2372 Assignment

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Personal request to markers

Dear Professor,

Every question is worked with maximum effort, except Question 5 c).

I have done all the work without anyone's help. I did minimal online research, mostly relying on the lecture notes.

Even though some working could inevitably appear complicated, please mark them with maximum patience, because I have worked very hard for each question and I look forward to receiving a full mark except Question 5 c).

I undoubtedly understand how stressful the process must be. My pdf is 20 pages long. I sincerely thank you for your time and effort for marking my work.

Thank you,

Yunbae Chae

Question 1

Q1 a

S'(t)

From question, $a, t > 0$.

$P(X \geq a) = P(X + t \geq a + t) = P((X + t)^2 \geq (a + t)^2)$

$\leq \frac{E[(X+t)^2]}{(a+t)^2}$ (Markov's inequality)

$= \frac{E[X^2+2tX+t^2]}{(a+t)^2}$ (Expansion)

$= \frac{E(X^2)+2tE(X)+E(t^2)}{(a+t)^2}$ (1)

From question, $E(X) = 0$ and $VarX = \sigma^2$. Then, $VarX = \sigma^2 = E(X^2) - E(X) = E(X^2) - 0$.

In this way, $E(X^2) = \sigma^2$.

(1)$= \frac{\sigma^2+2t\cdot 0+t^2}{(a+t)^2}$

$= \frac{\sigma^2+t^2}{(a+t)^2} = S(t)$.

To find the lower bound, minimize $S(t)$. Find $S'(t) = 0$.

Differentiation

Differentiate $\frac{\sigma^2+t^2}{(a+t)^2}$ $(S(t))$:

$S'(t) = \frac{(a+t)^2\cdot 2t-2(a+t)\cdot(\sigma^2+t^2)}{(a+t)^4}$

$= \frac{2a^2t+2at^2-2a\sigma^2-2t\sigma^2}{(a+t)^4}$

$= \frac{2at(a+t)-2\sigma^2(a+t)}{(a+t)^4}$

$= \frac{2(at-\sigma^2)(a+t)}{(a+t)^4}$

Find the t at which $S'(t) = 0$ (minimal point):

$at - \sigma^2 = 0, a + t = 0$

$t = \frac{\sigma^2}{a}$ but $t \neq -a$ (given $t > 0$)

Finish

Use $t = \frac{\sigma^2}{a}$.

$S(t) = \frac{\sigma^2 + \frac{\sigma^4}{a^2}}{(a+\frac{\sigma^2}{a})^2}$

$= \frac{\frac{a^2\sigma^2+\sigma^4}{a^2}}{\frac{(a^2+\sigma^2)^2}{a^2}}$

$= \frac{\sigma^2(a^2+\sigma^2)}{(a^2+\sigma^2)^2}$

$= \frac{\sigma^2}{a^2+\sigma^2}$

Back to the beginning, $P(X \geq a) = P(X+t \geq a+t) = P((X+t)^2 \geq (a+t)^2) \leq \frac{E[(X+t)^2]}{(a+t)^2} = \frac{\sigma^2}{a^2+\sigma^2}$

$$\therefore P(X \geq a) \leq \frac{\sigma^2}{a^2+\sigma^2}$$

Q1 b

$P(Y \geq \mu + a) \leq \frac{\sigma^2}{\sigma^2+a^2}$
$P(Y \leq \mu - a) \leq \frac{\sigma^2}{\sigma^2+a^2}$

results from $X = |Y - \mu|$.

Using the inequality in a) and $X = |Y - \mu|$,

$P(|Y - \mu| \geq a) \leq \frac{\sigma^2}{\sigma^2+a^2}$.

Here, $|Y - \mu| \geq a = Y - \mu \geq a, Y - \mu \leq -a$.

$\therefore Y \geq \mu + a$ and $Y \leq \mu - a$.

So,

$P(Y \geq \mu + a) \leq \frac{\sigma^2}{\sigma^2+a^2}$

$P(Y \leq \mu - a) \leq \frac{\sigma^2}{\sigma^2+a^2}$.

From $X = |Y - \mu|$, $Y - \mu = \pm X \Rightarrow Y = \mu \pm X$

$E(Y) = E(\mu \pm X) = E(\mu) \pm E(X)$ but given from question that, $E(X) = 0$.

Then, $E(Y) = E(\mu) = \mu$.

$varY = var\mu \pm varX = varX = \sigma^2$.

Question 2

Q2 a

i

$K_X(t) = log(M_X(t))$. So, I must get M_X first.

$M_X = E(e^{tX}) = \sum_x e^{tx} f_X(x)$ (Since X is discrete for Poisson.)

Now, $X = \frac{\lambda^x e^{-\lambda}}{x!}$.

$\sum_x e^{tx} \frac{\lambda^x e^{-\lambda}}{x!} = e^{-\lambda} \sum_x e^{tx} \frac{\lambda^x}{x!}$ (moved irrelevant element outside of summation.)

$= e^{-\lambda} \sum_x \frac{(\lambda e^t)^x}{x!}$ (Simplify.)

$\sum_x \frac{(\lambda e^t)^x}{x!}$ in the form of power series: $\sum_{k=0}^{\infty} \frac{z^k}{k!} = e^z$

$\sum_x \frac{(\lambda e^t)^x}{x!} = e^{\lambda e^t}$

$\therefore e^{-\lambda} \sum_x \frac{(\lambda e^t)^x}{x!} = e^{-\lambda} e^{\lambda e^t}$

$e^{-\lambda} e^{\lambda e^t} = e^{\lambda(e^t - 1)}$

$M_X(t) = e^{\lambda(e^t - 1)}$

Finally, $K_X(t) = log(M_X(t)) = log(e^{\lambda(e^t-1)})$

$= \lambda(e^t - 1) \cdot log(e)$

$= \lambda(e^t - 1)$

Easily, $K'_X(t) = \lambda e^t$.

$K''_X(t) = \lambda e^t$, $K^{(3)}_X(t) = \lambda e^t$, $K^{(4)}_X(t) = \lambda e^t$

$K^{(3)}_X(0) = \lambda e^0 = \lambda$, $K^{(4)}_X(0) = \lambda e^0 = \lambda$

From the text book, third and forth central moments are given that:

$\mu_3 = \kappa_3 = K^3_X(0) = \lambda$

$\mu_4 = \kappa_4 + 3\sigma^4 = \lambda + 3\lambda^2$

ii

From the question, $M_Y(t) = M_{\frac{X}{\sqrt{\lambda}}-\sqrt{\lambda}}(t)$.

From the notes, $M_{a+bX}(t) = e^{at}M_X(bt)$. $a = -\sqrt{\lambda}, b = \frac{1}{\sqrt{\lambda}}$.

$$\therefore M_Y(t) = M_{\frac{X}{\sqrt{\lambda}}-\sqrt{\lambda}}(t) = e^{-\sqrt{\lambda}t}M_X(\tfrac{t}{\sqrt{\lambda}}) \quad (1)$$

$M_X(\frac{t}{\sqrt{\lambda}}) = \sum_x e^{\frac{t}{\sqrt{\lambda}}x}\frac{\lambda^x e^{-\lambda}}{x!}$

$= e^{-\lambda}\sum_x \frac{(e^{\frac{t}{\sqrt{\lambda}}}\lambda)^x}{x!}$

$= e^{-\lambda} \cdot e^{\lambda e^{\frac{t}{\sqrt{\lambda}}}} = e^{\lambda(e^{\frac{t}{\sqrt{\lambda}}}-1)}$

Back to (1), $e^{-\sqrt{\lambda}t}M_X(\frac{t}{\sqrt{\lambda}}) = e^{-\sqrt{\lambda}t} \cdot e^{\lambda(e^{\frac{t}{\sqrt{\lambda}}}-1)}$

$= e^{\lambda e^{\frac{t}{\sqrt{\lambda}}}-\sqrt{\lambda}t-\lambda}$

iii

From $= e^{\lambda e^{\frac{t}{\sqrt{\lambda}}}-\sqrt{\lambda}t-\lambda}$,

Question prompts to substitute $\sum_{i=0}^{\infty}\frac{(\frac{t}{\sqrt{\lambda}})^i}{i!}$ into $e^{\frac{t}{\sqrt{\lambda}}}$. (e power series)

Then, $e^{\lambda(1+\frac{(\frac{t}{\sqrt{\lambda}})^1}{1!}+\frac{(\frac{t}{\sqrt{\lambda}})^2}{2!}+\frac{(\frac{t}{\sqrt{\lambda}})^3}{3!}+\cdots)-\sqrt{\lambda}t-\lambda}$.

$= e^{\lambda+\frac{t\cdot\lambda}{\sqrt{\lambda}\cdot 1!}+\frac{t^2\cdot\lambda}{\lambda\cdot 2!}+\frac{t^3}{\sqrt{\lambda}\cdot 3!}+\frac{t^4}{\lambda\cdot 4!}+\cdots-\sqrt{\lambda}t-\lambda}$

$= e^{\lambda-\lambda+\frac{t\cdot\lambda}{\sqrt{\lambda}\cdot 1!}-\sqrt{\lambda}t+\frac{t^2\cdot\lambda}{\lambda\cdot 2!}+\frac{t^3}{\sqrt{\lambda}\cdot 3!}+\frac{t^4}{\lambda\cdot 4!}+\cdots}$ (Rearrange)

$= e^{0+0+\frac{t^2}{2!}+\frac{t^3}{\sqrt{\lambda}\cdot 3!}+\frac{t^4}{\lambda\cdot 4!}+\cdots}$

As $\lambda \to \infty$, it happens all the fractions except $\frac{t^2}{2!}$ have λ in their denominator, so they all $\to 0$.

As in, $e^{0+0+\frac{t^2}{2!}+0+0+\cdots}$.

$$\lim_{\lambda\to\infty} M_Y(t) = e^{\frac{t^2}{2}}$$

This is a mgf of a standard normal distribution.

as $\lambda \to \infty$:

1. $M'_Y(t) = t \cdot e^{\frac{t^2}{2}}$

$M'_Y(0) = 0 \cdot e^{\frac{0^2}{2}} = 0$

2. $M''_Y(t) = 1 \cdot e^{\frac{t^2}{2}} + t^2 \cdot e^{\frac{t^2}{2}}$

$M''_Y(0) = 1 \cdot e^{\frac{0^2}{2}} + 0^2 \cdot e^{\frac{0^2}{2}}$

$= 1 \cdot 1 + 0 \cdot e^{\frac{0^2}{2}}$

$= 1$

The mean and variance of Y are 0 and 1 respectively. Also, it is given from lectures that the mgf of normal distribution is $e^{\mu_X t + \frac{1}{2}t^2 \frac{\sigma_X^2}{n}}$.

The standard normal is when $\mu_X = 0, \frac{\sigma_X^2}{n} = 1$. I.e. $e^{0 \cdot t + \frac{1}{2}t^2 \cdot 1} = e^{\frac{t^2}{2}}$.

The distribution function of Y converges to standard normal distribution function as $\lambda \to \infty$.

Q2 b

$$\lambda = 36$$

Need to get $(P \geq 45)$. This is a discrete setup. To get the probability of at least 45 accidents, get $\sum_{x=45}^{\infty} \frac{36^x e^{-36}}{x!}$.

It is same as $1 - P(X \leq 44)$ since $\sum_{x=0}^{\infty} \frac{36^x e^{-36}}{x!} = 1$.

$\therefore 1 - \sum_{x=0}^{44} \frac{36^x e^{-36}}{x!} =$

```
1-ppois(44,36)
```

```
## [1] 0.08186538
```

Q2 c

i

A Poisson process with rate 1 is of the distribution of time between events that follow a standard exponential distribution. The sum of the exponential random variables implies that no events occur at the same time. Each variables $V_i, i = 0, 1, 2, 3, \cdots, n$ are independent with equal rate 1. Therefore, conditions are met to make up a Poisson process.

$$X = max\{n : \sum_{i=1}^{n} Vi \leq \lambda\} \tag{0}$$

measures the number of events V_i in a fixed time frame λ. This is just a definition of a Poisson distribution with parameter λ.

ii

Since $V_i = -logU_i$, $-V_i = logU_i$.

Then $e^{-V_i} = U_i$.

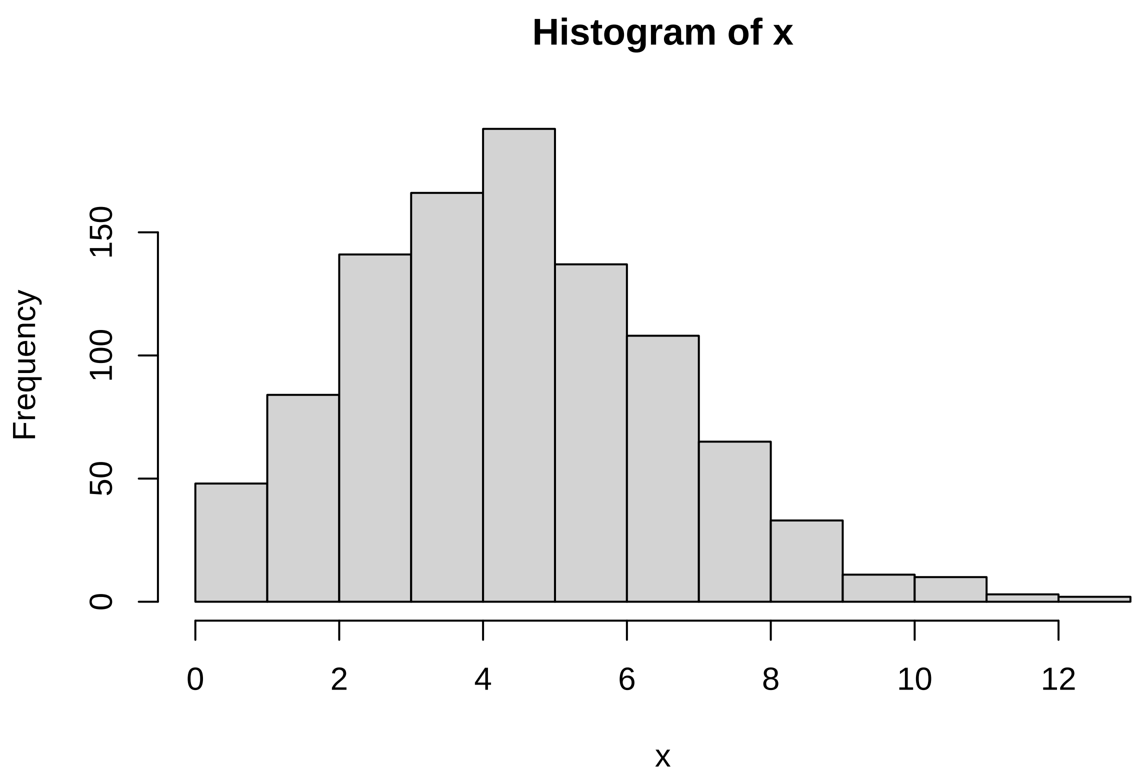
From (0) above, $X = max\{n : \sum_{i=1}^{n} -Vi \geq -\lambda\}$.

$X = max\{n : \sum_{i=1}^{n} e^{-Vi} \geq e^{-\lambda}\}$

$$X = max\{n : \sum_{i=1}^{n} U_i \geq e^{-\lambda}\} \qquad (e^{-V_i} = U_i)$$

iii

```
x <- 1
x[1:1000] <- 1
for (i in 1:1000){
  mass <- runif(1)
  while (mass>=exp(-5)){
    mass <- mass*runif(1)
    x[i] <- x[i]+1
  }
  x[i] <- x[i]-1
}
hist(x)
```




```r
mean(x)
```

```
## [1] 4.928
```

The theoretical counterpart is the mean of a Poisson distribution of parameter 5. The mean is λ, which is 5. The generated mean value should be something very close to 5.

Question 3

Q3 a

First, X and Y have same distribution:

$$f_X(x) = \begin{cases} 1 - \frac{1}{2} & x = 0 \\ \frac{1}{2} & x = 1 \\ 0 & otherwise \end{cases}$$

$$f_X(x) = \begin{cases} \frac{1}{2} & x = 0 \\ \frac{1}{2} & x = 1 \\ 0 & otherwise \end{cases}$$

$$f_Y(y) = \begin{cases} 1 - (1 - \frac{1}{2}) & y = 0 \\ 1 - \frac{1}{2} & y = 1 \\ 0 & otherwise \end{cases}$$

$$f_Y(y) = \begin{cases} \frac{1}{2} & y = 0 \\ \frac{1}{2} & y = 1 \\ 0 & otherwise \end{cases}$$

At $n \to \infty$, each value of X_n has probability of $\frac{1}{2}$ to be either 0 or 1. This is another way of saying $X_n \sim B(\frac{1}{2})$.

The cdf of X is same as that of Y so it is enough to consider only F_X from now:

$$F_X(x) = \begin{cases} 0 & x < 0 \\ \frac{1}{2} & 0 \leq x < 1 \\ 1 & x \geq 1 \end{cases}$$

In this way, $Ber(\frac{1}{2}) = F_X = F_Y$ and $X_n \xrightarrow{d} Ber(\frac{1}{2})$. Therefore, there is convergence in distribution for X_n, X and Y.

$\therefore X_n \xrightarrow{d} X$ and $X_n \xrightarrow{d} Y$.

Q3 b

First, consider if $\lim_{n\to\infty} P(|X_n - Y| \geq \epsilon) = 0$.

Each X_n and Y are either 0 or 1 with equal probability $\frac{1}{2}$.

$$f_{X_n - Y} = \begin{cases} \frac{1}{4} & X_n = 1, Y = 1 \\ \frac{1}{4} & X_n = 1, Y = 0 \\ \frac{1}{4} & X_n = 0, Y = 1 \\ \frac{1}{4} & X_n = 0, Y = 0 \end{cases}$$

This is equivalent to:

$$f_{X_n-Y} = \begin{cases} \frac{1}{2} & |X_n - Y| \geq \epsilon \\ \frac{1}{2} & |X_n - Y| < \epsilon \end{cases}$$

The above is an adequate argument that for $n \to \infty$, $\lim_{n\to\infty} P(|X_n - Y| \geq \epsilon) \neq 0$.

$X_n \overset{d}{\not\to} Y$. X_n does not converge to Y in probability.

Question 4

$$f(x_1, x_2) = \begin{cases} 4x_1x_2e^{-x_1^2} & x_1 > 0, x_2 > 0 \\ 0 & otherwise \end{cases}$$

A valid joint probability density function satisfies:

1. $f_{X,Y} \geq 0, \forall(x, y) \in \mathbb{R}^2$;
2. $\iint_{\mathbb{R}^{\neq}} f_{X,Y}(x, y)dxdy = 1$.

Q4 1.

Easily, $x_1 > 0$

$x_1^2 > 0$

$-x_1^2 < 0$

$e^{-x_1^2} < e^0$

$e^{-x_1^2} < 1$

Basically, $e^{-x_1^2}$ is positive around $x_1 = 0$ and then the limit: $\lim_{x_1 \to \infty} \frac{1}{e^{x_1^2}} \to \overset{+}{0}$

means $e^{-x_1^2}$ is positive and never 0.

$\therefore 0 < e^{-x_1^2} < 1$.

We have $4 \cdot x_1 \cdot x_2 > 0$ $\qquad (x_{1,2} > 0)$

$\therefore 4x_1x_2e^{-x_1^2} \geq 0, \forall(x_1, x_2) > 0$.

First condition satisfied.

Q4 2.

Solving the double integral must give 1:
$\int_0^\infty \int_0^\infty 4x_1x_2e^{-x_1^2}dx_1dx_2$

$= -2\int_0^\infty x_2[\int_0^\infty -2x_1e^{-x_1^2}dx_1]dx_2$

$= -2\int_0^\infty x_2[e^{-x_1^2}]_0^\infty dx_2$

$= -2\int_0^\infty x_2[0-1]dx_2$ (Let $e^{-\infty} = 0$)

$= \int_0^\infty 2x_2 dx_2$

$= [x^2]_0^\infty$

However, this is $\infty - 0 = \infty \neq 1$.

After all, since the integral is not equal to 1,

$$f(x_1, x_2) = \begin{cases} 4x_1x_2e^{-x_1^2} & x_1 > 0, x_2 > 0 \\ 0 & otherwise \end{cases}$$

is NOT a valid joint probability density function.

Question 5

Q5 a

$$T_X(z) = \sum_{x=a}^{b-1} z^x P(X > x)$$

$$= z^a P(X > a) + z^{a+1} P(X > a+1) + z^{a+2} P(X > a+2) + \cdots + z^{b-1} P(X > b-1). \quad (1)$$

Let $P(X > a) = \sum_{a+1}^{b} f_X(x), P(X > a+1) = \sum_{a+2}^{b} f_X(x)$ and so on (given from question X is discrete).

(1) $\times$ $(1 - z)$ gives $(1 - z)T_X(z) = (z^a - z^{a+1}) \sum_{a+1}^{b} f_X(x) + (z^{a+1} - z^{a+2}) \sum_{a+2}^{b} f_X(x) + (z^{a+2} - z^{a+3}) \sum_{a+3}^{b} f_X(x) + \cdots + (z^{b-2} - z^{b-1}) \sum_{b-1}^{b} f_X(x) + (z^{b-1} - z^b) f_X(b)$

Rearranging gives:

$= z^a \sum_{a+1}^{b} f_X(x) - z^{a+1}(\sum_{a+1}^{b} f_X(x) - \sum_{a+2}^{b} f_X(x)) - z^{a+2}(\sum_{a+2}^{b} f_X(x) - \sum_{a+3}^{b} f_X(x)) - \cdots - z^b f_X(b)$

$= z^a \sum_{a+1}^{b} f_X(x) - z^{a+1} f_X(a+1) - z^{a+2} f_X(a+2) - z^{a+3} f_X(a+3) - \cdots - z^b f_X(b)$

Add and subtract $z^a f_X(a)$:

$= z^a \sum_{a}^{b} f_X(x) - z^a f_X(a) - z^{a+1} f_X(a+1) - z^{a+2} f_X(a+2) - z^{a+3} f_X(a+3) - \cdots - z^b f_X(b)$

$\sum_{a}^{b} f_X(x) = 1$. So, RHS$= z^a \cdot 1 - \sum_x f_X(x) z^x$

$= z^a - G_X(z)$.

If X, who lives in $a \leq X \leq b$, is to be a non-negative discrete random variable $(0, 1, 2, 3, \cdots)$, then the lower bound $a = 0$.

$z^0 - G_X(z) = 1 - G_X(z)$.

Q5 b

E(X)

From lecture notes, $G'_X(1) = E(X)$.

From a), $(1-z)T_X(z) = 1 - G_X(z)$.

$\frac{d}{dz}(1-z)T_X(z) = \frac{d}{dz}(1 - G_X(z))$

$$= -1 \cdot T_X(z) + (1-z) \cdot T'_X(z) = -G'_X(z) \tag{1}$$

Let $z = 1$: $-1 \cdot T_X(1) + (1-1) \cdot T'_X(1) = -G'_X(1)$

$-T_X(1) = -G'_X(1)$

$\therefore G'_X(1) = T_X(1) = E(X)$

var(X)

From lecture notes, $$varX = G''_X(1) + G'_X(1) - G'_x(1)^2 \tag{2}$$

Bring (1) from above and differentiate it one more time:

$= \frac{d}{dz}(-1 \cdot T_X(z) + (1-z) \cdot T'_X(z)) = \frac{d}{dz}(-G'_X(z))$

$= -T'_X(z) - 1 \cdot T'_X(z) + (1-z) \cdot T''_X(z) = -G''_X(z)$ $\quad *(\frac{d}{dz}(1-z) \cdot T'_X(z) = -1 \cdot T'_X(z) + (1-z) \cdot T''_X(z))$

Let $z = 1$: $-T'_X(1) - T'_X(1) + (1-1) \cdot T''_X(1) = -G''_X(1)$

$\Rightarrow -2T'_X(1) = -G''_X(1)$

$G''_X(1) = 2T'_X(1)$.

Substituting $G'_X(1) = T_X(1), G''_X(1) = 2T'_X(1)$ in (2) gives:

$varX = 2T'_X(1) + T_X(1) - T_X(1)^2$.

Q5 c

Q5 d

From the provisions, $S_{\tau_n} \not\leq n$.

Consider 1. $S_j = n$ and 2. $S_j < n$:

1. When $S_j = n$, j is the number of Y_i that make the sum exactly equal to n which means at least $j+1$ many Y_i will make any value greater than n. The minimum number of Y_i in this setup is $j+1$. So, $\tau_n = j+1$.

2. When $S_j < n$, more Y_i are needed than the minimum required number of Y_i in the previous argument, which is $j+1$. In this way, $\tau_n > j+1$.

After all, $\tau_n \geq j+1$.

Q5 e

First, using c) and d) to define $P(\tau_n > j)$:

$P(S_j \leq n) = P(\tau_n \geq j+1)$ (from d)

$= P(\tau_n > j)$ (τ_n is discrete)

So, $P(\tau_n > j) = \frac{1}{n^j}\binom{n}{j}$ (1)

Second, expand $T_{\tau_n}(z)$:

$= z^0 P(\tau_n > 0) + z^1 P(\tau_n > 1) + z^2 P(\tau_n > 2) + \cdots + z^n P(\tau_n > n)$

Using (1), $= z^0 \cdot \frac{1}{n^0}\binom{n}{0} + z^1 \cdot \frac{1}{n^1}\binom{n}{1} + z^2 \cdot \frac{1}{n^2}\binom{n}{2} + \cdots + z^n \cdot \frac{1}{n^n}\binom{n}{n}$

This resembles a polynomial law case.

$= (\frac{z}{n})^0\binom{n}{0} + (\frac{z}{n})^1\binom{n}{1} + (\frac{z}{n})^2\binom{n}{2} + \cdots + (\frac{z}{n})^n\binom{n}{n}$

It is easily $(\frac{z}{n}+1)^n$ ($\sum_0^n \binom{n}{x} a^x b^{n-x} = (a+b)^n$)

$$\therefore T_{\tau_n}(z) = (\frac{z}{n}+1)^n$$

Q5 f

From b, $E(\tau_n) = T_{\tau_n}(1), var(\tau_n) = 2T'_{\tau_n}(1) + T_{\tau_n}(1) - T_{\tau_n}(1)^2$

E

From e, $T_{\tau_n}(1) = E(\tau_n) = (\frac{1}{n} + 1)^n$

Var

$var(\tau_n) = 2T'_{\tau_n}(1) + T_{\tau_n}(1) - T_{\tau_n}(1)^2$

Get $T'_{\tau_n}(1)$:

$T_{\tau_n}(1) = (\frac{1}{n} + 1)^n$

$ln(T_{\tau_n}(1)) = n \cdot ln(\frac{1}{n} + 1)$ (log both sides)

$\frac{d}{dn}[ln(T_{\tau_n}(1)) = n \cdot ln(\frac{1}{n} + 1)]$ (Differentiate both sides with respect to n)

LHS $= \frac{d}{dn} \cdot \frac{dT_{\tau_n}}{d} \frac{d}{dT_{\tau_n}} lnT_{\tau_n}$ (Implicit differentiation)

$= \frac{dT_{\tau_n}}{dn} \frac{1}{T_{\tau_n}}$

RHS $= \frac{d}{dn}(n \cdot ln(\frac{1}{n} + 1))$

$= 1 \cdot ln(\frac{1}{n} + 1) + n \cdot \frac{\frac{-1}{n^2}}{1+\frac{1}{n}}$

$= ln(\frac{1}{n} + 1) - \frac{1}{n+1}$

$\therefore \frac{dT_{\tau_n}}{dn} \frac{1}{T_{\tau_n}} = ln(\frac{1}{n} + 1) - \frac{1}{n+1}$

$\frac{dT_{\tau_n}}{dn} = T_{\tau_n} \cdot ln(\frac{1}{n} + 1) - \frac{1}{n+1}$

$T'_{\tau_n}(1) = (\frac{1}{n} + 1)^n[ln(\frac{1}{n} + 1) - \frac{1}{n+1}]$

$var(\tau_n) = 2(\frac{1}{n} + 1)^n[ln(\frac{1}{n} + 1) - \frac{1}{n+1}] + (\frac{1}{n} + 1)^n - (\frac{1}{n} + 1)^{2n}$

Q5 g

From a, $(1 - z)T_{\tau_n}(z) = 1 - G_{\tau_n}(z)$.

By rearranging, $G_{\tau_n}(z) = 1 - (1 - z)T_{\tau_n}(z)$.

$G_{\tau_n}(z) = 1 - (1 - z)(\frac{z}{n} + 1)^n$ ($T_{\tau_n}(z) = (\frac{z}{n} + 1)^n$ from e)

Q5 h

From lecture notes, $f_X(k) = \frac{1}{k!}\frac{d^k}{dz^k}G_X(z)\Big|_{z=0}$.

$$f_{\tau_n}(k) = \frac{1}{k!}\frac{d^k}{dz^k}[1-(1-z)(\tfrac{z}{n}+1)^n]\Big|_{z=0} \tag{1}$$

There is a pattern in the differentiation of $G_{\tau_n}(z)$, which can be generalized after observing up to 3^{rd} derivative.

$G'_{\tau_n}(z) = \frac{(n+1)z\cdot(\frac{z}{n}+1)^n}{z+n}$

$G''_{\tau_n}(z) = \frac{n\cdot(n+1)(z+1)\cdot(\frac{z}{n}+1)^n}{(z+n)^2}$

$G^{(3)}_{\tau_n}(z) = \frac{(n-1)n\cdot(n+1)(z+2)\cdot(\frac{z}{n}+1)^n}{(z+n)^3}$

$G^{(4)}_{\tau_n}(z) = \frac{(n-2)(n-1)n\cdot(n+1)(z+3)\cdot(\frac{z}{n}+1)^n}{(z+n)^4}$

Let $z=0$:

$G'_{\tau_n}(0) = \frac{(n+1)0\cdot(\frac{0}{n}+1)^n}{0+n}$

$G''_{\tau_n}(0) = \frac{n\cdot(n+1)(0+1)\cdot(\frac{0}{n}+1)^n}{(0+n)^2}$

$G^{(3)}_{\tau_n}(0) = \frac{(n-1)n\cdot(n+1)(0+2)\cdot(\frac{0}{n}+1)^n}{(0+n)^3}$

$G^{(4)}_{\tau_n}(0) = \frac{(n-2)(n-1)n\cdot(n+1)(0+3)\cdot(\frac{0}{n}+1)^n}{(0+n)^4}$

Back to $G'_{\tau_n}(0)$, simplify further:

$G'_{\tau_n}(0) = 0$

$G''_{\tau_n}(0) = \frac{n\cdot(n+1)}{n\cdot n}$

$G^{(3)}_{\tau_n}(0) = \frac{(n-1)n\cdot(n+1)2}{n\cdot n\cdot n}$

$G^{(4)}_{\tau_n}(0) = \frac{(n-2)(n-1)n\cdot(n+1)3}{n\cdot n\cdot n\cdot n}$

It is very easy to notice some obvious patterns. Generalization:

$$G^{(k)}_{\tau_n}(0) = (k-1)\cdot\overbrace{\frac{n+1}{n}\cdot\frac{n}{n}\cdot\frac{n-1}{n}\cdot\frac{n-2}{n}\cdots}^{\text{k number of instances}}$$

Back to (1), the probability function $f_{\tau_n}(k) = \frac{1}{k!}G^{(k)}_{\tau_n}(0)$.

$$= \frac{k-1}{k!} \cdot \overbrace{\frac{n+1}{n} \cdot \frac{n}{n} \cdot \frac{n-1}{n} \cdot \frac{n-2}{n} \cdots}^{\text{k number of instances}} \tag{2}$$

$f_{\tau_n}(k) = \frac{k-1}{k!} \frac{(n+1)!}{n^k \cdot (n+1-k)!}$ example: $((n+1)n(n-1) = \frac{(n+1)!}{(n-2)!})$

This can be further reduced because of $\frac{(n+1)!}{k! \cdot (n+1-k)!}$:

$f_{\tau_n}(k) = \frac{k-1}{n^k} \binom{n+1}{k}$

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From (2), $f_{\tau_n}(k) = \frac{k-1}{k!} \cdot \overbrace{\frac{n+1}{n} \cdot \frac{n}{n} \cdot \frac{n-1}{n} \cdot \frac{n-2}{n} \cdots}^{\text{k number of instances}}$

$= \frac{k-1}{k!} \cdot \frac{1+\frac{1}{n}}{1} \cdot \frac{1}{1} \cdot \frac{1-\frac{1}{n}}{1} \cdot \frac{1-\frac{2}{n}}{1} \cdots$ (divide both numerator and denominator by n)

$= \frac{k-1}{k!} \cdot \frac{1}{1} \cdot \frac{1}{1} \cdot \frac{1}{1} \cdot \frac{1}{1} \cdots$ (as $n \to \infty$)

$\therefore \lim_{n \to \infty} f_{\tau_n}(k) = \frac{k-1}{k!}$