Chapter 5 The Variational Approach to Optimal Control Problems

5.1 NECESSARY CONDITIONS FOR OPTIMAL CONTROL

The problem is to find an admissible control $\mathbf{u}^*$ that causes the system

$$\dot{\mathbf{x}}(t) = \mathbf{a}(\mathbf{x}(t), \mathbf{u}(t), t)$$

to follow an admissible trajectory $\mathbf{x}^*$ that minimizes the performance

measure

$$\mathbf{J}(u) = h(\mathbf{x}(t_f), t_f) + \int_{t_0}^{t_f} g(\mathbf{x}(t), \mathbf{u}(t), t)dt \,.$$

Assuming that h is differentiable function, we can write

$$h(\mathbf{x}(t_f), t_f) = \int_{t_0}^{t_f} \frac{d}{dt}[h(\mathbf{x}(t), t)]dt + h(\mathbf{x}(t_0), t_0)$$

So that performance measure can be expressed as

$$\mathbf{J}(u) = \int_{t_0}^{t_f} \left\{ (g(\mathbf{x}(t), \mathbf{u}(t), t) + \frac{d}{dt}[h(\mathbf{x}(t), t)] \right\} dt + h(\mathbf{x}(t_0), t_0)$$

Since the term $h(\mathbf{x}(t_0), t_0)$ is fixed, we minimize

$$\mathbf{J}(\mathbf{u}) = \int_{t_0}^{t_f} \left\{ (g(\mathbf{x}(t), \mathbf{u}(t), t) + \frac{d}{dt}[h(\mathbf{x}(t), t)] \right\} dt$$

Using chain rule of differentiation,

$$\mathbf{J}(\mathbf{u}) = \int_{t_0}^{t_f} \left\{ (g(\mathbf{x}(t), \mathbf{u}(t), t) + \left[\frac{\partial h}{\partial x}(\mathbf{x}(t), t) \right]^T \dot{\mathbf{x}}(t) + \frac{\partial h}{\partial t}(\mathbf{x}(t), t) \right\} dt$$

Using the Lagrange multipliers

$$\mathbf{J_a}(\mathbf{u}) = \int_{t_0}^{t_f} \left\{ (g(\mathbf{x}(t), \mathbf{u}(t), t) + \left[\frac{\partial h}{\partial \mathbf{x}}(\mathbf{x}(t), t) \right]^T \dot{\mathbf{x}}(t) + \frac{\partial h}{\partial t}(\mathbf{x}(t), t) \right.$$

$$\left. + \mathbf{p}^{\mathbf{T}}(t)[\mathbf{a}(\mathbf{x}(t), \mathbf{u}(t), t) - \dot{\mathbf{x}}(t)] \right\} dt$$

Define

$$g_a(\mathbf{x}(t), \dot{\mathbf{x}}(t), \mathbf{u}(t), \mathbf{p}(t), t) \triangleq g(\mathbf{x}(t), \mathbf{u}(t), t) + \mathbf{p}^{\mathbf{T}}(t)[\mathbf{a}(\mathbf{x}(t), \mathbf{u}(t), t) - \dot{\mathbf{x}}(t)]$$

$$+ \left[\frac{\partial h}{\partial \mathbf{x}}(\mathbf{x}(t), t) \right]^{\mathrm{T}} \dot{\mathbf{x}}(t) + \frac{\partial h}{\partial t}(\mathbf{x}(t), t)$$

So that

$$\mathbf{J_a}(u) = \int_{t_0}^{t_f} \left\{ g_a(\mathbf{x}(t), \dot{\mathbf{x}}(t), \mathbf{u}(t), \mathbf{p}(t), t) \right\} dt$$

Then necessary condition for minimization is

$$
\begin{aligned}
\delta \mathbf{J_a}(\mathbf{u}^*) = 0 = & \left[\frac{\partial g_a}{\partial \dot{\mathbf{x}}}(\mathbf{x}^*(t_f), \dot{\mathbf{x}}^*(t_f), \mathbf{u}^*(t_f), \mathbf{p}^*(t_f), t_f \right]^{\mathrm{T}} \delta \mathbf{x}_f \\
& + g_a(\mathbf{x}^*(t_f), \dot{\mathbf{x}}^*(t_f), \mathbf{u}^*(t_f), \mathbf{p}^*(t_f), t_f) \\
& - \left[\left[\frac{\partial g_a}{\partial \dot{\mathbf{x}}}(\mathbf{x}^*(t_f), \dot{\mathbf{x}}^*(t_f), \mathbf{u}^*(t_f), \mathbf{p}^*(t_f), t_f \right]^{\mathrm{T}} \dot{\mathbf{x}}^*(t_f) \right] \delta t_f \\
& + \int_{t_0}^{t_f} \left\{ \left[\left[\frac{\partial g_a}{\partial \mathbf{x}}(\mathbf{x}^*(t), \dot{\mathbf{x}}^*(t), \mathbf{u}^*(t), \mathbf{p}^*(t), t \right]^{\mathrm{T}} \right. \right. \\
& \qquad \left. - \frac{d}{dt} \left[\frac{\partial g_a}{\partial \dot{\mathbf{x}}}(\mathbf{x}^*(t), \dot{\mathbf{x}}^*(t), \mathbf{u}^*(t), \mathbf{p}^*(t), t \right]^{\mathrm{T}} \right] \delta \mathbf{x}(t) \\
& \qquad + \left[\frac{\partial g_a}{\partial \mathbf{u}}(\mathbf{x}^*(t), \dot{\mathbf{x}}^*(t), \mathbf{u}^*(t), \mathbf{p}^*(t), t \right]^{\mathrm{T}} \delta \mathbf{u}(t) \\
& \qquad \left. + \left[\frac{\partial g_a}{\partial \mathbf{p}}(\mathbf{x}^*(t), \dot{\mathbf{x}}^*(t), \mathbf{u}^*(t), \mathbf{p}^*(t), t \right]^{\mathrm{T}} \delta \mathbf{p}(t) \right\} dt
\end{aligned}
$$

Now consider the terms inside the integral which involve the function h

$$\frac{\partial}{\partial \mathbf{x}}\left[\left[\frac{\partial h}{\partial \mathbf{x}}(\mathbf{x}^*(t),t)\right]^{\mathrm{T}}\dot{\mathbf{x}}^*(t)+\frac{\partial h}{\partial t}(\mathbf{x}^*(t),t)\right]-\frac{d}{dt}\left\{\frac{\partial}{\partial \dot{\mathbf{x}}}\left[\left[\frac{\partial h}{\partial \mathbf{x}}(\mathbf{x}^*(t),t)\right]^{\mathrm{T}}\dot{\mathbf{x}}^*(t)\right]\right\}.$$

Partial differentiation yields

$$\left[\frac{\partial^2 h}{\partial \mathbf{x}^2}(\mathbf{x}^*(t),t)\right]\dot{\mathbf{x}}^*(t)+\left[\frac{\partial^2 h}{\partial t\partial \mathbf{x}}(\mathbf{x}^*(t),t)\right]-\frac{d}{dt}\left[\frac{\partial h}{\partial \mathbf{x}}(\mathbf{x}^*(t),t)\right]$$

After applying chain rule to the last term,

$$\left[\frac{\partial^2 h}{\partial \mathbf{x}^2}(\mathbf{x}^*(t),t)\right]\dot{\mathbf{x}}^*(t)+\left[\frac{\partial^2 h}{\partial t\partial \mathbf{x}}(\mathbf{x}^*(t),t)\right]-\left[\frac{\partial^2 h}{\partial \mathbf{x}^2}(\mathbf{x}^*(t),t)\right]\dot{\mathbf{x}}^*(t)$$
$$-\left[\frac{\partial^2 h}{\partial \mathbf{x}\partial t}(\mathbf{x}^*(t),t)\right]=0$$

Then integral term becomes

$$\int_{t_0}^{t_f}\left\{\begin{array}{l}\left[\left[\frac{\partial g}{\partial \mathbf{x}}(\mathbf{x}^*(t),\mathbf{u}^*(t),t)\right]^T+\mathbf{p}^{*\mathbf{T}}(t)\left[\frac{\partial a}{\partial \mathbf{x}}(\mathbf{x}^*(t),\mathbf{u}^*(t),t)\right]-\frac{d}{dt}\left[-\mathbf{p}^{*\mathbf{T}}(t)\right]\right]\delta\mathbf{x}(t)\\ +\left[\left[\frac{\partial g}{\partial \mathbf{u}}(\mathbf{x}^*(t),\mathbf{u}^*(t),t)\right]^{\mathrm{T}}+\mathbf{p}^{*\mathbf{T}}(t)\left[\frac{\partial a}{\partial \mathbf{u}}(\mathbf{x}^*(t),\mathbf{u}^*(t),t)\right]\right]\delta\mathbf{u}(t)\\ +\left[\mathbf{a}(\dot{\mathbf{x}}(t),\dot{\mathbf{u}}(t),t)-\dot{\mathbf{x}}^*(t)\right]^{\mathrm{T}}\delta\mathbf{p}(t)\end{array}\right\}dt$$

This integral must vanish on an extremal regardless of the boundary Condition. Also, the constraints

$$\dot{\mathbf{x}}(t)=\mathbf{a}(\dot{\mathbf{x}}(t),\dot{\mathbf{u}}(t),t)$$

must be satisfied so that the coefficients of $\delta\mathbf{p}(t)$ is zero. The Lagrange multiplier is selected to make the coefficients of $\delta\mathbf{x}$ is zero, that is

$$\dot{\mathbf{p}}^*(t) = -\left[\frac{\partial\mathbf{a}}{\partial\mathbf{x}}(\mathbf{x}^*(t),\mathbf{u}^*(t),t)\right]^{\mathrm{T}}\mathbf{p}^*(t) - \frac{\partial g}{\partial\mathbf{x}}(\mathbf{x}^*(t),\mathbf{u}^*(t),t)$$

(It is called costate equation and $\mathbf{p}(t)$ is called costate.)

The remaining variation $\delta\mathbf{u}$ is independent, so

$$0 = \frac{\partial g}{\partial\mathbf{u}}(\mathbf{x}^*(t),\mathbf{u}^*(t),t) + \left[\frac{\partial a}{\partial\mathbf{u}}(\mathbf{x}^*(t),\mathbf{u}^*(t),t)\right]^{\mathrm{T}}\mathbf{p}^*(t)$$

Since the variation $\delta\mathbf{J}_{\mathbf{a}}$ is zero and the integral term of $\delta\mathbf{J}_{\mathbf{a}}$ is also zero, so that the remaining terms are zero.

$$\left[\frac{\partial h}{\partial \mathbf{x}}(\mathbf{x}^*(t_f),t_f)-\mathbf{p}^*(t_f)\right]^{\mathrm{T}}\delta\mathbf{x}_f+\left[g(\mathbf{x}^*(t_f),\mathbf{u}^*(t_f),t_f)+\frac{\partial h}{\partial t}(\mathbf{x}^*(t_f),t_f)\right.$$
$$\left.+\mathbf{p}^{*\mathbf{T}}(t_f)\left[a(\mathbf{x}^*(t_f),\mathbf{u}^*(t_f),t_f)\right]\right]\delta t_f=0$$

Define Hamiltonian as;

$$\mathrm{H}(\mathbf{x}(t),\mathbf{u}(t),\mathbf{p}(t),t)\triangleq g(\mathbf{x}(t),\mathbf{u}(t),t)+\mathbf{p}^{\mathbf{T}}(t)[\mathbf{a}(\mathbf{x}(t),\mathbf{u}(t),t)]$$

$$\boxed{\left.\begin{cases}\dot{\mathbf{x}}^*(t)=\dfrac{\partial \mathrm{H}}{\partial \mathbf{p}}(\mathbf{x}^*(t),\mathbf{u}^*(t),\mathbf{p}^*(t),t)\\[2ex] \dot{\mathbf{p}}^*(t)=-\dfrac{\partial H}{\partial \mathbf{x}}(\mathbf{x}^*(t),\mathbf{u}^*(t),\mathbf{p}^*(t),t)\\[2ex] 0=\dfrac{\partial H}{\partial \mathbf{u}}(\mathbf{x}^*(t),\mathbf{u}^*(t),\mathbf{p}^*(t),t)\end{cases}\right\}\text{for all } \mathrm{t}\in[\mathrm{t}_0,t_f]}$$

$$
\left[\frac{\partial h}{\partial \mathbf{x}}(\mathbf{x}^*(t_f),t_f)-\mathbf{p}^*(t_f)\right]^T \delta \mathbf{x}_f + \Big[H(\mathbf{x}^*(t),\mathbf{u}^*(t),\mathbf{p}^*(t),t)
$$

$$
\boxed{+\frac{\partial h}{\partial t}(\mathbf{x}^*(t),t_f)\Big]\delta t_f = 0}
$$

*Boundary conditions

1. fixed final time

Case I. final state specified ($\delta x_f = 0,\ \delta t_f = 0$)

$\mathbf{x}^*(t_f) = \mathbf{x}_f$

Case II. Final state free ($\delta t_f = 0,\ \delta x_f =$ arbitrary)

$$\frac{\partial h}{\partial \mathbf{x}}(\mathbf{x}^*(t_f)) - \mathbf{p}^*(t_f) = 0$$

Case III. Final state lying on surface defined by $m(x(t)) = 0$

(note that t_f is fixed.)

EX

$$\frac{\partial m}{\partial \mathbf{x}}(\mathbf{x}^*(t_f)) = \begin{bmatrix} 2\left[x_1^*(t_f) - 3\right]^2 \\ 2\left[x_2^*(t_f) - 3\right]^2 \end{bmatrix}$$

$$\left[\frac{\partial m}{\partial \mathbf{x}}(\mathbf{x}^*(t_f))\right]^T \delta \mathbf{x}(t_f) = 2\left[x_1^*(t_f) - 3\right]\delta x_1(t_f) + \left[x_2^*(t_f) - 4\right]\delta x_2(t_f) = 0$$

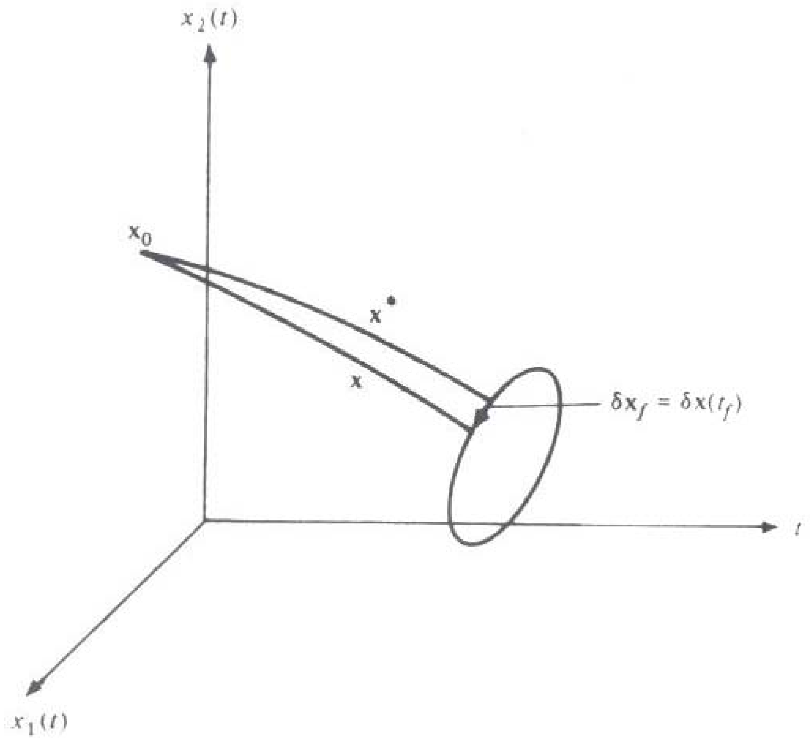


Figure 5-1 An extremal and a comparison curve that terminate on the curve $[x_1(t) - 3]^2 + [x_2(t) - 4]^2 - 4 = 0$ at the specified final time, t_f

$$\delta x_2(t_f) = \frac{-\left[x_1^*(t_f) - 3\right]}{\left[x_2^*(t_f) - 4\right]} \delta x_1(t_f)$$

$$\left[\frac{\partial h}{\partial \mathbf{x}}(\mathbf{x}^*(t_f)) - \mathbf{p}^*(t_f)\right]^T \begin{bmatrix} 1 \\ \dfrac{-\left[x_1^*(t_f)-3\right]}{\left[x_2^*(t_f)-4\right]} \end{bmatrix} = 0$$

$$m(\mathbf{x}^*(t_f)) = \left[x_1^*(t_f)-3\right]^2 + \left[x_2^*(t_f)-4\right]^2 - 4 = 0$$

In general situation,

$$m(\mathbf{x}(t)) = \begin{bmatrix} m_1(\mathbf{x}(t)) \\ \bullet \\ \bullet \\ \bullet \\ m_k(\mathbf{x}(t)) \end{bmatrix} = 0$$

$\delta x(t_f)$ is normal to each of

$$\frac{\partial m_1}{\partial \mathbf{x}}(\mathbf{x}^*(t_f)), \ \cdots \ , \frac{\partial m_k}{\partial \mathbf{x}}(\mathbf{x}^*(t_f)).$$

Since $\delta t_f = 0$,

$$\left[\frac{\partial h}{\partial \mathbf{x}}(\mathbf{x}^*(t_f)) - \mathbf{p}^*(t_f)\right]^T \delta\mathbf{x}(t_f) \triangleq \mathbf{v}^T \delta\mathbf{x}(t_f) = 0$$

It can be shown that this equation is satisfied iff the vector V is

A linear combination of gradient vectors.

$$\frac{\partial h}{\partial \mathbf{x}}(\mathbf{x}^*(t_f)) - \mathbf{p}^*(t_f) = d_1\left[\frac{\partial m_1}{\partial \mathbf{x}}(\mathbf{x}^*(t_f))\right] + \cdots + d_k\left[\frac{\partial m_k}{\partial \mathbf{x}}(\mathbf{x}^*(t_f))\right].$$

For 2n constants (state and costate) and $d_1, \cdots, d_k$, we have

n $X^*(t_0) = X_0$, n above equations and k following,

$$m(\mathbf{x}^*(t_f)) = 0$$

Ex.

$$m(\mathbf{x}(t)) = \left[x_1(t) - 3\right]^2 + \left[x_2(t) - 4\right]^2 - 4 = 0$$

$$\frac{\partial h}{\partial \mathbf{x}}(\mathbf{x}^*(t_f)) - \mathbf{p}^*(t_f) = d\begin{bmatrix} 2\left[x_1^*(t_f) - 3\right] \\ 2\left[x_2^*(t_f) - 4\right] \end{bmatrix},$$

$$m(\mathbf{x}^*(t_f)) = \left[x_1^*(t_f) - 3\right]^2 + \left[x_2^*(t_f) - 4\right]^2 - 4 = 0$$

Problems with Free Final Time

CASE 1. $$H(\mathbf{x}^*(t_f),\mathbf{u}^*(t_f),\mathbf{p}^*(t_f),t_f)+\frac{\partial h}{\partial t}(\mathbf{x}^*(t_f),t_f)=0$$ Final state fixed

CASE 2. $$\mathbf{p}^*(t_f)=\frac{\partial h}{\partial \mathbf{x}}(\mathbf{x}^*(t_f),t_f) \quad \text{(n equations)}$$ Final state free

$$H(\mathbf{x}^*(t_f),\mathbf{u}^*(t_f),\mathbf{p}^*(t_f),t_f)+\frac{\partial h}{\partial t}(\mathbf{x}^*(t_f),t_f)=0$$

h=0 $$\mathbf{p}^*(t_f)=0$$

$$H(\mathbf{x}^*(t_f),\mathbf{u}^*(t_f),\mathbf{p}^*(t_f),t_f)=0$$

CASE 3. $$\delta\mathbf{x}_f \doteq \left[\frac{d\boldsymbol{\theta}}{dt}(t_f)\right]\delta t_f$$ moving point

$$H(\mathbf{x}^*(t_f),\mathbf{u}^*(t_f),\mathbf{p}^*(t_f),t_f)+\frac{\partial h}{\partial t}(\mathbf{x}^*(t_f),t_f)+\left[\frac{\partial h}{\partial \mathbf{x}}(\mathbf{x}^*(t_f))-\mathbf{p}^*(t_f)\right]^T \times\left[\frac{d\boldsymbol{\theta}}{dt}(t_f)\right]=0$$

$$\mathbf{x}^*(t_f)=\boldsymbol{\theta}(t_f)$$

CASE 4. $$m(\mathbf{x}(t)) = \left[x_1(t) - 3\right]^2 + \left[x_2(t) - 4\right]^2 - 4 = 0$$

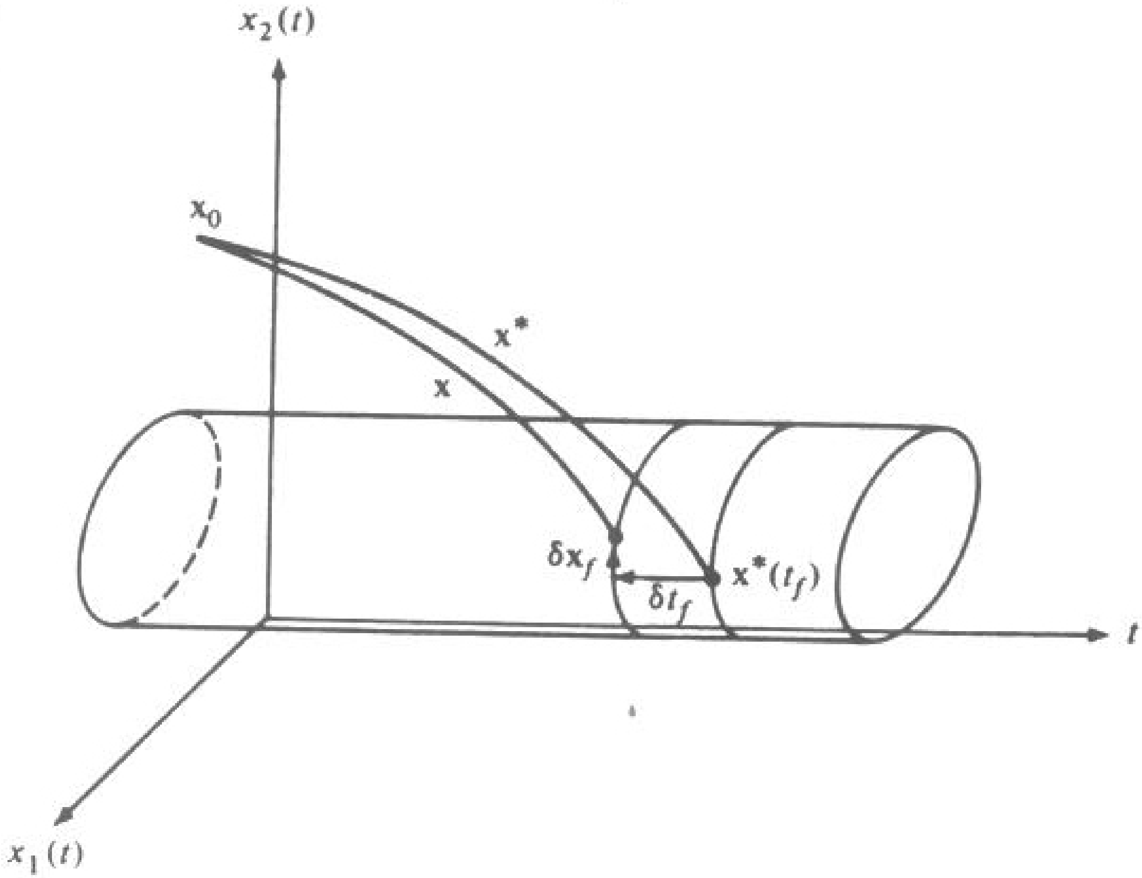


Figure 5-2 An extremal and a comparison curve that terminate on the surface $[x_1(t) - 3]^2 + [x_2(t) - 4]^2 - 4 = 0$

$$H(\mathbf{x}^*(t_f),\mathbf{u}^*(t_f),\mathbf{p}^*(t_f),t_f)+\frac{\partial h}{\partial t}(\mathbf{x}^*(t_f),t_f)=0$$

$$\frac{\partial m}{\partial \mathbf{x}}(\mathbf{x}^*(t_f))=\begin{bmatrix}\dfrac{2\left[x_1^*(t_f)-3\right]}{2\left[x_2^*(t_f)-4\right]}\end{bmatrix}$$

$$\left[\frac{\partial m}{\partial \mathbf{x}}(\mathbf{x}^*(t_f))\right]^T\delta\mathbf{x}_f=2\left[x_1^*(t_f)-3\right]\delta x_1+2\left[x_2^*(t_f)-4\right]\delta x_2=0$$

$$\delta x_2=\frac{-\left[x_1^*(t_f)-3\right]}{\left[x_2^*(t_f)-4\right]}\delta x_1,$$

$$\left[\frac{\partial h}{\partial \mathbf{x}}(\mathbf{x}^*(t_f),t_f)-\mathbf{p}^*(t_f)\right]^T\begin{bmatrix}1\\ \dfrac{-\left[x_1^*(t_f)-3\right]}{\left[x_2^*(t_f)-4\right]}\end{bmatrix}\delta x_{1f}=0$$

$$m(\mathbf{x}^*(t_f))=\left[x_1^*(t_f)-3\right]^2+\left[x_2^*(t_f)-4\right]^2-4=0$$

$$m(\mathbf{x}(t)) = \begin{bmatrix} m_1(\mathbf{x}(t)) \\ \bullet \\ \bullet \\ \bullet \\ m_k(\mathbf{x}(t)) \end{bmatrix} = 0$$

$$\frac{\partial m_1}{\partial \mathbf{x}}(\mathbf{x}^*(t_f), \ldots, \frac{\partial m_k}{\partial \mathbf{x}}(\mathbf{x}^*(t_f)),$$

2n+k+1 equations

$$\mathbf{x}^*(t_0) = x_0$$

$$\frac{\partial h}{\partial \mathbf{x}}(\mathbf{x}^*(t_f), t_f) - \mathbf{p}^*(t_f) = d_1\left[\frac{\partial m_1}{\partial \mathbf{x}}(\mathbf{x}^*(t_f))\right] + \cdots + d_k\left[\frac{\partial m_k}{\partial \mathbf{x}}(\mathbf{x}^*(t_f))\right]$$

$$m(\mathbf{x}^*(t_f)) = 0$$

$$H(\mathbf{x}^*(t_f), \mathbf{u}^*(t_f), \mathbf{p}^*(t_f), t_f) + \frac{\partial h}{\partial t}(\mathbf{x}^*(t_f), t_f) = 0$$

CASE 5. Final state lying on the moving surface

$$m(\mathbf{x}(t),t) = \left[x_1(t)-3\right]^2 + \left[x_2(t)-4-t\right]^2 - 4 = 0$$

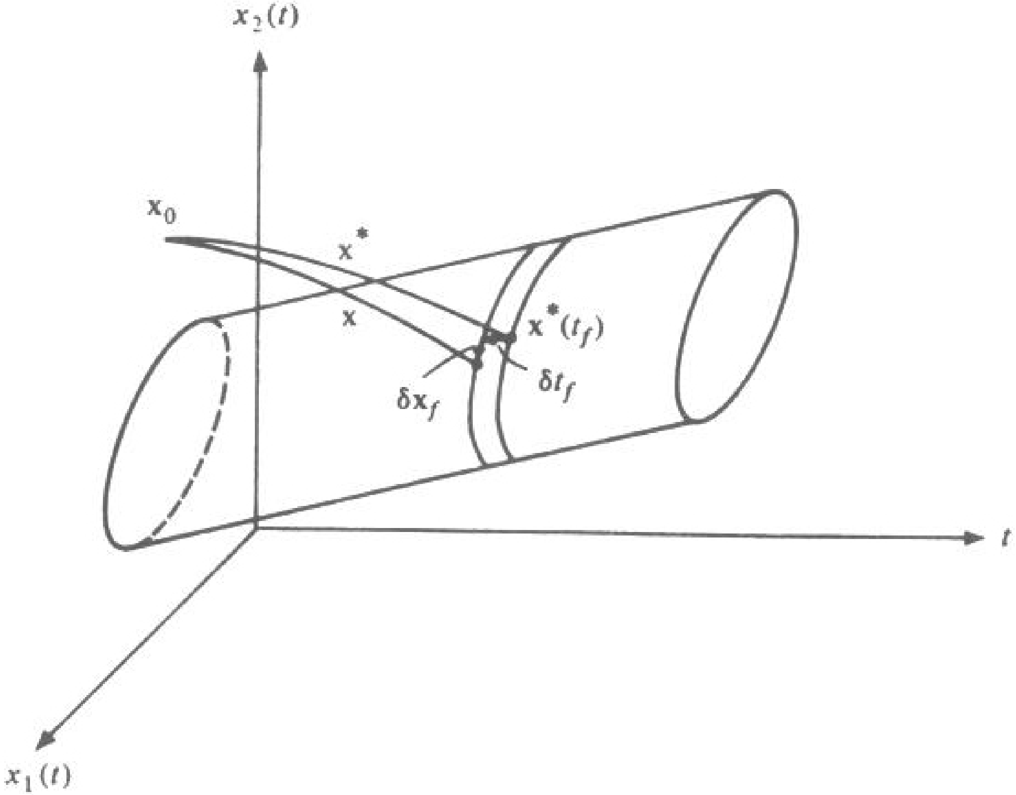


Figure 5-3 An extremal and a comparison curve that terminate on the surface $[x_1(t) - 3]^2 + [x_2(t) - 4 - t]^2 - 4 = 0$

$$\begin{bmatrix} \dfrac{\partial m}{\partial x_1}(\mathbf{x}^*(t_f),t_f) \\ \dfrac{\partial m}{\partial x_2}(\mathbf{x}^*(t_f),t_f) \\ \dfrac{\partial m}{\partial t}(\mathbf{x}^*(t_f),t_f) \end{bmatrix} \triangleq \begin{bmatrix} \dfrac{\dfrac{\partial m}{\partial x}(\mathbf{x}^*(t_f),t_f)}{\dfrac{\partial m}{\partial t}(\mathbf{x}^*(t_f),t_f)} \end{bmatrix}$$

$$\left[\frac{\partial m}{\partial x_1}(\mathbf{x}^*(t_f),t_f)\right]\delta x_1 + \left[\frac{\partial m}{\partial x_2}(\mathbf{x}^*(t_f),t_f)\right]\delta x_2 + \left[\frac{\partial m}{\partial t}(\mathbf{x}^*(t_f),t_f)\right]\delta t_f = 0$$

$$2\left[x_1^*(t_f)-3\right]\delta x_1 + 2\left[x_2^*(t_f)-4\right]\delta x_2 - \left[x_2^*(t_f)-4-t_f\right]\delta t_f = 0$$

$$\delta t_f = \frac{\left[x_1^*(t_f)-3\right]}{\left[x_2^*(t_f)-4-t_f\right]}\delta x_{1f} + \delta x_{2f}$$

$$\left[\frac{\partial h}{\partial x_1}(\mathbf{x}^*(t_f)) - p_1^*(t_f) + \Big[H(\mathbf{x}^*(t_f),\mathbf{u}^*(t_f),\mathbf{p}^*(t_f),t_f)\right.$$

$$\left.\left[+\frac{\partial h}{\partial t}(\mathbf{x}^*(t_f),t_f)\right]\left[\frac{\left[x_1^*(t_f)-3\right]}{\left[x_2^*(t_f)-4-t_f\right]}\right]\right]\delta x_{1f}$$

$$\left[\frac{\partial h}{\partial x_2}(\mathbf{x}^*(t_f),t_f) - p_2^*(t_f) + H(\mathbf{x}^*(t_f),\mathbf{u}^*(t_f),\mathbf{p}^*(t_f),t_f)\right.$$

$$\left.+\frac{\partial h}{\partial t}(\mathbf{x}^*(t_f),t_f)\right]\delta x_{2f} = 0$$

$$m(\mathbf{x}^*(t_f),t_f) = 0$$

$$m(\mathbf{x}(t),t) = m(\mathbf{x}(t)) = \begin{bmatrix} m_1(\mathbf{x}(t)) \\ \bullet \\ \bullet \\ \bullet \\ m_k(\mathbf{x}(t)) \end{bmatrix} = 0$$

$$\begin{bmatrix} \dfrac{\delta x_f}{\delta t_f} \end{bmatrix}$$

$$\begin{bmatrix} \dfrac{\dfrac{\partial m_1}{\partial \mathbf{x}}(\mathbf{x}^*(t_f),t_f)}{\dfrac{\partial m_1}{\partial t}(\mathbf{x}^*(t_f),t_f)} \end{bmatrix}, \cdots, \begin{bmatrix} \dfrac{\dfrac{\partial m_k}{\partial \mathbf{x}}(\mathbf{x}^*(t_f),t_f)}{\dfrac{\partial m_k}{\partial t}(\mathbf{x}^*(t_f),t_f)} \end{bmatrix},$$

$$\begin{bmatrix} \dfrac{\partial h}{\partial x}(\mathbf{x}^*(t_f),t_f)-\mathbf{p}^*(t_f) \\ \hline H(\mathbf{x}^*(t_f),\mathbf{u}^*(t_f),\mathbf{p}^*(t_f),t_f)+\dfrac{\partial h}{\partial t}(\mathbf{x}^*(t_f),t_f) \end{bmatrix}^T \begin{bmatrix} \delta\mathbf{x}_f \\ \hline \delta t_f \end{bmatrix} = 0 \triangleq \mathbf{v}^T \begin{bmatrix} \delta\mathbf{x}_f \\ \hline \delta t_f \end{bmatrix}$$

$$\mathbf{v} = d_1 \begin{bmatrix} \dfrac{\partial m_1}{\partial \mathbf{x}}(\mathbf{x}^*(t_f),t_f) \\ \hline \dfrac{\partial m_1}{\partial t}(\mathbf{x}^*(t_f),t_f) \end{bmatrix}, \cdots, d_k \begin{bmatrix} \dfrac{\partial m_k}{\partial \mathbf{x}}(\mathbf{x}^*(t_f),t_f) \\ \hline \dfrac{\partial m_k}{\partial t}(\mathbf{x}^*(t_f),t_f) \end{bmatrix},$$

$$\frac{\partial h}{\partial \mathbf{x}}(\mathbf{x}^*(t_f),t_f)-\mathbf{p}^*(t_f) = d_1 \left[\frac{\partial m_1}{\partial \mathbf{x}}(\mathbf{x}^*(t_f),t_f)\right] + \cdots + d_k \left[\frac{\partial m_k}{\partial \mathbf{x}}(\mathbf{x}^*(t_f),t_f)\right]$$

$$\mathrm{H}(\mathbf{x}^*(t_f),\mathbf{u}^*(t_f),\mathbf{p}^*(t_f),t_f)+\frac{\partial h}{\partial t}(\mathbf{x}^*(t_f),t_f)=d_1\left[\frac{\partial m_1}{\partial \mathbf{x}}(\mathbf{x}^*(t_f),t_f)\right]$$

$$+\cdots+d_k\left[\frac{\partial m_k}{\partial \mathbf{x}}(\mathbf{x}^*(t_f),t_f)\right].$$

$$\mathbf{m}(\mathbf{x}^*(t_f),t_f)=0$$

Example 5.1-1

$$\dot{x}_1(t)=x_2(t)$$

$$\dot{x}_2(t)=-x_2(t)+u(t)$$

$$\mathbf{J}(u)=\int_{t_0}^{t_f}\frac{1}{2}u^2(t)dt$$

$$H(\mathbf{x}\ (t),\mathbf{u}\ (t),\mathbf{p}\ (t))=\frac{1}{2}u^2(t)+p_1(t)x_{2(t)}-p_2(t)x_2(t)+p_2(t)u(t)$$

$$\dot{p}_1^*(t) = -\frac{\partial H}{\partial x_1} = 0$$

$$\dot{p}_2^*(t) = -\frac{\partial H}{\partial x_2} = -p_1^*(t) + p_2^*(t),$$

$$0 = \frac{\partial H}{\partial u} = u^*(t) + p_2^*(t).$$

$$\dot{x}_1^*(t) = \dot{x}_2^*(t)$$

$$\dot{x}_2^*(t) = -\dot{x}_2^*(t) - p_2^*(t).$$

a. x(0)=0, x(2)=[5 2]'

$$\dot{x}_1^*(t) = c_1 + c_2\left[1-\varepsilon^{-t}\right] + c_3\left[-t - \frac{1}{2}\varepsilon^{-t} + \frac{1}{2}\varepsilon^{t}\right] + c_4\left[1 - \frac{1}{2}\varepsilon^{-t} - \frac{1}{2}\varepsilon^{t}\right]$$

$$\dot{x}_2^*(t) = c_2\varepsilon^{-t} + c_3\left[-1 + \frac{1}{2}\varepsilon^{-t} + \frac{1}{2}\varepsilon^{t}\right] + c_4\left[\frac{1}{2}\varepsilon^{-t} - \frac{1}{2}\varepsilon^{t}\right]$$

$$p_1^*(t) = c_3$$

$$p_2^*(t) = c_3\left[1-\varepsilon^{t}\right] + c_4\varepsilon^{t}$$

$$5 = c_3\left[-2 - \frac{1}{2}\varepsilon^{-2} + \frac{1}{2}\varepsilon^{2}\right] + c_4\left[1 - \frac{1}{2}\varepsilon^{-2} - \frac{1}{2}\varepsilon^{2}\right]$$

$$2 = c_3\left[-1 + \frac{1}{2}\varepsilon^{-2} + \frac{1}{2}\varepsilon^{2}\right] + c_4\left[\frac{1}{2}\varepsilon^{-2} - \frac{1}{2}\varepsilon^{2}\right].$$

$$\dot{x}_1^*(t) = 7.289t - 6.103 + 6.69\varepsilon^{-t} - 0.593\varepsilon^{t}$$

$$\dot{x}_2^*(t) = 7.289 - 6.69\varepsilon^{-t} - 0.593\varepsilon^{t}$$

b. x(0)=0, x(2)=unspecified

$$\mathbf{J}(u)=\frac{1}{2}\left[x_1(2)-5\right]^2+\frac{1}{2}\left[x_2(2)-2\right]^2+\frac{1}{2}\int_0^2 u^2(t)dt.$$

Table 5-1 SUMMARY OF BOUNDARY CONDITIONS IN OPTIMAL CONTROL PROBLEMS

Problem	*Description*	*Substitution in Eq. (5.1-18)*	*Boundary-condition equations*	*Remarks*
t_f fixed	1. $\mathbf{x}(t_f)=\mathbf{x}_f$ specified final state	$\delta\mathbf{x}_f=\delta\mathbf{x}(t_f)=\mathbf{0}$ $\delta t_f=0$	$\mathbf{x}^*(t_0)=\mathbf{x}_0$ $\mathbf{x}^*(t_f)=\mathbf{x}_f$	$2n$ equations to determine $2n$ constants of integration
	2. $\mathbf{x}(t_f)$ free	$\delta\mathbf{x}_f=\delta\mathbf{x}(t_f)$ $\delta t_f=0$	$\mathbf{x}^*(t_0)=\mathbf{x}_0$ $\frac{\partial h}{\partial \mathbf{x}}(\mathbf{x}^*(t_f))-\mathbf{p}^*(t_f)=\mathbf{0}$	$2n$ equations to determine $2n$ constants of integration
	3. $\mathbf{x}(t_f)$ on the surface $\mathbf{m}(\mathbf{x}(t))=\mathbf{0}$	$\delta\mathbf{x}_f=\delta\mathbf{x}(t_f)$ $\delta t_f=0$	$\mathbf{x}^*(t_0)=\mathbf{x}_0$ $\frac{\partial h}{\partial \mathbf{x}}(\mathbf{x}^*(t_f))-\mathbf{p}^*(t_f)=\sum_{i=1}^{k} d_i\left[\frac{\partial m_i}{\partial \mathbf{x}}(\mathbf{x}^*(t_f))\right]$ $\mathbf{m}(\mathbf{x}^*(t_f))=\mathbf{0}$	$(2n+k)$ equations to determine the $2n$ constants of integration and the variables $d_1,\ldots,d_k$
t_f free	4. $\mathbf{x}(t_f)=\mathbf{x}_f$ specified final state	$\delta\mathbf{x}_f=\mathbf{0}$	$\mathbf{x}^*(t_0)=\mathbf{x}_0$ $\mathbf{x}^*(t_f)=\mathbf{x}_f$ $\mathscr{H}(\mathbf{x}^*(t_f),\mathbf{u}^*(t_f),\mathbf{p}^*(t_f),t_f)+\frac{\partial h}{\partial t}(\mathbf{x}^*(t_f),t_f)=0$	$(2n+1)$ equations to determine the $2n$ constants of integration and t_f
	5. $\mathbf{x}(t_f)$ free		$\mathbf{x}^*(t_0)=\mathbf{x}_0$ $\frac{\partial h}{\partial \mathbf{x}}(\mathbf{x}^*(t_f),t_f)-\mathbf{p}^*(t_f)=\mathbf{0}$ $\mathscr{H}(\mathbf{x}^*(t_f),\mathbf{u}^*(t_f),\mathbf{p}^*(t_f),t_f)+\frac{\partial h}{\partial t}(\mathbf{x}^*(t_f),t_f)=0$	$(2n+1)$ equations to determine the $2n$ constants of integration and t_f
	6. $\mathbf{x}(t_f)$ on the moving point $\boldsymbol{\theta}(t)$	$\delta\mathbf{x}_f=\left[\frac{d\boldsymbol{\theta}}{dt}(t_f)\right]\delta t_f$	$\mathbf{x}^*(t_0)=\mathbf{x}_0$ $\mathbf{x}^*(t_f)=\boldsymbol{\theta}(t_f)$ $\mathscr{H}(\mathbf{x}^*(t_f),\mathbf{u}^*(t_f),\mathbf{p}^*(t_f),t_f)+\frac{\partial h}{\partial t}(\mathbf{x}^*(t_f),t_f)$ $+\left[\frac{\partial h}{\partial \mathbf{x}}(\mathbf{x}^*(t_f),t_f)-\mathbf{p}^*(t_f)\right]^T\left[\frac{d\boldsymbol{\theta}}{dt}(t_f)\right]=0$	$(2n+1)$ equations to determine the $2n$ constants of integration and t_f

7. $\mathbf{x}(t_f)$ on the surface $\mathbf{m}(\mathbf{x}(t)) = \mathbf{0}$		$\mathbf{x}^*(t_0) = \mathbf{x}_0$ $\frac{\partial h}{\partial \mathbf{x}}(\mathbf{x}^*(t_f), t_f) - \mathbf{p}^*(t_f) = \sum_{i=1}^{k} d_i\left[\frac{\partial m_i}{\partial \mathbf{x}}(\mathbf{x}^*(t_f))\right]$ $\mathbf{m}(\mathbf{x}^*(t_f)) = \mathbf{0}$ $\mathscr{H}(\mathbf{x}^*(t_f), \mathbf{u}^*(t_f), \mathbf{p}^*(t_f), t_f) + \frac{\partial h}{\partial t}(\mathbf{x}^*(t_f), t_f) = 0$	$(2n + k + 1)$ equations to determine the $2n$ constants of integration, the variables $d_1, \ldots, d_k$, and t_f
8. $\mathbf{x}(t_f)$ on the moving surface $\mathbf{m}(\mathbf{x}(t), t) = \mathbf{0}$		$\mathbf{x}^*(t_0) = \mathbf{x}_0$ $\frac{\partial h}{\partial \mathbf{x}}(\mathbf{x}^*(t_f), t_f) - \mathbf{p}^*(t_f) = \sum_{i=1}^{k} d_i\left[\frac{\partial m_i}{\partial \mathbf{x}}(\mathbf{x}^*(t_f), t_f)\right]$ $\mathbf{m}(\mathbf{x}^*(t_f), t_f) = \mathbf{0}$ $\mathscr{H}(\mathbf{x}^*(t_f), \mathbf{u}^*(t_f), \mathbf{p}^*(t_f), t_f) + \frac{\partial h}{\partial t}(\mathbf{x}^*(t_f), t_f)$ $= \sum_{i=1}^{k} d_i\left[\frac{\partial m_i}{\partial t}(\mathbf{x}^*(t_f), t_f)\right]$	$(2n + k + 1)$ equations to determine the $2n$ constants of integration, the variables $d_1, \ldots, d_k$, and t_f.

$$p_1^*(t) = x_1^*(2) - 5$$

$$p_2^*(t) = x_2^*(2) - 2$$

$$\begin{bmatrix} 0.627 & -2.762 \\ 9.151 & -11.016 \end{bmatrix} \begin{bmatrix} c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$$

$$x_1^*(t) = 2.697t - 2.422 + 2.560\varepsilon^{-t} - 0.137\varepsilon^{t}$$

$$x_2^*(t) = 2.697 - 2.560\varepsilon^{-t} - 0.137\varepsilon^{t}$$

c. x(0)=0, x(2)=line

$$x_1(t)+5x_2(t)=15$$

$$x_1^*(t)+5x_2^*(t)=15$$

$$-p_1^*(2)=d$$

$$-p_2^*(2)=5d$$

$$\begin{bmatrix} 15.437 & -20.897 \\ 11.389 & -7.389 \end{bmatrix}\begin{bmatrix} c_3 \\ c_4 \end{bmatrix}=\begin{bmatrix} 15 \\ 0 \end{bmatrix}$$

$$x_1^*(t)=0.894t-1.379+1.136\varepsilon^{-t}-0.242\varepsilon^{t}$$

$$x_2^*(t)=0.894-1.136\varepsilon^{-t}-0.242\varepsilon^{t}$$

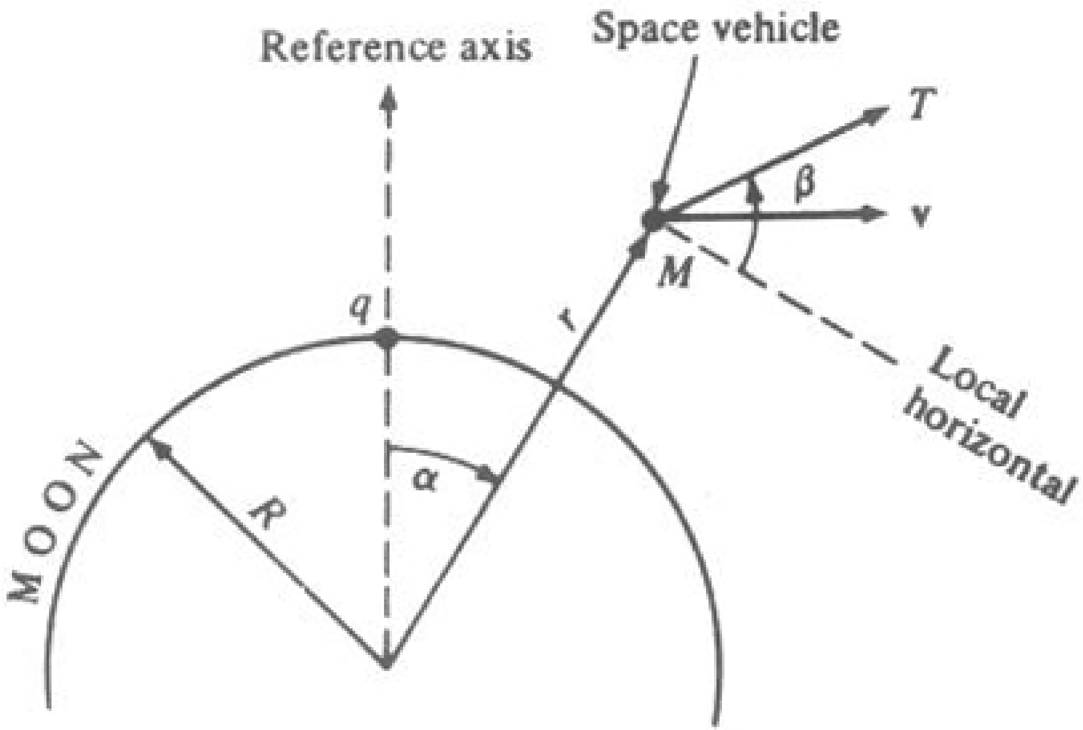


Figure 5-4 A space vehicle in the gravity field of the moon

$$\dot{x}_1(t) = x_3(t)$$

$$\dot{x}_2(t) = \frac{x_4(t)}{x_1(t)}$$

$$\dot{x}_3(t) = \frac{x_4^2(t)}{x_1(t)} - \frac{g_0 R^2}{x_1^2(t)} + \left[\frac{T}{M}\right] \sin\ u(t)$$

$$\dot{x}_4(t) = -\frac{x_3(t) x_4(t)}{x_1(t)} + \left[\frac{T}{M}\right] cos\ u(t)$$

Misson a, The space vehicle is to be launched from point q into a circoular orbit of altitude D in minimum time

$$\mathbf{J}(u) = \int_0^{t_f} dt$$

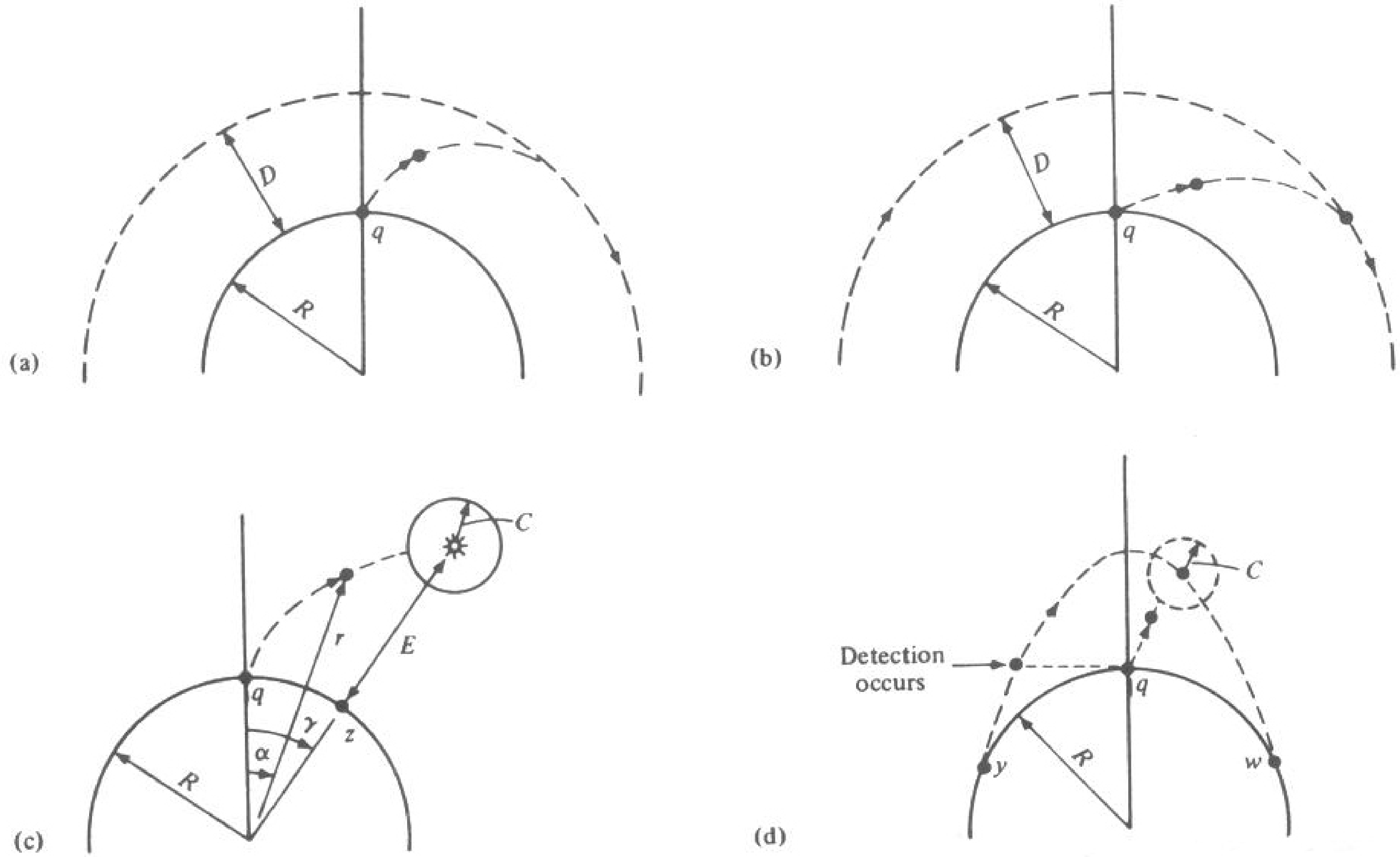


Figure 5-5 (a) Orbit injection. (b) Rendezvous. (c)Reconnaissance of synchronous satellite. (d) Reconnaissance of approaching spacecraft.

$$H(\mathbf{x}(t),\mathbf{u}(t),\mathbf{p}(t)) = 1 + p_1(t)x_3 9t) + \frac{p_2(t)x_4(t)}{x_1(t)}$$

$$+ p_3(t)\left[\frac{x_4^2(t)}{x_1(t)} - \frac{g_0R^2}{x_1^2(t)} + \left[\frac{T}{M}\right]\sin\ u(t)\right]$$

$$+ p_4(t)\left[-\frac{x_3(t)x_4(t)}{x_1(t)} + \left[\frac{T}{M}\right]cos\ u(t)\right]$$

$$\dot{p}_1^*(t) = -\frac{\partial H}{\partial x_1} = \frac{p_2^*(t)x_4^*(t)}{x_1^{*2}(t)} + p_3^*(t)\left[\frac{x_4^{*2}(t)}{x_1^{*2}(t)} - \frac{2g_0R^2}{x_1^{*3}(t)}\right] - \frac{p_4^*x_3^*(t)x_4^*(t)}{x_1^{*2}(t)}$$

$$\dot{p}_2^*(t) = -\frac{\partial H}{\partial x_2} = 0$$

$$\dot{p}_3^*(t) = -\frac{\partial H}{\partial x_3} = -p_1^*(t) + \frac{p_4^*(t)x_4^*(t)}{x_1^*(t)}$$

$$\dot{p}_4^*(t) = -\frac{\partial H}{\partial x_4} = -\frac{p_2^*(t)}{x_1^*(t)} - \frac{2p_3^*(t)x_4^*(t)}{x_1^*(t)} + \frac{p_4^*(t)x_3^*(t)}{x_1^*(t)}$$

$$\dot{\mathbf{x}}^*(t) = a(\mathbf{x}^*(t), \mathbf{u}^*(t))$$

$$0 = \frac{\partial H}{\partial \mathbf{u}} = \left[\frac{T}{M}\right]\left[p_3^*(t) cos\ \mathbf{u}^*(t) - p_4^*(t) \sin\ \mathbf{u}^*(t)\right].$$

$$u^*(t) = \tan^{-1}\theta \frac{p_3^*(t)}{p_4^*(t)},$$

$$\sin\ u^*(t) = \frac{p_3^*(t)}{\sqrt{p_3^{*2}(t) + p_4^{*2}(t)}}$$

$$cos\ u^*(t) = \frac{p_4^*(t)}{\sqrt{p_3^{*2}(t) + p_4^{*2}(t)}}$$

$$x_1^*(t_f) = R + D$$
$$p_2^*(t_f) = 0$$
$$x_3^*(t_f) = 0$$
$$x_4^*(t_f) = \sqrt{\frac{g_0 R^2}{[R + D]}}$$
$$H(\mathbf{x}^*(t_f), \mathbf{p}^*(t_f)) = 0$$

Misson b, The space vehicle is to rendezvous with another spacecraft that is in a fixed circular orbit D miles

$$\theta(t) = \begin{bmatrix} R + D \\ \operatorname{mod} ulo(\pi t) \\ {\scriptstyle 2\pi} \\ 0 \\ \pi[R + D] \end{bmatrix}$$

$$\delta x_{2f} = \left[\frac{d\theta_2}{dt}(t_f)\right]\delta t_f$$

$$= \pi\delta t_f$$

$$-\pi p_2^*(t_f) + H(\mathbf{x}^*(t_f, \mathbf{p}^*(t_f)) = 0$$

$$\mathbf{x}^*(t_f) = \begin{bmatrix} R+D \\ \underset{2\pi}{\mathrm{mod}\,ulo}(\pi t) \\ 0 \\ \pi\left[R+D\right] \end{bmatrix} = \boldsymbol{\theta}(t_f).$$

Misson c, a satellite is in synchronous orbit E miles

$$m(\mathbf{x}(t)) = \left[r(t)\cos\alpha(t) - \left[R+E\right]\cos\gamma\right]^2$$
$$+\left[r(t)\sin\alpha(t) - \left[R+E\right]\sin\gamma\right]^2 - C^2 = 0$$

$$-\mathbf{p}^*(t_f) = d\left[\frac{\partial m}{\partial \mathbf{x}}(\mathbf{x}^*(t_f))\right].$$

$$-\mathbf{p}^*(t_f) = d\begin{bmatrix} 2r^*(t_f) - 2[R+E]\cos(\alpha^*(t_f) - \gamma) \\ 2r^*(t_f)[R+E]\sin(\alpha^*(t_f) - \gamma) \\ 0 \\ 0 \end{bmatrix}$$

$$=2d\begin{bmatrix} x_1^*(t_f) - [R+E]\cos(x_2^*(t_f) - \gamma) \\ x_1^*(t_f)[R+E]\sin(\alpha^*(t_f) - \gamma) \\ 0 \\ 0 \end{bmatrix}$$

$$m(\mathbf{x}^*(t)) = \left[x_1^*(t_f)\cos x_2^*(t_f) - [R+E]\cos\gamma \right]^2$$

$$+\left[x_1^*(t_f)\sin x_2^*(t_f) - [R+E]\sin\gamma \right]^2 - C^2 = 0$$

$$H(\mathbf{x}^*(t_f), \mathbf{p}^*(t_f)) = 0$$

Misson d, The reconnaissance spacecraft is dispathced from point q

$$m(\mathbf{x}(t), t) = \left[r(t)\cos\alpha(t) - 2.78Rt + 6.95Rt^2 - R \right]^2$$

$$+\left[r(t)\sin\alpha(t) - 1.85Rt + 0.32R \right]^2 - C^2 = 0$$

$$-\mathbf{p}^*(t_f) = d\left[\frac{\partial m}{\partial \mathbf{x}}(\mathbf{x}^*(t_f), t_f) \right].$$

$$-\mathrm{p}_1^*(t_f) = 2d\left[x_1^*(t_f) + R\left\{\left[-2.78t_f + 6.95t_f^2 - 1\right]\cos x_2^*(t_f) + \left[-1.85t_f + 0.32\right]\sin x_2^*(t_f)\right\}\right]$$

$$-\mathrm{p}_2^*(t_f) = -2d\left[Rx_1^*(t_f)\left\{\left[-2.78t_f + 6.95t_f^2 - 1\right]\sin x_2^*(t_f) + \left[1.85t_f - 0.32\right]\cos x_2^*(t_f)\right\}\right]$$

$$-\mathrm{p}_3^*(t_f) = 0$$

$$-\mathrm{p}_4^*(t_f) = 0$$

$$\left[x_1^*(t_f)\cos x_2^*(t_f) - 2.78Rt_f + 6.95Rt_f^2 - R\right]^2$$

$$+\left[x_1^*(t_f)\sin x_2^*(t_f) - 1.85Rt_f + 0.32R\right]^2 - C^2 = 0$$

$$H(\mathbf{x}^*(t_f), \mathbf{p}^*(t_f)) = 2dR\left\{\left[-2.78 + 13.9t_f\right]\left[x_1^*(t_f)\cos x_2^*(t_f) - 2.78Rt_f + 6.95Rt_f^2 - R\right]\right.$$

$$\left. -1.85\left[x_1^*(t_f)\sin x_2^*(t_f) - 1.85Rt_f + 0.32R\right]\right\}$$

5.2 LINEAR REGULATOR PROBLEMS

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(\mathbf{t})\mathbf{u}(t)$$

$$J = \frac{1}{2}\mathbf{x}^T(t_f)\mathbf{H}\mathbf{x}(t_f) + \frac{1}{2}\int_{t_0}^{t_f}\left[\mathbf{x}^T(t)\mathbf{Q}(t)\mathbf{x}(t) + \mathbf{u}^T(t)\mathbf{R}(t)\mathbf{u}(t)\right]dt$$

$$H(\mathbf{x}(t), \mathbf{u}(t), \mathbf{p}(t), t) = \frac{1}{2}\mathbf{X}^T(t)\mathbf{Q}(t)\mathbf{x}(t) + \frac{1}{2}\mathbf{u}^T(t)\mathbf{R}(t)\mathbf{u}(t)$$
$$+\mathbf{p}^T\mathbf{A}(t)\mathbf{x}(t) + \mathbf{p}^T\mathbf{B}(t)\mathbf{u}(t)$$

$$\dot{\mathbf{x}}^*(t) = \mathbf{A}(t)\mathbf{x}^*(t) + \mathbf{B}(t)\mathbf{u}^*(t)$$

$$\dot{\mathbf{p}}^*(t) = -\frac{\partial H}{\partial \mathbf{x}} = -\mathbf{Q}(t)\mathbf{x}^*(t) - \mathbf{A}^T(t)\mathbf{p}^*(t)$$

$$0 = \frac{\partial H}{\partial \mathbf{u}} = \mathbf{R}(t)\mathbf{u}^*(t) + \mathbf{B}^T(t)\mathbf{p}^*(t).$$

$$\mathbf{u}^*(t) = -\mathbf{R}^{-1}(t)\mathbf{B}^T(t)\mathbf{p}^*(t)$$

$$\dot{\mathbf{x}}^*(t) = \mathbf{A}(t)\mathbf{x}^*(\mathbf{t}) - \mathbf{B}(t)\mathbf{R}^{-1}(t)\mathbf{B}^T(t)\mathbf{p}^*(t)$$

$$\begin{bmatrix} \dot{\mathbf{x}}^*(t) \\ \dot{\mathbf{p}}^*(t) \end{bmatrix} = \begin{bmatrix} \mathbf{A}(t) & -\mathbf{B}(t)\mathbf{R}^{-1}(t)\mathbf{B}^T(t) \\ -\mathbf{Q}(t) & -\mathbf{A}^T(t) \end{bmatrix} \begin{bmatrix} \mathbf{x}^*(t) \\ \mathbf{p}^*(t) \end{bmatrix}$$

$$\begin{bmatrix} \mathbf{x}^*(t_f) \\ \mathbf{p}^*(t_f) \end{bmatrix} = \varphi(t_f, t) \begin{bmatrix} \mathbf{x}^*(t) \\ \mathbf{p}^*(t) \end{bmatrix}$$

$$\begin{bmatrix} \mathbf{x}^*(t_f) \\ \mathbf{p}^*(t_f) \end{bmatrix} = \begin{bmatrix} \varphi_{11}(t_f,t) & \varphi_{12}(t_f,t) \\ \varphi_{21}(t_f,t) & \varphi_{22}(t_f,t) \end{bmatrix} \begin{bmatrix} \mathbf{x}^*(t) \\ \mathbf{p}^*(t) \end{bmatrix}$$

$$\mathbf{p}^*(t_f) = \mathbf{H}\mathbf{x}^*(t_f)$$

$$\mathbf{x}^*(t_f) = \varphi_{11}(t_f,t)\mathbf{x}^*(t) + \varphi_{12}(t_f,t)\mathbf{p}^*(t)$$

$$\mathbf{H}\mathbf{x}^*(t_f) = \varphi_{21}(t_f,t)\mathbf{x}^*(t) + \varphi_{22}(t_f,t)\mathbf{p}^*(t).$$

$$\mathbf{H}\varphi_{11}(t_f,t)\mathbf{x}^*(t) + \mathbf{H}\varphi_{12}(t_f,t)\mathbf{p}^*(t) = \varphi_{21}(t_f,t)\mathbf{x}^*(t) + \varphi_{22}(t_f,t)\mathbf{p}^*(t)$$

$$\mathbf{p}^*(t) = \left[\varphi_{22}(t_f, t) - \mathbf{H}\varphi_{12}(t_f, t)\right]^{-1}\left[\mathbf{H}\varphi_{11}(t_f, t) - \varphi_{21}(t_f, t)\right]\mathbf{x}^*(t).$$

$$\mathbf{p}^*(t) \triangleq \mathbf{K}(t)\mathbf{x}^*(t)$$

$$\boxed{\begin{aligned} \mathbf{u}^*(t) &= -\mathbf{R}^{-1}(t)\mathbf{B}^T(t)\mathbf{K}(t)\mathbf{x}(t) \\ &\triangleq \mathbf{F}(\mathrm{t})\mathbf{x}(\mathrm{t}), \end{aligned}}$$

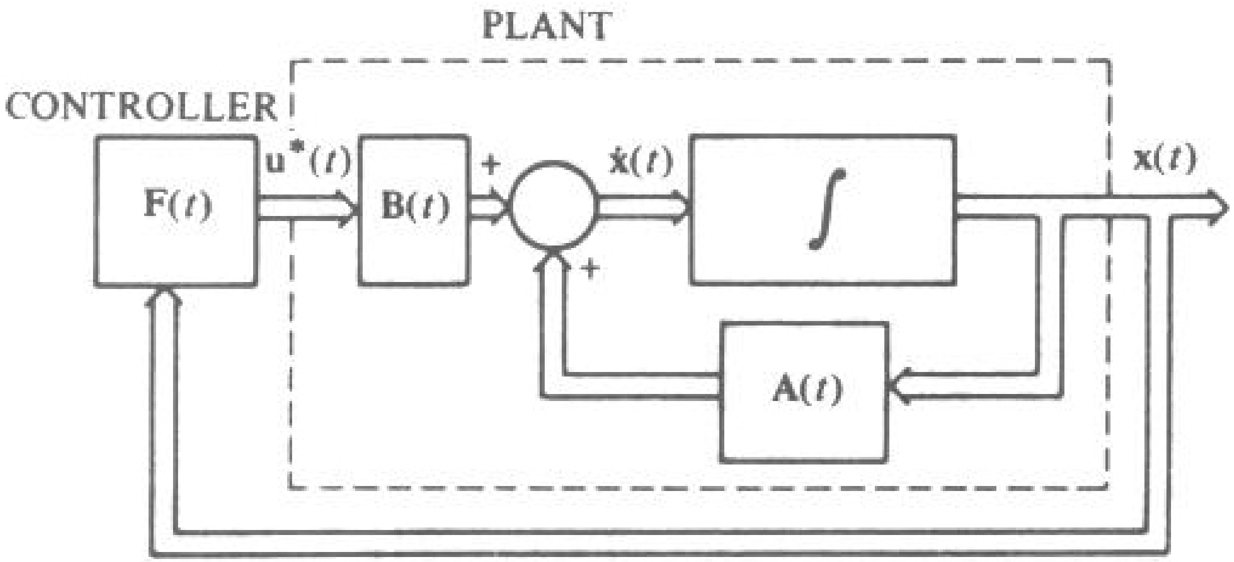


Figure 5-6 Plant and optimal feedback controller for linear regulator problems

$$\left\{ s\mathbf{I} - \begin{bmatrix} \mathbf{A} & -\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^{T} \\ -\mathbf{Q} & -\mathbf{A}^{T} \end{bmatrix} \right\}^{-1}$$

$$\dot{\mathbf{K}}(t) = -\mathbf{K}(t)\mathbf{A}(t) - \mathbf{A}^{\mathbf{T}}(t)\mathbf{K}(t) - \mathbf{Q}(t) + \mathbf{K}(t)\mathbf{B}(t)\mathbf{R}^{\mathbf{-1}}(t)\mathbf{B}^{\mathbf{T}}(t)\mathbf{K}(t)$$

Example 5.2-1.

$$\dot{x}(t) = ax(t) + u(t)$$

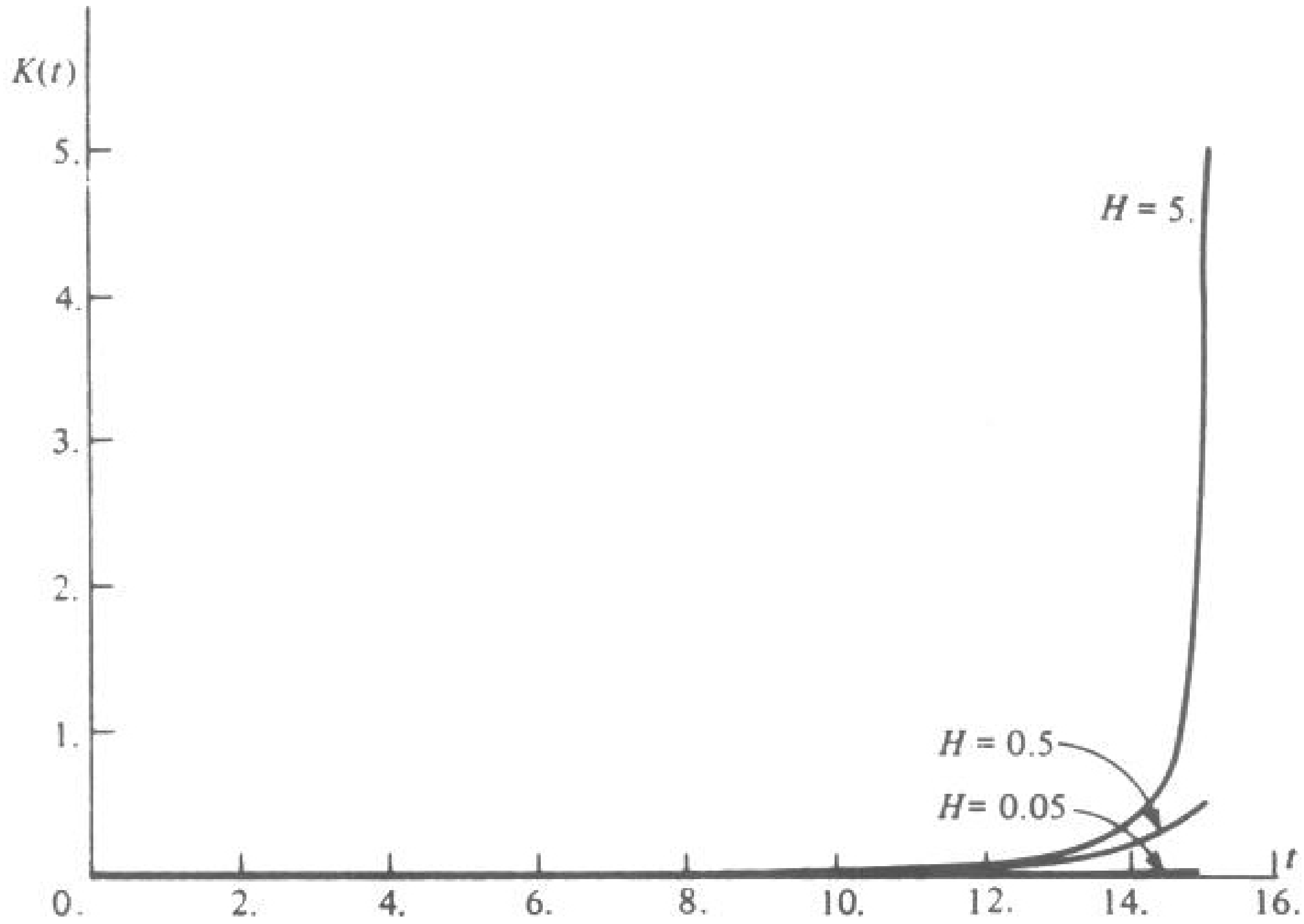
$$J(u) = \frac{1}{2}Hx^2(T) + \int_0^T \frac{1}{4}u^2(t)dt$$

$$\begin{bmatrix} \dot{x}^*(t) \\ \dot{p}^*(t) \end{bmatrix} = \begin{bmatrix} a & -2 \\ 0 & -a \end{bmatrix} \begin{bmatrix} x^*(t) \\ p^*(t) \end{bmatrix}$$

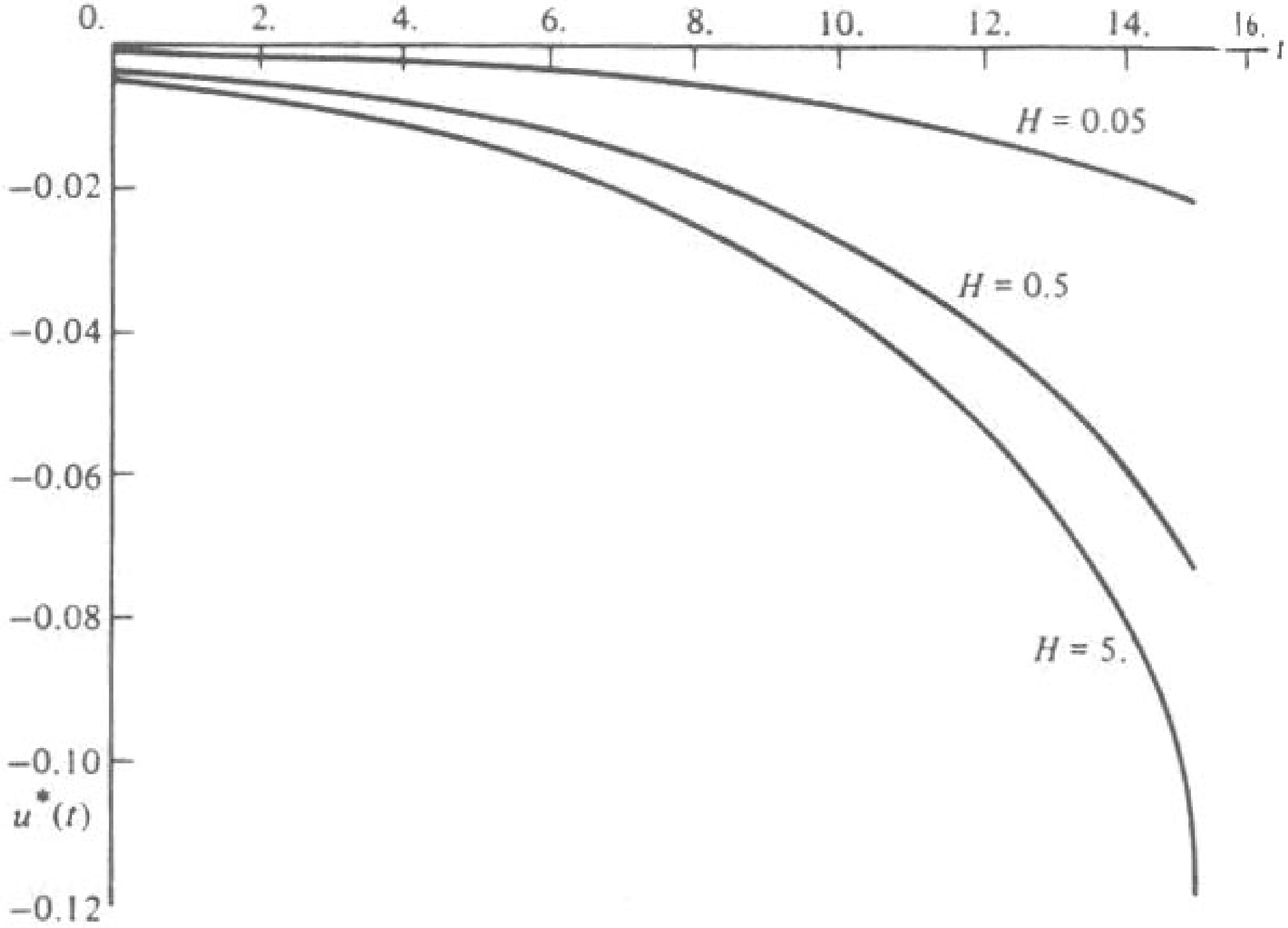
$$\varphi(t) = \begin{bmatrix} \varepsilon^{at} & \frac{1}{a}\varepsilon^{-at} - \frac{1}{a}\varepsilon^{at} \\ 0 & \varepsilon^{-at} \end{bmatrix}$$

$$K(t) = \left[\varepsilon^{-a(T-t)} - \frac{H}{a}\left[\varepsilon^{-a(T-t)} - \varepsilon^{a(T-t)} \right] \right]^{-1} \left[\mathbf{H}\varepsilon^{a(T-t)} \right]$$

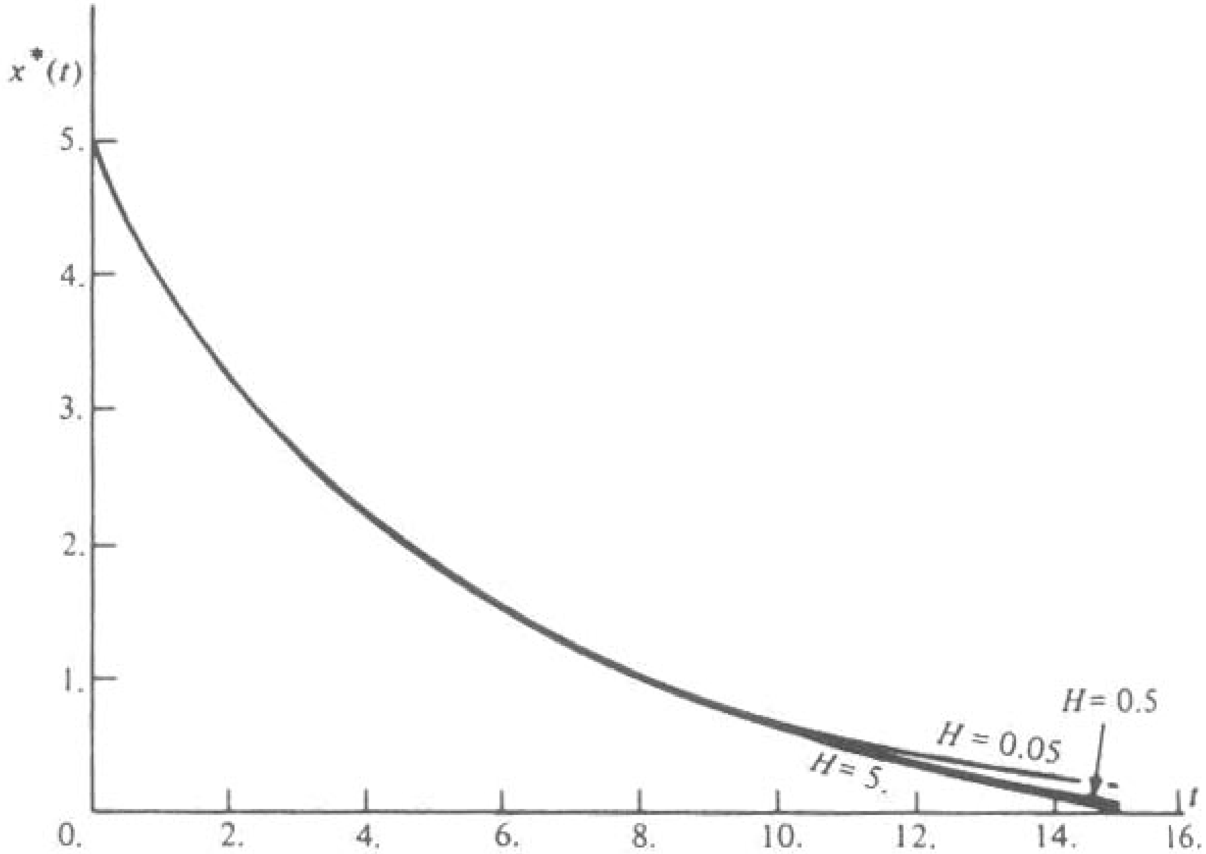
$$u^{*}(t) = -2K(t)x(t)$$



(a)

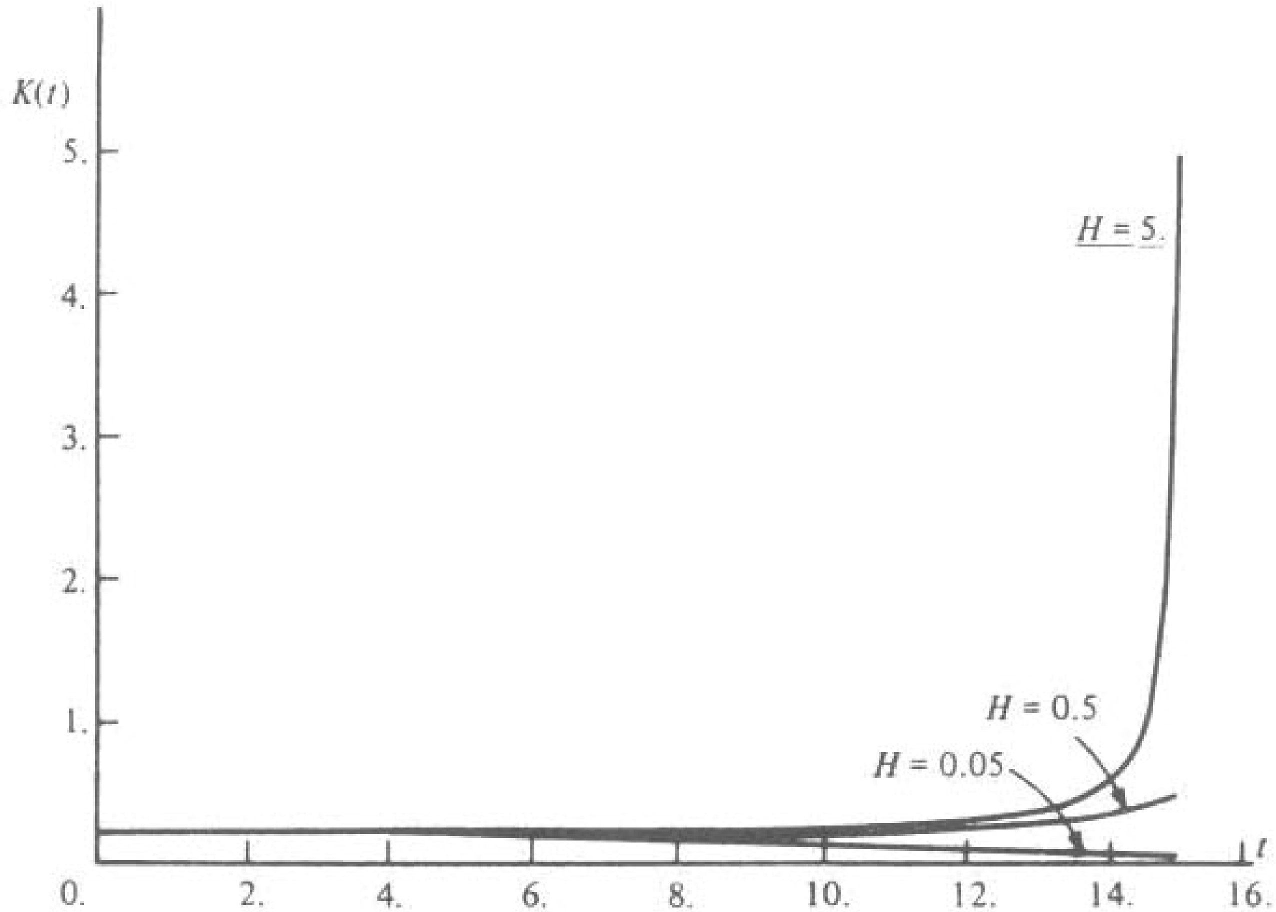


(b)

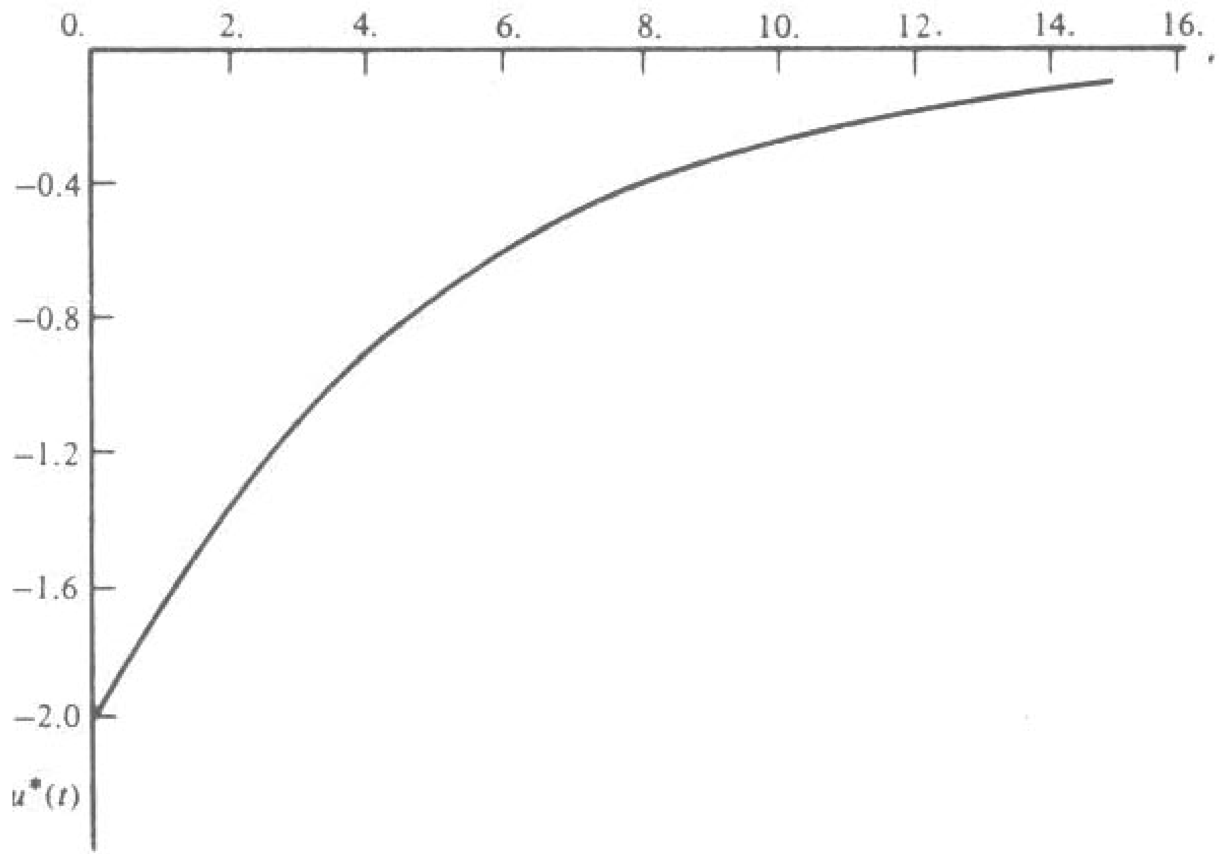


(c)

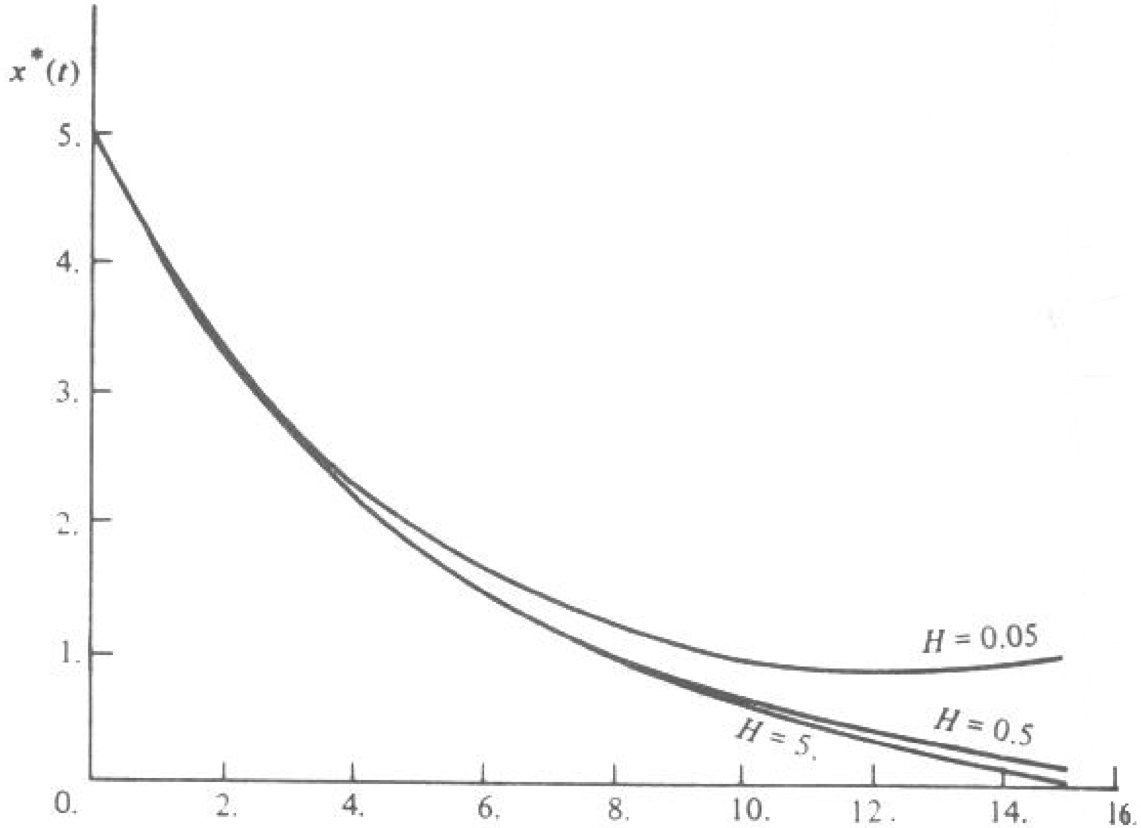
Figure 5-7 (a) Solution of the Riccati equation for $a = -0.2$, $H = 5, 0.5, 0.05$. (b) The optimal control histories for $a = -0.2$, $H = 5, 0.5, 0.05$. (c) The optimal trajectories for $a = -0.2$, $H = 5, 0.5$, and 0.05.



(a)



(b)



(c)

Figure 5-8 (a) Solution of the Riccati equation for $a = 0.2$, $H = 5, 0.5, 0.05$. (b) The optimal control histories for $a = 0.2$, $H = 5, 0.5, 0.05$. (c) The optimal trajectories for $a = 0.2$, $H = 5, 0.5, 0.05$.

Example 5.2-2.

$$\dot{x}_1(t) = x_2(t)$$

$$\dot{x}_2(t) = 2x_1(t) - x_2(t) + u(t)$$

$$J(u) = \int_0^T \left[x_1^2(t) + \frac{1}{2} x_2^2(t) + \frac{1}{4} u^2(t) \right] dt$$

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \mathbf{Q} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \quad R = \frac{1}{2}$$

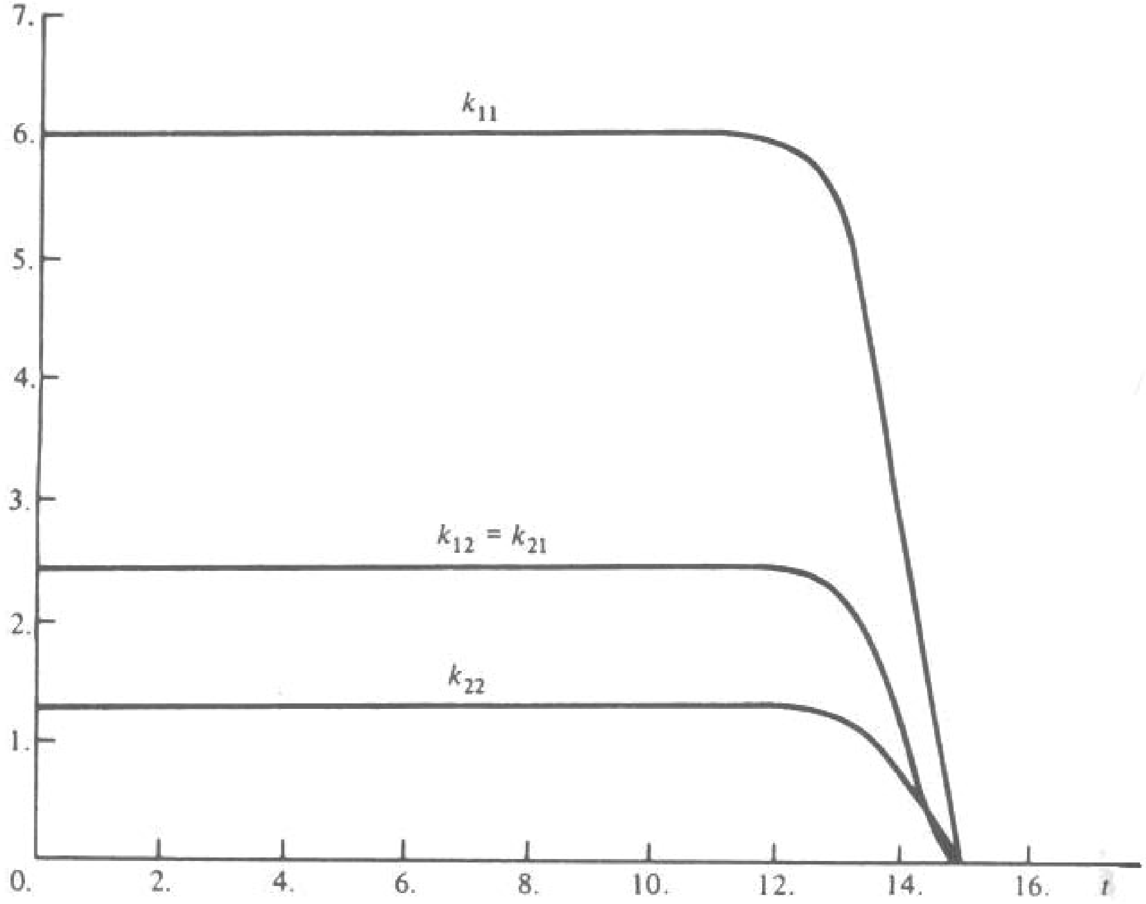
$$\dot{k}_{11}(t) = 2\left[k_{12}^2(t) - 2k_{12}(t) - 1\right]$$

$$\dot{k}_{12}(t) = 2k_{12(t)}k_{22(t)} - k_{11}(t) + k_{12}(t) - 2k_{22}(t)$$

$$\dot{k}_{22}(t) = 2k_{22(t)}^2 - 2k_{12}(t) + 2k_{22}(t) - 1$$

$$u^*(t) = -2\left[k_{12}(t) \quad \mathrm{k}_{22}(t)\right] x(t).$$

$$\mathbf{0 = -KA - A^T K - Q + KBR^{-1}B^T K}$$



(a)

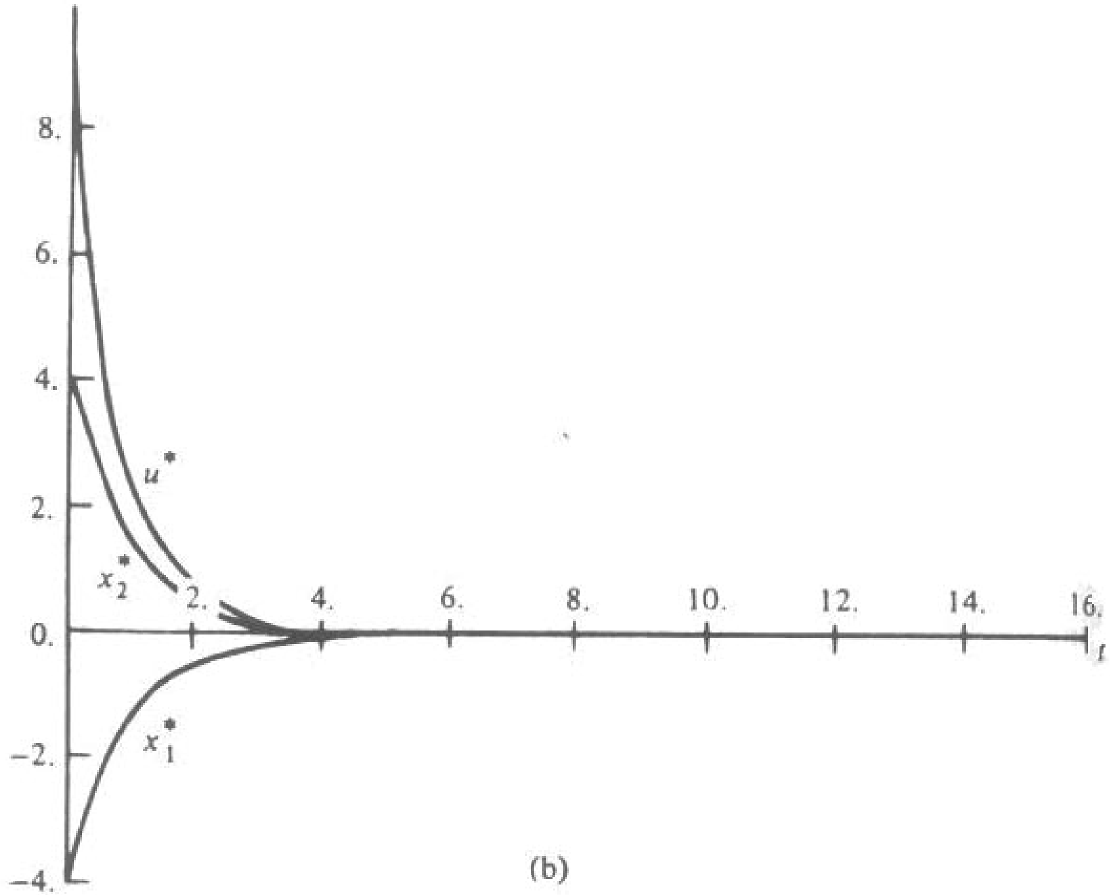


Figure 5-9 (a) The solution of the Riccati equation. (b) The optimal control and its trajectory

Linear Tracking Problems

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t)$$

$$J = \frac{1}{2}\left[\mathbf{x}(t_f) - \mathbf{r}(t_f)\right]^T \mathbf{H}\left[\mathbf{x}(t_f) - \mathbf{r}(t_f)\right] + \frac{1}{2}\int_{t_0}^{t_f} \left\{\left[\mathbf{x}(t) - \mathbf{r}(t)\right]^T \mathbf{Q}(t)\left[\mathbf{x}(t) - \mathbf{r}(t)\right]\right.$$

$$\left. + \mathrm{u}^{\mathrm{T}}(t)\mathbf{R}(t)u(t)\right\} dt$$

$$\triangleq \frac{1}{2}\left\|\mathbf{x}(t_f) - r(t_f)\right\|_{\mathbf{H}}^2 + \frac{1}{2}\int_{t_0}^{t_f} \left\{\left\|\mathbf{x}(t) - r(t)\right\|_{\mathbf{Q}(t)}^2 + \left\|\mathbf{u}(t)\right\|_{\mathbf{R}(t)}^2\right\} dt$$

$$H(\mathbf{x}(t), \mathbf{u}(t), \mathbf{p}(t), t) = \frac{1}{2}\left\|\mathbf{x}(t) - r(t)\right\|_{\mathbf{Q}(t)}^2 + \frac{1}{2}\left\|\mathbf{u}(t)\right\|_{\mathbf{R}(t)}^2$$

$$+ \mathbf{p}^{\mathbf{T}}(t)\mathbf{A}(t)\mathbf{x}(t) + \mathbf{p}^{\mathbf{T}}(t)\mathbf{B}(t)\mathbf{u}(t)$$

$$\dot{\mathbf{p}}^*(t) = -\frac{\partial H}{\partial \mathbf{x}} = -\mathbf{Q}(t)\mathbf{x}^*(t) - \mathbf{A}^{\mathbf{T}}(t)\mathbf{p}^*(t) + \mathbf{Q}(t)\mathbf{r}(t)$$

$$0=\frac{\partial H}{\partial \mathbf{u}}=\mathbf{R}(t)\mathbf{u}^*(t)+\mathbf{B}^{\mathbf{T}}(t)\mathbf{p}^*(t)$$

$$u^*(t)=-\mathbf{R}^{-1}(t)\mathbf{B}^{\mathbf{T}}(t)\mathbf{p}^*(t)$$

$$\begin{bmatrix}\dot{\mathbf{x}}^*(t)\\ \dot{\mathbf{p}}^*(t)\end{bmatrix}=\begin{bmatrix}A(t) & -\mathbf{B}(t)\mathbf{R}^{\mathbf{-1}}(t)\mathbf{B}^{\mathbf{T}}(t)\\ -Q(t) & -\mathbf{A}^{\mathbf{T}}(t)\end{bmatrix}\begin{bmatrix}\mathbf{x}^*(t)\\ \mathbf{p}^*(t)\end{bmatrix}+\begin{bmatrix}0\\ \mathbf{Q}(t)\mathbf{r}(t)\end{bmatrix}$$

$$\begin{bmatrix}\dot{\mathbf{x}}^*(t_f)\\ \dot{\mathbf{p}}^*(t_f)\end{bmatrix}=\varphi(t_f,t)\begin{bmatrix}\dot{\mathbf{x}}^*(t)\\ \dot{\mathbf{p}}^*(t)\end{bmatrix}+\int_t^{t_f}\varphi(t_f,\tau)\begin{bmatrix}0\\ \mathbf{Q}(\tau)\mathbf{r}(\tau)\end{bmatrix}d\tau$$

$$\begin{bmatrix}\mathbf{f}_1(t)\\ \mathbf{f}_2(t)\end{bmatrix}$$

$$\mathbf{x}^*(t_f)=\varphi_{11}(t_f,t)\mathbf{x}^*(t)+\varphi_{12}(t_f,t)\mathbf{p}^*(t)+\mathbf{f}_1(t)$$

$$\mathbf{p}^*(t_f)=\varphi_{21}(t_f,t)\mathbf{x}^*(t)+\varphi_{22}(t_f,t)\mathbf{p}^*(t)+\mathbf{f}_2(t)$$

$$\mathbf{p}^*(t_f)=\mathbf{H}\mathbf{x}^*(t_f)-\mathbf{H}\mathbf{r}(t_f).$$

$$
\begin{aligned}
&\mathbf{H}\left[\varphi_{11}(t_f,t)\mathbf{x}^*(t)+\varphi_{12}(t_f,t)\mathbf{p}^*(t)+\mathbf{f}_1(t)\right]-\mathbf{H}\mathbf{r}(t_f)\\
&\qquad=\varphi_{21}(t_f,t)\mathbf{x}^*(t)+\varphi_{22}(t_f,t)\mathbf{p}^*(t)+\mathbf{f_2}(t)
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{p}^*(t)&=\left[\varphi_{22}(t_f,t)-\mathbf{H}\varphi_{12}(t_f,t)\right]^{-1}\left[\mathbf{H}\varphi_{11}(t_f,t)-\varphi_{21}(t_f,t)\right]\mathbf{x}^*(t)\\
&\quad+\left[\left[\varphi_{22}(t_f,t)-\mathbf{H}\varphi_{12}(t_f,t)\right]^{-1}\right]\left[\mathbf{H}\mathbf{f}_1(t)-\mathbf{H}\mathbf{r}(t_f)-\mathbf{f}_2(t)\right]\\
&\triangleq\mathbf{K}(t)\mathbf{x}^*(t)+\mathbf{s}(t).
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{u}^*(t)&=-\mathbf{R}^{\mathbf{-1}}(t)\mathbf{B}^{\mathbf{T}}\mathbf{K}(t)\mathbf{x}(t)-\mathbf{R}^{\mathbf{-1}}(t)\mathbf{B}^{\mathbf{T}}\mathbf{s}(t)\\
&\triangleq\ \mathbf{F}(t)\mathbf{x}(t)+\mathbf{v}
\end{aligned}
$$

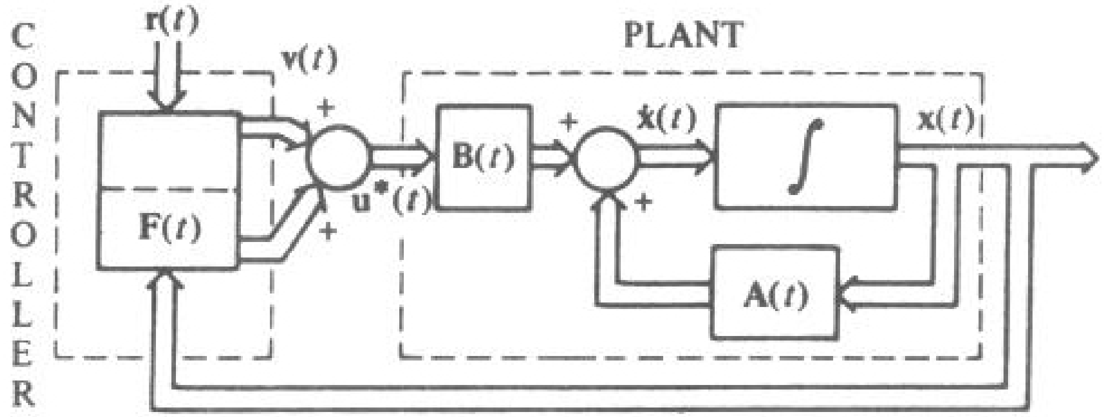


Figure 5-10 Plant and optimal feedback controller for linear tracking problems

$$\mathbf{p}^*(t) = \mathbf{K}(t)\mathbf{x}^*(t) + \mathbf{s}(t)$$

$$\dot{\mathbf{p}}^*(t) = \dot{\mathbf{K}}(t)\mathbf{x}^*(t) + \mathbf{K}(t)\dot{\mathbf{x}}^*(t) + \dot{\mathbf{s}}(t).$$

$$\left[\dot{\mathbf{K}}(t) + \mathbf{Q}(t) + \mathbf{K}(t)\mathbf{A}(t) + \mathbf{A}^{\mathbf{T}}(t)\mathbf{K}(t) - \mathbf{K}(t)\mathbf{B}(t)\mathbf{R}^{-1}(t)\mathbf{B}^{\mathbf{T}}(t)\mathbf{K}(t)\right]\mathbf{x}^*(t)$$
$$+\left[\dot{\mathbf{s}}(t) + \mathbf{A}^{\mathbf{T}}(t)\mathbf{s}(t) - \mathbf{K}(t)\mathbf{B}(t)\mathbf{R}^{-1}(t)\mathbf{B}^{\mathbf{T}}(t)\mathbf{s}(t) - \mathbf{Q}(t)\mathbf{r}(t)\right] = 0$$

$$\dot{\mathbf{K}}(t) = -\mathbf{K}(t)\mathbf{A}(t) - \mathbf{A}^{\mathbf{T}}(t)\mathbf{K}(t) - \mathbf{Q}(t) + \mathbf{K}(t)\mathbf{B}(t)\mathbf{R}^{\mathbf{-1}}(t)\mathbf{B}^{\mathbf{T}}(t)\mathbf{K}(t)$$

$$\dot{\mathbf{s}}(t) = -\left[\mathbf{A}^{\mathbf{T}}(t) - \mathbf{K}(t)\mathbf{B}(t)\mathbf{R}^{\mathbf{-1}}(t)\mathbf{B}^{\mathbf{T}}(t)\right]\mathbf{s}(t) + \mathbf{Q}(t)\mathbf{r}(t)$$

$$\mathbf{p}^*(t_f) = \mathbf{H}\mathbf{x}^*(t_f) - \mathbf{H}\mathbf{r}(t_f) = \mathbf{K}(\mathrm{t_f})\mathbf{x}^*(t_f) + \mathbf{s}(t_f).$$

$$\mathbf{K}(t_f) = \mathbf{H}$$

$$\mathbf{s}(t_f) = -\mathbf{H}\mathbf{r}(t_f)$$

Example 5.2-3

$$\dot{x}_1(t) = x_2(t)$$

$$\dot{x}_2(t) = 2x_1(t) - x_2(t) + u(t)$$

$$J(u) = \left[x_1(T) - 1 \right]^2 + \int_0^T \left\{ \left[x_1(t) - 1 \right]^2 + 0.0025u^2(t) \right\} dt$$

$$\dot{k}_{11}(t) = 2\left[100k_{12}^2(t) - 2k_{12}(t) - 1 \right]$$

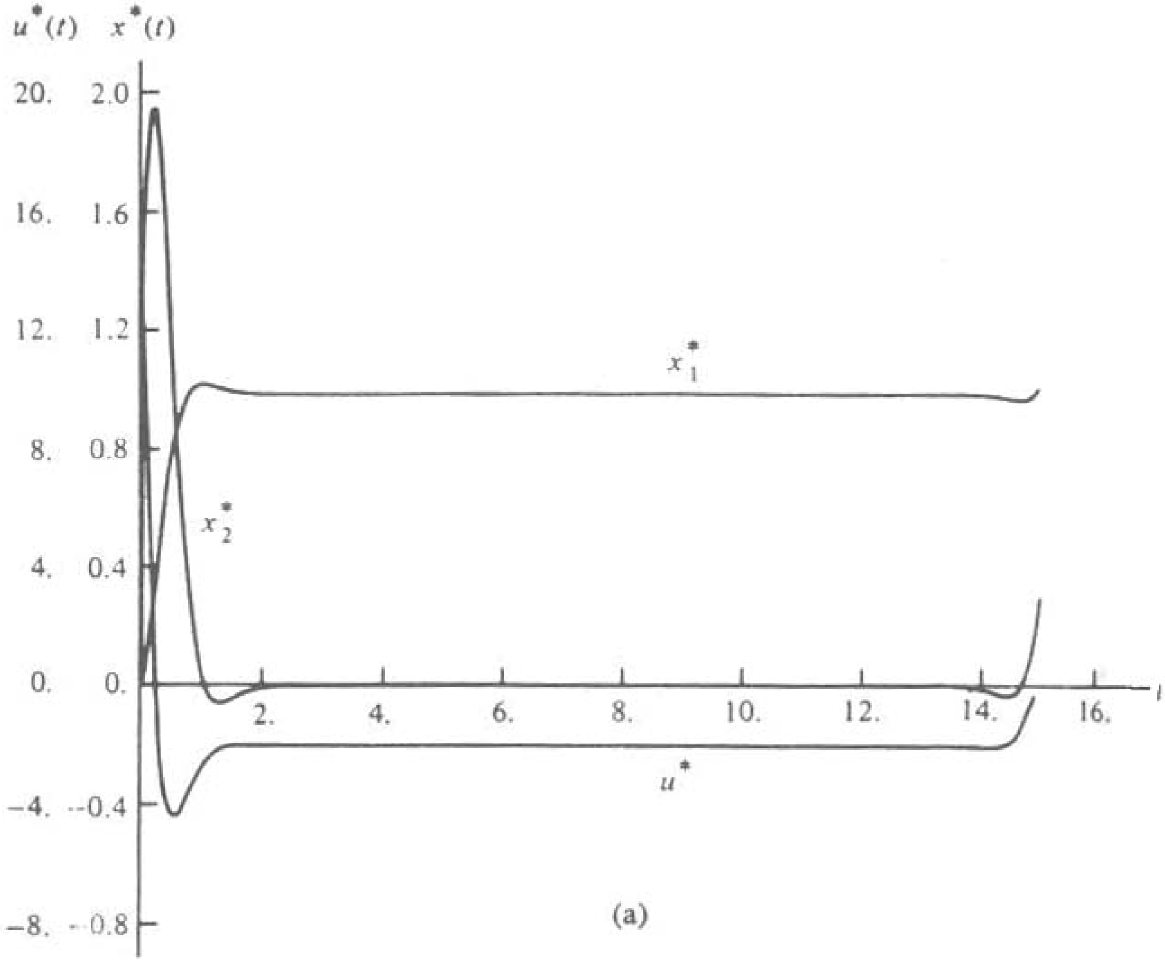
$$\dot{k}_{12}(t) = 200k_{12(t)}k_{22(t)} - k_{11}(t) + k_{12}(t) - 2k_{22}(t)$$

$$\dot{k}_{22}(t) = 200k_{22(t)}^2 - 2k_{12}(t) + 2k_{22}(t)$$

$$\dot{s}_1(t) = 2\left[100k_{12}(t) - 1 \right] s_2(t) + 2$$

$$\dot{s}_2(t) = -s_1(t) + \left[1 + 200k_{22}(t) \right] s_2(t)$$

$$u^*(t) = -200\left[k_{12}(t)x_1(t) + k_{22}(t)x_2(t) + s_2(t)\right]$$



(a)

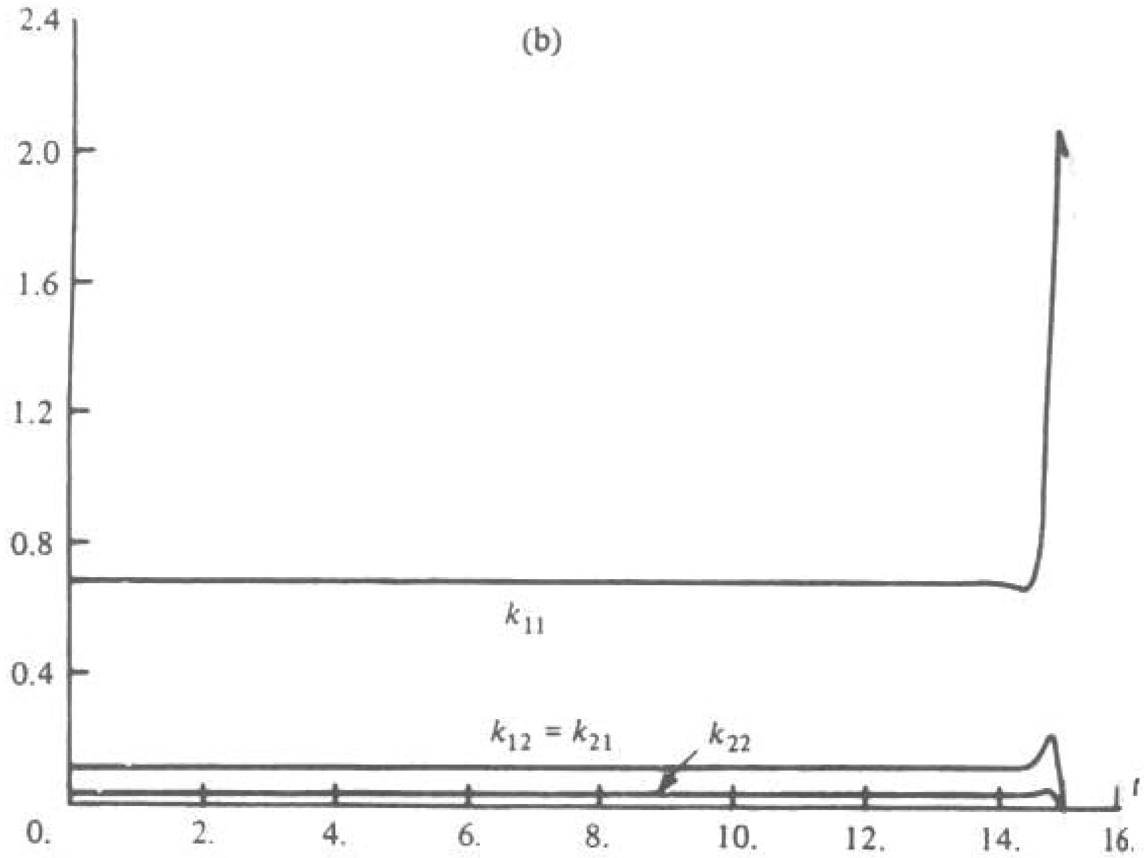
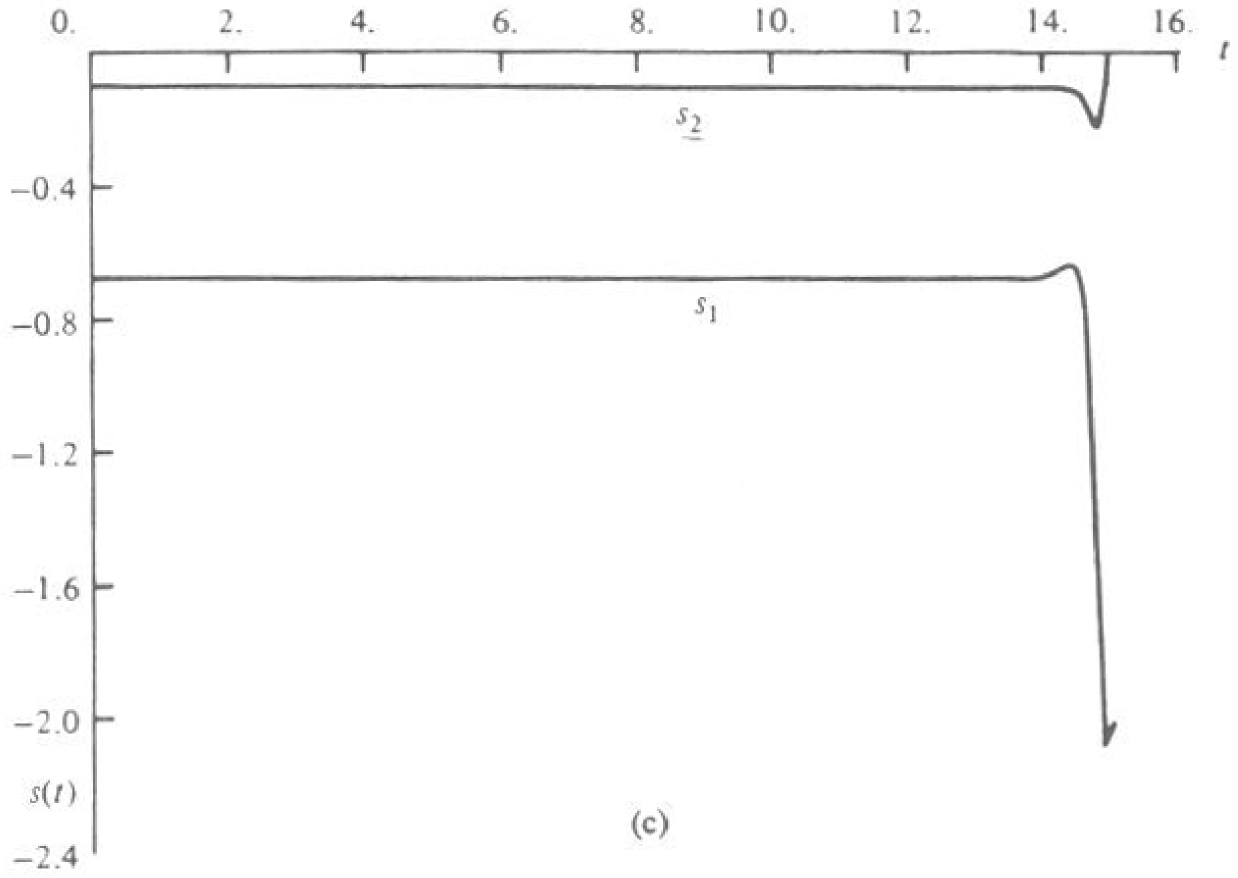


Figure 5-11 (a) The optimal control and trajectory for a linear tracking problem: $r_1(t) = 1.0$, $\mathbf{x}(0) = \mathbf{0}$. (b) Solution of the Riccati equation for Example 5.2-3. (c) s_1 and s_2 for Example 5.2-3



(c)

Figure 5-11 cont.

Example 5.2-4.

$$J(u) = \int_0^T \left\{ \left[x_1(t) - 0.2t \right]^2 + 0.025 u^2(t) \right\} dt.$$

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad \mathbf{Q} = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}, \qquad \mathbf{H} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$R = 0.05, \quad \text{and} \quad \mathbf{r}(t) = \begin{bmatrix} 0.2t \\ 0 \end{bmatrix}$$

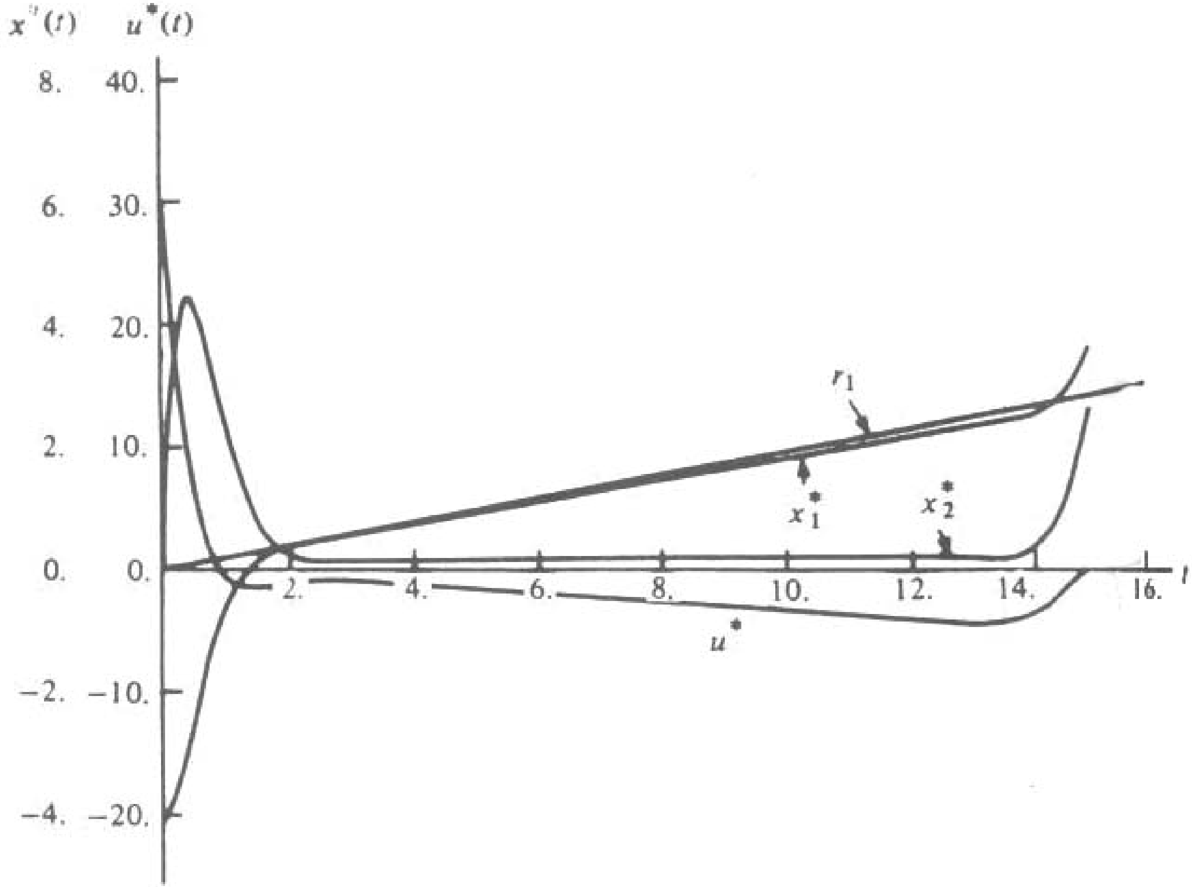
$$\dot{k}_{11}(t) = 20k_{12}^2(t) - 4k_{12}(t) - 2$$

$$\dot{k}_{12}(t) = 20k_{12}(t)k_{22}(t) - k_{11}(t) + k_{12}(t) - 2k_{22}(t)$$

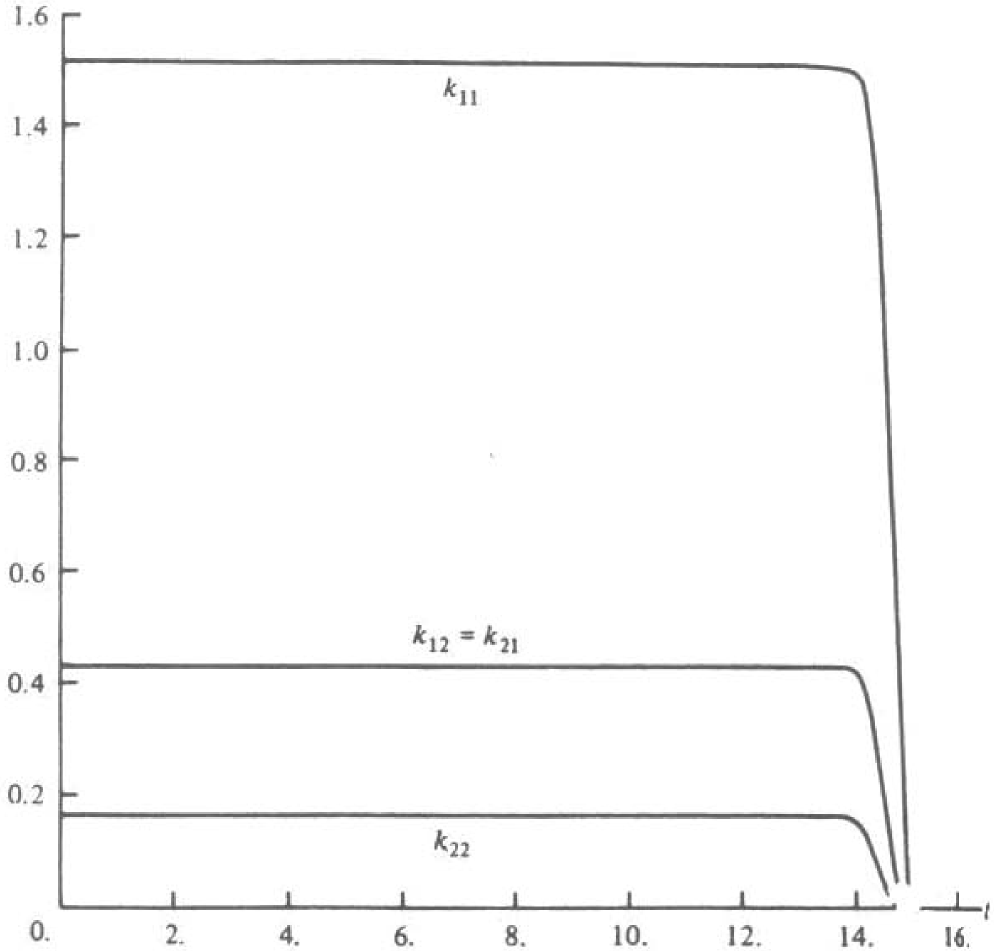
$$\dot{k}_{22}(t) = 20k_{22}^2(t) - 2k_{12}(t) + 2k_{22}(t)$$

$$\dot{s}_1(t) = 2\left[10k_{1\,2}(t) - 1\right] s_2(t) + 0.4t$$
$$\dot{s}_2(t) = -s_1(t) + \left[20k_{2\,2}(t) + 1\right] s_2(t).$$

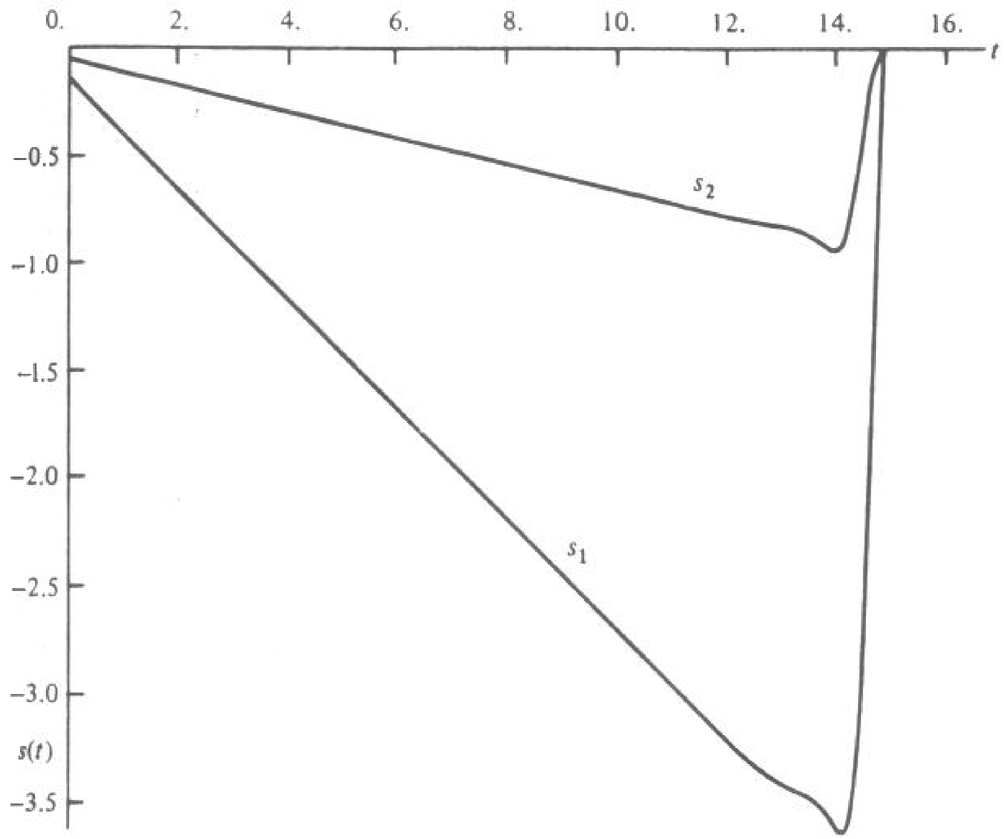
$$u^*(t) = -20\left[k_{1\,2}(t)x_1(t) + k_{2\,2}(t)x_2(t) + s_2(t)\right].$$



(a)



(b)



(c)

Figure 5-12 (a) The optimal control and trajectory for a linear tracking problem: $r_1(t) = 0.2t$, $\mathbf{x}(0) = [-4 \;\; 0]^T$. (b) Solution of the Riccati equation for Example 5.2-4. (c) s_1 and s_2 for Example 5.2-4

5.3 PONTRYAGIN`S MINIMUM PRINCIPLE AND STATE INEQUALITY CONSTRAINTS

Pontryagin's Minimum Principle

$$J(\mathbf{u}) - J(\mathbf{u}^*) = \Delta J \geq 0$$

$$\Delta J(\mathbf{u}^*, \delta\mathbf{u}) = \delta J(\mathbf{u}^*, \delta\mathbf{u}) + \text{higher-order terms;}$$

$$\delta J(\mathbf{u}^*, \delta\mathbf{u}) \geq 0$$

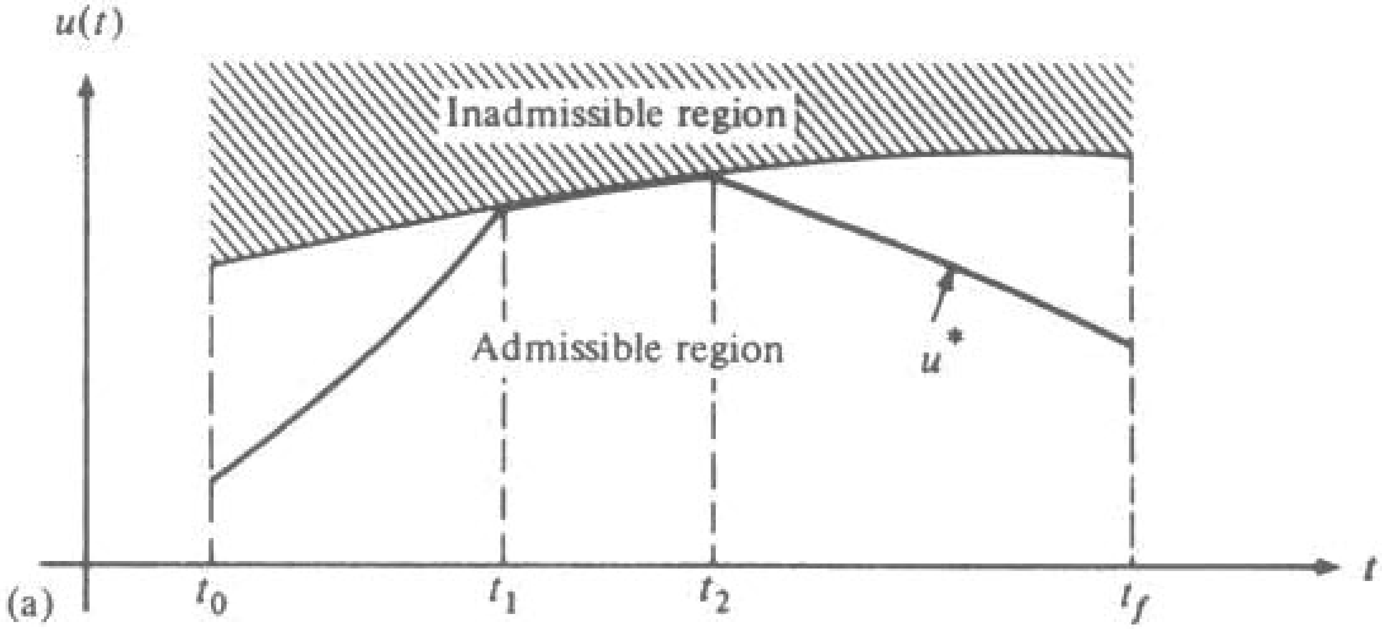


Figure 5-13 (a) An extremal control that is constrained by

a boundary

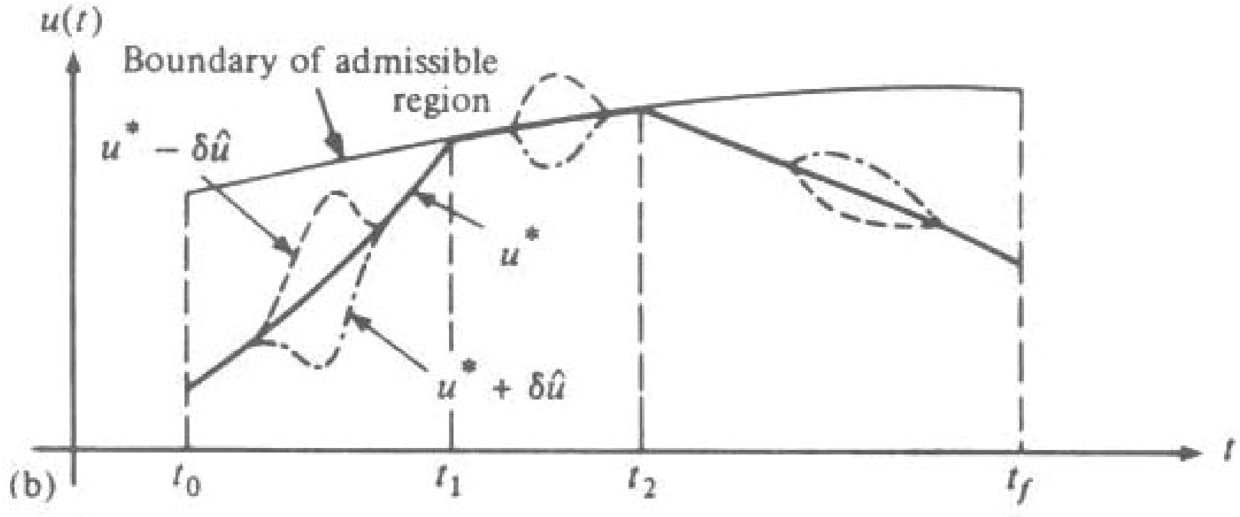


Figure 5-13 (b) An admissible variation $\delta\hat{u}$ for which $-\delta\hat{u}$ is not admissible.

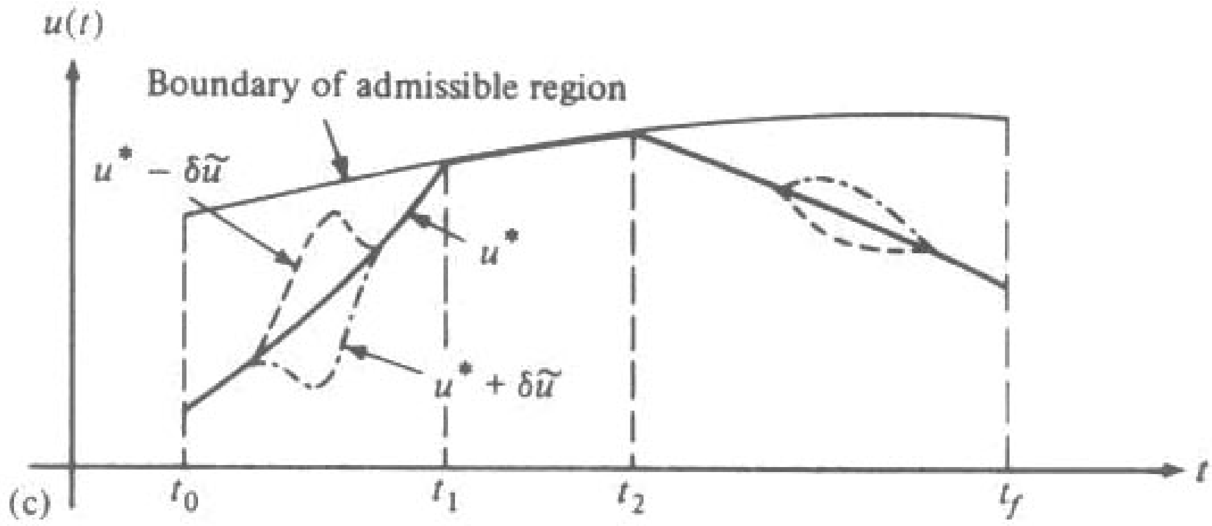


Figure 5-13 (c) An admissible variation $\delta\tilde{u}$ for which $-\delta\tilde{u}$ is admissible.

$df(t_0, \Delta t) \ge 0, \quad \text{admissible } \Delta t \ge 0$

$df(t_f, \Delta t) \ge 0, \quad \text{admissible } \Delta t \le 0$

$df(t, \Delta t) = 0$

Necessary condition for f to have a relative minimum at t_0 <t<t_f

Analogous necessar conditions for the control problem

$\delta J(\mathbf{u}^*, \delta\mathbf{u}) \geq 0$ if u* lies on the boundary during any portion of the time interval

$\delta J(\mathbf{u}^*, \delta\mathbf{u}) = 0$ if u* lies within the boundary

$$\Delta J(\mathbf{u}^*, \delta\mathbf{u}) = \left[\frac{\partial h}{\partial \mathbf{x}}(\mathbf{x}^*(t_f), t_f) - \mathbf{p}^*(t_f)\right]^T \delta\mathbf{x}_f + \left[H(\mathbf{x}^*(t_f), \mathbf{u}^*(t_f), \mathbf{p}^*(t_f), t_f) + \frac{\partial h}{\partial t}(\mathbf{x}^*(t_f), t_f)\right]\delta t_f$$

$$+\int_{t_0}^{t_f} \left\{\left[\dot{\mathbf{p}}^*(t) + \frac{\partial H}{\partial \mathbf{x}}(\mathbf{x}^*(t), \mathbf{u}^*(t), \mathbf{p}^*(t), t)\right]^T \delta\mathbf{x}(t) + \left[\frac{\partial H}{\partial \mathbf{u}}(\mathbf{x}^*(t), \mathbf{u}^*(t), \mathbf{p}^*(t), t)\right]^T \delta\mathbf{u}(t)\right.$$

$$\left. + \left[\frac{\partial H}{\partial \mathbf{p}}(\mathbf{x}^*(t), \mathbf{u}^*(t), \mathbf{p}^*(t), t) - \dot{\mathbf{x}}^*(t)\right]^T \delta\mathbf{p}(t)\right\} dt + \text{higher-order terms.}$$

$$\Delta J(\mathbf{u}^*, \delta\mathbf{u}) = \int_{t_0}^{t_f} \left[\frac{\partial H}{\partial \mathbf{u}} (\mathbf{x}^*(t), \mathbf{u}^*(t), \mathbf{p}^*(t), t) \right]^T \delta\mathbf{u}(t) dt$$

+ higher-order terms.

$$\left[\frac{\partial H}{\partial \mathbf{u}} (\mathbf{x}^*(t), \mathbf{u}^*(t), \mathbf{p}^*(t), t) \right]^T \delta\mathbf{u}(t)$$

$$\doteq H(\mathbf{x}^*(t), \mathbf{u}^*(t) + \delta\mathbf{u}(t), \mathbf{p}^*(t), t) - H(\mathbf{x}^*(t), \mathbf{u}^*(t), \mathbf{p}^*(t), t);$$

$$\Delta J(\mathbf{u}^*, \delta\mathbf{u}) = \int_{t_0}^{t_f} \Big[H(\mathbf{x}^*(t), \mathbf{u}^*(t) + \delta\mathbf{u}(t), \mathbf{p}^*(t), t)$$

$$-H(\mathbf{x}^*(t), \mathbf{u}^*(t), \mathbf{p}^*(t), t) \Big] dt$$

+higher-order terms.

$$\int_{\mathrm{t}_0}^{\mathrm{t}_\mathrm{f}} \left[H(\mathbf{x}^*(t), \mathbf{u}^*(t) + \delta\mathbf{u}(t), \mathbf{p}^*(t), t) - H(\mathbf{x}^*(t), \mathbf{u}^*(t), \mathbf{p}^*(t), t) \right] dt \geq 0$$

$$H(\mathbf{x}^*(t), \mathbf{u}^*(t) + \delta\mathbf{u}(t), \mathbf{p}^*(t), t) \geq H(\mathbf{x}^*(t), \mathbf{u}^*(t), \mathbf{p}^*(t), t)$$

$$\mathbf{u}(t) = \mathbf{u}^*(t); \quad t \notin [t_1, t_2]$$

$$\mathbf{u}(t) = \mathbf{u}^*(t) + \delta\mathbf{u}(t); \quad t \in [t_1, t_2],$$

$$\|\delta\mathbf{u}\| = \int_{t_0}^{t_f} \left[\sum_{i=1}^{m} |\delta u_i(t)| \right] dt$$

Suppose that

$$H(\mathbf{x}^*(t), \mathbf{u}(t), \mathbf{p}^*(t), t) < H(\mathbf{x}^*(t), \mathbf{u}^*(t), \mathbf{p}^*(t), t)$$

$$\int_{t_0}^{t_f} \left[H(\mathbf{x}^*(t), \mathbf{u}(t), \mathbf{p}^*(t), t) - H(\mathbf{x}^*(t), \mathbf{u}^*(t), \mathbf{p}^*(t), t) \right] dt$$

$$= \int_{t_1}^{t_2} \left[H(\mathbf{x}^*(t), \mathbf{u}(t), \mathbf{p}^*(t), t) - H(\mathbf{x}^*(t), \mathbf{u}^*(t), \mathbf{p}^*(t), t) \right] dt < 0$$

$H(\mathbf{x}^*(t), \mathbf{u}(t), \mathbf{p}^*(t), t) < H(\mathbf{x}^*(t), \mathbf{u}^*(t), \mathbf{p}^*(t), t)$ which makes dJ<0, contradicting the optimality of u*

Therefore $H(\mathbf{x}^*(t), \mathbf{u}^*(t), \mathbf{p}^*(t), t) \leq H(\mathbf{x}^*(t), \mathbf{u}(t), \mathbf{p}^*(t), t)$

$$\dot{\mathbf{x}}(t) = \mathbf{a}(\mathbf{x}(t), \mathbf{u}(t), t)$$

$$J(\mathbf{u}) = h(\mathbf{x}(t_f), t_f) + \int_{t_0}^{t_f} g(\mathbf{x}(t), \mathbf{u}(t), t)\,dt,$$

$$H(\mathbf{x}(t), \mathbf{u}(t), \mathbf{p}(t), t) \triangleq g(\mathbf{x}(t), \mathbf{u}(t), t) + \mathbf{p}^T(t)\left[\mathbf{a}(\mathbf{x}(t), \mathbf{u}(t), t)\right]$$

$$\boxed{\left.\begin{aligned}
&\dot{\mathbf{x}}^*(t) = \frac{\partial H}{\partial \mathbf{p}}(\mathbf{x}^*(t), \mathbf{u}^*(t), \mathbf{p}^*(t), t) \\
&\dot{\mathbf{p}}^*(t) = -\frac{\partial H}{\partial \mathbf{x}}(\mathbf{x}^*(t), \mathbf{u}^*(t), \mathbf{p}^*(t), t) \\
&H(\mathbf{x}^*(t), \mathbf{u}^*(t), \mathbf{p}^*(t), t) \le H(\mathbf{x}^*(t), \mathbf{u}(t), \mathbf{p}^*(t), t) \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{for all admissible } \mathbf{u}(t)
\end{aligned}\right\} \text{for all } t \in \left[t_0, t_f\right]}$$

$$\boxed{\begin{aligned}&\left[\frac{\partial h}{\partial \mathbf{x}}(\mathbf{x}^*(t_f),t_f)-\mathbf{p}^*(t_f)\right]^T \delta\mathbf{x}_f\\&+\left[H(\mathbf{x}^*(t_f),\mathbf{u}^*(t_f),\mathbf{p}^*(t_f),t_f)+\frac{\partial h}{\partial t}(\mathbf{x}^*(t_f),t_f)\right]\delta t_f=0\end{aligned}}$$

$$\frac{\partial H}{\partial \mathbf{u}}(\mathbf{x}^*(t),\mathbf{u}^*(t),\mathbf{p}^*(t),t)=0$$

$$\frac{\partial^2 H}{\partial u^2}(\mathbf{x}^*(t),\mathbf{u}^*(t),\mathbf{p}^*(t),t)$$

$$\begin{aligned}H(\mathbf{x}(t),\mathbf{u}(t),\mathbf{p}(t),t)=&\ f(\mathbf{x}(t),\mathbf{p}(t),t)\\&+\left[\mathbf{c}(\mathbf{x}(t),\mathbf{p}(t),t)\right]^T\mathbf{u}(t)+\frac{1}{2}\mathbf{u}^{\mathbf{T}}(t)\mathbf{R}(t)\mathbf{u}(t)\end{aligned}$$

$$\frac{\partial^2 H}{\partial \mathbf{u}^2}(\mathbf{x}^*(t),\mathbf{u}^*(t),\mathbf{p}^*(t),t)=\mathbf{R}(t);$$

$$\mathbf{u}^*(t)=-\mathbf{R}^{-1}(t)\mathbf{c}(\mathbf{x}^*(t),\mathbf{p}^*(t),t)$$

Example 5.3-1.

$$\dot{x}_1(t) = x_2(t)$$

$$\dot{x}_2(t) = -x_2(t) + u(t)$$

$$J(u) = \int_{t_0}^{t_f} \frac{1}{2}\left[x_1^2(t) + u^2(t) \right] dt;$$

t_f is specified, and the final state $\mathbf{x}(t_f)$ is free.

a. Find necessary conditions for an *unconstrained* control to minimize *J*.

$$H(\mathbf{x}(t), u(t), \mathbf{p}(t)) = \frac{1}{2}x_1^2(t) + \frac{1}{2}u^2(t) + p_1(t)x_2(t)$$
$$- p_2(t)x_2(t) + p_2(t)u(t),$$

$$\dot{p}_1^*(t) = -\frac{\partial H}{\partial x_1} = -x_1^*(t)$$

$$\dot{p}_2^*(t) = -\frac{\partial H}{\partial x_2} = -p_1^*(t) + p_2^*(t).$$

$$\frac{\partial H}{\partial u} = u^*(t) + p_2^*(t) = 0$$

$$\frac{\partial^2 H}{\partial u^2} = 1;$$

$$u^*(t) = -p_2^*(t)$$

$$\mathbf{p}^*(t_f) = 0$$

b. Find necessary conditions for optimal control if

$$-1 \le u(t) \le +1 \qquad \text{for all } t \in \left[t_0, t_f \right]$$

$$H(\mathbf{x}^*(t), u(t), \mathbf{p}^*(t)) = \frac{1}{2} x_1^{*2}(t) + \frac{1}{2} u^2(t) + p_1^*(t) x_2^*(t)$$
$$- p_2^*(t) x_2^*(t) + p_2^*(t) u(t)$$

$$\frac{1}{2} u^2(t) + p_2^*(t) u(t)$$

$$u^*(t) = -p_2^*(t)$$

$$u^*(t) = \begin{cases} -1, & \text{for } p_2^*(t) > 1 \\ +1, & \text{for } p_2^*(t) < -1 \end{cases}$$

$$u^*(t) = -p_2^*(t)$$

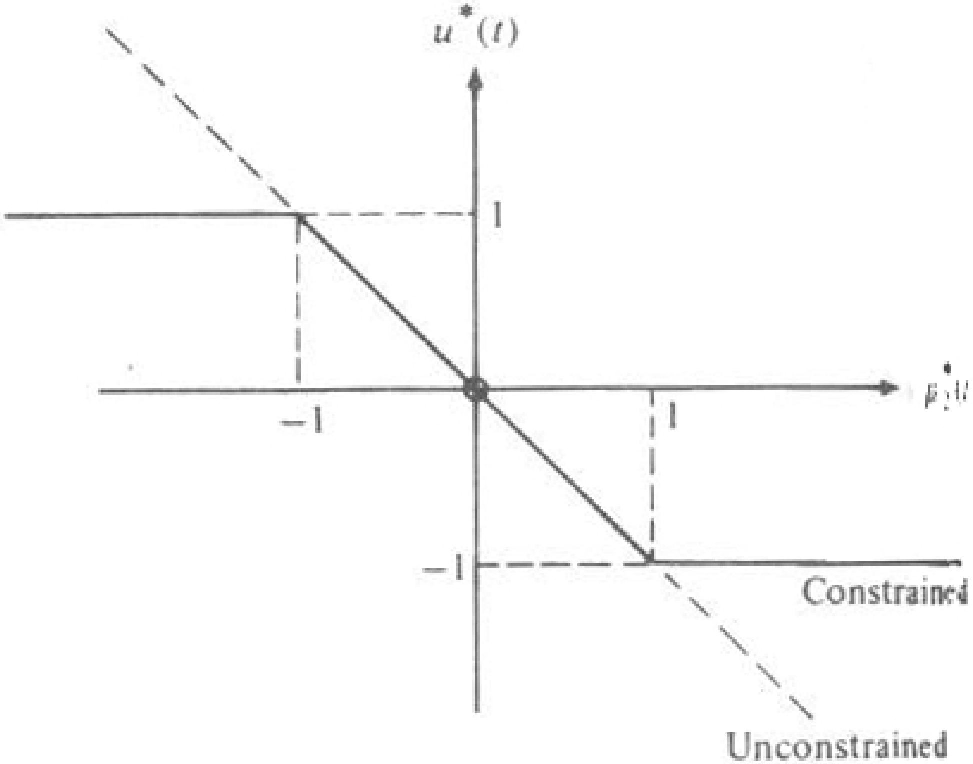


Figure 5-14 Constrained and unconstrained optimal controls for Example 5.3-1

$$u^*(t) = \begin{cases} -1, & \text{for } 1 < p_2^*(t) \\ -p_2^*(t), & \text{for } -1 \le p_2^*(t) \le 1 \\ +1, & \text{for } p_2^*(t) < -1. \end{cases}$$

(5.3-39a)

Additional Necessary Conditions

1. $H(\mathbf{x}^*(t), \mathbf{u}^*(t), \mathbf{p}^*(t)) = c_1 \quad \text{for } t \in [t_0, t_f]$ final time is fixed
2. $H(\mathbf{x}^*(t), \mathbf{u}^*(t), \mathbf{p}^*(t)) = 0 \quad \text{for } t \in [t_0, t_f]$ final time is free

State Variable Inequality Constraints

$\mathbf{f}(\mathbf{x}(t), t) \ge 0$

$$\dot{x}_{n+1}(t) \triangleq \left[f_1 \ (\mathbf{x}(t),t) \right]^2 \mathbb{1}(-f_1 \ ) + \left[f_2 \ (\mathbf{x}(t),t) \right]^2 \mathbb{1}(-f_2 \ )$$

$$+ \cdots + \left[f_l \ (\mathbf{x}(t),t) \right]^2 \mathbb{1}(-f_l)$$

$$\mathbb{1}(-f_i) = \begin{cases} 0, & \text{for } f_i \ (\mathbf{x}(t),t) \geq 0 \\ 1, & \text{for } f_i \ (\mathbf{x}(t),t) < 0, \end{cases}$$

$$x_{n+1}(t) = \int_{t_0}^{t} \dot{x}_{n+1}(t)dt + x_{n+1}(t_0),$$

$$J(\mathbf{u}) = h(\mathbf{x}(t_f),t_f) + \int_{t_0}^{t_f} g(\mathbf{x}(t),\mathbf{u}(t),t)dt$$

$$\dot{\mathbf{x}}(t) = \mathbf{a}(\mathbf{x}(t),\mathbf{u}(t),t)$$

$$\mathbf{f}(\mathbf{x}(t),t) \geq 0$$

$$
\begin{aligned}
H(\mathbf{x}(t),\mathbf{u}(t),\mathbf{p}(t),t) &= g(\mathbf{x}(t),\mathbf{u}(t),t) + p_1(t)a_1(\mathbf{x}(t),\mathbf{u}(t),t) \\
&\quad + \cdots + p_n(t)a_n(\mathbf{x}(t),\mathbf{u}(t),t) \\
&\quad + p_{n+1}(t)\left\{\left[f_1(\mathbf{x}(t),t)\right]^2 \mathbb{1}(-f_1) + \cdots + \left[f_i(\mathbf{x}(t),t)\right]^2 \mathbb{1}(-f_i)\right\} \\
&\triangleq g(\mathbf{x}(t),\mathbf{u}(t),t) + \mathbf{p}^T\mathbf{a}(\mathbf{x}(t),\mathbf{u}(t),t)
\end{aligned}
$$

$$
a_{n+1}(\mathbf{x}(t),t) \triangleq \left[f_1(\mathbf{x}(t),t)\right]^2 \mathbb{1}(-f_1) + \cdots + \left[f_i(\mathbf{x}(t),t)\right]^2 \mathbb{1}(-f_i)
$$

$$\left.\begin{aligned}
&\dot{x}_1^*(t) = a_1(\mathbf{x}^*(t), \mathbf{u}^*(t), t)\\
&\vdots\\
&\dot{x}_{n+1}^*(t) = a_{n+1}(\mathbf{x}^*(t), t);\\
&\dot{p}_1^*(t) = -\frac{\partial H}{\partial x_1}(\mathbf{x}^*(t), \mathbf{u}^*(t), \mathbf{p}^*(t), t)\\
&\vdots\\
&\dot{p}_{n+1}^*(t) = -\frac{\partial H}{\partial x_{n+1}}(\mathbf{x}^*(t), \mathbf{u}^*(t), \mathbf{p}^*(t), t) = 0;\\
&\textit{and}\\
&H(\mathbf{x}^*(t), \mathbf{u}^*(t), \mathbf{p}^*(t), t) \le H(\mathbf{x}^*(t), \mathbf{u}(t), \mathbf{p}^*(t), t)\\
&\text{for all admissible } \mathbf{u}(t)
\end{aligned}\right\}\text{for all } t \in \left[t_0, t_f\right]$$

$$\dot{x}_1(t) = x_2(t)$$

$$\dot{x}_2(t) = -x_2(t) + u(t)$$

$$J(u)=\int_{t_0}^{t_f}\frac{1}{2}\left[x_1^2(t)+u^2(t)\right]dt;$$

$$-1\le u(t)\le 1 \qquad \text{for all } t\in\left[t_0,t_f\right]$$

$$-2\le x_2(t)\le 2 \qquad \text{for all } t\in\left[t_0,t_f\right]$$

$$\left[x_2(t)+2\right]\ge 0$$

$$\left[2-x_2(t)\right]\ge 0$$

$$f_1\left(\mathbf{x}(t)\right)=\left[x_2(t)+2\right]\ge 0$$

$$f_2\left(\mathbf{x}(t)\right)=\left[2-x_2(t)\right]\ge 0$$

$$H(\mathbf{x}(t),u(t),\mathbf{p}(t)) = \frac{1}{2}x_1^2(t) + \frac{1}{2}u^2(t) + p_1(t)x_2(t)$$

$$- p_2(t)x_2(t) + p_2(t)u(t) + p_3(t)\Big\{\left[x_2(t)+2\right]^2 \mathbb{1}(-x_2(t)-2)$$

$$+\left[2-x_2(t)\right]^2 \mathbb{1}(x_2(t)-2)\Big\}$$

$$\dot{x}_1^*(t) = x_2^*(t), \quad x_1^*(t_0) = x_{1o}$$

$$\dot{x}_2^*(t) = -x_2^*(t) + u^*(t) \quad x_2^*(t_0) = x_{2o}$$

$$\dot{x}_3^*(t) = \left[x_2^*(t)+2\right]^2 \mathbb{1}(-x_2^*(t)-2) + \left[2-x_2^*(t)\right]^2 \mathbb{1}(x_2^*(t)-2), \quad x_3^*(t_0) = 0$$

$$\dot{p}_1^*(t) = -\frac{\partial H}{\partial x_1} = -x_1^*(t)$$

$$\dot{p}_2^*(t) = -\frac{\partial H}{\partial x_2} = -p_1^*(t) + p_2^*(t) - 2p_3^*\left[x_2^*(t) + 2\right]\mathbb{1}(-x_2^*(t) - 2)$$

$$+2\mathrm{p}_3^*(t)\left[2 - x_2^*(t)\right]\mathbb{1}(x_2^*(t) - 2)$$

$$\dot{p}_3^*(t) = -\frac{\partial H}{\partial x_3} = 0 \Rightarrow p_3^*(t) = \text{a constant}$$

$$u^*(t) = \begin{cases} -1, & \text{for } 1 < p_2^*(t) \\ -p_2^*(t), & \text{for } -1 \le p_2^*(t) \le 1 \\ +1, & \text{for } p_2^*(t) < -1. \end{cases}$$

$$\mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t) \ge 0$$

5.4 MINIMUM-TIME PROBLEMS

$$\dot{\mathbf{x}}(t) = \mathbf{a}(\mathbf{x}(t), \mathbf{u}(t), t)$$

$$\mathbf{J}(\mathbf{u}) = \int_{t_0}^{t_f} dt = t_f - t_0$$

$$\left|\mathbf{u}_i(t)\right| \le 1, \qquad i = 1, 2, \ldots m, \qquad t \in [t_0, t^*]$$

Example 5.4-1.

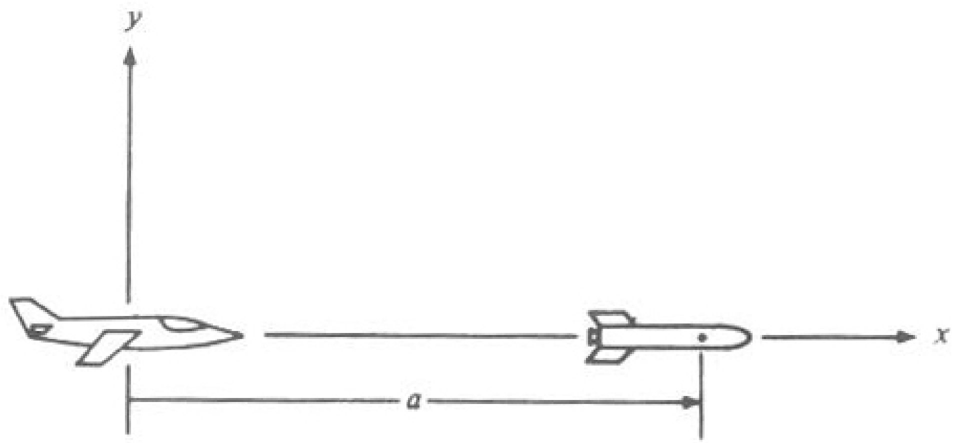


Figure 5-15 An intercept problem

$$x_M(t) = a + 0.1t^3$$
$$y_M(t) = 0$$

$$\ddot{x}(t) = u(t)$$

$$\dot{x}_1(t) = x_2(t)$$
$$\dot{x}_2(t) = u(t)$$

$$|u(t)| \le 1.0$$

$$\frac{1}{2}[t^*]^2 = a + 0.1[t^*]^3$$

The Set of Reachable States

DEFINITION 5-1

> If a system with initial state $\mathbf{x}(t_0) = \mathbf{x}_0$ is subjected to *all* admissible control histories for a time interval [*t0, t*], the collection of state values $\mathbf{x}(t)$ is

called the set of states that are reachable (from $\mathbf{x}_0$) at time t, or simply *the set of reachable states*.

Example 5.4-2.

$$\dot{x}(t) = u(t)$$

$$-1 \leq u(t) \leq 1$$

$$x(t) = x_0 + \int_{t_0}^{t} u(\tau) d\tau$$

$$x_0 - [t - t_0] \leq x(t) \leq x_0 + [t - t_0]$$

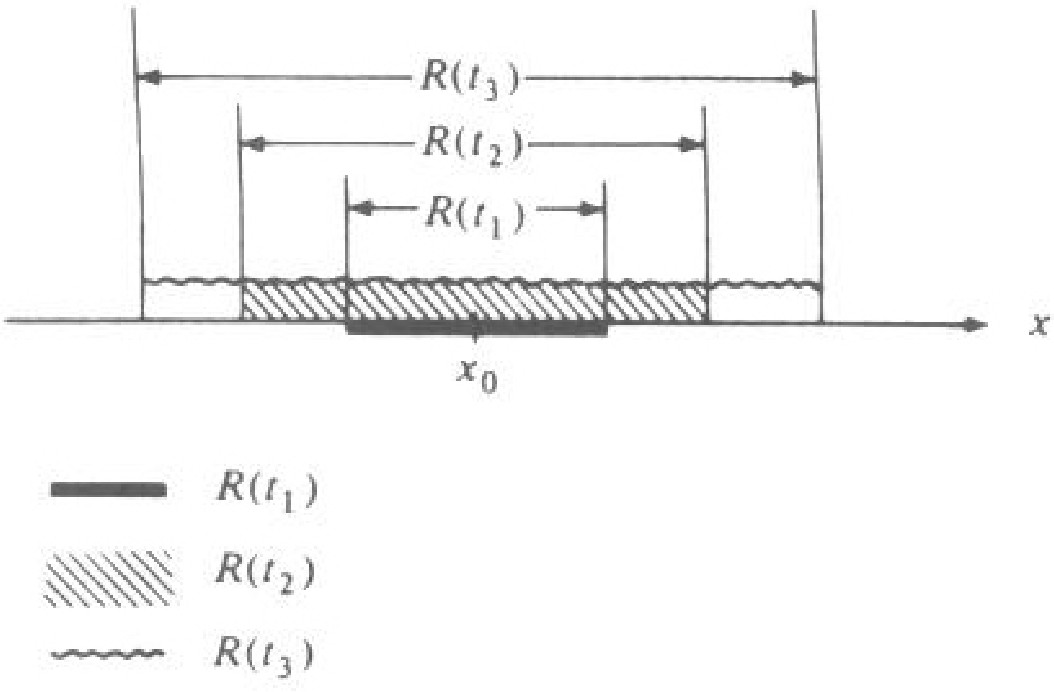


Figure 5-16 The reachable states for Example 5.4-2

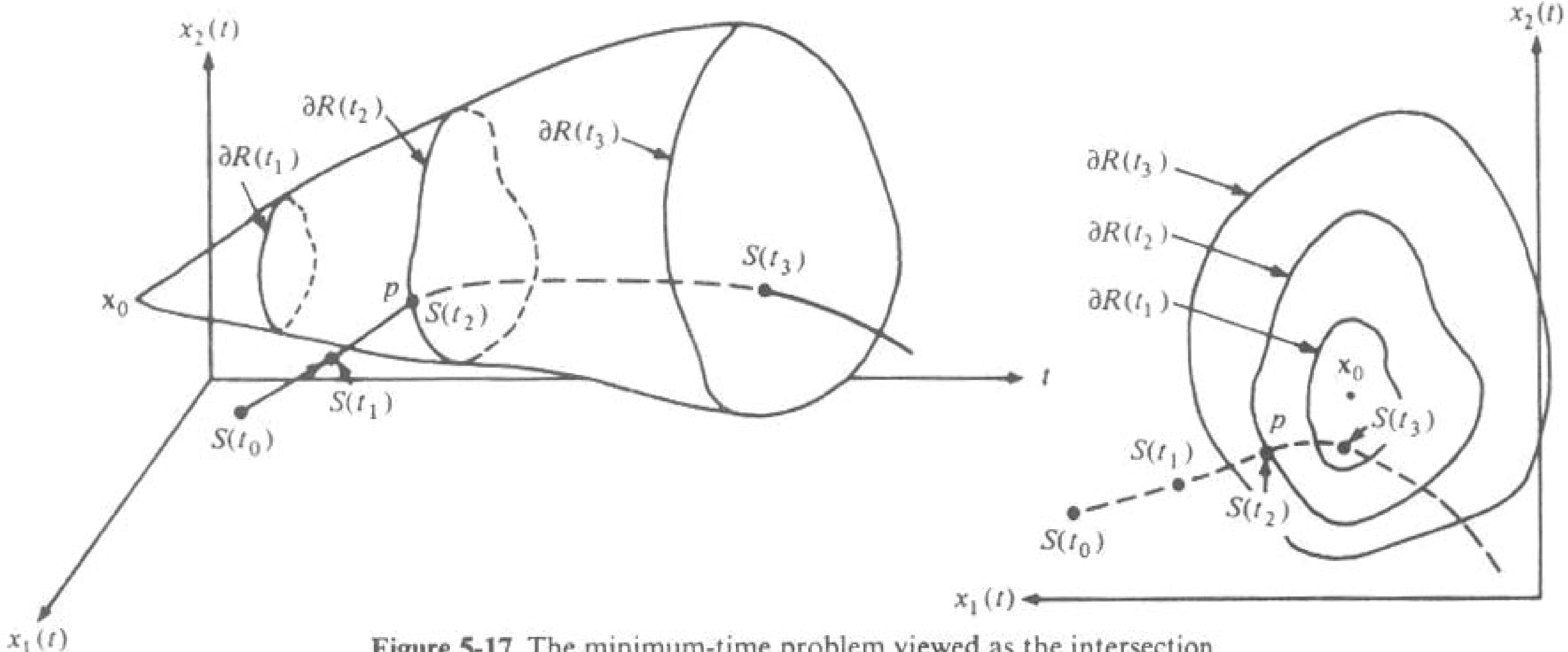


Figure 5-17 The minimum-time problem viewed as the intersection of a target set, $S(t)$, and the set of reachable states, $R(t)$

The Form of the Optimal Control for a Class of Minimum-Time Problems

$$\dot{\mathbf{x}}(t) = \mathbf{a}(\mathbf{x}(t),t) + \mathbf{B}(\mathbf{x}(t),t)\mathbf{u}(t)$$

$$M_{i-} \le u_i(t) \le M_{i+}, \qquad i = 1, 2, \ldots m, \qquad t \in [t_0, t^*];$$

$$\mathscr{H}(\mathbf{x}(t), \mathbf{u}(t), \mathbf{p}(t), t) = 1 + \mathbf{p}^{\mathbf{T}}(t)[\mathbf{a}(\mathbf{x}(t),t) + \mathbf{B}(\mathbf{x}(t),t)\mathbf{u}(t)]$$

$$\begin{aligned} &1 + \mathbf{p}^{*\mathbf{T}}(t)[\mathbf{a}(\mathbf{x}^*(t),t) + \mathbf{B}(\mathbf{x}^*(t),t)\mathbf{u}^*(t)] \\ &\le 1 + \mathbf{p}^{*\mathbf{T}}(t)[\mathbf{a}(\mathbf{x}^*(t),t) + \mathbf{B}(\mathbf{x}^*(t),t)\mathbf{u}(t)] \end{aligned}$$

$$\mathbf{p}^{*\mathbf{T}}(t)\mathbf{B}(\mathbf{x}^*(t),t)\mathbf{u}^*(t) \le \mathbf{p}^{*\mathbf{T}}(t)\mathbf{B}(\mathbf{x}^*(t),t)\mathbf{u}(t)$$

$$\mathbf{B}(\mathbf{x}^*(t),t) = \left[\mathbf{b_1}(\mathbf{x}^*(t),t) \vdots \mathbf{b_2}(\mathbf{x}^*(t),t) \vdots \cdots \vdots \mathbf{b_m}(\mathbf{x}^*(t),t)\right]$$

$$\mathbf{p}^{*\mathbf{T}}(t)\mathbf{B}(\mathbf{x}^*(t),t)\mathbf{u}(t) = \sum_{i=1}^{m} \mathbf{p}^{*\mathbf{T}}(t)[\mathbf{b}_i(\mathbf{x}^*(t),t)]u_i(t)$$

$$u_i^*(t) = \begin{cases} M_{i+}, & \text{for } \mathbf{p}^{*\mathbf{T}}(t)\mathbf{b}_i(\mathbf{x}^*(t),t) < 0 \\ M_{i-}, & \text{for } \mathbf{p}^{*\mathbf{T}}(t)\mathbf{b}_i(\mathbf{x}^*(t),t) > 0 \\ \text{Undetermined}, & \text{for } \mathbf{p}^{*\mathbf{T}}(t)\mathbf{b}_i(\mathbf{x}^*(t),t) = 0. \end{cases}$$

$$i = 1, 2, \ldots, m$$

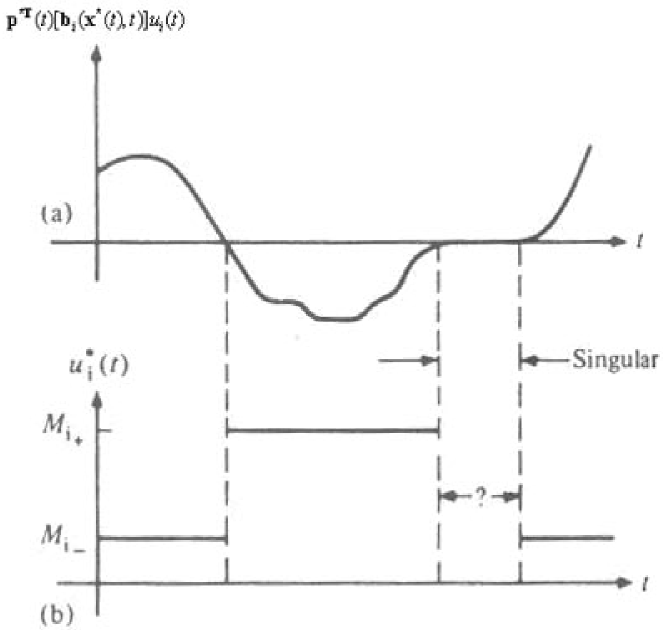


Figure 5-18 The relationship between a time-optimal control and its coefficient in the Hamiltonian

Example 5.4-3.

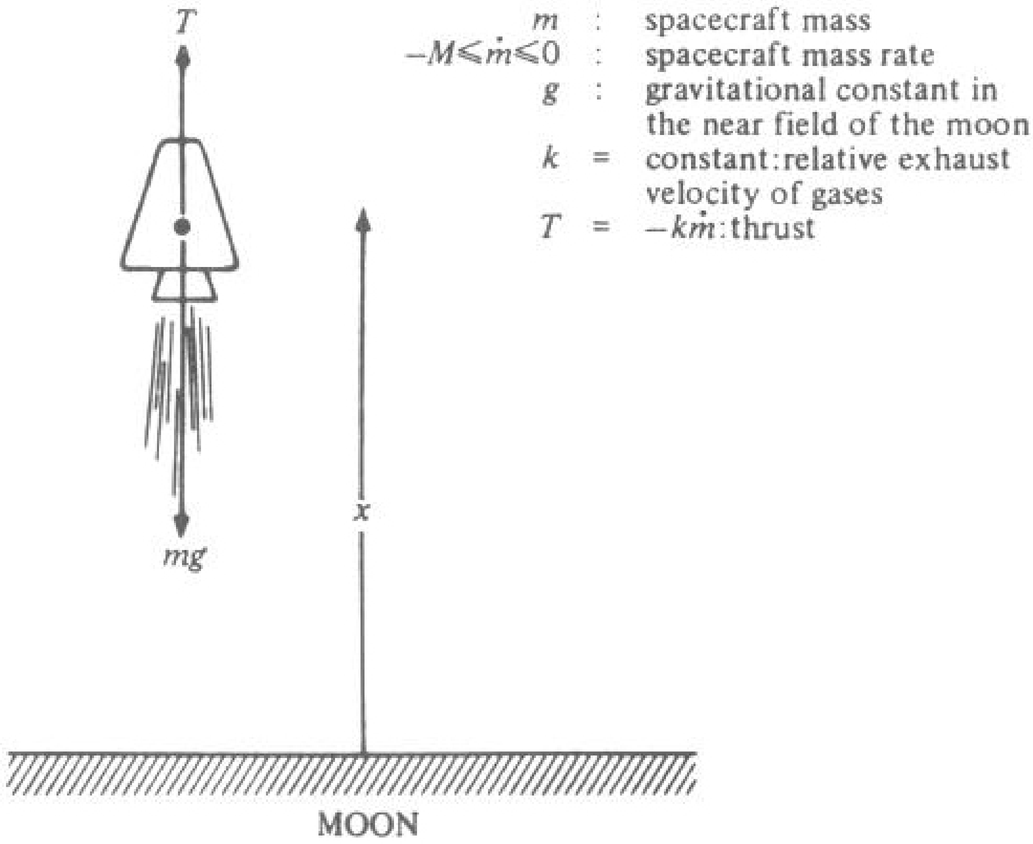


Figure 5-19 Lunar soft landing

a. Aerodynamic forces and gravitational forces of bodies other than the moon are negligible.

b. Lateral motion is ignored; thus, the descent trajectory is vertical and the thrust vector is tangent to the trajectory.

c. The acceleration of gravity is a constant, because of the nearness of the spacecraft to the moon.

d. The relative velocity of the exhaust gases with respect to the spacecraft is constant.

e. The mass rate is constrained by

$$-M \le \dot{m} \le 0$$

$$\begin{aligned} m(t)\ddot{x}(t) &= -gm(t) + T(t) \\ &= -gm(t) - k\dot{m}(t). \end{aligned}$$

$$\dot{x}_1(t) = x_2(t)$$

$$\dot{x}_2(t) = -g - \frac{k}{x_3(t)}u(t)$$

$$\dot{x}_3(t) = u(t)$$

$$\mathcal{H}(\mathbf{x}(t), \mathbf{u}(t), \mathbf{p}(t)) = 1 + p_1(t)p_2(t) - gp_2(t) - \frac{kp_2(t)u(t)}{x_3(t)} + p_3(t)u(t)$$

$$\mathcal{H}(\mathbf{x}^*(t), \mathbf{u}^*(t), \mathbf{p}^*(t)) \le \mathcal{H}(\mathbf{x}^*(t), u(t), \mathbf{p}^*(t))$$

$$u^*(t)\begin{cases} 0, & \text{for } p_3^*(t) - \dfrac{kp_2^*(t)}{x_3^*(t)} < 0 \\ -M, & \text{for } p_3^*(t) - \dfrac{kp_2^*(t)}{x_3^*(t)} > 0 \\ \text{Undetermined}, & \text{for } p_3^*(t) - \dfrac{kp_2^*(t)}{x_3^*(t)} = 0 \end{cases}$$

Minimum-Time Control of Time-Invariant Linear Systems

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t),$$

$$|u_i(t)| \le 1, \qquad i = 1, 2, \ldots, m.$$

THEOREM 5.4-1 (EXISTENCE)

If *all* of the eigenvalues of A have nonpositive real parts, then an optimal control exists that transfers any initial state $\mathbf{x}_0$ to the orgin.

THEOREM 5.4-2 (UNIQUENESS)

If an extremal control exists, then it is unique.

THEOREM 5.4-3 (NUMBER OF SWITCHINGS)

If the eigenvalues of A are all real, and a (unique) time-optimal control exists, then each control component can switch at most (n-1) times.

Example 5.4-4.

$$|u(t)| \le 1$$

$$\dot{x}_1(t) = x_2(t)$$
$$\dot{x}_2(t) = u(t)$$

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\mathscr{H}(\mathbf{x}(t), u(t), \mathbf{p}(t)) = 1 + p_1(t)x_2(t) + p_2(t)u(t)$$

$$p_2^*(t)u^*(t) \le p_2^*(t)u(t)$$

$$u^*(t) = \begin{Bmatrix} -1, & \text{for } p_2^* > 0 \\ +1, & \text{for } p_2^* < 0 \end{Bmatrix} \triangleq -\operatorname{sgn}(p_2^*(t))$$

$$p_1^*(t) = 0$$

$$p_2^*(t) = -p_1^*(t)$$

$$p_1^*(t) = c_1$$

$$p_2^*(t) = -c_1 t + c_2$$

$$u^*(t) = \begin{cases} +1, & \text{for all } t \in [t_0, t^*], \text{ or} \\ -1, & \text{for all } t \in [t_0, t^*], \text{ or} \\ +1, & \text{for } t \in [t_0, t_1), \text{ and } \quad -1, \quad \text{for } t \in [t_1, t^*], \text{ or} \\ -1, & \text{for } t \in [t_0, t_1), \text{ and } \quad +1, \quad \text{for } t \in [t_1, t^*] \end{cases}$$

$$x_2(t) = \pm t + c_3$$

$$x_1(t) = \pm \frac{1}{2} t^2 + c_3 t + c_4$$

$$x_1(t) = \frac{1}{2} x_2^2(t) + c_5, \qquad \text{for } u = +1$$

$$x_1(t) = -\frac{1}{2} x_2^2(t) + c_6, \qquad \text{for } u = -1$$

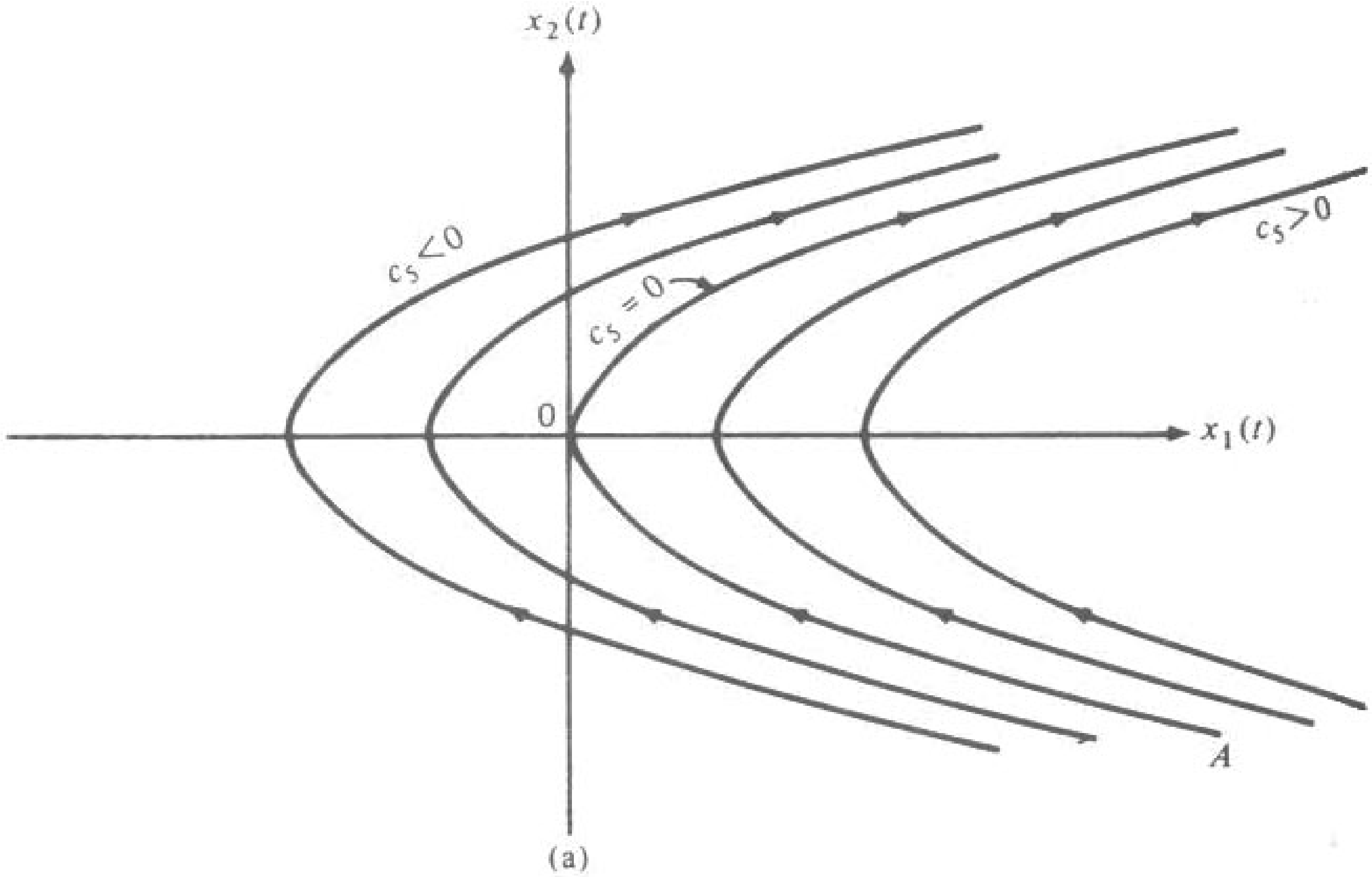


Figure 5-20 (a) Trajectories for u=+ 1

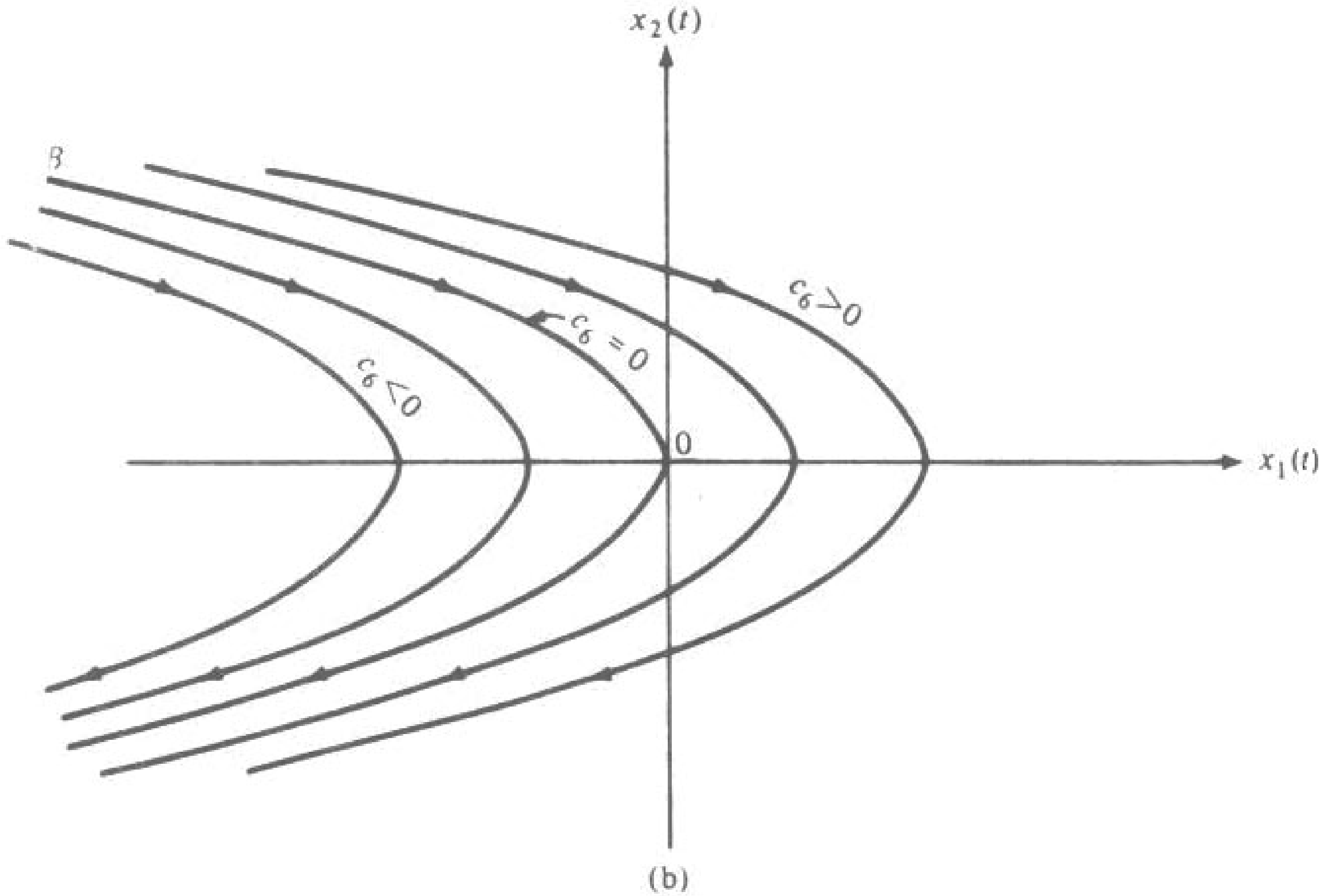


Figure 5-20 (b) Trajectories for u=-1

$$x_1(t) = -\frac{1}{2}x_2(t)\left|x_2(t)\right|.$$

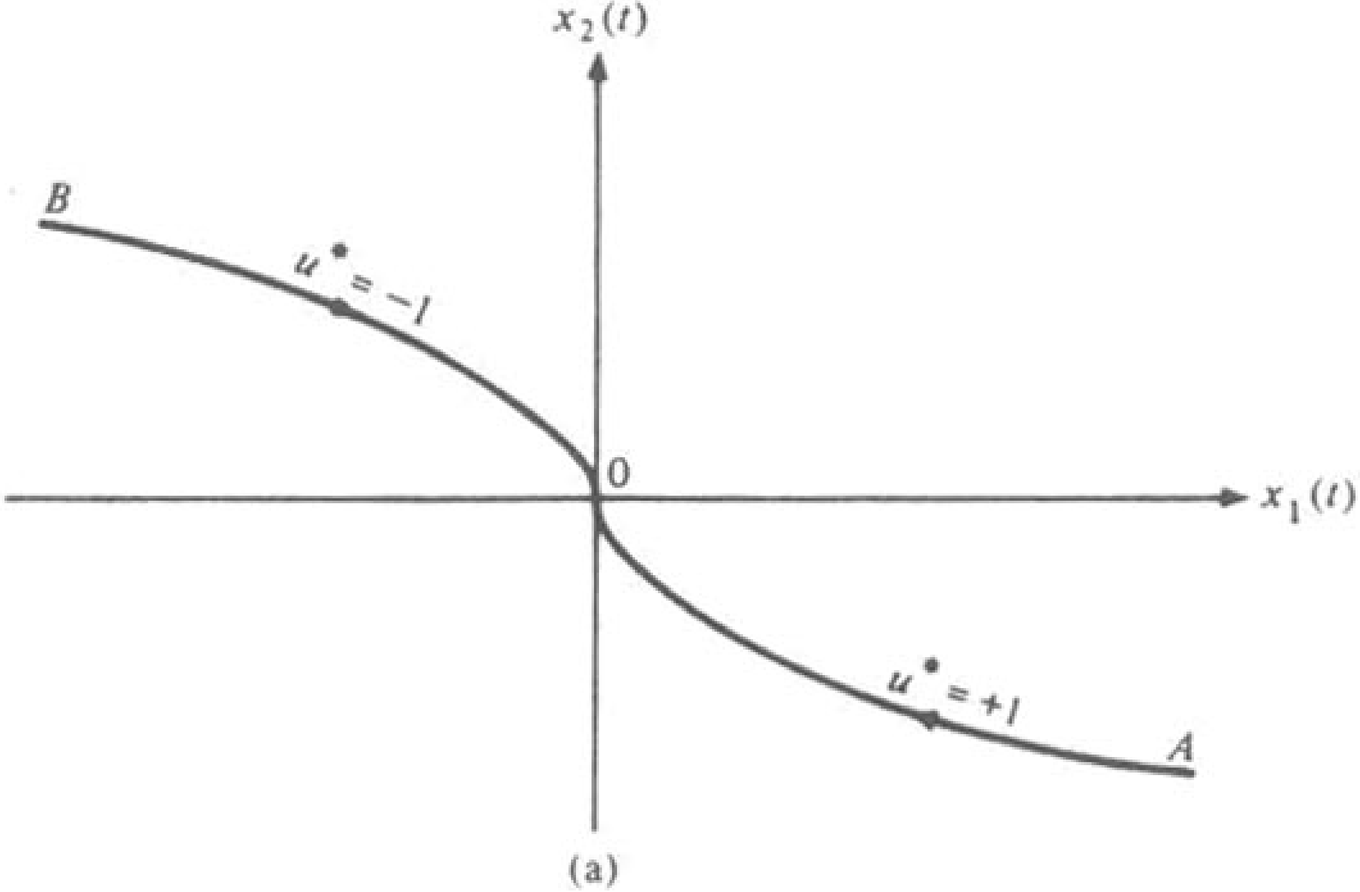
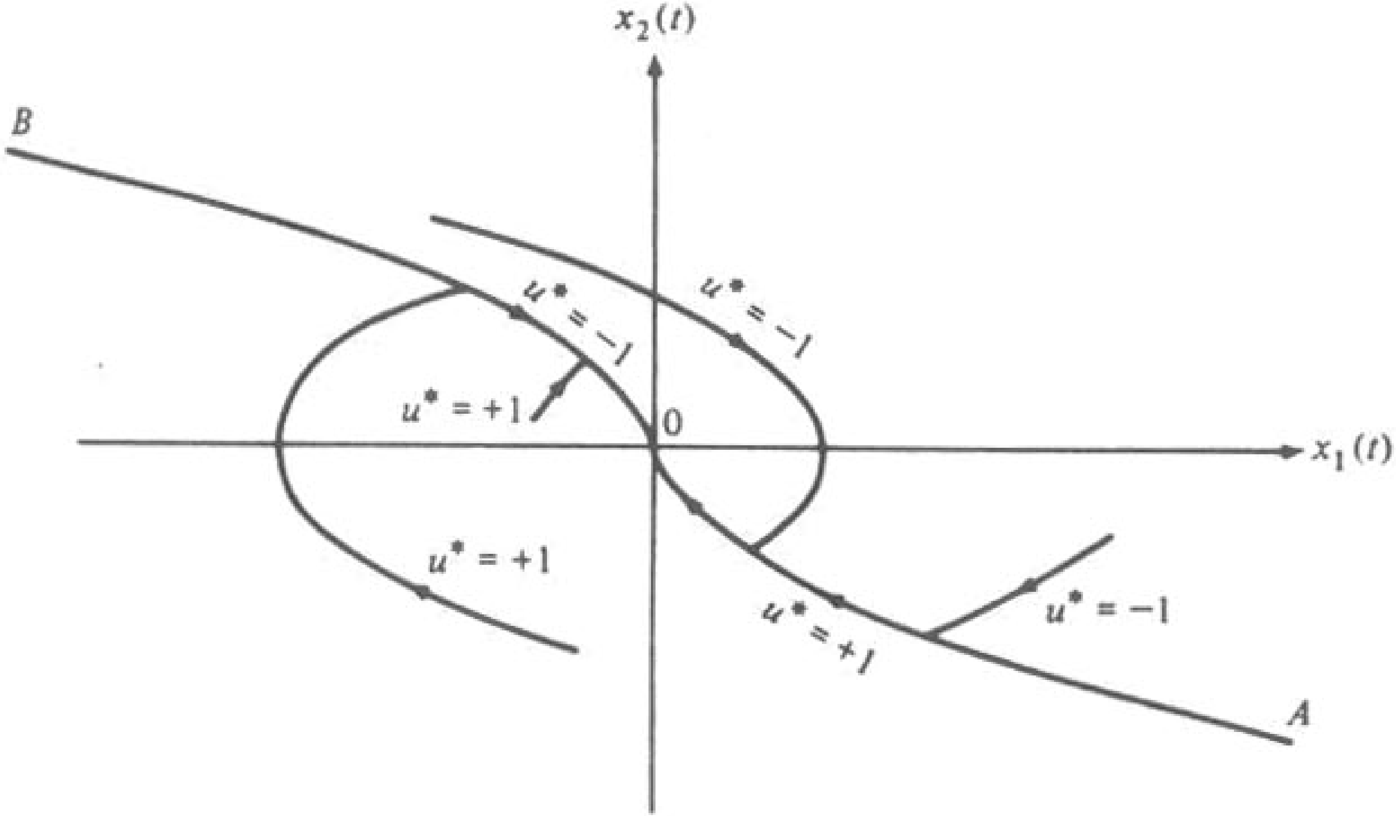


Figure 5-21 (a) The switching curve.



(b)

Figure 5-21 (b) Optimal trajectories for several initial state values.

$$s(\mathbf{x}(t)) \triangleq x_1(t) + \frac{1}{2} x_2(t)\left|x_2(t)\right|$$

$$u^*(t) = \begin{cases} -1, & \text{for } \mathbf{x}(t) \text{ such that } s(\mathbf{x}(t)) > 0 \\ +1, & \text{for } \mathbf{x}(t) \text{ such that } s(\mathbf{x}(t)) < 0 \\ -1, & \text{for } \mathbf{x}(t) \text{ such that } s(\mathbf{x}(t)) = 0 \text{ and } x_2(t) > 0 \\ +1, & \text{for } \mathbf{x}(t) \text{ such that } s(\mathbf{x}(t)) = 0 \text{ and } x_2(t) < 0 \\ 0, & \text{for } \mathbf{x}(t) = 0. \end{cases}$$

1. (a)

$$O_1 = O_+ \cup O_-$$

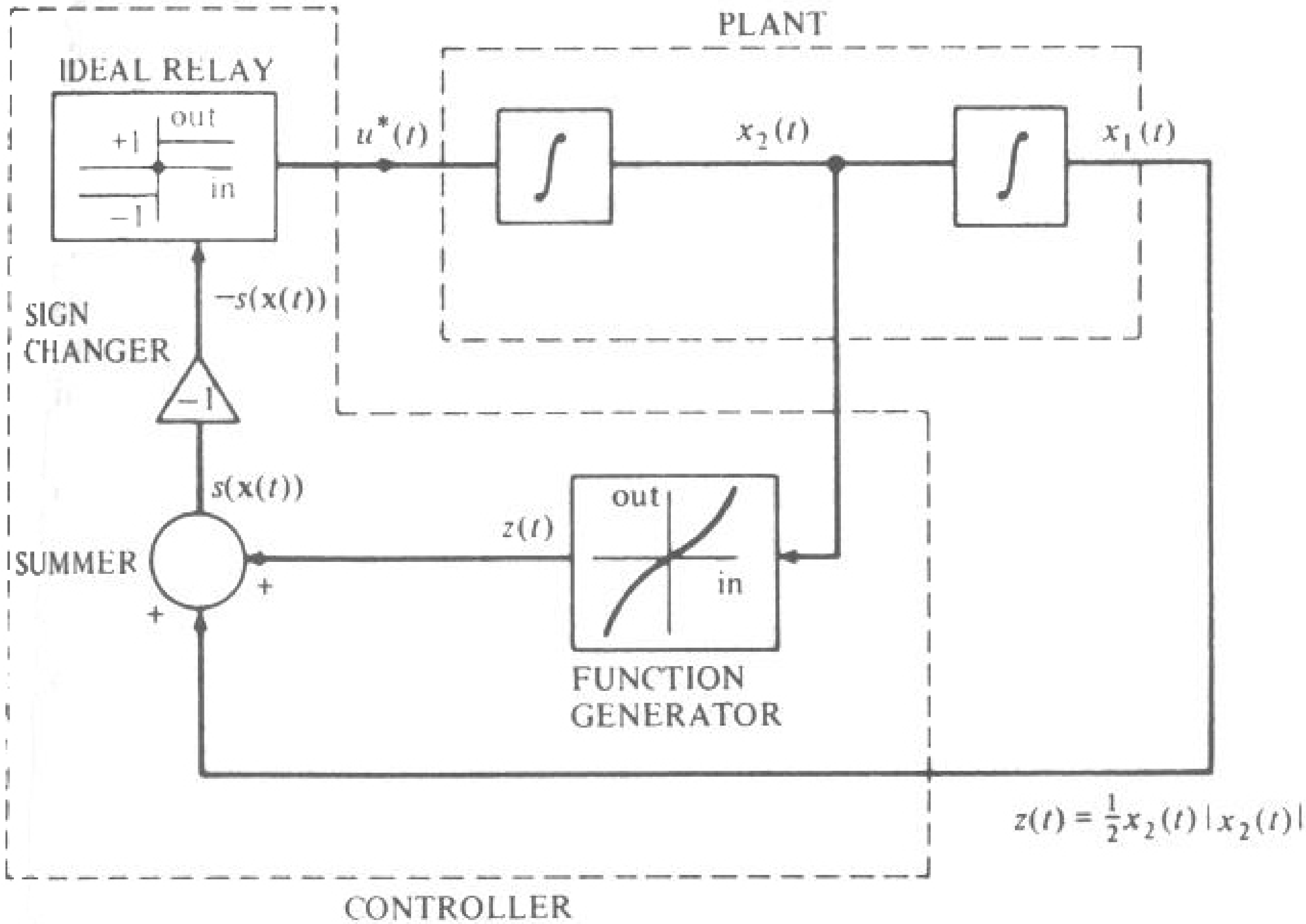


Figure 5-22 Implementation of the time-optimal control law for Example 5.4-4

(b)

$$O_2 = O_+ \cup O_- \cup O_{+-} \cup O_{-+}$$
$$= O_1 \cup O_{+-} \cup O_{-+}$$

(c)

$$s(\mathbf{x}(t)) = 0.$$

Example 5.4-5.

$$\dot{x}_1(t) = x_2(t)$$
$$\dot{x}_2(t) = -ax_2(t) + u(t)$$

$$\left|u(t)\right| \le 1,$$

$$x_2(t) = c_1\varepsilon^{-at} \pm \frac{1}{a}[1 - \varepsilon^{-at}]$$

$$x_1(t) = -\frac{c_1}{a}\varepsilon^{-at} \pm \frac{1}{a}t \pm \frac{1}{a^2}\varepsilon^{-at} + c_2$$

$$x_1(0) = 0 \qquad x_2(0) = 0$$

$$x_2(t) = \pm\frac{1}{a}[1-\varepsilon^{-at}]$$

$$x_1(t) = \pm\frac{1}{a}t \pm \frac{1}{a^2}\varepsilon^{-at} \mp \frac{1}{a^2}$$

$$x_1(t) = -\frac{1}{a^2}\ln\left(-a\left[x_2(t)-\frac{1}{a}\right]\right)-\frac{1}{a}x_2(t)$$ u=+1 일때

$$O_- = \left\{x_1(t), x_2(t) : x_1(t) = \frac{1}{a^2}\ln\left(a\left[x_2(t)+\frac{1}{a}\right]\right)-\frac{1}{a}x_2(t)\right\}$$

$$O_1 = \left\{x_1(t), x_2(t) : x_1(t) = \frac{x_2(t)}{|x_2(t)|}\frac{1}{a^2}\ln\left(a\left[|x_2(t)|+\frac{1}{a}\right]\right)-\frac{1}{a}x_2(t)\right\}$$

$$s(\mathbf{x}(t)) = x_1(t) - \frac{x_2(t)}{|x_2(t)|}\frac{1}{a^2}\ln\left(a\left[|x_2(t)| + \frac{1}{a}\right]\right) + \frac{1}{a}x_2(t)$$

$$u^*(t) = \begin{cases} -1, & \text{for } \mathbf{x}(t) \text{ such that } s(\mathbf{x}(t)) > 0 \\ +1, & \text{for } \mathbf{x}(t) \text{ such that } s(\mathbf{x}(t)) < 0 \\ -1, & \text{for } \mathbf{x}(t) \text{ such that } s(\mathbf{x}(t)) = 0 \text{ and } x_2(t) > 0 \\ +1, & \text{for } \mathbf{x}(t) \text{ such that } s(\mathbf{x}(t)) = 0 \text{ and } x_2(t) < 0 \\ 0, & \text{for } \mathbf{x}(t) = 0 \end{cases}$$

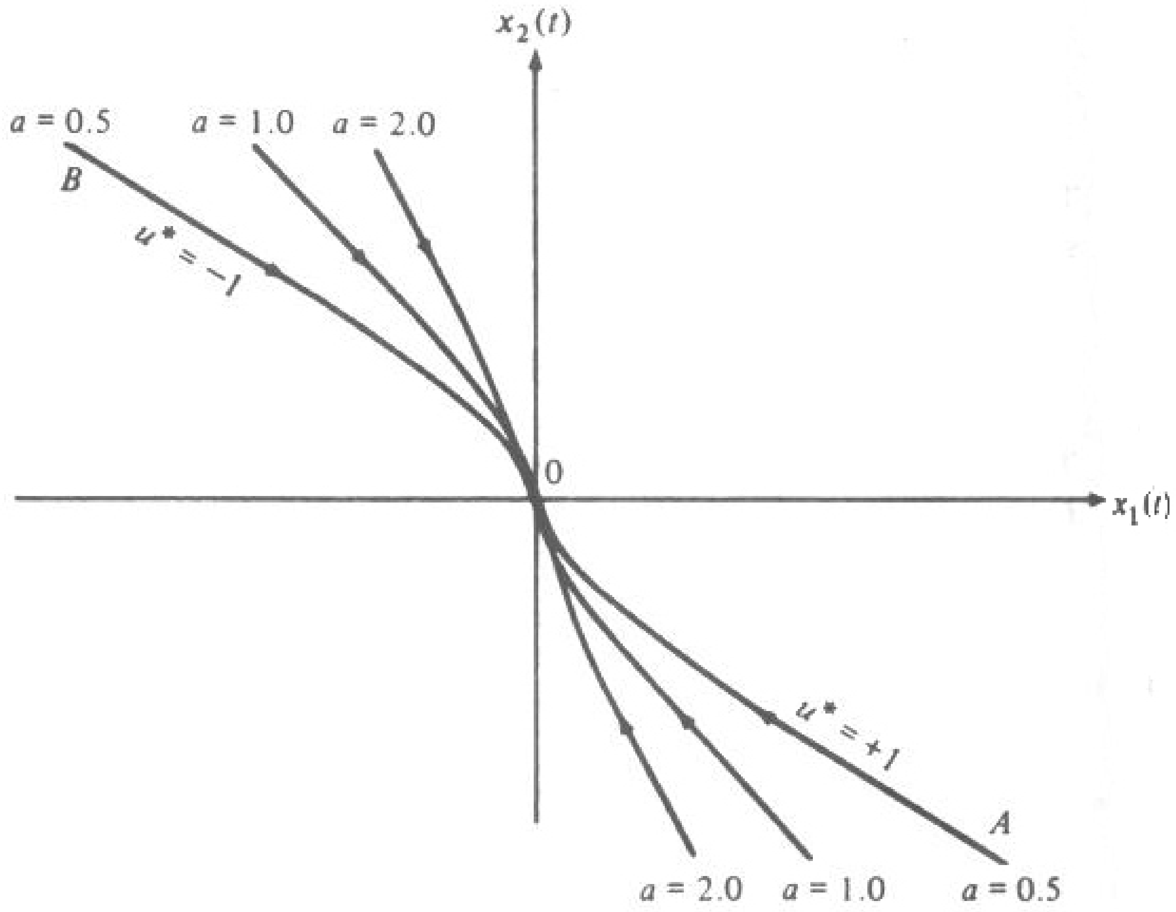


Figure 5-23 Time-optimal switching curves for Example 5.4-5 with $a = 0.5, 1.0, 2.0$

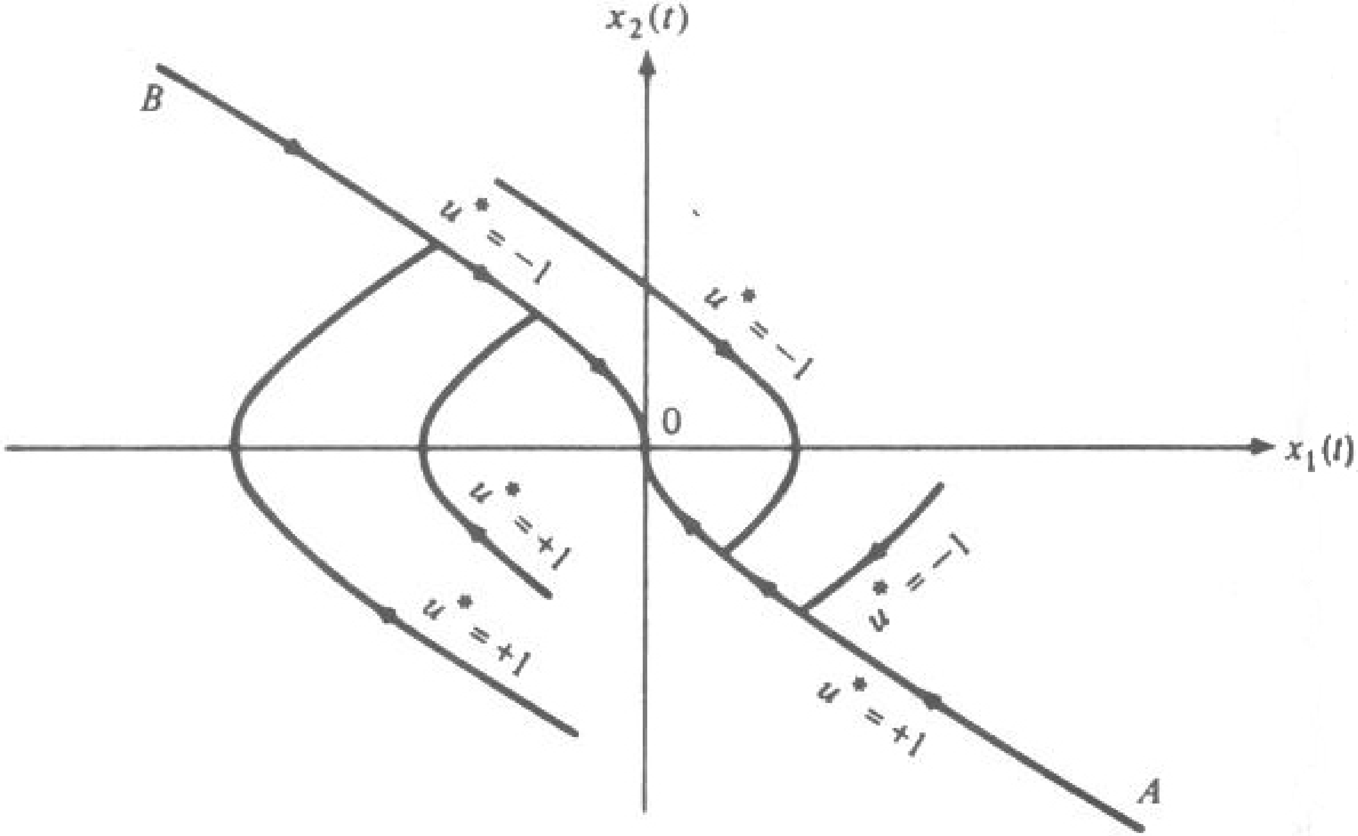


Figure 5-24 Several optimal trajectories for Example 5.4-5 with $a = 0.5$