MATH180C: Introduction to Stochastic Processes II

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Today: Processes generated by BM

> Q&A: December 7,9

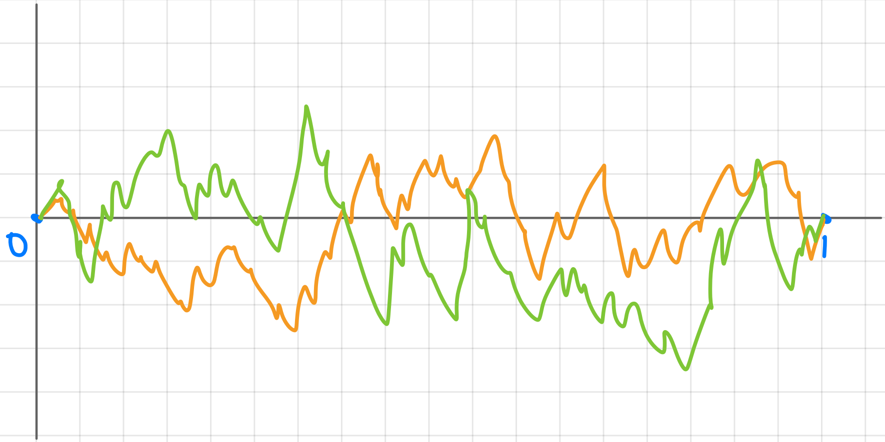
Next: Review

This week:

- Homework 8 (due THURSDAY, December 10)
- Homework 9 (do not submit, practice problems)
- Quiz 5 on Wednesday, December 9 (lectures 18-20)

Brownian bridge

Brownian bridge is constructed from a BM by conditioning on the event $\{B(0)=0, B(1)=0\}$.



Thm 1. Brownian bridge is a continuous Gaussian process on $[0,1]$ with mean 0 and covariance function

$\Gamma(s,t) =$

Conditioned multivariate normal distribution

<u>Lemma</u> Let (X, Y) be a random vector with multivariate normal distribution $N(0, \Sigma)$ with $\Sigma = \begin{pmatrix} \sigma_x^2 & \sigma_{xy} \\ \sigma_{xy} & 1 \end{pmatrix}$.

Then $$f_{X|Y}(x|0) =$$

i.e., $X|Y=0$ is Gaussian with mean 0 and variance $\sigma_x^2 - \sigma_{xy}^2$.

<u>Proof</u>. By definition of the joint normal distribution,

$$f_{X,Y}(x,0) =$$

Then $f_{X|Y}(x|0) = \dfrac{f_{X,Y}(x,0)}{f_Y(0)} =$

Now $(x, 0)\Sigma^{-1}\begin{pmatrix} x \\ 0 \end{pmatrix} =$, $\det \Sigma =$, therefore

$$f_{X|Y}(x|0) =$$

$\blacksquare$

Proof of Theorem 1 (1)

Let $(B_t)_{t\geq 0}$ be a standard BM. Denote by $(B^\circ_t)_{t\in[0,1]}$ the part of B on $[0,1]$ conditioned on the event $B_1 = 0$.

1) B° is continuous on $[0,1]$

2) In order to show that B° is Gaussian, we need to show that $\forall\, \alpha_i \in \mathbb{R}$ and $0 \leq t_1 < t_2 < \cdots < t_n \leq 1$

$$\sum_{i=1}^{n} \alpha_i B^\circ_{t_i} \text{ is Gaussian}$$

$$\Longleftrightarrow \qquad \text{is Gaussian}$$

B is Gaussian $\Rightarrow \forall\, \beta_1, \beta_2 \in \mathbb{R}$ is Gaussian

$$\Rightarrow \left(\sum_{i=1}^{n} \alpha_i B_{t_i}, B_1 \right) \text{ are jointly normal}$$

$$\overset{\text{Lemma}}{\Longrightarrow} \sum_{i=1}^{n} \alpha_i B_{t_i} \mid B_1 = 0 \text{ is Gaussian}$$

Proof of Theorem 1 (2)

3) From Lemma we also know that $E(B_t^{\circ})=0$.

To compute the covariance function, note that for $0<s<t<1$

$$f_{B_s,B_t,B_1}(x,y,0)=(2\pi)^{-3/2}(\det\Sigma)^{-1/2}\exp\left(-\frac{1}{2}(x,y,0)\Sigma^{-1}\begin{pmatrix}x\\y\\0\end{pmatrix}\right),$$

where $\Sigma=$. Also note that $f_{B_1}(0)=$.

If $\Sigma^{-1}=\begin{pmatrix} & & * \\ & & * \\ * & * & * \end{pmatrix}$, then $(x,y,0)\Sigma^{-1}\begin{pmatrix}x\\y\\0\end{pmatrix}=$

Now, $\det\Sigma=$ and $T=\frac{1}{s(t-s)(1-t)}\begin{pmatrix} t(1-t) & -s(1-t) \\ -s(1-t) & s(1-s)\end{pmatrix}$

$$\det T=\frac{1}{s(t-s)(1-t)} \quad \text{and} \quad T^{-1}=\begin{pmatrix} s(1-s) & s(1-t) \\ s(1-t) & t(1-t)\end{pmatrix}=:\tilde{\Sigma}$$

Finally, $\dfrac{f_{B_s,B_t,B_1}(x,y,0)}{f_{B_1}(0)}=$

$\Rightarrow \mathrm{Cov}(B_s^{\circ},B_t^{\circ})=$

Brownian bridge. Remark

Remark. Let $(B_t)_{t\geq 0}$ be a BM. Then the process

$$(X_t)_{t\in[0,1]}, \quad X_t = \qquad\qquad \text{for } t\in[0,1]$$

is a Brownian bridge.

Indeed: 1) $(X_t)_{t\in[0,1]}$ is continuous, $X_0 = 0$

2) $(X_t)_{t\in[0,1]}$ is Gaussian: $\forall\, \alpha_i,\ 0\leq t_1 < \dots < t_n \leq 1$

$$\sum_{i=1}^{n} \alpha_i X_{t_i} =$$

which is Gaussian since $(B_t)_{t\geq 0}$ is a Gaussian process

3) $\mathrm{Cov}(X_s, X_t) = \mathrm{Cov}(B_s - sB_1, B_t - tB_1) =$

$=$

Brownian motion with drift

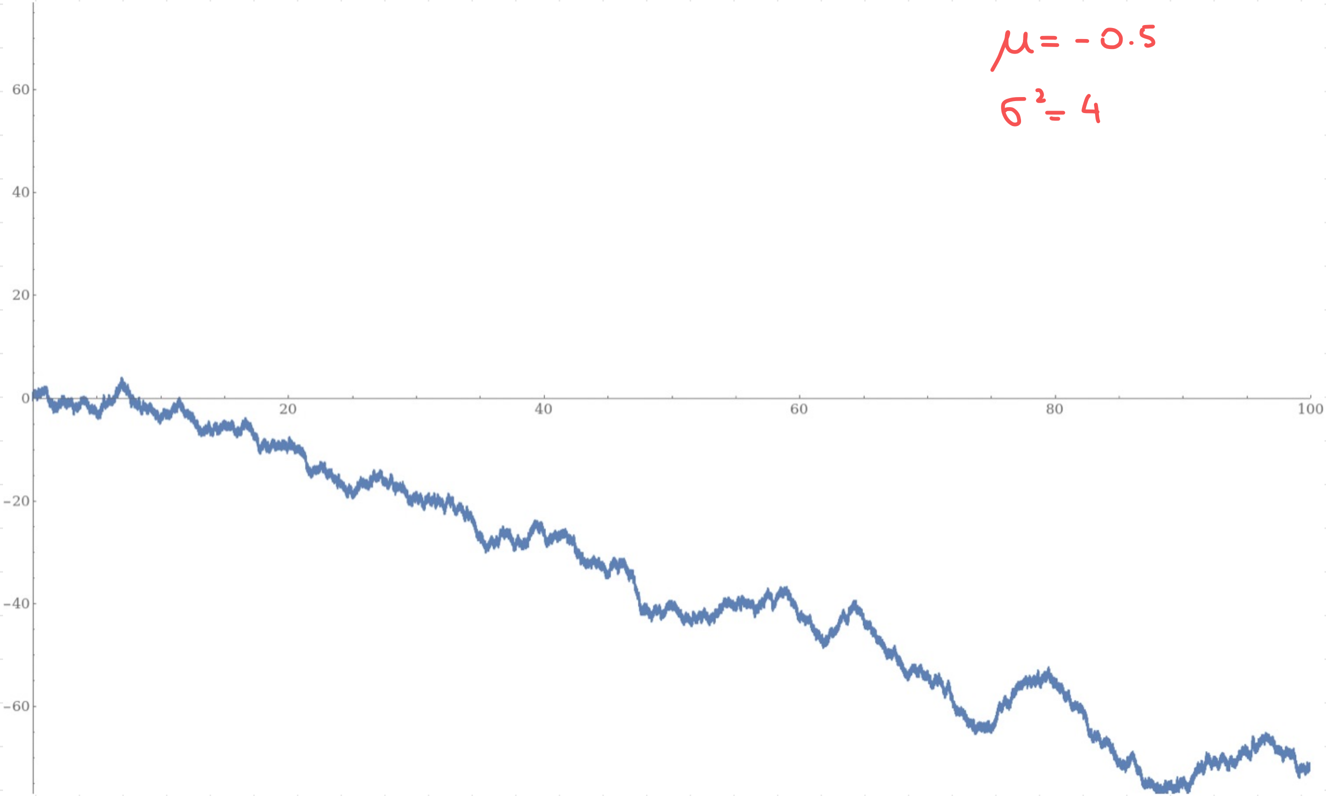
Def Let $(B_t)_{t\geq 0}$ be a standard BM. Then for $\mu \in \mathbb{R}$ and $\sigma > 0$ the process $(X_t)_{t\geq 0}$ with $X_t =$, $t \geq 0$ is called the Brownian motion with drift μ and variance paremeter σ^2.

Remark BM with drift μ and variance paremeter σ is a stochastic process $(X_t)_{t\geq 0}$ satisfying

1) $X_0 = 0$, $(X_t)_{t\geq 0}$ has continuous sample paths
2) $(X_t)_{t\geq 0}$ has independent increments
3) For $t > s$ $X_t - X_s \sim$

In particular, $X_t \sim$ $\Rightarrow X_t$ is not centered, not symmetric w.r.t. the origin

Brownian motion with drift



Gambler's ruin problem for BM with drift

Let $(X_t)_{t\geq 0}$ be a BM with drift $\mu \in \mathbb{R}$ and variance parameter $\sigma^2 > 0$. Fix $a < x < b$ and denote

$$T = T_{ab} = \min\{t \geq 0 : X_t = a \text{ or } X_t = b\}, \text{ and}$$

$$u(x) = P(X_T = b \mid X_0 = x).$$

Theorem.

(i) $u(x) =$

(ii) $E(T_{ab} \mid X_0 = x) =$

No proof

Example

Fluctuations of the price of a certain share is modeled by the BM with drift $\mu = 1/10$ and variance $\sigma^2 = 4$. You buy a share at 100 \$ and plan to sell it if its price increases to 110 \$ or drops to 95 \$.

(a) What is the probability that you will sell at profit?

(b) What is the expected time until you sell the share?

Denote by $(X_t)_{t \geq 0}$ a BM with drift $\frac{1}{10}$ and variance 4,

$x =$, $b =$, $a =$. Then $2\mu/\sigma^2 =$ and

(a) $P(X_T = 110 \mid X_0 = 100) =$

(b) $E(T \mid X_0 = 100) =$

Maximum of a BM with negative drift

Thm Let $(X_t)_{t\geq 0}$ be a BM with drift $\mu<0$, variance σ^2 and $X_0=0$. Denote $M=\max_{t\geq 0} X_t$. Then

Proof. $X_0=0$, therefore $M\geq 0$. For any $b>0$

$$P(M>b) =$$

$$=$$

$$=$$

$$P(M>b) =$$

Geometric BM

Def. Stochastic process $(Z_t)_{t\geq 0}$ is called a geometric Brownian motion with drift parameter α and variance σ^2 if $X_t =$ is a BM with drift $\mu = \alpha - \frac{1}{2}\sigma^2$ and variance σ^2.

In other words, $Z_t =$, where $(B_t)_{t\geq 0}$ is a standard BM and $z>0$ is the starting point $Z_0 = z$.

If $0 \leq t_1 < t_2 < \cdots < t_n$, then $\frac{Z_{t_i}}{Z_{t_{i-1}}} =$

Since B has independent increments

$\frac{Z_{t_1}}{Z_{t_0}}, \frac{Z_{t_2}}{Z_{t_1}}, \ldots, \frac{Z_{t_n}}{Z_{t_{n-1}}}$ are independent and

$\frac{Z_{t_n}}{Z_{t_0}} =$ $\leftarrow$ "relative change of price = product of indepentent relative changes"

Expectation of Geometric BM

Let $(Z_t)_{t\geq 0}$ be geometric BM with paremeters α and σ.

Then

$$E(Z_t \mid Z_0 = z) =$$

$$E(e^{\sigma B_t}) =$$

$$\Rightarrow E(Z_t \mid Z_0 = z) = z\, e^{(\alpha - \frac{1}{2}\sigma^2)t} e^{t\frac{\sigma^2}{2}} =$$

Remark

It can be shown that for $0 < \alpha < \frac{1}{2}\sigma^2$ $Z_t \to 0$ as $t \to \infty$

At the same time, for $\alpha > 0$ $E(Z_t) \to \infty$.

Variance of geometric BM

$E(Z_t^2 \mid Z_0 = z) =$

$=$

$\mathrm{Var}(Z_t \mid Z_0 = z) =$

Theorem.

Let $(Z_t)_{t \geq 0}$ be geometric BM with paremeters α and σ^2.

Then

(i) $E(Z_t \mid Z_0 = z) = z e^{\alpha t}$

(ii) $\mathrm{Var}(Z_t \mid Z_0 = z) = z^2 e^{2\alpha t}\left(e^{\sigma^2 t} - 1\right)$

Gambler's ruin for geometric BM

Let $(Z_t)_{t\geq 0}$ be geometric BM with paremeters α and σ^2.

Let $A<1<B$, and denote $T=\min\{t: \frac{Z_t}{Z_0}=A \text{ or } \frac{Z_t}{Z_0}=B\}$.

Theorem $P\left(\frac{Z_T}{Z_0}=B\right)=$

Example Fluctuations of the price are modeled by a geometric BM with drift $\alpha=0.1$ and variance $\sigma^2=4$. You buy a share at 100\$ and plan to sell it if its price increases to 110\$ or drops to 95\$.

Take $A=$, $B=$, $2\alpha/\sigma^2=$, $1-2\alpha/\sigma^2=$

$P(X_T=110 \mid X_0=100)=$

Black-Scholes option pricing formula

Call option gives the buyer the right (not obligation) to buy a block of shares at a specific price ← striking price at any time during a certain period. How much should you pay for it?

Example: For the premium of 6\$ the call allows you to buy 60\$ of shares during the period of one month. If at some point during this period the actual price of the shares becomes $x > 66\$$, you can buy the shares using the call option, then immediately sell it gaining $(x-66)\$$. Or you may opt not to buy the shares at all → lose 6\$.

Let z be the current value of the share and τ be the length of the time period. Denote $F(z,\tau)$ the value of the call.

Black-Scholes option pricing formula

Then $F(z,\tau) = e^{-r\tau} E\left((Z_\tau - a)^+ \mid Z_0 = z\right)$ [BS], where

- a is the striking price
- r is the return rate for riskless investments
- $(Z_t)_{t\geq 0}$ is a geometric BM with parameters r and σ^2
- σ^2 is the volatility (variance parameter) of the share price

Computing the conditional expectation gives

$$F(z,\tau) = z\,\Phi\left(\frac{\log\frac{z}{a} + \left(r+\frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}}\right) - a e^{-r\tau}\,\Phi\left(\frac{\log\frac{z}{a} + \left(r-\frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}}\right)$$

Black-Scholes formula