Application of Calculus of Variations to Optimal Control Problems

Optimal Control Problem

Problem :

Find an admissible control $U(t)$ which causes the system

$\dot{X} = f(X,U,t), \quad X(t_0) = X_0$ (fixed)

To follow an admissible trajectory that optimizes

the performance index

$$J = \varphi(X_f, t_f) + \int_{t_0}^{t_f} L(X,U,t)\,dt \quad \left[\text{No } \dot{X} \text{ term as } \dot{X} = f(X,U,t)\right]$$

while satisfying appropriate boundary conditions.

Necessary Conditions of Optimality: Path Equations

Problematic (incompatible) term : $\varphi(X_f, t_f)$

Let us consider:

$$\int_{t_0}^{t_f} \frac{d}{dt}\left[\varphi(X,t)\right]dt = \varphi(X_f,t_f) - \varphi(X_0,t_0)$$

$$\varphi(X_f,t_f) = \varphi(X_0,t_0) + \int_{t_0}^{t_f} \frac{d}{dt}\left[\varphi(X,t)\right]dt$$

However, since the initial condition is fixed $X(t_0) = X_0$, $\varphi(X_0,t_0)$ is a constant.

So, instead of optimizing J, it is equivalent to optimize

$$J_1 = \int_{t_0}^{t_f} \left[L(X,U,t) + \frac{d}{dt}\left[\varphi(X,t)\right] \right] dt$$

The problem now is compatible with the calculus of variations.

Necessary Conditions of Optimality: Path Equations

$$\bar{J} = \int_{t_0}^{t_f} \left\{ L(X,U,t) + \frac{d}{dt}\left[\varphi(X,t)\right] + \lambda^T(t)\left[f(X,t) - \dot{X}\right] \right\} dt$$

Define

Hamiltonian: $H \triangleq L(X,U,t) + \lambda^T f(X,U,t)$

Then

$$\bar{J} = \int_{t_0}^{t_f} \underbrace{\left[H + \frac{d\varphi}{dt} - \lambda^T \dot{X} \right]}_{L^*} dt = \int_{t_0}^{t_f} L^* \, dt$$

Necessary Conditions of Optimality: Path Equations

Define

$$L^* \triangleq \left[H + \frac{d\varphi}{dt} - \lambda^T \dot{X} \right], \quad H \triangleq L(X, U, t) + \lambda^T f(X, U, t)$$

Necessary Conditions (E - L Equations)

(1) $$\frac{\partial L^*}{\partial X} - \frac{d}{dt}\left(\frac{\partial L^*}{\partial \dot{X}} \right) = 0$$

(2) $$\frac{\partial L^*}{\partial U} - \frac{d}{dt}\left(\cancelto{0}{\frac{\partial L^*}{\partial \dot{U}}} \right) = 0$$

(3) $$\frac{\partial L^*}{\partial \lambda} - \frac{d}{dt}\left(\cancelto{0}{\frac{\partial L^*}{\partial \dot{\lambda}}} \right) = 0$$

Necessary Conditions of Optimality: Path Equations

Simplification :

$$\frac{\partial L^*}{\partial X} = \frac{\partial}{\partial X}\left(H + \frac{d\varphi}{dt} - \lambda^T \dot{X} \right)$$

$$= \frac{\partial H}{\partial X} + \frac{\partial}{\partial X}\left[\frac{\partial \varphi}{\partial t} + \left(\frac{\partial \varphi}{\partial X} \right)^T \dot{X} \right]$$

$$= \frac{\partial H}{\partial X} + \frac{\partial^2 \varphi}{\partial X \partial t} + \left[\frac{\partial^2 \varphi}{\partial X^2} \right] \dot{X}$$

$$\boxed{\begin{aligned} L^* &\triangleq \left[H + \frac{d\varphi(X,t)}{dt} - \lambda^T \dot{X} \right] \\ &= \left[H + \left\{ \frac{\partial \varphi}{\partial t} + \left(\frac{\partial \varphi}{\partial X} \right)^T \dot{X} \right\} - \lambda^T \dot{X} \right] \\ H &\triangleq L(X,U,t) + \lambda^T f(X,U,t) \end{aligned}}$$

$$\frac{d}{dt}\left(\frac{\partial L^*}{\partial \dot{X}} \right) = \frac{d}{dt}\left[\frac{\partial \varphi}{\partial X} - \lambda \right] = \frac{\partial}{\partial t}\left(\frac{\partial \varphi}{\partial X} \right) + \left[\frac{\partial}{\partial X}\left(\frac{\partial \varphi}{\partial X} \right) \right] \dot{X} - \dot{\lambda}$$

$$= \frac{\partial^2 \varphi}{\partial X \partial t} + \left[\frac{\partial^2 \varphi}{\partial X^2} \right] \dot{X} - \dot{\lambda}$$

Necessary Conditions of Optimality: Path Equations

$$\boxed{H \triangleq L(X,U,t) + \lambda^T f(X,U,t)}$$

(1) $$\frac{\partial L^*}{\partial X} - \frac{d}{dt}\left(\frac{\partial L^*}{\partial \dot{X}}\right) = 0$$

$$\frac{\partial H}{\partial X} + \cancel{\frac{\partial^2 \varphi}{\partial X \partial t}} + \cancel{\left[\frac{\partial^2 \varphi}{\partial X^2}\right]}\dot{X} - \cancel{\frac{\partial^2 \varphi}{\partial X \partial t}} - \cancel{\left[\frac{\partial^2 \varphi}{\partial X^2}\right]}\dot{X} + \dot{\lambda} = 0$$

$$\frac{\partial H}{\partial X} + \dot{\lambda} = 0 \Rightarrow \dot{\lambda} = -\left(\frac{\partial H}{\partial X}\right)$$ Costate/Adjoint Equation

(2) $$\frac{\partial L^*}{\partial U} = 0 \quad \Rightarrow \quad \left(\frac{\partial H}{\partial U}\right) = 0$$ Optimal Control/Stationary Equation

(3) $$\frac{\partial L^*}{\partial \lambda} = 0 \quad \Rightarrow \quad \frac{\partial H}{\partial \lambda} - \dot{X} = 0$$

$$\dot{X} = \left(\frac{\partial H}{\partial \lambda}\right) = f(X,U,t)$$ State Equation/System Dynamics

Necessary Conditions of Optimality: Boundary/Transversality Conditions

Summary :

Define $H \triangleq \left(L + \lambda^T f\right)$ and satisfy:

(1) $\dot{X} = f\left(X, U, t\right)$ (State Equation)

(2) $\dfrac{\partial H}{\partial U} = 0$ (Optimal Control Equation)

(3) $\dot{\lambda} = -\left(\dfrac{\partial H}{\partial X}\right)$ (Costate Equation)

Necessary Conditions of Optimality: Boundary/Transversality Conditions

$$\left(\frac{\partial L^*}{\partial \dot{X}}\right)_{t_f}^T \delta X_f + \left[L^* - \left(\frac{\partial L^*}{\partial \dot{X}}\right)^T \dot{X}\right]_{t_f} \delta t_f = 0$$

$$\boxed{\begin{aligned} L^* &\triangleq H + \frac{d\varphi(X,t)}{dt} - \lambda^T \dot{X} \\ &= H + \left[\frac{\partial \varphi}{\partial t} + \left(\frac{\partial \varphi}{\partial X}\right)^T \dot{X}\right] - \lambda^T \dot{X} \end{aligned}}$$

[Note: Both t_0 and $X(t_0)$ are assumed to be fixed!]

$$\left[\frac{\partial \varphi}{\partial X} - \lambda\right]_{t_f}^T \delta X_f + \left[L^* - \left(\frac{\partial \varphi}{\partial X} - \lambda\right)^T \dot{X}\right]_{t_f} \delta t_f = 0$$

$$\left[\frac{\partial \varphi}{\partial X} - \lambda\right]_{t_f}^T \delta X_f + \left[H + \frac{\partial \varphi}{\partial t} + \cancel{\left(\frac{\partial \varphi}{\partial X}\right)^T \dot{X}} - \cancel{\lambda^T \dot{X}} - \cancel{\left(\frac{\partial \varphi}{\partial X}\right)^T \dot{X}} + \cancel{\lambda^T \dot{X}}\right]_{t_f} \delta t_f = 0$$

$$\left[\frac{\partial \varphi}{\partial X} - \lambda\right]_{t_f}^T \delta X_f + \left[\frac{\partial \varphi}{\partial t} + H\right]_{t_f} \delta t_f = 0$$

Necessary Conditions of Optimality: Boundary/Transversality Conditions

Special Cases:

(1) t_f : fixed, X_f : free

$$\left[\frac{\partial \varphi}{\partial X} - \lambda\right]_{t_f}^T \delta X_f = 0$$

$$\lambda_f = \left[\frac{\partial \varphi}{\partial X}\right]_{t_f} = \left[\frac{\partial \varphi\left(X_f, t_f\right)}{\partial X_f}\right] \quad (n \text{ boundary conditions: TPBVP})$$

(2) t_f : free, X_f : fixed

$$H\left(t_f\right) = -\left[\frac{\partial \varphi}{\partial t}\right]_{t_f} = \frac{\partial \varphi\left(X_f, t_f\right)}{\partial t_f} \quad (1 \text{ boundary condition})$$

Optimal Control Formulation for a Class of Problems: An Alternate Approach

Optimal Control Problem

- Performance Index (to minimize / maximize):

$$J = \varphi\left(t_f, X_f\right) + \int_{t_0}^{t_f} L(t, X, U)\, dt$$

- Path Constraint:

$$\dot{X} = f\,(t, X, U)$$

- Boundary Conditions: $X\,(0) = X_0$: Specified

t_f : Fixed, $X\left(t_f\right)$: Free

Necessary Conditions of Optimality

- Augmented PI $\bar{J} = \varphi + \int_{t_0}^{t_f} \left[L + \lambda^T \left(f - \dot{X} \right) \right] dt$
- Hamiltonian $H \triangleq \left(L + \lambda^T f \right)$
- First Variation $\delta \bar{J} = \delta\varphi + \delta \int_{t_0}^{t_f} \left(H - \lambda^T \dot{X} \right) dt$

$$= \delta\varphi + \int_{t_0}^{t_f} \delta \left(H - \lambda^T \dot{X} \right) dt$$

Necessary Conditions of Optimality

- First Variation $\delta \bar{J} = \delta\varphi + \int_{t_0}^{t_f} \left(\delta H - \delta\lambda^T \dot{X} - \lambda^T \delta\dot{X} \right) dt$

- Individual terms

$$\delta\varphi\left(t_f, X_f\right) = \left(\delta X_f\right)^T \left(\frac{\partial\varphi}{\partial X_f}\right)$$

$$\delta H\left(t, X, U, \lambda\right) = \left(\delta X\right)^T \left(\frac{\partial H}{\partial X}\right) + \left(\delta U\right)^T \left(\frac{\partial H}{\partial U}\right) + \left(\delta\lambda\right)^T \left(\frac{\partial H}{\partial \lambda}\right)$$

Necessary Conditions of Optimality

$$\int_{t_0}^{t_f} \left(\lambda^T \, \delta \dot{X} \right) dt = \int_{t_0}^{t_f} \left(\lambda^T \frac{d(\delta X)}{dt} \right) dt$$

$$= \left[\lambda^T \, \delta X \right]_{t_0, \delta X_0}^{t_f, \delta X_f} - \int_{t_0}^{t_f} \left(\frac{d\lambda}{dt} \right)^T \delta X \, dt$$

$$= \left[\lambda_f^T \, \delta X_f - \lambda_0^T \underbrace{\delta X_0}_{\mathbf{0}} \right] - \int_{t_0}^{t_f} (\delta X)^T \dot{\lambda} \, dt$$

$$= \lambda_f^T \, \delta X_f - \int_{t_0}^{t_f} (\delta X)^T \dot{\lambda} \, dt$$

Necessary Conditions of Optimality

- First Variation

$$\delta \bar{J} = (\delta X_f)^T \left(\frac{\partial \varphi}{\partial X_f} \right) - (\delta X_f)^T \lambda_f$$

$$+ \int_{t_0}^{t_f} \left[(\delta X)^T \left(\frac{\partial H}{\partial X} \right) + (\delta U)^T \left(\frac{\partial H}{\partial U} \right) + (\delta \lambda)^T \left(\frac{\partial H}{\partial \lambda} \right) \right] dt$$

$$+ \int_{t_0}^{t_f} (\delta X)^T \dot{\lambda}\, dt - \int_{t_0}^{t_f} (\delta \lambda)^T \dot{X}\, dt$$

Necessary Conditions of Optimality

- First Variation

$$\delta \bar{J} = \left(\delta X_f\right)^T \left[\frac{\partial \varphi}{\partial X_f} - \lambda_f\right]$$

$$+ \int_{t_0}^{t_f} (\delta X)^T \left[\frac{\partial H}{\partial X} + \dot{\lambda}\right] dt + \int_{t_0}^{t_f} (\delta U)^T \left[\frac{\partial H}{\partial U}\right] dt$$

$$+ \int_{t_0}^{t_f} (\delta \lambda)^T \left[\frac{\partial H}{\partial \lambda} - \dot{X}\right] dt$$

$$= 0$$

Necessary Conditions of Optimality: Summary

- State Equation $\qquad \dot{X} = \dfrac{\partial H}{\partial \lambda} = f(t, X, U)$
- Costate Equation $\qquad \dot{\lambda} = -\left(\dfrac{\partial H}{\partial X}\right)$
- Optimal Control Equation $\qquad \dfrac{\partial H}{\partial U} = 0$
- Boundary Condition $\qquad \lambda_f = \dfrac{\partial \varphi}{\partial X_f} \qquad X(t_0) = X_0 : Fixed$

Necessary Conditions of Optimality: Some Comments

- State and Costate equations are dynamic equations. **If one is stable, the other turns out to be unstable!**
- Optimal control equation is a stationary equation
- Boundary conditions are split: it leads to **Two-Point-Boundary-Value Problem (TPBVP)**
- State equation develops forward whereas Costate equation develops backwards.
- It is known as "***Curse of Complexity***" in optimal control
- Traditionally, TPBVPs demand computationally-intensive iterative numerical procedures, which lead to "open-loop" control structure.

An Useful Theorem

Theorem:

If the Hamiltonian H is not an explicit function of time, then H is 'constant' along the optimal path.

Proof:

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} + \dot{X}^T \frac{\partial H}{\partial X} + \dot{U}^T \frac{\partial H}{\partial U} + \dot{\lambda}^T \frac{\partial H}{\partial \lambda}$$

$$= \frac{\partial H}{\partial t} + \dot{X}^T \underbrace{\left(\frac{\partial H}{\partial X} + \dot{\lambda} \right)}_{0} + \dot{U}^T \underbrace{\left(\frac{\partial H}{\partial U} \right)}_{0} \qquad \left(\because \ \frac{\partial H}{\partial \lambda} = \dot{X} \quad \text{and} \quad \dot{\lambda}^T \dot{X} = \dot{X}^T \dot{\lambda} \right)$$

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} \quad (\text{on optimal path})$$

$$= 0 \quad (\text{if } H \text{ is not an explicit function of } t). \ \text{Hence, the result!}$$

General Boundary/Transversality Condition

General condition:

$\left[\text{with } \left(t_0, X_0\right) \text{fixed}\right]$

$$\boxed{\left[\frac{\partial\Phi}{\partial X} - \lambda\right]_{t_f}^T \delta X_f + \left[\frac{\partial\Phi}{\partial t} + H\right]_{t_f} \delta t_f = 0}$$

Special Cases: 1) t_f : fixed, X_f : free

$$\left[\frac{\partial\Phi}{\partial X} - \lambda\right]_{t_f}^T \delta X_f = 0 \quad \Rightarrow \quad \lambda_f = \frac{\partial\Phi\left(t_f, X_f\right)}{\partial X_f}$$

2) t_f : free, X_f : fixed

$$\left[\frac{\partial\Phi}{\partial t} + H\right]_{t_f} \delta t_f = 0 \quad \Rightarrow \quad H\left(t_f\right) = \frac{\partial\Phi}{\partial t_f}$$

Example 1

Problem:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -x_2 + u \end{bmatrix}$$

$$J = \frac{1}{2}\left(x_{1_f} - 5\right)^2 + \frac{1}{2}\left(x_{2_f} - 2\right)^2 + \frac{1}{2}\int_{t_0}^{t_f} u^2\, dt$$

$$t_0 = 0, t_f = 2, \quad x_1(0) = x_2(0) = 0$$

Solution: $H = \left(u^2/2\right) + \lambda_1 x_2 + \lambda_2\left(-x_2 + u\right)$

Costate Eq. $\begin{bmatrix} \dot{\lambda}_1 \\ \dot{\lambda}_2 \end{bmatrix} = \begin{bmatrix} -\left(\partial H/\partial x_1\right) \\ -\left(\partial H/\partial x_2\right) \end{bmatrix} = \begin{bmatrix} 0 \\ -\lambda_1 + \lambda_2 \end{bmatrix}$

Optimal control Eq. $u + \lambda_2 = 0 \quad \Rightarrow \quad \boxed{u = -\lambda_2}$

Example 1

Boundary Conditions

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} \lambda_1(2) \\ \lambda_2(2) \end{bmatrix} = \begin{bmatrix} x_1(2)-5 \\ x_2(2)-2 \end{bmatrix}$$

Define $Z \triangleq \begin{bmatrix} x_1 & x_2 & \lambda_1 & \lambda_2 \end{bmatrix}^T$

$$\dot{Z} = AZ$$

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

Solution

$$Z(t) = e^{At}C$$

Example 1

Use the boundary condition at $t=0$

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

Use the boundary condition at $t_f = 2$

$$\begin{bmatrix} x_1(2) \\ x_2(2) \\ x_1(2)-5 \\ x_2(2)-2 \end{bmatrix} = e^{2A} \begin{bmatrix} 0 \\ 0 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 1 & 0.86 & 1.63 & -2.76 \\ 0 & 0.14 & 2.76 & -3.63 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -6.39 & 7.39 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ c_3 \\ c_4 \end{bmatrix}$$

Example 1

Four equations and four unknowns:

$$\begin{bmatrix} 1 & 0 & -1.63 & 2.76 \\ 0 & 1 & -2.76 & 3.63 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 6.39 & -7.39 \end{bmatrix} \begin{bmatrix} x_1(2) \\ x_2(2) \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 5 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} x_1(2) \\ x_2(2) \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1.63 & 2.76 \\ 0 & 1 & -2.76 & 3.63 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 6.39 & -7.39 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ 5 \\ 2 \end{bmatrix} = \begin{bmatrix} 2.30 \\ 1.33 \\ -2.70 \\ -2.42 \end{bmatrix}$$

Example 1

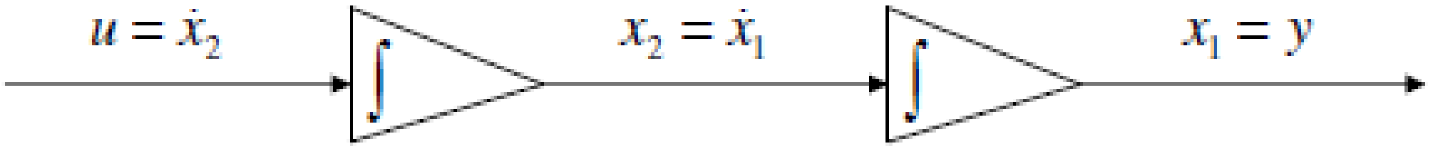
- Solution for State and Costate

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ \lambda_1(t) \\ \lambda_2(t) \end{bmatrix} = e^{At} \begin{bmatrix} 0 \\ 0 \\ -2.70 \\ 2.42 \end{bmatrix} \quad \text{where } A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

- Solution for Optimal Control

$$\boxed{u = -\lambda_2(t)}$$

Example – 2: Double Integrator Problem



Consider a double integrator problem as shown in the above figure.

Find such $u(t)$ that the system initial values $X(0) = [10 \quad 0]^T$ are driven to the origin by minimizing

$$J = t_f^{\,2} + \frac{1}{2}\int_0^{t_f} u^2 dt$$

Note : (1) t_f: unspecified

(2) Control variable $u(t)$ is unconstrained

Double Integrator Problem

Solution :

<u>System dynamics</u>

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{X} + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B} u = AX + Bu$$

$$y = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{C} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = CX \qquad \text{(not required)}$$

<u>Boundary Condition</u>

$$X(0) = \begin{bmatrix} 10 \\ 0 \end{bmatrix}, \quad X(t_f) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Double Integrator Problem

Controllability Check :

Controllability Matrix

$$M = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} & \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$|M| = -1 \neq 0$$

Hence, the system is controllable.

Double Integrator Problem

$$H = \frac{1}{2}u^2 + \lambda^T\left(AX + Bu\right)$$

(1) State Eq: $\dot{X} = AX + Bu$

(2) Optimal Control Eq: $\dfrac{\partial \mathrm{H}}{\partial u} = 0$

$$u + B^T\lambda = 0$$

$$u = -B^T\lambda = -\lambda_2$$

(3) Costate Eq: $\dot{\lambda} = -\dfrac{\partial H}{\partial X} = -A^T\lambda$

Necessary Conditions of Optimality

$$\begin{bmatrix} \dot{\lambda}_1 \\ \dot{\lambda}_2 \end{bmatrix} = -A^T\lambda = -\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} 0 \\ -\lambda_1 \end{bmatrix}$$

$$\dot{\lambda}_1 = 0 \Rightarrow \lambda_1 = c_1$$

$$\dot{\lambda}_2 = -\lambda_1 = -c_1$$

$$\lambda_2 = -c_1 t + c_2$$

$$\therefore \quad u = -\lambda_2 = c_1 t - c_2$$

Optimal Control Solution

However,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ u \end{bmatrix} = \begin{bmatrix} x_2 \\ c_1 t - c_2 \end{bmatrix}$$

Hence

$$x_2 = c_1 \frac{t^2}{2} - c_2 t + c_3$$

$$x_1 = \int x_2 dt = c_1 \frac{t^3}{6} - c_2 \frac{t^2}{2} + c_3 t + c_4$$

Optimal State Solution

Using the B.C. at $t = 0$:

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} c_4 \\ c_3 \end{bmatrix} = \begin{bmatrix} 10 \\ 0 \end{bmatrix}$$

$$\therefore \quad \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \frac{c_1}{6}t^3 - \frac{c_2}{2}t^2 + 10 \\ \frac{c_1}{2}t^2 - c_2 t \end{bmatrix}$$

Using the B.C at $t = t_f$:

$$\begin{bmatrix} x_1(t_f) \\ x_2(t_f) \end{bmatrix} = \begin{bmatrix} \frac{c_1}{6}t_f^{\ 3} - \frac{c_2}{2}t_f^{\ 2} + 10 \\ \frac{c_1}{2}t_f^{\ 2} - c_2 t_f \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Transversality Conditions (tf : free)

$$\left.\frac{\partial \varphi}{\partial t}\right|_{t_f} = -H\Big|_{t_f}$$

$$2t_f = -\left[\frac{u^2}{2} + \lambda^T\left(AX + Bu\right)\right]_{t_f}$$

$$= -\left[\frac{u^2}{2} + \begin{bmatrix}\lambda_1 & \lambda_2\end{bmatrix}\begin{bmatrix}x_2 \\ u\end{bmatrix}\right]_{t_f}$$

$$= -\left[\frac{\left(c_1 t_f - c_2\right)^2}{2} + \underbrace{\lambda_1\left(t_f\right)x_2\left(t_f\right)}_{0\ \text{(B.C.)}} - \left(c_1 t_f - c_2\right)^2\right]$$

$$= \frac{1}{2}\left(c_1 t_f - c_2\right)^2$$

$$4t_f = c_1^2 t_f^{\,2} - 2c_1 c_2 t_f + c_2^{\,2}$$

Transversality Conditions (tf : free)

In summary, we have to solve for c_1, c_2 and t_f from:

$$c_1 t_f^{\,3} - 3c_2 t_f^{\,2} + 60 = 0$$

$$c_1 t_f^{\,2} - 2c_2 t_f = 0$$

$$c_1^{\,2} t_f^{\,2} - \left(2c_1 c_2 + 4t_f\right) + c_2^{\,2} = 0$$

At this point, one can solve c_1, c_2 from first two equations in terms of t_f and subtitute them in the third equation. Then the resulting nonlinear equation in t_f can be solved (preferably in closed form). However, one must discard unrealistic solutions (e.g. $t_f \le 0$ is unrealistic).

Note : One may use numerical tehniques (like Newton-Raphson technique

Transversality Conditions (tf : free)

Finally, $\begin{bmatrix} c_1 \\ c_2 \\ t_f \end{bmatrix} = \begin{bmatrix} 2.025 \\ 3.95 \\ c_2^{\,2}/4 \end{bmatrix}$

Hence, the optimal solution is given by:

$u = c_1 t - c_2 = 2.025t - 3.95$

and it will take $t_f = \dfrac{(3.95)^2}{4} = 3.901$ time units to reach $X_f = [0 \quad 0]^T$,

starting from $X(0) = [10 \quad 0]^T$

Note: (1) It is an open-loop control law

(2) The application of control has to be terminated at t_f