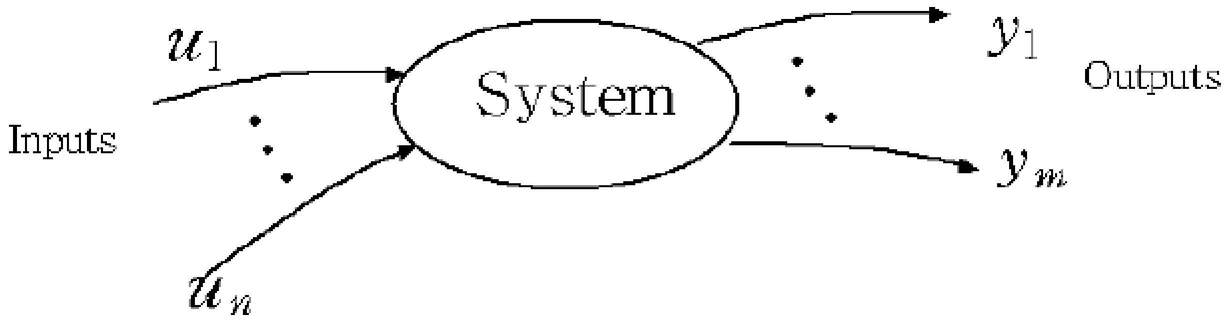
Mathematical Models of Systems

Contents

System / Control / Design

- System : A combination of components acting together to perform a certain objective



- Control: Applying inputs to the system to correct or limit deviation of the output values from desired values

Introduction

- Mathematical models
  - To analyze relationship between the system variables and
  - To obtain a mathematical model
- Differential equations
  - Systems under consideration are dynamic in nature
- Laplace transform
  - can be utilized, if the differential equations can be linearlized, to simplify the method of solution
- The approach to dynamic system problems
  - Define the system and its components.
  - Formulate the mathematical model and list the assumptions.
  - Write the differential equations describing the model.
  - Solve the equations for the desired output variables.
  - Examine the solutions and the assumptions.
  - If necessary,reanalyze or redesign the system.

Differential equations of physical systems

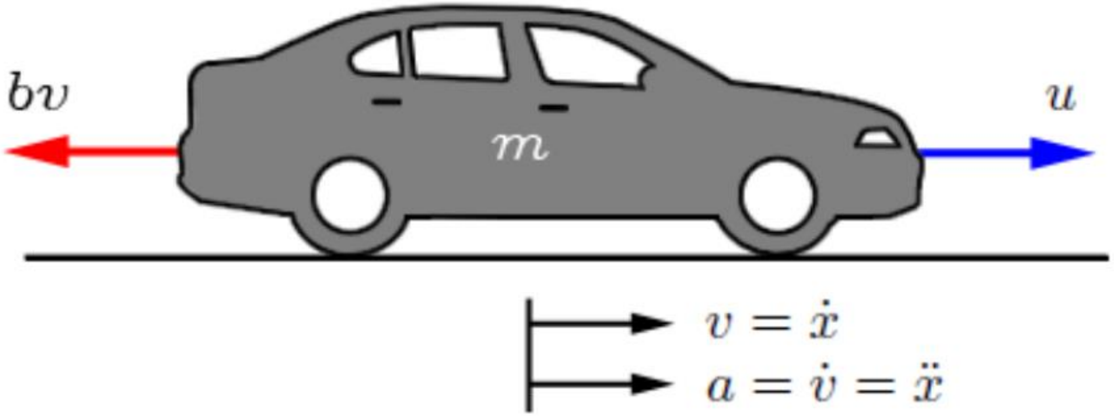
- The differential equations describing the dynamic performance of a physical system are obtained by utilizing the physical laws of the process.
  - For mechanical systems : Newton's laws
  - For electrical systems : Kirchhoff's voltage/current laws

- *Summary of describing differential equations for ideal elements*
  - Analogous variables and analogous systems
    - electrical, mechanical, thermal, and fluid systems

  - *Examples:*
    - *Simple spring-mass-damper mechanical system*
    - *RLC circuit*

Contents

Exercise



$$m\frac{dv}{dt} + bv = u,$$
$$y = v$$

$$m = 1000, b = 50Ns/m$$

Differential equations of physical systems

- First order differential equations solution (1)

(Ex) $\frac{dx(t)}{dt} + x(t) = 1$

Multiplying e^t

$$\boxed{\begin{aligned} \frac{d(f(t)g(t))}{dt} &= \frac{df(t)}{dt}g(t) + f(t)\frac{dg(t)}{dt} \\ \frac{de^{at}}{dt} &= ae^{at} \end{aligned}}$$

$$e^t \frac{dx(t)}{dt} + e^t x(t) = \frac{d\left[e^t x(t)\right]}{dt} = e^t$$

$$\longrightarrow \int_0^t \frac{d\left[e^\tau x(\tau)\right]}{d\tau} d\tau = e^t x(t) - x(0) = \int_0^t e^\tau d\tau$$

$$\longrightarrow x(t) = e^{-t}x(0) + e^{-t}\int_0^t e^\tau d\tau = e^{-t}x(0) + 1 - e^{-t}$$

Differential equations of physical systems

- First order differential equations solution (2)

$$\text{(Ex) } y' + p(x)y = 0$$

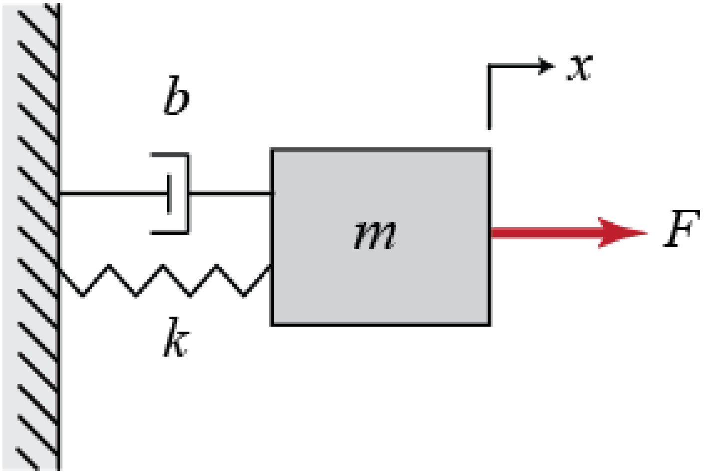
$$\frac{dy}{dx} = -p(x)y$$

$$\frac{dy}{y} = -p(x)dx$$

$$\ln|y| = -\int p(x)dx + c^*$$

$$y = ce^{-\int p(x)dx}$$

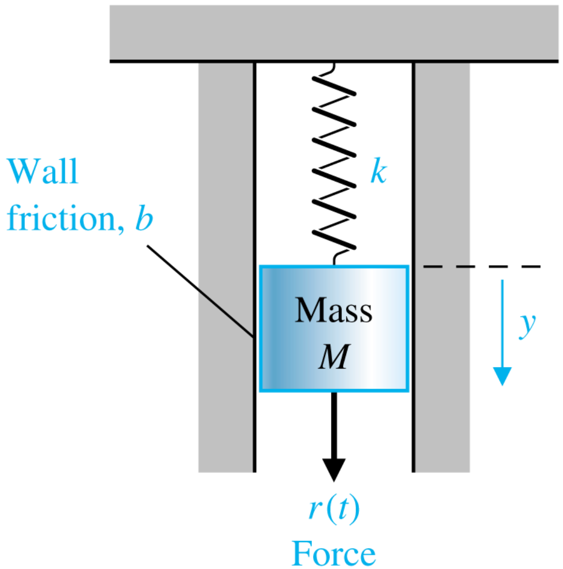
Mass spring damper system

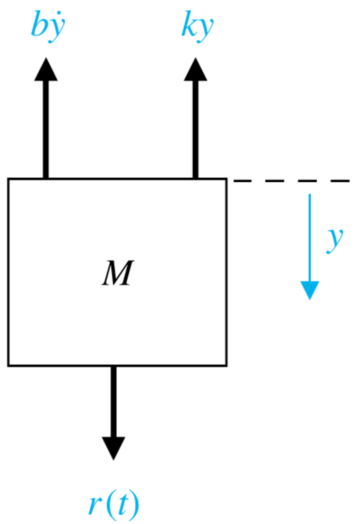


$$\sum F = ma \quad \Rightarrow \quad F - kx - b\dot{x} = m\ddot{x}$$

Differential equations of physical systems

- *Simple spring-mass-damper mechanical system*





Force-displacement (Newton's 2nd law)

$$M\frac{d^2y(t)}{dt^2}+b\frac{dy(t)}{dt}+ky(t)=r(t)$$

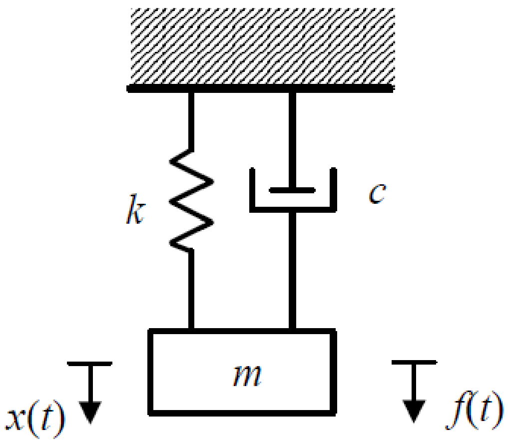
Force-velocity($v(t)=\dfrac{dy(t)}{dt}$)

$$M\frac{dv(t)}{dt}+bv(t)+k\int_0^t v(t)=r(t)$$

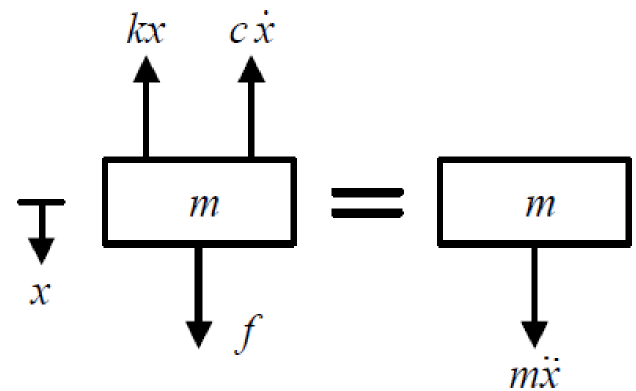
The solution : (underdamped)

$$y(t)=K_1e^{-\alpha_1 t}\sin(\beta_1 t+\theta_1)$$

Differential equations of physical systems

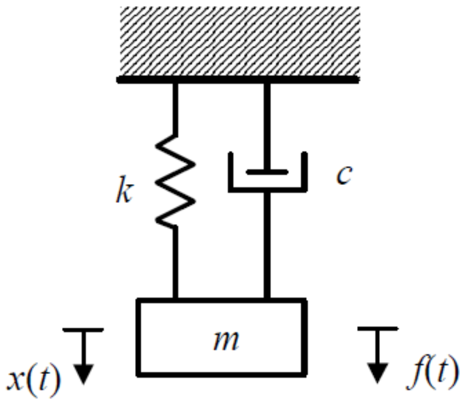


m =1kg
c =3Ns/m
k =2N/m
initial conditions
$x(0)=1.2$, $\dot{x}(0)=0.1$



$$m\ddot{x} + c\dot{x} + kx = f(t) = 5u(t)$$

Differential equations of physical systems



m =1kg
c =3Ns/m
k =2N/m
initial conditions
$x(0)=1.2$, $\dot{x}(0)=0.1$

$$ms^2x(s) + csx(s) + kx(s) = 5u(s)$$

$$s^2x(s) + 3sx(s) + 2x(s) = 5u(s)$$

$$y(s) = x(s)$$

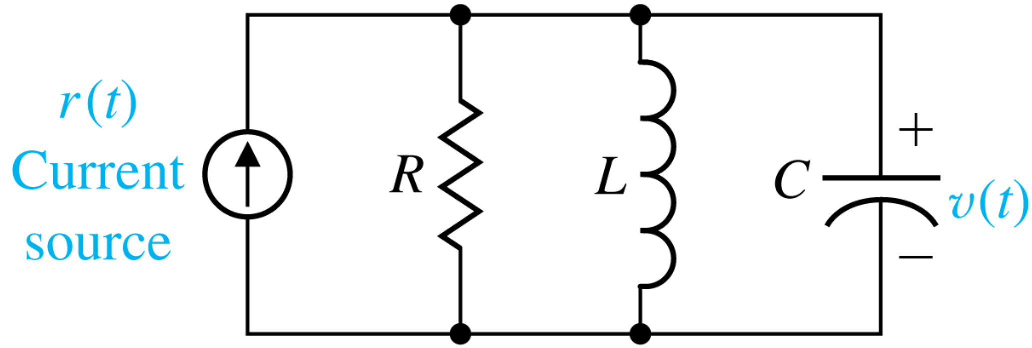
$$\frac{y(s)}{u(s)} = \frac{5}{s^2 + 3s + 2}$$

Differential equations of physical systems

- *RLC circuit*

Kirchhof's current law

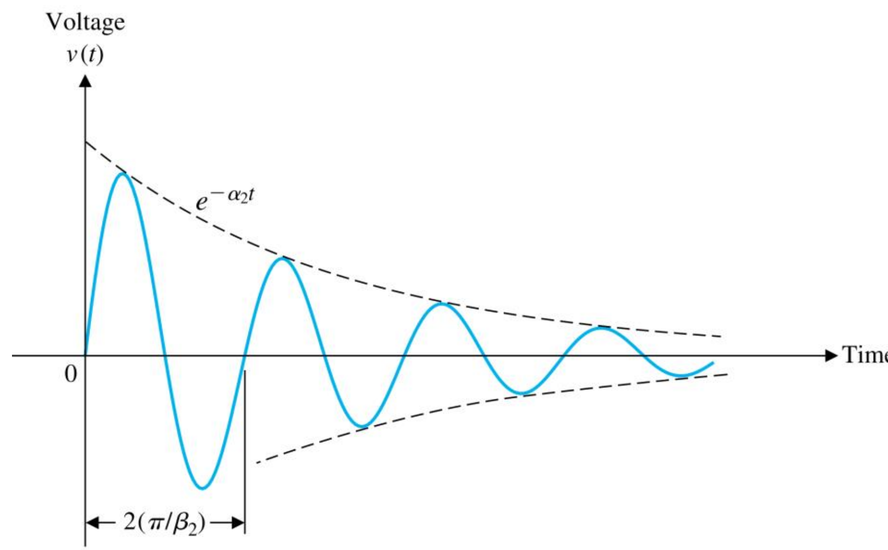
$$\frac{v(t)}{R} + C\frac{dv(t)}{dt} + \frac{1}{L}\int_0^t v(t) = r(t)$$



Spring-mass-damper와 비교

$$M\frac{dv(t)}{dt} + bv(t) + k\int_0^t v(t) = r(t)$$

Analogous variables: $v(t)$ – velocity vs voltage

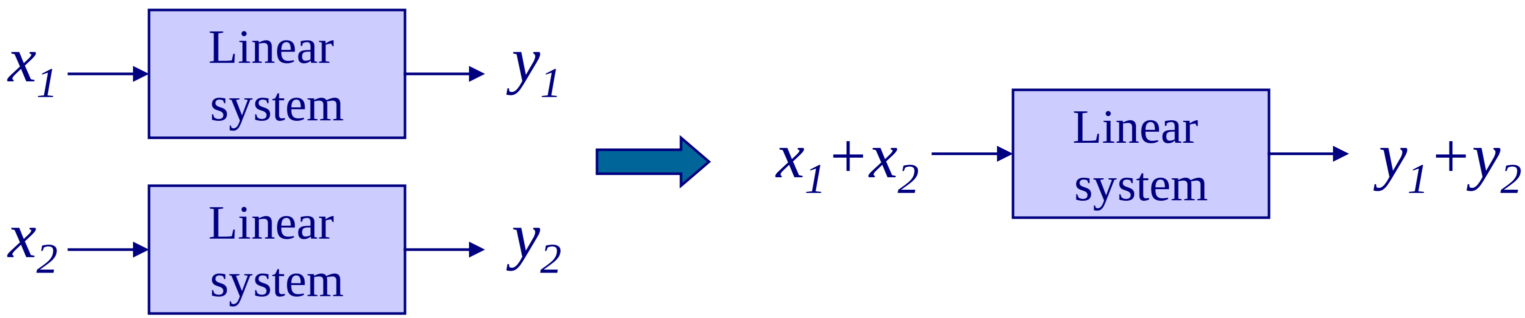


$$v(t) = K_2 e^{-\alpha_2 t}\cos(\beta_2 t + \theta_2)$$

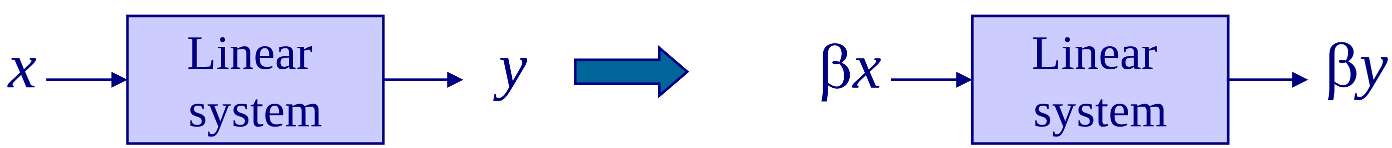
(underdamped)

Linear approximations of physical systems

- *Linear* in terms of the system excitation and response
- A linear system satisfies the properties of
  - superposition and



  - homogeneity.



Linear approximations of physical systems

- Linearization
  - The relationship of the two variables is given as

$$y(t) = g(x(t))$$

  - A Taylor series expansion about the normal operating point x_0

$$y = g(x) = g(x_0) + \left.\frac{dg}{dx}\right|_{x=x_0} \frac{(x-x_0)}{1!} + \left.\frac{d^2 g}{dx^2}\right|_{x=x_0} \frac{(x-x_0)}{2!} + \cdots$$

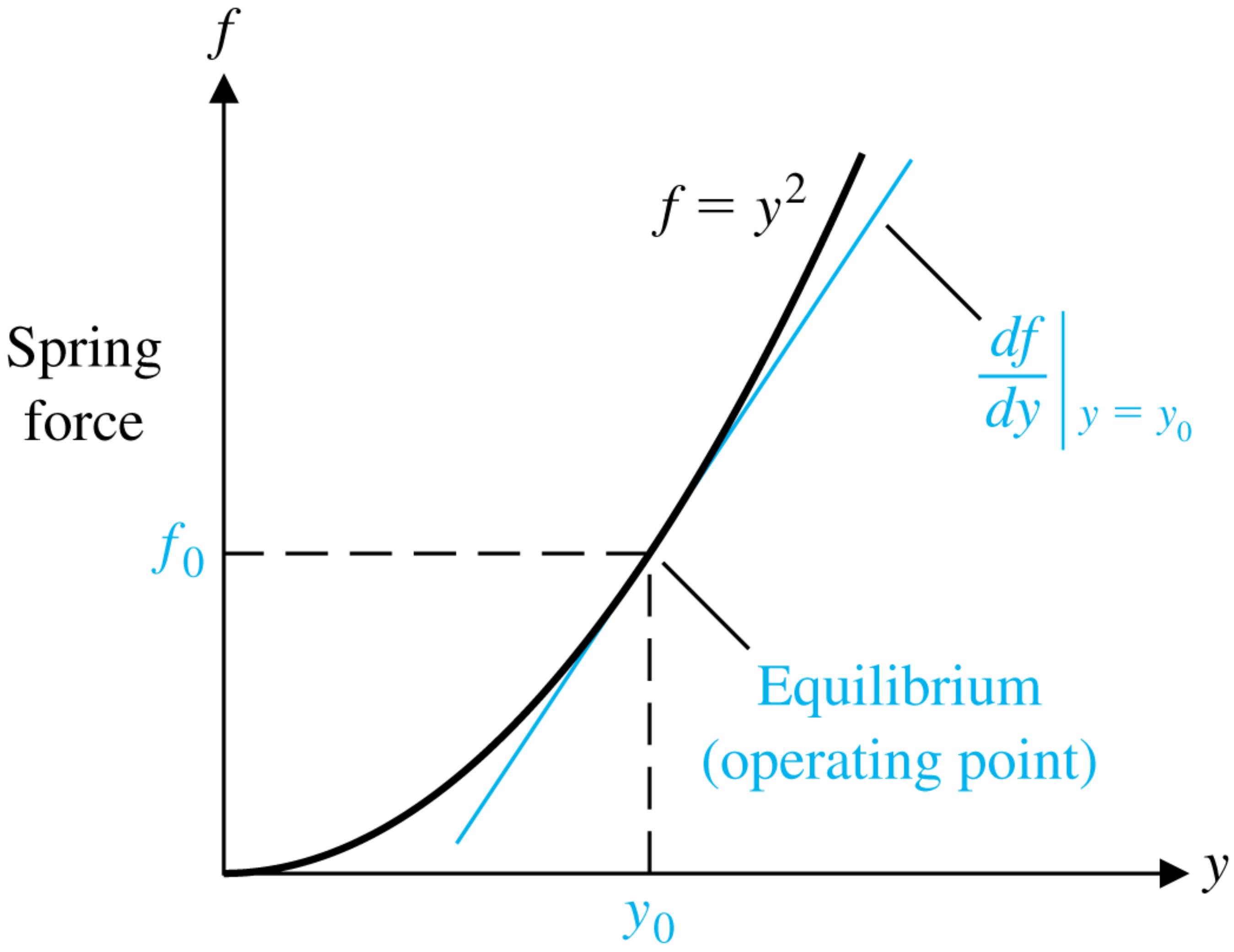
  - The slope at the operating point is a good approximation to the curve over a small range of $(x\text{-}x_0)$.

$$y = g(x_0) + \left.\frac{dg}{dx}\right|_{x=x_0} (x-x_0) = y_0 + m(x-x_0)$$

$$\Rightarrow \quad \begin{aligned} y - y_0 &= m(x-x_0) \\ \Delta y &= m\Delta x \end{aligned}$$

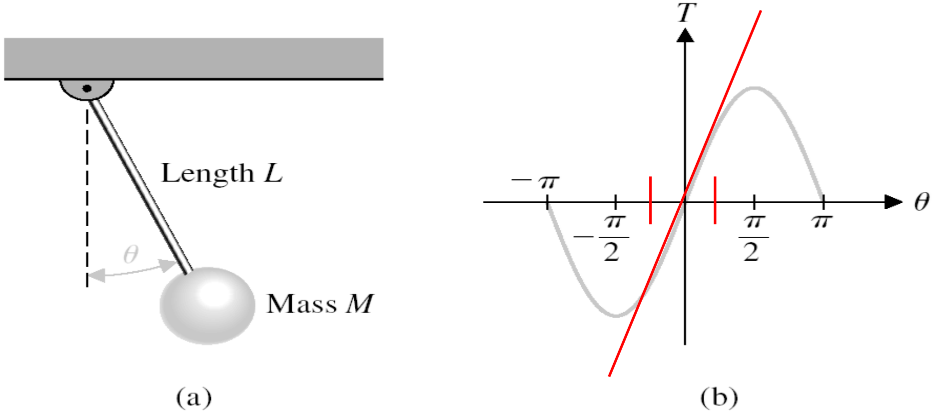
Linear equation

Linear approximations of physical systems



Linear approximations of physical systems

- *Pendulum oscillator*



Newton's 2nd law $\tau = MgLsin\theta$

$$\tau = I\alpha = ML^2\alpha \longrightarrow \frac{dw}{dt} = \frac{g}{L}sin\theta$$

$$\longrightarrow \frac{d\theta^2}{dt^2} = \frac{g}{L}\theta$$

$$(-\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4})$$

Linearization at operating point $(0°)$

$$sin\theta \approx \sin(0°) + \cos(0°)\,(\theta - 0°) + \cdots$$

The transfer function of linear systems

Transfer function of a linear system:

The ratio of the Laplace transform of the output variable to the Laplace transform of the input variable, with all initial conditions assumed to be zero.

$$G(s) = \frac{Y(s)}{U(s)}$$

$$Y(s) = G(s)U(s)$$

The transfer function of linear systems

The dynamic system represented by the differential equation

$$\frac{d^n y(t)}{dt^n} + a_{n-1}\frac{d^{n-1}y(t)}{dt^{n-1}} + \cdot\ \cdot\ \cdot + a_1\frac{dy(t)}{dt} + a_0 y(t)$$
$$= b_m\frac{d^m u(t)}{dt^m} + b_{m-1}\frac{d^{m-1}u(t)}{dt^{m-1}} + \cdot\ \cdot\ \cdot + b_1\frac{du(t)}{dt} + b_0 u(t)$$

$$(s^n + a_{n-1}s^{n-1} + \cdot\ \cdot\ \cdot + a_1 s + a_0)Y(s)$$
$$= (b_m s^m + b_{m-1}s^{m-1} + \cdot\ \cdot\ \cdot + b_1 s + b_0)U(s)$$

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_m s^m + b_{m-1}s^{m-1} + \cdot\ \cdot\ \cdot + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \cdot\ \cdot\ \cdot + a_1 s + a_0}$$

The transfer function of linear systems

- Second order differential equation example

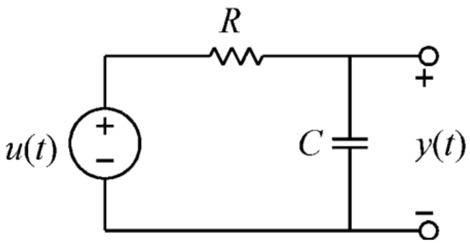
$$a_2\ddot{y}(t) + a_1\dot{y}(t) + a_0 y(t) = r(t)$$

$$Y(s) = \frac{1}{a_2 s^2 + a_1 s + a_0} R(s)$$

$$\Rightarrow \mathrm{G(s)} = \frac{Y(s)}{R(s)} = \frac{1}{a_2 s^2 + a_1 s + a_0}$$

The transfer function of linear systems

- RC circuit example



$$RC\frac{dy(t)}{dt} + y(t) = u(t)$$

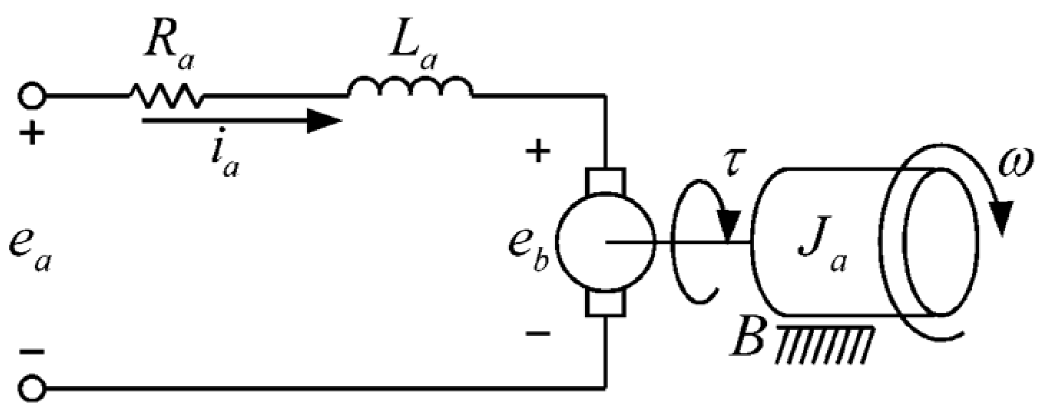
$$RC(sY(s) - y(0)) + Y(s) = U(s)$$

$$Y(s) = \frac{1}{RCs+1}(U(s) + y(0))$$

$$G(s) = \frac{1}{RCs+1}$$

The transfer function of linear systems

- DC servo motor example



$$e_a = R_a i_a + L_a \frac{di_a}{dt} + e_b = R_a i_a + L_a \frac{di_a}{dt} + K_b \frac{d\theta}{dt}$$

$$\tau = K_t i_a = J_a \frac{d^2\theta}{dt^2} + B\frac{d\theta}{dt}$$

$$e_b = K_b \frac{d\theta}{dt} = K_b \omega$$

$\Rightarrow$

$$E_a(s) = R_a I_a(s) + L_a s I_a(s) + K_b s \Theta(s)$$

$$\frac{\Theta(s)}{E_a(s)} = \frac{K_t}{\left(J_a s^2 + Bs\right)\left(L_a s + R_a\right) + K_t K_b s}$$

The transfer function of linear systems

- Solution of a differential equation

$$\frac{d^2y}{dt^2}+4\frac{dy}{dt}+3y=2r(t) \quad \text{with initial conditions of } \; y(0)=1, \frac{dy}{dt}(0)=0, \; \text{and } r(t)=1, t\geq 0$$

Taking the Laplace transform:

$$[s^2Y(s)-s\underbrace{\boxed{y(0)}}_{\boxed{=1}}]+4[sY(s)-\underbrace{\boxed{y(0)}}_{\boxed{=1}}]+3Y(s)=2\underbrace{\boxed{R(s)}}_{\boxed{=1/s}}$$

$$\Longrightarrow \quad Y(s)=\frac{(s+4)}{(s^2+4s+3)}+\frac{2}{s(s^2+4s+3)}$$

By the partial fraction expansion

$$Y(s)=\left[\frac{3/2}{(s+1)}+\frac{-1/2}{(s+3)}\right]+\left[\frac{-1}{(s+1)}+\frac{1/3}{(s+3)}\right]+\frac{2/3}{s}$$

Hence the response is

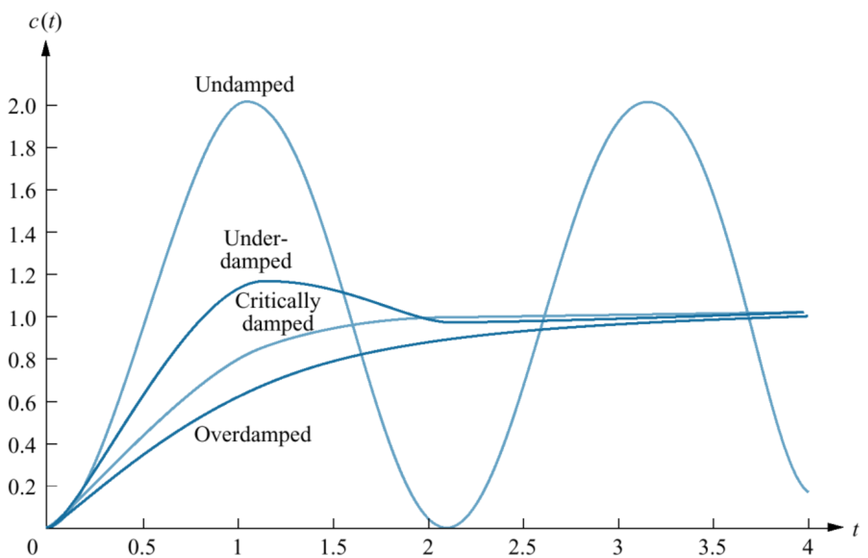
$$y(t)=\left[\frac{3}{2}e^{-t}-\frac{1}{2}e^{-3t}\right]+\left[-e^{-t}+\frac{1}{3}e^{-3t}\right]+\frac{2}{3}$$

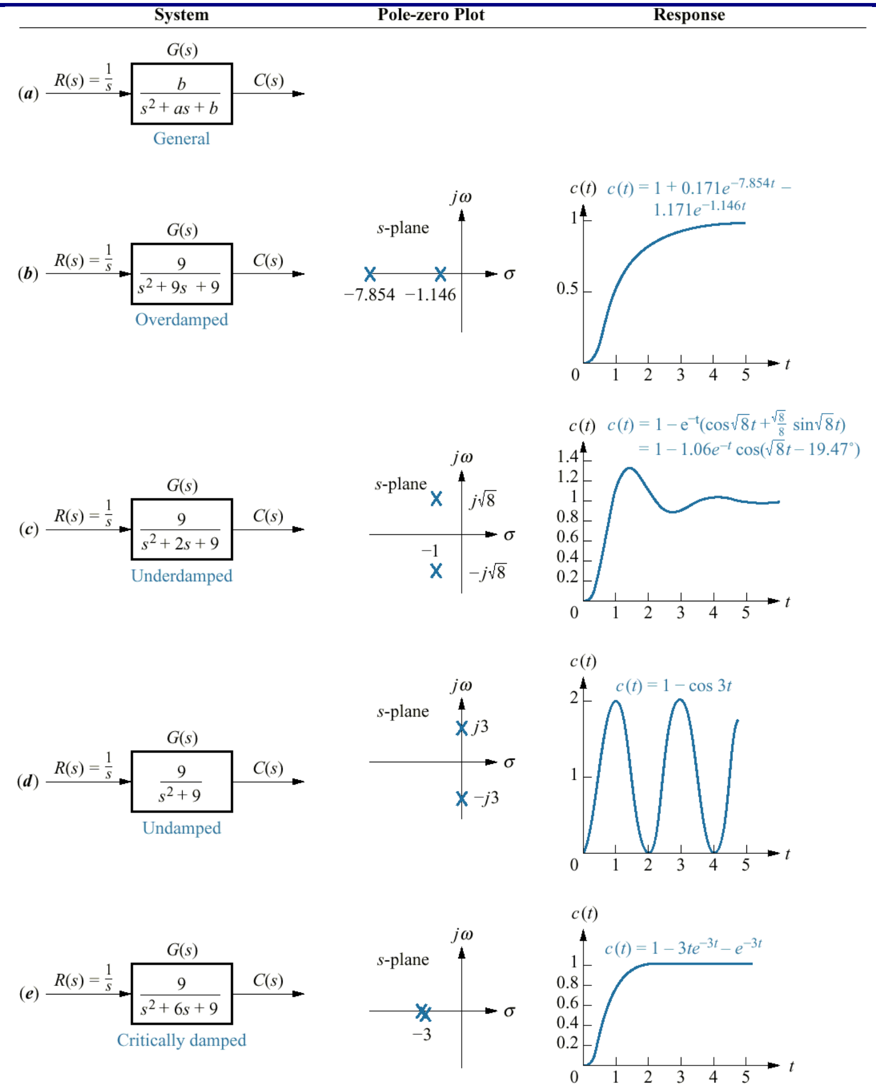
the steady state response is

$$\lim_{t\to\infty} y(t)=\frac{2}{3}$$

The Laplace transform (2nd order system)

- Overdamped response (b)
- Underdamped response (c)
- Undamped response (d)
- Critically damped response (e)



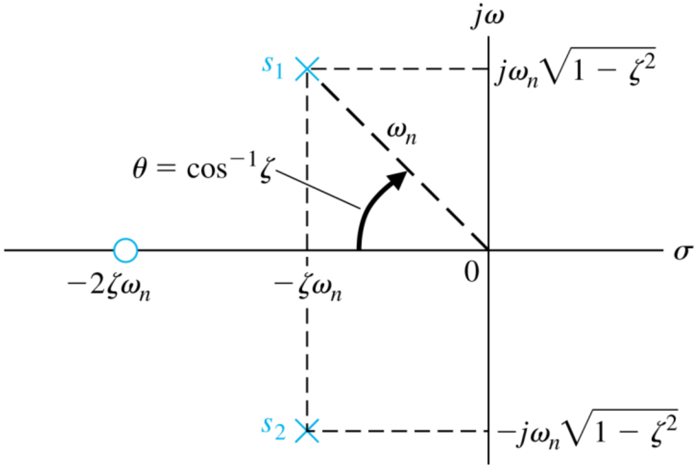


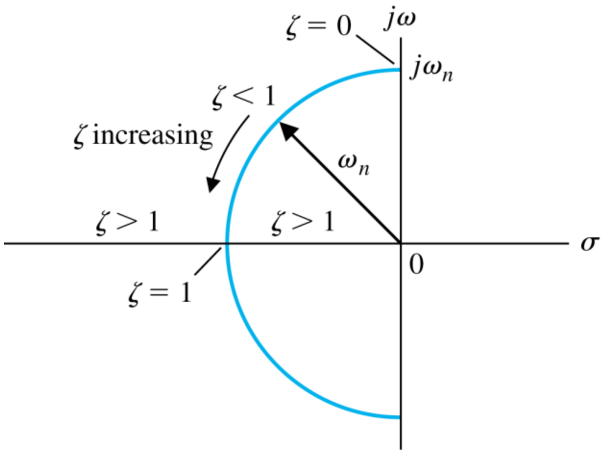
The Laplace transform (2nd order system)

- The general Second-order System
  - Natural frequency(ω_n): the frequency of oscillation of the system without damping.
  - Damping ratio(ζ): $$\zeta = \frac{\text{Exponential decay frequency}}{\text{Natural frequency (rad/sec)}}$$

General second-order transfer function $$G(s) = \frac{\omega_n^{\,2}}{s^2 + 2\zeta\omega_n s + \omega_n^{\,2}}$$

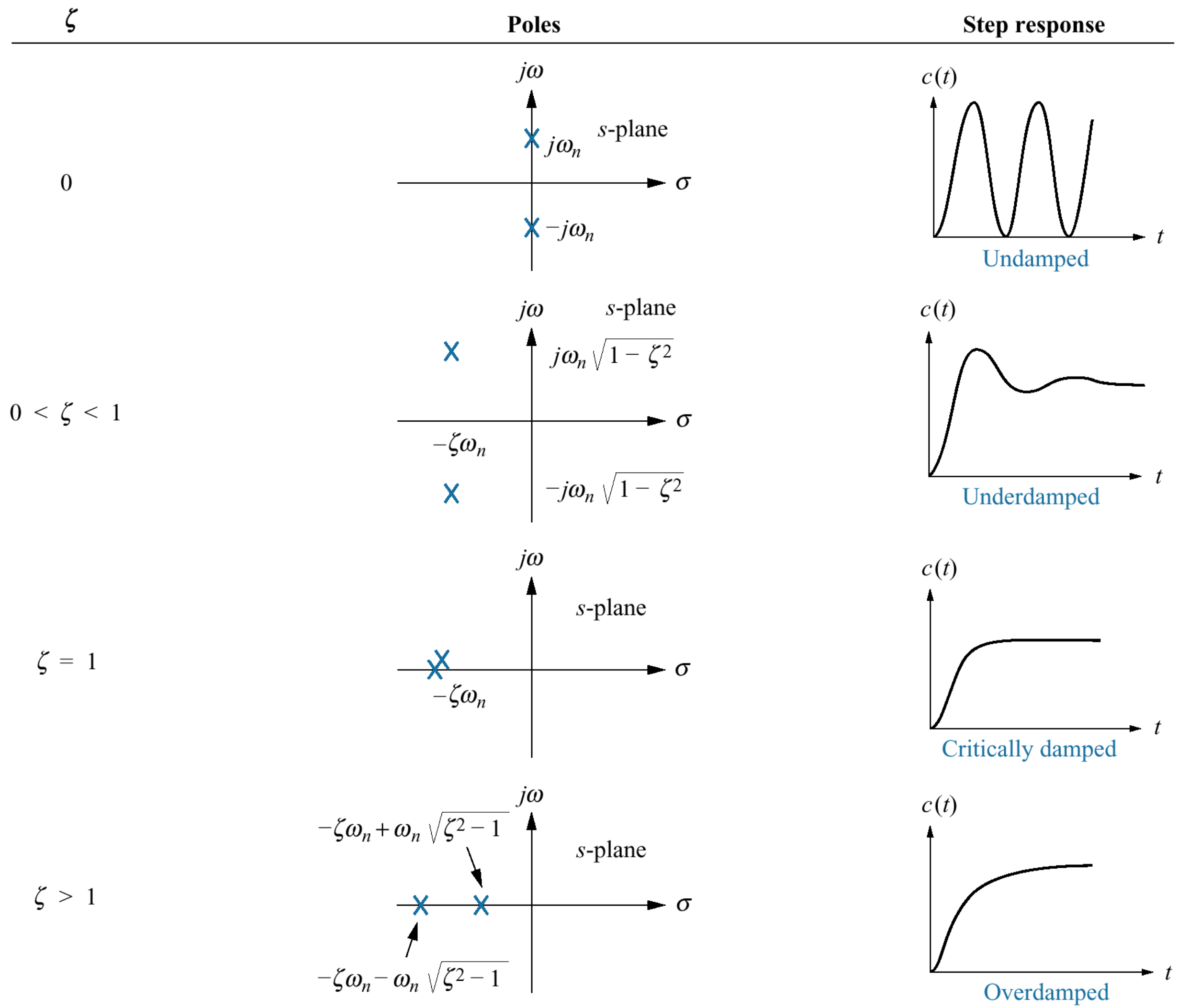
The poles of the transfer function: $$s_{1,2} = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$$





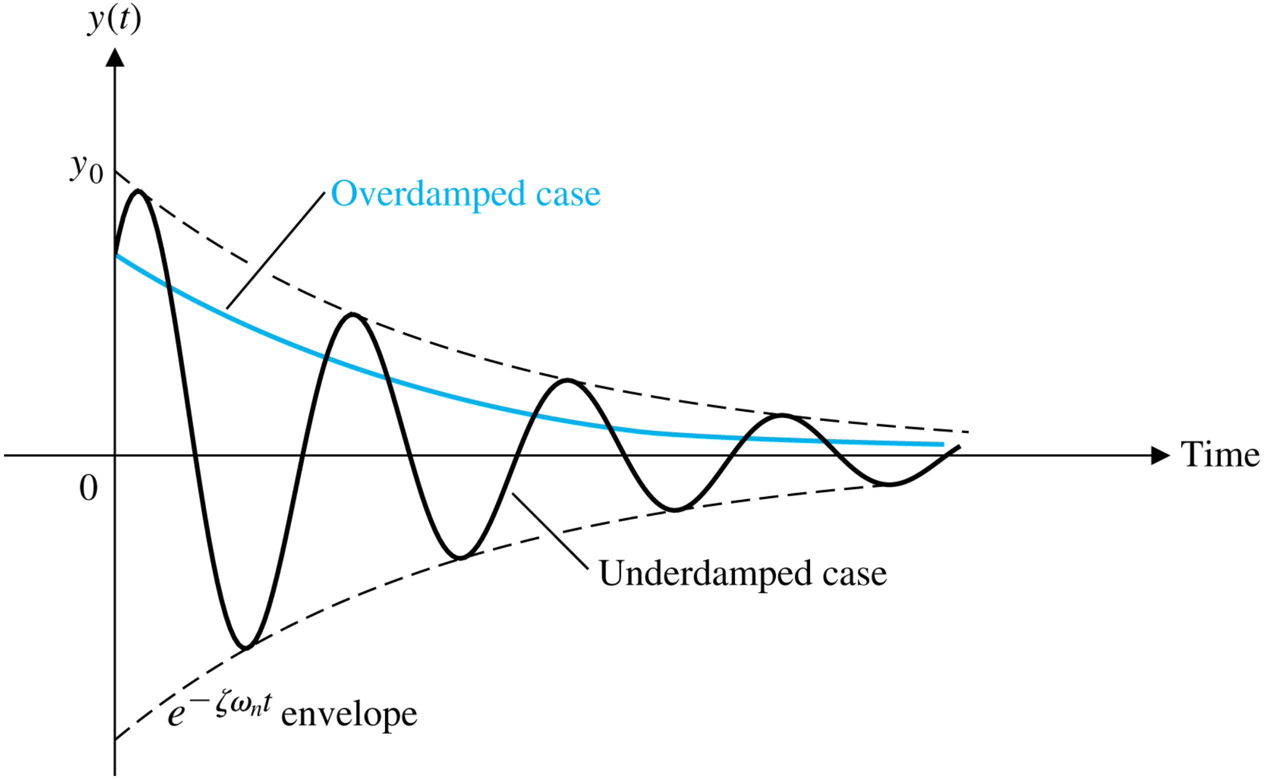
The Laplace transform (2nd order system)

Second-order response as a function of damping ratio



The Laplace transform (2nd order system)

Response of the spring-mass-damper system



State variable and state vector

- A system is represented in state space by the following equations:

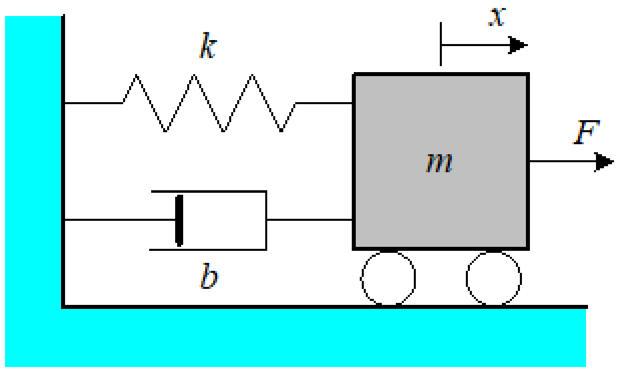
$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

State variable: The smallest set of linearly independent system variables such that the values of the members f the set at time t_0 along with known forcing functions completely determine the value of all system variables for all $t \geq t_0$

State vector: A vector whose elements are the state variables.

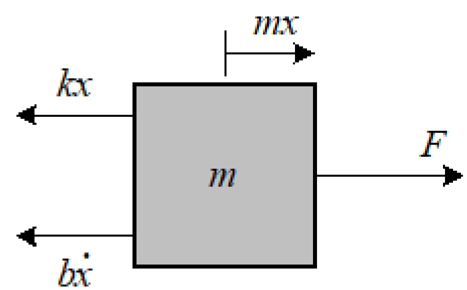
State variable and state vector

- Mass-spring-damper system



$$\Longleftrightarrow$$

$$m\ddot{x} + b\dot{x} + kx = F$$



$x_1(t) = x(t)$: position

$x_2(t) = \dot{x}(t)$: velocity

$$\dot{x}_1(t) = \dot{x}(t) = x_2(t)$$

$$\dot{x}_2(t) = \ddot{x}(t) = -\frac{k}{m}x_1 - \frac{b}{m}x_2 + \frac{1}{m}F$$

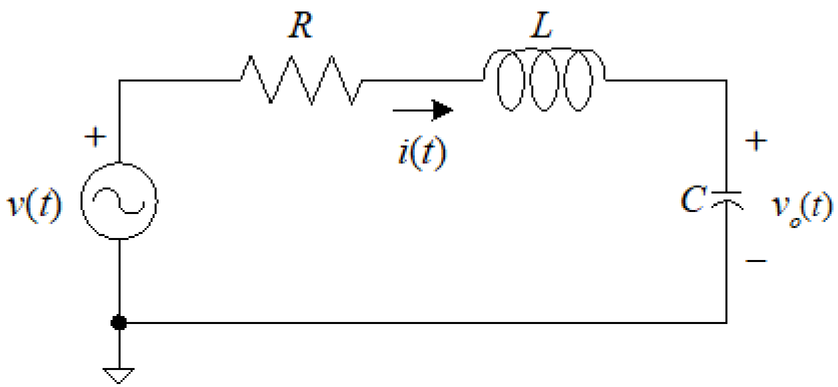
$$\Rightarrow \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} F$$

State variables $= [x(t) \quad v(t)]$

Input $= u(t) = F$

State variable and state vector

- RLC circuit system



$$
\begin{aligned}
v(t) &= R\,i(t) + L\frac{d}{dt}\,i(t) + v_o(t) \\
i(t) &= C\frac{d}{dt}\,v_o(t)
\end{aligned}
$$

$$
\begin{aligned}
\dot{x}(t) &= \begin{bmatrix} 0 & \frac{1}{C} \\ -\frac{1}{L} & -\frac{R}{L} \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix} u(t) \\
y(t) &= [1 \quad 0]\, x(t)
\end{aligned}
$$

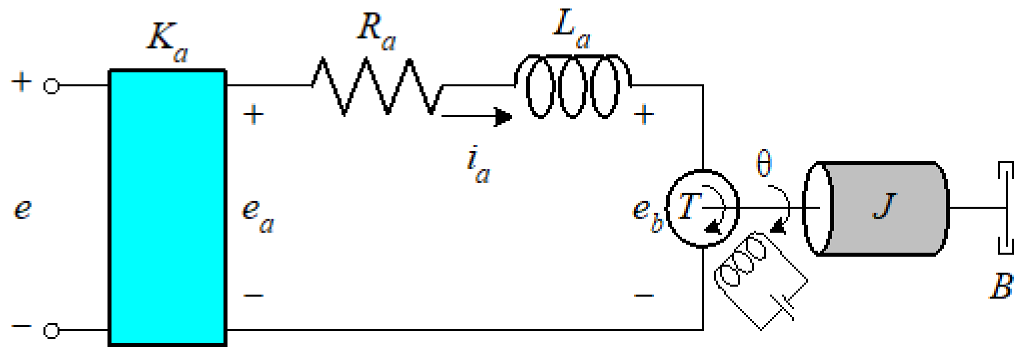
State variables$=[v_0(t) \quad i(t)]$

Input $= u(t) = v(t)$

Output $= y(t) = v_0(t)$

State variable and state vector

- DC servo motor



$$e_a = K_a e = R_a i_a + L_a \frac{di_a}{dt} + e_b$$

$$e_b = K_b \omega = K_b \frac{d\theta}{dt}$$

$$\frac{d}{dt}\begin{bmatrix} i_a \\ \dot{\theta} \\ \theta \end{bmatrix} = \begin{bmatrix} -R_a/L_a & -K_b/L_a & 0 \\ K_t/J & -B/J & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} i_a \\ \dot{\theta} \\ \theta \end{bmatrix} + \begin{bmatrix} K_a/L_a \\ 0 \\ 0 \end{bmatrix} e$$

$$y = [0 \;\; 0 \;\; 1]\begin{bmatrix} i_a \\ \dot{\theta} \\ \theta \end{bmatrix}$$

State variables $= [i_a \quad \dot{\theta} \quad \theta]$

Input $= u(t) = e$

Output $= y(t) = \theta$

Derivation of transfer function from S.E.

$$
\begin{aligned}
\dot{x}(t) &= Ax(t) + Bu(t) \\
y(t) &= Cx(t) + Du(t)
\end{aligned}
$$

Laplace transformation

$$
\begin{aligned}
sX(s) - x(0) &= AX(s) + BU(s) \\
Y(s) &= CX(s) + DU(s)
\end{aligned}
$$

Transfer function is ratio of input $U(s)$ and $Y(s)$ with $x(0) = 0$

$$
\begin{aligned}
(sI - A)X(s) &= BU(s) \\
X(s) &= (sI - A)^{-1}BU(s) \\
Y(s) &= [C(sI - A)^{-1}B + D]U(s) \\
\Rightarrow \quad G(s) &= \frac{Y(s)}{U(s)} = C(sI - A)^{-1}B + D
\end{aligned}
$$

Derivation of transfer function from S.E.

- S.E $\Rightarrow$ Transfer Function

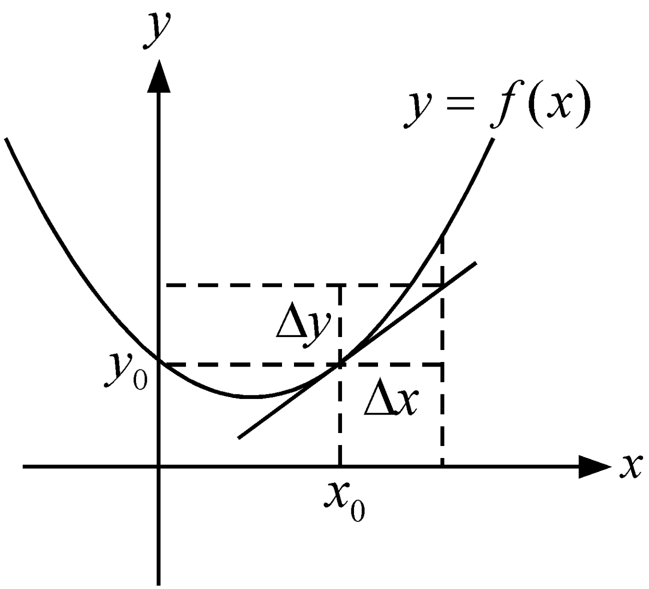
$$\dot{x}(t) = \begin{bmatrix} -1 & -2 \\ 1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t)$$
$$y(t) = \begin{bmatrix} 2 & 1 \end{bmatrix} x(t) + 3u(t)$$

$$A = \begin{bmatrix} -1 & -2 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & 1 \end{bmatrix}, \quad D = 3$$

$$\begin{aligned} \Rightarrow \quad G(s) &= C(sI - A)^{-1}B + D \\ &= \begin{bmatrix} 2 & 1 \end{bmatrix} \begin{bmatrix} s+1 & 2 \\ -1 & s \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 3 \\ &= \frac{1}{s^2+s+2} \begin{bmatrix} 2 & 1 \end{bmatrix} \begin{bmatrix} s & -2 \\ 1 & s+1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 3 \\ &= \frac{3s-2}{s^2+s+2} + 3 \end{aligned}$$

Linearization

$$y = f(x)$$



$$\Delta y = \left(\left. \frac{df}{dx} \right|_{x=x_0} \right) \cdot \Delta x = K \cdot \Delta x$$

$$y_0 = f\left(x_0\right)$$

Linearization

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} f_1(x_1, x_2, \cdots, x_n, u) \\ f_2(x_1, x_2, \cdots, x_n, u) \\ \vdots \\ f_n(x_1, x_2, \cdots, x_n, u) \end{bmatrix}$$

Nonlinear dynamic eq.

$$(x_{10}, x_{20}, \cdots, u_0)$$

Operating point

$$\dot{x}_{i0} = f_i(x_{10}, x_{20}, \cdots, u_0),\ i = 1, 2, \cdots, n$$

$$x_i = x_{i0} + \Delta x_i,\ i = 1, 2, \cdots n$$

$$u = u_0 + \Delta u$$

Linearization

Step 1) Suppose

If the input u changes slightly by δu in the vicinity of the equilibrium point, state x and output y change finely by δx and δy.

Step 2) Taylor series expansion at (x_0, u_0, y_0)

$$
\begin{aligned}
\frac{d}{dt}[x_0 + \delta x(t)] &= f(x_0 + \delta x, u_0 + \delta u) \\
&= f(x_0, u_0) + \frac{\partial}{\partial x} f(x_0, u_0)\delta x(t) + \frac{\partial}{\partial u} f(x_0, u_0)\delta u(t) + O(\delta x, \delta u) \\
&\approx f(x_0, u_0) + \frac{\partial}{\partial x} f(x_0, u_0)\delta x(t) + \frac{\partial}{\partial u} f(x_0, u_0)\delta u(t)
\end{aligned}
$$

$$
y_0 + \delta y(t) \approx g(x_0, u_0) + \frac{\partial}{\partial x} g(x_0, u_0)\delta x(t) + \frac{\partial}{\partial u} g(x_0, u_0)\delta u(t)
$$

Linearization

$$\dot{x}_i = \dot{x}_{i0} + \Delta\dot{x}_i = f_i\left(x_1, x_2, \cdots, x_n, u\right)$$

$$\approx f_i\left(x_{10}, x_{20}, \cdots, u_0\right) + \left(\left.\frac{\partial f_i}{\partial x_1}\right|_{x_1=x_{10}, x_2=x_{20}, \cdots, x_n=x_{n0}, u=u_0}\right) \cdot \Delta x_1$$

$$+ \left(\left.\frac{\partial f_i}{\partial x_2}\right|_{x_1=x_{10}, x_2=x_{20}, \cdots, x_n=x_{n0}, u=u_0}\right) \cdot \Delta x_2$$

$$\vdots$$

$$+ \left(\left.\frac{\partial f_i}{\partial x_n}\right|_{x_1=x_{10}, x_2=x_{20}, \cdots, x_n=x_{n0}, u=u_0}\right) \cdot \Delta x_n$$

$$+ \left(\left.\frac{\partial f_i}{\partial u}\right|_{x_1=x_{10}, x_2=x_{20}, \cdots, x_n=x_{n0}, u=u_0}\right) \cdot \Delta u, \ \ i = 1, 2, \cdots, n$$

Linearization

$$\begin{bmatrix} \Delta\dot{x}_1 \\ \Delta\dot{x}_2 \\ \vdots \\ \Delta\dot{x}_n \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \\ \vdots \\ \Delta x_n \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \Delta u$$

$$a_{ij} = \left.\frac{\partial f_i}{\partial x_j}\right|_{x_1=x_{10},\, x_2=x_{20},\cdots,x_n=x_{n0},\, u=u_0} ,\, b_i = \left.\frac{\partial f_i}{\partial u}\right|_{x_1=x_{10},\, x_2=x_{20},\cdots,x_n=x_{n0},\, u=u_0}$$

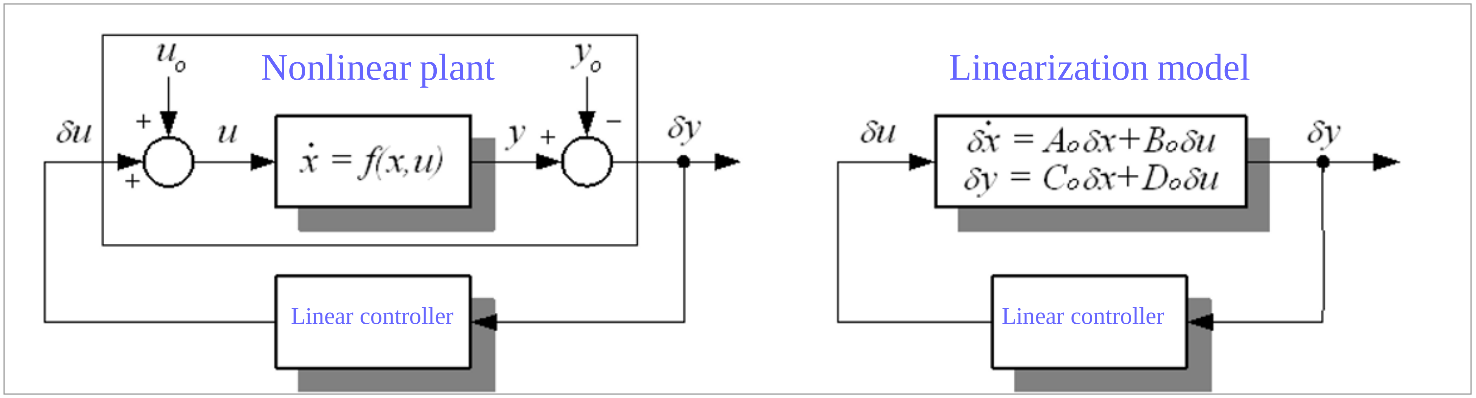
Linearization

Step 3) Linearization model

$$\begin{aligned} \delta\dot{x}(t) &\approx A_0\delta x(t) + B_0\delta u(t) \\ \delta y(t) &\approx C_0\delta x(t) + D_0\delta u(t) \end{aligned}$$

$$\begin{aligned} A_0 &= \frac{\partial}{\partial x} f(x_0, u_0), \quad & B_0 &= \frac{\partial}{\partial u} f(x_0, u_0) \\ C_0 &= \frac{\partial}{\partial x} g(x_0, u_0), \quad & D_0 &= \frac{\partial}{\partial u} g(x_0, u_0) \end{aligned}$$

$$\delta u = u - u_0, \quad \delta x = x - x_0, \quad \delta y = y - y_0$$



Linearization

- Nonlinear system operating point and linearization

$$
\begin{aligned}
\dot{x}(t) &= -x^2(t) - u^2(t) + 1 \\
y(t) &= x(t)u(t)
\end{aligned}
$$

1) Operating point

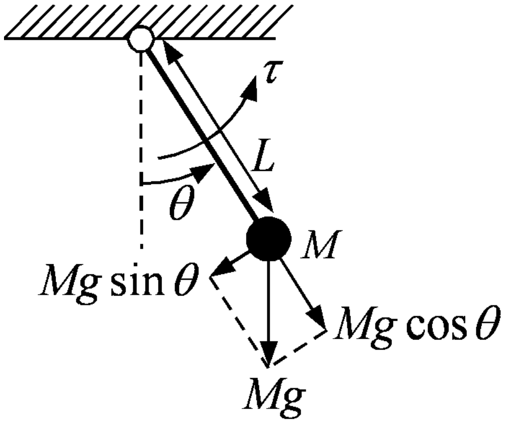
$$
f(x_0, u_0) = -x_0^2 - u_0^2 + 1 = 0
$$

2) Taylor series

$$
\begin{aligned}
\delta\dot{x}(t) &\approx A_0\delta x(t) + B_0\delta u(t) \\
\delta y(t) &\approx C_0\delta x(t) + D_0\delta u(t)
\end{aligned}
$$

$$
\begin{aligned}
A_0 &= \frac{\partial f}{\partial x}(1,0) = -2, \quad & B_0 &= \frac{\partial f}{\partial u}(1,0) = 0 \\
C_0 &= \frac{\partial g}{\partial x}(1,0) = 0, \quad & D_0 &= \frac{\partial g}{\partial u}(1,0) = 1
\end{aligned}
$$

Ex 1



$$\tau - LMg\sin\theta = ML^2\frac{d^2\theta}{dt^2}$$

$$x_1 = \theta,\ x_2 = \dot{\theta},\ u = \tau$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} f_1\left(x_1, x_2, u\right) \\ f_2\left(x_1, x_2, u\right) \end{bmatrix} = \begin{bmatrix} x_2 \\ -\dfrac{g}{L}\sin x_1 + \dfrac{1}{ML^2}u \end{bmatrix}$$

Ex 1

$$\left.\frac{\partial f_1}{\partial x_1}\right|_{x_1=0,x_2=0,u=0} = 0,$$

$$\left.\frac{\partial f_1}{\partial x_2}\right|_{x_1=0,x_2=0,u=0} = 1,$$

$$\left.\frac{\partial f_2}{\partial x_1}\right|_{x_1=0,x_2=0,u=0} = -\frac{g}{L}\cos x_1\Big|_{x_1=0,x_2=0,u=0} = -\frac{g}{L},$$

$$\left.\frac{\partial f_2}{\partial x_2}\right|_{x_1=0,x_2=0,u=0} = 0$$

$$\left.\frac{\partial f_1}{\partial u}\right|_{x_1=0,x_2=0,u=0} = 0,$$

$$\left.\frac{\partial f_2}{\partial u}\right|_{x_1=0,x_2=0,u=0} = \frac{1}{ML^2}$$

Ex 1

$$\begin{bmatrix} \Delta\dot{x}_1 \\ \Delta\dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\dfrac{g}{L} & 0 \end{bmatrix} \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \dfrac{1}{ML^2} \end{bmatrix} \Delta u$$

$$s\Delta X_1(s) = \Delta X_2(s)$$

$$s\Delta X_2(s) = -\frac{g}{L}\Delta X_1(s) + \frac{1}{ML^2}\Delta U(s)$$

$$\frac{\Delta X_1(s)}{\Delta U(s)} = \frac{1}{ML^2}\frac{1}{s^2 + (g/L)}$$