1. (25 points) Suppose that you are waiting for a bus whose arrival time is distributed as an exponential random variable with mean 1 hour. Once the bus arrives, it takes 1 hour to drive you home. However, if you wait 1 hour for the bus and it still has not arrived, you decide to give up on the bus and walk home, which takes 10 hours. Let Y be the amount of time (in hours) that it takes for you to get home including the time spent waiting for the bus.

(a) (15 points) Calculate the CDF of Y.

Solution. If $X \sim \text{Exp}(1)$, then

$$Y = \begin{cases} X+1 & \text{if } X < 1; \\ 10+1 = 11 & \text{if } X \geq 1. \end{cases}$$

So,

$$F_Y(t) = \begin{cases} \mathbb{P}(X \leq t-1) = 0 & \text{if } t < 1; \\ \mathbb{P}(X \leq t-1) = 1 - \mathbb{P}(X > t-1) = 1 - e^{-(t-1)} & \text{if } 1 \leq t < 2 \\ \mathbb{P}(X < 1) = 1 - e^{-(2-1)} = 1 - e^{-1} & \text{if } 2 \leq t < 11 \\ \mathbb{P}(Y \leq 11) = 1 & \text{if } t \geq 11. \end{cases}$$

Note that Y is neither continuous nor discrete (you did not need to say this to get full credit). $\square$

(b) (10 points) Calculate the expected value $\mathbb{E}[Y]$.

Solution. We use the fact that Y is a function of a continuous random variable with a known density. So,

$$\mathbb{E}[Y] = \int_0^1 (x+1)e^{-x}\,dx + \int_1^\infty 11e^{-x}\,dx.$$

The first integral can be computed using integration by parts:

$$\int (x+1)e^{-x}\,dx = -(x+1)e^{-x} + \int e^{-x}\,dx = -(x+1)e^{-x} - e^{-x} + C.$$

So,

$$\int_0^1 (x+1)e^{-x}\,dx = -(x+1)e^{-x} - e^{-x}\Big|_0^1 = -2e^{-1} - e^{-1} + 2 = 2 - 3e^{-1}.$$

The second integral can be computed using the tail probability for the exponential distribution:

$$\int_1^\infty 11e^{-x}\,dx = 11\mathbb{P}(X \geq 1) = 11e^{-1}.$$

So,

$$\mathbb{E}[Y] = 2 - 3e^{-1} + 11e^{-1} = 2 + 8e^{-1}.$$

$\square$

2. (25 points) Let $X \sim \mathrm{Geom}(p)$, where $p \in (0,1)$. Compute

$$\mathbb{E}\left[\frac{1}{X!}\right],$$

where we recall that $X!$ is the factorial. To receive full credit, your final answer should not contain an infinite series.

Solution. This is a direct computation:

$$\begin{aligned}
\mathbb{E}\left[\frac{1}{X!}\right] &= \sum_{k=1}^{\infty} \frac{1}{k!}(1-p)^{k-1}p \\
&= \frac{p}{1-p}\sum_{k=1}^{\infty} \frac{(1-p)^k}{k!} \\
&= \frac{p}{1-p}\left(\sum_{k=0}^{\infty} \frac{(1-p)^k}{k!} - 1\right) \\
&= \frac{p}{1-p}\left(e^{1-p} - 1\right).
\end{aligned}$$

$\square$

3. (25 points) 250000 randomly chosen individuals were interviewed to estimate the unknown fraction $p \in (0,1)$ of the population that like bagels. The resulting estimate is $\hat{p}$. Suppose that we want to construct a 98% confidence interval $(\hat{p}-\varepsilon, \hat{p}+\varepsilon)$. How large must we choose ε? You may leave your answer in terms of the inverse Φ^{-1} of the CDF of the standard normal.

Solution. Recall the equation

$$\mathbb{P}(|\hat{p} - p| < \varepsilon) \geq 2\Phi(2\varepsilon\sqrt{n}) - 1.$$

So, we want

$$2\Phi(2\varepsilon\sqrt{n}) - 1 \geq .98,$$

where $n = 250000$. Solving for ε in the above, we get

$$\varepsilon \geq \frac{\Phi^{-1}(.99)}{2\sqrt{n}} = \frac{\Phi^{-1}(.99)}{2\sqrt{250000}}.$$

This can be simplified to

$$\varepsilon \geq \frac{\Phi^{-1}(.99)}{1000},$$

but this was not necessary for full credit. □

4. (25 points) Let X be the random variable with density

$$f(x) = \begin{cases} 1 & \text{if } x \in (0,1); \\ 0 & \text{if } x \notin (0,1). \end{cases}$$

Let $Y = \ln(\sqrt{X})$.

(a) (15 points) Compute the moment generating function $M_Y(t)$ of Y. Hint: do not try to compute the density of Y.

Solution. By definition, the moment generating function is

$$M_Y(t) = \mathbb{E}[e^{tY}] = \mathbb{E}[e^{t\ln(\sqrt{X})}] = \mathbb{E}[\sqrt{X}^t] = \mathbb{E}[X^{t/2}].$$

We compute the latter expectation as a function of a random variable that we have a density for:

$$E[X^{t/2}] = \int_0^1 x^{t/2}\, dx = \begin{cases} \left.\dfrac{x^{\frac{t}{2}+1}}{\frac{t}{2}+1}\right|_0^1 = \dfrac{1}{\frac{t}{2}+1} & \text{if } t \neq 0 \text{ and } t > -2; \\ 1 & \text{if } t = 0; \\ +\infty & \text{if } t \leq -2. \end{cases}$$

Note that since $\frac{1}{\frac{t}{2}+1}$ at $t = 0$ is 1, we can simply write this as

$$M_Y(t) = \begin{cases} \dfrac{1}{\frac{t}{2}+1} & \text{if } t > -2; \\ +\infty & \text{if } t \leq -2. \end{cases}$$

□

(b) (10 points) Use the moment generating function to compute the nth moment of Y.

Solution. We need to compute the Taylor series of $\frac{1}{\frac{t}{2}+1}$. Rather than computing derivatives, we use the fact that it can written as a geometric series:

$$\frac{1}{\frac{t}{2}+1} = \frac{1}{1-(-\frac{t}{2})} = \sum_{n=0}^{\infty}\left(-\frac{t}{2}\right)^n = \sum_{n=0}^{\infty}\frac{(-1)^n n!}{2^n}\frac{t^n}{n!}$$

for $|t| < 2$. So, the nth moment is

$$\frac{(-1)^n n!}{2^n}.$$

As a fun exercise, you can show that $-\log(X^r) \sim \mathrm{Exp}(1/r)$ for $r > 0$. Try to think of how you might do this using part (a). $\square$