In this notebook we compute the limiting spectral distribution for a variety of noncommutative self-adjoint polynomials in random matrices

11/16/23 16:38:54 In[12]:=

```
SetDirectory[NotebookDirectory[]]; (*set directory*)
```

Add the folder(s) containing the Random Matrix package as well as NC package to $Path and load the packages

11/16/23 16:38:58 In[13]:=

```
AppendTo[$Path,
  "/Users/yumish/Dropbox/a_recherche/mathematica_all/mathematica_packages/"];
<< NC` (*load NCAlgebra*)
<< NCAlgebra`
<< PolynomialsOfRandomMatrices`
```

* Function ***DoSplot[Polynomial, {E1,E2},Optional:LinearizationType->{"Minimal","Big"}, MinimizationType->{"MachinePrecision","Exact","HighPrecision"}, MinimizationPrecision->MachinePrecision, SolutionAccuracy->0.001, Iterations->5000, ImaginaryPart→0.001, GridNumber→1000, Rescaling->1]*** returns the plot of the density of states for the *Polynomial* on the interval [*E1,E2*] with *ImaginaryPart* equal to 0.001. We have also several optional arguments (default values in blue) as well as the plot options

11/17/23 02:27:49 In[158]:=

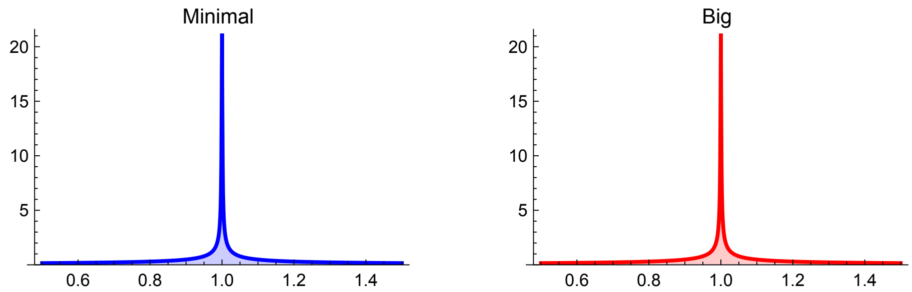
```
Options[DoSplot] =
  Join[{LinearizationType → "Minimal", MinimizationType → "MachinePrecision",
    MinimizationPrecision → MachinePrecision, SolutionAccuracy → 0.001, Iterations →
     5000, ImaginaryPart → 0.001, GridNumber → 1000, Rescaling → 1}, Options[Plot]];
DoSplot[Poly_, Interval_, opts : OptionsPattern[]] :=
  Block[{LLinearization, VVariableList, LL0, LL, JJ, SSOperator, MM},
   Switch[OptionValue[LinearizationType],
    "Big", LLinearization = BigLinearization[Poly];,
    "Minimal", LLinearization = MinimalLinearization[Poly,
       OptionValue[MinimizationType], OptionValue[MinimizationPrecision]];
   ];
   MM = IterativelySolveMDEonInterval2[LLinearization, z , Interval⟦1⟧,
     Interval⟦2⟧, OptionValue[ImaginaryPart], OptionValue[GridNumber],
     OptionValue[Iterations], OptionValue[SolutionAccuracy]];

   Return[ListLinePlot[
     Table[{MM⟦i, 1⟧, Im[MM⟦i, 2⟧⟦1, 1⟧] / Pi * OptionValue[Rescaling]},
      {i, Length[MM]}], FilterRules[{opts}, Options[Plot]],
     PlotRange → {0, 1.02 Max[Table[Im[MM⟦i, 2⟧⟦1, 1⟧] / Pi, {i, Length[MM]}]]},
     Ticks → Automatic, PlotLegends → Automatic, ImageSize → 270,
     PlotStyle → {Blue, Thick}, Filling → Bottom]];
  ];
```

Basic example:

11/17/23 02:27:52 In[160]:=

```
GraphicsRow[
 {DoSplot[1 + x ** x ** x - z, {0.5, 1.5}, PlotLabel → "Minimal"], DoSplot[1 + x ** x ** x - z,
   {0.5, 1.5}, LinearizationType → "Big", PlotLabel → "Big", PlotStyle → {Red}]}]
```

11/17/23 02:27:53 Out[160]=



Limiting eigenvalue distribution for randomly generated noncommutative polynomials of degree 3 in two Wigner matrices

Example of a generated polynomial

11/17/23 02:28:38 In[161]:=

```
1 - z + GeneratePolynomial[{x, y}, {2, 4, 8}, "Real"]
```

11/17/23 02:28:38 Out[161]=

```
1 - 1.52537 x + 1.97101 y - z + 8.38746 x ** x - 0.151023 x ** y -
 0.151023 y ** x - 3.65231 y ** y - 9.36504 x ** x ** x - 3.31932 x ** x ** y +
 0.00796495 x ** y ** x + 0.388089 x ** y ** y - 3.31932 y ** x ** x -
 3.59479 y ** x ** y + 0.388089 y ** y ** x + 1.92581 y ** y ** y
```

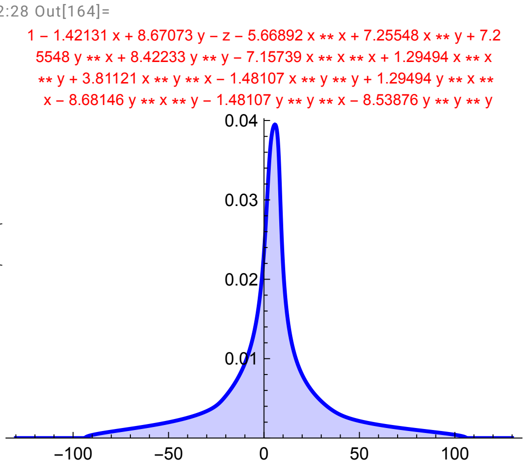
We generate noncommutative self-adjoint polynomials. For each polynomial we compute its minimal linearization, solve the Dyson equation for the linearization on the interval [-130, 130], and recover the self-consistent density of states for the polynomial. We then plot the corresponding density. Notice that some polynomials have bimodal structure (we don’t see this very often)

11/17/23 02:31:12 In[162]:=

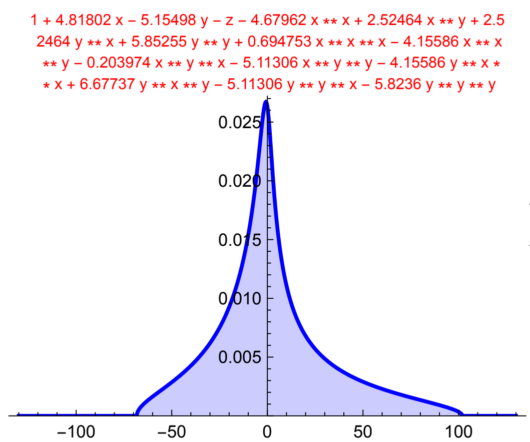
```
p = {};
Do[Poly = 1 - z + GeneratePolynomial[{x, y}, {2, 4, 8}, "Real"];

  PolyLabel = StringJoin[Characters[ToString[Poly]][[1 ;;
        Ceiling[Length[Characters[ToString[Poly]]] / 4]]] <> "\n " <> StringJoin[
     Characters[ToString[Poly]][[Ceiling[Length[Characters[ToString[Poly]]] / 4] + 1
        ;; Ceiling[2 * Length[Characters[ToString[Poly]]] / 4]]] <>
    "\n " <> StringJoin[Characters[ToString[Poly]][[Ceiling[
          2 * Length[Characters[ToString[Poly]]] / 4] + 1 ;;
        Ceiling[3 * Length[Characters[ToString[Poly]]] / 4]]] <> "\n " <> StringJoin[
     Characters[ToString[Poly]][[Ceiling[3 * Length[Characters[ToString[Poly]]] / 4] +
         1 ;; Ceiling[Length[Characters[ToString[Poly]]]]]]];

  p = Append[p, DoSplot[Poly, {-130, 130}, GridNumber → 1000,
     PlotLabel → Style[PolyLabel, FontColor → Red, FontSize → 8]]]
  , {ind, 10}];
Table[Table[p[[ind * 2 + j]], {j, 2}], {ind, 0, 4}]
```

11/17/23 02:32:28 Out[164]=

{{

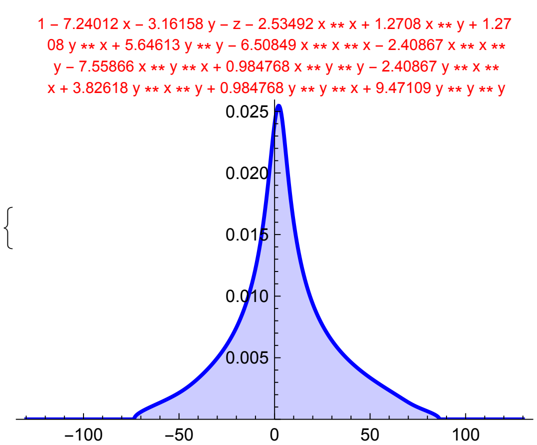


,

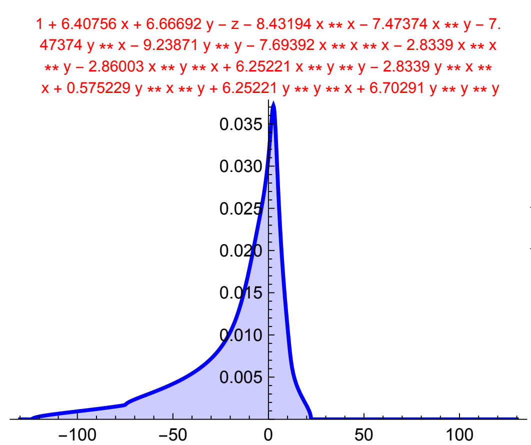


},

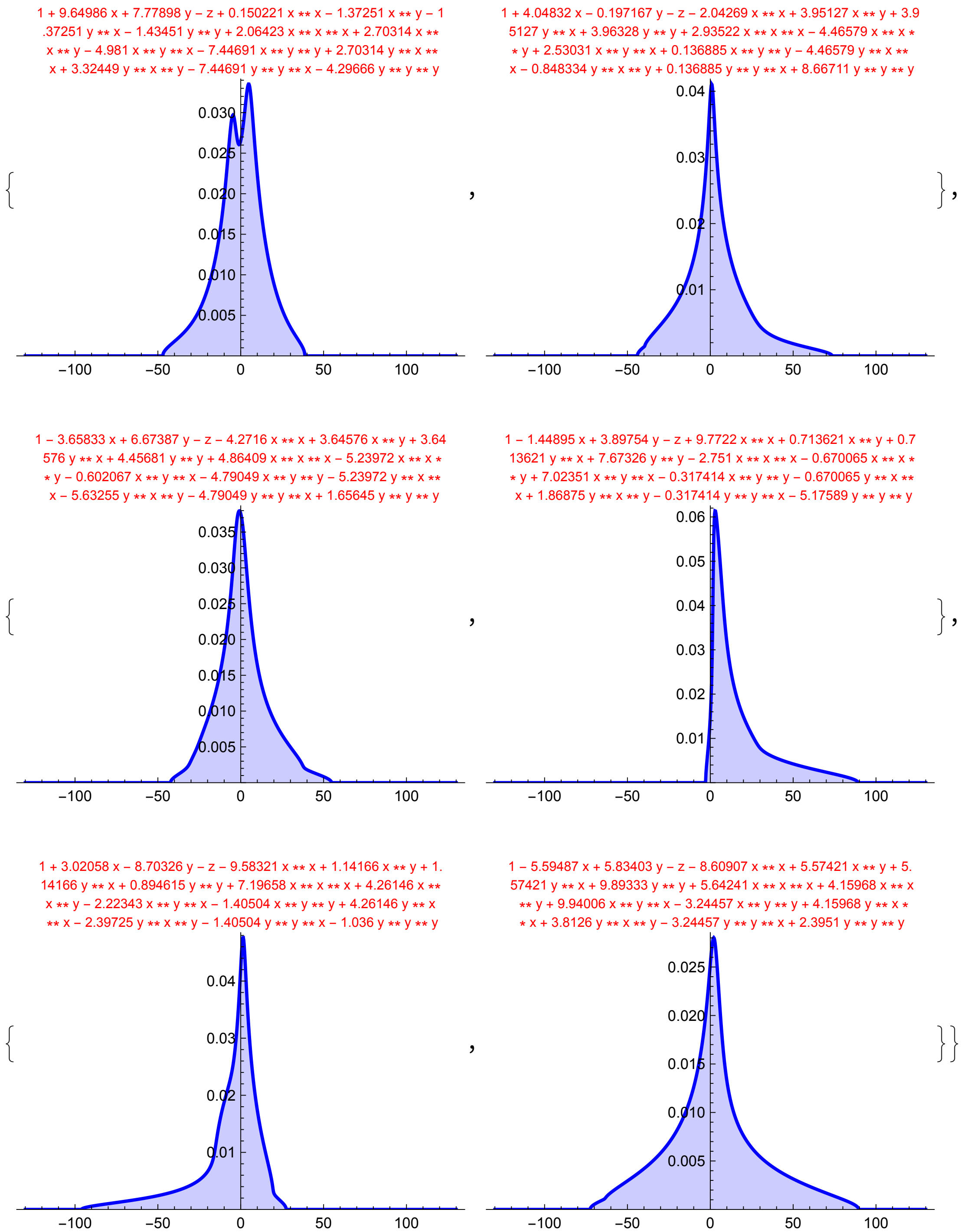
{



,



},



Diagonal entries of Im M

The conjecture is that all diagonal entries of Im M vanish if and only if the (1,1) entry of Im M vanishes. In other words, Im M either has full rank or is equal to zero. In order to check this numerically for a concrete polynomial, we can plot the self-consistent density of states, and then plot the diagonal entries of Im M in the vicinity of the edge of the support (see example below)

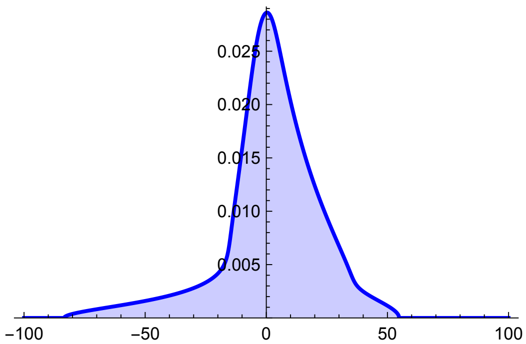
11/17/23 02:35:04 In[165]:=

```
Poly = 1 + GeneratePolynomial[{x, y}, {2, 4, 8}, "Real"];
```

11/17/23 02:35:09 In[166]:=

```
DoSplot[1 + Poly - z, {-100, 100}]
```

11/17/23 02:35:14 Out[166]=



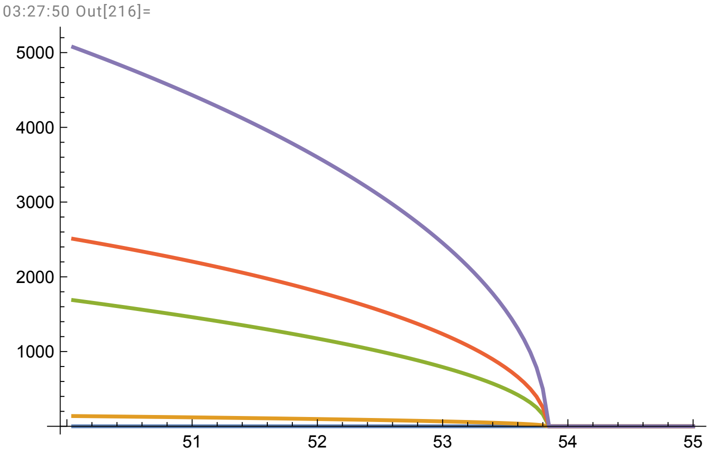
11/17/23 03:18:46 In[213]:=

```
Linearization = MinimalLinearization[Poly - z, "MachinePrecision"];
KK = 500 000; ϵϵ = 0.000001;
DData =
  Table[Table[{50 + j * 5 / 100, Sort[Table[Im[IterativelySolveMDE2[Linearization, z,
          50 + j * 5 / 100 + 0.000001 * i, i * IdentityMatrix[Length[Linearization]],
          KK, ϵϵ][[ind, ind]]], {ind, 5}]][[k]]}, {j, 100}], {k, 5}];
```

11/17/23 03:27:50 In[216]:=

```
ListLinePlot[DData, PlotRange → All]
```

11/17/23 03:27:50 Out[216]=

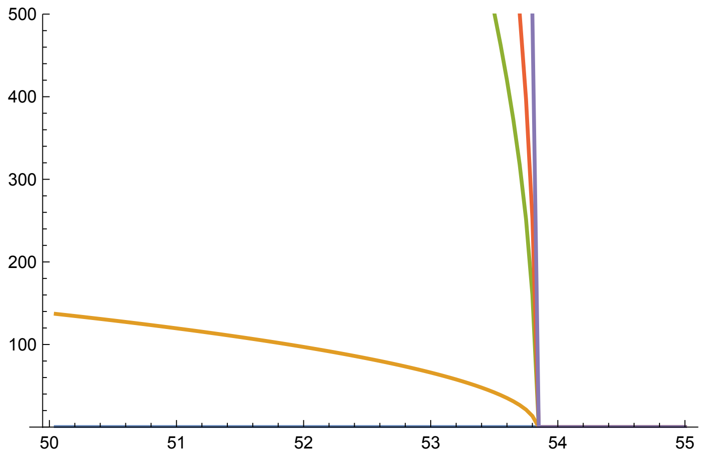


It looks like one of the diagonal entries is zero...

11/17/23 03:29:04 In[220]:=

```
ListLinePlot[DData, PlotRange → {-1, 500}]
```

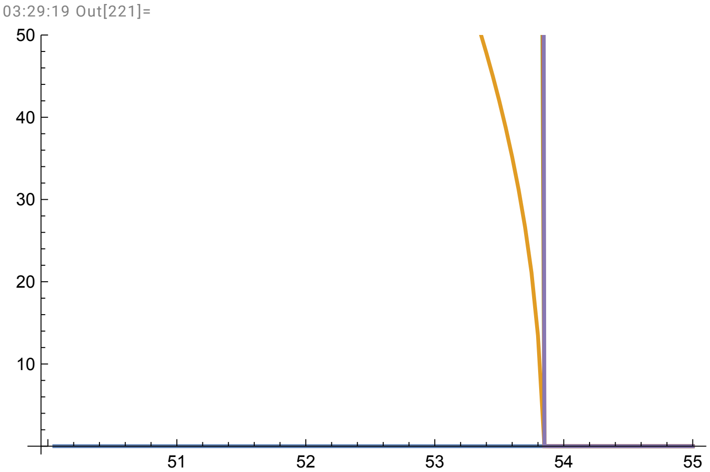
11/17/23 03:29:04 Out[220]=



11/17/23 03:29:19 In[221]:=

```
ListLinePlot[DData, PlotRange → {-1, 50}]
```

11/17/23 03:29:19 Out[221]=

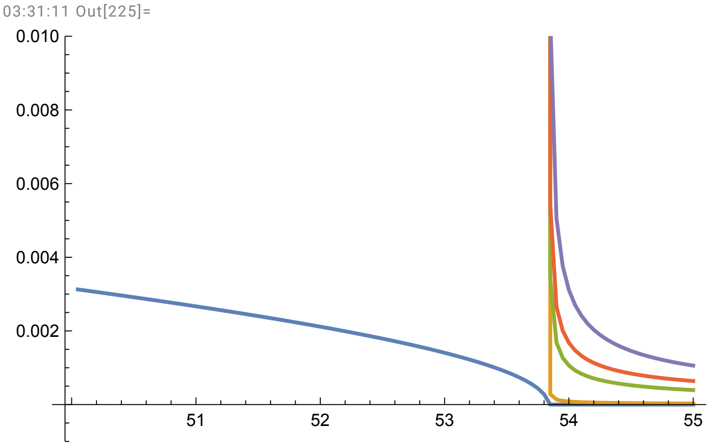


... but in fact this is just the (1,1) entry which is much smaller than all other entries.

11/17/23 03:31:11 In[225]:=

```
ListLinePlot[DData, PlotRange → {-0.001, 0.01}]
```

11/17/23 03:31:11 Out[225]=



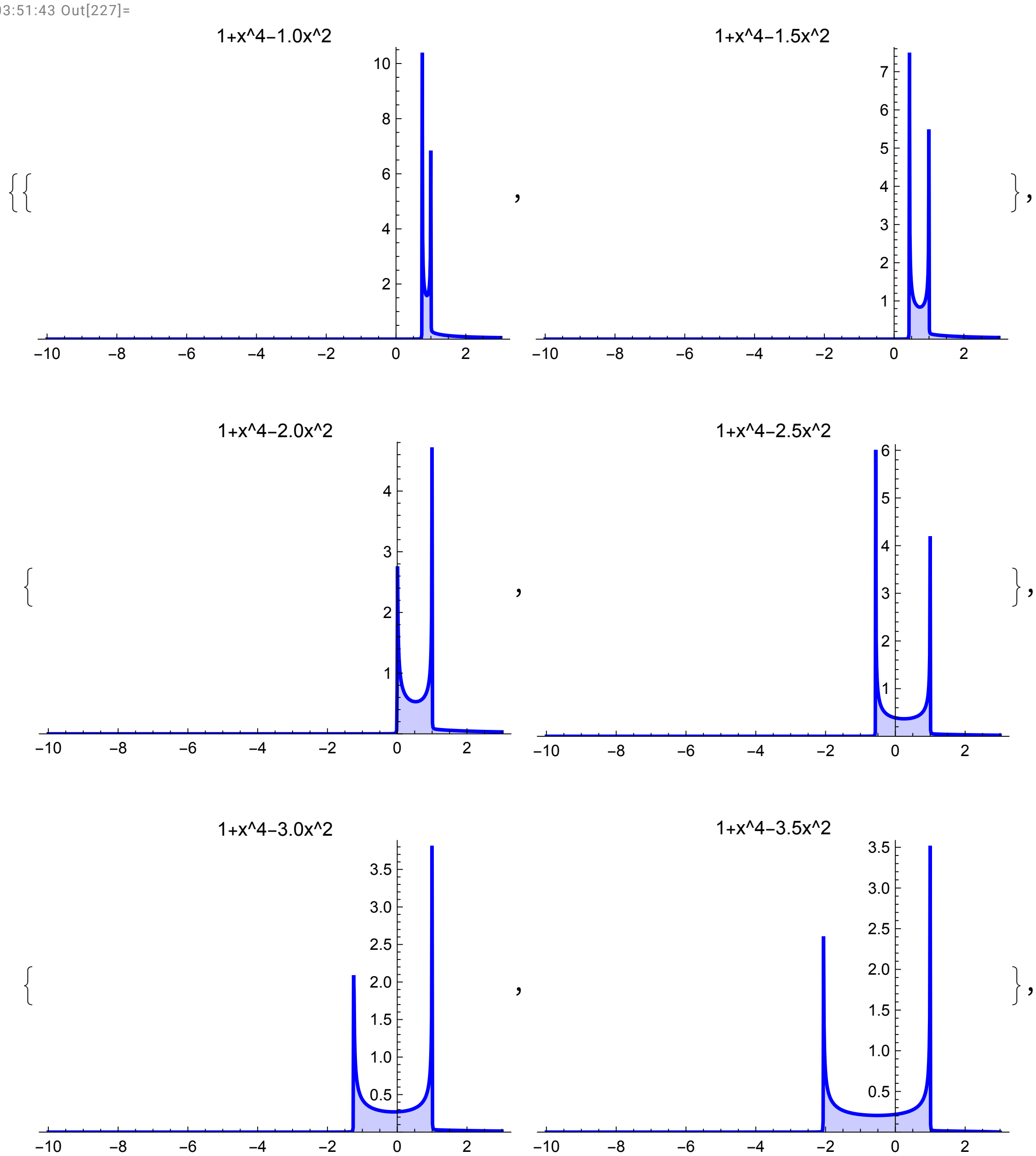
Family of polynomials with bimodal densities

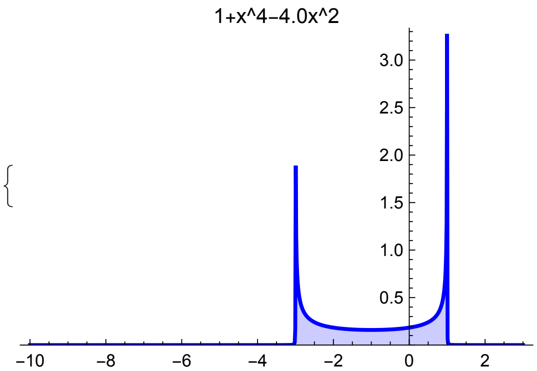
Here we plot the density of states for a family of polynomials 1+x^4+αx^2. We see that for α>0 and α<0 the density looks differently. More importantly, for α<0 the density blows up at two different point, which we haven’t seen before.

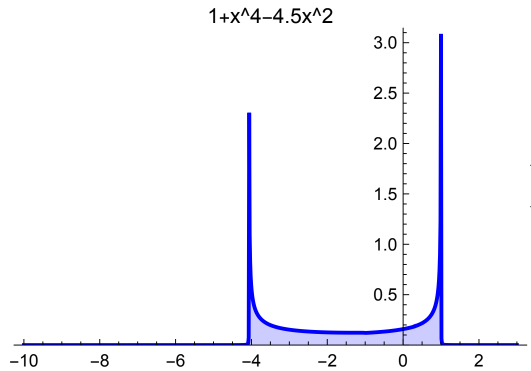
11/17/23 03:50:49 In[226]:=

```
p = Table[DoSplot[1 + x ** x ** x ** x - (1 + j / 2) * x ** x - z, {-10, 3},
    PlotLabel → "1+x^4-" <> ToString[N[1 + j / 2, 2]] <> "x^2"], {j, 0, 9}];
Table[Table[p⟦2 * ind + j⟧, {j, 2}], {ind, 0, 4}]
```

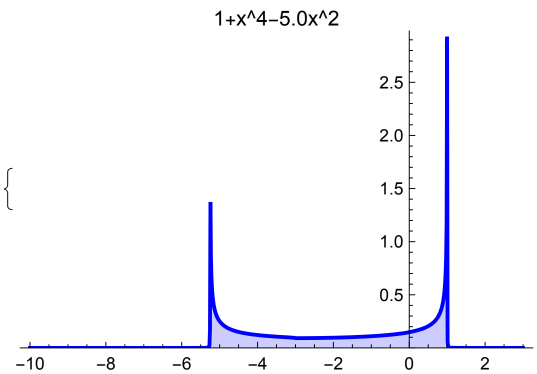
11/17/23 03:51:43 Out[227]=

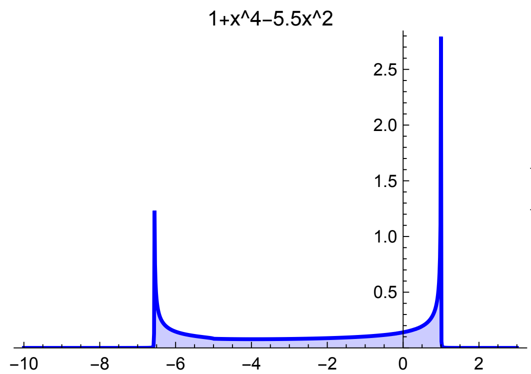


{ 

, 

},

{ 

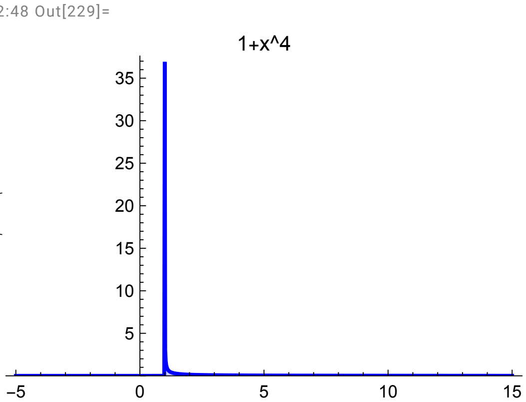
, 

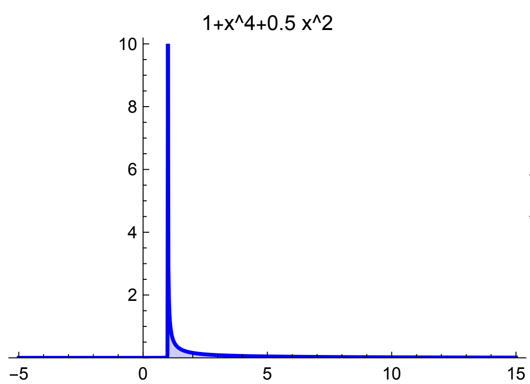
}}

11/17/23 03:52:33 In[228]:=

```
q = Join[{DoSplot[1 + x ** x ** x ** x - z, {-5, 15}, PlotLabel → "1+x^4"],
    DoSplot[1 + x ** x ** x ** x + 0.5 * x ** x - z, {-5, 15}, PlotLabel → "1+x^4+0.5  x^2"]},
   Table[DoSplot[1 + x ** x ** x ** x + (j / 2) * x ** x - z, {-5, 15},
     PlotLabel → "1+x^4+" <> ToString[N[j / 2, 2]] <> "  x^2"], {j, 2, 9}]];
Table[Table[q[[2 * ind + j]], {j, 2}], {ind, 0, 4}]
```

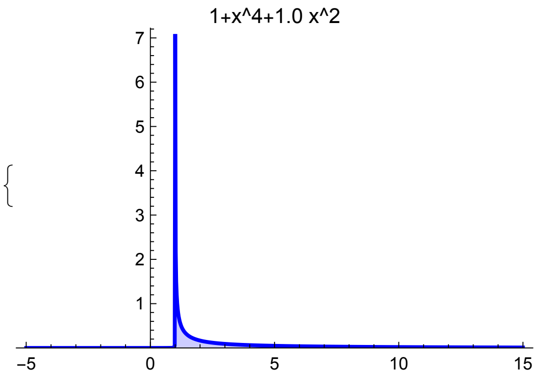
11/17/23 03:52:48 Out[229]=

{{ 

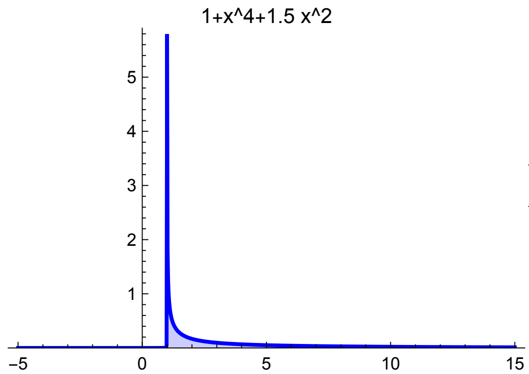
, 

},

{

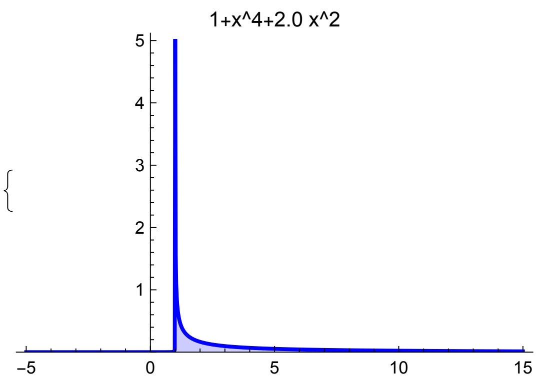

1+x^4+1.0 x^2
7
6
5
4
3
2
1
-5
0
5
10
15

,

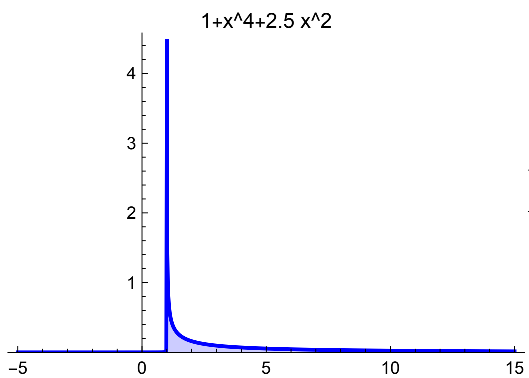

1+x^4+1.5 x^2
5
4
3
2
1
-5
0
5
10
15

},

{

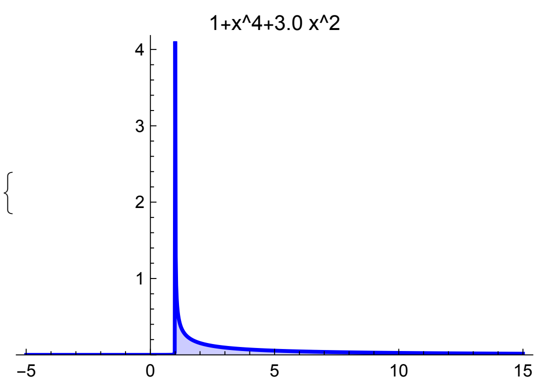

1+x^4+2.0 x^2
5
4
3
2
1
-5
0
5
10
15

,

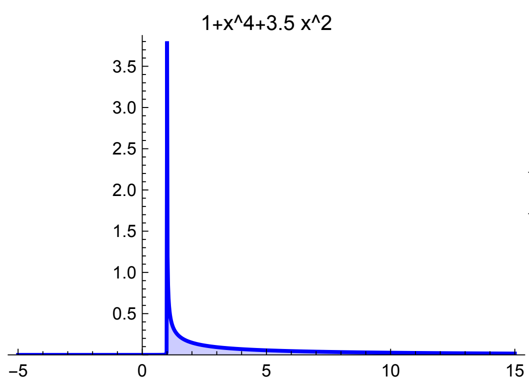

1+x^4+2.5 x^2
4
3
2
1
-5
0
5
10
15

},

{

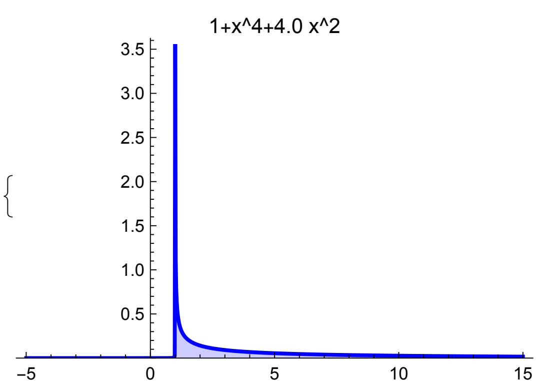

1+x^4+3.0 x^2
4
3
2
1
-5
0
5
10
15

,

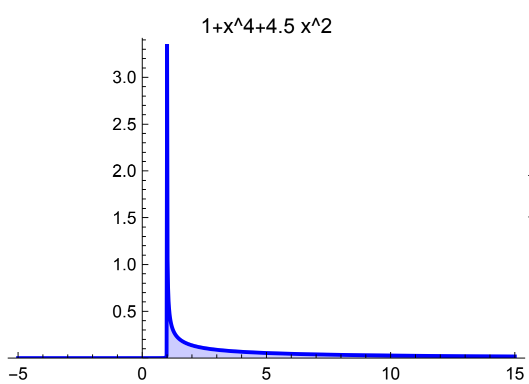

1+x^4+3.5 x^2
3.5
3.0
2.5
2.0
1.5
1.0
0.5
-5
0
5
10
15

},

{


1+x^4+4.0 x^2
3.5
3.0
2.5
2.0
1.5
1.0
0.5
-5
0
5
10
15

,


1+x^4+4.5 x^2
3.0
2.5
2.0
1.5
1.0
0.5
-5
0
5
10
15

}}

Compare the smallest eigenvalues of Im M for the big and the minimal linearization

The conjecture is that for the minimal linearization all eigenvalues of Im M vanish together (Im M is either full rank or a zero matrix), while for the big linearization Im M may not be full rank.

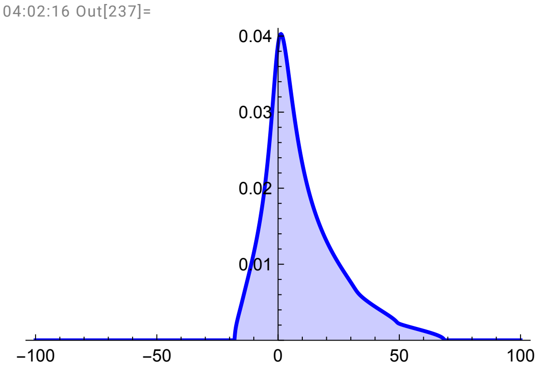
Conclusion : λ (Im M) can be equivalent to Tr (Im M) **only for minimal** linearization

Example 1:

11/17/23 04:02:11 In[236]:=

```
Poly = 1 + GeneratePolynomial[{x, y}, {2, 4, 8}, "Real"];
DoSplot[1 + Poly - z, {-100, 100}]
```

11/17/23 04:02:16 Out[237]=



We start by plotting two smallest eigenvalues of Im M near the edge for the Minimal linearization

11/17/23 04:09:03 In[238]:=

```
MatrixIm[M_] := (M - ConjugateTranspose[M]) / (2 * ⅈ);
```

11/17/23 04:26:09 In[262]:=

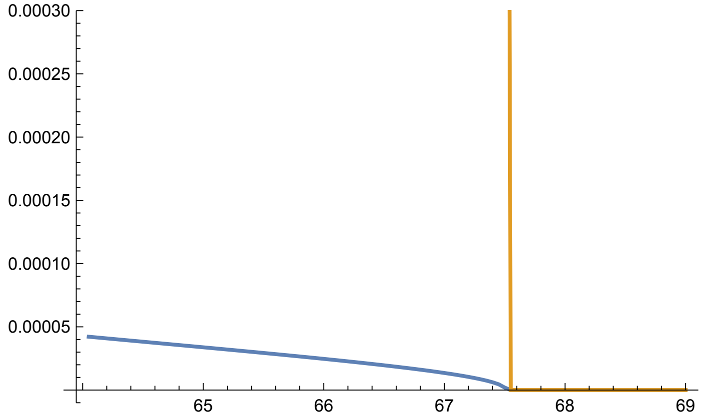
```
Linearization = MinimalLinearization[Poly - z, "MachinePrecision"];
KK = 500 000; ϵϵ = 0.00001;
DData = Table[Table[
    {N[64 + j * 5 / 100], Eigenvalues[MatrixIm[IterativelySolveMDE2[Linearization, z,
        64 + j * 5 / 100 + 0.000001 * ⅈ, ⅈ * IdentityMatrix[Length[Linearization]],
        KK, ϵϵ]]][[-ind]]}, {j, 100}], {ind, 2}];
```

```
100 0010.0332368

100 0010.0332368
```

11/17/23 04:27:25 In[265]:=

```
ListLinePlot[DData, PlotRange → {-0.00001, 0.0003}]
```

11/17/23 04:27:25 Out[265]=



Here we plot three smallest eigenvalues of Im M near the edge for the Big linearization

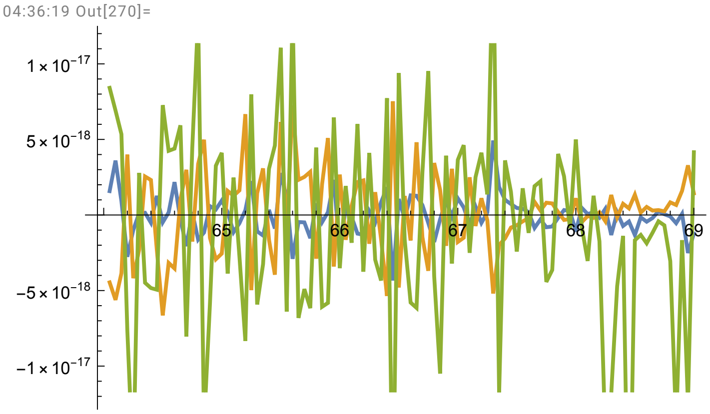
11/17/23 04:29:03 In[267]:=

```
Linearization = BigLinearization[Poly - z];
KK = 500 000; ϵϵ = 0.00001;
DData = Table[Table[
    {N[64 + j * 5 / 100], Eigenvalues[MatrixIm[IterativelySolveMDE2[Linearization, z,
          64 + j * 5 / 100 + 0.000001 * i, i * IdentityMatrix[Length[Linearization]],
          KK, ϵϵ]]][[-ind]]}, {j, 100}], {ind, 3}];
```

11/17/23 04:36:19 In[270]:=

```
ListLinePlot[DData]
```

11/17/23 04:36:19 Out[270]=

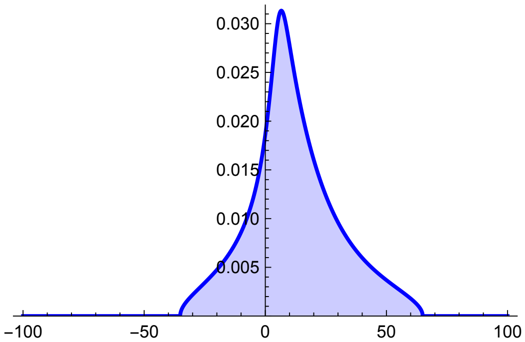


Example 2:

11/17/23 04:36:43 In[271]:=

```
Poly = 1 + GeneratePolynomial[{x, y}, {2, 4, 8}, "Real"];
DoSplot[1 + Poly - z, {-100, 100}]
```

11/17/23 04:36:46 Out[272]=



Again, we plot two smallest eigenvalues of Im M near the edge for the Minimal linearization

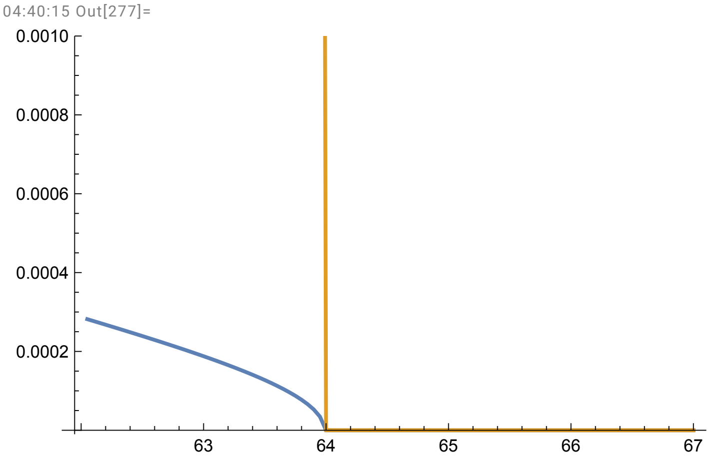
11/17/23 04:38:44 In[273]:=

```
Linearization = MinimalLinearization[Poly - z, "MachinePrecision"];
KK = 500 000; ϵϵ = 0.00001;
DData = Table[Table[
    {N[62 + j * 5 / 100], Eigenvalues[MatrixIm[IterativelySolveMDE2[Linearization, z,
        62 + j * 5 / 100 + 0.000001 * ⅈ, ⅈ * IdentityMatrix[Length[Linearization]],
        KK, ϵϵ]]][[-ind]]}, {j, 100}], {ind, 2}];
```

11/17/23 04:40:15 In[277]:=

```
ListLinePlot[DData, PlotRange → {-0.00001, 0.001}]
```

11/17/23 04:40:15 Out[277]=



We plot three smallest eigenvalues of Im M near the edge for the Big linearization

11/17/23 04:42:30 In[282]:=

```
Linearization = BigLinearization[Poly - z];
KK = 500 000; ϵϵ = 0.00001;
DData = Table[Table[
    {N[62 + j * 5 / 100], Eigenvalues[MatrixIm[IterativelySolveMDE2[Linearization, z,
        62 + j * 5 / 100 + 0.000001 * ⅈ, ⅈ * IdentityMatrix[Length[Linearization]],
        KK, ϵϵ]]][[-ind]]}, {j, 100}], {ind, 3}];
```

11/17/23 04:46:09 In[285]:=

```
ListLinePlot[DData]
```

11/17/23 04:46:10 Out[285]=