# MATH180C: Introduction to Stochastic Processes II

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Today: General continuous time MC. Q-matrices. Matrix exponentials

> Q&A: October 16

Next: PK 6.3, 6.6, Durrett 4.2

Week 2:

- No homework!

# Q-matrices (infinitesimal generators)

Let $S=\{0,1,\dots,N\}$. We call $Q=(q_{ij})_{i,j=0}^{N}$ a Q-matrix if $Q$ satisfies the following conditions:

(a) $0\le -q_{ii}<\infty$ for all $i$

(b) $q_{ij}\ge 0$ for all $i\neq j$

(c) $\sum_{j} q_{ij}=0$ for all $i$

Denote $q_i=\sum_{j\neq i} q_{ij}$

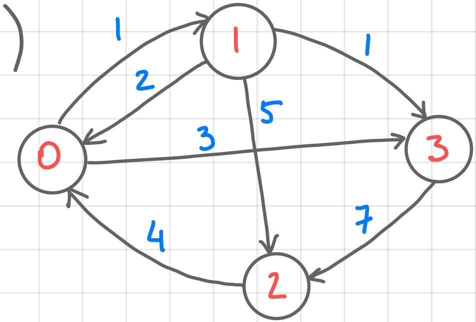
then $q_{ii}=-q_i$

## Examples

(a)

$$Q=\begin{pmatrix} -2 & 1 & 1 \\ 2 & -7 & 5 \\ 0 & 2 & -2 \end{pmatrix}$$

(b)




$$\begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & & & & \\ 1 & & & & \\ 2 & & & & \\ 3 & & & & \end{array}
\quad
\begin{array}{cc} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{pmatrix} -4 & 1 & 0 & 3 \\ 2 & -8 & 5 & 1 \\ 4 & 0 & -4 & 0 \\ 0 & 0 & 7 & -7 \end{pmatrix} \end{array}$$

# Matrix exponentials

Let $Q=(q_{ij})_{i,j=1}^N$ be a matrix. Then the series $\sum_{k=0}^{\infty}\frac{Q^k}{k!}$ converges componentwise, and we denote its sum $\sum_{k=0}^{\infty}\frac{Q^k}{k!} =: e^Q$, the matrix exponential of Q.

In particular, we can define $e^{tQ}=\sum_{k=0}^{\infty}\frac{Q^k t^k}{k!}$ for $t\geq 0$.

__Thm.__ Define $P(t)=e^{tQ}$. Then

(i) $P(t+s)=P(t)P(s)$ for all $s,t$

(ii) $(P(t))_{t\geq 0}$ is the unique solution to the equations

$$\begin{cases}\frac{d}{dt}P(t)=P(t)Q \\ P(0)=I\end{cases}, \quad \text{and} \quad \begin{cases}\frac{d}{dt}P(t)=QP(t) \\ P(0)=I\end{cases}$$

# Main theorem

Let $P(t)$ be a matrix-valued function $t \geq 0$.

Consider the following properties

(a) $P_{ij}(t) \geq 0$, $\sum_j P_{ij}(t) = 1$ for all $i$, $t \geq 0$

(b) $P(0) = I$

(c) $P(t+s) = P(t)P(s)$ for all $t, s \geq 0$

(d) $\lim_{t \downarrow 0} P(t) = I$ (continuous at $0$)

<u>Theorem A.</u> $P(t)$ satisfies (a)–(d)

if and only if

$$P(t) = e^{tQ} \text{ for some } Q\text{-matrix } Q$$

# Main theorem. Remarks

This theorem establishes one-to-one correspondance between matrices $P(t)$ satisfying (a)-(d) and the Q-matrices of the same dimension.

## Remarks

1. Conditions (a)-(d) imply that $P(t)$ is differentiable

2. If $P(t) = e^{tQ}$, then $P(h) = I + Qh + o(h)$ as $h \to 0$

$$P(h) = I + Qh + \underbrace{\sum_{k \geq 2} \frac{Q^k h^k}{k!}}_{o(h)}$$

# Q-matrices and Markov chains

Let $(X_t)_{t\geq 0}$ be a continuous time MC, $X_t \in \{0,1,\dots,N\}$ with right-continuous sample paths

Denote $P_{ij}(t) = P(X_t = j \mid X_0 = i)$ , $i,j \in \{0,1,\dots,N\}$

stationary

Then

- $P_{ij}(t) \geq 0$ , $\sum_{j=0}^{N} P_{ij}(t) = 1 \quad \left( = \sum_{j=0}^{N} P(X_t = j \mid X_0 = i) \right)$
- $P_{ij}(0) = \delta_{ij} \quad \left( P(X_0 = j \mid X_0 = i) = \begin{cases} 1, & j = i \\ 0, & j \neq i \end{cases} \right) \Leftrightarrow P(0) = I$
- $P_{ij}(t+s) = P(X_{t+s} = j \mid X_0 = i) = \sum_{k=0}^{N} P_{kj}(s) P_{ik}(t)$

  $= \sum_{k=0}^{N} P(X_{t+s} = j \mid X_0 = i, X_t = k) P(X_t = k \mid X_0 = i)$
- $\lim_{h \downarrow 0} P(X_h = j \mid X_0 = i) = \delta_{ij}$


$P(h) \to I, \; h \downarrow 0$

# Q-matrices and Markov chains (cont.)

$P(t)$ satisfies properties (a)-(d) from Theorem A.

$\Longrightarrow$ there is a Q-matrix $Q$ such that

$$P(t) = e^{tQ}$$

$$p_{ij}(h) = q_{ij}h + o(h) \quad i \neq j$$

In particular,

$$p_{ii}(h) = 1 + q_{ii}h + o(h)$$

$$P(h) = I + Qh + o(h)$$

This implies the one-to-one correspondance between Q-matrices and continuous time MC with right-continuous sample paths.

$Q$ is called the infinitesimal generator of $(X_t)_{t\geq 0}$

# Infinitesimal description of cont. time MC

Let $Q = (q_{ij})_{i,j=0}^{N}$ be a Q-matrix, let $(X_t)_{t\geq 0}$ be right-continuous stochastic process, $X_t \in \{0, 1, \dots, N\}$.

We call $(X_t)_{t\geq 0}$ a Markov chain with generator Q, if

(i) $(X_t)_{t\geq 0}$ satisfies the Markov property

(ii) $$P(X_{t+h} = j \mid X_t = i) = \begin{cases} q_{ij} h + o(h), & i \neq j \\ 1 + q_{ii} h + o(h), & i = j \end{cases} \quad h \to 0$$

## Example

Pure death process

- $P_{i,i-1}(h) = \mu_i h + o(h)$
- $P_{ii}(h) = 1 - \mu_i h + o(h)$
- $P_{ij}(h) = o(h)$ for $j \notin \{i-1, i\}$
- $P_{0j}(h) = 0$ for $j \neq 0$

The corresponding Q-matrix

$$Q = \begin{array}{c} 0 \\ 1 \\ 2 \\ \vdots \\ N \end{array} \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots \\ \mu_1 & -\mu_1 & & & \\ & \mu_2 & -\mu_2 & & \\ & & \ddots & \ddots & \\ & & & \mu_N & -\mu_N \end{pmatrix}$$

# Sojourn time description

Let $Q = (q_{ij})_{i,j=0}^{N}$ be a Q-matrix. Denote $q_i = \sum_{j \neq i} q_{ij}$

so that

$$Q = \begin{pmatrix} -q_0 & q_{01} & q_{02} & \cdots \\ q_{10} & -q_1 & q_{12} & \cdots \\ q_{20} & q_{21} & -q_2 & \cdots \\ \vdots & \vdots & & \ddots \end{pmatrix}$$

$q_0 = \sum_{i \neq 0} q_{0i}$

$\vdots$

$(X_t)_{t \geq 0}$ is MC with infinit. generator $Q$.

Denote $Y_k := X_{w_k}$ (jump chain).

Then the MC with generator matrix $Q$ has the following equivalent jump and hold description

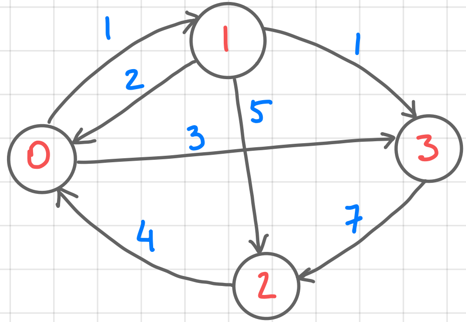
- sojourn times $S_k$ are independent r.v. with $P(S_k > t \mid Y_k = i) = e^{-q_i t}$ $\left(S_k \sim \mathrm{Exp}(q_{Y_k})\right)$

- transition probabilities $P(Y_{k+1} = j \mid Y_k = i) = \dfrac{q_{ij}}{q_i}$

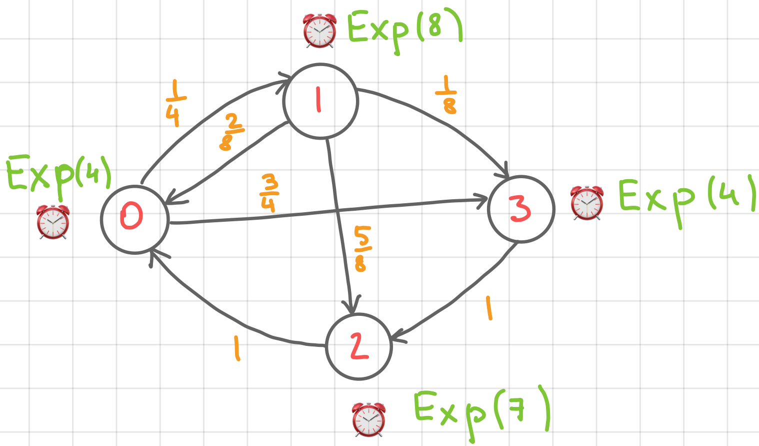
# Example

$$
\begin{array}{c|cccc}
 & 0 & 1 & 2 & 3 \\
\hline
0 & -4 & 1 & 0 & 3 \\
1 & 2 & -8 & 5 & 1 \\
2 & 4 & 0 & -4 & 0 \\
3 & 0 & 0 & 7 & -7
\end{array}
$$

## Rate diagram of $(X_t)_{t \geq 0}$

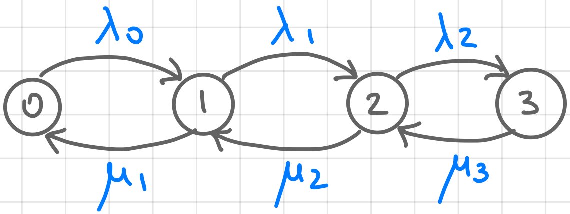



$$\textcircled{i} \xrightarrow{\alpha} \textcircled{j} \quad \text{"="} \quad P_{ij}(h) = \alpha h + o(h)$$



# Example

Birth and death process on $\{0,1,2,3\}$




$$Q = \begin{pmatrix} -\lambda_0 & \lambda_0 & & \\ \mu_1 & -(\lambda_1+\mu_1) & \lambda_1 & \\ & \mu_2 & -(\lambda_2+\mu_2) & \lambda_2 \\ & & \mu_3 & -\mu_3 \end{pmatrix}$$