Assignment 1
Formal proofs, Alphabets and Words

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1 Task 1

Prove using induction that

$$\sum_{k=1}^{n} k(k+1) = \frac{n(n+1)(n+2)}{3}$$

I'm going to prove this using the five steps of Proof by Induction:
Step 1 (State the property P):

$$P(n) \equiv \sum_{k=1}^{n} k(k+1) = \frac{n(n+1)(n+2)}{3}$$

Step 2 (State method) : I'm going to prove this by using mathematical induction.

Step 3 (State and prove base cases): Base case : P(0).
Proof of base case :

$$P(0) \equiv \sum_{k=1}^{0} k(k+1) = \frac{0(0+1)(0+2)}{3}$$

Which gives $0 = 0$, since taking a sum from k n, where $n < k$, results in a sum over nothing, which is equal to 0. Hence, the base case holds.

Step 4 (State the Inductive Hypothesis): Assume $m \in \mathbb{N}$.
Assume

$$P(m) \equiv \sum_{k=1}^{m} k(k+1) = \frac{m(m+1)(m+2)}{3} = \frac{m^3+3m^2+2m}{3}$$

Step 5 (State and prove inductive steps): Prove that

$$P(m+1) \equiv \sum_{k=1}^{m+1} k(k+1) = \frac{(m+1)(m+2)(m+3)}{3} = \frac{m^3+6m^2+11m+6}{3}$$

This polynomial algebra will help me later in on in proving the induction.

The first step now is to split the sum into two parts, in order to use my IH. First first sum will be the original sum used in my IH, while the second sum will simply be the (m+1):th term.

$$\sum_{k=1}^{m}(k(k+1) + \sum_{m+1}^{m+1}(k(k+1) = \frac{m^3+6m^2+11m+6}{3}$$

$$\sum_{k=1}^{m}(k(k+1)+(m+1)(m+2)=\frac{m^3+6m^2+11m+6}{3}$$

$$\frac{m^3+3m^2+2m}{3}+m^2+3m+2=\frac{m^3+6m^2+11m+6}{3}$$

The next step will be two write the left hand side on the same fraction, so I will extend the second half of the LHS by 3.

$$\frac{m^3+3m^2+2m}{3}+\frac{3m^2+9m+6}{3}=\frac{m^3+6m^2+11m+6}{3}$$

$$\frac{m^3+6m^2+11m^2+6}{3}=\frac{m^3+6m^2+11m+6}{3}$$

Closure: By having proven P(0) and $\forall m \in \mathbb{N}.P(m) \Rightarrow P(m+1)$, we obtain $\forall n \in \mathbb{N}.P(n)$ by mathematical induction.$\square$

2 Task 2

Given definitions:
$f, g, h : \mathbb{N} \to \mathbb{N}$

$f(0) = 0$
$f(n+1) = 2 * h(n) + g(n) + 4$

$g(0) = 0$
$g(n+1) = h(n) + 1$

$h(0) = 1$
$h(n+1) = g(n) + 3$

a) Computations of f, g and h.

- $f(1)$ (i.e. n = 0) $= 2 * h(0) + g(0) + 4 = 2 + 4 = 6$
- $g(1) = h(0) + 1 = 1 + 1 = 2$
- $h(1) = g(0) + 3 = 0 + 3 = 3$
- $f(2) = 2 * h(1) + g(1) + 4 = 2 * 3 + 2 + 4 = 12$
- $g(2) = h(1) + 1 = 3 + 1 = 4$
- $h(2) = g(1) + 3 = 2 + 3 = 5$
- $f(3) = 2 * h(2) + g(2) + 4 = 2 * 5 + 4 + 4 = 18$
- $g(3) = h(2) + 1 = 5 + 1 = 6$
- $h(3) = g(3) + 3 = 4 + 3 = 7$

b) Prove $\forall n \in \mathbb{N}. f(n) = 3 * g(n)$ by proving $\forall n \in \mathbb{N}. f(n) = 3 * g(n) \wedge h(n) = 1 + g(n)$. Again I will use the five steps of Proof by Induction.

Step 1: Property

$$P(n) \equiv \forall n \in \mathbb{N}. f(n) = 3 * g(n) \wedge h(n) = 1 + g(n)$$

Step 2: This time I will use mutual induction in order to prove this.

Step 3: My base cases are what is given to me already, that is, f(0), g(0) and h(0).
Proof:

$$f(0) = 3 * g(0) \wedge h(0) = 1 + g(0)$$

$$0 = 0 \wedge 1 = 1$$

So the base case holds.

Step 4: Assume $m \in \mathbb{N}$. Assume

$$P(m) \equiv f(m) = 3 * g(m) \wedge h(m) = 1 + g(m)$$

Step 5: Prove P(m+1).

$$f(m+1) = 3 * g(m+1) \wedge h(m+1) = 1 + g(m+1)$$

And by using the definitions of f,g,h we will have

$$2 * h(m) + g(m) + 4 = 3(h(m) + 1) \wedge g(m) + 3 = 1 + h(m) + 1$$

$$2 * h(m) + g(m) + 4 = 3 * h(m) + 3 \wedge g(m) + 3 = 1 + h(m) + 1$$

And now, on the LHS of the AND sign, I will move all g(m) to one side, and all h(m) to one side. On the RHS of the AND sign, I will do the same.

$$g(m) + 1 = h(m) \wedge g(m) + 1 = h(m)$$

Because we assume that $g(m) + 1 = h(m)$ in our IH, then this will be true even for this case.

Since by induction $\forall m \in \mathbb{N}.P(m)$ is true, then $\forall n \in \mathbb{N}.P(n)$ is also true.$\square$

3 Task 3

The assignment describes that the most simplest group in the set GO is simply the pair A and B (will be denoted as Pair(A,B). You can then **extend** a group x by adding another organism A and another organism B, forming a new group. Finally, you can also form a new group by **merging** two groups x and y **and** extend by adding one organism A and two organisms B.

a) I will inductively describe this way to form groups into GO by first stating the base case and the inductive step, before moving on to the closure.

Base case: $Pair(A, B) \in GO$

Inductive step: If $x, y \in GO$ then

$$\forall x \in GO.Extend(x, A, B) \in GO$$

and

$$\forall x, y \in GO.Merge(x, y, A, B, B) \in \ GO$$

Closure: There are no other ways to form groups in GO.

b) For this assignment I will define the functions nrA and nrB, and I will start doing this by first defining nrA:

Base case: $nrA(Pair(A, B)) = 1$
Recursive cases: Given $x, y, ... \in GO$, then

$$nrA(Extend[x, A, B]) = nrA(x) + 1;$$

$$nrA(Merge[x, y, A, B, B]) = nrA(x) + nrA(y) + 1;$$

And then I define nrB:

Base case: $nrB(Pair(A, B)) = 1$
Recursive cases: Given $x, y, ... \in GO$, then

$$nrB(Extend[x, A, B]) = nrB(x) + 1;$$

$$nrB(Merge[x, y, A, B, B]) = nrB(x) + nrB(y) + 2;$$

c) Prove by induction that $\forall x \in GO.nrA(x) \leq nrB(x) \leq 2 * nrA(x)$
Again, I will prove this using the five steps of induction. The only difference here is that I will have two different induction steps, since nrA and nrB can

either be counting an extended or merged group (or a simple pair but this will be my base case).

Step 1: Property

$$P(x) \equiv \forall x \in GO.nrA(x) \leq nrB(x)$$

Step 2: I'll use the method called structural induction here.

Step 3: Base case:

$P(Pair[A,B]) \equiv nrA(Pair[A,B]) \leq nrB(Pair[A,B]) \leq 2{*}nrA(Pair[A,B]) = 1 \leq 1 \leq 2$

and so the base case holds.

Step 4: My induction hypothesis is as follows:

$$P(y) \equiv nrA(y) \leq nrB(y) \leq 2 * nrA(y)$$

Step 5: Assuming the IH, I prove P(Extend(y,A,B)):

$$nrA(Extend[y,A,B]) \leq nrB(Extend[y,A,B]) \leq 2 * nrA(Extend[y,A,B])$$

$$nrA(y) + 1 \leq nrB(y) + 1 \leq 2 * (nrA(y) + 1)$$

$$nrA(y) \leq nrB(y) \leq 2 * nrA(y) + 1$$

Because the +1 on the right hand side only increases this value further, this expression corresponds to the IH and therefore this inductive step holds.

Assuming the IH, I prove P(Merge[x,y,A,B,B]):

$$nrA(Merge[x,y,A,B,B]) \leq nrB(Merge[x,y,A,B,B]) \leq 2{*}nrA(Merge[x,y,A,B,B])$$

$$nrA(x) + nrA(y) + 1 \leq nrB(x) + nrB(y) + 2 \leq 2 * (nrA(x) + nrA(y) + 1)$$

$$nrA(x) + nrA(y) + 1 \leq nrB(x) + nrB(y) + 2 \leq 2 * nrA(x) + 2 * nrA(y) + 2$$

$$nrA(x) + nrA(y) \leq nrB(x) + nrB(y) + 1 \leq 2 * nrA(x) + 2 * nrA(y) + 1$$

According to the IH, nrB will always be at least as large as nrA. So even if we have two of each, nrB will be larger than nrA, which means we can ignore at least one of each term on either side. Likewise, since 2*nrA will always be at least as large as nrB, we can also neglect one of the terms on this side, leaving us with

$$nrA(y) \leq nrB(y) + 1 \leq 2 * nrA(y) + 1$$

Again, the +1 terms can be ignored since they only increase their values further. Correspondingly, this expression corresponds to the IH, and therefore this inductive step holds as well.

Closure: $\forall x \in GO.nrA(x) \leq nrB(x) \leq 2 * nrA(x) \square$.

4 Task 4

Here the task states to prove using induction that $\forall x \in \Sigma^*.|rev(x)| = |x|$. I am allowed to assume all properties from lecture 4, but I will state those I am going to use below.

Definition length of string: The length function $|\_|\colon\ \Sigma^* \to \mathbb{N}$ is defined as

$$|\varepsilon| = 0$$

$$|ax| = 1 + |x|$$

Definition reverse of string: The reverse function $\text{rev}(\_) : \Sigma^* \to \Sigma^*$ is defined as

$$rev(\varepsilon) = \varepsilon$$

$$rev(ax) = rev(x)a$$

And now for the proof.

Step 1: Property

$$P(x) \equiv \forall x \in \Sigma^*.|rev(x)| = |x|$$

Step 2: I am going to use structural induction.

Step 3: Base case

$$P(\varepsilon) \equiv |rev(\varepsilon)| = |\varepsilon|$$

$$|\varepsilon| = |\varepsilon|$$

so the base holds.

Step 4: Assuming $a \in \Sigma$ and $k \in \Sigma^*$. Assume

$$|rev(k)| = |k|$$

Step 5: Then prove

$$P(ak) = |rev(ak)| = |ak|$$

$$|rev(k)a| = 1 + |k|$$

$$1 + |rev(k)| = 1 + |k|$$

$$rev(k) = |k|$$

assuming the IH is true, the proof holds.$\square$