# MATH180C: Introduction to Stochastic Processes II

www.math.ucsd.edu/~ynemish/teaching/180c

## Today: Poisson process as a renewal process

> Q&A: November 4

## Next: PK 7.2-7.3, Durrett 3.1

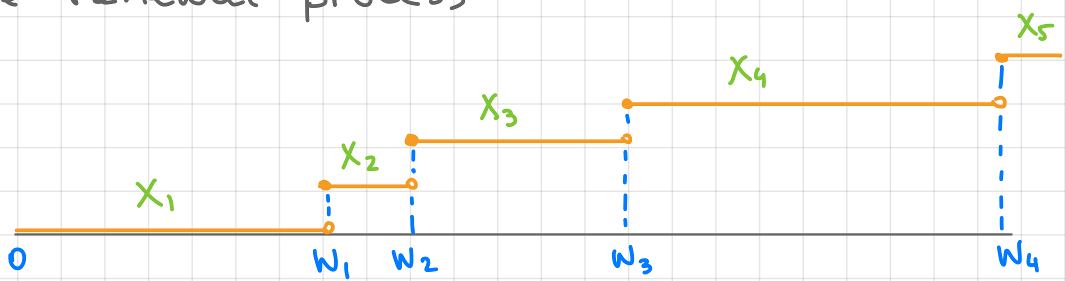
This week:

- Quiz 3 (November 4)
- Homework 4 (due Friday, November 6, 11:59 PM)

# Poisson process as a renewal process

The Poisson process $N(t)$ with rate $\lambda > 0$ is a renewal process with $F(x) = 1 - e^{-\lambda x}$.

- sojourn times $S_i$ are i.i.d., $S_i \sim Exp(\lambda)$
- $S_i$ represent intervals between two consecutive events (arrivals of customers)
- $W_n = \sum_{i=0}^{n-1} S_i$
- we can take $X_i = S_{i-1}$ in the definition of the renewal process



# Poisson process as a renewal process

We know that $N(t) \sim \mathrm{Pois}(\lambda t)$, so in particular

$$E(N(t)) = \lambda t$$

Example Compute $M(t) = \sum_{n=1}^{\infty} F^{*n}(t)$ for PP

$F_2(t) =$

Denote $\varphi_k(t) = \frac{(\lambda t)^k}{k!} e^{-\lambda t}$:

$$\varphi_k * F(t) =$$

$F * F(t) =$

$F^{*3}(t) =$

$\vdots$

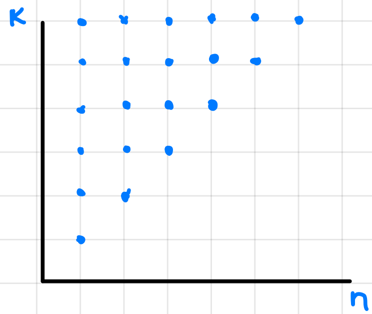
$F^{*n}(t) =$

# Poisson process as a renewal process (cont.)

$$\sum_{n=1}^{\infty} F^{*n}(t) =$$

$$=$$

$$=$$




$$M(t) =$$

## Renewal density

<u>Proposition</u> Let $N(t)$ be a renewal process with continuous interrenewal times $X_i$ having density $f(x)$. Denote

$m(t) = \sum_{n=1}^{\infty} f^{*n}(t)$. Then

and $(*)$

renewal density

<u>Proof</u>: $\frac{d}{dt} F^{*n}(t) =$ $\blacksquare$

<u>Example</u>: Compute the renewal density for PP using $(*)$.

$f(x) = \lambda e^{-\lambda x}$, so $(*)$ becomes

$m(t) =$

$=$

(cont.)

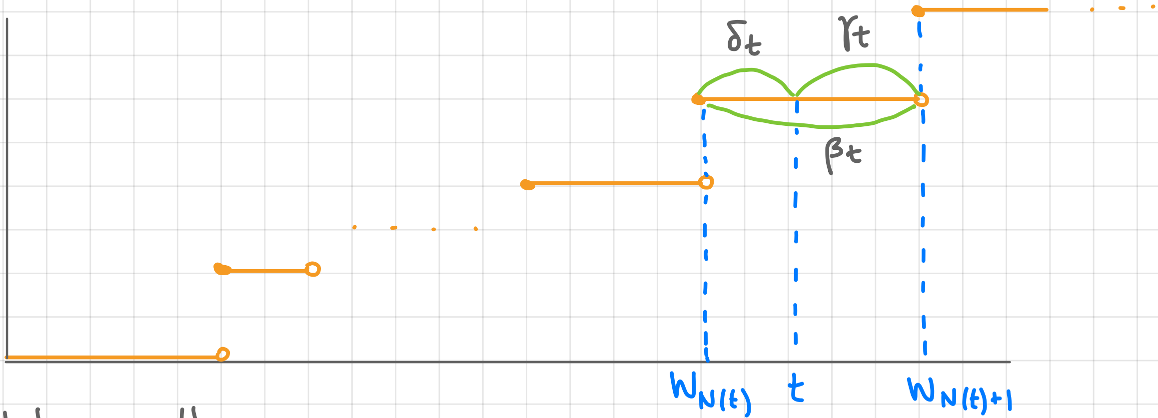
$e^{\lambda t} m(t) =$ $\leftarrow$ differentiate

$\Rightarrow$

Indeed,

# Excess life and current life of PP (summary)

Recall: Let $N(t)$ be a renewal process.




Def. We call

- $\gamma_t := W_{N(t)+1} - t$ the excess (or residual) lifetime
- $\delta_t := t - W_{N(t)}$ the current life (or age)
- $\beta_t := \gamma_t + \delta_t$ the total life

Remarks
1) $\gamma_t > h \geq 0$ iff $N(t+h) = N(t)$
2) $t \geq h$ and $\delta_t \geq h$ iff $N(t-h) = N(t)$

# Excess life and current life of PP

Let $N(t)$ be a PP. Then

- excess life

$P(\gamma_t > x) =$

- current life $\delta_t$

$$P(\delta_t > x) = \begin{cases} & \\ & \end{cases}$$

- total life $\beta_t = \gamma_t + \delta_t$

$E(\gamma_t + \delta_t) =$

$=$

# Excess life and current life of PP (cont.)

- Joint distribution of $(\gamma_t, \delta_t)$

$$P(\gamma_t > x, \delta_t > y) = \left\{ \vphantom{\begin{matrix} a \\ b \\ c \end{matrix}} \right.$$

$\Rightarrow$