MATH 142A: Introduction to Analysis

www.math.ucsd.edu/~ynemish/teaching/142a

Today: Derivative of the inverse. L'Hôpital's rule > Q&A: March 1

Next: Ross § 31

- Homework 8 (due Sunday, March 7)
- CAPE at www.cape.ucsd.edu

Derivative of the inverse

$$f: I \rightarrow J$$
, $f^{-1}: J \rightarrow I$, $\forall x \in I$ $f^{-1}\circ f(x) = x$, $\forall y \in J$ $f \circ f^{-1}(y) = y$
If $f \in D(I)$, $f \in D(J)$, then differentiating both sides gives
$$\forall x \in I \quad (f^{-1}\circ f)'(x) = I \quad \forall y \in J \quad (f \circ f^{-1})'(y) = I$$

 $\forall x \in I$ $(f^{-1} \circ f)'(x) = I$, $\forall y \in J$ $(f \circ f^{-1})'(y) = I$ By the chain rule $(f \circ f^{-1})'(y) = f'(f^{-1}(y))(f^{-1})'(y) = I$

$$\Rightarrow (f')'(y) = \frac{1}{f'(f'(y))}$$
If f' exists and f and f' are differentiable, then
$$(f')'$$
 is given by $(*)$.

(f')' is given by (*). Suppose $f: I \rightarrow J$, $f: J \rightarrow I$ exists and f is differentiable at $x \in I$. Does this imply that f is differentiable at $y = f(x \circ)$?

Derivative of the inverse

Thm. 29.9. Let f: I - J be one-to-one and continuous on I.

Fix E>0.

$$(x_{\circ}) \neq 0$$

Proof

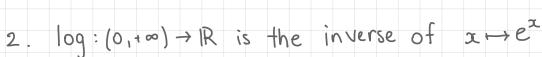
Examples

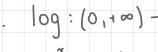
1. arcsin = sin ,

Sin:
$$(-\frac{\pi}{2}, \frac{\pi}{2}) \rightarrow (-1, 1)$$
 is a bijection (strictly increasing)

$$\forall x \in (-\frac{\pi}{2}, \frac{\pi}{2}) \quad \sin(x) =$$
Let $y \in (-1, 1)$ and let $x \in (-\frac{\pi}{2}, \frac{\pi}{2})$ s.t. $\sin x = y$







$$\log:(0,+\infty)\rightarrow$$

$$e^{x} \in D(IR), (e^{x})' = e^{x}, e^{x} > 0$$

$$\Rightarrow \forall y \in (0, +\infty) \log is$$
and $(\log y)' =$

$$\Rightarrow \forall y \in (0, +\infty) \text{ log is differentiable at } y$$

Examples

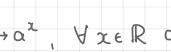
So

3.
$$f: \mathbb{R} \to (0, +\infty)$$
, $f(x) = \alpha^{x}$ $(\alpha > 0, \alpha \neq 1)$
 $f(x) = \Rightarrow \forall x \in \mathbb{R}$ $f(x) =$

$$f(x) = \Rightarrow$$

4.
$$\log_a : (0, +\infty) \to \mathbb{R}$$
 is the inverse of $x \mapsto a^2$, $\forall x \in \mathbb{R}$ $a^2 > 0$,

(logay) =



L'Hôpital's rule

Consider the limit $\lim_{x\to a} \frac{f(x)}{g(x)}$, $a \in \mathbb{R} \cup \{+\infty, -\infty\}$, $S \in \mathbb{R}$

• if
$$\lim_{S \ni x \to a} f(x) = F \in \mathbb{R}$$
, $\lim_{S \ni x \to a} g(x) = G \in \mathbb{R} \setminus \{0\}$, then
$$\lim_{S \ni x \to a} \frac{f(x)}{g(x)} = \frac{F}{G}$$

$$F=0$$
 and $G=0$ $\frac{0}{0}$

Generalized mean value theorem (Cauchy's Thm) $\frac{\text{Thm 30.1}}{f,g \in C([a,b])} \Rightarrow f,g \in D((a,b))$] x \((a, b) 5.t. Proof Consider h(z)= he C([a,b]) $h \in D((a,b))$ $h(\alpha) =$ h(b) =

If
$$q(b) \neq g(a)$$
, $g'(x) \neq 0$, then

L'Hôpital's Rule

Thm 30.2 Let a & R and s signify a, at, at, to or - ...

Suppose that f and g are differentiable (on appropriately Chosen intervals) and that $\lim_{x\to s} \frac{f'(z)}{g'(x)} = L$ exists.

Then if
$$(i) \lim_{x \to s} f(x) = \lim_{x \to s} g(x) = 0$$

$$x \to s \quad x \to s \quad \Rightarrow$$

 \Rightarrow

$$\frac{OR}{(ii) \lim_{x \to s} |g(x)| = \infty}$$

Proof of L'Hôpital's rule (1) Suppose $-\infty < L \le +\infty$. Take $L_1 < L$. $\lim_{x \to s} \frac{f'(x)}{g'(x)}$ exists \Rightarrow

Take KE (L1, L).
$$\lim_{x \to s} \frac{f'(z)}{g'(x)} = L > K \Rightarrow$$

By Cauchy's thm \(\([x,y] \) \((d,s) \) \(\)

$$(f(y)-f(x))g'(z)=(g(y)-g(x))f'(z) \Rightarrow$$

If (i) holds, take

$$\Rightarrow \forall [x,y]c(d,s) \frac{f(y)-f(z)}{g(y)-g(z)} \cdot \frac{g(y)-g(z)}{g(y)}$$

$$\Rightarrow \frac{f(y)}{g(y)} =$$

Take the limit (for any fixed
$$x \in (d_{1}, s)$$
)
$$\lim_{x \to \infty} \frac{f(x) - kg(x)}{g(x)} = \Rightarrow \exists d_{2} \in (d_{1}, s) \ s.t. \ \forall \ y \in (d_{2}, s)$$

$$\lim_{y \to s} \frac{f(x) - kg(x)}{g(y)} = \Rightarrow \exists \lambda_2 \in (\lambda_1, s)$$

(2) If - 00 & L 4+00, then

(3) Suppose
$$L \in \mathbb{R}$$
. Fix $\varepsilon > 0$. Take $L_1 = L - \varepsilon$, $L_2 = L + \varepsilon$
(A) \Rightarrow

Suppose L=+ . Fix M>0. Take L1= M.

$$(A) \Rightarrow$$







Suppose
$$L=-\infty$$
. Fix M>0. Take $L_2=-M\Rightarrow \lim_{x\to s}\frac{f(x)}{g(x)}=-\infty$



3.
$$\lim_{x\to 0} \frac{\sin x}{x} =$$