CS/ECE 374 Spring 2023 William Cheng (shihuac2@illinois.edu)
Homework 1 Problem 1

Solution: (a) **Claim:** *For* $a \geq max(\frac{\gamma}{1-(c_1^2+c_2^2+c_3^2)}, 1)$[1] and $b = 0$, and for all $n \geq 1$, $T(n) \leq an^2 + b$.

Proof. **Base case:** For $1 \leq n \leq \frac{1}{c_1}$, $T(n) = 1 \leq an^2 + b$ for $a \geq 1$ by definition.
Inductive hypothesis: Let $n > \frac{1}{c_1}$. Assume $T(k) \leq ak^2 + b$ for all $1 \leq k < n$.
Inductive step:

$$\begin{aligned}
T(n) &= T(\lfloor c_1 n \rfloor) + T(\lfloor c_2 n \rfloor) + T(\lfloor c_3 n \rfloor) + \gamma n^2 \\
&\leq a(\lfloor c_1 n \rfloor)^2 + a(\lfloor c_2 n \rfloor)^2 + a(\lfloor c_3 n \rfloor)^2 + 3b + \gamma n^2 && \text{by induction} \\
&\leq a(c_1 n)^2 + a(c_2 n)^2 + a(c_3 n)^2 + 3b + \gamma n^2 && \text{by definition of floor operation} \\
&\leq ((c_1^2 + c_2^2 + c_3^2)a + \gamma)n^2 + 3b \leq an^2 + b
\end{aligned}$$

provided that

$$\begin{aligned}
&((c_1^2 + c_2^2 + c_3^2)a + \gamma) \leq a \iff a \geq \frac{\gamma}{1 - (c_1^2 + c_2^2 + c_3^2)} \\
&3b \leq b \iff b = 0 && \text{since } b \geq 0
\end{aligned}$$

Hence, $T(n) \leq an^2 + b$ for any $a \geq max(\frac{\gamma}{1-(c_1^2+c_2^2+c_3^2)}, 1)$ and $b = 0$ for all $n \geq 1$. Thus, $T(n) = O(n^2)$ for all $n \geq 1$.

(b) The asymptotic upper bound is determined by the rightmost leaf node of the recursion tree. The value of the node is $c_3^k n$ where k is the depth. $c_3^k n = 1$ since it's the leaf node, which gives $k = \log_{\frac{1}{c_3}} n$. Hence, the upper bound of the tree depth is $\log_{\frac{1}{c_3}} n$.

(c) $a \geq max(\frac{\gamma}{1-\sum_{i=1}^{k} c_i^2}, 1)$
The upper bound of the depth of the recursion tree is $\log_{\frac{1}{c_k}} n$

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[1] For the induction step, the condition $a \geq \frac{\gamma}{1-(c_1^2+c_2^2+c_3^2)}$ is sufficient, but $a \geq 1$ is necessary for the base case where $n = 1$.

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Homework 1 Problem 2

Solution: (a) **Claim:** *For any $w \in L_1$ with $n = |w| \geq 0$, $w \in L_{ee}$.*

Proof. **Base case:** For $n = |w| = 0$, $w = \epsilon$. $\#(0, \epsilon) = 0$ and $\#(1, \epsilon) = 0$. Since 0 is an even number, $w \in L_{ee}$.

Inductive hypothesis: Let $n > 0$. Assume that all strings $x \in L_1$ with $0 \leq |x| < n$ are in L_{ee}.

Inductive step: Let w be a string of length n in L_1. By the last property of L_1, w can only be generated from a string $z \in L_1$. Consider the case where w is generated by inserting a 00 or 11 into some string $z \in L_1$. Then $|z| = n - 2$, which implies $z \in L_{ee}$ by induction. Then z has even number of 0's and even number of 1's by definition of L_{ee}. Adding exactly two ones or zeros would still make an even number of 0's and 1's, therefore $w \in L_2$ by definition. Consider the other case where w is generated by concatenating some string $x \in L_1$ with 0101 or 1010. Then $|x| = n - 4$, which implies $x \in L_{ee}$. Then $\#(0, x)$ and $\#(1, x)$ are even. $\#(0, w) = \#(0, x) + 2$ and $\#(1, w) = \#(1, x) + 2$ are also even, therefore $w \in L_{ee}$. Since $w \in L_1$ implies $w \in L_{ee}$, $L_1 \subseteq L_{ee}$.

(b) **Claim:** *For any $w \in L_{ee}$ with $n = |w| \geq 0$, $w \in L_1$.*

Proof. **Base case:** For $n = |w| = 0$, $w = \epsilon$, $\#(0, w) = \#(1, w) = 0$. Since 0 is an even number, $w \in L_{ee}$. By definition, $\epsilon \in L_1$, therefore $w \in L_1$.

Inductive hypothesis: Let $n > 0$. Assume that all strings $x \in L_{ee}$ with $0 \leq |x| < n$ are in L_1.

Inductive step: Let w be a string of length n in L_{ee}. By definition, the number of 1's and the number of 0's in w are even. Consider the case where w contains at least two consecutive 1's or 0's. Then w can be written as $x11y$ or $x00y$. $xy \in L_{ee}$ because taking exactly 2 ones or zeros out of w would still make an even number of ones and zeros. Then $xy \in L_1$ by induction. By definition of L_1, $xy \in L_1$ implies $x00y \in L_1$ and $x11y \in L_1$, thus $w \in L_1$. Then consider the case where w does not contain two consecutive 1's or 0's. In this case w must be alternating 0's and 1's, and $|w| \geq 4$ to ensure even number of 0's and 1's. Thus, w is in the form $z0101$ or $z1010$ where $|z| = |w| - 4$. Since $\#(0, z) = \#(0, w) - 2$ and $\#(1, z) = \#(1, w) - 2$, z must have even number of 1's and even number of 0's, therefore $z \in L_{ee}$, which implies $z \in L_1$ by induction. By definition of L_1, $z \in L_1$ implies $z0101 \in L_1$ and $z1010 \in L_1$, therefore $w \in L_1$. Since $w \in L_{ee}$ implies $w \in L_1$ in all cases, $L_{ee} \subseteq L_1$.

(c)
- A string in $L_{eo} - L_2$: 010
- $L' = 010(1010)^*$.

Proof. $L' \subseteq L_{eo}$**:** For any $w \in L'$, $\#(0, w) = 2 + 2n$ which is even and $\#(1, w) = 1 + 2n$ which is odd, where n is the number of 1010's after 010 in w, therefore $w \in L_{eo}$. $L' \cap L_2 = \emptyset$**:** Let w be an arbitrary string in L'. Assume $w \in L_2$. Then 010 must be in L_2 since w can only be obtained by concatenating 010 with some number of 1010's by definition of L_2. However, 010 cannot be in L_2 since it is not in the form $x00y$ or $x11y$ or $x1010$ or $x0101$. Hence, $w \in L'$ implies $w \notin L_2$.

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