ELEC ENG 3CL4: Introduction to Control Systems Lab 3: Proportional Control & Proportional With Velocity Feedback Control of DC Motor Servomechanism

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Objective

To design a proportional controller and a proportional controller with velocity feedback for the DC motor servomechanism, and explore trade-offs involved in the selection of the controller parameters and their impact on transient and steady-state responses of the control system.

Assessment

This laboratory is conducted in groups of no more than two students. Students are required to attend their assigned lab section. The assessment of this lab will occur based on your answer to the pre-lab questions, in-lab activities, and a written laboratory report. Each group is required to submit their answers to the pre-lab questions electronically through Avenue-to-Learn by 12:01pm on the day of the lab. Pre-labs submitted after 12:01pm but before 2:30pm will be subject to a penalty of 50%. No marks will be awarded to pre-labs submitted after 2:30pm. You will earn a maximum of 100 marks from Lab 3 activities. Lab 3 will contribute to a maximum of 25% of your total lab grade for this course. The components of the assessment are:

- Pre-lab Questions 1-15, which must be completed before the lab (45 marks);
- Three experiments (35 marks);
- A laboratory report (20 marks).

Your performance of the experiments will be evaluated during the lab by the TA's. The marks for each component are clearly indicated in this document.

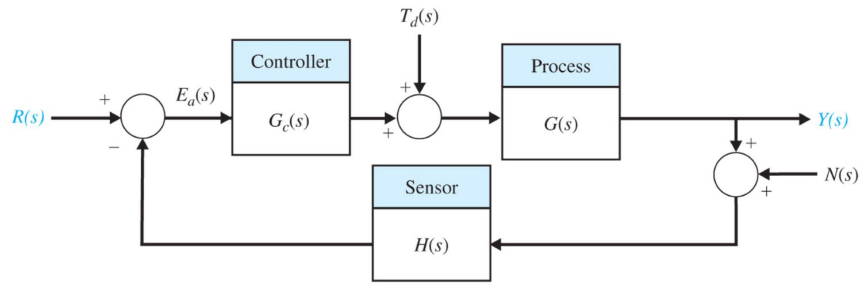

Figure 1: Feedback system with $y(t) = \theta(t)$. We will consider the case in which $H(s) = 1$. The disturbance signal $T_d(s)$ is used to model friction torque in the motor, but it could also model any addition external torque applied to the shaft. We will neglect the measurement noise signal $N(s)$.

1 Proportional Control of DC Motor

Recall from Lab. 2 that the operation of the servomotor that we are considering can be approximated by the transfer function

$$G(s) = \frac{A}{s(s\tau_m + 1)}. \tag{1}$$

Now, consider the closed-loop control system in Fig. 1, with the motor as the process, $H(s) = 1$, and the proportional controller $G_c(s) = k_p$.

Pre-Lab Question 1 **(2 marks)** *Provide a complete derivation of the following expression for the closed-loop poles:*

$$p_{1,2} = -\frac{1}{2\tau_m} \pm \frac{1}{2\tau_m}\sqrt{1 - 4k_pA\tau_m} \tag{2}$$

Pre-Lab Question 2 **(4 marks)** *Draw a picture of the path in the s-plane along which these closed-loop poles move as k_p increases from zero towards infinity.*

Pre-Lab Question 3 **(4 marks)** *Rewrite the closed loop transfer function in the form of standard second order system and show that:*

$$\zeta\omega_n = \frac{1}{2\tau_m}, \quad \omega_n = \sqrt{\frac{k_pA}{\tau_m}}, \quad \zeta = \frac{1}{2\sqrt{k_pA\tau_m}} \tag{3}$$

where ζ is the damping ratio and ω_n is the natural frequency.

Pre-Lab Question 4 **(1 mark)** *Show a controller gain that would yield a critically damped closed loop is given by*

$$k_p = \frac{1}{4A\tau_m} \tag{4}$$

Pre-Lab Question 5 **(2 marks)** *Show that if $k_p > \frac{1}{4A\tau_m}$, then the pole positions can be written as*

$$p_{1,2} = \frac{1}{2\tau_m}\left(-1 \pm j\tan(\phi)\right) \tag{5}$$

with $\phi = \cos^{-1}(\zeta)$.

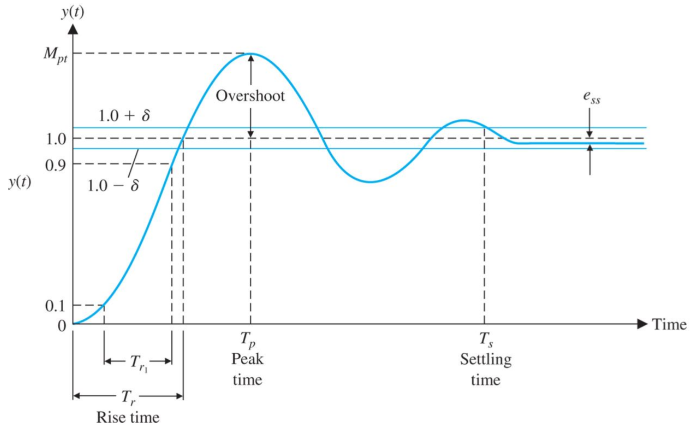

Figure 2: Generic step response of an under-damped second-order system (Figure 5.7 of Dorf and Bishop, *Modern Control Systems*, 11th edition, Prentice Hall, 2008).

2 Trade-offs in Proportional Control of a Servomotor: Theoretical Insight

Recall that the closed loop transfer function of system in Fig. 1 with $H(s) = 1$, $G_c(s) = k_p$ and the motor as the process is

$$T(s) = \frac{\Theta(s)}{R(s)} = \frac{k_p G(s)}{1 + k_p G(s)} \tag{6a}$$

$$= \frac{k_p A}{s(s\tau_m + 1) + k_p A} \tag{6b}$$

$$= \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \tag{6c}$$

where $\omega_n = \sqrt{\frac{k_p A}{\tau_m}}$ and $\zeta = \frac{1}{2\omega_n \tau_m}$. When $k_p > \frac{1}{4A\tau_m}$, the system is underdamped, and servomechanisms are often operated in this mode. When $k_p > \frac{1}{4A\tau_m}$, the step response of the system takes the form

$$\theta(t) = 1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin\left(\omega_n \sqrt{1-\zeta^2}\, t + \phi\right) \tag{7}$$

where $\cos(\phi) = \zeta$. A typical step response is shown in Fig. 2.

When we design a simple proportional controller for this type of servo mechanism, we often consider the following transient response performance criteria:

1. The 2% settling time, T_s;

2. The percentage overshoot;

3. The 10% to 90% rise time, T_{r1}.

Typically, we would like design k_p so as to have the settling time, the percentage overshoot and the rise time all be small. The purpose of this section is to assess the extent to which this can be achieved.[1]

[1] If there are fundamental restrictions on this goal that cannot be overcome, then we need to consider more sophisticated controllers, such as those that involve integration and differentiation of the error signal, as well as just amplification.

To begin the design process, we would like to determine the relationship between the three design quantities of interest and our design parameter, the gain k_p. To do so, recall from lectures that for a generic underdamped second-order system with a transfer function of the form in (6c) with $0 < \zeta < 1$, the 2% settling time is approximately four time constants. That is,

$$T_s \approx \frac{4}{\zeta\omega_n}. \tag{8}$$

The percentage overshoot of such a system is

$$\text{percentage overshoot} = 100 \exp\left(-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}\right), \tag{9}$$

and for moderately underdamped systems for which $0.3 \leq \zeta \leq 0.8$, the 10%–90% rise time can be approximated by

$$T_{r1} \approx \frac{2.16\zeta + 0.6}{\omega_n}. \tag{10}$$

Pre-Lab Question 6 **(3 marks)** *Use those expressions to show that when the closed loop system in Fig. 1 is (moderately) underdamped*

$$T_s \approx 8\tau_m \tag{11}$$

$$\text{percentage overshoot} = 100 \exp\left(-\frac{\pi}{\sqrt{4k_pA\tau_m - 1}}\right) \tag{12}$$

$$T_{r1} \approx \frac{2.16 + 1.2\sqrt{k_pA\tau_m}}{2k_pA}. \tag{13}$$

Pre-Lab Question 7 **(2 marks)** *How do these terms change with increasing k_p?*

Pre-Lab Question 8 **(2 marks)** *What are the implications for design?*

In addition to the transient response characteristics considered above, we are often interested in the steady-state response of the system to a step reference input $R(s) = \frac{\theta_d}{s}$ and constant disturbance input $T_d(s) = \frac{\tau_d}{s}$. The disturbance input is often used to model the friction in the motor, and a variety of other effects.

Pre-Lab Question 9 **(3 marks)** *Show that total output response to the reference and disturbance inputs in the Laplace domain is given by:*

$$\Theta(s) = \frac{\frac{k_pA}{\tau_m}}{s^2 + \frac{1}{\tau_m}s + \frac{k_pA}{\tau_m}} R(s) + \frac{\frac{A}{\tau_m}}{s^2 + \frac{1}{\tau_m}s + \frac{k_pA}{\tau_m}} T_d(s) \tag{14}$$

Pre-Lab Question 10 **(3 marks)** *What is the steady-state error in response to a step input $R(s) = \frac{\theta_d}{s}$ in the absence of disturbance, i.e, $T_d(s) = 0$. What is the impact of the feedback gain k_p on this error?*

Pre-Lab Question 11 **(3 marks)** *What is the steady-state error in response to a step input $R(s) = \frac{\theta_d}{s}$ in the presence of a constant disturbance, i.e, $T_d(s) = \frac{\tau_d}{s}$? What is the impact of the feedback gain k_p on this error?*

3 Experiment: Qualitative Trade-offs in Rise-time, Steady-State Error, Overshoot, and Settling Time for a Proportional Controller (10 marks)

The proportional controller has one free parameter k_p. The pre-lab exercises examined the impact of this gain on important performance measures of rise time, steady-sate error, overshoot, and settling time in the presence of a constant disturbance to the proportionally-controlled servomechanism. In particular, an inherent trade-off was exposed between the desire to have a small rise time and small steady-state error due to the disturbance, while also having a small overshoot. In this experiment you will observe this trade-off with multiple different values of k_p, using the following procedure:

i) Open Matlab, and allow it to fully open.

ii) Download the simulink file `qube_servo2_EE3CL4_Lab3.slx` from Avenue-to-Learn, and open it. Once having completed those two steps, open Quanser Interactive Labs, and proceed to the "Servo Workspace" as described in Lab. 1.

iii) Observe the addition of the disturbance signal in the simulink diagram. We have added this explicitly because the (model of the) Quanser motor has so little friction. Note that the step disturbance is modelled as a square wave of very low frequency. Note also that we have dropped the frequency of the input square wave to 0.2Hz.

iv) Set the amplitude of the disturbance to $\tau_d = 0.1745$. Since $r(t)$ and $\theta(t)$ are measured in radians (see the Simulink diagram), when $k_p = 1$, this disturbance would generate a steady-state error of 10° if the model were perfectly linear. That represents a significant disturbance.

v) Set the proportional gain to be $k_p = 1$.

vi) Perform the following measurements:

- Measure the peak overshoot (with respect to the final value)
- Measure the steady-state error due to the disturbance
- Measure the settling time
- Measure the rise time. The "Bilevel Measurements" tool under the ruler icon may assist you in this task.

vii) Repeat the above measurements for $k_p = 0.5, 2, 3, 4, 5$.

Remark: To earn marks for the performance of the experiment, you need to demonstrate the step responses of the system to your TA and explain the impact of your choice of k_p on rise time, overshoot, settling time, and the steady-state error due to the disturbance. Your report should include plots of the step response and your observations and analysis of the system response in the context of predictions made through the theoretical analysis. In particular,

- *You should plot the positions of the poles of the closed-loop transfer function from $R(s)$ to $\Theta(s)$, for each of the chosen values of k_p, and you should comment on how the transient performance of the system can be predicted from these closed-loop pole positions. To assist you in plotting the closed-loop pole positions, observe that we can plot an* x *at the point $a+jb$ in the s-plane using the Matlab command* `plot(a,b,'x')`. *If you wish to plot another point on the same figure, you should type* `hold on`. *Type* `help plot` *to see a list of possible symbols.*
- *You should analyse the relationship between the predicted steady-state errors due to disturbances, and the measured errors. You should comment on any discrepancies and relate them to observations that you have made in earlier labs.*

4 Experiment: Proportional Controller Design (10 marks)

The one design parameter in the proportional controller, k_p, allows you to independently adjust only one of the closed-loop step response characteristics. (If the closed loop is underdamped, it does not enable control of the settling time.) In this part, you will use the model that you identified in Lab. 2 to select k_p to achieve a desired damping ratio in the following scenarios.

- To produce an *underdamped* step response with a maximum overshoot of 50%, 55%, 60% and 65%.

That is, using the parameters of the model of the motor, A and τ_m, that you identified in Lab. 2, and the derivations in the pre-lab, for each of the above specifications, choose value for k_p that achieves the design goal. You may wish to write a short Matlab script to automate these calculations

Note, that you may wish to check the measurements and calculations that you performed in Lab. 2 before taking on the designs. In particular, if your time-domain and frequency-domain measurements gave significantly different answers, you may wish to check the consistency of your measurements and calculations. In particular, you may wish to check that the model with the parameters identified in the time domain yields results that are consistent with your measurements in the frequency domain, and that the model with the parameters identified in the frequency domain yields results that are consistent with your measurements in the time domain. If you identify any inconsistencies, you may wish to perform the model identification from Lab. 2 again.

Now we seek to verify the design using the Quanser simulation model. In particular, we will implement the proportional controller and observe the closed loop step response following the procedure below. You need to examine the response of the system for each of the values of k_p calculated above.

i) Close and re-open Quanser Interactive Labs.

ii) For each of your designed values for k_p,

   - Use the theoretical models to predict the peak value of the step response and the steady-state error due to the disturbance.
   - Set the proportional gain to your proposed design value
   - Set the amplitude of the step disturbance to zero.
   - Measure the peak value of the response and the settling time, and save a plot of the step response. Calculate the actual percentage overshoot.
   - Set the amplitude of the step disturbance to $\tau_d = 0.1745$.
   - Measure the steady-state error, and save a plot of the step response.

Remark: To earn marks for the performance of the experiment, you must present the computations of the controller gains to the TA. You must also demonstrate the step responses of the closed-loop system with these gains to your TA. Your report must include a brief description of the control design, a plot of the closed-loop pole positions, plots of the step response of the system, and a brief analysis of the results in the context of predictions made through the theoretical analysis. You should explain any potential discrepancy between the experimental results and theoretical predictions.

5 Proportional Controller with Velocity Feedback

The theoretical and experimental developments with the proportional controller in the previous sections revealed some limitations of this controller for the servo-mechanism. The choice of only one design parameter k_p leads to a trade-off between competing design objectives of fast rise time, small error to a constant disturbance, and a small overshoot in the step response. Furthermore, the settling time is unaffected by the controller gain. To help alleviate these shortcomings in the proportional control of the servo-mechanism, a

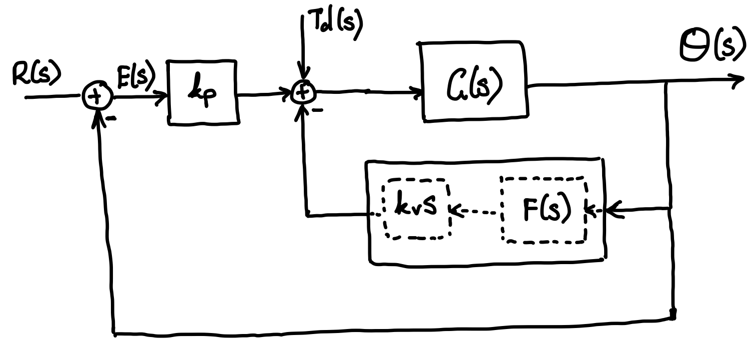

Figure 3: Proportional control of the servomotor with additional velocity feedback, where $G(s) = \frac{A}{s(s\tau_m+1)}$ is the transfer function of the motor.

velocity feedback loop is added to the system as shown in Figure 3. Unlike in a conventional Proportional-Derivative (PD) controller where the differentiator acts on the error $e(t) = r(t) - y(t)$, the derivative is only applied to a filtered version $y(t)$ in the controller in Figure 3. In this case of a step reference $r(t)$, this modified controller arrangement avoids generating an unbounded control signal due to $\dot{r}(0)$ being unbounded. The filter $F(s) = \frac{1}{1+s/150}$ reduces the high-frequency noise in the measured signal before differentiation. Note that the cut-off frequency of the filter, which is $150\,\text{rad/s} \approx 24\,\text{Hz}$, is significantly larger than the bandwidth of the motor, which was implicitly identified in Lab. 2. That means that over the bandwidth of the motor, the frequency response of the filter is approximately 1.

Pre-Lab Question 12 **(5 marks)** *Using the block diagram in Figure 3, and the approximation that $F(s) \approx 1$, show that for values of s such that $|s| \ll 150$, the total output response to the reference and disturbance inputs can be approximated by:*

$$\Theta(s) \approx \frac{\frac{k_p A}{\tau_m}}{s^2 + \frac{1+k_v A}{\tau_m}s + \frac{k_p A}{\tau_m}} R(s) + \frac{\frac{A}{\tau_m}}{s^2 + \frac{1+k_v A}{\tau_m}s + \frac{k_p A}{\tau_m}} T_d(s) \tag{15}$$

Pre-Lab Question 13 **(2 marks)** *Show that the magnitude of steady-state error due to a step disturbance $T_d(s) = \frac{\tau_d}{s}$ is given by:*

$$|e_{ss}| = \frac{\tau_d}{k_p}. \tag{16}$$

Pre-Lab Question 14 **(3 marks)** *Show that the damping ratio of the closed-loop poles is given by:*

$$\zeta = \frac{1 + k_v A}{2\sqrt{k_p A \tau_m}}. \tag{17}$$

Pre-Lab Question 15 **(6 marks)** *How do the controller parameters k_p and k_v affect the settling time, the rise time, the maximum overshoot, and the steady-state error to a constant disturbance? Hint: In addition to the above formulae, the (dominant) closed-loop pole positions may help to guide your answers.*

6 Experiment: Joint Design of Proportional Controller and Velocity Feedback (15 marks)

In this part, using the parameters A and τ_m that you identified in Lab. 2, the controller gains k_p and k_v are chosen together to simultaneously satisfy the following two design requirements:

- The magnitude of the steady-state error due to the constant disturbance input $T_d(s) = \frac{\tau_d}{s}$ is less than $2°$. (Recall from the simulink diagram that $r(t)$ and $\theta(t)$ are measured in radians, but are displayed on the 'scope' in degrees.) In practice, we might use $T_d(s)$ to model the friction in the motor, and in that case the actual value of τ_d for the motor you are working on would not be known. Therefore, the gain k_p that achieves this objective would need to be determined experimentally. However, the Quanser motor, and its model, have quite low friction, and hence we will introduce a disturbance of a known size $\tau_d = 0.1745$. If the model was perfectly linear, then when $k_p = 1$, that disturbance would generate a steady-state error of $10°$. We will leave the value of θ_d at $\pi/2$, representing a step change in angular position from $-45°$ to $+45°$.
- The percentage maximum overshoot in the step response is less than 10%.

Complete the experiment in the following way:

i) Using the mathematical models employed in the pre-lab, and the parameters of the motor that you obtained in Lab. 2, design the controller gains k_p and k_d to simultaneously achieve the two performance objectives. By observing the nature of (16) and (17) you can see that we can perform this design process sequentially. In particular, you can do this by first designing a proportional-gain-only controller with $k_v = 0$ that achieves the steady-state error objective, and then, for that value for k_p, selecting a value for k_v so that the proportional-plus-velocity-feedback controller will achieve both the steady-state error and overshoot objectives.

ii) Close and re-open Quanser Interactive Labs.

iii) Set the amplitude of the disturbance to $\tau_d = 0.1745$.

iv) Set the proportional gain to the value you have designed and the velocity feedback gain to zero. Run the simulink model and measure the steady-state error, the overshoot and the settling time of the proportional-gain only controller.

v) While the simulation continues to run, change the velocity feedback gain to the value you have designed, and wait for the transients to decay. Then measure the steady-state error, the overshoot and the settling time of the proportional-plus velocity feedback controller.

Remark: To earn marks for the performance of the experiment, you must present to your TA the values of the control gains and explain how they have been obtained. You must also show the step response of the closed loop system. Your report must include a brief description of the control design approach, plots of the closed-loop pole positions for both your design, plots of the step response of the closed-loop system for both your designs, and a brief analysis of the results in the context of predictions made through the theoretical analysis. You should explain any potential discrepancy between the experimental results and theoretical predictions.

7 Laboratory Report (25 marks)

Each group must submit an electronic report through Avenue-to-Learn due by 11:59pm one week from the day of the lab. For example if your lab is on Monday, your report would be due by next Monday at midnight. Reports that are late up to 24 hours will receive a penalty of 50. No marks will be awarded to reports that are more than 24 hours late. The report should be formatted in single-column, single-spaced, using Times New Roman 12 or equivalent font. The group members should clearly state their individual contributions to the report in a statement in the beginning of the report. The laboratory report must include plots of the results of the experiments followed with a brief analysis of the results, as instructed throughout this document. All computations not included in the pre-lab report must be presented in the the final report. You should also briefly in 1-2 paragraph(s) compare the two approaches of proportional control and proportional control with velocity feedback for the servomechanism.