Variational Problems in Multiple Dimensions: Without Constraints

Multiple Dimension Problems

Problem: Optimize $J = \int_{t_0}^{t_f} L\left[X(t), \dot{X}(t), t\right] dt$ by appropriate selection of $X(t)$.

where $X \triangleq \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}^T$

Solution: Make sure $\delta J = 0$ for arbitrary $\delta X(t)$

t_0, t_f : Fixed

Necessary Conditions:

1) Euler – Lagrange (E-L) Equation

$$\boxed{\frac{\partial L}{\partial X} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{X}}\right) = 0}$$

2) Transversality (Boundary) Condition

$$\boxed{\left[\left(\frac{\partial L}{\partial \dot{X}}\right)^T \delta X\right]_{t_0}^{t_f} = 0}$$

Proof

Necessary condition: $\delta J = \int_{t_0}^{t_f} \left[\left(\frac{\partial L}{\partial X} \right)^T \delta X + \left(\frac{\partial L}{\partial \dot{X}} \right)^T \delta \dot{X} \right] dt = 0$

However,

$$\int_{t_0}^{t_f} \left[\left(\frac{\partial L}{\partial \dot{X}} \right)^T \delta \dot{X} \right] dt = \int_{t_0}^{t_f} \left[\left(\frac{\partial L}{\partial \dot{X}} \right)^T \frac{d(\delta X)}{dt} \right] dt$$

$$= \left[\left(\frac{\partial L}{\partial \dot{X}} \right)^T \delta X \right]_{t_o}^{t_f} - \int_{t_0}^{t_f} \left[\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{X}} \right) \right]^T \delta X \, dt$$

Proof

$$\delta J = \int_{t_0}^{t_f} \left[\left(\frac{\partial L}{\partial X} \right)^T \delta X + \left(\frac{\partial L}{\partial \dot{X}} \right)^T \delta \dot{X} \right] dt$$

$$= \int_{t_0}^{t_f} \left(\frac{\partial L}{\partial X} \right)^T \delta X \, dt + \left[\left(\frac{\partial L}{\partial \dot{X}} \right)^T \delta X \right]_{t_o}^{t_f} - \int_{t_0}^{t_f} \left[\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{X}} \right)^T \right] \delta X \, dt$$

$$= \int_{t_0}^{t_f} \left[\left(\frac{\partial L}{\partial X} \right) - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{X}} \right) \right]^T \delta X \, dt + \left[\left(\frac{\partial L}{\partial \dot{X}} \right)^T \delta X \right]_{t_o}^{t_f}$$

$$= 0 \quad \text{(Necessary condition of optimality)}$$

Proof

- **Necessary Conditions:**

1) **Euler – Lagrange (E-L) Equation**

$$\boxed{\left(\frac{\partial L}{\partial X}\right)-\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{X}}\right)=0}$$

2) **Transversality (Boundary) Condition**

$$\boxed{\left[\left(\frac{\partial L}{\partial \dot{X}}\right)^T \delta X\right]_{t_0}^{t_f}=0}$$

Transversality Condition

- General condition

$$\left[\left(\frac{\partial L}{\partial \dot{X}}\right)^{T} \delta X\right]_{t_0}^{t_f} + \left[\left\{L - \dot{X}^{T}\left(\frac{\partial L}{\partial \dot{X}}\right)\right\} \delta t\right]_{t_0}^{t_f} = 0$$

- Special case: Similar to scalar case

Variational Problems in Multiple Dimensions: With Constraints

Variational Problems with Constraints

$$\text{Optimize}: J = \int_{t_0}^{t_f} L\left(X, \dot{X}, t\right) dt$$

$$\text{Subject to}: \quad \Phi\left(X, \dot{X}, t\right) = 0$$

where

$$X \triangleq \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}^T, \quad \Phi \triangleq \begin{bmatrix} \varphi_1 & \varphi_2 & \cdots & \varphi_{\tilde{n}} \end{bmatrix}^T$$

Variational Problems with Constraints

- Lagrange's Existence Theorem

$\exists\, \lambda_{\bar{n}\times 1}(t)$: The above constrained optimization problem leads to the same solution as the following unconstrained cost functional

$$\bar{J} = \int_{t_0}^{t_f} \left[L\left(X, \dot{X}, t\right) + \lambda^T \Phi\left(X, \dot{X}, t\right) \right] dt$$

$$\text{Let } L^*\left(X, \dot{X}, \lambda, t\right) = L\left(X, \dot{X}, t\right) + \lambda^T \Phi\left(X, \dot{X}, t\right)$$

Variational Problems with Constraints

- Necessary Conditions of Optimality:

(1) E-L Equations:

$$\text{(a)}\ \frac{\partial L^*}{\partial X} - \frac{d}{dt}\left[\frac{\partial L^*}{\partial \dot{X}}\right] = 0 \ \ (\ n \text{ equations})$$

$$\text{(b)}\ \frac{\partial L^*}{\partial \lambda} - \frac{d}{dt}\cancel{\left[\frac{\partial L^*}{\partial \dot{\lambda}}\right]} = 0 \ \ (\ \tilde{n} \text{ equations})$$

$$\left(\text{Note: } \frac{\partial L^*}{\partial \dot{\lambda}} = 0 \text{ as there is no } \dot{\lambda} \text{ term in } L^*\right)$$

Variational Problems with Constraints

- (2) Transversality Conditions

$$\text{(a)}\ \left[\left(\frac{\partial L^*}{\partial \dot{X}}\right)^T \delta X\right]_{t_o}^{t_f} + \left[\left\{L^* - \dot{X}^T\left(\frac{\partial L^*}{\partial \dot{X}}\right)\right\}\delta t\right]_{t_o}^{t_f} = 0$$

$$\text{(b)}\ \left[\cancel{\left(\frac{\partial L^*}{\partial \dot{\lambda}}\right)^T} \delta \lambda\right]_{t_o}^{t_f} + \left[\left\{L^* - \dot{\lambda}^T\cancel{\left(\frac{\partial L^*}{\partial \dot{\lambda}}\right)}\right\}\delta t\right]_{t_o}^{t_f} = 0$$

Variational Problems with Constraints

Varaibles: $n \quad + \quad \tilde{n} \quad + 1$
$\qquad (X) \quad (\lambda) \quad (t_f)$
Boundary Conditions : $n + \tilde{n} + 1$

E-L Equations:

1) (a) $\left(\dfrac{\partial L^*}{\partial X}\right) - \dfrac{d}{dt}\left(\dfrac{\partial L^*}{\partial \dot{X}}\right) = 0$

(b) $\left(\dfrac{\partial L^*}{\partial \lambda}\right) = \Phi\left(X, \dot{X}, t\right) = 0$ (same constraint equation)

2)Transversality Conditions: $\left(t_0, X_0\right)$ fixed, $\left(t_f, X_f\right)$ free

(a) $\left(\dfrac{\partial L^*}{\partial \dot{X}}\right)^T_{t_f} \delta X_f + \left[L^* - \dot{X}^T \left(\dfrac{\partial L^*}{\partial \dot{X}}\right)\right]_{t_f} \delta t_f = 0$ ($\tilde{n}$ equations)

(b) $L^*_{t_f} \delta t_f = 0$ However t_f is free $\Rightarrow$ $\delta t_f \neq 0$

so $L^*_{t_f} = 0$ (1 equation)

Constraint Equations

- Nonholonomic constraints

$$\Phi\left(X, \dot{X}, t\right) = 0$$

- Isoperimetric constraints

$$\int_{t_0}^{t_f} q\left(X, \dot{X}, t\right) dt = k$$

One way to get rid of Isoperimetric constraints is to convert them into Nonholonomic constraints.

Isoperimetric Constraints

- Define $\dot{x}_{n+1} = q\left(X, \dot{X}, t\right)$
- Then

$$\int_{t_0}^{t_f} \dot{x}_{n+1} dt = \int_{t_0}^{t_f} q\left(X, \dot{X}, t\right) dt = k$$

$$x_{n+1}(t_f) - x_{n+1}(t_0) = k$$

Choose one of $x_{n+1}(t_f)$ or $x_{n+1}(t_0)$ and fix the other

Let $x_{n+1}(t_0) = 0$

$$x_{n+1}(t_f) = k$$

Isoperimetric Constraints

- Summary

The following additional non-holonomic Constraint is introduced: $\dot{x}_{n+1} = q\left(X, \dot{X}, t\right)$

with boundary conditions:

$$x_{n+1}(t_0) = 0$$

$$x_{n+1}(t_f) = k$$

The original problem is augmented with this information and solved.

Example: Constrained Problem

$$\text{Minimize } J = \int_0^1 (x_1^2 + x_2^2)\,dt$$

with $x_1(0) = 1,\ x_1(1) = 0$

Subject to: $\dot{x}_1 = -x_1 + x_2$

$$\begin{bmatrix} \text{Note: Here } x_2(t) \text{ can be considered as } u(t) \\ \text{i.e. like a control variable.} \end{bmatrix}$$

Example: Constrained Problem

- Method-1: Direct substitution

$$x_2 = \dot{x}_1 + x_1$$

$$J = \int_0^1 (x_1^2 + (\dot{x}_1 + x_1)^2)dt$$

$$L = x_1^2 + (\dot{x}_1 + x_1)^2$$

E-L Equation :

$$\frac{\partial L}{\partial x_1} - \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{x}_1}\right] = 0$$

$$2x_1 + 2(\dot{x}_1 + x_1) - \frac{d}{dt}[2(\dot{x}_1 + x_1)] = 0$$

Example: Constrained Problem

$$2x_1 + \cancel{\dot{x}_1} - \ddot{x}_1 - \cancel{\dot{x}_1} = 0$$

$$\ddot{x}_1 - 2x_1 = 0$$

Characteristic equation:

$$\lambda^2 - 2 = 0$$

$$\lambda = \pm\sqrt{2}$$

$$x_1(t) = c_1 e^{\sqrt{2}\,t} + c_2 e^{-\sqrt{2}\,t} \;\Rightarrow\; x_2 = \dot{x}_1 + x_1$$

Boundary condition:

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{pmatrix} 1 & 1 \\ e^{\sqrt{2}} & e^{-\sqrt{2}} \end{pmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

Example: Constrained Problem

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \left(\frac{1}{e^{-\sqrt{2}} - e^{\sqrt{2}}} \right) \begin{pmatrix} e^{-\sqrt{2}} & 1 \\ -e^{\sqrt{2}} & 1 \end{pmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Note: This method may appear simple. However, it is not always convenient. (especially for higher dimensional problem)

Example: Constrained Problem

- Method-2 (Lagrange approach)

$$L^* = (x_1^2 + x_2^2) + \lambda(\dot{x}_1 + x_1 - x_2)$$

E-L Equation

$$\frac{\partial L^*}{\partial x_1} - \frac{d}{dt}\left[\frac{\partial L^*}{\partial \dot{x}_1}\right] = 0$$

$$\frac{\partial L^*}{\partial x_2} - \frac{d}{dt}\cancel{\left[\frac{\partial L^*}{\partial \dot{x}_2}\right]} = 0 \quad \Rightarrow \quad \frac{\partial L^*}{\partial x_2} = 0$$

$$\frac{\partial L^*}{\partial \lambda} - \frac{d}{dt}\cancel{\left[\frac{\partial L^*}{\partial \dot{\lambda}}\right]} = 0 \quad \Rightarrow \quad \frac{\partial L^*}{\partial \lambda} = 0$$

Example: Constrained Problem

These equations lead to:

$$(2x_1 + \lambda) - \dot{\lambda} = 0 \qquad (1a)$$

$$2x_2 - \lambda = 0 \qquad (1b)$$

$$\dot{x}_1 = -x_1 + x_2 \qquad (1c)$$

(1b) & (1c) $\Rightarrow$

$$\lambda = 2x_2 = 2(\dot{x}_1 + x_1) \qquad (2)$$

$$\dot{\lambda} = 2(\ddot{x}_1 + \dot{x}_1) = 2x_1 + 2(\dot{x}_1 + x_1)$$

$$\ddot{x}_1 = 2x_1$$

Same equation as before. Hence, proceed the same way!

Example: Constrained Problem

Finally:

$$x_1(t) = c_1 e^{\sqrt{2}\,t} + c_2 e^{-\sqrt{2}\,t}$$

$$x_2(t) = \dot{x}_1 + x_1 = \left(\sqrt{2}+1\right) c_1 e^{\sqrt{2}\,t} + \left(-\sqrt{2}+1\right) c_1 e^{-\sqrt{2}\,t}$$

- Note:

If $x_2(t) = u(t)$ (a control variable), then
we have actually solved an "optimal control problem"!

Maximum Radius Orbit Transfer at a Given Time

Problem :

Given a constant thrust (T) rocket engine operating for a fixed t_f,

find the thrust direction history $\varphi(t)$ to transfer the rocket vehicle

from a given initial circular orbit to the largest possible circular orbit

Solution :

u : radial component of velocity

v : tangential component of velocity

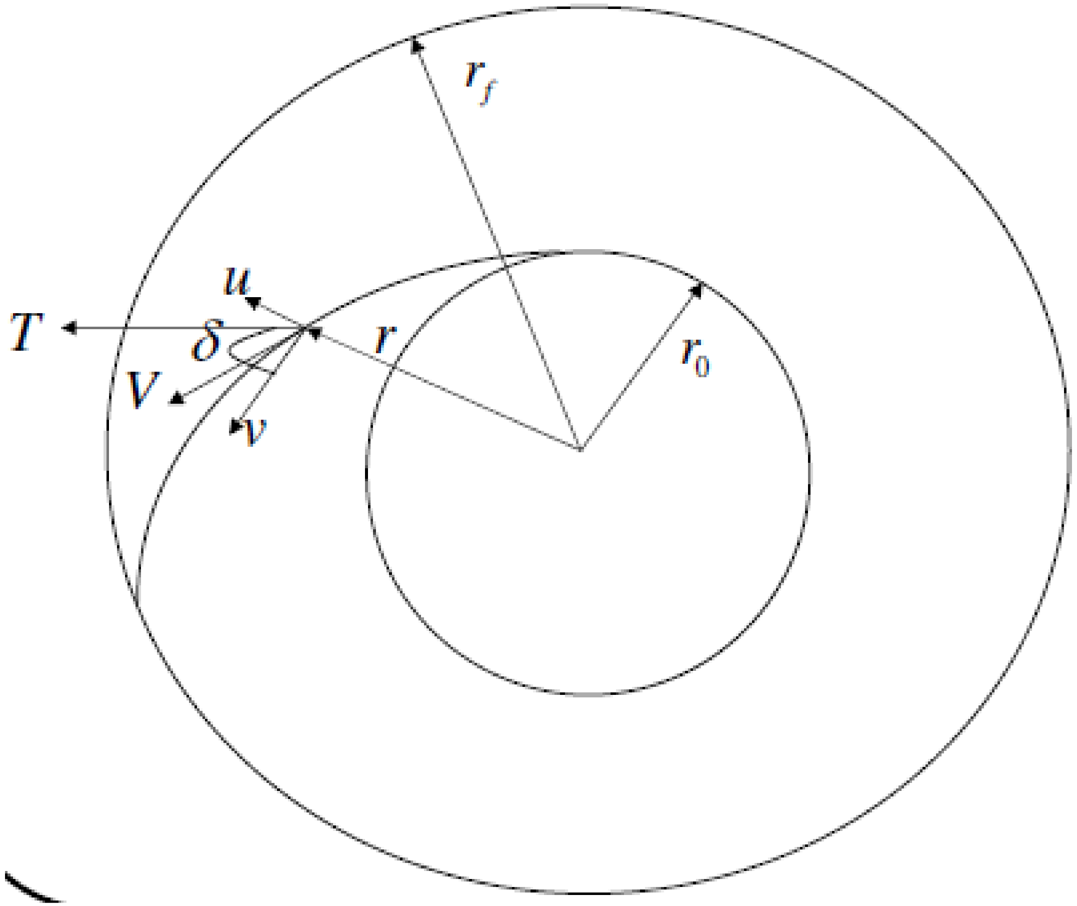
m : mass of vehicle $= m_0 - \dot{m}t$

μ : gravitational constant of attracting centre

r : radial distance of space craft from attracting centre

δ : thrust deffection angle

Maximum Radius Orbit Transfer at a Given Time



$$\dot{u} = \frac{v^2}{r} - \frac{\mu}{r^2} + \frac{T\sin\delta}{m_0 - \dot{m}t}$$

$$\dot{v} = -\frac{uv}{r} + \frac{T\cos\delta}{m_0 - \dot{m}t}$$

System Dynamics and B.C.

System dynamics

$$\dot{r} = u$$

$$\dot{u} = \frac{v^2}{r} - \frac{\mu}{r^2} + \frac{T\sin\delta}{m_0 - \dot{m}t}$$

$$\dot{v} = -\frac{uv}{r} + \frac{T\cos\delta}{m_0 - \dot{m}t}$$

Boundary conditions

At $t = t_0$

$$\begin{bmatrix} r(0) \\ u(0) \\ v(0) \end{bmatrix} = \begin{bmatrix} r_0 \\ 0 \\ \sqrt{\mu / r_0} \end{bmatrix}$$

At $t = t_f$

$$\Psi_1 = u_f = 0$$

$$\Psi_2 = \left(v_f - \sqrt{\mu / r_f}\right) = 0$$

Performance index

Cost function: $J = r_f$ (to be maximized)

Solution:

$$H = \lambda_r u + \lambda_u \left(\frac{v^2}{r} - \frac{\mu}{r^2} + \frac{T \sin \delta}{m_0 - \dot{m}t} \right) + \lambda_v \left(-\frac{uv}{r} + \frac{T \cos \delta}{m_0 - \dot{m}t} \right)$$

$$\Phi = r_f + \upsilon_1 u_f + \upsilon_2 \left[v_f - \sqrt{\mu / r_f} \right]$$

Necessary Condition

(1) State Eq.

$$\dot{r} = u$$

$$\dot{u} = \frac{v^2}{r} - \frac{\mu}{r^2} + \frac{T\sin\delta}{m_0 - \dot{m}t}$$

$$\dot{v} = -\frac{uv}{r} + \frac{T\cos\delta}{m_0 - \dot{m}t}$$

(3) Costate Eq.

$$\dot{\lambda}_r = -\frac{\partial H}{\partial r} = -\lambda_u\left(-\frac{v^2}{r^2} + \frac{2\mu}{r^3}\right) - \lambda_v\left(\frac{uv}{r^2}\right)$$

$$\dot{\lambda}_u = -\frac{\partial H}{\partial u} = -\lambda_r + \lambda_v\left(\frac{v}{r}\right)$$

$$\dot{\lambda}_v = -\frac{\partial H}{\partial v} = -\lambda_u\frac{2v}{r} + \lambda_v\left(\frac{u}{r}\right)$$

(2) Optimal Control Eq.

$$\frac{\partial H}{\partial \delta} = (\lambda_u\cos\delta - \lambda_v\sin\delta)\left(\frac{T}{m_0 - \dot{m}t}\right) = 0$$

This leads to: $\tan\delta = \left(\frac{\lambda_u}{\lambda_v}\right)$

$$\delta = \tan^{-1}\left(\frac{\lambda_u}{\lambda_v}\right)$$

Necessary Condition

(4) Boundary Condition:

$$\text{At } t = t_0 \quad \begin{bmatrix} r(0) \\ u(0) \\ v(0) \end{bmatrix} = \begin{bmatrix} r_0 \\ 0 \\ \sqrt{\mu / r_0} \end{bmatrix} : \text{Known}$$

$$\text{At } t = t_f, \quad \lambda_{r_f} = 1 + \frac{\upsilon_2 \sqrt{\mu}}{2 r_f^{3/2}}$$

$$\lambda_{u_f} = \upsilon_1$$

$$\lambda_{v_f} = \upsilon_2$$

$$u_f = 0$$

$$v_f = \sqrt{\mu / r_f} \qquad \text{(sufficient boundary conditions exist)}$$

However, this is a complex problem and needs numerical algorithms to solve!

Minimum-drag nose shape in hypersonic flow