The Proof For Fair Openning Of Game

Group 11

Defination: Items

Player Sets: $P = \{p_1,...,p_j,...p_k\}$

Grids: n places on board(7 on the right). e.g. (a,b,c,d...)

Feasible State Set: $S=\{s_1,...,s_i,...s_m\}$, $m=C^k_n$. e.g. $s_i = (b,d,e)$ for 3 player

Players’ game level: x,y,z… (MC-GRAVE,Flat Monte Carlo...)

$S_{xyz} = S_{x'y'z'}$ iff $x,y,z = x', y', z'$

Indenpent: Each player fight for himself for higher winning rate.

Compact: Each player knows all game states.

Comparable: Same level, so $s(b,d,e) = s(d,e,b)=...$

Defination:Fairness

Weak-d Fair Openning Condition:

Judge a state is weak fair state:

$\min_s d(s) = \min_s |1/k, \min_j s_j|$ for all s in S.

A dual metrics is Weak-D: $\min_s D(s) = \min_s |1/k, \max_j s_j|$ for all s in S.

Strong Fair Openning Condition:

$\bar D = \min_s \min_j |1/k, s_j| = \min_s \max(d(s), D(s))$ for all s in S

50	25	25

D=50-33=17 d=33-25=8

40	30	30

D=40-33=7 d=33-30=3

35	35	30

D=35-33=2 d=33-30=3

Weak-d to Weak-D:

s_j => (1-s_j) - (k-2)/k

Vitrual State, may not in S.

The probability of winning =>the probability of not winning and normalized.

40	20	20	20
30	30	30	10

d=5,D=15=>D=5,d=15

Defination:Fairness

Hard to say which state is more fair, the weak one or the strong one.

(20,20,20,40) and (15,15,35,35)

Complete Fair: $\min_{s \in S} ||1/K,s||_{L1}$ $C_1=30$, $C_2=40$

Semi-Complete Fair: $\min_{s \in S} \max_{i,j} |s_i-s_j|$ $SC_1 = 20$, $SC_2 = 20$.

Selfish: Each player want to maximum his own winning rate, but not balance it.

(20,30,50) and (18,37,45)

Most of time, we can only get a Weak-d Fairness.

Matrix Game and Board Game

Both can be described as a set of states represented by the probability of winning.

Infinite,Continuous/Finite,Discrete

The latter player's choice will not affect the winning rate of the previous player's choice if the previous player has completed his moving/will affect. Each element in the state can only be determined after the last player has made his move

Matrix game: 40; 40,20; 40,20,40

Board game: b; b,e; b,e,a 18,37,45 b,e,c 40,35,25

Inplace Previous Rule for Matrix Game

To force the player who split the winning rates will only get the **smallest** one.

- two-player swap rule
  - A:min(p,1-p) -----> p is close to 0.5
- in place previous of 3 players
  - A first choose p as his win chance
  - If B do not swap, B and C plays a 2-player swap game. So he can only get 0.5*(1-p) at most. So if $p > \frac{1}{3}$, B will swap.
  - Then A and C plays a 2-player swap game.
  - In any case A will get the smallest winning rates, so p is close to 0.33.
  - It is a good swap rule to extend to k+1 players because it make sure the first player in k players can only get minimum.

Board Game: 2 Player Swap Rule

Only 3 grids, a,b,c, with states:

{'ab':90,10; 'ba':10,90; 'ac':70,30; 'ca':30,70; 'bc':40,60; 'cb':60,40}

Swap Rule: Player 1 moves first, Player 2 decides whether to swap. If he swaps, he will become the first player to move with the occupied grid and Player 1 will move another grid.

Note: not the swap rule: Player 1 moves first and Player 2 moves next. Then Player 2 decides whether to swap their moves. It seems like "cut and choose", which is a simple strategy and we will discuss in the end.

Board Game: 2 Player Swap Rule

{‘ab’:90,10; ‘ba’:10,90; ‘ac’:70,30; ‘ca’:30,70; ‘bc’:40,60; ‘cb’:60,40}

1:a->2:swap->1:a (30,70)

1:b->2:a (10,90)

1:c->2:a (30,70)

The final openning will be (30,70)

The ideal openning of this game is (40,60)

The Role of Swap Rule

When player i fully considered all the strategies in the presence of swap rule and completed his move, he will not be swapped by any other player j, with $i<j$.

1.The person who chooses first can only guarantee a lower win rate: If not in 4 players game, Player 1 can guarantee the second lowest win rate, he will choose a strategy with (33,0,33,33).

2.A k-player swap rule can degenerate into k-1 player swap rule when the first player complete his move. (Iterative solution: Tower of Hanoi)

Only these two characteristics can filter the states in S, $s_i<s_j$ if $i<j$.

Selfish: the last player will choose the option that is best for him.

{‘ab’:90,10; ‘ba’:10,90; ‘ac’:70,30; ‘ca’:30,70; ‘bc’:40,60; ‘cb’:60,40}

3-Player Board Game

Operators:

P_1(x),P_2(y),P_3(z): The projection of S if player 1/2/3 moves x/y/z.

max_x max_y max_z: Maximum corresponding position in s.

Pure game without swap rule:

max_x max_y max_z P_1(x) P_2(y) P_3(z), x,y,z can be all possible grid can x!=y!=z.

Game with “Inplace with Previous One” swap rule:

S(i,i+1): if s_{i+1} < s_i, change the order.

max_x S(1,2) max_y S(2,3) max_z P_1(x) P_2(y) P_3(z), x!=y!=z

3-Player Board Game

Game with “Inplace with Any One”swap rule:

S(i): reorder the state s after ith items from lower to higher.

max_x S(1) max_y S(2) max_z P_1(x) P_2(y) P_3(z), x!=y!=z

The two are equivalent ！

It is asymmetric because there is no constraint on the last player, just like 2-player one. You can check the experiment result in github:S

https://github.com/yumao-gu/MaastrichtAI-FairOpenning

Equal Status

Swap rule should also achieve the same status of each player.

Using “Inplace with Previous One” and the first player can swap with the last player.

{‘ab’:90,10; ‘ba’:10,90; ‘ac’:70,30; ‘ca’:30,70; ‘bc’:40,60; ‘cb’:60,40}

```
1:a->2:b->1:ns 'ab':(90,10)
          ->1:s->2:a 'ba':(10,90)
          ->1:s->2:c 'bc':(40,60)
   ->2:c->1:ns 'ac':(70,30)
          ->1:s->2:a 'ca':(30,70)
          ->1:s->2:b 'cb':(60,40)
   ->2:s->1:b 'ba':(10,90)
          ->1:c  'ca':(30,70)
```

```
1:b->2:a->1:ns 'ba':(10,90)
          ->1:s->2:b 'ab':(90,10)
          ->1:s->2:c  'ac':(70,30)
   ->2:c->1:ns 'bc':(40,60)
          ->1:s->2:a 'ca':(30,70)
          ->1:s->2:b 'cb':(60,40)
   ->2:s->1:a 'ab':(90,10)
          ->1:c  'cb':(60,40)
```

```
1:c->2:a->1:ns 'ca':(30,70)
          ->1:s->2:c 'ac':(70,30)
          ->1:s->2:b 'ab':(90,10)
   ->2:b->1:ns 'cb':(60,40)
          ->1:s->2:a 'ba':(10,90)
          ->1:s->2:c 'bc':(40,60)
   ->2:s->1:a 'ac':(70,30)
          ->1:b  'bc':(40,60)
```

Proof

In 2-player case, Player 1 try to move on x and Player 2 on y (X,Y):

max_X max_Y(max_1(_,X),max_1((X,Y),max_2(Y,_)))

max_1((X,Y),max_2(Y,_))<=max_1((X,Y),(Y,X)) ==>

max_X max_Y(max_1(_,X),max_1((X,Y),max_2(Y,_))) >=max_X max_Y max_1((X,Y),(Y,X))

It is too complicated to analyze, but we can simplify it. Suppose any triple (X,Y,Z), if the swap rule works and finds the most fair state of all 6 states. Then for 2-player board game, the swap rule should work no matter how many grids there are.

max_1 is (X,Y),(X,Z)   max_1 is (Y,X),(Z,X)   max_1 is (Y,X),(X,Z)

Proof

Thing goes bad again when it comes to multi-players board game, because $s_1 + s_k \neq 1$ and they can be low or high together.{(10,50),(20,60),(0,70)}.

However, we find the swap rule can guarantee a maximum of min s (20,60) for any triple.

Since the s_1,s_k is the lowest and highest item in s, we can get a maximum of s_1 in the end. That is to say, we can get a weak-d fairness!

“Cut And Choose”

The first player determines k grids as the opening, All players can only choose the opening from these k grids, and the first player chooses the last.

It means Player 1 can choose any s from S, so it can achieve any degree of fairness. At least it will get a weak-d fairness.

{‘ab’:90,10; ‘ba’:10,90; ‘ac’:70,30; ‘ca’:30,70; ‘bc’:40,60; ‘cb’:60,40}

It does not even require that each player has the same level. That is, when the same grids are occupied, the order is different, and the components of the state s are different. $C^k_n \Rightarrow A^k_n$.