MATH180C: Introduction to Stochastic Processes II

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Today: Martingales

> Q&A: November 25

Next: PK 8.1

This week:

- Thanksgiving
- Next homework deadline: December 2 (HW 7)

Martingales

Definition. A stochastic process $(X_n, n \geq 0)$ is a martingale if for $n = 0, 1, \ldots$

(a) $E(|X_n|) < \infty \quad \forall n$

(b) $E(X_{n+1} \mid X_0, \ldots, X_n) = X_n$

After taking the expectation of both sides of (b) we get that

$$E(X_{n+1}) = E(X_n)$$

$(X_n)_{n \geq 0}$ is a martingale $\Rightarrow E(X_n) = E(X_0) \quad \forall n$

- submartingale: $E(X_{n+1} \mid X_0, \ldots, X_n) \geq X_n$ (increases)
- supermartingale: $E(X_{n+1} \mid X_0, \ldots, X_n) \leq X_n$ (decreases)

Examples of martingales

(i) Let $X_1, X_2, \dots$ be independent RV's with $E(|X_k|) < \infty$ and $E(X_k) = 0$. Define $S_n = X_1 + \dots + X_n$, $S_0 = 0$.

Then
$$E(S_{n+1} \mid S_0, \dots, S_n) = E(S_n + X_{n+1} \mid S_0, \dots, S_n)$$
$$= E(S_n \mid S_0, \dots, S_n) + E(X_{n+1} \mid S_0, \dots, S_n)$$
$$= S_n + E(X_{n+1}) = S_n$$

$\Rightarrow$ $(S_n)_{n \geq 0}$ is a martingale with $E(S_0) = E(S_n) = 0$

(ii) Let $X_1, X_2, \dots$ be independent RV with $X_k \geq 0$, $E(|X_k|) < \infty$ and $E(X_k) = 1$. Define $M_n = X_1 X_2 \cdots X_n$, $M_0 = 1$.

Then
$$E(M_{n+1} \mid M_0, \dots, M_n) = E(M_n \cdot X_{n+1} \mid M_0, \dots, M_n)$$
$$= M_n E(X_{n+1} \mid M_0, \dots, M_n) = M_n \cdot E(X_{n+1}) = M_n$$

$\Rightarrow$ $(M_n)_{n \geq 0}$ is a martingale with $E(M_0) = E(M_n) = 1$

Example

Stock prices in a perfect market

Let X_n be the closing price at the end of day n of a certain publicly traded security such as a share or stock. Many scholars believe that in a perfect market these price sequences should be martingales. (see PK page 73 for more details).

History and gambling

Let $(X_n)_{n\geq 0}$ be a stochastic process describing your total winnings in n games with unit stake.

Think of $X_n - X_{n-1}$ as your net winnings per unit stake in game n, $n\geq 1$, in a series of games, played at times $n=1,2,\ldots$.

In the martingale case

$$E(X_n - X_{n-1} \mid X_0,\ldots,X_{n-1}) = E(X_n \mid X_0,\ldots,X_{n-1}) - E(X_{n-1} \mid X_0,\ldots,X_{n-1})$$

$$= X_{n-1} - X_{n-1} = 0 \quad \text{(fair game)}$$

Some early works of martingales was motivated by gambling. Note that there exists a betting strategy called the "martingale system" ← doubling bets after losses

Some basic properties

Let $(X_n)_{n\geq 0}$ be a martingale.

- $E(X_m \mid X_0, \dots, X_n) = X_n$

  Proof $\quad X_n = E(X_{n+1} \mid X_0, \dots, X_n)$

  $X_{n+1} = E(X_{n+2} \mid X_0, \dots, X_{n+1})$

$$X_n = E(X_{n+1} \mid X_0, \dots, X_n) = E(E(X_{n+2} \mid X_0, \dots, X_{n+1}) \mid X_0, \dots, X_n)$$
$$= E(X_{n+2} \mid X_0, \dots, X_n)$$

Exercise: $E(E(X \mid Y, Z) \mid Z) = E(X \mid Z)$ (show for disrete r.v.)

- Markov inequality: If $X_n \geq 0 \ \forall n$, then for any $\lambda > 0$

$$P(X_n \geq \lambda) \leq \frac{E(X_n)}{\lambda} = \frac{E(X_0)}{\lambda}$$

$\Rightarrow$ For all n $\quad P(X_n \geq \lambda) \leq \dfrac{E(X_0)}{\lambda} \quad \forall \lambda > 0$

Maximal inequality for nonnegative martingales

Thm. Let $(X_n)_{n\geq 0}$ be a martingale with nonnegative values. For any $\lambda>0$ and $m\in\mathbb{N}$

$$P\left(\max_{0\leq n\leq m} X_n \geq \lambda\right) \leq \frac{E(X_0)}{\lambda} \qquad (1)$$

and

$$P\left(\max_{n\geq 0} X_n \geq \lambda\right) \leq \frac{E(X_0)}{\lambda} \qquad (2)$$

Proof. We prove (1), (2) follows by taking the limit $m\to\infty$.
Take the vector $(X_0, X_1, \dots, X_m)$ and partition the sample space wrt the index of the first r.v. rising above λ

$$1 = \mathbb{1}_{X_0\geq\lambda} + \mathbb{1}_{X_0<\lambda, X_1\geq\lambda} + \dots + \mathbb{1}_{X_0<\lambda, \dots, X_{m-1}<\lambda, X_m\geq\lambda} + \mathbb{1}_{X_0<\lambda, \dots, X_m<\lambda}$$

Compute $E(X_m) = E(X_m \cdot 1)$ using the above partition

Proof of the maximal inequality

$$E(X_m) = \sum_{n=0}^{m} E\left(X_m \mathbb{1}_{X_0<\lambda,\dots,X_{n-1}<\lambda,\, X_n \geq \lambda}\right) + E\left(X_m \mathbb{1}_{X_1<\lambda,\dots,X_m<\lambda}\right)$$

$$\geq \sum_{n=0}^{m} E\left(X_m \mathbb{1}_{X_0<\lambda,\dots,X_{n-1}<\lambda,\, X_n \geq \lambda}\right)$$

Compute $E(X_m \mathbb{1}_{X_0<\lambda,\dots,X_{n-1}<\lambda,X_n\geq\lambda})$ by conditioning on $X_0, X_1, \dots, X_{n-1}, X_n$:

$$E(X_m \mathbb{1}_{X_0<\lambda,\dots,X_{n-1}<\lambda,X_n\geq\lambda})$$

$$= E\left(E\left(X_m \mathbb{1}_{X_0<\lambda,\dots,X_{n-1}<\lambda,X_n\geq\lambda} \mid X_0,\dots,X_n\right)\right)$$

$$= E\left(\mathbb{1}_{X_0<\lambda,\dots,X_{n-1}<\lambda,X_n\geq\lambda}\, E(X_m \mid X_0,\dots,X_n)\right)$$

$$= E\left(\mathbb{1}_{X_0<\lambda,\dots,X_{n-1}<\lambda,X_n\geq\lambda}\, X_n\right) \geq \lambda\, P(X_0<\lambda,\dots,X_{n-1}<\lambda,X_n\geq\lambda)$$

Sum for all n

$$E(X_m) \geq \lambda \sum_{n=0}^{m} P(X_0<\lambda,\dots,X_{n-1}<\lambda,X_n\geq\lambda) = \lambda \cdot P\left(\max_{0\leq n\leq m} X_n \geq \lambda\right) \qquad \blacksquare$$

Example

A gambler begins with a unit amount of money and faces a series of independent fair games. In each game the gamblers bets fraction p of his current fortune, wins with probability $\frac{1}{2}$, loses with probability $\frac{1}{2}$. Estimate the probability that the gambler ever doubles the initial fortune.

Denote by $Z_n, n \geq 0$, the gambler's fortune after n-th game.

Denote $\{Y_i\}_{i=1}^{\infty}$ i.i.d. r.v.s with $P(Y_i = 1+p) = P(Y_i = 1-p) = \frac{1}{2}$

Then $Z_n = Y_1 \cdot Y_2 \cdots Y_n$

$E(Y_i) = 1 \Rightarrow (Z_n)_{n \geq 0}$ is a nonnegative martingale

$$\Rightarrow P(\max_n Z_n \geq 2) \leq \frac{E(Z_0)}{2} = \frac{1}{2}$$

Martingale transform

In the previous example the stake in n-th game is $p Z_{n-1}$. What if we choose another strategy?

<u>Def</u> Let $(X_n)_{n\geq 0}$ be a nonnegative martingale, and let $(C_n)_{n\geq 0}$ be a stochastic process with $C_n = f_n(X_0, \dots, X_{n-1})$. Then the stochastic process

$$\sum_{k=1}^{n} C_k (X_k - X_{k-1}) =: (C \bullet X)_n , \quad (C \bullet X)_0 = 0$$

is called the martingale transform of X by C

Think of
- $X_k - X_{k-1}$ as the winning per unit stake in k-th game
- C_k as your stake in k-th game
  decision is made based on the previous history
- $(C \bullet X)_n$ as total winnings up to time n

Martingale transform

Prop. Let $Z_n = X_0 + (C \bullet X)_n$. Let $C_k > 0$ bounded if $Z_{k-1} > 0$ and $C_k = 0$ if $Z_{k-1} = 0$. Then $(Z_n)_{n \geq 0}$ is a martingale

Proof: $E(Z_{n+1} \mid Z_0, \dots, Z_n) = E(Z_n + C_{n+1}(X_{n+1} - X_n) \mid Z_0, \dots, Z_n)$

$$= Z_n + E(C_{n+1}(X_{n+1} - X_n) \mid Z_0, \dots, Z_n)$$

Note that $Z_n - Z_{n-1} = C_n(X_n - X_{n-1})$, $Z_0 = X_0$.

If $Z_n > 0$, then $C_1 > 0, \dots, C_n > 0$,

$$X_1 = (Z_1 - Z_0)C_1^{-1} + Z_0, \quad X_n = (Z_n - Z_{n-1})C_n^{-1} + X_{n-1}$$ and

$$E(Z_{n+1} \mid Z_0, \dots, Z_n) = Z_n + E(C_{n+1}(X_{n+1} - X_n) \mid X_0, \dots, X_n)$$

$$= Z_n + C_{n+1}(E(X_{n+1} \mid X_0, \dots, X_n) - X_n) = Z_n$$

If $Z_n = 0$, then $C_{n+1} = 0$ and $E(Z_{n+1} \mid Z_0, \dots, Z_n) = 0 = Z_n$

$\blacksquare$

Gambling example:

Start from the initial fortune $X_0 = 1$. Define

$$Z_n = 1 + (C \bullet X)_n \geq 0$$

fortune after n-th game with strategy C

Then $(Z_n)_{n \geq 0}$ is a nonnegative martingale, $E(Z_0) = 1$

$$\Rightarrow \quad P\left(\max_n Z_n \geq 2\right) \leq \frac{1}{2}$$

Convergence of nonnegative martingales

Thm.

If $(X_n)_{n\geq 0}$ is a nonnegative (super)martingale, then with probability 1

$$\exists \lim_{n\to\infty} X_n =: X_\infty$$

and

$$E(X_\infty) \leq E(X_0)$$

Example

An urn initially contains one red ball and one green ball. Choose a ball and return it to the urn together with another ball of the same color. Repeat. Denote by X_n the fraction of red ball after n iterations.

Example (cont.)

(i) $(X_n)_{n\geq 0}$ is a martingale

Denote by R_n the number of red balls after n-th iteration

$$R_n = X_n \cdot (n+2)$$

Then

$$E(X_{n+1} \mid X_0, \dots, X_n) = \frac{R_n+1}{n+3} X_n + \frac{R_n}{n+3}(1-X_n)$$

$$= \frac{1}{n+3}(X_n + R_n) = \frac{1}{n+3}(X_n + X_n(n+2)) = X_n$$

(ii) X_n is nonnegative $\Rightarrow$ $\exists \lim_{n\to\infty} X_n = X_\infty$

(iii) Compute the distribution of X_∞

$$P\left(X_n = \frac{k}{n+2}\right) = \frac{1}{n+1} \quad \text{for } k \in \{1, 2, \dots, n+1\}$$

$$P(X_\infty \leq x) = x, \quad x \in (0,1) \Rightarrow X_\infty \sim \mathrm{Unif}[0,1)$$