MATH 180A: Introduction to Probability

Lecture A00 (Au)

www.math.ucsd.edu/~bau/w21.180a

Lecture B00 (Nemish)

www.math.ucsd.edu/~ynemish/teaching/180a

Today: Random sampling.
Infinitely many outcomes.
Properties of probability.

Video: Prof. Todd Kemp, Fall 2019

Next: ASV 2.1-2.2

Week 1:

- Homework 0 (due Friday, January 8)
- Join Piazza

Combinatorics

* selecting k objects from among n, with replacement:

$$\# \text{ways} = n^k$$

* selecting k objects from among n, without replacement; order matters:

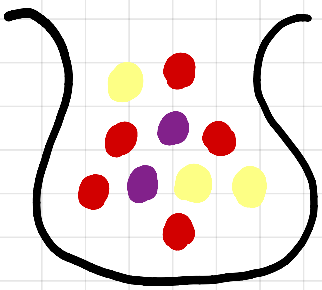
$$\# \text{ways} = n(n-1)(n-2)\cdots(n-k+1) \quad (k \le n)$$

* selecting k objects from among n, without replacement; order doesn't matter:

$$\# \text{ways} = \binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!}$$

$$= \frac{n!}{(n-k)!\,k!} = \binom{n}{n-k}$$

Sampling with Replacement (order doesn't matter)

E.g. 

An urn contains 10 balls: $\rightarrow b_1, b_2, b_3, b_4, b_5$
$b_6\ b_7\ b_8\ b_9\ b_{10}$

2 blue

3 yellow

5 red

Problem: 3 balls are chosen without replacement.

$$P(2 \text{ yellow}, 1 \text{ red})$$

$$\Omega = \{\{b_1, b_2, b_3\} : b_i \neq b_j \text{ if } i \neq j\} \qquad \#\Omega = \binom{10}{3}$$

order matters

$$A = \{2 \text{ are yellow}, 1 \text{ red}\} \qquad \#A = \binom{5}{1} \cdot \binom{3}{2}$$

$$\therefore\ P(A) = \frac{\binom{5}{1} \cdot \binom{3}{2}}{\binom{10}{3}} = \frac{15}{120} = \frac{1}{8} = 12.5\%$$

1.3

What if $\#\Omega = \infty$?

Then we need a different notion of uniform.

Eg. A random real number is chosen in $[0,1]$.

(a) What is the probability it is ≥ 0.7 ?

(b) What is the probability it is $= \frac{1}{2}$?

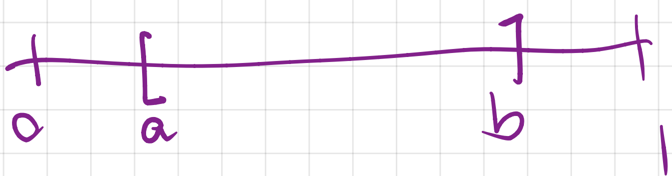
$\underbrace{\text{the probability}}_{\text{must define!}}$

(a) $\mathbb{P}([0.7,1])$
$= 1 - 0.7 = 0.3$

(b) $\mathbb{P}([\frac{1}{2},\frac{1}{2}])$
$= \frac{1}{2} - \frac{1}{2} = 0.$

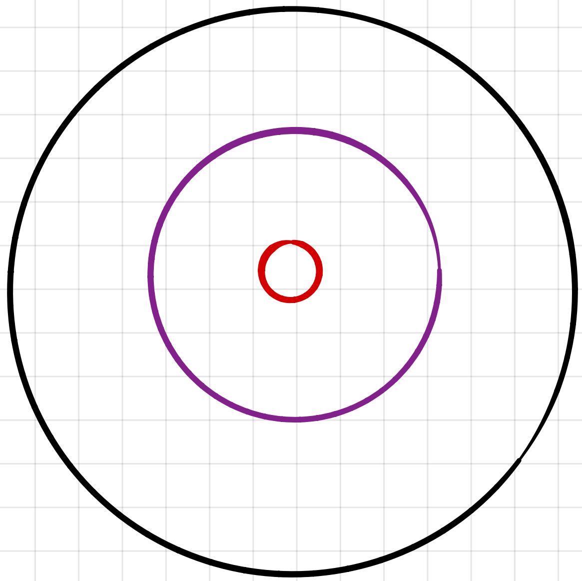
$(\Omega, \mathcal{F}, \mathbb{P})$

$\Omega = [0,1]$, $\mathcal{F} = \mathcal{B}(\cancel{\mathbb{R}})$ $[0,1]$, $\mathbb{P}([a,b]) := b - a.$



$$\mathbb{P}\left([0,0.3) \cup [\tfrac{1}{2}, 0.96)\right)$$
$$= \mathbb{P}([0,0.3)) + \mathbb{P}([\tfrac{1}{2}, 0.96)) = 0.3 + 0.46$$
$$= 0.76$$

E.g.



An archery target is a disk 50 cm in diameter.

A blue disk in the center is 25 cm in diameter.

A red disk in the center is 5 cm in diameter.

Given that you hit the target (randomly), what are the chances of hitting the blue disk? The red disk?

Ω = target

$\mathcal{F}$ = { subsets that have "area" }

$$\mathbb{P}(A) = \frac{\text{Area}(A)}{\text{Area}(\Omega)}$$

$\mathbb{P}$(bullseye) = 1%

Decompositions

1.4

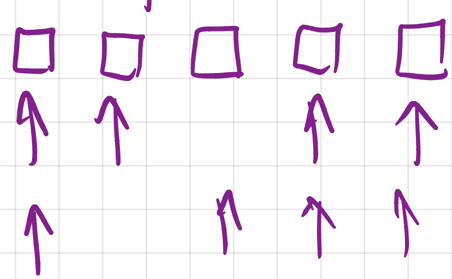
Eg. A fair coin is tossed 5 times. What is the probability that at least 3 tosses come up tails?

$A = \{\text{at least 3 tails}\} = A_3 \cup A_4 \cup A_5$

$A_k := \{\text{exactly } k \text{ tails}\}$

$$\mathbb{P}(A) = \mathbb{P}(A_3) + \mathbb{P}(A_4) + \mathbb{P}(A_5)$$

$\mathbb{P}(A_5) = \binom{5}{5}\frac{1}{2^5}$



A_4: # configurations $= \binom{5}{4}$

$\mathbb{P}(A_4) = \binom{5}{4} \cdot \frac{1}{2^5}$.

$\mathbb{P}(A_3) = \binom{5}{3}\frac{1}{2^5}$

$$\therefore \mathbb{P}(A) = \frac{1}{2^5}\left(\binom{5}{3} + \binom{5}{4} + \binom{5}{5}\right) = \frac{1}{2^5}(1+5+10) = \frac{16}{32}$$

$= 50\%$

E.g. A fair die is rolled 4 times. What is the probability of at least one double?

$A = \{\text{some number comes up at least two times}\}$

$A_k = \{k \text{ comes up at least two times}\}$

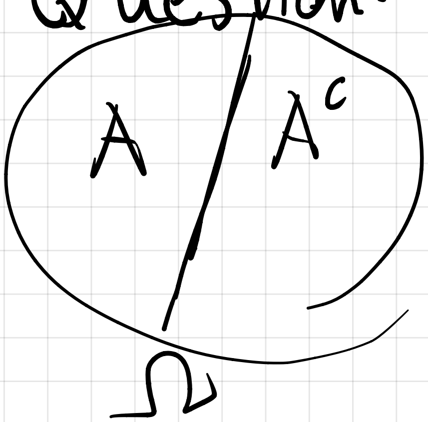
$A = A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5 \cup A_6$ not disjoint

$A_k^m = \{k \text{ comes up exactly } m \text{ times}\}$ ~~zillions of~~ 30 scenarios.

$A_1 = A_1^2 \cup A_1^3 \cup A_1^4 \cup A_1^5 \cup A_1^6$

$\vdots$

Question: Are all these events disjoint? No!



$$P(A^c) = \frac{6 \cdot 5 \cdot 4 \cdot 3}{6^4} = \frac{5}{18}$$

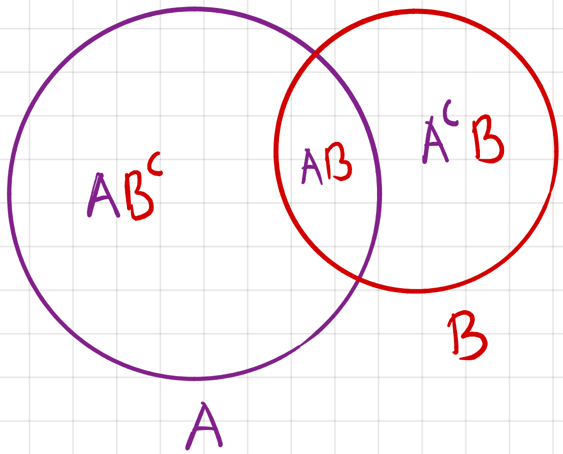
$$1 = P(\Omega) = P(A) + P(A^c)$$

$$\therefore P(A) = 1 - P(A^c) = 1 - \frac{5}{18} = \frac{13}{18}$$

Sometimes, you can't avoid lack of disjointness so easily. You have to take intersections into account.

Notation: $A \cap B = \{\text{all outcomes in both } A \text{ and } B\}$

$$= AB$$

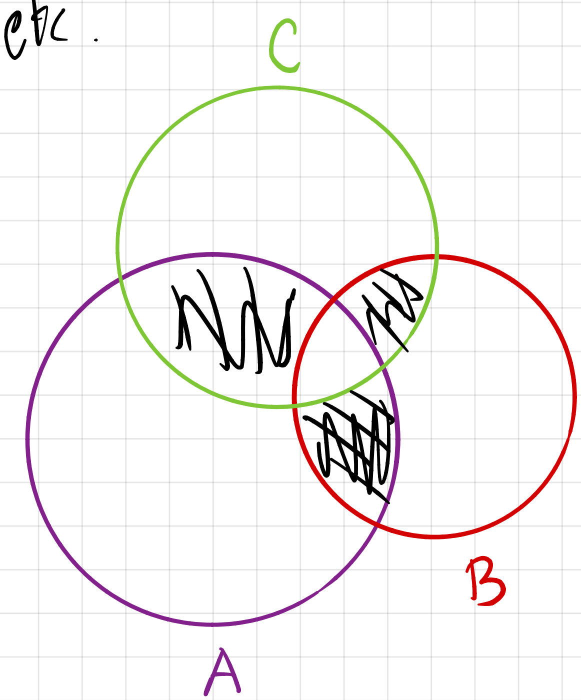


$$A \cup B = AB^c \cup AB \cup A^cB \quad \leftarrow \text{disjoint}$$

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$$

Principle of Inclusion / Exclusion

The probability of a union can be computed by adding the probabilities, then subtracting off the intersection(s) overcounted. If you have more sets, you have to keep going and re-add back in pieces that you over-subtracted, etc.



$$\mathbb{P}(A \cup B \cup C)$$
$$= \mathbb{P}(A) + \mathbb{P}(B) + \mathbb{P}(C)$$
$$- \mathbb{P}(AB) - \mathbb{P}(AC) - \mathbb{P}(BC)$$
$$+ \mathbb{P}(ABC)$$

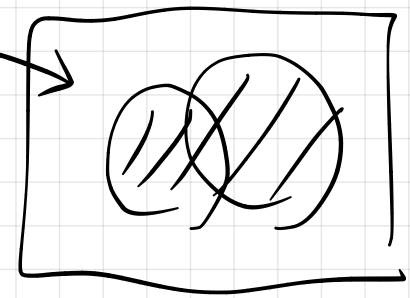
E.g. 20% of the population own cats.
25% of the population own dogs.
5% of the population own both.
What is the probabality that a random persan owns neither?

$C \quad D$

$\mathbb{P}(C) = 0.2$

$\mathbb{P}(D) = 0.25$

$\mathbb{P}(CD) = 0.05$



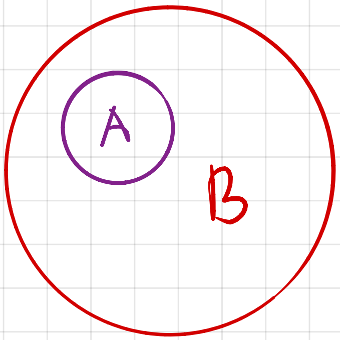
$$
\begin{aligned}
\mathbb{P}(C^c D^c) &= \mathbb{P}((C \cup D)^c) \\
&= 1 - \mathbb{P}(C \cup D) \\
&= 1 - (\mathbb{P}(C) + \mathbb{P}(D) - \mathbb{P}(CD)) \\
&= 1 - (0.2 + 0.25 - 0.05) \\
&= 0.6.
\end{aligned}
$$

Monotonicity

If $A \subseteq B$ then $B = A \cup A^c B$ is a disjoint union

$$\therefore \mathbb{P}(B) = \mathbb{P}(A) + \underbrace{\mathbb{P}(A^c B)}_{\geq 0}$$

$$\geq \mathbb{P}(A)$$



Eg 90% of your friends like the xiao long bao at Din Tai Fung.
80% of your friends like the xiao long bao at Shanghai Saloon.
What is the smallest possible proportion of your friends who like the xiao long bao at both restaurants?