☐ **Write your name and PID on the top of EVERY PAGE.**

☐ **Write the solutions to each problem on separate pages. CLEARLY INDICATE on the top of each page the number of the corresponding problem. Different parts of the same problem can be written on the same page (for example, part (a) and part (b))**

☐ **Remember this exam is graded by a human being. Write your solutions NEATLY AND COHERENTLY, or they risk not receiving full credit.**

☐ **All steps of the proofs should be INCLUDED in your solutions. Provide references to the theorem/examples from the lectures/textbook used in your proofs.**

☐ **You are allowed to use the textbook, lecture notes and your personal notes. You are not allowed to use the electronic devices (except for accessing the online version of the textbook) or outside assistance. Outside assistance includes but is not limited to other people, the internet and unauthorized notes.**

1. (25 points) Let $(a_n)$ be a sequence of real numbers satisfying

$$|a_{n+1} - a_n| \le \frac{1}{n(n+1)}$$

for all $n \in \mathbb{N}$. Prove that $(a_n)$ is a Cauchy sequence.

For any $m, n \in \mathbb{N}$, $n > m$, using the triangle inequality gives

$$|a_n - a_m| \le |a_{m+1} - a_m| + \cdots + |a_n - a_{n-1}| \le \frac{1}{m} - \frac{1}{m+1} + \frac{1}{m+1} - \frac{1}{m+2} + \cdots + \frac{1}{n-1} - \frac{1}{n} < \frac{1}{m}.$$

Fix $\varepsilon > 0$. Then for any $m, n \in \mathbb{N}$, $n > m$, $m > \lfloor \frac{1}{\varepsilon} \rfloor \ge \frac{1}{\varepsilon}$

$$|a_n - a_m| < \frac{1}{m} \le \varepsilon,$$

which means that $(a_n)$ is a Cauchy sequence.

2. (25 points) Let $(x_n)$ be a sequence of real numbers given by

$$x_n = \sqrt[n]{1 + 2^{n(-1)^n}}$$

for $n \in \mathbb{N}$. Determine the set of the subsequential limits of $(x_n)$, $\limsup x_n$ and $\liminf x_n$.

Notice that for any $n \in \mathbb{N}$

$$1 + 2^{2n(-1)^{2n}} = 1 + 2^{2n}, \qquad 1 + 2^{(2n+1)(-1)^{2n+1}} = 1 + \frac{1}{2^{2n+1}}.$$

Therefore,

$$\lim_{n\to\infty} x_{2n} = \lim_{n\to\infty} \sqrt[2n]{1 + 2^{2n}} = \lim_{n\to\infty} 2 \sqrt[2n]{\frac{1}{2^{2n}} + 1} = 4,$$

and

$$\lim_{n\to\infty} x_{2n+1} = \lim_{n\to\infty} \sqrt[2n+1]{1 + \frac{1}{2^{2n+1}}} = 1.$$

Therefore the set of the subsequential limits of $(x_n)$ contains the set $1, 2$.

Let $(x_{n_k}) \subset (x_n)$ be a subsequence. If $(x_{n_k})$ has only finitely many even terms, then $\lim_{k\to\infty} x_{n_k} = 1$. If $(x_{n_k})$ has only finitely many odd terms, then $\lim_{k\to\infty} x_{n_k} = 2$. If $(x_{n_k})$ has infinitely many even terms and infinitely many odd terms, then it has a subsequence converging to 1 and a subsequence converging to 2, and therefore by Theorem 11.8 $(x_{n_k})$ is not convergent. We conclude that the set of the subsequential limits of $(x_n)$ is equal to $\{1, 2\}$.

By Theorem 11.8, $\limsup x_n = \sup\{1, 2\} = 2$, $\liminf x_n = \inf\{1, 2\} = 1$.

3. (25 points) Determine if the series

$$\sum_{n=1}^{\infty} \frac{3^n (n!)^2}{(2n)!}$$

converges.

Use the Ratio test:

$$\lim_{n\to\infty} \frac{3^{n+1}((n+1)!)^2}{(2n+2)!} \cdot \frac{(2n)!}{3^n (n!)^2} = \lim_{n\to\infty} \frac{3(n+1)^2}{(2n+1)(2n+2)} = \frac{3}{4} < 1.$$

Therefore, by the Raio test $\sum_{n=1}^{\infty} \frac{3^n (n!)^2}{(2n)!} < \infty$

4. (25 points) Use the Cantor-Heine theorem to prove that the function

$$f(x) = \frac{x}{4 - x^2}$$

is uniformly continuous on the interval $[-1, 1]$.

Function $x \mapsto 4 - x^2$ is continuous and strictly positive on the interval $[-1, 1]$. Therefore, by Theorem 17.4 $f$ is continuous on $[-1, 1]$. By the Cantor-Heine theorem, the function that is continuous on a closed interval is also uniformly continous on that interval. We conclude that $f$ is uniformly continous on $[-1, 1]$.