MATH180C: Introduction to Stochastic Processes II

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Today: Brownian Motion

> Q&A: November 30

Next: PK 8.1-8.2

This week:

- Homework 7 (due THURSDAY, December 3)
- HW6 regrades (until Wednesday, December 2, 11 PM)

Brownian motion. History

- Critical observation: Robert Brown (1827), botanist, movement of pollen grains in water
- First (?) mathematical analysis of Brownian motion: Louis Bachelier (1900), modeling stock market fluctuations
- Brownian motion in physics: Albert Einstein (1905) and Marian Smoluchowski (1906), explained the phenomenon observed by Brown
- First rigorous construction of mathematical Brownian motion: Norbert Wiener (1923)

Brownian motion = Wiener process
in mathematics

Brownian motion. Motivation

- almost all interesting classes of stochastic processes contain Brownian motion: BM is a
  - martingale
  - Markov process
  - Gaussian process
  - Lévy process (independent stationary increments)
- BM allows explicit calculations, which are impossible for more general objects
- BM can be used as a building block for other processes
- BM has many beautiful mathematical properties

Brownian motion. Definition

Def. **Brownian motion** with diffusion coefficient σ^2 is a continuous time stochastic process $(B_t)_{t\geq 0}$ satisfying

(i)

(ii)

(iii)

$\sigma^2 = 1 \leftarrow$ standard BM

BM as a continuous time continuous space Markov process

Recall: continuous time discrete space MC $(X_t)_{t\geq 0}$ is characterized by the transition probability function

$$P_{ij}(t) =$$

($(X_t)_{t\geq 0}$ has stationary transition probability functions)

In particular, $P(X_{s+t} \in A \mid X_s = i) =$

In the continuous state space case the transition probabilities are described by the transition density

(i)

(ii) $P(X_{s+t} \in A \mid X_s = x) =$ for any $x \in \mathbb{R}$, $A \subset \mathbb{R}$

density of X_{s+t} given $X_s = x$

BM as a continuous time continuous space Markov process

<u>Propotition.</u> Let $(B_t)_{t\geq 0}$ be a standard BM.
Then $(B_t)_{t\geq 0}$ is a with transition density

<u>Informal explanation</u>: Independent stationary increments imply that $(B_t)_{t\geq 0}$ is Markov with stationary transition density. Given $B_s = x$, information before time s is irrelevant.

$$P(B_{s+t} \leq u \mid B_s = x) =$$

BM as a continuous time continuous space Markov process

Let $t_1 < t_2 < \cdots < t_n < \infty$, $(a_i, b_i) \subset \mathbb{R}$. Then

$P(B_{t_1} \in (a_1, b_1), B_{t_2} \in (a_2, b_2)) =$

$=$

$=$

$=$

More generally,

$$P(B_{t_1} \in (a_1, b_1), B_{t_2} \in (a_2, b_2), \dots, B_{t_n} \in (a_n, b_n))$$

$$= \int \cdots \int_{(a_1, b_1) \times \cdots \times (a_n, b_n)} p_{t_1}(0, x_1)\, p_{t_2 - t_1}(x_1, x_2) \cdots p_{t_n - t_{n-1}}(x_{n-1}, x_n)\, dx_1 \cdots dx_n$$

Diffusion equation. Transition semigroup. Generator

Let $(X_t)_{t\geq 0}$ be a Markov process,

Suppose we want to know how the distribution of X_t evolves in time:

We call $(P_t)_{t\geq 0}$ the transition semigroup $\left[P_{s+t} f(x) \overset{\text{CK}}{=} P_s(P_t f(x)) \right]$

<u>Proposition</u> Let $(P_t)_{t\geq 0}$ be the transition semigroup of BM.

Then (i) the "infinitesimal generator" of $P(t)$ is given by

(ii) density p_t satisfies [K backward]

(iii) density p_t satisfies [K forward]

↑ diffusion equation