

# Math 180A: Introduction to Probability

Lecture A00 (Au)

[math.ucsd.edu/~bau/w21.180a](http://math.ucsd.edu/~bau/w21.180a)

Lecture B00 (Nemish)

[math.ucsd.edu/~ynemish/teaching/180a](http://math.ucsd.edu/~ynemish/teaching/180a)

## Today: ASV 3.3 (Expected value)

Video: Prof. Todd Kemp, Fall 2019

## Next: ASV 3.4, 3.5

Week 5: Quiz 3 (Wednesday, Feb 3, covering Lectures 8-10)

Homework 4 (due Sunday, Feb 7)

## Expectation

Definition: Let  $X$  be a discrete random variable with possible values  $t_1, t_2, t_3, \dots$ . The expectation or expected value of  $X$  is

$$E(X) := \sum_j t_j P(X=t_j)$$

It is also called the **mean** of  $X$ , and is often denoted  $\mu$ .

E.g. Let  $X$  be a discrete random variable with probability mass fn.

$$P_X(k) = \frac{c}{k(k+1)}, \quad k = 1, 2, 3, \dots$$

$$\sum_{k=1}^{\infty} p_X(k) = 1$$

$$\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$$

Find  $c$ , and compute  $E(X)$ .

$$\begin{aligned} 1 &= \sum_{k=1}^{\infty} \frac{c}{k(k+1)} = c \left( \sum_{k=1}^{\infty} \frac{1}{k(k+1)} \right) \\ &= c \left( \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) + \dots \right) = c \end{aligned}$$

$$E(X) = \sum_{k=1}^{\infty} k \cdot P_X(k) = \sum_{k=1}^{\infty} k \cdot \frac{1}{k(k+1)} = \infty.$$

# Expectation of Continuous Random Variable (with density)

X discrete,  $X \in \{t_1, t_2, t_3, \dots\}$

$$\mathbb{E}(X) = \sum_j t_j \underbrace{\mathbb{P}(X=t_j)}_{\sum_t t \cdot p_X(t)}$$

X continuous ( $\mathbb{P}(X=t)=0$   
for each  $t \in \mathbb{R}$ )  
probability density  $f_X(t)$

Def:  $\mathbb{E}(X) := \int_{-\infty}^{\infty} t f_X(t) dt$

E.g. Let  $U \sim \text{Unif}([a, b])$ .  $f_U(t) = \begin{cases} \frac{1}{b-a} & a \leq t \leq b \\ 0 & \text{otherwise} \end{cases} \leftarrow 0 \text{ except on } [a, b]$

$$\begin{aligned} \mathbb{E}(U) &= \int_{-\infty}^{\infty} t f_U(t) dt = \int_a^b t \cdot \frac{1}{b-a} dt = \frac{1}{b-a} \cdot \frac{1}{2} t^2 \Big|_{t=a}^{t=b} \\ &= \frac{1}{2(b-a)} (b^2 - a^2) = \frac{a+b}{2}. \end{aligned}$$

## Question:

Shoot an arrow at a circular target of radius 1. What is the expected distance of the arrow from the center?

- (a) 1
- (b)  $\frac{2}{3}$
- (c)  $\frac{1}{2}$
- (d)  $\frac{1}{4}$
- (e) 0

$X = \text{dist. from center}$

$$F_X(r) = \begin{cases} 0 & r \leq 0 \\ r^2 & 0 \leq r \leq 1 \\ 1 & r \geq 1 \end{cases}$$

$$f_X(r) = \begin{cases} 0 & r < 0 \\ 2r & 0 \leq r \leq 1 \\ 0 & r > 1 \end{cases}$$

$$\begin{aligned} E(X) &= \int_{-\infty}^{\infty} r f_X(r) dr \\ &= \int_0^1 r \cdot 2r dr = \frac{2}{3} r^3 \Big|_0^1 \\ &= \frac{2}{3} \end{aligned}$$

$$\left( \frac{\text{Area}(D_{\frac{2}{3}})}{\text{Area}(D_1)} = \frac{\pi(\frac{2}{3})^2}{\pi(1)^2} = \frac{4}{9} \right)$$

E.g.  $f(t) = \begin{cases} 0 & t \leq 1 \\ 1/t^2 & t > 1 \end{cases}$  is a probability density.

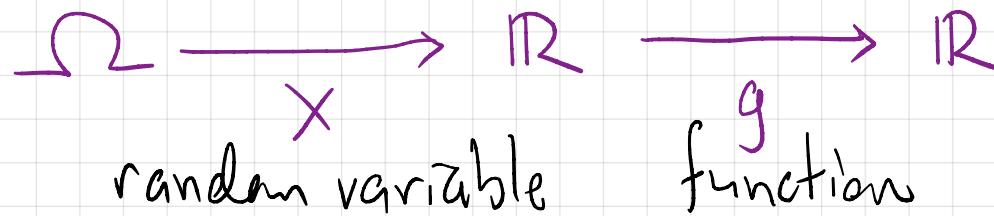
$\underbrace{\quad}_{\geq 0 \checkmark}$

$$\int_{-\infty}^{\infty} f(t) dt = \int_1^{\infty} \frac{1}{t^2} dt = -\frac{1}{t} \Big|_1^{\infty} = \left( \frac{-1}{\infty} - \left( -\frac{1}{1} \right) \right) = 1 \quad \checkmark$$

If  $X$  is a continuous random variable with  $f_X = f$ , what is  $E(X)$ ?

$$\begin{aligned} E(X) &= \int_{-\infty}^{\infty} t \cdot f(t) dt = \int_1^{\infty} t \cdot \frac{1}{t^2} dt = \int_1^{\infty} \frac{1}{t} dt \\ &= \ln|t| \Big|_1^{\infty} \\ &= \ln|\infty| - \ln|1| \\ &= \infty \end{aligned}$$

# Expectations of Functions of Random Variables



$g(X) = g \circ X$  is a new random variable.

E.g.  $X \sim \text{Bin}(n, p)$  is the number of successes in  $n$  trials

$g(X) = X/n$  is the proportion of successful trials.

$$g(X) \in \left\{ 0, \frac{1}{n}, \frac{2}{n}, \frac{3}{n}, \dots, \frac{n}{n} = 1 \right\}$$

$$\mathbb{E}(g(X)) = \sum_t t \cdot P(g(X)=t) \quad \begin{aligned} &= \sum_{k=0}^n \frac{k}{n} \cdot P\left(\frac{X}{n} = \frac{k}{n}\right) \\ &\quad \xrightarrow{\text{---}} \quad \xrightarrow{\text{---}} \\ &= \sum_{k=0}^n \frac{k}{n} \cdot P(g(X) = \frac{k}{n}) \\ &= \sum_{k=0}^n \frac{k}{n} \cdot P(X=k) \end{aligned}$$

In general: for a discrete random variable

Propositions:  $\mathbb{E}(g(X)) = \sum_s g(s) P(X=s)$ .

Note, by definition,  $\mathbb{E}(g(X)) = \sum_t t \cdot P(g(X)=t)$ .

Propositions: for a discrete random variable

$$\mathbb{E}(g(X)) = \sum_s g(s) P(X=s).$$

Pf. We know

$$\begin{aligned} & \sum_t t \cdot P(g(X)=t) \\ &= \sum_t t \cdot \sum_{\substack{s: \\ g(s)=t}} P(X=s) \\ &= \sum_t \sum_{\substack{s: \\ g(s)=t}} t \cdot P(X=s) \\ &= \underbrace{\sum_t}_{s} \sum_{\substack{s: \\ g(s)=t}} g(s) P(X=s) \end{aligned}$$

$\{g(X)=t\} = \bigcup_{\substack{s: \\ g(s)=t}} \{X=s\}$   
(disjoint)



Proposition: For a continuous random variable  $X$  with probability density  $f_X$ ,

$$\mathbb{E}(g(X)) = \int_{-\infty}^{\infty} g(t) f_X(t) dt.$$

Eg. Let  $U$  be a uniform random variable on  $[a, b]$ . Then

$$\begin{aligned}\mathbb{E}(U^2) &= \int_a^b t^2 f_U(t) dt \leftarrow \int_a^b \frac{1}{b-a} t^2 dt = \frac{1}{b-a} \cdot \frac{1}{3} t^3 \Big|_a^b \\ f_U(t) &= \begin{cases} \frac{1}{b-a} & a \leq t \leq b \\ 0 & \text{otherwise} \end{cases} \\ &= \frac{b^3 - a^3}{3(b-a)} \\ &= \frac{a^2 + ab + b^2}{3}.\end{aligned}$$

$$(a=0, b=1), \quad \mathbb{E}(U_{[0,1]}^2) = \frac{1}{3}.$$

E.g. Recall the car accident /insurance example.

An accident causes  $\$Y$  of damage to your car, where

Your insurance deductible is  $\$500$ .

What is the expected amount you pay?

$$Y \sim \text{Unif}([100, 1500]).$$

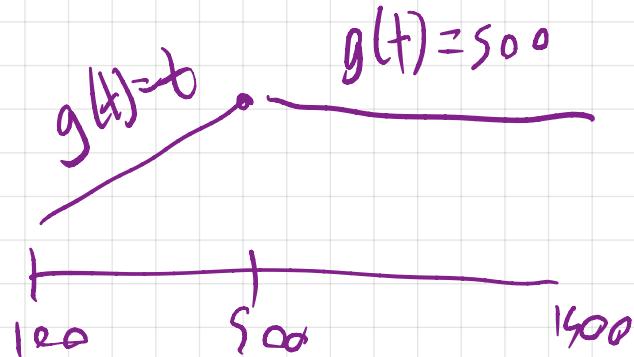
$X$  = amount you pay

(neither continuous  
nor discrete r.v.)

$$= \min(Y, 500)$$

$$= g(Y)$$

$$E(X) = E(g(Y)) = \int_{100}^{1500} \min(t, 500) \frac{1}{1400} dt$$



$$\begin{aligned} &= \int_{100}^{500} t \cdot \frac{1}{1400} dt + \int_{500}^{1500} 500 \cdot \frac{1}{1400} dt \\ &= \frac{3100}{7} = 442.86. \end{aligned}$$