几何空间与n维向量空间疑难分析

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一. 向量组线性相关(无关)及线性表出的基本概念

1. 向量组$\alpha_1,\alpha_2,\cdots,\alpha_m$线性相关

$\Leftrightarrow$存在不全为零的数$k_1,k_2,\cdots,k_m$,使得$k_1\alpha_1+k_2\alpha_2+\cdots+k_m\alpha_m=0$

2. 向量组$\alpha_1,\alpha_2,\cdots,\alpha_m$线性无关

$\Leftrightarrow$ "$k_1\alpha_1+k_2\alpha_2+\cdots+k_m\alpha_m=0\Rightarrow k_1=k_2=\cdots=k_m=0$"

3. $A=(\alpha_1,\alpha_2,\cdots,\alpha_n)$　　　$R\{\alpha_1,\alpha_2,\cdots,\alpha_n\}$ $\|$

(1)$\alpha_1,\alpha_2,\cdots,\alpha_n(\in R^m)$线性相关$\Leftrightarrow R(A)<n\Leftrightarrow Ax=0$有非零解

(2)$\alpha_1,\alpha_2,\cdots,\alpha_n(\in R^m)$线性无关$\Leftrightarrow R(A)=n\Leftrightarrow Ax=0$只有零解

4. β可由$\alpha_1,\alpha_2,\cdots,\alpha_n$线性表出

$\Leftrightarrow$存在$k_1,k_2,\cdots,k_n$,使得$\beta=k_1\alpha_1+k_2\alpha_2+\cdots+k_n\alpha_n$

5. b可由$\alpha_1,\alpha_2,\cdots,\alpha_n$线性表出$\Leftrightarrow Ax=b$有解$\Leftrightarrow R(A)=R(A|b)$

6. 向量组$\alpha_1,\alpha_2,\cdots,\alpha_n$线性相关

$\Leftrightarrow$存在一个向量可由其余的线性表出

7. 向量组(I)$:\alpha_1,\alpha_2,\cdots,\alpha_s$可由向量组(II)$:\beta_1,\beta_2,\cdots,\beta_t$线性表出

$\Rightarrow R(\mathrm{I})\le R(\mathrm{II})$.

8. 向量组(I)$:\alpha_1,\alpha_2,\cdots,\alpha_s$与向量组(II)$:\beta_1,\beta_2,\cdots,\beta_t$等价

$\Rightarrow R(\mathrm{I})=R(\mathrm{II})$.

1.设$\alpha_1,\alpha_2,\cdots,\alpha_r$可由$\beta_1,\beta_2,\cdots,\beta_s$线性表示,则$r$与$s$的关系是(　)

(A) $r\le s$　(B) $r\ge s$　(C) $r<s$　(D) r与s无关.

答案:(D).

注:

$\alpha_1,\alpha_2,\cdots,\alpha_r$可由$\beta_1,\beta_2,\cdots,\beta_s$线性表出$\begin{cases}(1)\ \alpha_1,\alpha_2,\cdots,\alpha_r\text{线性无关}\Rightarrow r\le s\\(2)\ r>s\Rightarrow\alpha_1,\alpha_2,\cdots,\alpha_r\text{线性相关}\end{cases}$

例:

(1):(I) $\begin{pmatrix}1\\2\end{pmatrix},\begin{pmatrix}2\\1\end{pmatrix}$; (II) $\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix}$.

(2):(I) $\begin{pmatrix}1\\2\end{pmatrix},\begin{pmatrix}2\\1\end{pmatrix},\begin{pmatrix}1\\1\end{pmatrix}$; (II) $\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix}$.

(3):(I) $\begin{pmatrix}1\\2\end{pmatrix},\begin{pmatrix}2\\1\end{pmatrix}$; (II) $\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix},\begin{pmatrix}1\\1\end{pmatrix}$.

2.设$\alpha_1,\alpha_2,\alpha_3$线性相关,$\alpha_2,\alpha_3,\alpha_4$线性无关,**证明:**

(1).α_1能由α_2,α_3线性表出;

(2).α_4不能由$\alpha_1,\alpha_2,\alpha_3$线性表出.

证明:

(1)$\alpha_2,\alpha_3,\alpha_4$线性无关$\Rightarrow$ α_2,α_3线性无关 $\Rightarrow$ α_1可由α_2,α_3线性表出

又$\alpha_1,\alpha_2,\alpha_3$线性相关 $\Rightarrow \alpha_1=l_2\alpha_2+l_3\alpha_3$

(2)若α_4可由$\alpha_1,\alpha_2,\alpha_3$线性表出$\Leftrightarrow \alpha_4=k_1\alpha_1+k_2\alpha_2+k_3\alpha_3 \Rightarrow$

$\alpha_4=k_1(l_2\alpha_2+l_3\alpha_3)+k_2\alpha_2+k_3\alpha_3=(k_1l_2+k_2)\alpha_2+(k_1l_3+k_3)\alpha_3$,矛盾!

3. 已知$\alpha_1,\alpha_2,\alpha_3$线性无关, **证明:** $\alpha_1+\alpha_2,\alpha_2+\alpha_3,\alpha_3+\alpha_1$线性无关.

证(法1): $\boxed{令\ k_1(\alpha_1+\alpha_2)+k_2(\alpha_2+\alpha_3)+k_3(\alpha_3+\alpha_1)=0}$

$$\Leftrightarrow (k_1+k_3)\alpha_1+(k_1+k_2)\alpha_2+(k_2+k_3)\alpha_3=0$$

$\alpha_1,\alpha_2,\alpha_3$线性无关 $\Updownarrow$

$$\begin{cases} k_1+\qquad k_3=0 \\ k_1+k_2\qquad =0 \\ \qquad k_2+k_3=0 \end{cases}, \left(\begin{vmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix}=2\neq 0\right) \Rightarrow \boxed{k_1=k_2=k_3=0}$$

$\Rightarrow \alpha_1+\alpha_2,\alpha_2+\alpha_3,\alpha_3+\alpha_1$线性无关.

(P_{128}:Ex9; P_{131}:Ex4)

3. 已知$\alpha_1,\alpha_2,\alpha_3$线性无关,**证明:** $\alpha_1+\alpha_2,\alpha_2+\alpha_3,\alpha_3+\alpha_1$线性无关.

分析: 令$\beta_1=\alpha_1+\alpha_2,\beta_2=\alpha_2+\alpha_3,\beta_3=\alpha_3+\alpha_1$

"β_1,β_2,β_3线性无关 $\Leftrightarrow$ $B=(\beta_1,\beta_2,\beta_3), R(B)=3$"

(相关) (<)

证(法2): $B=(\beta_1,\beta_2,\beta_3)=(\alpha_1+\alpha_2,\alpha_2+\alpha_3,\alpha_3+\alpha_1)=\underbrace{(\alpha_1,\alpha_2,\alpha_3)}_{A}\underbrace{\begin{pmatrix}1&0&1\\1&1&0\\0&1&1\end{pmatrix}}_{P}$

"$|P|=2\neq 0\Leftrightarrow P$可逆" $\Downarrow$

$$R(B)=R(AP)=R(A)$$

$\alpha_1,\alpha_2,\alpha_3$线性无关 $\Leftrightarrow R(A)=3\Leftrightarrow R(B)=3\Leftrightarrow \beta_1,\beta_2,\beta_3$线性无关

$\boxed{\alpha_1,\alpha_2,\alpha_3\text{线性无关}\Leftrightarrow\beta_1,\beta_2,\beta_3\text{线性无关}}$

4. 已知$\alpha_1,\alpha_2,\cdots,\alpha_m;\beta_1=\alpha_1+\alpha_2,\beta_2=\alpha_2+\alpha_3,\cdots,\beta_m=\alpha_m+\alpha_1$.

证明:(1)$m=2k \Rightarrow \beta_1,\beta_2,\cdots,\beta_m$线性相关;

(2)$m=2k+1$,

$\alpha_1,\alpha_2,\cdots,\alpha_m$线性无关$\Leftrightarrow \beta_1,\beta_2,\cdots,\beta_m$线性无关.

分析:"$\beta_1,\beta_2,\cdots,\beta_m$线性无关$\Leftrightarrow B=(\beta_1,\beta_2,\cdots,\beta_m),R(B)=m$"

(相关) (<)

证:$B=(\beta_1,\beta_2,\cdots,\beta_m)=(\alpha_1+\alpha_2,\alpha_2+\alpha_3,\cdots,\alpha_m+\alpha_1)$

$$=(\alpha_1,\alpha_2,\cdots,\alpha_m)\begin{pmatrix}1 & 0 & \cdots & 0 & 1\\ 1 & 1 & \ddots & & 0\\ 0 & 1 & \ddots & \ddots & \vdots\\ \vdots & \ddots & \ddots & 1 & 0\\ 0 & \cdots & 0 & 1 & 1\end{pmatrix}=AP$$

$$|P| = \begin{vmatrix} 1 & 0 & \cdots & 0 & 1 \\ 1 & 1 & \ddots & & 0 \\ 0 & 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 & 0 \\ 0 & \cdots & 0 & 1 & 1 \end{vmatrix} = 1 + (-1)^{m+1} = \begin{cases} 0, & m = 2k \\ 2, & m = 2k+1 \end{cases}$$

(1)$m = 2k \Rightarrow P$不可逆 $\Leftrightarrow R(P) < m \Rightarrow R(B) = R(AP) \le R(P) < m$

$\Leftrightarrow \beta_1, \beta_2, \cdots, \beta_m$线性相关

(2)$m = 2k+1 \Rightarrow P$可逆 $\Rightarrow R(B) = R(AP) = R(A)$

$\Leftrightarrow \beta_1, \beta_2, \cdots, \beta_m$与$\alpha_1, \alpha_2, \cdots, \alpha_m$有相同的相关与无关性

注：$m = 2k+1$时，

$\alpha_1, \alpha_2, \cdots, \alpha_m$线性无关 $\Leftrightarrow R(A) = m \Leftrightarrow R(B) = m \Leftrightarrow \beta_1, \beta_2, \cdots, \beta_m$线性无关

$\alpha_1, \alpha_2, \cdots, \alpha_m$线性相关 $\Leftrightarrow R(A) < m \Leftrightarrow R(B) < m \Leftrightarrow \beta_1, \beta_2, \cdots, \beta_m$线性相关

思考. 已知$\alpha_1,\alpha_2,\cdots,\alpha_m;\beta_1=\alpha_1,\beta_2=\alpha_1+\alpha_2,\cdots,\beta_m=\alpha_1+\alpha_2+\cdots+\alpha_m$.

证明: $\alpha_1,\alpha_2,\cdots,\alpha_m$线性无关$\Leftrightarrow\beta_1,\beta_2,\cdots,\beta_m$线性无关.

5. 设向量组$\alpha_1,\alpha_2,\cdots,\alpha_n$中$\alpha_1\neq 0$,且每个$\alpha_i(i=1,2,\cdots,n)$都不能由它前面的$i-1$个向量$\alpha_1,\alpha_2,\cdots,\alpha_{i-1}$线性表出,**求证:**向量组$\alpha_1,\alpha_2,\cdots,\alpha_n$线性无关.

分析: $\alpha_1\neq 0\Leftrightarrow$ "$k_1\alpha_1=0\Leftrightarrow k_1=0$"

证: 令$k_1\alpha_1+k_2\alpha_2+\cdots+k_n\alpha_n=0$

如果$k_n\neq 0\Rightarrow \alpha_n=-\frac{k_1}{k_n}\alpha_1-\frac{k_2}{k_n}\alpha_2-\cdots-\frac{k_{n-1}}{k_n}\alpha_{n-1}$

α_n可由$\alpha_1,\alpha_2,\cdots,\alpha_{n-1}$线性表出,与假设矛盾.

因此$k_n=0\Rightarrow k_1\alpha_1+k_2\alpha_2+\cdots+k_{n-1}\alpha_{n-1}=0$

同理 $k_{n-1}=0$, 以此类推$k_{n-2}=k_{n-3}=\cdots=k_2=k_1=0$

$\Rightarrow$向量组$\alpha_1,\alpha_2,\cdots,\alpha_n$线性无关

6.设A是n阶方阵,若存在正整数k,使$A^k x=0$有解向量α,且$A^{k-1}\alpha\neq 0$,

证明: $\alpha, A\alpha, A^2\alpha, \cdots, A^{k-1}\alpha$线性无关.

分析: (1) $A^k\alpha=0 \Rightarrow A^m\alpha=0 (m\geq k)$

(2) $A^{k-1}\alpha\neq 0 \Rightarrow$ " $lA^{k-1}\alpha=0 \Leftrightarrow l=0$ "

证: 令 $l_0\alpha+l_1A\alpha+l_2A^2\alpha+\cdots+l_{k-1}A^{k-1}\alpha=0$

$$\Rightarrow A^{k-1}(l_0\alpha+\underline{l_1A\alpha+l_2A^2\alpha+\cdots+l_{k-1}A^{k-1}}\alpha)=0 \Rightarrow l_0=0$$

$$\Rightarrow A^{k-2}(l_1A\alpha+\underline{l_2A^2\alpha+\cdots+l_{k-1}A^{k-1}}\alpha)=0 \Rightarrow l_1=0$$

$$\Rightarrow l_i=0(i=0,1,\ldots,k-1)$$

$\Rightarrow \alpha, A\alpha, A^2\alpha, \cdots, A^{k-1}\alpha$线性无关.

$6'$. 设A是n阶方阵,$\alpha_1,\alpha_2,\alpha_3$为$n$维向量组,其中$\alpha_1\neq 0$,且满足:

$A\alpha_1=2\alpha_1, A\alpha_2=\alpha_1+2\alpha_2, A\alpha_3=\alpha_2+2\alpha_3$,**证明**:$\alpha_1,\alpha_2,\alpha_3$线性无关.

分析: $\alpha_1\neq 0 \Leftrightarrow$ "$l\alpha_1=0\Leftrightarrow l=0$"

$$\begin{cases} A\alpha_1=2\alpha_1 \Leftrightarrow (A-2I)\alpha_1=0; \\ A\alpha_2=\alpha_1+2\alpha_2 \Leftrightarrow (A-2I)\alpha_2=\alpha_1; \\ A\alpha_3=\alpha_2+2\alpha_3 \Leftrightarrow (A-2I)\alpha_3=\alpha_2. \end{cases} \boxed{\Rightarrow \begin{cases} (A-2I)^2\alpha_1=0 \\ (A-2I)^2\alpha_2=0 \\ (A-2I)^2\alpha_3=\alpha_1 \end{cases}}$$

证:令 $k_1\alpha_1+k_2\alpha_2+k_3\alpha_3=0(1)$,则

$$(A-2I)(k_1\alpha_1+k_2\alpha_2+k_3\alpha_3)=0\Rightarrow k_2\alpha_1+k_3\alpha_2=0(2)$$

$$(A-2I)(k_2\alpha_1+k_3\alpha_2)=0\Rightarrow k_3\alpha_1=0(3)$$

又$\alpha_1\neq 0$ $\Rightarrow k_3=0$

将$k_3=0$代入(2)、(1) $\Rightarrow k_1=k_2=k_3=0\Rightarrow\alpha_1,\alpha_2,\alpha_3$线性无关.

思考: $A\alpha_1,A\alpha_2,A\alpha_3$线性属性如何?

7. 设η^*是非齐次线性方程组$Ax=b$的一个解,$\xi_1,\xi_2,\cdots,\xi_{n-r}$是对应齐次线性方程组的一个基础解系,

证明: (1)$\eta^*,\xi_1,\xi_2,\cdots,\xi_{n-r}$线性无关;

(2)$\eta^*,\eta^*+\xi_1,\eta^*+\xi_2,\cdots,\eta^*+\xi_{n-r}$线性无关.

证:(1)(法1) $\left.\begin{array}{l}\xi_1,\xi_2,\cdots,\xi_{n-r}\text{线性无关}\\ \text{若}\ \eta^*,\xi_1,\xi_2,\cdots,\xi_{n-r}\text{线性相关}\end{array}\right\}\Rightarrow \eta^*=k_1\xi_1+k_2\xi_2+\cdots+k_{n-r}\xi_{n-r}$

即η^*为$Ax=0$的解. 矛盾!

$\Rightarrow\ \eta^*,\xi_1,\xi_2,\cdots,\xi_{n-r}$线性无关.

证：(法2)

令 $k_0\eta^* + k_1\xi_1 + k_2\xi_2 + \cdots + k_{n-r}\xi_{n-r} = 0$

若 $k_0 \neq 0 \Rightarrow \eta^*$ 可由 $\xi_1, \xi_2, \cdots, \xi_{n-r}$ 线性表出 $\Rightarrow \eta^*$ 为 $Ax = 0$ 的解. 矛盾！

故 $k_0 = 0 \Rightarrow k_1 = k_2 = \cdots = k_{n-r} = 0 \Rightarrow \eta^*, \xi_1, \xi_2, \cdots, \xi_{n-r}$ 线性无关.

证：(法3) $\boxed{A\eta^* = b, A\xi_i = 0(i = 1, 2, \cdots, n-r)}$

令 $k_0\eta^* + k_1\xi_1 + k_2\xi_2 + \cdots + k_{n-r}\xi_{n-r} = 0$

$\Rightarrow A(k_0\eta^* + k_1\xi_1 + k_2\xi_2 + \cdots + k_{n-r}\xi_{n-r}) = 0 \Rightarrow k_0 b = 0 (b \neq 0)$

$\Rightarrow k_0 = 0$

$\Rightarrow k_1 = k_2 = \cdots = k_{n-r} = 0 \Rightarrow \eta^*, \xi_1, \xi_2, \cdots, \xi_{n-r}$ 线性无关.

证:(2)(法1) $k_0\eta^*+k_1(\eta^*+\xi_1)+k_2(\eta^*+\xi_2)+\cdots+k_{n-r}(\eta^*+\xi_{n-r})=0$

$$\Rightarrow (k_0+k_1+\cdots+k_{n-r})\eta^*+k_1\xi_1+k_2\xi_2+\cdots+k_{n-r}\xi_{n-r}=0$$

$$\Rightarrow k_0+k_1+\cdots+k_{n-r}=0,\ k_1=k_2=\cdots=k_{n-r}=0$$

$$\Rightarrow k_0=k_1=k_2=\cdots=k_{n-r}=0$$

$\eta^*,\eta^*+\xi_1,\eta^*+\xi_2,\cdots,\eta^*+\xi_{n-r}$线性无关

(法2)

$$B=(\eta^*,\eta^*+\xi_1,\eta^*+\xi_2,\cdots,\eta^*+\xi_{n-r})=(\eta^*,\xi_1,\xi_2,\cdots,\xi_{n-r})\begin{pmatrix}1&1&1&\cdots&1\\0&1&0&\cdots&0\\0&0&1&\cdots&0\\\vdots&\vdots&\vdots&&\vdots\\0&0&0&\cdots&1\end{pmatrix}=AP$$

P可逆 $\Rightarrow R(B)=R(AP)=R(A)=n-r+1 \Rightarrow \eta^*,\eta^*+\xi_1,\eta^*+\xi_2,\cdots,\eta^*+\xi_{n-r}$线性无关.

注:$\eta^*,\eta^*+\xi_1,\eta^*+\xi_2,\cdots,\eta^*+\xi_{n-r}$线性无关$\Leftrightarrow\eta^*,\xi_1,\xi_2,\cdots,\xi_{n-r}$线性无关

8. 设$\alpha_1,\alpha_2,\cdots,\alpha_n\in R^n$, **证明:**

$$\alpha_1,\alpha_2,\cdots,\alpha_n\text{线性无关}\Leftrightarrow D=\begin{vmatrix}\alpha_1^{\mathrm T}\alpha_1 & \alpha_1^{\mathrm T}\alpha_2 & \cdots & \alpha_1^{\mathrm T}\alpha_n\\ \alpha_2^{\mathrm T}\alpha_1 & \alpha_2^{\mathrm T}\alpha_2 & \cdots & \alpha_2^{\mathrm T}\alpha_n\\ \vdots & \vdots & & \vdots\\ \alpha_n^{\mathrm T}\alpha_1 & \alpha_n^{\mathrm T}\alpha_2 & \cdots & \alpha_n^{\mathrm T}\alpha_n\end{vmatrix}\neq 0.$$

证: $A=(\alpha_1,\alpha_2,\cdots,\alpha_n)$,则

$$\text{"}\alpha_1,\alpha_2,\cdots,\alpha_n\text{线性无关}\Leftrightarrow \det A\neq 0\text{"}$$

$$A^{\mathrm T}A=\begin{pmatrix}\alpha_1^{\mathrm T}\\ \alpha_2^{\mathrm T}\\ \vdots\\ \alpha_n^{\mathrm T}\end{pmatrix}(\alpha_1,\alpha_2,\cdots,\alpha_n)=\begin{pmatrix}\alpha_1^{\mathrm T}\alpha_1 & \alpha_1^{\mathrm T}\alpha_2 & \cdots & \alpha_1^{\mathrm T}\alpha_n\\ \alpha_2^{\mathrm T}\alpha_1 & \alpha_2^{\mathrm T}\alpha_2 & \cdots & \alpha_2^{\mathrm T}\alpha_n\\ \vdots & \vdots & & \vdots\\ \alpha_n^{\mathrm T}\alpha_1 & \alpha_n^{\mathrm T}\alpha_2 & \cdots & \alpha_n^{\mathrm T}\alpha_n\end{pmatrix}$$

$$\alpha_1,\alpha_2,\cdots,\alpha_n\text{线性无关}\Leftrightarrow \det A\neq 0\Leftrightarrow 0\neq(\det A)^2=\det(A^{\mathrm T}A)=D.$$

二. 关于向量组的秩及矩阵秩的结论

1. $A_{m\times n}, B_{m\times p}$, 证明: $\max\{R(A), R(B)\} \le R(A,B) \le R(A)+R(B)$.

2. $A_{m\times n}, B_{m\times n}$, 证明: $R(A+B) \le R(A,B) \le R(A)+R(B)$.

3. $A_{m\times n}, B_{n\times p}$, 证明: $R(AB) \le \min\{R(A), R(B)\}$.(待续)

4. $A_{m\times n}, B_{n\times p}$ 且 $AB=O$, 证明: $R(A)+R(B) \le n$.

5.导出结论:

(1) $R(A^*) = \begin{cases} n, & R(A)=n \\ 0, & R(A)<n-1 \\ 1, & R(A)=n-1 \end{cases}$

(2) $A_{n\times n}$, 且$A^2=A \Rightarrow R(A)+R(A-I)=n$.

$A_{n\times n}$, 且$A^2+A=O \Rightarrow R(A)+R(A+I)=n$.

$A_{n\times n}, B_{n\times n}$,且 $ABA=B^{-1} \Rightarrow R(AB+I)+R(AB-I)=n$.

1. proof： $A_{m\times n}$, $B_{m\times p}$, 证明： $\max\{R(A),R(B)\}\le R(A,B)\le R(A)+R(B)$.

证(法1): 令 $A=(\alpha_1,\alpha_2,\cdots,\alpha_n)$, $B=(\beta_1,\beta_2,\cdots,\beta_p)$, 则

$$(A,B)=(\underbrace{\alpha_1\alpha_2,\cdots,\alpha_n}_{\text{(I)}},\underbrace{\beta_1,\beta_2,\cdots,\beta_p}_{\text{(II)}})$$

记

$$C=(\underbrace{\alpha_{i_1},\alpha_{i_2},\cdots,\alpha_{i_s}}_{\text{(I)的极大无关组}},\underbrace{\beta_{j_1},\beta_{j_2},\cdots,\beta_{j_t}}_{\text{(II)的极大无关组}})$$

则 (A,B) 的列向量组与 C 的列向量组等价.

$$\Rightarrow \left\{\begin{array}{l} R(A) \\ R(B) \end{array}\right. \le R(A,B)=R(C)\le s+t=R(A)+R(B)$$

$$\Rightarrow \max\{R(A),R(B)\}\le R(A,B)\le R(A)+R(B).$$

1. proof： $A_{m\times n}$, $B_{m\times p}$, 证明： $\max\{R(A),R(B)\}\le R(A,B)\le R(A)+R(B)$.

证(法2)： $(A,B)=(I_m,I_m)\begin{pmatrix}A & \\ & B\end{pmatrix}\Rightarrow R(A,B)\le R\begin{pmatrix}A & \\ & B\end{pmatrix}=R(A)+R(B)$

$$\left.\begin{cases} A=(A,B)\begin{pmatrix}I_n\\0\end{pmatrix}\Rightarrow R(A)\le R(A,B) \\ B=(A,B)\begin{pmatrix}0\\I_p\end{pmatrix}\Rightarrow R(B)\le R(A,B)\end{cases}\right\}\Rightarrow \max\{R(A),R(B)\}\le R(A,B)$$

$$\Rightarrow \max\{R(A),R(B)\}\le R(A,B)\le R(A)+R(B).$$

注： $\boxed{A_{m\times n},\ B_{p\times n}\ \Rightarrow \max\{R(A),R(B)\}\le R\begin{pmatrix}A\\B\end{pmatrix}\le R(A)+R(B).}$

2. proof: $A_{m\times n}$, $B_{m\times n}$, 证明: $R(A+B)\le R(A,B)\le R(A)+R(B)$.

证(法1): 令$A=(\alpha_1,\alpha_2,\cdots,\alpha_n),B=(\beta_1,\beta_2,\cdots,\beta_n)$,则

$$A+B=(\alpha_1+\beta_1,\alpha_2+\beta_2,\cdots,\alpha_n+\beta_n)$$

$$(A,B)=(\alpha_1,\alpha_2,\cdots,\alpha_n,\beta_1,\beta_2,\cdots,\beta_n)$$

$\underline{\alpha_1+\beta_1,\alpha_2+\beta_2,\cdots,\alpha_n+\beta_n}$可由$\underline{\alpha_1\alpha_2,\cdots,\alpha_n},\underline{\beta_1,\beta_2,\cdots,\beta_n}$线性表出

$$\Rightarrow R\{\alpha_1+\beta_1,\alpha_2+\beta_2,\cdots,\alpha_n+\beta_n\}\le R\{\alpha_1\alpha_2,\cdots,\alpha_n,\beta_1,\beta_2,\cdots,\beta_n\}$$

$$\Rightarrow R(A+B)\le R(A,B)\ (\le R(A)+R(B))$$

证(法2): $(A,B)\xrightarrow{c_2+c_1}(A+B,B)\Rightarrow R(A+B)\le R(A+B,B)=R(A,B)$

$$(\le R(A)+R(B))$$

证(法3): $A+B=(A,B)\begin{pmatrix} I_n \\ I_n \end{pmatrix}\Rightarrow R(A+B)\le R(A,B)\ \ (\le R(A)+R(B))$

4. $A_{m\times n}$，$B_{n\times p}$ 且 $AB=O$，证明：$R(A)+R(B)\le n$.

($A_{n\times n}$，$B_{n\times n}$ 且 $AB=O$，证明：$R(A)+R(B)\le n$. (P_{143}:Ex4))

分析：$R(A)+R(B)\le n \Leftrightarrow R(B)\le n-R(A)$

$$n-R(A) = \dim W \Leftarrow W=\{x \mid Ax=0, \forall x\in R^n\}$$

证：(1). 若$B=0$,则$R(A)+R(B)=R(A)+0=R(A)\le n$.

(2). 设$B=(b_1,b_2,\cdots,b_p)\ne O$,则 $b_l\ne 0$

$$(Ab_1,Ab_2,\cdots,Ab_p)=A(b_1,b_2,\cdots,b_p)=AB=O=(0,0,\cdots,0)$$

$Ab_i=0(i=1,2,\cdots,p)$ 且 $Ab_l=0(b_l\ne 0)$

$\Downarrow$ $\Downarrow$

$b_i(i=1,2,\cdots,p)$为$Ax=0$的解 有基础解系$\xi_1,\xi_2,\cdots,\xi_{n-r}(r=R(A))$

$$R(B)=R\{b_1,b_2,\cdots,b_p\}\le R\{\xi_1,\xi_2,\cdots,\xi_{n-r}\}=n-r=n-R(A)$$

$\Leftrightarrow R(A)+R(B)\le n$.

(Sylvester公式): $A_{m\times n}, B_{n\times p} \Rightarrow R(A)+R(B)-n \le R(AB)$

证: $R(A)=r \Rightarrow$ 存在P,Q可逆, 使得$PAQ=\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$

$$PAB = PAQ\underbrace{Q^{-1}B}_{C} = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} C = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} C_1 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{cases} (1)\ R(AB) = R(PAB) = R\begin{pmatrix} C_1 \\ 0 \end{pmatrix} = R(C_1), \\ (2)\ R(C_2) \le n-r \\ (3)\ R(B) = R(C) = R\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} \le R(C_1)+R(C_2) \le R(AB)+n-r = R(AB)+n-R(A) \end{cases}$$

$$\Rightarrow R(B)+R(A)-n \le R(AB)$$

Sylvester推论: $A_{m\times n}, B_{n\times p}$ 且 $AB=O \Rightarrow R(A)+R(B) \le n$

问题1. 设A, B为$AB=O$的任意两个非零矩阵,则必有( )

(A) A的列向量组线性相关, B的行向量组线性相关.

(B) A的列向量组线性相关, B的列向量组线性相关.

(C) A的行向量组线性相关, B的行向量组线性相关.

(D) A的行向量组线性相关, B的列向量组线性相关.

分析: $A_{m\times n}$, $B_{n\times s}$

(1) $AB=O \Rightarrow R(A)+R(B)\le n$

(2) A, B非零$\Rightarrow R(A)\ge 1, R(B)\ge 1$

$\Rightarrow 1\le R(A)<n, 1\le R(B)<n.$

A的列向量组线性相关 $\Uparrow$ $R(\alpha_1,\alpha_2,\cdots,\alpha_n) = R(A)$

$R(B) = R\begin{pmatrix}\beta_1\\ \beta_2\\ \vdots\\ \beta_n\end{pmatrix}$

B的行向量组线性相关 $\Leftarrow R\begin{pmatrix}\beta_1\\ \beta_2\\ \vdots\\ \beta_n\end{pmatrix}$

问题2. $A_{m\times n}$, $R(A)=m<n$,则( )

(A) A的任意m个列向量所成向量组线性无关.

(B) A的任意一个m阶子式不为零

(C) 若$BA=O$,则$B=O$

(D) 通过行初等变换,必可化为(E_m,O)

解:(法1) $$\left.\begin{aligned} BA=O \Rightarrow R(B)+R(A)\le m \\ R(A)=m \end{aligned}\right\} \Rightarrow R(B)\le 0 \Rightarrow R(B)=0 \Leftrightarrow B=O$$

(法2) $$BA=O \Leftrightarrow \begin{pmatrix} \boxed{b_{11} & \cdots & b_{1m}} \\ b_{21} & \cdots & b_{2m} \\ \cdots & \cdots & \cdots \\ b_{r1} & \cdots & b_{rm} \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_m \end{pmatrix} = \begin{pmatrix} \boxed{0} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \Leftrightarrow \sum_{j=1}^{m} b_{ij}\alpha_j = 0(\forall i)$$

$$R(A)=m \Leftrightarrow \alpha_1,\alpha_2,\cdots,\alpha_m\text{线性无关} \Rightarrow b_{ij}=0(\forall i,j) \Leftrightarrow B=O$$

导出结论(2).proof：$A_{n\times n}$ 且 $A^2=A\Rightarrow R(A)+R(A-I)=n.$

证: $A^2=A\Leftrightarrow A(A-I)=O$

$\Rightarrow$ (1) $R(A)+R(A-I)\le n.$

(2) $R(A)+R(A-I)=R(A)+R(I-A)\ge R(A+I-A)=R(I)=n.$

$\Rightarrow R(A)+R(A-I)=n.$

三. 如何判断向量组线性表出关系?

(1) $Ax=b$有解 $\Leftrightarrow R(A)=R(A|b)\Leftrightarrow b$可由$\alpha_1,\alpha_2,\cdots,\alpha_n$线性表出

$\Leftrightarrow \alpha_1,\alpha_2,\cdots,\alpha_n$与$\alpha_1,\alpha_2,\cdots,\alpha_n,b$等价

(2) $\begin{cases}(\mathrm{I}):\alpha_1,\alpha_2,\cdots,\alpha_n;(\mathrm{II}):b_1,b_2,\cdots,b_l\\ A=(\alpha_1,\alpha_2,\cdots,\alpha_n);B=(b_1,b_2,\cdots,b_l)\end{cases}$, (II)可由(I)表出 $\Leftrightarrow R(A)=R(A|B)$

证: (II)可由(I)表出 $\Leftrightarrow \forall b_j(j=1,2,\cdots,l),b_j$可由$\alpha_1,\alpha_2,\cdots,\alpha_n$表出 $\Leftrightarrow R(A)=R(A|b_j)$

$\Leftrightarrow R(A)=R(A|b_1)=R(A|b_2)=\cdots=R(A|b_l)\Leftrightarrow R(A)=R(A|b_1,b_2,\cdots,b_l)=R(A|B)$

(3) $\begin{cases}(\mathrm{I}):\alpha_1,\alpha_2,\cdots,\alpha_n;(\mathrm{II}):b_1,b_2,\cdots,b_l\\ A=(\alpha_1,\alpha_2,\cdots,\alpha_n);B=(b_1,b_2,\cdots,b_l)\end{cases}$, (II)与(I)等价 $\Leftrightarrow R(A)=R(B)=R(A|B)$

证: (II)与(I)等价 $\Leftrightarrow \begin{cases}(\mathrm{II})\text{可由}(\mathrm{I})\text{表出}\Leftrightarrow R(A)=R(A|B)\\ (\mathrm{I})\text{可由}(\mathrm{II})\text{表出}\Leftrightarrow R(B)=R(B|A)\end{cases}\Leftrightarrow R(A)=R(B)=R(A|B)$

1.设向量组：$a_1=(1,-1,1,-1)^{\mathrm T}$、$a_2=(3,1,1,3)^{\mathrm T}$；

向量组：$b_1=(2,0,1,1)^{\mathrm T}$、$b_2=(1,1,0,2)^{\mathrm T}$、$b_3=(3,-1,2,0)^{\mathrm T}$.

证明：向量组a_1、a_2与b_1、b_2、b_3等价.

证：令 $A=(a_1,a_2),B=(b_1,b_2,b_3)$

两向量组等价 $\Leftrightarrow R(A)=R(B)=R(A\,|\,B)$

$$(A\,|\,B)=\left(\begin{array}{cc:ccc}1&3&2&1&3\\-1&1&0&1&-1\\1&1&1&0&2\\-1&3&1&2&0\end{array}\right)\xrightarrow[r_1+r_2]{\substack{r_3+r_4\\r_2+r_3}}\left(\begin{array}{cc:ccc}1&3&2&1&3\\0&4&2&2&2\\0&2&1&1&1\\0&4&2&2&2\end{array}\right)\xrightarrow[-\frac{1}{2}r_2+r_3]{-r_2+r_4}\left(\begin{array}{cc:ccc}1&3&2&1&3\\0&2&1&1&1\\0&0&0&0&0\\0&0&0&0&0\end{array}\right),$$

$$\Rightarrow\left\{\begin{array}{l}(1)\,R(A)=R(A\,|\,B)=2\\(2)\begin{vmatrix}2&1\\1&1\end{vmatrix}=1\neq 0\Rightarrow R(B)=2\end{array}\right\}\Rightarrow R(A)=R(B)=R(A\,|\,B)$$

$\Leftrightarrow$ 两向量组等价

四. 如何从向量组线性表出的观点认识两矩阵的乘积?

1. $A=(a_{ij})_{m\times n}$, $B=(b_{ij})_{n\times p}$ 且 $C=AB$, 则

(1)矩阵 $C=AB$ 的列向量组能由 A 的列向量组线性表出.

(2)矩阵 $C=AB$ 的行向量组能由 B 的行向量组线性表出.

证: (1) $A=(\alpha_1,\alpha_2,\cdots,\alpha_n)$, $C=(c_1,c_2,\cdots,c_p)$, 有

$$\boxed{C=AB}\Leftrightarrow(\boxed{c_1},c_2,\cdots,c_p)=\boxed{(\alpha_1,\alpha_2,\cdots,\alpha_n)}\begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2p} \\ \vdots & \vdots & & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{np} \end{pmatrix}$$

$$\Leftrightarrow c_j=\sum_{i=1}^{n} b_{ij}\alpha_i=b_{1j}\alpha_1+\cdots+b_{nj}\alpha_n\ (j=1,2,\cdots,p)$$

$\Leftrightarrow$ 矩阵 $C=AB$ 的列向量组能由 A 的列向量组线性表出.

$\Rightarrow$ 矩阵 $C^T==B^T A^T$ 的列向量组能由 B^T 的列向量组线性表出.

(2) 矩阵 $C=AB$ 的行向量组能由 B 的行向量组线性表出.

1. 设$A_{m\times n}$, $B_{n\times p}$, **证明：** $R(AB)\le\min\{R(A),R(B)\}$.

证(法1): 令 $C_{m\times p}=AB$

$$\Rightarrow (c_1,\cdots,c_p)=(\alpha_1,\cdots,\alpha_n)\begin{pmatrix} b_{11} & \cdots & b_{1p} \\ b_{21} & \cdots & b_{2p} \\ \cdots & \cdots & \cdots \\ b_{n1} & \cdots & b_{np} \end{pmatrix}$$

$$\Rightarrow c_k=b_{1k}\alpha_1+b_{2k}\alpha_2+\cdots+b_{nk}\alpha_n,\ (k=1,\ldots,p)$$

$\Rightarrow$ 矩阵$C=AB$的列向量组能由A的列向量组线性表出.

(1) $R(AB)=R(C)=R\{c_1,\cdots,c_p\}\le R\{\alpha_1,\cdots,\alpha_n\}=R(A)$

(2) $R(AB)=R(C)=R(C^{\mathrm{T}})=R(B^{\mathrm{T}}A^{\mathrm{T}})\le R(B^{\mathrm{T}})=R(B)$

$\Rightarrow R(AB)\le\min\{R(A),R(B)\}$.

1.设$A_{m\times n}, B_{n\times p}$,**证明：** $R(AB)\le\min\{R(A),R(B)\}$

证(法2): **设**$R(A)=r$,则存在P,Q可逆,使得 $PAQ=\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$

$$\Rightarrow PAB=PAQ\underbrace{Q^{-1}B}_{C}=\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}C=\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} C_1 \\ C_2 \end{pmatrix}=\begin{pmatrix} C_1 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{cases} (1)\ R(AB)=R(PAB)=R\begin{pmatrix} C_1 \\ 0 \end{pmatrix}\le r=R(A) \\ (2)\ R(AB)=R((AB)^{\mathrm{T}})=R(B^{\mathrm{T}}A^{\mathrm{T}})\le R(B^{\mathrm{T}})=R(B) \end{cases}$$

$$\Rightarrow R(AB)\le\min\{R(A),R(B)\}$$

2. $A_{n\times m}$, $B_{m\times n}$(其中$n<m$), $I_{n\times n}$. 若$AB=I$,证明:B的列向量组线性无关.

证(法1): 设$B=(\beta_1,\beta_2,\cdots,\beta_n)$,

令$x_1\beta_1+x_2\beta_2+\cdots+x_n\beta_n=0$, 即$(\beta_1,\beta_2,\cdots,\beta_n)\begin{pmatrix}x_1\\x_2\\\vdots\\x_n\end{pmatrix}=0$

$\Leftrightarrow Bx=0 \Rightarrow x=Ix=ABx=A0=0 \Rightarrow \beta_1,\beta_2,\cdots,\beta_n$线性无关.

证(法2): $\left.\begin{cases}(1)R(B)\le n\\(2)R(B)\ge R(AB)=R(I)=n\end{cases}\right\}$ $\Rightarrow \underset{\substack{\parallel\\R(\beta_1,\beta_2,\cdots,\beta_n)}}{R(B)}=n$

$\Leftrightarrow \beta_1,\beta_2,\cdots,\beta_n$线性无关.

3.设A,B为n阶矩阵,则下列结论正确的是(   )

(A) $R(A,AB)=R(A)$; (B) $R(A,BA)=R(A)$;

(C) $R(A,B)=\max\{R(A),R(B)\}$; (D) $R(A,B)=R(A^{\mathrm{T}},B^{\mathrm{T}})$.

注:(1). $(A,AB)=A(I,B)\Rightarrow \left.\begin{array}{l} R(A,AB)\le R(A) \\ 又\ R(A)\le R(A,AB)\end{array}\right\}\Rightarrow R(A,AB)=R(A)$

(另).记$A=(\alpha_1,\alpha_2,\cdots,\alpha_n),C=AB=(\gamma_1,\gamma_2,\cdots,\gamma_n)\Rightarrow$

$$R(A,AB)=R\{\alpha_1,\alpha_2,\cdots,\alpha_n,\gamma_1,\gamma_2,\cdots,\gamma_n\}=R\{\alpha_1,\alpha_2,\cdots,\alpha_n\}=R(A)$$

(2).令$A=\begin{pmatrix}1&1\\0&0\end{pmatrix},B=\begin{pmatrix}0&1\\1&0\end{pmatrix}$,则$BA=\begin{pmatrix}0&0\\1&1\end{pmatrix}\Rightarrow(A,BA)=\begin{pmatrix}1&1&0&0\\0&0&1&1\end{pmatrix}$

$\Rightarrow 1=R(A)\ne R(A,BA)=2$

(3,4).令$A=\begin{pmatrix}1&0\\0&0\end{pmatrix},B=\begin{pmatrix}0&0\\1&0\end{pmatrix}\Rightarrow\begin{cases}R(A,B)=R\begin{pmatrix}1&0&0&0\\0&0&1&0\end{pmatrix}=2\\ R(A^{\mathrm{T}},B^{\mathrm{T}})=R\begin{pmatrix}1&0&0&1\\0&0&0&0\end{pmatrix}=1\end{cases}$

$\max\{R(A),R(B)\}=1$

五."矩阵A与B等价"与"向量组$\alpha_1,\alpha_2,\cdots,\alpha_n$与$\beta_1,\beta_2,\cdots,\beta_n$等价"的关系?

(1)矩阵A与B等价 $\Leftrightarrow A_{m\times n}\xrightarrow{\text{有限次初等变换}}B_{m\times n}$

$\Leftrightarrow$ 存在初等矩阵$E_1,\cdots,E_s,F_1,\cdots,F_t$,使得$E_s\cdots E_1AF_1\cdots F_t=B$

$\Leftrightarrow$ 存在可逆矩阵P,Q,使得$PAQ=B \Leftrightarrow R(A)=R(B)$

(2)矩阵A与B行等价 $\Leftrightarrow A\xrightarrow{\text{有限次行初等变换}}B$

$\Leftrightarrow$ 存在可逆矩阵P,使得$PA=B$

$\Leftrightarrow A=P^{-1}B=QB$ $\Rightarrow$ A与B的行向量组等价

(3)矩阵A与B列等价 $\Leftrightarrow A\xrightarrow{\text{有限次列初等变换}}B$

$\Leftrightarrow$ 存在可逆矩阵Q,使得$AQ=B$

$\Leftrightarrow A=BQ^{-1}=BP$ $\Rightarrow$ A与B的列向量组等价

注：1^0. 矩阵A与B等价 $\Leftrightarrow A_{m\times n} \xrightarrow{\text{有限次初等变换}} B_{m\times n}$

$\Rightarrow$ A与B的行向量组未必等价,A与B的列向量组未必等价.

举例：(1)$A=\begin{pmatrix}1&2\\0&0\end{pmatrix}=\begin{pmatrix}\alpha_1\\\alpha_2\end{pmatrix}$, $B=\begin{pmatrix}0&0\\1&3\end{pmatrix}=\begin{pmatrix}\beta_1\\\beta_2\end{pmatrix}$

$\Rightarrow$ A与B等价，但α_1,α_2与β_1,β_2不等价.

(2)$A=\begin{pmatrix}1&2\\0&0\end{pmatrix}=(\alpha_1,\alpha_2)$, $B=\begin{pmatrix}0&0\\1&3\end{pmatrix}=(\beta_1,\beta_2)$

$\Rightarrow$ A与B等价, 但α_1,α_2与β_1,β_2不等价.

2^0. 设两列向量组等价(m维), 若它们所含向量个数不相同,则它们对应的两个矩阵不同型,因而不等价;若它们所含向量个数相同(如都为n个)那么它们对应的两个$m\times n$矩阵列等价,从而一定等价,但不一定行等价.

六. 矩阵的行初等变换对列向量组和行向量组各有什么作用?

$A_{m\times n} \xrightarrow{\text{行初等变换}} B_{m\times n}$ ($\Leftrightarrow$ 矩阵A与B的行等价)

$\Rightarrow \begin{cases} \text{(1)矩阵}A\text{与}B\text{的行向量组等价.} \\ \text{(2)矩阵}A\text{与}B\text{的列向量组有相同的线性(相关与无关)关系.} \end{cases}$

注: $A \xrightarrow{\text{行初等变换}} B$(行阶梯形或简化行阶梯形)

(求最大无关组及用最大无关组表出其它向量的理论基础)

(3) $R(A) = A$的行秩$=A$的列秩.

1. 求向量组：

$$\alpha_1=(1,2,3,4)^T,\alpha_2=(2,3,4,5)^T,\alpha_3=(3,4,5,6)^T,\alpha_4=(4,5,6,7)^T$$

的秩与一个最大无关组,并用所求最大无关组表示其余向量.

解: $$(\alpha_1,\alpha_2,\alpha_3,\alpha_4)=\begin{pmatrix}1&2&3&4\\2&3&4&5\\3&4&5&6\\4&5&6&7\end{pmatrix}\xrightarrow[(-1)r_1+r_2]{(-1)r_3+r_4 \atop (-1)r_2+r_3}\begin{pmatrix}1&2&3&4\\1&1&1&1\\1&1&1&1\\1&1&1&1\end{pmatrix}$$

$$\xrightarrow[(-1)r_1+r_2]{(-1)r_3+r_4 \atop (-1)r_2+r_3}\begin{pmatrix}1&2&3&4\\0&-1&-2&-3\\0&0&0&0\\0&0&0&0\end{pmatrix}\xrightarrow[(-1)r_2]{2r_2+r_1}\begin{pmatrix}1&0&-1&-2\\0&1&2&3\\0&0&0&0\\0&0&0&0\end{pmatrix}\Rightarrow R(A)=2$$

α_1,α_2为一个最大无关组;

$$\begin{cases}\alpha_3=-\alpha_1+2\alpha_2\\ \alpha_4=-2\alpha_1+3\alpha_2\end{cases}$$

七. 相关与无关两个对偶结论

1. (I): $\alpha_1,\alpha_2,\cdots,\alpha_s(\in R^n)$; (II) $\alpha_1,\alpha_2,\cdots,\alpha_s,\alpha_{s+1},\cdots,\alpha_m(\in R^n)$;

(I)相关 $\xrightarrow{\text{个数}\uparrow}$ (II)相关; (II)无关 $\xrightarrow{\text{个数}\downarrow}$ (I)无关. (P_{129} : Th3)

"部分相关$\Rightarrow$整体相关" "整体无关$\Rightarrow$部分无关"

2. (I): $\alpha_1,\alpha_2,\cdots,\alpha_s(\in R^r)$ $\xrightarrow[\beta_1,\beta_2,\cdots,\beta_s\in R^{n-r}]{}$ $\begin{pmatrix}\alpha_1\\ \beta_1\end{pmatrix},\begin{pmatrix}\alpha_2\\ \beta_2\end{pmatrix},\cdots,\begin{pmatrix}\alpha_s\\ \beta_s\end{pmatrix}\in R^n$

$$\begin{array}{cccc} \| & \| & & \| \\ \gamma_1 & \gamma_2 & ,\cdots, & \gamma_s\in R^n \end{array}$$

则 $\begin{cases}(1)\ \alpha_1,\alpha_2,\cdots,\alpha_s\text{线性无关}\Rightarrow\gamma_1,\gamma_2,\cdots,\gamma_s\text{线性无关};\\ (2)\ \gamma_1,\gamma_2,\cdots,\gamma_s\text{线性相关}\Rightarrow\alpha_1,\alpha_2,\cdots,\alpha_s\text{线性相关}.\end{cases}$ (P_{131} : $Ex10$)

分析: "$\alpha_1,\alpha_2,\cdots,\alpha_s$线性无关$\Leftrightarrow$ $A=(\alpha_1,\alpha_2,\cdots,\alpha_s)$, $Ax=0$只有零解"

(相关) (有非零解)

证：(I)：$(\alpha_1,\alpha_2,\cdots,\alpha_s)\begin{pmatrix}x_1\\x_2\\\vdots\\x_s\end{pmatrix}=0$，(II)：$(\gamma_1,\gamma_2,\cdots,\gamma_s)\begin{pmatrix}x_1\\x_2\\\vdots\\x_s\end{pmatrix}=0\Leftrightarrow\begin{pmatrix}\alpha_1,\alpha_2,\cdots,\alpha_s\\\beta_1,\beta_2,\cdots,\beta_s\end{pmatrix}\begin{pmatrix}x_1\\x_2\\\vdots\\x_s\end{pmatrix}=0$

$\Updownarrow$ 令$A=(\alpha_1,\alpha_2,\cdots,\alpha_s)$

(I)：$Ax=0$

$\Updownarrow$ 令$B=(\beta_1,\beta_2,\cdots,\beta_s)$

(II)：$\begin{pmatrix}A\\B\end{pmatrix}x=0\Leftrightarrow\begin{cases}Ax=0\\Bx=0\end{cases}$

则(II)的前r个方程就是(I)的方程$\Rightarrow$(II)的解必是(I)的解,即

$$\boxed{\{0\}\subset\{\text{(II)的解集}\}\subset\{\text{(I)的解集}\}}$$

(1)$\alpha_1,\alpha_2,\cdots,\alpha_s$线性无关$\Leftrightarrow$(I)只有零解$\Rightarrow$(II)只有零解

$\Leftrightarrow\gamma_1,\gamma_2,\cdots,\gamma_s$线性无关.

(2)$\gamma_1,\gamma_2,\cdots,\gamma_s$线性相关$\Leftrightarrow$(II)有非零解$\Rightarrow$(I)有非零解

$\Leftrightarrow\alpha_1,\alpha_2,\cdots,\alpha_s$线性相关.

1.设有向量组$\gamma_i=(a_i,a_i^2,\cdots,a_i^n)^{\mathrm{T}},(i=1,2,\cdots,m)(m\le n)$,试证:向量组$\gamma_1,\gamma_2,\cdots,\gamma_m$线性无关,(其中:$a_1,a_2,\cdots,a_m$为$m$个互不相等且不为零的常数).

证:$A=(\gamma_1,\cdots,\gamma_m)=\begin{pmatrix} a_1 & a_2 & \cdots & a_m \\ a_1^2 & a_2^2 & \cdots & a_m^2 \\ \vdots & \vdots & & \vdots \\ \hline a_1^n & a_2^n & \cdots & a_m^n \end{pmatrix}(m\le n)$,由$A$的前$m$行与$m$列构成的子式

$$D=\begin{vmatrix} a_1 & a_2 & \cdots & a_m \\ a_1^2 & a_2^2 & \cdots & a_m^2 \\ \vdots & \vdots & & \vdots \\ a_1^m & a_2^m & \cdots & a_m^m \end{vmatrix}=a_1a_2\cdots a_m\begin{vmatrix} 1 & 1 & \cdots & 1 \\ a_1 & a_2 & \cdots & a_m \\ \vdots & \vdots & & \vdots \\ a_1^{m-1} & a_2^{m-1} & \cdots & a_m^{m-1} \end{vmatrix}$$

$\beta_1\quad\beta_2\ ,\cdots,\ \beta_m\in R^m$

$$=(\prod_{i=1}^{m}a_i)\{\prod_{1\le i<j\le m}^{m}(a_j-a_i)\}\ne 0,$$

令$\beta_i=(a_i,a_i^2,\cdots,a_i^m)^{\mathrm{T}}\Rightarrow\beta_1,\beta_2,\cdots,\beta_m$线性无关$\Rightarrow\gamma_1,\gamma_2,\cdots,\gamma_m$线性无关.

八. 线性方程组解的结构

1.设$R(A)=r<n$,则$Ax=0$有基础解系且所含向量个数为$n-r$,

即$\dim W=n-r$,(其中n为方程组未知量的个数,$W=\{x\,|\,Ax=0\}$).

2.若$R(A)=n$,则$Ax=0$只有零解,无基础解系.

3.$Ax=0$的通解:设$\xi_1,\xi_2,\cdots,\xi_{n-r}$为$Ax=0$一个基础解系,则

$\forall\alpha(Ax=0$的解),

$\alpha=k_1\xi_1+k_2\xi_2+\cdots+k_{n-1}\xi_{n-r},\forall k_1,k_2,\cdots,k_{n-1}\in R.$

4.$Ax=b$的通解:设η_0为$Ax=b$一个特解,$\xi_1,\xi_2,\cdots,\xi_{n-r}$为其导出组的一个基础解系,则

$\forall\alpha(Ax=b$的解),

$\alpha=\eta_0+k_1\xi_1+k_2\xi_2+\cdots+k_{n-1}\xi_{n-r},\forall k_1,k_2,\cdots,k_{n-1}\in R.$

1.设矩阵$A=(\alpha_1,\alpha_2,\alpha_3,\alpha_4)$,其中$\alpha_2,\alpha_3,\alpha_4$线性无关,$\alpha_1=2\alpha_2-\alpha_3$.

如果$\beta=\alpha_1+\alpha_2+\alpha_3+\alpha_4$,求非齐次线性方程组$Ax=\beta$的通解.

解:(1)$\alpha_1=2\alpha_2-\alpha_3 \Rightarrow \alpha_1,\alpha_2,\alpha_3$线性相关$\Rightarrow \alpha_1,\alpha_2,\alpha_3,\alpha_4$线性相关

$\Rightarrow Ax=0$有非零解$\Leftrightarrow Ax=0$有基础解系.

(2)$\alpha_2,\alpha_3,\alpha_4$线性无关,$\alpha_1,\alpha_2,\alpha_3,\alpha_4$线性相关$\Rightarrow R\{\alpha_1,\alpha_2,\alpha_3,\alpha_4\}=3$

$\Leftrightarrow R(A)=3 \Rightarrow$基础解系含向量的个数为$4-R(A)=4-3=1$.

(3)$\alpha_1=2\alpha_2-\alpha_3 \Rightarrow 1\alpha_1-2\alpha_2+1\alpha_3+0\alpha_4=0 \Leftrightarrow (\alpha_1,\alpha_2,\alpha_3,\alpha_4)\begin{pmatrix}1\\-2\\1\\0\end{pmatrix}=0$

$\Rightarrow Ax=0$的基础解系为$(1,-2,1,0)^{\mathrm{T}}$.

(4)$\beta=\alpha_1+\alpha_2+\alpha_3+\alpha_4=(\alpha_1,\alpha_2,\alpha_3,\alpha_4)\begin{pmatrix}1\\1\\1\\1\end{pmatrix}=A\begin{pmatrix}1\\1\\1\\1\end{pmatrix} \Rightarrow (1,1,1,1)^{\mathrm{T}}$为$Ax=\beta$的特解.

$\Rightarrow Ax=\beta$的通解为:$(1,1,1,1)^T+k(1,-2,1,0)^T,\forall\in R$.

2.证明: $R(A^{\mathrm{T}}A)=R(A)$.

分析: $A_{m\times n}\Rightarrow(A^{\mathrm{T}}A)_{n\times n}$. $\quad R(A^{\mathrm{T}}A)=R(A)\Leftrightarrow \underset{\dim W_2}{\underset{\|}{n-R(A^{\mathrm{T}}A)}}=\underset{\dim W_1}{\underset{\|}{n-R(A)}}$

令 $W_1=\{x\,|\,Ax=0,x\in R^n\}$,

$W_2=\{x\,|\,(A^{\mathrm{T}}A)x=0,x\in R^n\}$

证:

(1) $\forall x\in W_1, Ax=0\Rightarrow A^{\mathrm{T}}(Ax)=A^{\mathrm{T}}0=0\Rightarrow x\in W_2\Rightarrow W_1\subset W_2$.

(2) $\forall x\in W_2, A^{\mathrm{T}}Ax=0\Rightarrow x^{\mathrm{T}}A^{\mathrm{T}}Ax=x^{\mathrm{T}}0=0\Rightarrow Ax=0\Rightarrow x\in W_1\Rightarrow W_2\subset W_1$.

$$\Rightarrow\quad W_1=W_2\Rightarrow \underset{n-R(A)}{\underset{\|}{\dim W_1}}=\underset{n-R(A^{\mathrm{T}}A)}{\underset{\|}{\dim W_2}}\Rightarrow R(A^{\mathrm{T}}A)=R(A).$$

注: $\boxed{R(A^{\mathrm{T}}A)=R(A)=R(AA^{\mathrm{T}})}$

$2'$.设$A_{m\times p}$，$B_{p\times n}$，证明：$R(AB)\le \min\{R(A),R(B)\}$.

分析：$A_{m\times p}$，$B_{p\times n}\Rightarrow (AB)_{m\times n}$：

$$R(AB)\le R(B) \Leftrightarrow \underbrace{n-R(AB)}_{=\dim W_2}\ge \underbrace{n-R(B)}_{=\dim W_1}$$

令 $W_1=\{x\,|\,Bx=0, x\in R^n\}$,

$W_2=\{x\,|\,(AB)x=0, x\in R^n\}$

证：$\forall x\in W_1 \Rightarrow Bx=0 \Rightarrow ABx=A0=0 \Rightarrow x\in W_2$

$\Rightarrow W_1\subset W_2 \Rightarrow n-R(B)=\dim W_1\le \dim W_2=n-R(AB)$

$\Leftrightarrow$ (1) $R(AB)\le R(B)$

$\Rightarrow$ (2) $R(AB)=R[(AB)^T]=R(B^TA^T)\le R(A^T)=R(A)$

$\Rightarrow$ $R(AB)\le \min\{R(A),R(B)\}$.

2″.证明: $A^{\mathrm{T}}Ax = A^{\mathrm{T}}b$有解.

分析: $A^{\mathrm{T}}Ax = A^{\mathrm{T}}b$有解 $\Leftrightarrow R(A^{\mathrm{T}}A) = R(A^{\mathrm{T}}A, A^{\mathrm{T}}b)$

证: $A_{m\times n}$, $b \in R^m$

$$(A^{\mathrm{T}}A, A^{\mathrm{T}}b) = A^{\mathrm{T}}(A,b) \Longrightarrow (1)\quad R(A^{\mathrm{T}}A, A^{\mathrm{T}}b) = R[A^{\mathrm{T}}(A,b)]$$

$$\le R(A^{\mathrm{T}}) = R(A) = R(A^{\mathrm{T}}A)$$

$$(2)\quad R(A^{\mathrm{T}}A) \le R(A^{\mathrm{T}}A, A^{\mathrm{T}}b)$$

$$\Longrightarrow \quad R(A^{\mathrm{T}}A) = R(A^{\mathrm{T}}A, A^{\mathrm{T}}b)$$

$$\Longrightarrow \quad A^{\mathrm{T}}Ax = A^{\mathrm{T}}b \text{有解.}$$

九. 几何空间

1.设$\alpha\times\beta+\beta\times\gamma+\gamma\times\alpha=0$,**证明:**$\alpha,\beta,\gamma$共面.

证: $\alpha\bullet(\alpha\times\beta+\beta\times\gamma+\gamma\times\alpha)=\alpha\bullet 0=0$

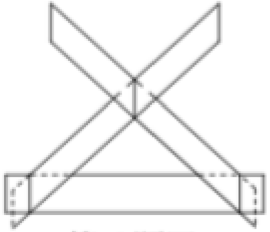
$$\alpha\bullet(\alpha\times\beta)+\alpha\bullet(\beta\times\gamma)+\alpha\bullet(\gamma\times\alpha)=\alpha\bullet 0=0$$

即: $[\alpha\beta\gamma]=\alpha\bullet(\beta\times\gamma)=0 \Rightarrow \alpha,\beta,\gamma$共面

2.有3个平面两两相交且交线相互平行,它们的方程为:

$$a_{i1}x+a_{i2}y+a_{i3}z=d(i=1,2,3)$$

组成的线性方程组的系数矩阵和增广矩阵分别为$A,\bar{A}$,则



(A) $R(A)=2,R(\bar{A})=3$; (B) $R(A)=2,R(\bar{A})=2$;

(C) $R(A)=1,R(\bar{A})=2$; (D) $R(A)=1,R(\bar{A})=1$.

注: 令$A=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, x=(x,y,z)^{\mathrm{T}}, b=(d_1,d_2,d_3)^{\mathrm{T}}$.

(1). 3个平面无公共交点$\Leftrightarrow Ax=b$无解$\Leftrightarrow R(A)<R(\bar{A})\leq 3$

(2). 记$n_i=(a_{i1},a_{i2},a_{i3})(i=1,2,3)\Rightarrow A=\begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix}$

$$\pi_i \cap \pi_j (j \neq i) \Leftrightarrow n_i \not\parallel n_j \Leftrightarrow n_i, n_j \text{线性无关} \Rightarrow R(A) \geq 2$$

又 $R(A) < R(\bar{A}) \leq 3$

$$\Rightarrow 2 \leq R(A) < R(\bar{A}) \leq 3 \Leftrightarrow 2 = R(A) < R(\bar{A}) = 3$$

3.已知直线$L_1:\dfrac{x-a_2}{a_1}=\dfrac{y-b_2}{b_1}=\dfrac{z-c_2}{c_1}$与$L_2:\dfrac{x-a_3}{a_2}=\dfrac{y-b_3}{b_2}=\dfrac{z-c_3}{c_2}$相交于一点,记向量$\alpha_i=(a_i,b_i,c_i),i=1,2,3$,则

(A) α_1可由α_2,α_3线性表出; (B) α_2可由α_1,α_3线性表出;

(C) α_3可由α_1,α_2线性表出; (D) $\alpha_1,\alpha_2,\alpha_3$线性无关.

注: $\left.\begin{array}{r}\alpha_i=(a_i,b_i,c_i)\text{为}L_i\text{的方向向量}(i=1,2)\\ L_1\text{与}L_2\text{相交}\end{array}\right\}\Rightarrow\alpha_1,\alpha_2$线性无关

$$\left.\begin{array}{c}P_1(a_2,b_2,c_2)\in L_1,P_2(a_3,b_3,c_3)\in L_2\\ L_1\text{与}L_2\text{相交}\Rightarrow L_1,L_2\text{共面}\end{array}\right\}\Rightarrow\alpha_1,\alpha_2,\overrightarrow{P_1P_2}\text{共面}$$

$$\Leftrightarrow[\alpha_1,\alpha_2,\overrightarrow{P_1P_2}]=0$$

$$\Leftrightarrow [\alpha_1,\alpha_2,\overrightarrow{P_1P_2}] = \begin{vmatrix} a_1 & a_2 & a_3-a_2 \\ b_1 & b_2 & b_3-b_2 \\ c_1 & c_2 & c_3-c_2 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0$$

$$\left.\begin{array}{l} \Leftrightarrow \alpha_1,\alpha_2,\alpha_3\text{线性相关} \\ \quad\text{又}\ \alpha_1,\alpha_2\text{线性无关} \end{array}\right\} \Rightarrow \alpha_3\text{可由}\alpha_1,\alpha_2\text{线性表出.}$$