closed in G_v (otherwise, we would replace it by its closure), and consequently identifying $U(i)_v$ with the homogeneous space determined by G'_v and the stabilizer of one of its points in G'_v. But the definition of the gauge θ_i in Theorem 2 of Chapter V, n^{o}44, shows that it is invariant under G_v, up to a factor ± 1; consequently, the measure $|\theta_i|_v$ is invariant under G_v, and for stronger reason under G'_b. The known theorems on the uniqueness of the invariant measure in homogeneous spaces (cf. for ex. [4],§2,n^{o}6) then show that ν_v can only differ from $|\theta_i|_v$ by a scalar factor of $c(\Phi')$. The general case where we don't assume $\Phi' \geq 0$ is immediately reduced to the previous one.

Lemma 23. *Let $\widetilde{E}$ be a positive tempered measure, invariant under T_k and sum of measures $\widetilde{\mu}_i$ respectively carried by $U(i)_{\mathbb{A}}$ for $i \in I(X)_k$. Suppose there exists a place v of k, and a subgroup G'_v of G_v acting transitively on $U(i)_v$ for any $i \in I(X)_k$ such that $\widetilde{E}$ is invariant under G'_v. Then the function $S \to \widehat{E}(S\Phi)$ is bounded on $T'_{\mathbb{A}}$, uniformly in Φ on any compact subset of $\S(X_{\mathbb{A}})$.*

Let $\widehat{E}_\alpha$, for $0 \leq \alpha \leq n$, be the sum of $\widehat{\mu}_i$ for $i \in I_k^{(\alpha)}$; we will have $\widehat{E} = \widehat{E}_0 + \ldots + \widehat{E}_n$. If $t \in T_k$, $\overline{\lambda}_t$ determines a permutation on each of the sets $I_k^{(\alpha)}$, so that each of measures $\widehat{E}_\alpha$ is invariant under T_k. On the other hand, $G_{\mathbb{A}}$ leaves invariant each of the sets $i_X^{-1}(\{i\})$; with the hypotheses of the statement, it follows that G'_v leaves invariant each of the measures $\widehat{\mu}_i$, and thus also each of $\widehat{E}_\alpha$; these therefore satisfy the same hypotheses as $\widehat{E}$, and we are reduced to the proof of the $\widehat{E}_\alpha$.

Let α be such that $0leq\alpha \leq n$. There is a constant q, equal to 1 is v is a place at infinity and to $N(\mathfrak{p}_v)$ otherwise, such that there exists, for all $\tau \in \mathbb{R}_+^*$, an element y of k_v satisfying $\tau \leq |y|_v \leq q\tau$; and there exists a compact subset C of I_k such that for all elements t of I_k satisfying that $1 \geq |t|_{\mathbb{A}} \geq q^{-1}$ can be written in the form ρc with $\rho \in k$, $c \in C$; we will denote C^n the compact subset of $T_{\mathbb{A}}$ formed by elements $(c_1, \ldots, c_n)$ with $c_\beta \in C$ for $1 \leq \beta \leq n$. Let us also give ourselves a compact subset C_0 of $\mathcal{S}(X_{\mathbb{A}})$, and apply Lemma 6 of Chapter I, n^{o}13, to the space $X_k = M_{m,n}(\mathfrak{k})$ considered as the product of spaces

$$X_k^{(1)} = M_{m,\alpha+1}(\mathfrak{k}), \quad X_k^{(2)} = \ldots = X_k^{(n-\alpha)} = M_{m,1}(\mathfrak{k})$$

in such a way that the projection of $x = (x_1, \ldots, x_n)$ on these spaces are respectively $(x_1, \cdots, x_{\alpha+1})$, $x_{\alpha+2}, \ldots, x_n$; we will take $Y_k = M_{n,\alpha+1}(\mathfrak{k})$, and p will be the morphism from X to Y given by

$$p(x) = {}^t x' \cdot h \cdot (x_1, \cdots, x_{\alpha+1}).$$

We conclude that there exists $\Phi_0 \in \mathcal{S}(X_{\mathbb{A}})$ such that we have

$$|\mathfrak{d}(\lambda_\theta)\mathfrak{d}(\lambda_c)\Phi(x)| \leq \Phi_0(x)$$

whenever $x \in X_{\mathbb{A}}$, $i_X(x) \in I_k^{(\alpha)}$, $c \in C^m$, $\Phi \in C_0$, and that θ belongs to the set Θ'_α of elements $(a_{\tau_1}, \ldots, \alpha_{\tau_n})$ of $\Theta(T)$ which satisfy the condition

$$\tau_1 = \ldots = \tau_{\alpha+1} \geq \ldots \geq \tau_n \geq 1.$$

Moreover, we can assume that Φ_0 has been taken of the form $\Phi_v(x_v)\Phi'(x')$, with

$$\Phi_v \in \mathcal{S}(X_v), \quad \Phi' \in \mathcal{S}(X').$$

Now let $t = (t_1, \ldots, t_n)$ be an element of $T'_{\mathbb{A}}$. For $1 \leq \beta \leq \alpha$, let $y_\beta \in k_v$ be such that $|y_\beta|_v$ is between $|t_\beta t_{\alpha+1}^{-1}|_{\mathbb{A}}$ and $q|t_\beta t_{\alpha+1}^{-1}|_{\mathbb{A}}$; let $y_\beta = 1$ for $\beta \geq \alpha+1$; we will have $|y_\beta|_v \geq 1$ for $1 \leq \beta \leq n$. On the other hand, for $\beta \geq \alpha+1$, let $\tau_\beta \in \mathbb{R}_+^*$ such that $|a_{\tau_\beta}|_{\mathbb{A}} = |t_\beta|_{\mathbb{A}}$, and let $\tau_\beta = \tau_{\alpha+1}$ for $1 \leq \beta \leq \alpha$. For all β, we have

$$1 \geq |t_\beta y_\beta^{-1} a_{\tau_\beta}^{-1}|_{\mathbb{A}} \geq q^{-1},$$

so that we can write $t_\beta = \rho_\beta y_\beta a_{\tau_\beta} c_\beta$, with $\rho_\beta \in k$, $c_\beta \in C$, for $1 \leq \beta \leq n$. By setting

$$y = (y_1, \ldots, y_n), \quad \theta = (a_{\tau_1}, \ldots, a_{\tau_n}),$$

we will thus have $t = \rho y \theta c$ with $\rho \in T_k$, $y \in T_v$, $\theta \in \Theta'_\alpha$, and $c \in C^m$. As $\widehat{E}_\alpha$ is invariant under T_k, it follows that we have

$$|\widehat{E}_\alpha(\mathbf{d}(\lambda_t))\Phi| \leq \widehat{E}_\alpha(\mathbf{d}(\lambda_t)\Phi_0)$$

whenever $\Phi \in C_0$, Φ_0 being chosen as above.

To evaluate the right hand side of the inequality, we will apply Lemma 22 to each of the measures $\widehat{\mu}_i$ for $i \in I_k^{(\alpha)}$; denoting by $c_i(\Phi')$ the constant that appears in the right hand side of (37) when we substitute $\widehat{\mu}_i$ for μ in the left hand side, we obtain:

$$\widehat{E}_\alpha(\mathbf{d}(\lambda_y)\Phi_0) = \sum_{i \in I_k^{(\alpha)}} c_i(\Phi') \int_{U(i)_v} \mathbf{d}(\lambda_y)\Phi_v \cdot |\theta_i|_v.$$

As we have $y_\beta = 1$ for $\beta \geq \alpha + 1$, it follows from the definition of $I_k^{(\alpha)}$ that the automorphism $\overline{\lambda}_v$ of $I(X)_v$ determines by λ_y leaves invariant all elements of $I_k^{(\alpha)}$. By applying to the terms of the right hand side of the above relation formulas (35) and (36) of n^{o}46, we then obtain

$$\widehat{E}_\alpha(\mathbf{d}(\lambda_y)\Phi_0) = |y_1 \ldots y_\alpha|_v^{(-m+2n+4\varepsilon-2)\delta/2} \widehat{E}_\alpha(\Phi_0).$$

As we have assumed (B) is satisfied, the exponent of the right hand side is < 0. As we have $|y_\beta|_v \geq 1$ for any β, we finally obtain

$$|\widehat{E}_\alpha(\mathbf{d}(\lambda_t)\Phi)| \leq \widehat{E}_\alpha(\Phi_0),$$

this inequality being valid whenever $t \in T'_{\mathbb{A}}$ and $\Phi \in C_0$. This finish the proof.

50. We are now in a position to proof the main result we had in this chapter.

Theorem 4. *Let $\mathcal{A}_k$ an algebra of type (I) and X_k an left $\mathcal{A}_k$-module, satisfying condition (B), that is to say $m > 2n + 4e - 2$. Let v a place of k such that $U(0)_v$ is not empty, and G'_v a subgroup of G_v acting transitively on $U(i)_v$ for any $i \in I(X)_k$. Let E' be a positive tempered measure on $X_{\mathbb{A}}$, invariant by $Ps_k = Ps(X_k/\mathcal{A}_k)$ and under G'_v, and such that $E' - E$ is a sum of measures carried by the $U(i)_{\mathbb{A}}$ for $i \in I(X)_k$. Then we have $E' = E$.*

With the notations of Theorem 2 and 3 of Chapter V, n^{o}44, we have $E = \sum E_Z$, the summation being extended to all submodules Z_k of X_k; E_X is the sum of measures $|\theta_i|_{\mathbb{A}}$ respectively carried by the $U(i)_{\mathbb{A}}$, while E_Z has its support contained in $Z_{\mathbb{A}}$ for any $Z_k \neq X_k$. On the universal domain, let U be the set of points of X of maximal rank; it is k-open; it is an orbit for the group of invertible elements of $\mathcal{A}$; for any $i \in I(X)_k$, $U(i)$ is a subvariety of U and is therefore k-closed in U. Let $F = X - U$, it is a k-closed subset of X, invariant under the action of the group of invariable elements of $\mathcal{A}$ and for stronger reason under G, and which contains Z whenever $Z_k \neq X_k$. Consequently, $F_{\mathbb{A}}$ is a closed subset of $X_{\mathbb{A}}$, invariant under $G_{\mathbb{A}}$ and evidently also under $\mathrm{Aut}(X_k)$, which contains $Z_{\mathbb{A}}$ for $Z_k \neq X_k$, and which has no common point with $U(i)_{\mathbb{A}}$ for any $i \in I(X)_k$. It follows that E_X is the restriction of E to the open set $X_{\mathbb{A}} - F_{\mathbb{A}}$, and that the sum $\sum E_Z$, extended to all $Z_k \neq X_k$, is the restriction of E to $F_{\mathbb{A}}$ The hypothesis made on E' then implies that the restriction $\widehat{E}$ of E' to $X_{\mathbb{A}} - F_{\mathbb{A}}$ is a sum of measures $\widehat{\mu}_i$ respectively carried by the $U(i)_{\mathbb{A}}$ for $i \in I(X)_k$, and that the restriction of E' to $F_{\mathbb{A}}$ is the same as that of E, so that we have $E' - \widehat{E} = E - E_X$; moreover, as E' and $F_{\mathbb{A}}$ are invariant under G'_v and under $\mathrm{Aut}(X_k)$, the same is true of $\widehat{E}$, which therefore satisfies the hypotheses of Lemma 23 of n^{o}49.

According to this lemma, the function $S \to \widehat{E}(\Phi)$ is therefore bounded on $T'_{\mathbb{A}}$ uniformly in Φ on any compact subset of $\mathcal{S}(X_{\mathbb{A}})$. This conclusion applies in particular to E_X, which is deduced from E as $\widehat{E}$ is from E'; it therefore also applies to the tempered measure E'' given by

$$E'' - E' - E = \widehat{E} - E_X.$$

But this is invariant under Ps_k, since this is the case for E, and also, by hypothesis, for E'; we can therefore apply Lemma 20 of n^{o}47 to it, which shows that the function $S \to E''(S\Phi)$ is bounded on $\mathrm{Mp}_{\mathbb{A}}$, for any $\Phi \in \mathcal{S}(X_{\mathbb{A}})$. For any $\Phi \in \mathcal{S}(X_{\mathbb{A}})$, we will denote by $M(\Phi)$ the upper bound of $|E''(S\Phi)|$ for $S \in \mathrm{Mp}_{\mathbb{A}}$; we will have $M(S\Phi) = M(\Phi)$ for any $S \in \mathrm{Mp}_{\mathbb{A}}$.

The measure E'' is the sum of measures $\mu''_i = \widehat{\mu}_i - \mu_i$, where μ_i again denotes the measures $|\theta_i|_{\mathbb{A}}$ carried by $U(i)_{\mathbb{A}}$. We have therefore, for $\Phi \in \mathcal{S}(X_{\mathbb{A}})$:

$$E''(\Phi) = \sum_{i \in I(X)_k} \int \Phi d{\mu''}_i;$$

in this formula, the series of the right hand side is absolutely convergent, uniformly in Φ on any compact subset of $\mathcal{S}(X_{\mathbb{A}})$, since this is evidently the case for the analogues series formed by means of the positive measures $\widehat{\mu}_i$ and μ_i. Let $i^* \in I(X)^*_{\mathbb{A}}$; let q be the elements of $Q(X/\mathcal{A})_{\mathbb{A}}$ which corresponds to u^* by the virtue of the isomorphisms of Chapter II, n^{o}15, that is to say which is defined by $q(x) = [i_X(x), i^*]$ for $x \in X_{\mathbb{A}}$. We then have, for $\Phi \in \mathcal{S}(X_{\mathbb{A}})$:

$$\mathbf{t}(q)\Phi(x) = \Phi(x)\chi(q(x)) = \Phi(x)\chi([i_X(x), i^*]),$$

and consequently

$$E''(\mathbf{t}(q)\Phi) = \sum_{i \in I(X)_k} \chi([i, i^*]) \int \Phi d\mu''_i.$$

We can consider this formula as giving the development of the left hand side as a Fourier series on the compact group $I(X)^*_{\mathbb{A}}/I(X)^*_k$. As the left hand side, in absolute value, is $\leq M(\Phi)$, we will have, by the virtue of Fourier formula

$$|\int \Phi d{\mu_i}''| \leq M(\Phi),$$

and consequently, replacing Φ by $S\Phi$:

$$|\int S\Phi \cdot d{\mu_i}''| \leq M(\Phi), \tag{38}$$

this inequality being valid for any $S \in \mathrm{Mp}_{\mathbb{A}}$, $i \in I(X)_k$ and $\Phi \in \mathcal{S}(X_{\mathbb{A}})$.

Let us take Φ of the form $\Phi_v(x_v)\Phi'(x')$, with $\Phi_v \in \mathcal{S}(X_v)$, $\Phi' \in \mathcal{S}(X')$. By virtue of the hypotheses made on E', the $\widehat{\mu}_i$ are invariant under G'_v, and the same is obviously true for the μ_i; we can therefore apply Lemma 22 of n^{o}49. Consequently, we can write

$$\int \Phi d{\mu_i}'' = c_i(\Phi') \int_{U(i)_v} \Phi h_v \cdot |\theta_i|_v.$$

In this formula, let us replace Φ by $\mathbf{d}(\lambda_t)\Phi$, with $t \in T_v$; this amounts to not changing Φ' and replacing Φ_v by $\mathbf{d}(\lambda_t)\Phi_v$, this last function being given by the formula analogous to (36), relatively to X_v. By setting $i' = i\overline{\lambda}_t$, this gives, according to (35):

$$\int \mathbf{d}(\lambda_t)\Phi \cdot d\mu_i{}'' = c_i(\Phi')|t_1 \cdots t_n|_v^{(-m+2n+4\varepsilon-2)\delta/2} \int_{U(i')_v} \Phi_v \cdot |\theta_{i'}|_v. \tag{39}$$

Let us denote by $F(i')$ for the integral appears in the right hand side; Proposition 6 of Chapter III, n^{o}37, shows that it is a continuous function of i' in $I(X)_v$, so that $F(i')$ tends towards $F(0)$ when we make all the $|t_\alpha|_v$ toward 0 As the exponent of $|t_1 \dots t_n|_v$, in the right hand side of (39), is < 0 by the virtue of condition (B), and that, according to (38), this right hand side must remain bounded for any $t \in T_v$, we conclude that we must have $c_i(\Phi')F(0) = 0$. But $F(0)$ is given by

$$F(0) = \int_{U(0)_v} \Phi_v \cdot |\theta_0|_v,$$

and, by hypothesis, $U(0)_v$ is not empty; we can therefore choose Φ_v so that $F(0)$ is nonzero. We thus have $c_i(\Phi') = 0$, and consequently $\int \Phi d\mu'' = 0$ for any Φ is of the form $\Phi_v(x_v)\Phi'(x')$. This obviously implies $\mu_i'' = 0$. As this is true for any $i \in I(X)_k$, we thus have $E'' = 0$, that is to say $E' = E$.

We will observe that Theorem 4 provides a characterization of the measure E_X; by recurrence on the rank n of X_k, starting from $E_0 = \delta_0$.

VI Siegel's Formula

51. The **Siegel's formula**, in the sense we understand it here, gives the relation between the **Eisenstein-Siegel series** $E(\Phi)$ that we introduced and studied in Chapter V, and certain integral of type considered in Chapter I, n^{os}7, 8 and 12.

THe involution algebra $\mathcal{A}_k$ and the module X_k being given as before, we will continue to denote by $\mathcal{A}$ the extension of $\mathcal{A}_k$ to the unversal domain and by G the reductive group of elements u of $\mathcal{A}$ such that $u \cdot u^\iota = 1$. We will denote by ρ the representation of G (in the group of automorphisms of the underlying vector space X) which is given by $\rho(u)x = ux$. We will again consider the subgroup G_1 of G which was defined in n^{o}28 of Chapter II; we will denote φ_1 the canonical injection of G_1 into G, and ρ_1 the representation of G_1 defined by $\rho_1 = \rho \circ \varphi_1$. When $\mathcal{A}_k$ is of type (I_4) with $m \geq 3$, or of type (I_3) when $m \geq 2$, we will denote by $\widetilde{G}$ the simply connected covering of G_1 (the **spin** group), by $\widetilde{\varphi}_1$ the canonical homomorphism of $\widetilde{G}$ onto G_1 (whose kernel is a group with two elements), and we will set $\widetilde{\varphi} = \varphi_1 \circ \widetilde{\varphi}_1$, $\widetilde{\rho} = \rho \circ \widetilde{\varphi}$.

With these notations, let us write, for $\Phi \in \mathcal{S}(X_{\mathbb{A}})$:

$$I(\Phi) = \int_{G_{\mathbb{A}}/G_k} \sum_{\xi \in X_k} \Phi(\rho(u)\xi) \cdot d\mu(u),$$

where μ is a Haar measure on $G_{\mathbb{A}}$. For $\Phi \geq 0$, $\Phi(0) = 1$, this integral is $\geq \mu(G_{\mathbb{A}}/G_k)$; for it to converge whatever $\Phi \in \mathcal{S}(X_{\mathbb{A}})$, it is necessary that $G_{\mathbb{A}}/G_k$ is of finite measure. Under these conditions, we will agree to normalize μ by the condition $\mu(G_{\mathbb{A}}/G_k) = 1$.

We will denote by $I_1(\Phi)$ and $\widetilde{I}(\Phi)$ the integrals obtained by substituting, on the one hand G_1 and ρ_1, on the other hand (when $\widetilde{G}$ is defined) $\widetilde{G}$ and $\widetilde{\rho}$ for G and ρ.

We will first determine the cases where these integrals satisfy the sufficient conditions of convergence contained in Lemma 5 of Chapter I, n^{o}12; we will limit ourselves for this to the cases where $\mathcal{A}_k$ is of tpe (I) or of type (II). In the case (I), with our usual notations, the involution ι on $\mathcal{A}_k$ is defined by means of an η-hermitian form h on $A \times A$, A being a right vector space of dimension m over $\mathfrak{k}$; we will denote by r the index of h, that is the dimension over $\mathfrak{k}$ of a maximal totally isotropic subsapce for h in A.

Proposition 8. *If $\mathcal{A}_k$ is of type (I), the integral $I(\Phi)$ is absolutely convergent, whatever $\Phi \in \mathcal{S}(X_{\mathbb{A}})$, each time that $r = 0$ or that $m - r > n + 2e - 1$; the same is true of $I_1(\Phi)$, and of $\widetilde{I}(\Phi)$ when $\widetilde{G}$ is defined. If $\mathcal{A}_k$ is of type (II), $I_1(\Phi)$ is absolutely convergent for any $\Phi \in \mathcal{S}(X_{\mathbb{A}})$, each time that $m = 1$ or that $m > p + q$.*

The assertion is trivial when G (resp. G_1, $\widetilde{G}$) is **anisotropic**, that is to say contains no non-trivial torus, since then $G_{\mathbb{A}}/G_k$, or the analogous space for G_1 or $\widetilde{G}$, is compact. This is case, as we know, for $r = 0$ in case (I), and for $m = 1$, concerning the group G_1, in case (II). Leaving these cases aside, let us first consider case (I); h being of index $r > 0$, we know that we can choose a basis in A for which h is given by a matrix of the form

$$h = \begin{pmatrix} 0 & 0 & 1_r \\ 0 & h_0 & 0 \\ \eta \cdot 1_r & 0 & 0 \end{pmatrix},$$

where h_0 is the matrix (or order $m - 2r$) of an η-hermitian form of index 0 (or **anisotropic**); this implies that we have $m \geq 2r$. For $t = (t_1, \cdots, t_r) \in (\mathbb{G}_m)^r$, let us denote by $f(t)$ the diagonal matrix of order m whose diagonal elements are

$$(t_1, \cdots, t_r, 1, \cdots, 1, t_1^{-1}, \cdots, t_r^{-1}).$$

It is well known that f is then an isomorphism of $(\mathbb{G}_m)^r$ onto a maximal trivial torus T of G; T is also a maximal trivial torus of G_1, and a maximal trivial torus $\widetilde{T}$ of $\widetilde{G}$, in the case where $\widetilde{G}$ is defined, is given by $\widetilde{T} = \widetilde{\varphi}^{-1}(T)$. Let us identify T with $(\mathbb{G}_m)^r$ by means of f, and denote by δ, as always, the dimension of $\mathfrak{k}$ over k. Then it is easy to see that we can order the roots of G so that the strictly positive roots (i.e., $\neq 1$ and $\succ 1$) are the following: $t_i t_j^{-1}$ and $t_i t_j$ for $1 \leq i < j \leq r$, each with multiplicity δ; t_i and t_i^2 for $1 \leq i \leq r$, with respective multiplicities $\delta(m - 2r)$ and $\delta(1 - \varepsilon)$. The weights of ρ are the t_i and the $t)i^{-1}$ for $1 \leq i \leq r$, each with multiplicity δm. The roots of G_1 and the weights of ρ_1 are the same as those of G and of ρ; those of $\widetilde{G}$ and of $\widetilde{\rho}$ are deduced by the homomorphism of $\widetilde{T}$ onto T induced by $\widetilde{\varphi}$. The application of Lemma 5 of Chapter I, n^{o}12, then reduces to a calculation that presents no difficulty, and gives the announced result. The verification is done similarly in case (II).

Since we have $m \geq 2r$, hence $m - r \geq m/2$, in case (I), we see that condition (B) of Chapter III, n^{o}38 (i.e., $m > 2n + 4e - 2$ in case (I) and $m > p + q$ in case (II)) always implies the convergence condition of Proposition 8, in the case (I) as well as in the case (II); it is even equivalent to it, in case (I), if $m = 2r$, and, in case (II), if $m \neq 1$.

52. We can now prove the **Siegel formula**; it is contained in the following theorem:

Theorem 5. *Let $\mathcal{A}_k$ be a algebra of type (I) and X_k a left $\mathcal{A}_k$-module, satisfies the condition (B), i.e., $m > 2n + 4\varepsilon - 2$. Let v be a place of k such that $U(0)_v$ is not empty, and G'_v a subgroup of*

G_v that operates transitively on $U(i)_v$ for any $i \in I(X)_k$. Let ν a positive measure over $G_{\mathbb{A}}/G_k$, invariant by G'_v, such that $\nu(G_{\mathbb{A}}/G_k) = 1$ and that the integral

$$I_v(\Phi) = \int_{G_{\mathbb{A}}/G_k} \sum_{\xi \in X_k} \Phi(\rho(u)\xi) \cdot d\nu(u)$$

is absolutely convergent for any $\Phi \in \mathcal{S}(X_{\mathbb{A}})$. Then we have $I_v(\Phi) = E(\Phi)$, and, for all $i \in I(X)_k$:

$$\int_{G_{\mathbb{A}}/G_k} \sum_{\xi \in U(i)_k} \Phi(\rho(u)\xi) \cdot d\nu(u) = \int \Phi d\mu_i, \tag{40}$$

where μ_i is the measure $|\theta_i|_{\mathbb{A}}$ determined on $U(i)_{\mathbb{A}}$ by the gauge θ_i defined in Theorem 2 of Chapter V, n^{o}44. These results remain valid when G and ρ are replaced by G_1 and ρ_1, or, in cases (I_3), (I_4), by $\widetilde{G}$ and $\widetilde{\rho}$. They are valid in particular when ν is taken to be the Haar measure on G normalized by $\nu(G_{\mathbb{A}}/G_k) = 1$, or the Haar measure on G_1 (or, in cases (I_3), (I_4), on $\widetilde{G}$) normalized similarly.

We will proceed by recurrence on the rank n of X_k; for this, we will first show that the hypotheses made on v and ν relative to X_k imply that v and ν have the analogous properties relative to any $\mathcal{A}_k$-module X'_k of rank $n' \leq n$. All $\mathcal{A}_k$-modules of the same rank being isomorphic, it suffices to proof this assertion when X'_k is a submodule of X_k. Regarding the conditions imposed to ν, the asseration is evident. Regarding v, let us agree, for all $i' \in I(X')_k$, to denote by $U'(i')_v$ the set of elements x' of X'_v, of maximal rank in X'_v, which satisfy $i_{X'}(x') = i'$. Let us first consider the hypothesis $U(0)_v \neq \emptyset$, which we want to show implies $U'(0)_v \neq \emptyset$. Let us use the notation of n^{o}26 of Chapter II, taking $K = k_v$ and writting m_v, n_v instead of m_K, n_K. Suppose first that $\mathcal{A}_v$ is of type (I); as in n^{o}26 of Chapter II, let us write $\mathcal{A}_v$ in the form $M_{m_v}(\mathfrak{K})$, where $\mathfrak{K}$ is a division algebra over k_v and where $m_v = m\nu$; the involution ι on $\mathcal{A}_v$ is also determined by an involution on $\mathfrak{K}$, and by a matrix h_v with m_v rows and m_v columns over $\mathfrak{K}$, η_v-hermitian with respect to this involution. Saying that $U(0)_v \neq \emptyset$ amounts to saying that h_v is of index $\geq n_v = n\nu$, and $U'(0)_v \neq \emptyset$ amounts to saying that this index is $\geq n'_v = n'\nu$; for $n' \leq n$, the first assertion obviously implies the second. If $\mathcal{A}_k$ is of type (II), $U(0)_v \neq \emptyset$ and $U'(0)_v \neq \emptyset$ are respectively equivalent to $m_v \geq 2n_v$ and to $m_v \geq 2n'_v$, according to n^{o}23 of Chapter II, and we draw the same conclusion; we can note moreover that, in this case, we have $\varepsilon = \frac{1}{2}$ according to n^{o}26 of Chapter II, thus $m > 2n$ according to (B), so that we certainly have $U(0)_v \neq \emptyset$ and $U'(0)_v \neq \emptyset$. Let us move on to the transitivity of G'_v on the sets $U(i)_v$, $U'(i')_v$. Let us identify X_k, as usual, with $M_{m,n}(\mathfrak{k})$, and X'_k with the module of elements of $M_{m,n}(\mathfrak{k})$ of the form $(x_1, \ldots, x_{n'}, 0, \ldots, 0)$; let us denote by X''_k the module of elements of $M_{m,n}(\mathfrak{k})$ of the form $(0, \ldots, 0, x_{n'+1}, \cdots, x_n)$, so that we have $X_k = X'_k \oplus X''_k$. Let us identify $I(X_k)$ and $I(X'_k)$ with the spaces of η-hermitian matrices over $\mathfrak{k}$, of order n and order n', respectively, as explained in n^{o}19 of Chapter II. Let $i' \in I(X'_k)$, and let i be the element of $I(X_k)$ given by the matrix $\begin{pmatrix} i' & 0 \\ 0 & 0 \end{pmatrix}$ with n rows and n columns. Suppose first that $\mathcal{A}_v$ is of type (I); with the same notation as above, we can identify $I(X_v)$ with the space of η_v-hermitian matrices with n_v rows and n_v columns over $\mathfrak{K}$, and do the same for $I(X'_v)$. The canonical isomorphism of $I(X_k) \otimes k_v$ onto $I(X_v)$, whose definition results from n^{o}15 of Chapter II, induces on $I(X_k)$ a k-linear map from $I(X_k)$ to $I(X_v)$; it would be easy to make this explicit by introducing some supplement notation, but that is unnecessary for our purpose. Let i_v be the element of $I(X_v)$, image of i by this map; if i'_v is element of $I(X'_v)$ similarly deduced from i', we will have $i_v = \begin{pmatrix} i'_v & 0 \\ 0 & 0 \end{pmatrix}$. According to n^{o}19 of

Chapter II, $U(i)_v$ is identified with the set of matrices x with m_v rows and n_v columns over $\mathfrak{K}$, of maximal rank (that is to say, equal to n_v), which satisfy $h_v[x] = i_v$; we have an analogous assertion for $U'(i')_v$. By hypothesis, $U(0)_v$ is not empty, which means that h_v is of index $\geq n_v$; we easily deduce from this, nu elementary reasoning, that $U(i)_v$ is not empty; let $a \in U(i)_v$. Then, if a', a'' are the projections of a onto X'_v, X''_v for the decomposition $X_v = X'_v \oplus X''_v$ of X_v as a direct sum, we will have

$$h_v[(a', a'')] = i_v = \begin{pmatrix} i'_v & 0 \\ 0 & 0 \end{pmatrix},$$

thus $h_v[a'] = u'_v$; moreover, as a is of maximal rank (equal to n_v) in X_V, a' must be of maximal rank (equal to n'_v) in X'_v; we therefore have $a' \in U'(i')_v$. Now let $x' \in U'(i')_v$. According to Proposition 3 of Chapter II, n^{o}22, there is an $u \in G_v$ such that $x' = ua'$. Then a and ua both belong to $U(i)_v$, so that, by hypothesis, there is $u' \in G'_v$ such that $ua = u'a$, hence $x' = u'a'$. This shows indeed that G'_v acts transitively on $U'(i')_v$. The proof is done in a completely analogous manner when $\mathcal{A}_v$ is of type (II).

Now let v and ν satisfy the hypothese of Theorem 5 relative to a module X_k of rank n, and let us first consider the case of integral I_v formed by means of G of ρ. For $n = 0$, the the assertion of the theorem reduce to $I_v = \delta_0$, which is an obvious consequence of the hypothesis $\nu(G_{\mathbb{A}}/G_k) = 1$. Let us proceed by recurrence on n, and suppose $n \geq 1$. As $I_v(\Phi0$ is convergent by hypothesis for any $\Phi \in \mathcal{S}(X_{\mathbb{A}})$, the Lemma 2 of Chapter I, n^{o}2, joint with Lemma 5 of [14], n^{o}41, immediately shows that I_v is a positive tempered measure. The Theorem 6 of [14], n^{o}41, and the Proposition 9 of [14], n^{o}51, then show that I_v is invariant by Ps_k; it is obviously also invariant by G'_v. For the same reason, if we denote by $I_{v,i}(\Phi)$ the left hand side of (40), $I_{v,i}$ is a positive tempered measure. Let $I_{v,X}$ be the sum of $I_{v,i}$ for $i \in I(X)_k$; we can consider $I_{v,X}$ as defined by the integral analogous to the one that defines I_v, but where the summation is restricted to element ξ of X_k that are of maximal rank in X_k. Similarly, for any submodule Z_k of X_k, let us denote by $I_{v,Z}$ the positive tempered measure defined by the integral analogous to the one that defines I_v, but where the summation is restricted to element ξ of Z_k that are of maximal rank in Z_k, or in other words to elements ξ of X_k that satisfy $\mathcal{A}_k \cdot \xi = Z_k$. Taking into account Theorem 2 of Chapter V, n^{o}44, we see that Theorem 5 for X_k implies that $I_{v,X} = E_X$; consequently, the recurrence hypothesis implies that $I_{v,Z} = E_Z$ for any submodule Z_k of X_k, other than X_k. We therefore have, by the virtue of this hypothesis:

$$I_v = \sum_{i \in I(X)_k} I_{v,i} + \sum_{Z_K \neq X_K} E_Z. \tag{41}$$

According to Theorem 3 of Chapter V, n^{o}44, the second sum of right hand side is nothing but $E - E_X$. On the other hand, according to Proposition 3 of Chapter II, n^{o}22, those $U(i)_k$ which are not empty are orbits of G_k in X_k; taking into account the change of notation, formula (10) of Chapter I, n^{o}7, then shows that the measures $I_{v,i}$ are respectively carried by $U(i)_{\mathbb{A}}$, in the sense in which this term was defined in n^{o}49 of Chapter V. As a consequently, I_v satisfies all the hypotheses of Theorem 4 of Chapter V, n^{o}50. According to this theorem, we therefore have $I_v = E$, and consequently $I_{v,X} = E_X$ according to (41). As $I_{v,i}$ and μ_i are the restrictions of $I_{v,X}$ and of E_X, respectively, to the set $i_X^{-1}(\{i\})$, it follows that we have $I_{v,i} = \mu_i$ for any $i \in I(X)_k$. To see that the result thus obtained applies when we take for v the Haar measure on $g_{\mathbb{A}}$, it suffices to take $G'_v = G_v$, the place v being chosen so that $U(0)_v$ is not empty; mow this is the case for almost all v, for example according to the results recalled in n^{o}4 of Chapter I. This complete the proof as far as G and ρ are concerned.

There would be nothing to change to what precedes, as far as G_1 and ρ_1 (resp. $\widetilde{G}$ and $\widetilde{\rho}$) are concerned if we were assured, in all cases, that $(G_1)_K$, operating on X_K by means of ρ_1 (resp. $\widetilde{G}_K$ operating by means of $\widetilde{\rho}$), had the same orbits as G_K in X_K, for any $K \supset k$; but this question does not always seem easy to decide, and we can be content with partial results in this regard.

The question does not arise if $\mathcal{A}_k$ is of type (I_0) or (I_1), since then $\widetilde{G}$ is not defined and $G_1 = G$. If $\mathcal{A}_K$ is of type (II), $\mathcal{A}_k$ must be of type (I_2) according to n^{o}26 of Chapter II, so that we have $\varepsilon = \frac{1}{2}$, hence $m > 2n$ according to (B), hence $m_K > 2n_K$; by means of the results of n^{o}23 of Chapter II, we then easily verify that $(G_1)_K$ operates transitively on the orbits G_K in X_K. This will be the case, in particular, if $\mathcal{A}_k$ is of type (I_2) and we take for K the universal domain. When $\mathcal{A}_K$ is of type (I_4), we similarly verify that $(G_1)_K$ operates transitively on the orbits G_K in X_K provided that $m_K > 2n_K$; (B) being always assumed satisfied, this will be the case whenever $\mathcal{A}_k$ is of type (I_3) or (I_4) and K is the universal domain. Finally, the homomorphism $\widetilde{\varphi}_1$ of $\widetilde{G}$ onto G_1 determines a homomorphism of $\widetilde{G}_K$ onto $(G_1)_K$ whenever K is algebraically closed; thus $\widetilde{G}_K$ then operates transitively on the orbits of $(G_1)_K$ in X_K, and this is the case in particular when K is the universal domain. This being established, if we take up the proof given above of the first part of Theorem 5 for G, we see that it also applies to G_1 (resp. to $\widetilde{G}$) provided we know that the $I_{v,i}$ are measures respectively carried by the $U(i)_{\mathbb{A}}$. Now $U(i)_k$ is in any case a union of orbits of $(G_1)_k$ (resp. $\widetilde{G}_k$), to each of which we can apply formula (10) of Chapter I, n^{o}7; as, in this formula, $H(\omega)$ denotes the orbit of ξ_ω on the universal domain, we see that it suffices here to know that $U(i)$ is the orbit of any points of $U(i)_k$ by G_1 (resp. $\widetilde{G}$) on the unversal domain; now this results follows from what precedes.

It remains only to verify the last assertion of Theorem 5 concerning G_1 (resp. $\widetilde{G}$); for this, it suffices to show that we can choose v so that $U(0)_v$ is not empty and that $(G_1)_v$ (resp. $\widetilde{G}_v$) operates transitively on the $U(i)_v$. If $\mathcal{A}_k$ is of type (I_0) or (I_1), there is nothing to prove. If $\mathcal{A}_k$ is of type (I_2), it suffices, according to what precedes and according to n^{o}23 of Chapter II, to choose v to that $\mathcal{A}_v$ is of type (II). Let $\mathcal{A}_k$ is of type (I_4), and consequently, according to what precedes, $G'_v = (G_1)_v$ will have the desired properties. Concerning $\widetilde{G}_v$, let us choose v as we have just said, assuming moreover that v is not a place at infinite. The condition (B) then implies $m_v - 2n_v \geq 3$; making use of the results of n^{o}23 of Chapter II and the knwon properties use of the results of n^{o}23 of Chapter II and the known properties of the **spinorial norm**, we easily conclude that $\widetilde{G}_v$ then operates transitively on the orbits of $(G_1)_v$ in X_v. This completes the proof of Theorem 5.

When we take for ν the Haar measure in $G_{\mathbb{A}}$, the formula $I_v = E$ is the same one that was announced in the previous memoir ([14], n^{o}52).

53. Applying now Theorem 5 of n^{o}52 to the group G, and to the Haar measure ν on G, we will extract from formula (40) some arithmetic results that are contained therein. The first, of a qualitative nature, is obtained by observing that the tempered measure of (40) is 0 whenever $U(i)_k$ is empty, while the measure μ, which appears in the right hand side can be 0, by the virtue of its definition, only if $U(i)_{\mathbb{A}}$ is empty. Consequently, *when* (B) *is satisfied, the assertion* $U(i)_k \neq \emptyset$ *and* $U(i)_{\mathbb{A}} \neq \emptyset$ *is equivalent*. Taking into account the results mentioned in n^{o}4 of Chapter I, it amounts to saying that, if (B) is satisfied, $U(i)_k \neq \emptyset$ is equivalent to the assertion $U(i)_v \neq \emptyset$ **for any** v. In this form, we recognize the **Hasse principle**, except that this principle, as we know, remains valid even for certain values of m and n that do not satisfy the condition (B); for example, if $\mathcal{A}_k$ is of type (I_4), it is valid for any m and n. However, the remarks at the end of Chapter III, n^{o}38, combined with what has just been proved, show that, if (B) is satisfied, $U(i)_k \neq \emptyset$ is equivalent even to the assertion $U(i)_v \neq \emptyset$ **whatever the place at infinity** v **of** k. For example, for $k = \mathbb{Q}$, $\mathcal{A}_k$ of type (I_4), $n = 1$ and $i = 0$, we thus obtain the classical Meyer theorem, according to which any indefinite quadratic form in $m \geq 5$ variables, over $\mathbb{Q}$ **represents** 0.

From (40), we will now derived quantitative results concerning the **Tamagawa numbers** of classical groups. Let us first observe that by the virtue of Proposition 3 of Chapter II, n^{o}22, and its corollary, the condition (W) of n^{o}8 of Chapter I is well satisfied by G operating on X by means of the representation ρ, so that we can apply the results of n^{o}8 of Chapter I to the left hand side of (40). Let ξ be a element of maximal rank of X_k; let $i = i_X(\xi)$, and let g the stabilizer of ξ in G; according to n^{o}8 of Chapter I, we can identify $U(i)$ with G/g and $U(i)_{\mathbb{A}}$ with $G_{\mathbb{A}}/g_{\mathbb{A}}$; moreover, according to this n^o, $g_{\mathbb{A}}/g_k$ is of finite measure, which is moreover easy to verify directly by means of the results of n^{o}23 of Chapter II which give the structure of g. Let ν_0 be the Haar measure on g, normalized by the condition $\nu_0(g_{\mathbb{A}}/g_k) = 1$, n^{o}8 of Chapter I then shows that the left hand side of (40) is nothing but the measure ν/ν_0 carried by $G_{\mathbb{A}}/g_{\mathbb{A}} = U(i)_{\mathbb{A}}$; (40) can therefore be expressed by saying that $\nu/\nu_0 = |\theta_i|_{\mathbb{A}}$.

Let du, dv be invariant gauges on G and on g, respectively; we deduce, as we know (cf. [13], Th. 2.4.1) a invariant gauge du/dv on $U(i)$, which must therefore coincide with θ_i up to a scalar factor; by multiplication such factor du if necessary, we can arrange to have $du/dv = \theta_i$. Let λ be a system of convergence factors for G; then, according to a general result ([13], Theorem 2.4.3) combined with Lemma 19 of Chapter V, n^{o}43, λ is also a system of convergence factors for g, and we have

$$|\theta_i|_{\mathbb{A}} = |\lambda du|_{\mathbb{A}}/|\lambda dv|_{\mathbb{A}}.$$

For any λ a system of convergence factors for a group G; we will agree to denote by $\tau_\lambda(G)$ the measure of $G_{\mathbb{A}}/G_k$ for the Haar measure of $|\lambda du|_{\mathbb{A}}$ on $G_{\mathbb{A}}$, du being an invariant gauge on G; for $\lambda = 1$, $\tau_\lambda(G)$ is by definition the **Tamagawa number** of G. With this notation, the measures ν, ν_0 defined above, on $G_{\mathbb{A}}$ and on $g_{\mathbb{A}}$ respectively, can be written:

$$d\nu(u) = \tau_\lambda(G)^{-1}|\lambda du|_{\mathbb{A}}, \quad d\nu_0(v) = \tau_\lambda(g)^{-1}|\lambda dv|_{\mathbb{A}}.$$

Taking into account what precedes, we see that (40) is equivalent to the formula

$$\tau_\lambda(g) = \tau_\lambda(G), \tag{42}$$

which is therefore valid whenever (B) is satisfied and g is the stabilizer of a point ξ of X_k of maximal rank in X_k.

54. Let us first place ourselves in the case (I_0); then G is the symplectic group $\mathrm{Sp}(m)$ with m variables, m being necessarily even. We can then take $\lambda = 1$. Let us take $n = 1$; then we have $I(X)_k = \{0\}$, and n^{o}23 of Chapter II shows that g is isomorphic to a semi-direct product of $\mathrm{Sp}(m-2)$ and a unipotent group. Consequently, (42) shows, for $m > 2$, that this number has the value 1.

If $\mathcal{A}_k$ is not of type (I_0), we can, by choosing a suitable basis, arrange for the η-hermitian form h to be given by a diagonal matrix; let $a_1, \cdots, a_m$ be the diagonal elements of this matrix; let us agree to write $G(a_1, \cdots, a_m)$ instead of G, and $\tau_\lambda(a_1, \cdots, a_m)$ instead of $\tau_\lambda(G)$. Let us take $n = 1$, X_k then identifying with $M_{m,1}(\mathfrak{k})$, and apply (42) to the stabilizer g of $\xi = (0, \ldots, 0, 1)$. Either according to n^{o}23 of Chapter II, or directly, we immediately see that g is isomorphic to $G(a_1, \cdots, a_{m-1})$; we therefore have

$$\tau_\lambda(a_1, \ldots, a_{m-1}) = \tau_\lambda(a_1, \ldots, a_m)$$

whenever $m > 4\varepsilon$. Similarly, let us apply (42) to the case where we have taken

$$G = G(a_1, \cdots, a_{m-2}, 1, -1), \quad \xi = (0, \cdots, 0, 1, 1);$$

as we then have $i_X(\xi) = 0$, n^{o}23 of Chapter II shows that g is isomorphic to the semi-direct product of $G(a_1, \cdots, a_{m-2})$ and a unipotent group; we therefore have

$$\tau_\lambda(a_1, \cdots, a_{m-2}) = \tau_\lambda(a_1, \cdots, a_{m-2}, 1, -1)$$

whenever $m > 4\varepsilon$, provided we agree that the left hand side has the value 1 for $m = 2$.

In the case (I_1), we can take $\lambda = 1$, and it follows from the above that the Tamagawa number of G is always 1. In the cases (I_2), (I_3), (I_4), we can only conclude that $\tau_\lambda(a_1, \cdots, a_m)$ is independent of a_μ and of m provided that $m > 4\varepsilon - 2$. This result is new, it seems, in the case (I_3), and also in the case (I_2) if f is not commutative.

Let us recall that we can determine $\tau_\lambda(G)$ directly, by means of **canonical isomorphisms**, for $m = 3$ in the case (I_4), for $m = 2$ in the case (I_3), and also $m = 1$ in the case (I_2) when $\mathfrak{k}$ is commutative or is an algebra of quaternions over its center. Consequently, in all cases, we thus obtain, in substance, the complete determination of the Tamagawa numbers.

Finally, when the group G_1 and $\widetilde{G}$, introduced above, satisfy condition (W) of Chapter I, n^{o}8, we can apply the same method to them, and deduce analogous consequences.

55. Let $\mathcal{G}$ denote one of the groups considered above, that is, G, G_1, or $\widetilde{G}$. Let $\mathcal{G}'$ an open subgroup of $\mathcal{G}_\mathbb{A}$, and let $\mathcal{G}'_k = \mathcal{G}_k \cap \mathcal{G}'$. We can, in ana analogous way, identify $\mathcal{G}'/\mathcal{G}'_k$ with $\mathcal{G}'\mathcal{G}_k/\mathcal{G}_k$, that is to say with the image of $\mathcal{G}'$ in $\mathcal{G}_\mathbb{A}/\mathcal{G}_k$; this is an open and closed subset of $\mathcal{G}_\mathbb{A}/\mathcal{G}_k$. If ν is again denotes the Haar measure in $G_\mathbb{A}$, normalized by $\nu(\mathcal{G}_\mathbb{A}/\mathcal{G}_k) = 1$, we will therefore have:

$$- < \nu(\mathcal{G}'\mathcal{G}_k/\mathcal{G}_k) \leq 1$$

Let $\alpha = \nu(\mathcal{G}_\mathbb{A}/\mathcal{G}_k) = 1$, and let f a characterize function of the subset $\mathcal{G}'\mathcal{G}_k$ in $\mathcal{G}_\mathbb{A}$. In $\mathcal{G}_\mathbb{A}$, consider the measure ν_1 given by

$$d\nu_1(u) = \alpha^{-1} f(u) d\nu(u). \tag{43}$$

This measure is right-invariant by $\mathcal{G}_k$ and determines, by passage to the quotient, a measure on $\mathcal{G}_\mathbb{A}/\mathcal{G}_k$, which we will still denote by ν_1; this induces on $\mathcal{G}'\mathcal{G}_k/\mathcal{G}_k$, or which amounts to the same thing, on $\mathcal{G}'\mathcal{G}_k/\mathcal{G}_k$, or, which amounts to the same thing, on $\mathcal{G}'/\mathcal{G}'_k$, a measure ν', left-invariant by $\mathcal{G}'$ and satisfying $\nu'(\mathcal{G}'/\mathcal{G}'_k) = 1$; it is clear that ν' is completely determined by these later conditions.

Let $\mathcal{J}(\Phi)$ denote the integral analogous to $I_v(\Phi)$, where G is replaced by $\mathcal{G}$ (and ρ by ρ, ρ_1 or $\widetilde{\rho}$ on each case); according to Theorem 5, we have $\mathcal{J} = E$. We can also consider $\mathcal{J}'(\Phi)$ as being the analogous integral to $I_v(\Phi)$ taken on $\mathcal{G}'\mathcal{G}'_k/\mathcal{G}'_k$, or which amounts to the same thing, taken on $\mathcal{G}_\mathbb{A}/\mathcal{G}_k$ by means of ν_1. This latter definition shows that, for $\Phi \geq 0$, $\mathcal{J}'(\Phi)$ is majorized by $\alpha^{-1}\mathcal{J}(\Phi)$, so that this integral is absolutely convegent for any $\Phi \in \mathcal{D}(X_\mathbb{A})$.

Siegel's work on indefinite quadratic forms leads one to ask under which condition we have $\mathcal{J}' = E$. According to Theorem 5, it suffices for this that there is a place v of k such that $U(0)_v$ is not empty and that the measure ν_1, or, which amounts to the same thing, the set $\mathcal{G}'\mathcal{G}_k$, be left-invariant by a subgroup pf $\mathcal{G}_v$ which operates transitively on $U(i)_v$ for any $i \in I(X)_k$.

This will obviously be the acse if there exists v such that $U(0)_v$ is not empty and $\mathcal{G}_v \cap \mathcal{G}'$ has the transitivity property in question. Moreover, when this is the case, the Theorem 5 shows that the analogue of (40) is valid for $\mathcal{J}'$. It follows in particular that, under these conditions, the support of the measure defined by the left hand side of (40) is $U(i)_{\mathbb{A}}$; consequently, *the set of elements* $\rho(u)\xi$, *for* $u \in \mathcal{G}'$, $\xi \in U(i)_k$, *is everywhere dense in* $U(i)_{\mathbb{A}}$ when the preceding conditions are satisfied (including condition (B). This is the **approximation theorem** that was announced in the introduction.

56. It would be appropriate to compare the previous result with the appriximation theorems that have been obtained in recent years by arithmetic methods, and primarily with that of M. Kneser ([8]) which seems to include all of them as special cases. We will not undertake this comparison here; but we wil apply Kneser's theorem to solve Siegel's problem in cases where we cannot proof as we did above.

For this, suppose we have taken $\mathcal{G}'$ of the form $\prod \mathcal{G}'_v$, where $\mathcal{G}'_v$ is, for all v, an open subgroup of $\mathcal{G}_v$, with $\mathcal{G}'_v = \mathcal{G}^\circ_v$ for almost all v; suppose moreover that there is a place v for which we have $\mathcal{G}'_v = \mathcal{G}_v$ and for which $\mathcal{G}_v$ is not compact. We will see that then we have $\mathcal{J}' = E$.

Let $\widetilde{\mathcal{G}}$ denote the group G when $\mathcal{A}_k$ is of type (I_0) or (I_1), the group G_1 when $\mathcal{A}_k$ is of type (I_2), and the group $\widetilde{G}$ when $\mathcal{A}_k$ is of type (I_3) or (I_4). In all cases, $\widetilde{G}$ is therefore connected and simply connected, and we can apply Kneser's theorem to it. In particular, if $\mathcal{G} = \widetilde{\mathcal{G}}$, this shows that with the above hypotheses we have $\mathcal{G}'\mathcal{G}_k = \mathcal{G}_{\mathbb{A}}$, therefore $\mathcal{J}' = \mathcal{H}$, hence the announced result. In all other cases, let ψ be the homomorphism of $\widetilde{\mathcal{G}}$ in $\mathcal{G}$ defined in n^{o}51 (that is to say φ_1 in the case (I_2) if $\mathcal{G} = G$, $\widetilde{\varphi}$ in the case (I_3), (I_4) if $\mathcal{G} = G$, and $\widetilde{\varphi}_1$ in the similar case if $\mathcal{G} = G_1$); ψ determines a homomorphism $\psi_{\mathbb{A}}$ of $\widetilde{\mathcal{G}}_{\mathbb{A}}$ in $\mathcal{G}_{\mathbb{A}}$. Let $\widetilde{G} = \psi_{\mathbb{A}}^{-1}(\mathcal{G}')$; according to Kneser's theorem, we have $\widetilde{G}'\widetilde{G}_k = \widetilde{G}_{\mathbb{A}}$.

We will show that then the measure ν_1 defined by (43) is left-invariant by $\widetilde{\mathcal{G}}_{\mathbb{A}}$ operating on $\mathcal{G}_{\mathbb{A}}$ by $\psi_{\mathbb{A}}$; for this, it suffices to show that this is the case for the set $\mathcal{G}'\mathcal{G}_k$, that is to say that we have $\psi_{\mathbb{A}}(g)xy \in \mathcal{G}'\mathcal{G}_k$ for any $g \in \widetilde{\mathcal{G}}_{\mathbb{A}}$, $x \in \mathcal{G}'$, $y \in \mathcal{G}_k$. Now we easily verify, in each of the cases we have to consider here, that $\psi_{\mathbb{A}}(\widetilde{\mathcal{G}}_{\mathbb{A}})$ is an invariant subgroup of $\mathcal{G}_{\mathbb{A}}$. It follows that, if $g \in \widetilde{\mathcal{G}}_{\mathbb{A}}$ and $x \in \mathcal{G}_{\mathbb{A}}$, $x^{-1}\psi_{\mathbb{A}}(g)x$ belongs to $\psi_{\mathbb{A}}(\widetilde{\mathcal{G}}_{\mathbb{A}})$ and is therefore of the form $\psi_{\mathbb{A}}(\widetilde{x}\widetilde{y})$ with $\widetilde{x} \in \widetilde{\mathcal{G}}'$, $\widetilde{y} \in \widetilde{\mathcal{G}}_k$ by Kneser's theorem. We therefore have

$$\psi_{\mathbb{A}}(g)xy = x\psi_{\mathbb{A}}(\widetilde{x}) \cdot \psi_{\mathbb{A}}(\widetilde{y})y;$$

the right hand side indeed belongs to $\mathcal{G}'\mathcal{G}_k$. We have thus shown that ν_1 is invariant by $\widetilde{\mathcal{G}}_{\mathbb{A}}$, therefore by $\widetilde{\mathcal{G}}_v$ for any v. As we have shown at the end of the proof of Theorem 5 that there always exists v such that $U(0)_v$ is not empty and that $\widetilde{\mathcal{G}}_v$ operates transitively on the $U(i)_v$, Theorem 5 applies to ν_1, which completes the proof.

We have thus recovered, somewhat generalized, all the results proved by Siegel in the course of his work on quadratic forms, as well as those stated at the end of [12], with the following exception. First of all, we have not addressed the study of the case where $m = 2n + 4\varepsilon - 2$; now the results obtained by Siegel in the case where $\mathcal{A}_k$ is of type (I_4) over $k = \mathbb{Q}$ or $k = \mathbb{Q}(\sqrt{-1})$, and where we take $m = 4$, $n = 1$, suggests that, if $m = 2n + 4\varepsilon - 2$, the Eisenstein-Siegel series ceases to be absolutely convergent, but that we can nevertheless, when the convergence condition of Proposition 8 of n^{o}51 is satisfied, **sum** the Eisenstein-Siegel series by means of suitably chosen **summation factors**, and thus recover the analogue of Siegel's formula. It would also remain to examine, from the point of view that has been ours in this work, Siegel's results on the zeta functions of indefinite quadratic forms, which are closed related to the questions we have studied here.

57. The case where $\mathcal{A}_k$ is of type (II) does not seem to lend itself at present to results as general or as simple as the cse (I). However, if $\mathcal{A}_k$ is of type (II) and X_k of rank $(1,1)$, we can proceed as follows. As always, assume (B) satisfied, which here amounts to $m > 2$. Let us consider the integral $I_1(\Phi)$ which appears in Proposition 8 of n^{o}51; we will decompose it according to the orbit of $(G_1)_k$ in X_k, by applying the results of n^{os}7 − 8 of Chapter I.

Modifying our usual notations slightly, let us write X_k as a direct sum of two modules Y_k, Z_k of respective ranks $(1,0)$ and $(0,1)$; there are respectively isomorphic to $M_{m,1}(\mathfrak{k})$ and $M_{m,1}(\mathfrak{k}')$. The groups G_k, $(G_1)_k$ are identified respectively with $\mathrm{GL}(m,\mathfrak{k})$ and $\mathrm{SL}(m,\mathfrak{k})$; G and G_1 are thus identified with the corresponding algebraic groups, which we will denote for brevity as GL_m and SL_m; for brevity also, we will write F instead of $(\mathrm{SL}_m)_{\mathbb{A}}/(\mathrm{SL}_m)_k$. Then $I_1(\Phi)$ is written:

$$I_1(\Phi) = \int_F \sum_{\eta,\xi} \Phi(u\eta, {}^t u'^{-1}\xi) \cdot d\mu(u),$$

where the summation extends to $\eta \in Y_k$, $\xi \in Z_k$, and where μ is the Haar measure on $(\mathrm{SL}_m)_{\mathbb{A}}$, normalized by $\mu(F) = 1$. As we know, 1 is a system of factors for SL_m; we will set $\tau_m = \tau_1(\mathrm{SL}_m)$; this is the Tamagawa number of SL_m; if du is an invariant gauge on SL_m, we have $d\mu(u) = \tau m^{-1}|du|_{\mathbb{A}}$.

According to Proposition 3 of Chapter II, n^{o}22, G_k has two orbits in Y_k, namely $\{0\}$ and the orbit $U'_k = Y_k = \{0\}$ of points of maximal rank in Y_k. If we extend $\mathcal{A}_k$ and Y_k to the universal domain, and denote by U' the set of points of maximal rank in Y, U' is an orbit of G, and U'_k is the set of points of U' which are rational over k. Similarly, for any field $K \supset k$, the set U'_k of points of U' which are rational over K coincides with the set of points of maximal rank of Y_K and is therefore an orbit of G_K. Moreover, we immediately verify that the group $(G_1)_K$ operates transitively on U'_K. Similarly, if U'' is the set of points of maximal rank of Z, U''_K is an orbit of G_K, and also of $(G_1)_K$, whatever $K \supset k$, and we have $U''_k = Z_k - \{0\}$.

According to n^{o}21 of Chapter II, we can here identify $I(X)_k$ with $\mathfrak{k}$. According to Proposition 3 of Chapter II, n^{o}22, and its corollary, the orbits of G_k in X_k are on one hand the $U(i)_k$ for $i \in I(X)_k$, and on the other hand the three orbits $U'_k \times \{0\}$, $\{0\} \times U''_k$ and $\{0\}$. We immediately verify that $(G_1)_k$ operates transitively on these orbits and that they satisfy the condition (W) of Chapter I, n^{o}8. The determination of stabilizer of the points of orbits in G_1 is done without difficulty, either by means of n^{o}23 of Chapter II, or directly; these groups are respectively isomorphic to SL_{m-1} (for the points of $U(i)_k$), and to a semi-direct product of SL_{m-2} and a unipotent group (for the points of $U'_K \times \{0\}$ and of $\{0\} \times U''_k$). The results of Chapter I, n^{os}7 − 8, then give, for the same reason as in n^{o}53:

$$I_1(\Phi) = \Phi(0,0) + c' \int_{U'_{\mathbb{A}}} \Phi(y,0)|dy|_{\mathbb{A}} + c'' \int_{U''_{\mathbb{A}}} \Phi(0,z)|dz|_{\mathbb{A}} + \sum_{i\in\mathfrak{k}} c_i \int_{U(i)_{\mathbb{A}}} \Phi \cdot |\theta_i|_{\mathbb{A}},$$

with $c_i = \tau_{m-1}/\tau_m$ for $i \neq 0$, $c_0 = \tau_{m-2}/\tau_m$, and $c' = c'' = \tau_{m-1}/\tau_m$.

The uniqueness theorems relating to the cases where we place ourselves here, theorem which were alluded to at the beginning of Chapter VI, would allow us to conclude from this that $c_i = 1$ for all i, therefore that τ_m is independent of m for $m \geq 1$. As it is in fact well known that we even have $\tau_m = 1$ for all m (cf. [13], Chap. III), this would not teach us anything new. In whatever way we proceed, we see that the coefficients c_i, c', c'' of the above formula all have the value 1. As we also know that $U'_{\mathbb{A}}$, $U''_{\mathbb{A}}$ are the complements of negligible sets in $Y_{\mathbb{A}}$ and $Z_{\mathbb{A}}$, for the measures

$|dy|\_BA$ and $|dz|_{\mathbb{A}}$ respectively (cf. [13], Lemma 3.4.1), we therefore obtain definitively[10]

$$I_1(\Phi) = E(\Phi) + \int_{Y_{\mathbb{A}}} \Phi(y,0)|dy|_{\mathbb{A}} + \int_{Z_{\mathbb{A}}} \Phi(0,z)|dz_{\mathbb{A}}|.$$

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[10] I take this opportunity to indicate, at Siegel's request, that the formula of [12], p.379, line 4 from the bottom, should be corrected as follows: in the right hand side, replace the exponent $-m$ to $-m/2$, add a term $+2y^{m/2}$, and replace the condition $m > 1$ by $m > 2$. After this rectification, which was communicated to me by Siegel some time ago, his formula becomes a special case of the one in the text.