

# MATH 142A: Introduction to Analysis

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Today: Sequences and their limits  
> Q&A: January 15

Next: Ross § 9

Week 2:

- Quiz 1 (Wednesday, January 13) - Lectures 1-2
- homework 1 (due Friday, January 15)

## Symbols $+\infty$ and $-\infty$

Often it is convenient to work with  $\mathbb{R} \cup \{+\infty, -\infty\}$

Extend  $\leq$  to this set using rules:

- $\forall x \in \mathbb{R} \quad -\infty \leq x \leq +\infty$
- $-\infty \leq +\infty, \quad -\infty \leq -\infty, \quad +\infty \leq +\infty$

Use  $\pm\infty$  to denote unbounded intervals

$$[a, +\infty) := \{x \in \mathbb{R} : x \geq a\}, \quad (a, +\infty) := \{x \in \mathbb{R} : x > a\}$$

$$(-\infty, b] := \{x \in \mathbb{R} : x \leq b\}, \quad (-\infty, b) := \{x \in \mathbb{R} : x < b\}$$

$$(-\infty, +\infty) := \mathbb{R}$$

We define

- $\sup S = +\infty$  if  $S$  is not bounded above
- $\inf S = -\infty$  if  $S$  is not bounded below

$$\sup \mathbb{N} = +\infty, \quad \inf \mathbb{Z} = -\infty$$

## Sequences

Function, mapping: Let  $X$  and  $Y$  be two sets. We say that there is a function defined on  $X$  with values in  $Y$ , if via some rule  $f$  we associate to each element  $x \in X$  an (one) element  $y \in Y$ . We write  $f: X \rightarrow Y$ ,  $x \mapsto y$  ( $y = f(x)$ )

$X$  is called the domain of definition of the function,  $y = f(x)$  is called the image of  $x$ . E.g.:  $f: [0,1] \rightarrow [0,1]$ ,  $x \mapsto x^2$

Def (Sequence) A function  $f: \mathbb{N} \rightarrow X \subset \mathbb{R}$ , whose domain of definition is the set of natural numbers, is called a **sequence**.

$$f(1), f(2), f(3), f(4), \dots$$

$$f(m), f(m+1), f(m+2), \dots$$

$$\overset{''}{f(m-1+1)}, \overset{''}{f(m-1+2)}, \overset{''}{f(m-1+3)}, \dots$$

$$g(n) := f(m-1+n)$$

$$\text{Notation: } (s_1, s_2, s_3, s_4, \dots) = (s_n)_{n=1}^{\infty}.$$

$$(s_n)_{n \geq 1}, (s_n)_{n \in \mathbb{N}}, \{s_1, s_2, s_3, \dots\}, \{s_n\}_{n=1}^{\infty}$$

$$(s_m, s_{m+1}, s_{m+2}, \dots) = (s_n)_{n=m}^{\infty} = (s_n)_{n \geq m}$$

## Examples of sequences

- $(a_n)_{n=1}^{\infty}$ ,  $a_n = 0$   $(a_n)_{n=1}^{\infty} = (0, 0, 0, 0, \dots)$
- $(a_n)_{n=1}^{\infty}$ ,  $a_n = n$   $(a_n)_{n=1}^{\infty} = (1, 2, 3, 4, \dots)$
- $(a_n)_{n=1}^{\infty}$ ,  $a_n = \frac{1}{n}$   $(a_n)_{n=1}^{\infty} = (1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots)$
- $b_n = \frac{1}{2^n}$ ,  $n \in \{0, 1, 2, 3, \dots\}$   $(b_n)_{n=0}^{\infty} = (1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots)$
- $\sin(\frac{n\pi}{2})$ ,  $n \in \mathbb{N}$   $(1, 0, -1, 0, 1, 0, \dots)$
- $b_n = (1 + \frac{1}{n})^n$ ,  $n \in \mathbb{N}$   $(b_n)_{n=1}^{\infty} = (2, 2.25, 2.3704, 2.4414, 2.5216, \dots)$
- $a_n = n^2 \cdot \sin(\frac{1}{n^2})$   $(a_n)_{n=1}^{\infty} = (0.84, 0.98, 0.997, 0.9993, 0.9997, \dots)$

## Convergence

Def 7.1. A sequence  $(s_n)$  of real numbers is said to **converge** to the real number  $s$  if

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} \quad \forall n > N \quad (|s_n - s| < \varepsilon)$$

Notation:  $\lim_{n \rightarrow \infty} s_n = s$  or  $s_n \rightarrow s, n \rightarrow \infty$

Def A sequence that does not converge is said to **diverge**.  
**convergent / divergent**

Remark  $N \in \mathbb{N}$  in the definition depends on  $\varepsilon$ .

$$s_n = \frac{n+1}{n}, n \in \mathbb{N}, \lim_{n \rightarrow \infty} s_n = 1, |s_n - n| = \left| \frac{n+1}{n} - 1 \right| = \frac{1}{n}$$

$$|s_n - s| < 1 \text{ for all } n > 1$$

$$|s_n - s| < 0.1 \text{ for all } n > 10$$

$$|s_n - s| < 0.01 \text{ for all } n > 100 \quad (\text{also for all } n > 101, n > 1000)$$

## Examples of sequences

- $(a_n)_{n=1}^{\infty}$ ,  $a_n = 0$   $(a_n)_{n=1}^{\infty} = (0, 0, 0, 0, \dots)$   $\lim_{n \rightarrow \infty} a_n = 0$
- $(a_n)_{n=1}^{\infty}$ ,  $a_n = n$   $(a_n)_{n=1}^{\infty} = (1, 2, 3, 4, \dots)$  diverges
- $(a_n)_{n=1}^{\infty}$ ,  $a_n = \frac{1}{n}$   $(a_n)_{n=1}^{\infty} = (1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots)$   $\lim_{n \rightarrow \infty} a_n = 0$
- $b_n = \frac{1}{2^n}$ ,  $n \in \{0, 1, 2, \dots\}$   $(b_n)_{n=0}^{\infty} = (1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots)$   $\lim_{n \rightarrow \infty} b_n = 0$
- $\sin(\frac{n\pi}{2})$ ,  $n \in \mathbb{N}$   $(1, 0, -1, 0, 1, \dots)$  diverges
- $b_n = \left(1 + \frac{1}{n}\right)^n$ ,  $n \in \mathbb{N}$   $(b_n)_{n=1}^{\infty} = (2, 2.25, 2.3704, 2.4414, 2.5216, \dots)$   
 $\lim_{n \rightarrow \infty} b_n = e$
- $a_n = n^2 \sin(\frac{1}{n^2})$ ,  $n \in \mathbb{N}$   $(a_n)_{n=1}^{\infty} = (0.84, 0.98, 0.997, 0.9993, 0.9997, \dots)$   
 $\lim_{n \rightarrow \infty} n^2 \sin(\frac{1}{n^2}) = 1$

## Uniqueness of limit

Prop. Let  $(s_n)_{n=1}^{\infty}$  be a convergent sequence. Then

$$\lim_{n \rightarrow \infty} s_n = s \wedge \lim_{n \rightarrow \infty} s_n = t \Rightarrow s = t$$

Proof. Fix  $\varepsilon > 0$ . Then

$$\textcircled{1} \quad \lim_{n \rightarrow \infty} s_n = s \Rightarrow \exists N_1 \quad \forall n > N_1 \quad |s_n - s| < \frac{\varepsilon}{2}$$

$$\textcircled{2} \quad \lim_{n \rightarrow \infty} s_n = t \Rightarrow \exists N_2 \quad \forall n > N_2 \quad |s_n - t| < \frac{\varepsilon}{2}$$

$$\textcircled{3} \quad \textcircled{1} \text{ and } \textcircled{2} \Rightarrow \forall n > \max\{N_1, N_2\}$$

Tr. Ineq.

$$|s - t| \leq |s_n - s| + |s_n - t| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$



$$\Rightarrow |s - t| < \varepsilon \text{ for any } \varepsilon > 0 \Rightarrow |s - t| = 0 \Rightarrow s = t$$



## Example

Let  $p \in \mathbb{Z}$ . Then

$$\lim_{n \rightarrow \infty} n^p = \begin{cases} 0, & p < 0 \\ 1, & p = 0 \\ \text{diverges}, & p > 0 \end{cases}$$

(a)      (b)      (c)

Proof (b)  $n^0 = 1 \Rightarrow \forall \varepsilon > 0 \ \forall n \in \mathbb{N} \quad |n^0 - 1| = 0 < \varepsilon$ . (take  $N=1$ )

(c) Suppose  $\exists s \in \mathbb{R}$  s.t.  $\lim_{n \rightarrow \infty} n^p = s$ . Then (take  $\varepsilon = 1$ )

$$\exists N \in \mathbb{N} \ \forall n > N \quad |n^p - s| < 1$$

$$\Rightarrow \forall n > N \quad n^p < 1+s \Rightarrow \forall n > N \quad n < \sqrt[p]{1+s}$$

(contradiction AP)

$$\Rightarrow n^p \text{ is divergent}$$

(a) Fix  $\varepsilon > 0$ , denote  $q = -p \in \mathbb{N}$ . {find  $N$  s.t.  $\forall n > N \quad \frac{1}{n^q} < \varepsilon \Leftrightarrow n^q > \frac{1}{\varepsilon}$ }

Take  $N = \left[ \sqrt[q]{\frac{1}{\varepsilon}} \right]$ . Then for  $n > N$

$$n^q > \frac{1}{\varepsilon} \Leftrightarrow \frac{1}{n^q} < \varepsilon \Leftrightarrow n^p < \varepsilon \Leftrightarrow |n^p - 0| < \varepsilon$$

## Example

$$\lim_{n \rightarrow \infty} \frac{5n^4 - n - 10}{7n^4 - n^2} = \frac{5}{7}$$

$$\frac{5n^4 - n - 10}{7n^4 - n^2} = \frac{n^4(5 - \frac{1}{n^3} - \frac{10}{n^4})}{n^4(7 - \frac{1}{n^2})}$$

$$\left| \frac{5n^4 - n - 10}{7n^4 - n^2} - \frac{5}{7} \right| = \left| \frac{35n^6 - 7n^5 - 70n^4 - 35n^4 - 5n^2}{49n^4 - 7n^2} \right| = \left| \frac{-5n^2 - 7n - 70}{49n^4 - 7n^2} \right| < \varepsilon$$

$$\forall n > 10 \quad 5n^2 + 7n + 70 < 7n^2 \Rightarrow \left| \frac{-5n^2 - 7n - 70}{7n^2(7n^2 - 1)} \right| < \frac{7n^2}{7n^2(7n^2 - 1)} = \frac{1}{7n^2 - 1} < \frac{1}{n^2} < \varepsilon$$

$$N = \max\{10, \lceil \frac{1}{\varepsilon} \rceil\}$$

Proof Fix  $\varepsilon > 0$ . Then  $\forall n > \max\{10, \lceil \frac{1}{\varepsilon} \rceil\} = N$

$$\left| \frac{5n^4 - n - 10}{7n^4 - n^2} - \frac{5}{7} \right| = \left| \frac{-5n^2 - 7n - 10}{7n^2(7n^2 - 1)} \right| < \frac{7n^2}{7n^2(7n^2 - 1)} < \frac{1}{6n^2} < \frac{1}{n^2},$$

$$\text{and } n > \lceil \frac{1}{\varepsilon} \rceil \Rightarrow n > \frac{1}{\varepsilon} \Rightarrow n^2 > \frac{1}{\varepsilon} \Rightarrow \frac{1}{n^2} < \varepsilon$$