ACST3007 Assinment

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Message to markers

Thank you for your time to mark my assignment. I have set out my answers in a way that I might get a full mark.

However, I am wary of losing marks as I am not fully aware of the depth of explaining that must be shown in my workings in each questions. Please review the below outline as my endeavor to find some middle ground where we can agree up to how much is assumed (therefore not explicitly shown in my workings).

$$W_t \sim N(0, t)$$

$$W_t - W_s \sim N(0, t - s)$$

$$Var[z(W_t - W_s)] = z^2 \cdot Var(W_t - W_s) = z^2(t - s)$$

$$e^{W_t - W_s} \sim LN(e^{\mu + \frac{\sigma^2}{2}}, [e^{\sigma^2} - 1] \cdot e^{2\mu + \sigma^2})$$

$$dW_t^2 = dt$$

The above principles were covered in earlier lectures regarding Standard Brownian Motions. I am assuming going through the principles again or explaining them is not necessary in the assignment! On top this, where it seemed very obvious and reasonable to skip calculations, I made my personal decisions as to which information to skip and which to include in my workings. Please take note of these and I hope you could provide some leniency if I did happen to skip some details that were rather expected to be provided!

Thank you!
Yunbae Chae

Question i

$$M_t = e^{2(aW_t - t)} + \frac{1}{a+1}W_t$$

The value of a should be found that makes

$$E[M_t|F_s] = M_s$$

$$= e^{2(aW_s - s)} + \frac{1}{a+1}W_s$$

Start with $E[M_t|F_s]$:

$$E[M_t|F_s] = E[e^{2(aW_t - t)} + \frac{1}{a+1}W_t|F_s]$$

$$= E[e^{2(a[W_s + (W_t - W_s)] - t)} + \frac{1}{a+1}(W_s + (W_t - W_s))|F_s]$$

$$= e^{2(aW_s - t)} \cdot E[e^{2a(W_t - W_s)}|F_s] + \frac{1}{a+1} \cdot (W_s + E[(W_t - W_s)|F_s])$$

Above using $E(W_s|F_s) = W_s,\ E(e^{W_s}|F_s) = e^{W_s}$ (Basic principles).

We know $2a(W_t - W_s) \sim N(0, 4a^2(t-s))$ (Explained in the Inroduction section*). Let $Y_t = W_t - W_s$, then $e^{Y_t} = Z_t$ follows a log-normal distribution:

$$e^{W_t - W_s} \sim LN(e^{\mu + \frac{\sigma^2}{2}}, [e^{\sigma^2} - 1] \cdot e^{2\mu + \sigma^2})$$

$$e^{2a(W_t - W_s)} \sim LN(e^{2a^2(t-s)}, [e^{\sigma^2} - 1] \cdot e^{2\mu + \sigma^2})$$

Please note variance is not calculated as it is not relevant.

$$E[e^{2a(W_t - W_s)}|F_s] = e^{2a^2(t-s)},\ E[(W_t - W_s)|F_s] = 0$$

$$E[M_t|F_s] = e^{2(aW_s-t)} \cdot e^{2a^2(t-s)} + \frac{1}{a+1} \cdot W_s$$

The value of a such that

$$E[M_t|F_s] = e^{2(aW_s-t+a^2(t-s))} + \frac{1}{a+1} \cdot W_s = e^{2(aW_s-s)} + \frac{1}{a+1} W_s$$

This leads to solving

$$aW_s - t + a^2(t-s) = aW_s - s$$

$$a^2(t-s) = t-s$$

$$a^2 = 1$$

$$a = \pm 1$$

but $a \neq -1$ regarding $\frac{1}{a+1}$.

$$\therefore a = 1$$

Question ii

$$dX_t = (4 - 2X_t)dt + dW_t$$

$$\int_0^t dX_s = \int_0^t (4 - 2X_s)ds + \int_0^t dW_s$$

$$X_s|_0^t = [4s - 2sX_s]_0^t + W_s|_0^t$$

$$X_t - X_0 = 4t - 2tX_t + W_t - W_0$$

$$X_t + 2tX_t = 1 + 4t + W_t$$

$$\therefore X_t = \frac{1 + 4t + W_t}{1 + 2t}$$

$$E(X_t) = \frac{1}{1 + 2t} E[1 + 4t + W_t]$$

$$= \frac{1 + 4t}{1 + 2t}$$

Because $E(W_t) = 0$. *Moving on, by manipulating the fraction,*

$$\frac{1 + 4t}{1 + 2t} = \frac{1 + 2t + 2t}{1 + 2t}$$

$$= 1 + \frac{2t}{1 + 2t}$$

$$= 1 + \frac{2}{\frac{1}{t} + 2}$$

$$\lim_{t\to\infty} \frac{1}{t} = 0$$

$$\therefore \lim_{t\to\infty} E(X_t) = 1 + \frac{2}{0+2} = 2$$

$$Var(X_t) = \frac{1}{(1+2t)^2} Var(1 + 4t + W_t)$$

$$= \frac{t}{(1+2t)^2}$$

Because $Var(W_t) = t$.

$$\frac{t}{1 + 4t + 4t^2} = \frac{1}{\frac{1}{t} + 4 + 4t}$$

As $t \to \infty$, this must approach zero as 4t remains in the denominator as the only ever increasing variable as t gets bigger forever.

$$\therefore \lim_{t\to\infty} Var(X_t) = 0$$

Question iii

The assumption of constant volatility is often breached in real market situations. This is because market conditions always change. In real life, in light of changing market conditions, volatility is not constant, but it rather exhibits clustering patterns. It is most commonly observed that periods of high volatility are followed by periods of low volatility and vice versa. For example, when there is a negative news release out of blue, it is most commonly observed that the high volatility continues for a while. As the economy stabilises, the asset prices start to pick up again and a period of stable (low volatility) price rally begins, which holds up for a while before another market crash happens. This is high volatility clustering during a bad market condition (larger price changes mostly downside), followed by a cluster of low volatility during a recovery period (smaller fall and rise of prices), rather than constant range of price rises and falls all the time.

Another Black-Scholes assumption that does not always hold is that the markets should always be continuous and efficient. The ideal Black-Scholes set up is that there is no time gap in trading and all information is available and reflected in stock prices instantaneously. On top of this, there should not be costs in transactions like brokerage, bid-ask spreads and liquidity constraints. Under this real life situation, the trades cannot deal at purely theoretical prices. Prevailing variants in real life markets are trading hours, holidays, the prices not always instantaneously reflecting all information as well as the aforementioned random costs within transactions that make it hard to gauge true theoretical prices.

Question iv

European Call Option

The Black-Scholes Call options formula:

$$C_t = S_t\Phi(d_1) - Ke^{-r(T-t)}\Phi(d_2)$$

$$\frac{dC_t}{dK} = S_t\frac{d\Phi(d_1)}{dK} - e^{-r(T-t)}\Phi(d_2) - Ke^{-r(T-t)}\frac{d\Phi(d_2)}{dK}$$

$$\frac{d\Phi(d_1)}{dK} = \frac{d\Phi(d_1)}{dd_1}\cdot\frac{dd_1}{dK} = \phi(d_1)\frac{dd_1}{dK}$$

Same applies to d_2 as well.

$$d_1 = \frac{ln(\frac{S_t}{K}) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

$$\frac{dd_1}{dK} = \frac{1}{\sigma\sqrt{T-t}}\cdot\frac{-\frac{S_t}{K^2}}{\frac{S_t}{K}} = -\frac{1}{K\sigma\sqrt{T-t}}$$

Here, $d_2 = d_1 - \sigma\sqrt{T-t}$ so $\frac{dd_2}{dK} = \frac{dd_1}{dK}$.

$$X_t = \frac{dC_t}{dK} = -\frac{S_t\phi(d_1)}{K\sigma\sqrt{T-t}} - e^{-r(T-t)}\Phi(d_2) + \frac{e^{-r(T-t)}}{\sigma\sqrt{T-t}}$$

$$X_t = \frac{dC_t}{dK} = -\frac{S_t\phi(d_1)}{K\sigma\sqrt{T-t}} - e^{-r(T-t)}\left[\Phi(d_2) - \frac{1}{\sigma\sqrt{T-t}}\right]$$

European Put Option

The Black-Scholes Put options formula:

$$P_t = -S_t\Phi(-d_1) + Ke^{-r(T-t)}\Phi(-d_2)$$

$$\frac{dP_t}{dK} = -S_t\frac{d\Phi(-d_1)}{dK} + e^{-r(T-t)}\Phi(-d_2) + Ke^{-r(T-t)}\frac{d\Phi(-d_2)}{dK}$$

$$\frac{d\Phi(-d_1)}{dK} = \frac{d\Phi(-d_1)}{dd_1}\cdot\frac{dd_1}{dK} = \phi(d_1)\frac{dd_1}{dK}$$

Because $\frac{d\Phi(z)}{dz} = \frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}} \Rightarrow \frac{d\Phi(-z)}{dz} = \frac{1}{\sqrt{2\pi}}e^{-\frac{(-z)^2}{2}} = \frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}} = \frac{d\Phi(z)}{dz}$

Same applies to d_2 as well.

$$d_1 = \frac{ln(\frac{S_t}{K}) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

$$\frac{dd_1}{dK} = \frac{1}{\sigma\sqrt{T-t}}\cdot\frac{-\frac{S_t}{K^2}}{\frac{S_t}{K}} = -\frac{1}{K\sigma\sqrt{T-t}}$$

Here, $-d_2 = -d_1 + \sigma\sqrt{T-t}$ so $\frac{dd_2}{dK} = \frac{dd_1}{dK}$.

$$X_t = \frac{dP_t}{dK} = \frac{S_t\phi(d_1)}{K\sigma\sqrt{T-t}} + e^{-r(T-t)}\Phi(d_2) - \frac{e^{-r(T-t)}}{\sigma\sqrt{T-t}}$$

$$X_t = \frac{dP_t}{dK} = \frac{S_t\phi(d_1)}{K\sigma\sqrt{T-t}} + e^{-r(T-t)}\left[\Phi(d_2) - \frac{1}{\sigma\sqrt{T-t}}\right]$$

Question v

Question v a

$$dS_t = S_t \mu dt + S_t \sigma dW_t$$

Let $X_t = lnS_t$.

$$\frac{\partial X_t}{\partial t} = 0$$

$$\frac{\partial X_t}{\partial S_t} = \frac{1}{S_t}$$

$$\frac{\partial^2 X_t}{\partial {S_t}^2} = -\frac{1}{S_t^2}$$

Using Using Itô's Lemma,

$$dX_t = 0 + \frac{1}{S_t} dS_t - \frac{1}{2}\frac{1}{S_t^2}(dS_t)^2$$

$$(dS_t)^2 = (\mu S_t dt + \sigma S_t dW_t)^2 = \mu^2 S_t^2 dt^2 + 2\mu\sigma S_t^2 dt dW_t + \sigma^2 S_t^2 dW_t^2$$

dt^2 and $dtdW_t$ are negligibly small. $dW_t^2 = dt$ (Basic principle)

$$\therefore (dS_t)^2 = \sigma^2 S_t^2 dt$$

Using $dS_t = \mu S_t dt + \sigma S_t dW_t$ again,

$$dX_t = \frac{1}{S_t}(\mu S_t dt + \sigma S_t dW_t) - \frac{1}{2}\frac{1}{S_t^2}\sigma^2 S_t^2 dt$$

$$dlnS_t = (\mu - \frac{1}{2}\sigma^2)dt + \sigma dW_t$$

$$\int_0^t dlnS_s = \int_0^t (\mu - \frac{1}{2}\sigma^2)ds + \int_0^t \sigma dW_s$$

$$ln\frac{S_t}{S_0} = (\mu - \frac{1}{2}\sigma^2)t + \sigma W_t$$

$$\Rightarrow \frac{S_t}{S_0} = e^{(\mu-\frac{1}{2}\sigma^2)t+\sigma W_t}$$

$$S_t = S_0e^{(\mu-\frac{1}{2}\sigma^2)t+\sigma W_t}$$

$$\tilde{S}_t = e^{qt}S_t$$

$$= S_0e^{(\mu+q-\frac{1}{2}\sigma^2)t+\sigma W_t}$$

Using given information on parameters (Under measure P),

$$\tilde{S}_t = 100e^{(0.05+0.01-\frac{1}{2}0.15^2)t+0.15W_t}$$

$$\tilde{S}_t = 100e^{0.04875\cdot t+0.15\cdot W_t}$$

Question v b

From Question v a,

$$\tilde{S}_t = S_0 e^{(\mu + q - \frac{1}{2}\sigma^2)t + \sigma W_t}$$

Call the discounted value of $\tilde{S}_t$, Z_t.

$$\Rightarrow Z_t = e^{-rt}\tilde{S}_t$$

Using Itô's Lemma again, $dZ_t = \frac{\partial Z_t}{\partial t}dt + \frac{\partial Z_t}{\partial \tilde{S}_t}d\tilde{S}_t + \frac{\partial^2 Z_t}{\partial \tilde{S}_t^{\,2}}(d\tilde{S}_t)^2$.

$$\frac{\partial Z_t}{\partial t} = -re^{-rt}\tilde{S}_t = -rZ_t$$

$$\frac{\partial Z_t}{\partial S_t} = e^{-rt}$$

From basic principles, $d\tilde{S}_t = \tilde{S}_t[(\mu + q)dt + \sigma dW_t]$:

$$dZ_t = -rZ_t dt + e^{-rt}\tilde{S}_t[(\mu + q)dt + \sigma dW_t]$$

$$= -rZ_t dt + Z_t(\mu + q)dt + Z_t \sigma dW_t$$

$$= Z_t[(\mu + q - r)dt + \sigma dW_t]$$

Using the CMG theorem (W_t under measure P, W_t^Q under measure Q),

$$dW_t^Q = dW_t + \gamma_t dt$$

$$\Rightarrow dW_t = dW_t^Q - \gamma_t dt$$

$$dZ_t = Z_t(\mu + q - r)dt + Z_t\sigma(dW_t^Q - \gamma_t dt)$$

$$dZ_t = Z_t(\mu + q - r - \gamma_t\sigma)dt + Z_t\sigma dW_t^Q$$

Using the same method in solving for Question v a,

$$Z_t = Z_0 e^{(\mu+q-r-\gamma_t\sigma-\frac{1}{2}\sigma^2)t+\sigma W_t^Q}$$

Under measure Q, Z_t is a martingale.

$$E^Q(Z_t|F_0) = E^Q(Z_0 e^{(\mu+q-r-\gamma_t\sigma-\frac{1}{2}\sigma^2)t+\sigma W_t^Q}|F_0)$$

$$= Z_0 e^{(\mu+q-r-\gamma_t\sigma-\frac{1}{2}\sigma^2)t} E^Q(e^{\sigma W_t^Q}|F_0)$$

$$= Z_0 e^{(\mu+q-r-\gamma_t\sigma)t}$$

$E^Q(Z_t|F_0)$ has to equal to Z_0.

$$\Rightarrow \mu + q - r - \gamma_t\sigma = 0$$

$$\Rightarrow \gamma_t = \frac{\mu + q - r}{\sigma} = \frac{0.04}{0.15}$$

$$dW_t = dW_t^Q - \gamma_t dt$$

$$dS_t = S_t(\mu dt + \sigma dW_t)$$

$$= S_t[\mu dt + \sigma(dW_t^q - \gamma_t dt)]$$

$$= S_t[(\mu - \gamma_t\sigma)dt + \sigma dW_t^Q]$$

Therefore the SDE satisfied by the stock price S_t under the risk-neutral measure Q is:

$$dS_t = S_t[(\mu - \gamma_t \sigma)dt + \sigma dW_t^Q]$$

Question v c

From b, the expectation under Q is martingale. In this way, the expectation of the stock price under Q at any time s is the stock price at that time s:

$$E^Q(S_t|F_s) = S_s$$

Question v d

from c,

$$E^Q(S_t|F_s) = S_s$$

Using the sample paths, each S_s's can be estimated using sample mean.

$$\Rightarrow \hat{S}_s = \frac{1}{100}\sum_{n=1}^{100} S_{s,n}$$

Each means at times 0, 0.01, 0.02, $\cdots$, 1 are collected and plotted in the Excel file.

Question v e

from a,

$$S_t = S_0 e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t}$$

$$E^P(S_t) = S_0 e^{(\mu - \frac{1}{2}\sigma^2)t} E^P(e^{\sigma W_t})$$

Using results already generated in previous questions regarding $E^P(e^{\sigma W_t})$,

$$E^P(S_t) = S_0 e^{\mu t}$$

This is also set out and plotted in Excel.

Question v f

The formula in Excel is set up based on the general Black-Scholes formula. Using different volatility figures (0.15, 0.16, $\cdots$), I found that the call value of 6 (dollars) lies between $\sigma = 0.23$ and 0.24. Using interpolation, more accurate value was found and displayed in cell J228.