Delayed Differential Equation from Replica-mean-field Limit of Exponential Firing Model

Luyan Yu[†], Thibaud Taillefumier[‡]

[†]Department of Physics, University of Texas at Austin

[‡]Department of Mathematics and Department of Neuroscience, University of Texas at Austin

luyan.yu@utexas.edu

Modeling

Stochastic intensity formulation

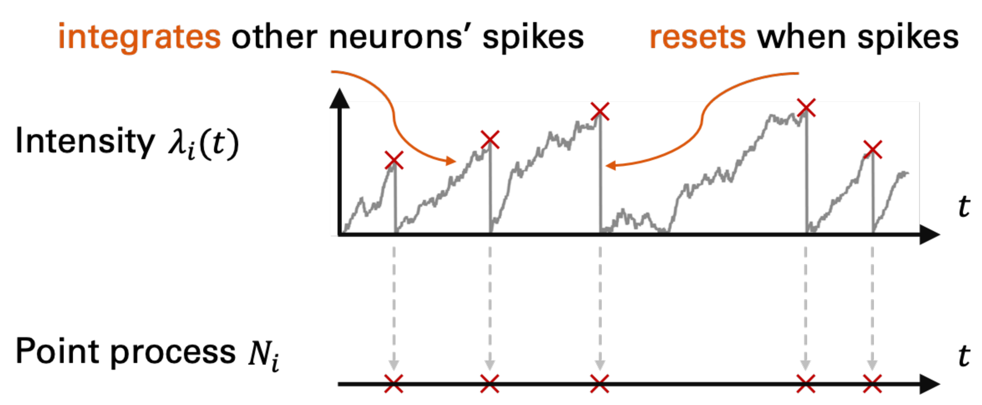

Figure 1: *Schematic figure of stochastic intensity formulation. A neuron i is treated as an object with associated intensity λ_i and spiking events modeled as point process N_i.*

Exponential firing model

In the exponential firing model, neuron i generates spikes according to a time-dependent Poissonian rate

$$\lambda_i(t) = h_i e^{a_i x_i(t)}, \tag{1}$$

where h_i, a_i are positive parameters and x_i is a (continuous) **leaky** internal variable counting all the incoming spikes.

A network consisting of K neurons are fully characterized by μ_{ji} representing the synaptic strengths from neuron j to i.

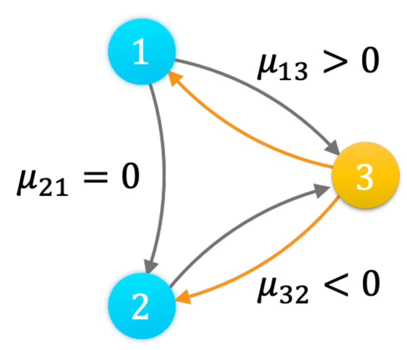

Figure 2: *The sign of μ_{ji} is determined by the role of neuron j: $\mu_{ji} > 0$ if it is excitatory; $\mu_{ji} < 0$ if it is inhibitory; and $\mu_{ji} = 0$ if neuron j and i are not connected.*

The dynamics of the network consist of three components:

Spiking: when neuron j spikes, $x_i \to x_i + \mu_{ji}, \forall i \neq j$;

Resetting: when neuron j spikes, $x_j \to 0$;

Relaxation: when there is no spike, $x_i'(t) = -x_i(t)/\tau_i, \forall i$

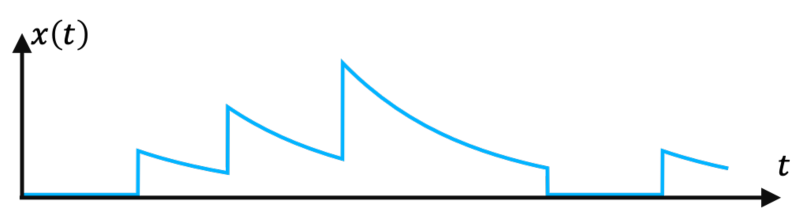

Figure 3: *Relaxation to zero on the internal variable x*

The Replica-mean-field Limit

RMF limit treats the network by making an infinite amount of replicas of the original network and each neuron sends its spikes to the connected neurons in a uniformly and independently chosen replica.

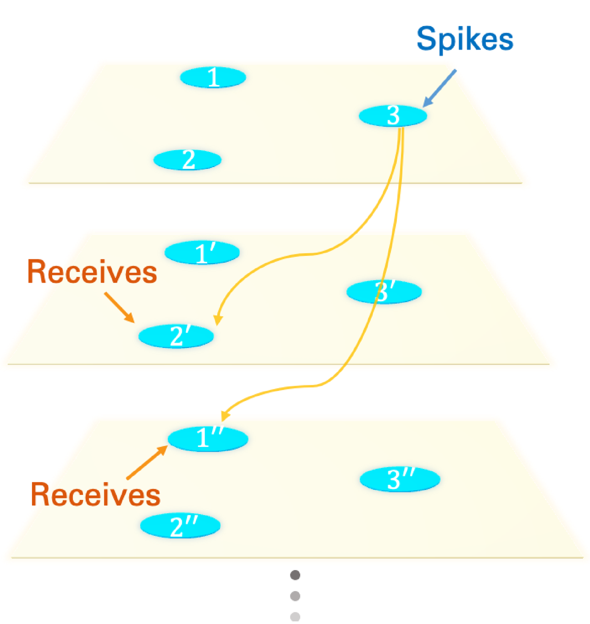

Figure 4: *Schematic illustration of how neurons communicate with each other in the replica-mean-field limit.*

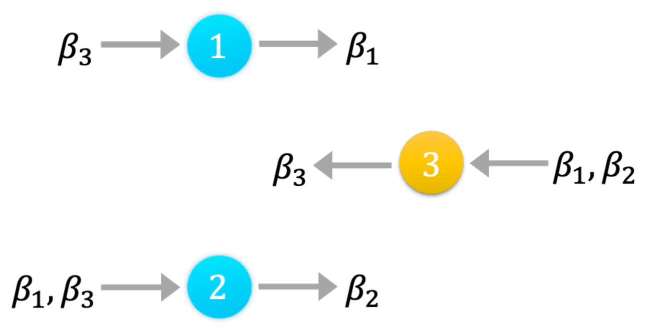

Figure 5: *In RMF limit, the probability of any specific pair of neurons talking to each other is infinitesimal, eliminating the correlation between neurons. Each neuron receives Poissonian inputs and generates Poissonian output.*

The Delayed Differential Equations

The moment generating function (MGF) of x_i, which characterizes the stationary state of the neuron, is defined as:

$$L_i(u) = \mathbb{E}\left[e^{u x_i}\right]. \tag{2}$$

We write down the dimensionless DDEs in the RMF limit for MGF:

$$u L_i'(u) - V_i(u; \{\beta_j\}) L_i(u) - [\beta_i - h_i L_i(u + a_i)] = 0 \tag{3}$$

$$\beta_i = h_i L_i(a_i)$$

$$V_i(u; \{\beta_j\}) = \sum_{j \neq i} \beta_j (e^{\mu_{ji} u} - 1)$$

Iterative Scheme for Solving the DDEs

Eq. (3) does not have the usual initial conditions as other DDEs have. Instead, we have global regularization conditions to the solutions. We adapted the **resolvent formalism** to our case.

For neuron i, the following equation relates the output mean firing rate β_i with other neurons input mean firing rates β_j's.

$$\frac{1}{\beta_i} = \frac{1}{h_i}\left[1 + \sum_{m=0}^{\infty} (-h_i)^m Q_i^{(m)}(-a_i; \{\beta_j\})\right] \tag{4}$$

where

$$Q_i^{(m)}(u; \{\beta_j\}) = \frac{1}{q_i(u + a_i)} \int_{a_i}^{u + a_i} \frac{q_i(v)}{v} Q_i^{(m-1)}(v; \{\beta_j\}) \, \mathrm{d}v$$

with $Q_i^{(0)}(u; \{\beta_j\}) = \dfrac{1}{q_i(u + a_i)} - 1$ and $q_i(u) = e^{-\int_{a_i}^{u} \frac{V_i(v; \{\beta_j\})}{v} \mathrm{d}v}$. (5)

Results

Single neuron with constant inputs

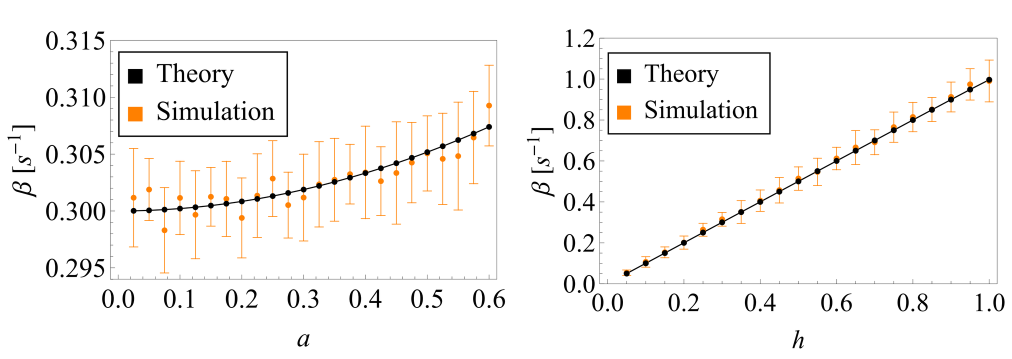

Figure 6: *The mean firing rates of a single neuron varying the parameter a **(left)** and h **(right)**. The neuron is under balanced inputs $\beta_e = \beta_i = 1.2\,s^{-1}$ and $\mu_e = -\mu_i = 0.5$. Other parameters: $\tau = 1.0\,s$, $h = 0.3\,s^{-1}$ and $a = 0.1$ (if not varying).*

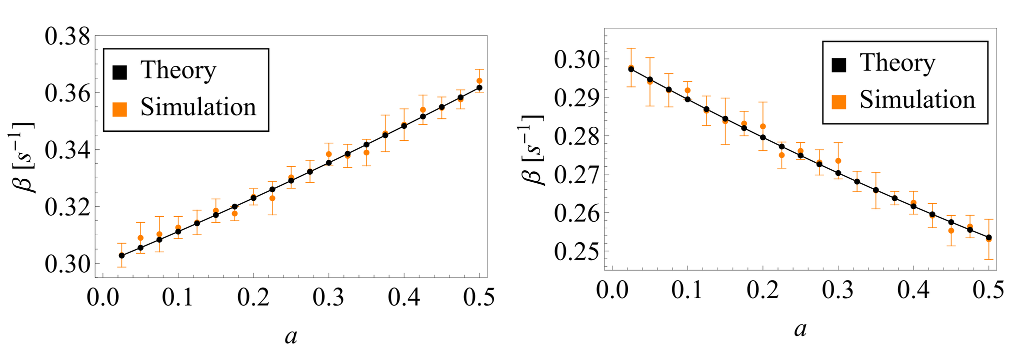

Figure 7: *The mean firing rates of a single neuron varying the parameter a.* **Left:** Excitatory input is stronger with $\beta_e = 1.2\,\mathrm{s}^{-1}, \beta_i = 0.5\,\mathrm{s}^{-1}$ and $\mu_e = 0.6, \mu_i = -0.5$. **Right:** Inhibitory input is stronger with $\beta_e = 0.5\,\mathrm{s}^{-1}, \beta_i = 1.2\,\mathrm{s}^{-1}$ and $\mu_e = 0.5, \mu_i = -0.6$. Other parameters: $\tau = 1.0\,\mathrm{s}$ and $h = 0.3\,\mathrm{s}^{-1}$.

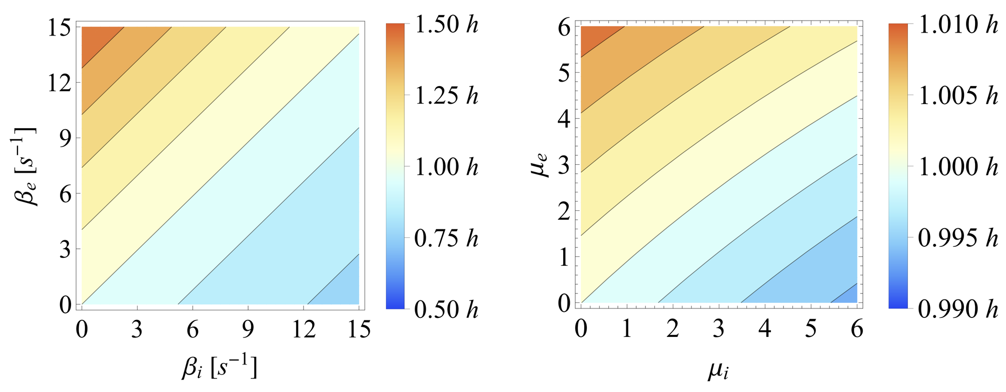

Figure 8: *Density plot of the mean firing rates of a single neuron.* **Left:** *Varying the excitatory and inhibitory input rates β_e, β_i. The neuron synaptic strengths are $\mu_e = -\mu_i = 0.5$. Other parameters: $\tau = 1.0$ s, $h = 0.002$ and $a = 0.1$.* **Right:** *Varying the excitatory and inhibitory synaptic strengths μ_e, μ_i. The neuron synaptic strengths are $\beta_e = \beta_i = 0.01$. Other parameters: $\tau = 1.0$ s, $h = 0.002$ and $a = 0.1$.*

Fully connected neuron network

We need to self-consistently solve Eq. (4) for all neurons.

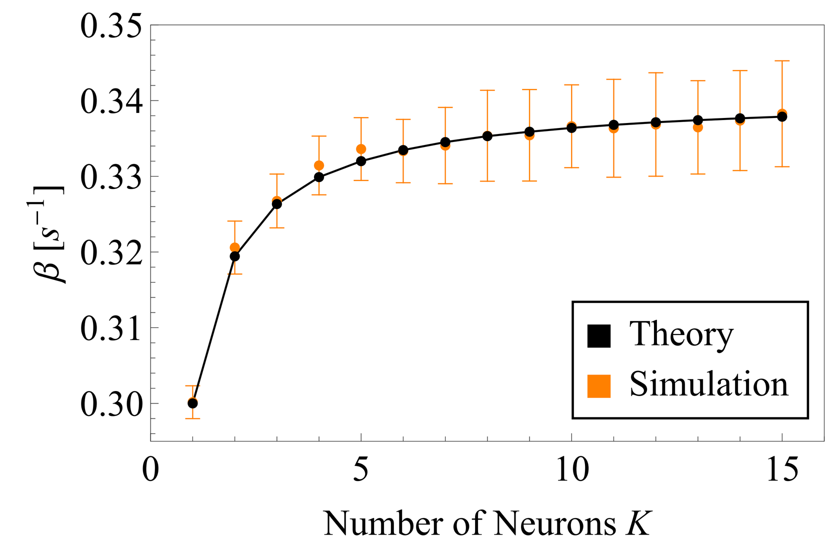

Figure 9: *The mean firing rates of excitatory neurons in a fully connected, homogeneous network varying the number of neurons K. The neuron synaptic strengths are $\mu = 5/K$. Other parameters: $\tau = 1.0$ s, $h = 0.3\,s^{-1}$ and $a = 0.1$.*

Balanced neuron network

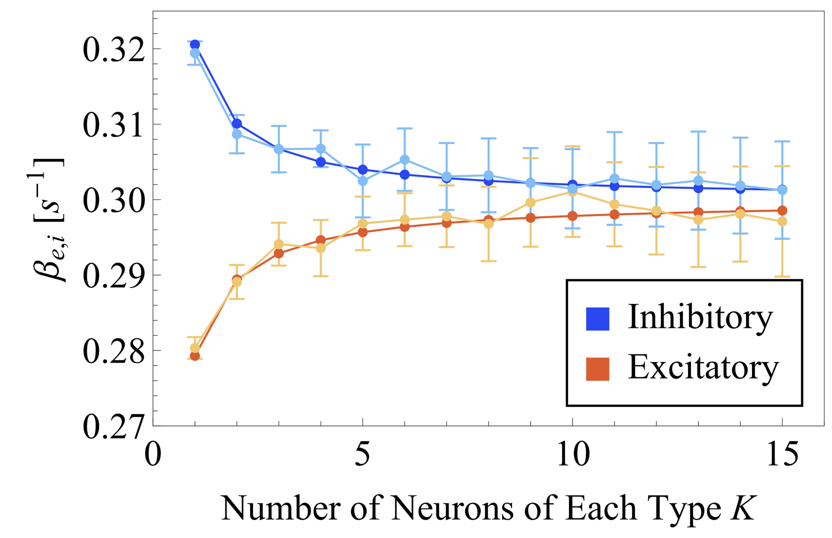

Figure 10: *The mean firing rates of excitatory and inhibitory neurons in a fully connected network with equal number of two types of neurons varying the number of each type of neurons $K = K_e = K_i$. The neuron synaptic strengths are $\mu = 3/K$. Other parameters: $\tau = 1.0$ s, $h = 0.3\,s^{-1}$ and $a = 0.1$.*

Conclusions

- Finite size effects are preserved in RMF limit of the networks, whose stationary states are fully characterized by the mean firing rates. We provided a scheme that can solve these mean firing rates.
- This approach enables the future studies of meta-stable systems that are not present in thermodynamic-mean-field limit.

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References

[1] François Baccelli and Thibaud Taillefumier. Replica-mean-field limits for intensity-based neural networks. *SIAM Journal on Applied Dynamical Systems*, 18(4):1756–1797, 2019.