Midterm 2 **Numerical Analysis · Fall 2021 · Boffi**

Name: ______________________________

Rules (READ THESE FIRST):

1. Calculators **are** allowed.
2. Other resources (human or otherwise) **are not** allowed.
3. If you use lemmas, theorems, or corollaries from our textbook **cite them clearly**. Do **not** quote from any other source or from examples or exercises in the textbook.
4. Clearly mark your final answers for full credit.
5. Work must be shown for full credit (unless otherwise indicated).

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4					

Total	

1. Consider a matrix $A \in \mathbb{R}^{n\times n}$.

   (a) Write down the definition of the induced matrix norm $\|A\|$ for a norm $\|\cdot\| : \mathbb{R}^n \to \mathbb{R}$ defined on vectors.

$$\max_{\|v\|=1} \|Av\|$$

   (b) Write down the expressions for $\|A\|_1$, $\|A\|_2$, and $\|A\|_\infty$.

$$\begin{aligned}
\|A\|_\infty &= \max_{i=1,\cdots,n} \sum_{j=1}^{n} |a_{i,j}|, \\
\|A\|_1 &= \max_{j=1,\cdots,n} \sum_{i=1}^{n} |a_{i,j}|, \\
\|A\|_2 &= \sqrt{\lambda_{\max}(A^\top A)}.
\end{aligned}$$

   (c) Write down the definition of the (relative) matrix condition number $\kappa(A)$ given a norm $\|\cdot\| : \mathbb{R}^{n\times n} \to \mathbb{R}$ defined on matrices.

$$\kappa(A) = \|A\| \left\|A^{-1}\right\|$$

2. Consider solving the linear system $Ax = b$ for the solution vector $x \in \mathbb{R}^n$ given a matrix $A \in \mathbb{R}^{n\times n}$ and an input vector $b \in \mathbb{R}^n$.

   (a) After running Gaussian elimination on the matrix A, the matrix may be factorized in a certain way. Write down this factorization. How can it be used to solve the system $Ax = b$?

   After running Gaussian elimination, we obtain the LU factorization

   $$A = LU$$

   where L is lower triangular and U is upper-triangular. Then the linear system may be written

   $$LUx = b.$$

   We may solve $Ly = b$ for $y = Ux$, and then solve $Ux = y$ for x. Because L and U are lower- and upper-triangular, respectively, these systems can easily be solved using forward substitution and backward subsitution.

   (b) Consider solving m large linear systems $Ax_i = b_i$ for $i = 1, \cdots, m$ with a fixed matrix $A \in \mathbb{R}^{n\times n}$. How would you leverage the fact that the matrix A is shared across each system to make this efficient? Justify your answer.

   We can run Gaussian elimination once to compute a single LU factorization $A = LU$. The complexity of computing this factorization is $O(n^3)$. Solving the resulting linear systems $LUx_i = b_i$ has cost $O(n^2)$ each, so that the bulk of the expense comes from computing the original LU decomposition.

   (c) What is the overall scaling of the operation count with n and m? How many systems m would you have to solve before the cost of solving all systems together starts to significantly contribute to the overall cost?

   The cost of the overall procedure is $O(n^3 + mn^2)$. If m is on the order of n, then the cost of solving all the systems becomes comparable to the original LU factorization, and starts to contribute to the overall computational expense. If m is constant, then the n^2 term is negligible in comparison to the n^3 term.

3. Consider estimating the derivative $f'(x)$ of a function $f : \mathbb{R} \to \mathbb{R}$ at a point $x \in \mathbb{R}$. We've now seen two-point forward, backward, and centered difference schemes for computing the first derivative. Here we will analyze a three-point backwards difference scheme.

   (a) Write the Taylor expansions of $f(x-h)$ and $f(x-2h)$ around $f(x)$ for a small step size $h > 0$. Keep terms up to $O(h^4)$. You do not need to keep track of an exact remainder.

   We may write

$$
\begin{aligned}
f(x-h) &= f(x) - hf'(x) + \frac{h^2}{2} f''(x) - \frac{h^3}{6} f'''(x) + \frac{h^4}{24} f^{(4)}(x) + O(h^5), \\
f(x-2h) &= f(x) - 2hf'(x) + 2h^2 f''(x) - \frac{4}{3} h^3 f'''(x) + \frac{2}{3} h^4 f^{(4)}(x) + O(h^5).
\end{aligned}
$$

   (b) Find a combination of your expansions in part (a) to cancel terms at order $O(h^2)$.

   The combination $4f(x-h) - f(x-2h)$ gives

$$
4f(x-h) - f(x-2h) = 3f(x) - 2hf'(x) + \frac{2}{3} h^3 f'''(x) - \frac{1}{2} h^4 f^{(4)}(x) + O(h^5).
$$

(c) Re-arrange your answer from (c) to find a three-point stencil for $f'(x)$. What is the order of accuracy?

$$f'(x) = \frac{3f(x) - 4f(x-h) + f(x-2h)}{2h} + \frac{1}{3}h^2 f'''(x) - \frac{1}{4}h^3 f^{(4)}(x) + O(h^4).$$

The stencil is second-order accurate, as the leading error term is $O(h^2)$.

(d) Perform one step of Richardson extrapolation on your scheme. What is the order of accuracy of the new scheme?

Define the scheme with step size h as

$$\phi_0(h) = \frac{3f(x) - 4f(x-h) + f(x-2h)}{2h}.$$

Then we may write that

$$f'(x) = \phi_0(h) + \frac{1}{3}h^2 f'''(x) - \frac{1}{4}h^3 f^{(4)}(x) + O(h^4),$$
$$f'(x) = \phi_0(h/2) + \frac{1}{12}h^2 f'''(x) - \frac{1}{32}h^3 f^{(4)}(x) + O(h^4).$$

Taking a new scheme as

$$\phi_1(h) = \frac{4\phi_0(h/2) - \phi_0(h)}{3}$$

cancels the terms of order h^2. The terms of order h^3 remain, so that the new scheme is third-order accurate.

4. Consider estimating the integral $\int_a^b f(x)dx$ for a function $f : \mathbb{R} \to \mathbb{R}$ via a Newton-Cotes quadrature rule on a set of equally spaced nodes $\{x_i\}_{i=0}^n$.

   (a) Write down the Lagrange interpolant at the points x_i, and specify the Lagrange basis functions.

$$\phi_i(x) = \prod_{j \neq i} \frac{x - x_j}{x_i - x_j},$$
$$p(x) = \sum_{i=0}^{n} f(x_i)\phi_i(x).$$

   (b) Write down integral expressions for the quadrature weights A_i in a Newton-Cotes rule at the nodes x_i.

$$A_i = \int_a^b \phi_i(x)dx.$$

   (c) Up to what order of polynomial will the quadrature rule integrate exactly?

   n-th degree polynomials.

   (d) Via the method of undetermined coefficients, write down a set of relations that the quadrature weights A_i must satisfy that can be solved to obtain the A_i, rather than computing the integrals from part (b).

   For $k = 0, \cdots, n$, we have that

$$\int_a^b x^k dx = \frac{b^{k+1} - a^{k+1}}{k+1} = \sum_{i=0}^{n} A_i x_i^{k+1}.$$