Berry phase in the quantum Hamilton-Jacobi theory

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Abstract

In this paper, we propose a Berry geometric phase in the quantum Hamilton-Jacobi theory in the complex space and the real space. In the complex space, according to the quantum index theorem, the winding number which we can get by Cauchy's argument principle is the index of wave function. On the other hand, in canonical quantum mechanics, the single valuedness of the wave function implies that the Berry phase going back to the initial point along a path must be equal to an integer multiple of 2π. The results of both are very similar, so we will make a further connection. Because the quantum Hamilton-Jacobi theory contains the path, the Berry phase should be included in the formula of the quantum Hamilton-Jacobi theory. We theoretically derive the Berry phase in the quantum Hamilton-Jacobi theory. Then, we give some examples such that single electron fixed magnetic field motion and Aharonov-Bohm effect, etc. Based on theory and simulation, we conclude that, in canonical quantum mechanics, the wave function derived from Schrodinger's equation is added to the Berry phase correction, which is equivalent to the wave function solution of quantum Hamilton-Jacobi theory, and no further correction is required.

Keywords: quantum, quantum Hamilton-Jacobi theory, quantum index theory, quantum phase, Berry phase, Cauchy argument principle

1. Introduction

Bohm first proposed Bohmian mechanics to describe quantum mechanics in 1952. [1] A wave function described with an amplitude and an action $\psi = \psi_0 \exp iS/\hbar$. Substituting this wave function to Schrodinger equation. [2] This equation is called the quantum Hamilton-Jacobi equation.

$$-\frac{\partial S}{\partial t} = \frac{1}{2m}(\nabla S)^2 + \frac{\hbar}{2mi}\nabla^2 S + V \tag{1}$$

Where V represents potential. In recent, Chia-Chun Chou and Robert E. Wyatt proposed an article that tries to describe wave function in complex space with the quantum Hamilton-Jacobi equation.[3] C. D. Yang et al. also use the quantum Hamilton-Jacobi theory for describing wave functions with complex variables.[4][5][6][7] In 2007, the quantum

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Hamilton-Jacobi theory was also verified with another point of view in quantum mechanics, path integration.[8] It also obtained corresponding results in many examples of quantum mechanics.[9][10][11][12][13][14][15]. An important difference in the theory of the quantum Hamilton-Jacobi theory is that the quantum Hamilton-Jacobi theory uses an action to describe wavefunction. It directly gives an information of path. Because this property, we think we can get an additional information of Berry's phase directly rather than canonical quantum mechanics for a wave function of the same Schrodinger equation. If the path returns to the initial point, the wave function is a single valuedness[16], but the quantum Hamilton-Jacobi theory has different wave functions. In 2009, C.D. Yang and T. Y. Chang proposed the quantum index theorem[17]. They use Cauchy's argument principle in a closed path to obtain the winding number. The winding number can represent the Berry phase. In canonical quantum mechanics, it gets Berry phase, an integer multiple of 2π, when the quantum system comes back to the initial point along a path. This result is similar to the winding number of $2\pi i$ given by Cauchy's argument principle.
Therefore, we try to find out whether the difference obtained when the wave function in the quantum Hamilton-Jacobi theory returns to the initial point along a path can also interpret this Berry geometric phase. Before considering the Berry phase in the quantum Hamilton-Jacobi theory, let us give a brief introduction to the development of the Berry phase.

In 1984, Berry first proposed the concept of Berry geometric phase, which can be obtained when a quantum system performs adiabatic evolution in an arbitrary closed path.[18] Simon gives an interpretation of Berry's geometric phase due to holonomy in a line bundle over the parameter space in 1983.[19] Then, Anandan and Stodolsky also show that the Berry phase can be obtained from the holonomy in a vector bundle.[20] Wilczek et al. began a series of generalization to Berry geometric phase. Wilczek and Zee generalize a degenerate Hamiltonian of the Berry phase. [21] Segert and Mead also made similar points. [22] [23] Aharonov and Anandan proposed a generalization which removes the restriction to adiabatic evolution in Hilbert space. [24] D. Page extended Berry phase in terms of natural geometrical structures on the space of rays. [25] In 1987, Berry developed a third generalization of Berry phase about that non-degenerate adiabatic scenario in which the system returns to its original state not exactly but in a close approximation.[26] The generalization provided by Aharonov and Anandan is particularly important for this study. In the complex mechanical structure, it is not necessary to satisfy the adiabatic closed path. Based on this, Aharonov and Anandan generalize to the closed path to calculate the quantum geometric phase. In quantum Hamilton-Jacobi theory, it is not necessary to emphasize that the Hamiltonian that the quantum system follows is an adiabatic approximation.

This article aims to demonstrate that the quantum phase has a more direct interpretation in the quantum Hamilton-Jacobi theory. We divide this article into the following sections: First, in section 2, we quickly demonstrate how to obtain the Berry phase in canonical quantum mechanics and write it as an analogical form, and then get the Berry phase in the quantum Hamilton-Jacobi theory by the viewpoint that the variable is complex in the quantum Hamilton-Jacobi theory. Lastly, compare the different Berry phase of both, which including the gauge transformation. Based on the conclusion of section 2, we show a two-dimensional example in section 3. When the motion trajectory returns to the starting point in the quantum Hamilton-Jacobi theory, the Berry phase satisfies that it is an integer multiple of 2π. Besides, the example of the Aharonov-Bohm effect can give Dirac quantization condition. From the above sections, we can conclude that in the quantum Hamilton-Jacobi theory, the geometric phase has a more direct interpretation, and also gives the quantum Hamilton-Jacobi theory a more complete physical image.

2. The relationship between Berry's phase in canonical quantum mechanics and the quantum Hamilton-Jacobi theory

In this section, we theoretically derive the relationship between the Berry geometric phase in the quantum Hamilton-Jacobi theory and the Berry geometric phase in canonical quantum mechanics. As shown in 1, the wave function solved by the quantum Hamilton-Jacobi theory contains the wave function solution of the Schrodinger equation in canonical quantum mechanics and adding the Berry geometric phase. The wave function itself has path information in the quantum Hamilton-Jacobi theory. However, in canonical quantum mechanics, we need to add the Berry geometric phase so that the wave function evolving with the adiabatic path. Thus, we can completely describe the wave function by adding the concept of the quantum Hamilton-Jacobi theory, the variables of the wave function are complex, to quantum mechanics.

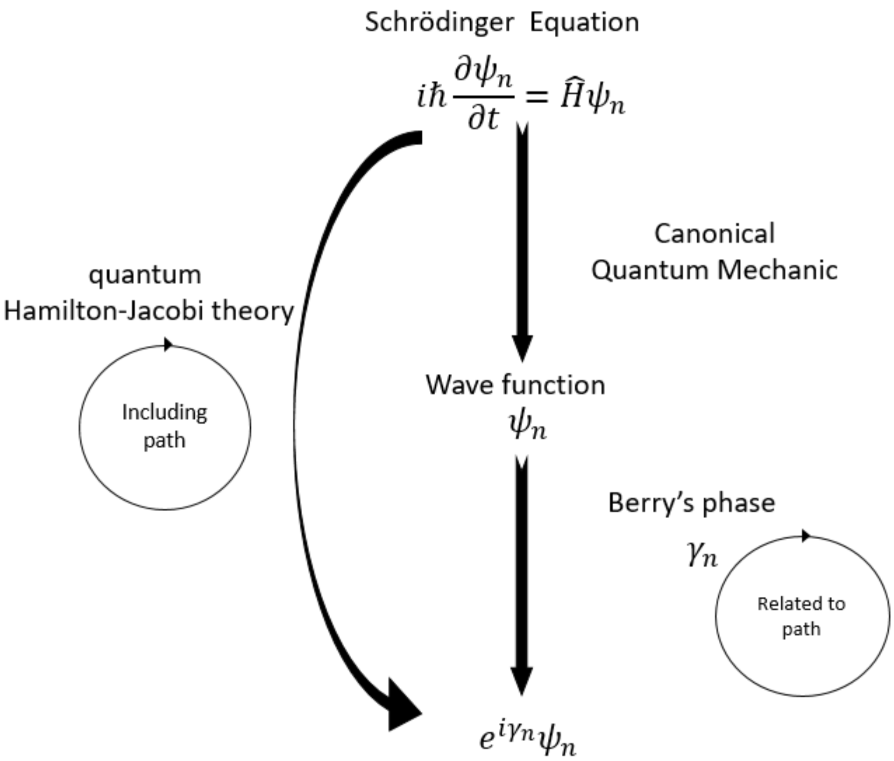

Figure 1: An overview for Sec. 2

First, let us review the derivation of Berry's phase in canonical quantum mechanics. We set a Hamiltonian $\hat{H}$ changed with a path $\mathbf{R}(t)$. This path $\mathbf{R}(t)$ depends on the coordinates we selected. For instance, if we choose the Cartesian coordinates, the path $\mathbf{R}(t)$ depends on (x, y, z). Besides, we assume $\mathbf{R}(t = T) = \mathbf{R}(t = 0)$ so that the path is a closed path. Lastly, we assume its eigenstates and energy eigenvalues are $|n(\mathbf{R}(t))\rangle$ and E_n, they satisfy

$$\hat{H}(\mathbf{R}(t))|n(\mathbf{R}(t))\rangle = E_n(\mathbf{R}(t))|n(\mathbf{R}(t))\rangle \tag{2}$$

From the paper which Berry proposed in 1984, [18] we know that a system evolves with Hamiltonian $\hat{H}$ from the state $|n(\mathbf{R}(0))\rangle$. At time t, the state $|\psi(t)\rangle$ is

$$|\psi(t)\rangle = \exp\left\{-\frac{i}{\hbar}\int_0^t dt' E_n(t')\right\}\exp(i\gamma_{q,n}(t))|n(\mathbf{R}(t))\rangle \tag{3}$$

Where $\gamma_{q,n}(t)$ means Berry geometric phase in canonical quantum mechanics. Besides, define a dynamic phase $\theta_n := -i/\hbar \int_0^t E_n(t')dt'$.[27]
We usually use the form of eigenfunction to show that the eigenfunction has the properties of complex variables in the quantum Hamilton-Jacobi theory, so we use eigenfunction $\psi_n(\mathbf{R}(t))$ of E_n instead of eigenstate $|n(\mathbf{R}(t))\rangle$ in eq.(3).

$$\psi(t) = \exp\left\{-\frac{i}{\hbar}\int_0^t dt' E_n(t')\right\}\exp(i\gamma_{q,n}(t))\psi_n(\mathbf{R}(t)) \tag{4}$$

Substitute eq.(4) into Schrodinger's equation $(i\hbar\frac{\partial}{\partial t} - \hat{H})|\psi(t)\rangle = 0$, we can get

$$\frac{\partial \exp(i\gamma_{q,n}(t))}{\partial t}\psi_n(\mathbf{R}(t)) + \exp(i\gamma_{q,n}(t))\frac{d}{dt}\psi_n(\mathbf{R}(t)) = 0 \tag{5}$$

Therefore, the Berry geometric phase $\gamma_{q,n}(t)$ in canonical quantum mechanics is

$$\gamma_{q,n}(t) = i\int_0^t \frac{\dot{\psi}_n(\mathbf{R}(t'))}{\psi_n(\mathbf{R}(t'))}dt' \tag{6}$$

Assume that the path $\mathbf{R}$ in N-dimensional coordinates can be separated as $R_1(t)$, $R_2(t)$, $R_3(t)$, ..., $R_N(t)$ and using the chain rule to get:

$$\frac{\partial \psi_n}{\partial t} = \frac{\partial \psi_n}{\partial R_1}\frac{dR_1}{dt} + \frac{\partial \psi_n}{\partial R_2}\frac{dR_2}{dt} + ... + \frac{\partial \psi_n}{\partial R_N}\frac{dR_N}{dt} \tag{7}$$

Suppose the $\nabla_{\mathbf{R}}$ to be the gradient with respect to these parameters, and if the Hamiltonian returns to its original form after a time T

$$\gamma_n(T) = i\oint \frac{\nabla_{\mathbf{R}}\psi_n}{\psi}d\mathbf{R} \tag{8}$$

In the following context, we use the concept of the quantum Hamilton-Jacobi theory, the variables of the wave function have the properties of complex variables, and the Cauchy

argument theorem to calculate the Berry geometric phase in the quantum Hamilton-Jacobi theory. In the quantum Hamilton-Jacobi theory, we know one of the biggest differences is that we have the concept of the path which a quantum system evolves according to. Therefore, in canonical quantum mechanics, the wave function derived from the Schrodinger equation with the phase affected by the path, that is Berry Phase. It returns a complete wave function containing Berry geometric phase. In the quantum Hamilton-Jacobi theory, we can get it directly from the solution of the wave equation, because our wave function itself contains path information.
We assume that the eigenfunction evolves along the path and returns to the origin.

$$\psi_n(\mathbf{R}(T),T) = e^{i\gamma_{c,n}} e^{i\theta_n} \psi_n(\mathbf{R}(0),T) \tag{9}$$

Expand it into real and imaginary parts

$$\psi_n(\mathbf{R}(T),T) = |\psi_n(\mathbf{R}(0),0)| e^{i(\theta_{\psi_n(\mathbf{R}(0))} + \gamma_{c,n} + \theta_n)} \tag{10}$$

$$\psi_n(\mathbf{R}(T),T) = |\psi_n(\mathbf{R}(T),T)| e^{i\theta_{\psi_n(\mathbf{R}(T))}} \tag{11}$$

Where $\theta_{\psi_n(\cdot)} = \arctan(\frac{\Im(\psi_n(\cdot))}{\Re(\psi_n(\cdot))})$
Because eq.(10) (11) must be exactly the same, comparing the two forms to get

$$\tan(\theta_{\psi_n(\mathbf{R}(0),0)} + \gamma_{c,n} + \theta_n) = \tan(\theta_{\psi_n(\mathbf{R}(T),T)}) \tag{12}$$

$$\theta_{\psi_n(\mathbf{R}(T),T)} = \theta_{\psi_n(\mathbf{R}(0),0)} + \gamma_{c,n} + 2n\pi + \theta_n \tag{13}$$

$$|\psi_n(\mathbf{R}(T),T)| = |\psi_n(\mathbf{R}(0),0)| \tag{14}$$

We use the Cauchy argument principle and assume the wave function ψ can be Separation of variables with $(R_1(t), R_2(t), ..., R_N(t))$, then we get the eq.(15)

$$\begin{aligned}
\oint \frac{\nabla_{\mathbf{R}}\psi_n(\mathbf{R}(t),t)}{\psi_n(\mathbf{R}(t),t)} d\mathbf{R} &= \oint \frac{\partial\psi_n(R_1(t),t)/\partial R_1(t)}{\psi_n(R_1(t),t)} dR_1 + \oint \frac{\partial\psi_n(R_2(t),t)/\partial R_2(t)}{\psi_n(R_2(t),t)} dR_2 \\
&+ ... + \oint \frac{\partial\psi_n(R_N(t),t)/\partial R_N(t)}{\psi_n(R_N(t),t)} dR_N \\
&= \oint d\ln|\psi_n(R_1(t),t)| + \oint d\ln|\psi_n(R_2(t),t)| \\
&+ ... + \oint d\ln|\psi_n(R_N(t),t)| \\
&+ i\oint d\theta_{\psi_n(R_1(t),t)} + i\oint d\theta_{\psi_n(R_2(t),t)} + ... + i\oint d\theta_{\psi_n(R_N(t),t)} \\
&= i\oint d\theta_{\psi_n(R_1(t),t)} + i\oint d\theta_{\psi_n(R_2(t),t)} + ... + i\oint d\theta_{\psi_n(R_N(t),t)}
\end{aligned} \tag{15}$$

For the other part, partial differential time of the wavefunction, $\partial\psi_n(\mathbf{R}(t),t)/\partial t$. The basic assumption of the quantum Hamilton-Jacobi theory, the wavefunction ψ can be expressed with action. Besides, from the Hamilton-Jacobi equation $\frac{\partial S}{\partial t} = -H$. Substitute it to $\partial\psi_n(\mathbf{R}(t),t)/\partial t$. We can get that dynamic phase equal

$$\theta_n = -i\int_0^T \frac{\partial\psi_n(\mathbf{R}(t),t)/\partial t}{\psi_n(\mathbf{R}(t),t)} \tag{16}$$

From eq.(13)(15), we can get the Berry phase in quantum Hamiltonian Jacobi theory

$$\begin{aligned}\gamma_{c,n} + 2n\pi + \theta_n &= \theta_{\psi_n(\mathbf{R}(T),T)} - \theta_{\psi_n(\mathbf{R}(0),0)} \\ &= -i\oint \frac{\nabla_{\mathbf{R}}\psi_n(\mathbf{R}(t))}{\psi_n(\mathbf{R}(t))}d\mathbf{R} - i\int_0^T \frac{\partial\psi_n(\mathbf{R}(t),t)/\partial t}{\psi_n(\mathbf{R}(t),t)} \\ &= -i\oint \frac{\nabla_{\mathbf{R}}\psi_n(\mathbf{R}(t))}{\psi_n(\mathbf{R}(t))}d\mathbf{R}\end{aligned} \tag{17}$$

Let's discuss the possible values of n. We know that when the argument is 0, the Berry phase is also 0. Because they are at the same original point and there should be no phase difference, n must be 0. In addition, we consider that the Berry geometric phase changes, continuously. First, we imagine that this state doesn't go through any path, that is, it stops at the origin. Its Berry phase is 0. For any state on the closed path, it must also satisfy that the Berry phase is 0 if the state stops at its point. That is to say, during the evolution of Berry phase on any closed path, it is continuously changing without any suddenly increase of $2n\pi$.Thus, the value of n must be 0.

$$\gamma_{c,n} = -i\oint \frac{\nabla_{\mathbf{R}}\psi_n(\mathbf{R}(t))}{\psi_n(\mathbf{R}(t))}d\mathbf{R} \tag{18}$$

Therefore, in the quantum Hamiltonian Jacobi theory, we can define the Berry geometric phase as eq.(18). Although this equation is similar to the Berry phase in canonical quantum mechanics, eq.(6). In fact, in the quantum Hamiltonian Jacobi theory, we assume that the trajectory is complex and calculate Berry phase following the complex trajectory, while traditional quantum mechanics follows the trajectory in real. Let's take a thought to illustrate their difference. Considering that $R_1(t)$ returns to the origin in real or complex, we can assume that the evolution of $R_1(t)$ is as shown in Fig. (2)

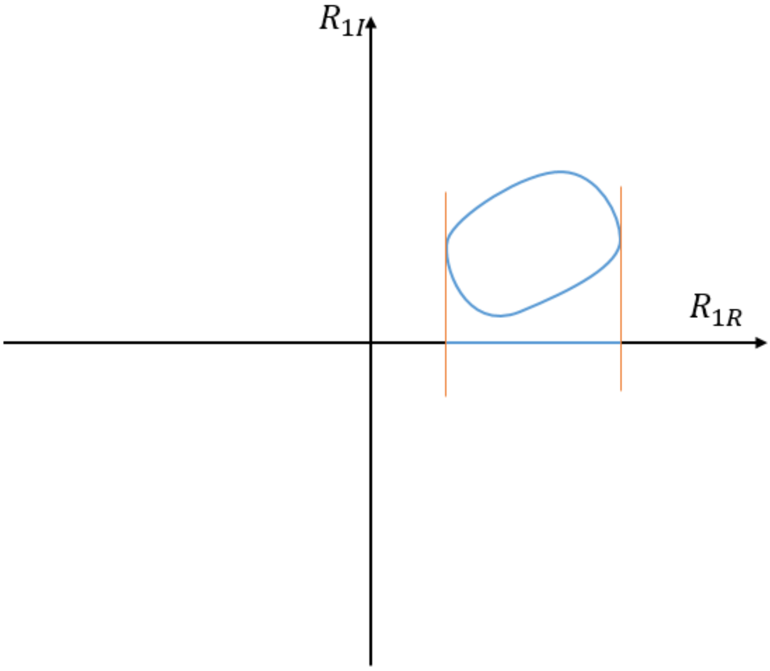

Figure 2: A path for $R_1(t)$ in complex space

Obviously, for one-dimensional parameters, the real path definitely only give 0. In contrast,in the complex path, if there are zeros in the ring path, it would give a non-zero Berry phase. Thus, we can know Berry phase in quantum Hamiltonian Jacobi theory is greater than or equal to Berry phase in canonical quantum mechanics. However, whether at the theoretical or experimental level, we just consider Berry Phase through a real path. Thus, if we want to verify the result. We need to redesigned to make the wave function have a complex path.

Next, given that quantum mechanics generally does not consider complex trajectories, we will derive a form just following a real path from eq.(9). The key to this derivation is that we do not use the Cauchy argument principle, but use the definition of quantum Hamiltonian Jacobi theory.

From the assumption in quantum Hamiltonian Jacobi theory, $\psi = \psi e^{iS/\hbar}$, Now, we consider t from 0 to T. From eq.(12)(13)(14), for a total changing phase, we can write $\gamma_{c,n} + \theta_n = \frac{1}{\hbar}(S_f - S_i)$. Where S_f means the final action and S_i means the original action. The assumption says that the action S is the phase of wavefunction. Besides, the eq. (14) show the amplitude is same between final and origin. Thus, we just calculate the changing of phase. Consider action S through the path $\mathbf{R}(t)$, we write that the changing of phase is $S_f - S_i = \int_i^f dS$. The i means that the wavefunction in the initial state, it is at $\mathbf{R}(0)$. The f means that the wavefunction in the final state, it is at $\mathbf{R}(T)$. The path $\mathbf{R}(t)$ changes with time t. We change the form to be

$$S_f - S_i = \int_0^T \frac{dS}{dt} dt \tag{19}$$

Besides, according to Hamilton Jacobi equation. The total phase is

$$\gamma_{c,n} + \theta_n = \frac{1}{\hbar}\left(\int_i^f \nabla_{\mathbf{R}} S dR + \int_0^T \frac{\partial S}{\partial t} dt\right) \tag{20}$$

We deal that the problem is the path is a closed path. Thus

$$\gamma_{tot} = \frac{1}{\hbar} \oint \nabla_{\mathbf{R}} S d\mathbf{R} + \int_0^T \frac{\partial S}{\partial t} dt) \tag{21}$$

The last term from Hamilton Jacobi equation, we can get the dynamic phase $\theta_n = \int_0^T \frac{\partial \psi/\partial t}{\psi} dt$, again. Then, since $\gamma_{n,c} = \gamma_{tot} - \theta_n$, we get Berry phase

$$\gamma_{n,c} = -i \oint \frac{\nabla_{\mathbf{R}} \psi}{\psi} d\mathbf{R} \tag{22}$$

We don't use the Cauchy argument principle. The path can be defined in complex or real. When the path is complex, the result is same as which we get from Cauchy argument principle. When the wavefunction follows the real path. You can find it has a relation with the Berry phase in canonical quantum mechanics. Besides, according to the single valueness, this result also give the basic principle of old quantum theory, $2\pi m\hbar = \oint p_i dR_i$. Where m is integer, p_i is momentum corresponding to the parameter R_i.
Finally, we find the relation, a negative sign for the difference, between Berry geometric phase in canonical quantum mechanics and Berry geometric phase in the quantum Hamilton-Jacobi theory. Therefore, the Berry geometric phase $\gamma_{q,n}$ in canonical quantum mechanics and Berry geometric phase $\gamma_{c,n}$ in the quantum Hamilton-Jacobi theory

$$\gamma_{q,n} = -\gamma_{c,n} \tag{23}$$

These results show that the quantum Hamilton-Jacobi theory can more directly interpret quantum mechanics and be a more complete interpretation of Schrodinger's equation without adding to Berry phase. In the complex path, because we use Cauchy argument principle, we can have convenient and quick mathematical tricks to tell us what the Berry phase should be. In fact, through Cauchy argument principle, we only need to find the poles and zeros of the evolution path which the quantum system according to and then we can find the Berry phase.
We set any wave function to expand into the following form

$$\psi(x) = \frac{(x-z_1)^{k_1}(x-z_2)^{k_2}...}{(x-p_1)^{m_1}(x-p_2)^{m_2}...} F(x) \tag{24}$$

Where $p_1, p_2, ..., z_1, z_2, ..., m_1, m_2, ..., k_1, k_2, ...$and $F(x)$ are the poles, zeros, powers of the poles, powers of the zeros, and functions without the poles and zeros.
Through eq.(24), we divide the first derivative of the wave function$d\psi/dx$ by the wave function ψ

$$\frac{d\psi/dx}{\psi} = (\frac{k_1}{x - z_1} + \frac{k_2}{x - z_2} + ...) - (\frac{m_1}{x - p_1} + \frac{m_2}{x - p_2} + ...) + \frac{F'}{F} \tag{25}$$

$$\oint \frac{\psi'(x)}{\psi(x)} dx = 2\pi i(k_1 + k_2 + ...) - 2\pi i(m_1 + m_2 + ...) \tag{26}$$

Let's set $Z_\psi = k_1 + k_2 + ...$and $P_\psi = m_1 + m_2 + ...$to be the total number of zeros and the total number of poles. Thus, we can directly find Berry phase by the total number of zeros Z_ψ and the total number of poles P_ψ. Section 3 will show how effective it is.

2.1. Corresponds in gauge transformation

In this subsection, we discuss whether the Berry phase with gauge transformation in the quantum Hamilton-Jacobi theory is the same as the Berry phase with gauge transformation in canonical quantum mechanics. The single quote in this section represents the new function selected after gauge transformation. The wave function $\psi_n(\mathbf{R}(t))$ undergoes gauge transformation by modifying an additional phase $\chi(t)$ to get a new wave function $\psi'_n(\mathbf{R}(t))$

$$\psi'_n(\mathbf{R}(t)) = e^{i\chi(t)}\psi_n(\mathbf{R}(t)) \tag{27}$$

Substitute eq.(27) to calculate the Cauchy argument principle, the new wave function gives the following relationship with the original wave function.

$$i\frac{\dot{\psi}'_n(\mathbf{R}(t))}{\psi'_n(\mathbf{R}(t))} = i\frac{\dot{\psi}_n(\mathbf{R}(t))}{\psi_n(\mathbf{R}(t))} - \frac{d\chi(t)}{dt} \tag{28}$$

Let's consider the evolution of a closed path, the quantum system goes back to origin at the time of T, so we can get the relationship between the Berry geometric with gauge transformation and the original Barry phase.

$$i\oint \frac{\dot{\psi}'_n(\mathbf{R}(t))}{\psi'_n(\mathbf{R}(t))} dt = i\oint \frac{\dot{\psi}_n(\mathbf{R}(t))}{\psi_n(\mathbf{R}(t))} dt - (\chi(T) - \chi(0)) \tag{29}$$

$$\gamma_{q,n} = \gamma'_{q,n} - (\chi(T) - \chi(0)) \tag{30}$$

In the quantum Hamilton-Jacobi theory, we can also get the same result. First, we also consider a quantum system evolves according to the closed path, but this wave function is a new wave function $\psi'_n(R(t))$ with gauge transformation evolves as:

$$e^{i\chi(T)}\psi_n(\mathbf{R}(T), T) = e^{i\gamma'_{c,n} i\theta_n} e^{i\chi(0)}\psi_n(\mathbf{R}(0), 0) \tag{31}$$

$$\psi_n(\mathbf{R}(T), T) = e^{i(\gamma'_{c,n} + \theta_n + \chi(0) - \chi(T))}\psi_n(\mathbf{R}(0), 0) \tag{32}$$

From eq.(13)(32)(17)(18)), we can get

$$\theta_{\psi_n(\mathbf{R}(T),T)} = \theta_{\psi_n(\mathbf{R}(0),0)} + \gamma'_{c,n} + \theta_n - (\chi(T) - \chi(0)) + 2n\pi \tag{33}$$

$$\gamma'_{c,n} - (\chi(T) - \chi(0)) = -i \oint \frac{\nabla_{\mathbf{R}}\psi_n(\mathbf{R}(t), t)}{\psi_n(\mathbf{R}(t), t)} d\mathbf{R} \tag{34}$$

$$\gamma'_{c,n} - (\chi(T) - \chi(0)) = \gamma_{c,n} \tag{35}$$

From the relationship eq.(23), you can find that the relationship eq.(30) in canonical quantum mechanics and the relationship eq.(35) in the quantum Hamilton-Jacobi theory are the same, so the property of that Berry geometric phase described in canonical quantum mechanics is gauge invariant and that the Berry phase under gauge transformation must be an integer multiple of 2π, which is described in the quantum Hamilton-Jacobi theory. Berry phases in the quantum Hamilton-Jacobi theory are perfectly equivalent with canonical quantum mechanics.

3. Example

3.1. Motion of a single electron in a constant magnetic field

We use the example proposed by C.D. Yang in 2006 to demonstrate how to use the poles and zeros to determine the Berry phase and how to track their trajectories. [12] First, we know that the wave function of a single electron moving in a constant magnetic field from his article is

$$\psi_{n_\rho,n_\theta}(\rho, \theta) = Ce^{in_\theta\theta}\rho^{|n_\theta|}e^{-\rho^2/2}F(-n_\rho, |n_\theta| + 1, \rho^2) \tag{36}$$

This wave function in his article is calculated in dimensionless form. Here, ρ represents the dimensionless radius of polar coordinate, θ represents the dimensionless angle of polar coordinate, C is a constant Coefficient, and the F function is a confluent hyper-geometric function.In addition, we split ψ_{n_ρ,n_θ} into ψ_ρ, ψ_θ with the way, separation of variables, for convenience of discussion. When $n_\rho = 0, n_\theta = 0$, we have

$$\psi_{n_\rho,n_\theta} = e^{-\rho^2/2} \tag{37}$$

According to the article, we have the time evolution equation of dimensionless radius and angle ρ, θ

$$\frac{d\rho}{d\tau} = \frac{1}{i}(-\rho + \frac{1}{2\rho}), \qquad \frac{d\theta}{d\tau} = 1 \tag{38}$$

From this time evolution equation, eq.(38), we can get the function of $\rho(\tau), \theta(\tau)$, and know the quantum trajectory in the quantum Hamilton-Jacobi theory As shown in fig.3(a)(b), it shows that ρ's path and θ' s path in complex coordinate, and then take their real part to find the appropriate trajectory. Here, we only care about the path which returns to the origin. We can see that Γ_1, Γ_2 in the fig.3 all return to the origin. In this case, $n_\rho = 0, n_\theta = 0$, $e^{in_\theta\theta}$

does not have any contribution. Thus, we just need to consider the poles and zeros of $e^{-\rho^2/2}$, we can find that the function does not have any poles and zeros, so we can use the eq.(26) to get that the Berry phase of the Γ_1, Γ_2 path is 0. No matter what the path is, it can also be seen from fig.3 (d) that the Berry phase is all 0 when it finally returns to the origin. Similarly, we can calculate the wave function and the time evolution equation of ρ, θ for $n_\rho = 1$. We can draw the corresponding quantum trajectory path and the variable ρ, θ's trajectory to determine the poles and zeros. The calculated result is shown in the table (1). From this method, we can find that when the path contains all the zeros, we can quickly get the zeros from ψ_ρ. Because

$$\oint \frac{\psi_\theta'}{\psi_\theta} d\theta = \oint \pm i\theta d\theta = \pm 2\pi i \tag{39}$$

Thus, the effect of the Berry phase given by the ψ_θ term is $\pm 2\pi$, and we can obtain the Berry phase with the zero point of ψ_ρ. This example shows the path in complex. We use the poles and zeros to calculate Berry phase, quickly. If we consider a real path projected from the path in complex. Obviously, the parameter ρ always give 0. The geometric phase decided from parameter θ

<table>
<tr><td>n_ρ</td><td colspan="2">0</td><td colspan="9">1</td></tr>
<tr><td>n_θ</td><td colspan="2">0</td><td colspan="3">-1</td><td colspan="3">0</td><td colspan="3">1</td></tr>
<tr><td>ψ_ρ</td><td colspan="2">$e^{-\rho^2/2}$</td><td colspan="3">$-\frac{1}{2}\rho(\rho^2-2)e^{-\rho^2/2}$</td><td colspan="3">$-\rho(\rho^2-1)e^{-\rho^2/2}$</td><td colspan="3">$-\frac{1}{2}\rho(\rho^2-2)e^{-\rho^2/2}$</td></tr>
<tr><td>ψ_θ</td><td colspan="2">1</td><td colspan="3">$e^{-i\theta}$</td><td colspan="3">1</td><td colspan="3">$e^{i\theta}$</td></tr>
<tr><td>$\frac{d\rho}{d\tau}$</td><td colspan="2">$\frac{1}{i}(-\rho+\frac{1}{2\rho})$</td><td colspan="3">$\frac{1}{i}(-\frac{\rho^4-5\rho^2+2}{\rho(\rho^2-2)}+\frac{1}{2\rho})$</td><td colspan="3">$\frac{1}{i}(-\frac{\rho(\rho^2-3)}{\rho^2-1}+\frac{1}{2\rho})$</td><td colspan="3">$\frac{1}{i}(-\frac{\rho^4-5\rho^2+2}{\rho(\rho^2-2)}+\frac{1}{2\rho})$</td></tr>
<tr><td>$\frac{d\theta}{d\tau}$</td><td colspan="2">1</td><td colspan="3">$e^{-i\theta}$</td><td colspan="3">1</td><td colspan="3">$e^{i\theta}$</td></tr>
<tr><td>c</td><td>Γ_1</td><td>Γ_2</td><td>Γ_1</td><td>Γ_2</td><td>Γ_3</td><td>Γ_1</td><td>Γ_2</td><td>Γ_3</td><td>Γ_1</td><td>Γ_2</td><td>Γ_3</td></tr>
<tr><td>γ</td><td>0</td><td>0</td><td></td><td>2</td><td>2</td><td></td><td>2</td><td>2</td><td></td><td>4</td><td>4</td></tr>
</table>

Table 1: Berry phase given by a single electron in a constant magnetic field and corresponding information, where c represents different paths

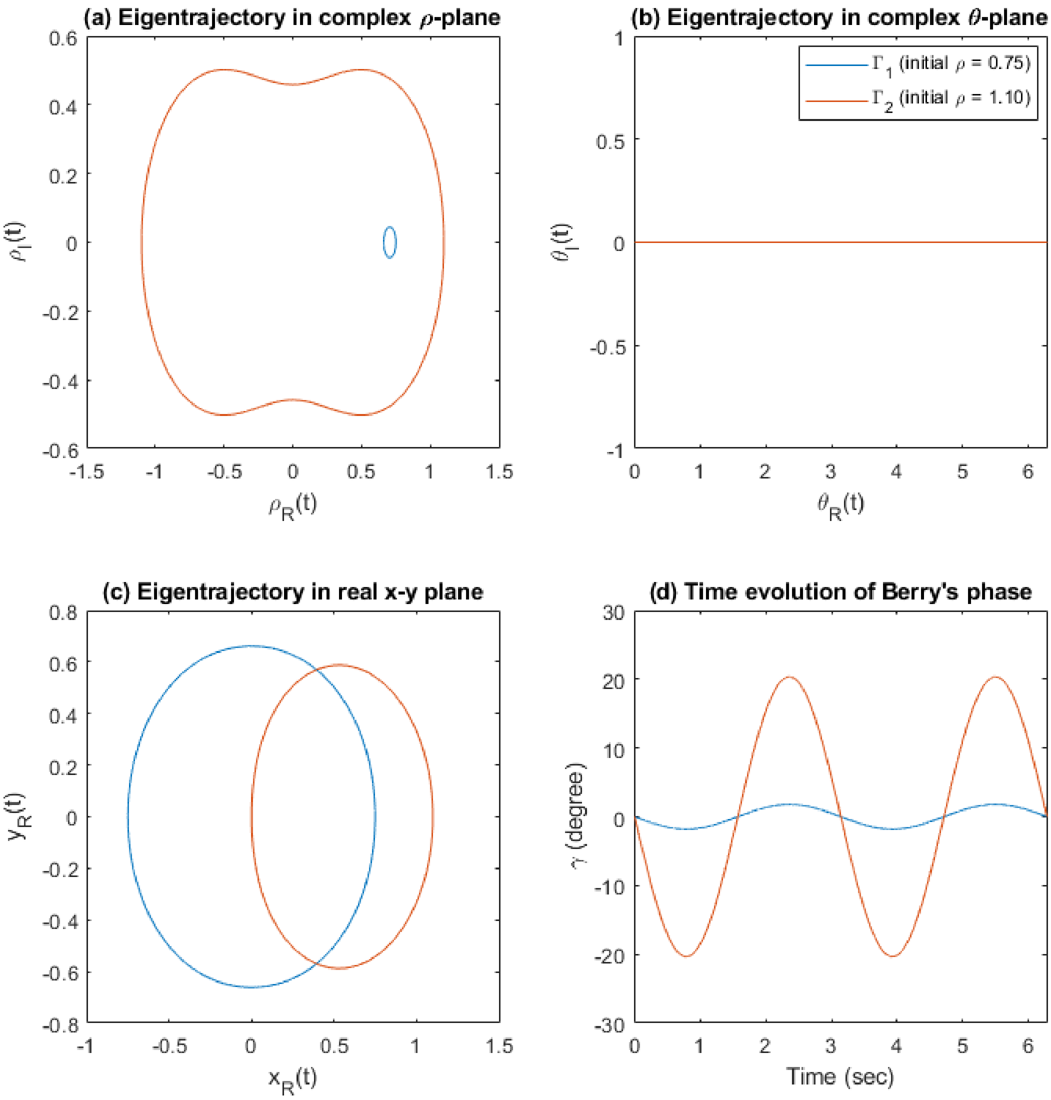

Figure 3: Energy state by $n_\rho = 0, n_\theta = 0$

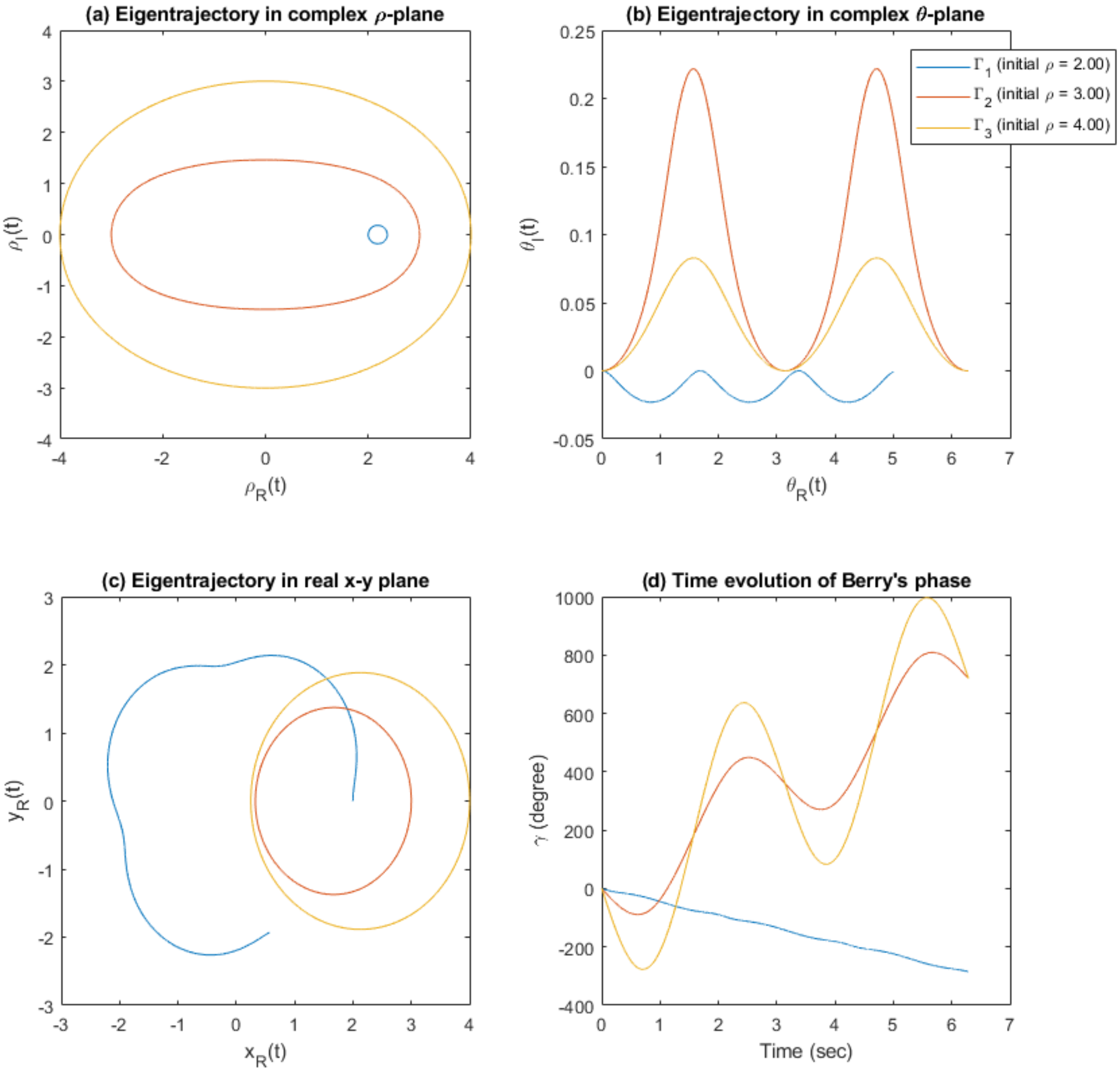

Figure 4: Energy state by $n_\rho = 1, n_\theta = -1$

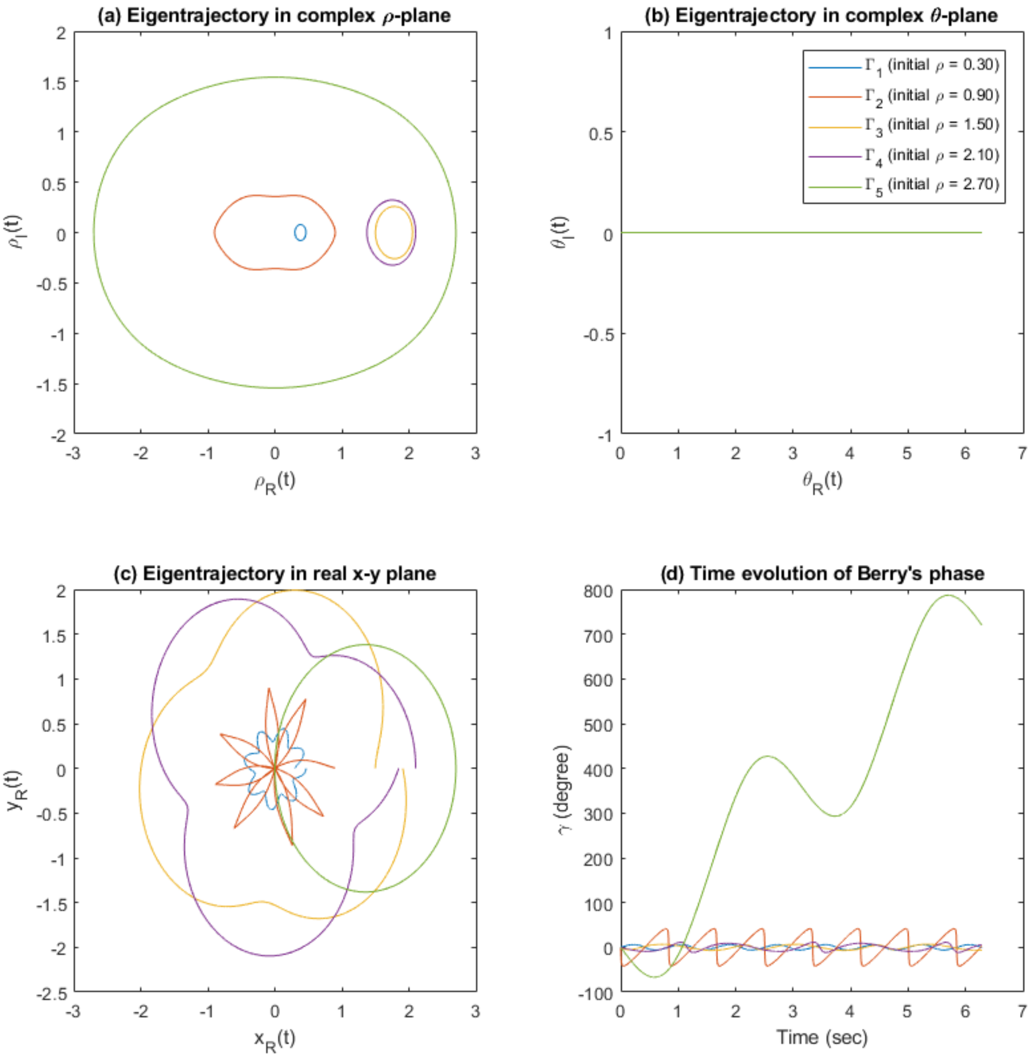

Figure 5: Energy state by $n_\rho = 1, n_\theta = 0$

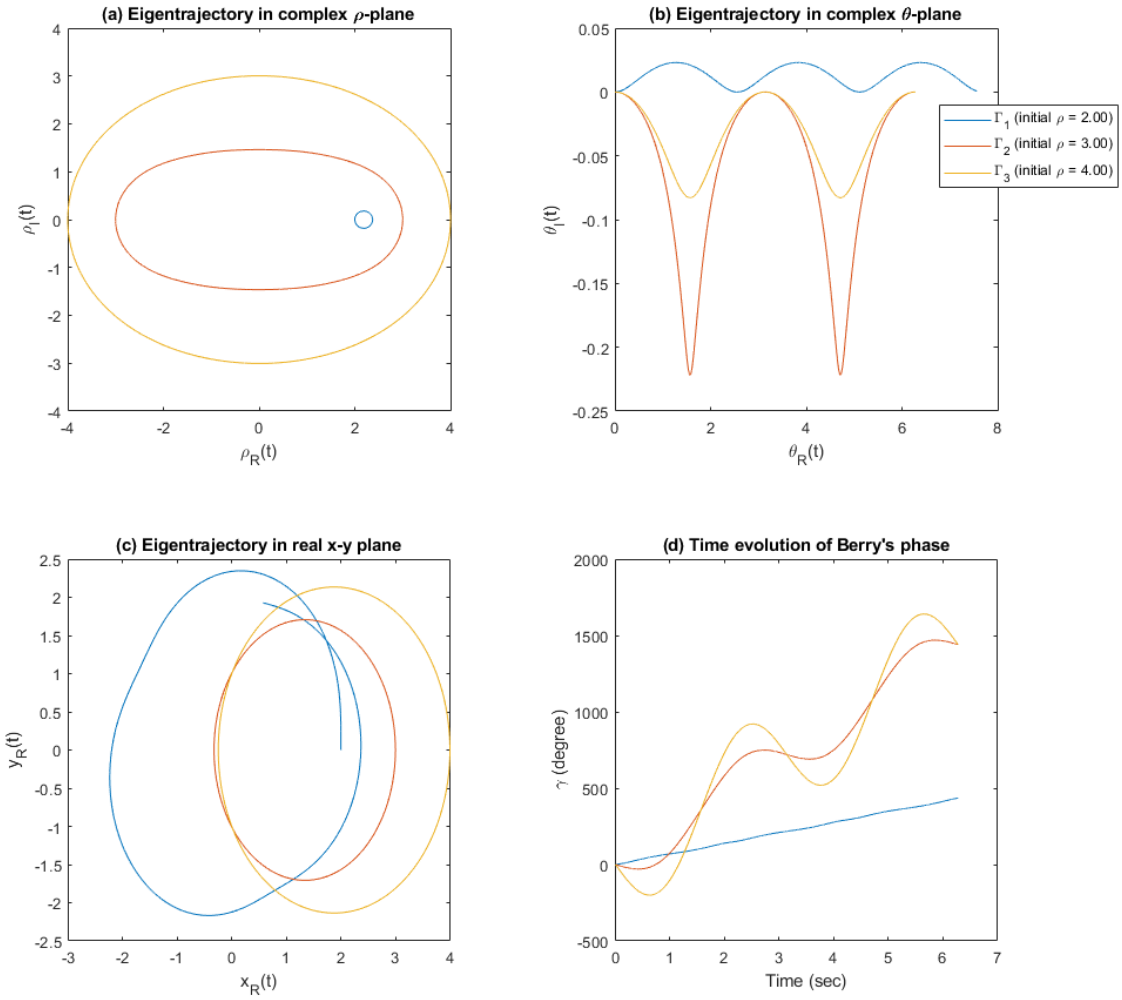

Figure 6: Energy state by $n_\rho = 1, n_\theta = 1$

3.2. Two-dimensional simple harmonic motion

In all examples of quantum mechanics, the simplest example is the simple harmonic motion. We use the concept of the the quantum Hamilton-Jacobi theory and Cauchy argument principle, eq.(18), to get Berry phase in a two-dimensional simple harmonic system. We can learn from the subsection(3.1) that the Berry phase can be obtained directly in the closed loop path. Therefore, we consider the number of zeros contained in the wave function to get Berry geometric phase. The general wave function for two-dimensional simple harmonic motion in dimensionless is [10]

$$\psi_{m,n}(x,y) = \frac{1}{\sqrt{2^{n+m}m!n!\pi}} e^{-(x^2+y^2)/2} H_m(x) H_n(y) \tag{40}$$

Where m, n is integer and $H(\cdot)$ is Hermite polynomial. We establish table. (2) to show Berry phase and the corresponding information in these complex paths. In the other hand, the geometric phase give 0 in the real path projected from these complex paths.

<table>
<tr><td>m</td><td>0</td><td colspan="2">0</td><td colspan="2">1</td><td colspan="4">1</td></tr>
<tr><td>n</td><td>0</td><td colspan="2">1</td><td colspan="2">0</td><td colspan="4">1</td></tr>
<tr><td>ψ_x</td><td>$\frac{1}{(\pi)^{1/4}}e^{-x^2/2}$</td><td colspan="2">$\frac{1}{(\pi)^{1/4}}e^{-x^2/2}$</td><td colspan="2">$(\frac{4}{\pi})^{1/4}xe^{-x^2/2}$</td><td colspan="4">$(\frac{4}{\pi})^{1/4}xe^{-x^2/2}$</td></tr>
<tr><td>ψ_y</td><td>$\frac{1}{(\pi)^{1/4}}e^{-y^2/2}$</td><td colspan="2">$(\frac{4}{\pi})^{1/4}ye^{-y^2/2}$</td><td colspan="2">$\frac{1}{(\pi)^{1/4}}e^{-y^2/2}$</td><td colspan="4">$(\frac{4}{\pi})^{1/4}ye^{-y^2/2}$</td></tr>
<tr><td>Z_x</td><td>0</td><td colspan="2">0</td><td>0</td><td>1</td><td colspan="2">0</td><td colspan="2">1</td></tr>
<tr><td>Z_y</td><td>0</td><td>0</td><td>1</td><td colspan="2">0</td><td>0</td><td>1</td><td>0</td><td>1</td></tr>
<tr><td>γ</td><td>0</td><td>0</td><td>1</td><td>0</td><td>1</td><td>0</td><td>1</td><td>1</td><td>2</td></tr>
</table>

Table 2: Berry phase of two-dimensional simple harmonic motion and corresponding information, where Z represents the zero point contained in the closed path

3.3. Aharonov–Bohm effect

For the first two examples, in canonical quantum mechanics, we don't seem to deliberately consider berry phase in canonical quantum mechanics. Thus, in the last and most important example, we want to apply to Aharonov–Bohm effect. It is a commonly example to say the phase according the different path. First, let's quickly review Berry phase in canonical quantum mechanics.[27] Let the radius of solenoid is a. When $r > a$. In gauge condition. A can be expressed $\Phi/(2\pi r)\hat{\phi}$ in cylindrical coordination.$\Phi = \pi a^2 B$. Then, the Berry phase can be calculated by

$$\gamma_{q,n} = \frac{q}{\hbar}\oint A(\mathbf{r}) \cdot d\mathbf{r} = \frac{q}{\hbar}\oint (A) \cdot d\mathbf{a} = \frac{q\Phi}{\hbar} \tag{41}$$

where the $\mathbf{r}$ is the path which electron evolves and the $\mathbf{a}$ is area which the path encircles. This result also gotten from that the phase difference of different path in the Aharonov–Bohm experiment. Thus, we have a connection between Aharonov–Bohm experiment and Beery phase.

In the following, let's calculate the Berry phase in real path with quantum Hamilton-Jacobi theory. First, for a circular orbital motion without a magnetic field, we know that the wave function $\psi(\theta)$ is [16]

$$\psi(\theta) = \frac{1}{\sqrt{2\pi}}e^{in\theta} \tag{42}$$

Where n represents an integer, and θ is the angle of the polar coordinates. Berry phase is

$$\gamma = i\oint \frac{\psi'}{\psi}d\theta = i\oint in d\theta = -2n\pi \tag{43}$$

Next, we consider that a magnetic flux Φ passes, and the magnetic flux cause the vector potential caused is$A_r = 0, A_\theta = \Phi/2\pi R$, R, which is the radius of the spiral. The differential equation is [16]

$$\frac{1}{2mR^2}(\frac{\hbar}{i}\frac{\partial}{\partial\theta}-\frac{q\Phi}{2\pi})^2\psi=E_n\psi \tag{44}$$

The corresponding eigenfunction and the energy eigenvalue

$$\psi=\frac{1}{\sqrt{2\pi}}e^{in\theta}\exp\{i\frac{\Phi}{\Phi_0}\int_0^\theta d\theta'\} \tag{45}$$

$$E_n=\frac{\hbar^2}{2mR^2}n^2 \tag{46}$$

The $\Phi_0=2\pi\hbar/q$ can be calculated by eq.(18). We can also directly use the gauge transformation, eq. (35) to obtain the same result.

$$\gamma=i\oint\frac{\psi'}{\psi}d\theta=i\oint(in+i\frac{\Phi}{\Phi_0})d\theta=-2(n+\frac{\Phi}{\Phi_0})\pi \tag{47}$$

Therefore, relative to the Berry phase (43) given by the original closed path function, the Berry phase given by our magnetic flux is

$$\gamma=-\frac{2\pi\Phi}{\Phi_0}=-\frac{q\Phi}{\hbar} \tag{48}$$

Note that in canonical quantum mechanics, we don't care about which path of the original ring gives Berry phase, eq. (43), so in canonical quantum mechanics, Berry phase given by the Aharonov–Bohm effect. Berry phase is eq.(43) and due to the single value condition $\psi(2\pi)=\psi(0)$, it can be found that $\frac{\Phi}{\Phi_0}$ is an integer. If there is a Dirac string, we still need to satisfy the single value condition, and we can give the Dirac quantization condition.

4. Discussion and Conclusion

We have established the relationship between the Berry phase in canonical quantum mechanics and the Berry phase in the quantum Hamilton-Jacobi theory. This relationship shows that the Berry phase in the quantum Hamilton-Jacobi theory is already included in the wave function, so we can establish eq.(23). Besides, we show different derivations to get Berry phase in the complex path and the real path. In complex path, we can quickly predict the Berry phase from the Cauchy argument theorem. In real path, we get a negative Berry phase in canonical quantum mechanics. We assume it is a Berry phase in the quantum Hamilton-Jacobi theory. In the real path, we can use the eq.(18) to get the Dirac quantization condition. It can be shown clearly that in the quantum Hamilton-Jacobi theory. It does have the information of Berry phase, because we just follow the assumption of the quantum Hamilton-Jacobi theory.

In the quantum Hamilton-Jacobi theory, we usually consider the path is in complex space, so we try to get Berry phase with the complex path in the quantum Hamilton-Jacobi theory. However, there is no complex path in any experiment. The complex trajectory in the

quantum Hamilton-Jacobi theory just seems a calculation result from the eq.(1). We can't make a complex path in real world.
Until now, we have already got the relation between Berry's phase and solid angle.[28][29][30] They can observe Berry phase from this relation in canonical quantum mechanics. Berry's phase is already observed in their experiments. According to our derivation of Berry's phase. The results of their experiments show that quantum states follow the real path. That is to say, generally, quantum state don't follow a complex path naturally. A complex trajectory in the quantum Hamilton-Jacobi theory seems to be treated as an idea or a mathematical trick rather than a real phenomenon. As the fig.(2) illustrates that Berry's phase in a complex path is larger than Berry's phase in a real path when we choose separable variables. Until now, the experiment didn't give larger Berry's phase as Berry's phase complex path provides. Although we can't say exactly that a complex trajectory doesn't exist, as the experiments show and we show different values about Berry's phase. We can conclude that we don't consider a phenomenon of complex path for quantum state naturally.
If we consider the variable to be a complex variable, we can see from C.D. Yang's early article on the establishment of the quantum Hamilton-Jacobi theory. We can describe the quantum trajectory in the complex space. When taking the probability distribution of the wave function in the quantum Hamilton-Jacobi theory, we even take the real part functions do satisfy the results of canonical quantum mechanics, but the extra information, the imaginary part, obtained in the quantum Hamilton-Jacobi theory. If this imaginary part is meaningless, it seems that the quantum Hamilton-Jacobi theory is just a mathematical technique to describe quantum mechanics. In this article, we try to give the meaning of the imaginary part. It seems really to change the value of Berry's phase when the quantum state evolves in the complex space. We exactly find the influence of the imaginary part to the quantum state, but it doesn't coincide to the experiment. Therefore, from mathematical derivation, we have shown that complex paths should not exist naturally.
In summary, this article shows Berry's phase in the quantum Hamilton-Jacobi theory. Besides, we argue whether the complex trajectories exist or not. This tells that we can't still observe a complex trajectories. In the quantum Hamilton-Jacobi theory, we can still this concept to calculate the evolution of wavefunction. It seems a method rather than a phenomenon. In the other respect, we can directly attain Berry's phase at wavefunction in the quantum Hamilton-Jacobi theory. It shows the wavefunction has the complete format including path. We learn the quantum Hamilton-Jacobi theory is more complete because it also includes paths. We also give an argument about the complex trajectory. the quantum Hamilton-Jacobi theory is interesting theory and seems powerful. It is worth more research and we hope to complete it.

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