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Problem 1

(a)

```
function T = tridiag(A)
m = length(A);

for k=1:m-2
    tmp = eye(m-k); e1 = tmp(:,1);
    x = A(k+1:m,k);
    vk = sign(x(1)) * norm(x) * e1 + x;
    vk = vk / norm(vk);

    A(k+1:m, k:m) = A(k+1:m, k:m) - 2 * vk * (vk'* A(k+1:m, k:m));
    A(1:m, k+1:m) = A(1:m, k+1:m) - 2 * (A(1:m, k+1:m) * vk) * vk';
end

for k=1:m-1
     A(k+2:m, k) = 0;
     A(k, k+1:m) = A(k+1:m, k)';
end
T = A;
end
```

Test for A=hilb(4)

```
%%%%% Problem 1 %%%%%
A = hilb(4);

%(a)
tridiag(A);
```

yields

```
ans =

    1.0000   -0.6509         0         0
   -0.6509    0.6506    0.0639         0
         0    0.0639    0.0253   -0.0012
         0         0   -0.0012    0.0003
```

(b)

```
function [Tnew, sawtooth] = qralg(T)
% input : tridiagonal matrix T
m = length(T);
counter = 1;
tm = zeros(1,1000);
while 1
    tm(1, counter) = T(m, m-1);
    if abs(T(m,m-1)) < 1e-12
        Tnew = T;
        sawtooth = tm(1,1:counter);
        break
    end

    [q, r] = qr(T);
    T = r*q;
    for k=1:m-1
        T(k+2:m, k) = 0;
        T(k, k+1:m) = T(k+1:m, k)';
    end
    counter = counter + 1;
end
end
```

For (c), I made this function with one more output (sawtooth), which is not described in (b). The following is the test for A=hilb(4).

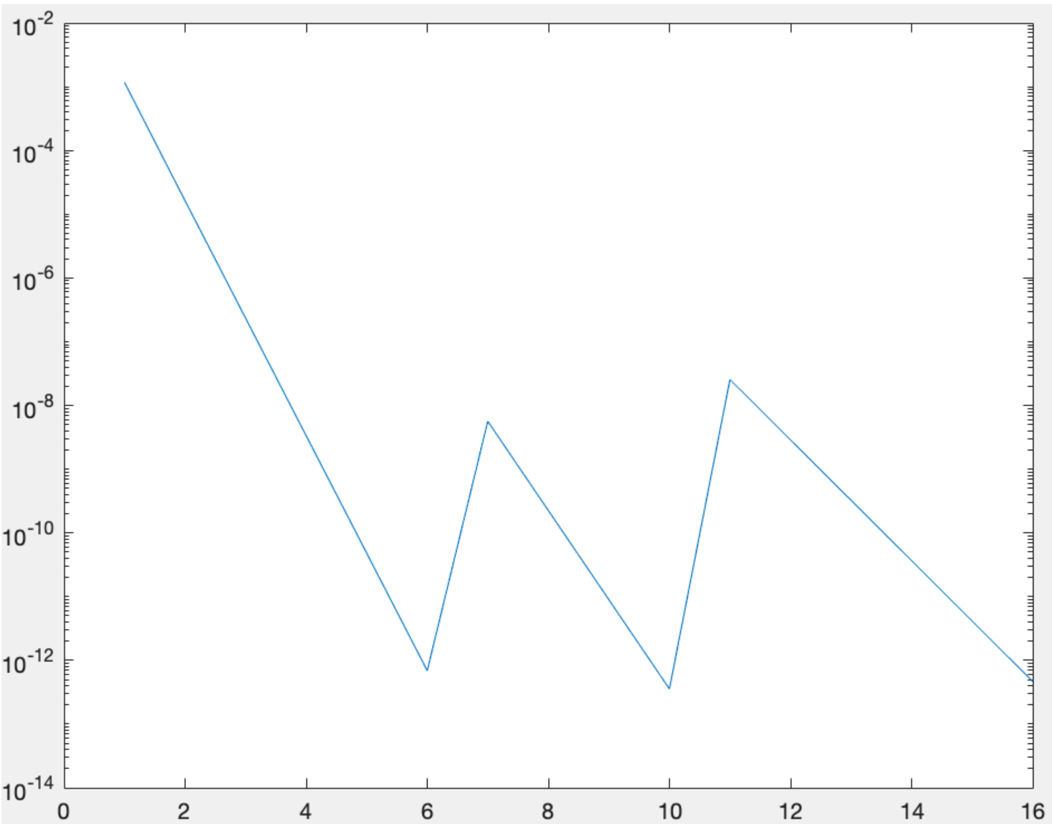
```
%(b)
[T, ~] = qralg(tridiag(A)); T;
ans =

    1.5002   -0.0000         0         0
   -0.0000    0.1691    0.0000         0
         0    0.0000    0.0067    0.0000
         0         0    0.0000    0.0001
```

(c)

```
%(c)
m = length(A); reduced_A = tridiag(A);
ew = zeros(1, m);
sawtooth = [0];
for i=1:m-1
    [newT, new_sawtooth] = qralg(reduced_A);
    sawtooth = [sawtooth, new_sawtooth];
    len = length(newT);
    ew(1, i) = newT(len, len);
    reduced_A = newT(1:len-1, 1:len-1);
end
ew(1, m) = newT(1,1); ew
figure;
semilogy(sawtooth(1, 2:end));
```

The output：eigenvalues (ew) and sawtooth plot

```
ew =

    0.0001    0.0067    0.1691    1.5002
```



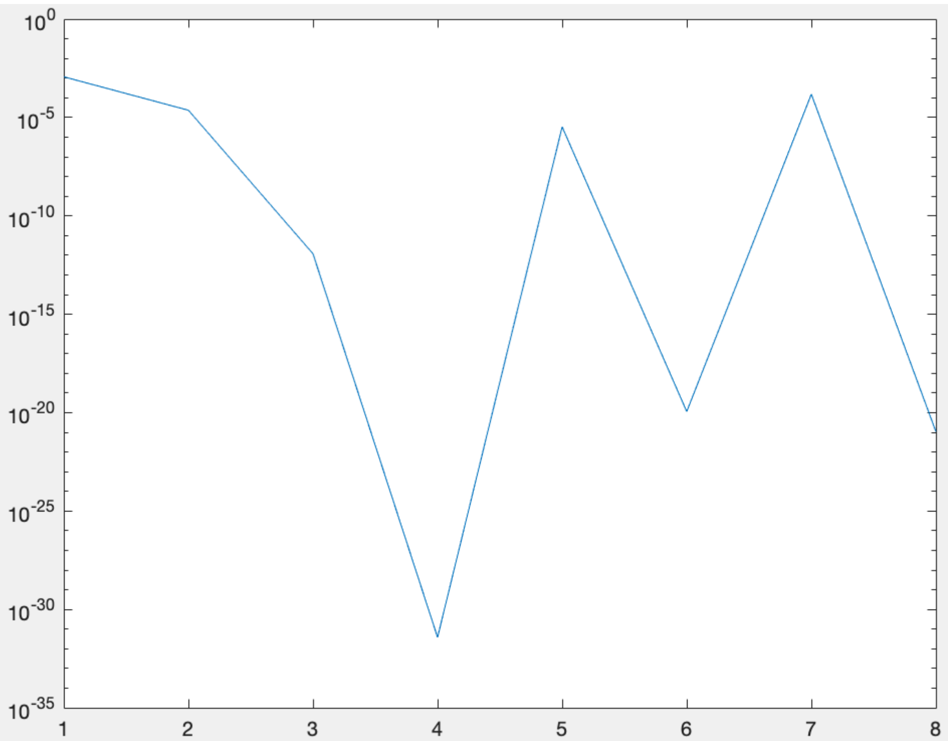
(d)

```
function [Tnew, sawtooth] = Wqralg(T)
% input : tridiagonal matrix T
m = length(T);
counter = 1;
tm = zeros(1,1000);
while 1
    tm(1, counter) = abs(T(m, m-1));
    if abs(T(m,m-1)) < 1e-12
        Tnew = T;
        sawtooth = tm(1,1:counter);
        break
    end

    shift = Wikinson(T(m-1:m, m-1:m));
    [q, r] = qr(T - shift * eye(m));
    T = r*q + shift * eye(m);
    for k=1:m-1
        T(k+2:m, k) = 0;
        T(k, k+1:m) = T(k+1:m, k)';
    end
    counter = counter + 1;
end
end
```

```
function shift = Wikinson(M)
% input : 2 x 2 matrix M
d = (M(1, 1) - M(2,2))/2;
sgn = sign(d);
if d == 0
    sgn = 1;
end
denom = abs(d)+sqrt(d^2+M(2,1)^2);
shift = M(2,2)-sgn*M(2,1)^2/denom;
end
```

Rather than modifying qralg, I just implemented qralg with Wikinson shift (Wqralg). The following is the result. (eigenvalues and Sawtooth plot)

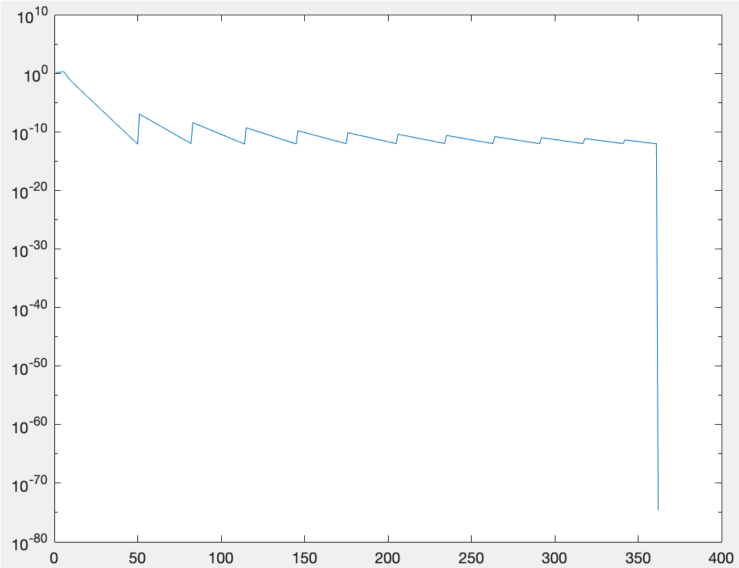
```
ew =

    0.0001    0.0067    0.1691    1.5002
```



(e)

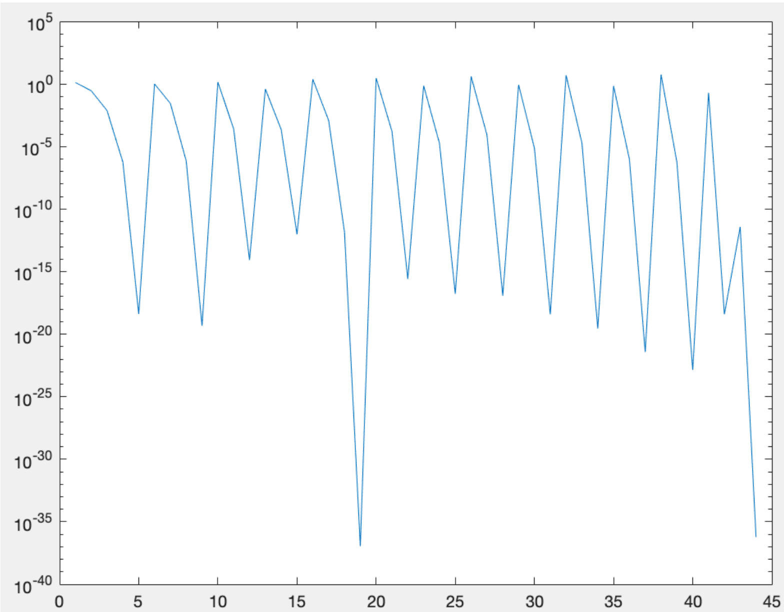
(qralg)

```
ew =

  Columns 1 through 10

    6.3629    7.3871    5.3390    8.4121    9.4387   10.4679    4.3143   11.5010    3.2878   12.5402

  Columns 11 through 15

    2.2570   13.5901    1.2147   14.6641   24.2231
```



(Wqralg)

```
ew =

  Columns 1 through 10

    1.2147    2.2570    3.2878    4.3143    5.3390    6.3629    7.3871    8.4121    9.4387   10.4679

  Columns 11 through 15

   11.5010   12.5402   13.5901   14.6641   24.2231
```



Here, note that the order in ew is bit different between two methods, but they yield the same set of eigenvalues.

When we take a look at the Sawtooth graph from simple qralg, the semi-log value of $|t_{m,m-1}|$ decreases linearly during each tooth (interval). As seen in the graph, the slope of decreases getting smaller along with each tooth (interval). Hence, depending on the distribution of the eigenvalues, the rates of convergence changes from superlinear(possibly starting with sublinear) to sublinear.

In the case of Wilkinson shift QR (Wqralg), during each tooth, the semi-log value decreases in the shape of parabola, which means that the rates of convergence is at least quadratic. Roughly, the power of $|t_{m,m-1}|$ (the "k" from 10^k) almost triples the previous power. Hence, I guess it's cubic.

For A=hilb(4), the number of QR iteration per eigenvalue (call it #) for qralg is 4 and # for Wqralg is 2. For A in (e), # for qralg is 24 and # for Wqralg is 3. From the #, we can roughly guess the rates of convergence.