MATH180C: Introduction to Stochastic Processes II

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Today: Conditioning on continuous random variables

> Q&A: October 30

Next: PK 7.1, Durrett 3.1

This week:

- Homework 3 (due Saturday, October 31, 11:59 PM)

Conditioning on continuous r.v.

<u>Def.</u> Let X and Y be jointly distributed continuous random variables with joint probability density function $f_{X,Y}(x,y)$. We call the function

the conditional probability density function of X given $Y=y$.

The function

is called conditional distribution of X given $Y=y$

Conditional expectation

Def. Let X and Y be jointly distributed continuous random variables, let $f_{X|Y}(x|y)$ be a conditional distribution of X given $Y=y$ and let $g: \mathbb{R} \to \mathbb{R}$ be a function for which $E(|g(X)|) < \infty$.

Then we call

the conditional expectation of $g(X)$ given $Y=y$.

In particular, if

Remark

If Y is a continuous random variable, then

Therefore, we cannot define $P(X \in A \mid Y=y)$ as

$$P(X \in A \mid Y=y) =$$

On the other hand consider example:

Intuitive explanation / derivation

$$P(X \in [x, x+\Delta x], Y \in [y, y+\Delta y])$$

$$=$$

Using the multiplication rule ($f_Y(y) > 0$ on $[y, y+\Delta y]$)

$$P(X \in [x, x+\Delta x], Y \in [y, y+\Delta y])$$

$$=$$

$$P(X \in [x, x+\Delta x] \mid Y \in [y, y+\Delta y])$$

Properties of conditional probability/expectation

1) $P(a < X < b, c < Y < d) =$

2) $P(a < X < b) =$

3) $E(g(X)) =$

Further properties of conditional expectation (PK, p.50)

4) $E(c_1 g_1(X_1) + c_2 g_2(X_2) \mid Y=y) = c_1 E(g_1(X_1) \mid Y=y) + c_2 E(g_2(X_2) \mid Y=y)$

5) $E(\nu(X,Y) \mid Y=y) =$

In particular, $E(\nu(X,Y)) =$

6) $$E(g(X)h(Y)) = \int_{-\infty}^{+\infty} h(y) E(g(X) \mid Y=y) f_Y(y)\, dy$$

$$= E(h(Y) E(g(X) \mid Y))$$

7) $E(g(X) \mid Y=y) = E(g(X))$ if X and Y are independent

Example 1

Let (X,Y) be jointly continuous r.v.s with density

$$f_{X,Y}(x,y) = \frac{1}{y} e^{-\frac{x}{y} - y}, \quad x, y > 0$$

Compute the conditional density of X given $Y=y$.

1) Compute the marginal density of Y

2) Compute the conditional density

Example 1 (cont.)

Suppose that $Y \sim \text{Exp}(1)$, and X has exponential distribution with parameter $\frac{1}{Y}$. Compute $E(X)$

$$E(X) = \int_0^\infty \int_0^\infty x \frac{1}{y} e^{-\frac{x}{y} - y} \, dx \, dy$$

Example 2: continuous and discrete r.v.s

Let $N \in \mathbb{N}$, $P \sim \mathrm{Unif}[0,1]$, $X \sim \mathrm{Bin}(N, P)$

What is the distribution of X?

$$P(X=k) =$$

$$=$$

$$=$$

$$=$$

Example 3

Let X and Y be i.i.d. $\text{Exp}(\lambda)$ r.v.

Define $Z = \frac{X}{Y}$. Compute the density of Z.

- If $X \sim \text{Exp}(\lambda)$, then for $\alpha > 0$ $\alpha X \sim \text{Exp}\left(\frac{\lambda}{\alpha}\right)$

- .