N.A. Fall 2021 Homework 2 Solutions

The following are partial solutions for homework 2, which should give you an idea of whether or not you were on the right track. Code listings are not included, but you should feel free to ask your TA or professor about them.

2a) In the bisection method, the width of the interval halves each time. So, if we have an interval of length $a \approx 10^{-6}$, then after N more steps, we will have an interval of length approximately $10^{-6}/2^N$. Setting $N = \text{ceil}(6 \log_2(10)) = 20$, where $\text{ceil}(x)$ is the function returning the smallest integer that is at least x, makes this less than 10^{-12}.

b) It should take Newton one iteration to go from an error of 10^{-8} to an error of 10^{-16}. This is a manifestation of the quadratic convergence of Newton's method.

First, we note that the root of this function is at 0. To see this, we note that f is smooth and $f'(0)$ is not zero, Newton's method is quadratically convergent, i.e. $|x_{k+1}| \approx C|x_k|^2$, for $|x_k|$ small enough. A linear approximation of f around it's root then tell us that

$$|f(x_{k+1})| \approx |f'(0)||x_{k+1}| \approx \frac{C}{|f'(0)|}(|f'(0)||x_k|)^2 \approx \frac{C}{|f'(0)|}|f(x_k)|^2. \tag{1}$$

Since the constant in the last expression will be relatively close to one, we can expect $|f(x_{k+1})|$ to be about $10^{-8 \cdot 2} = 10^{-16}$.

c) The secant method has an order of convergence of $q = \frac{1+\sqrt{5}}{2} \approx 1.6$. As in part b), the number of correct digits will be multiplied by q after each iteration, so we will need 2 iterations to get an error less than 10^{-16} after we have an error of 10^{-8}.

3) Since $f(x) = x^{-1} - R$, $f'(x) = -x^{-2}$. Thus the iterations are

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} = x_k + x_k^2(x_k^{-1} - R) = 2x_k - x_k^2 R. \tag{2}$$

In the case where $R = 3$ and $x_0 = 0.5$, the first few iterations are 0.5, 0.25, 0.3125, 0.3320, 0.3333, ... If we had chosen $x_0 = 1$, farther from the root, then Newton's method will over shoot the answer, and then escape to infinity.

For Newton's method to converge, it is sufficient for the function, g, mapping one iteration to the next to be a contraction mapping on an interval I, i.e. $g(I) \subset I$ and $\max_{x \in I} |g'(x)| < 1$. We know that $g(x) = 2x - x^2 R$ and $g'(x) = 2 - 2xR$. The absolute value of the derivative of g is thus less than 1 exactly when $x \in I = [\frac{1}{2R} - \epsilon, \frac{3}{2R} + \epsilon]$, for some small $\epsilon > 0$. It is also easy to see that $g(I) \subset (\frac{3}{4R}, \frac{1}{R}) \subset I$, so this interval is a suitable choice.

4) This problem is solved by applying Newton's method to $f(x) = x^2 - 2$. If you start with $x_0 = 1$, you achieve the desired accuracy in 4 iterations.

6a) We can use $f(x) = h'(x) = x^3 - 3$. The Newton iterations with this function are

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} = x_k - \frac{x_k^3 - 3}{3x_k^2}. \tag{3}$$

b) If $x_0 = 1$, then $x_1 = 1 - \frac{1-3}{3} = \frac{5}{3}$.

c) The bisection method will choose the intervals $[0, 2]$ then $[1, 2]$.

Section B: 9) The bisection method will fail on this function, since it is positive on either side of its roots. Newton's method can be used to find the roots. To see this, note that if the iteration starts in the concave up region around any root, it will converge, and that if it starts outside such a region, it will eventually end up in one. The convergence will be linear though, since the derivative is zero at the root.

Section A: 14) See Homework 3 solutions.

Section A: 16) See Homework 3 solutions.