

MATH180C: Introduction to Stochastic Processes II

www.math.ucsd.edu/~ynemish/teaching/180c

Today: Reflection principle

> Q&A: December 4

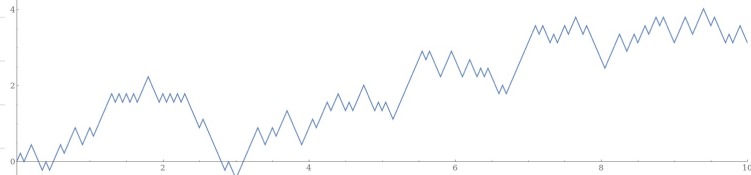
Next: PK 8.3

This week:

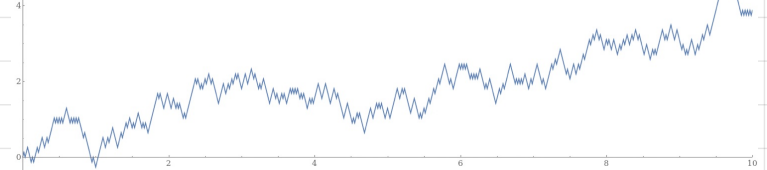
- Homework 7 (due THURSDAY, December 3)
- Homework 8 (due THURSDAY, December 10)
- CAPE at www.cape.ucsd.edu

Approximating a BM with random walks X_t^n

$n=20$



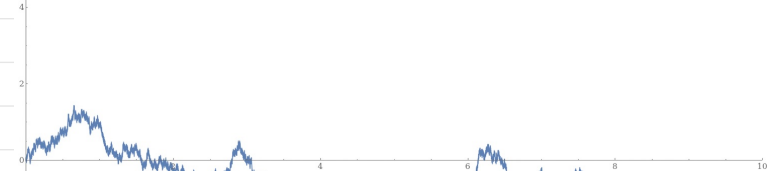
$n=100$



$n=500$



$n=1000$



Stopping times and the strong Markov property (lec. 3)

Def (Informal). Let $(X_t)_{t \geq 0}$ be a stochastic process and let $T \geq 0$ be a random variable. We call T a **stopping time** if the event

$$\{T \leq t\}$$

can be determined from the knowledge of the process up to time t (i.e., from $\{X_s : 0 \leq s \leq t\}$)

Examples: Let $(X_t)_{t \geq 0}$ be right-continuous

1. $\min\{t \geq 0 : X_t = x\}$ is a stopping time
2. $\sup\{t \geq 0 : X_t = x\}$ is not a stopping time

Stopping times and the strong Markov property (lec. 3)

Theorem (no proof)

Let $(X_t)_{t \geq 0}$ be a Markov process, let T be a stopping time of $(X_t)_{t \geq 0}$. Then, conditional on $T < \infty$ and $X_T = x$,

$$(X_{T+t})_{t \geq 0}$$

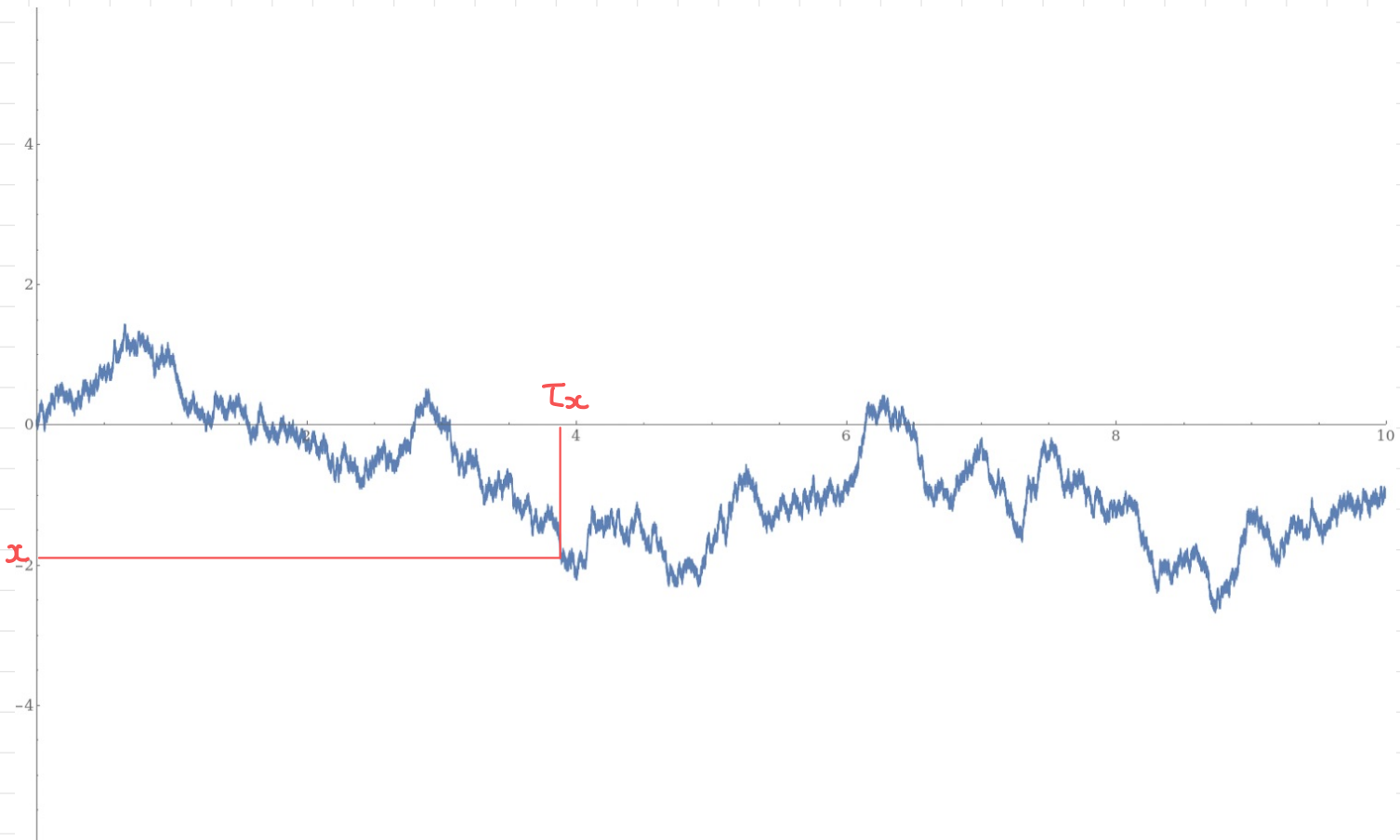
(i) is independent of $\{X_s, 0 \leq s \leq T\}$

(ii) has the same distribution as $(X_t)_{t \geq 0}$ starting from x

Example $(B_t)_{t \geq 0}$ is Markov. For any $x \in \mathbb{R}$ define

$\tau_x = \min \{t: B_t = x\}$. Then

- $(B_{t+\tau_x} - B_{\tau_x})_{t \geq 0}$ is a BM starting from 0
- $(B_{t+\tau_x} - B_{\tau_x})_{t \geq 0}$ is independent of $\{B_s, 0 \leq s \leq \tau_x\}$
(independent of what B was doing before it hit x)



Reflection principle

Thm. Let $(B_t)_{t \geq 0}$ be a standard BM. Then for any $t \geq 0$ and $x > 0$

$$P(\max_{0 \leq u \leq t} B_u > x) = P(|B_t| > x)$$

$\Downarrow S_t$

~~$(S_t)_{t \geq 0} \stackrel{(d)}{=} (|B_t|)_{t \geq 0}$~~

Proof. Let $\tau_x = \min\{t : B_t = x\}$. Note that τ_x is a stopping time and is uniquely determined by $\{B_u, 0 \leq u \leq \tau_x\}$

From the definition of τ_x , $\max_{0 \leq u \leq t} B_u \geq x \Leftrightarrow \tau_x \leq t$. Then

$$\begin{aligned} P(\max_{0 \leq u \leq t} B_u \geq x, B_t < x) &= P(\tau_x \leq t, B_{(t-\tau_x)+\tau_x} - B_{\tau_x} < 0) \\ &\stackrel{\text{sym}}{=} \frac{1}{2} P(\tau_x \leq t) = \frac{1}{2} P(\max_{0 \leq u \leq t} B_u \geq x) \end{aligned}$$

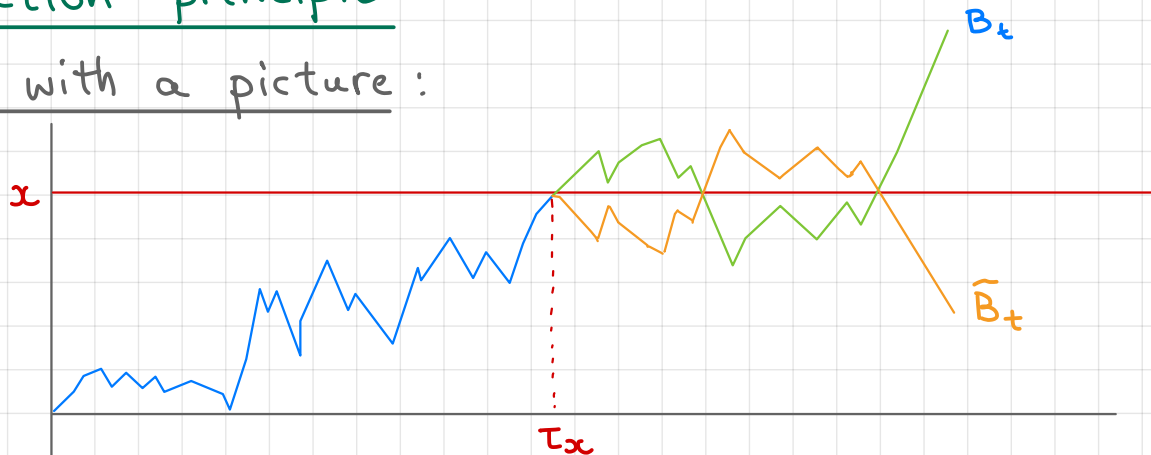
$\stackrel{=x}{\phantom{B_{(t-\tau_x)+\tau_x} - B_{\tau_x}}}$

$$\text{Now } P(\max_{0 \leq u \leq t} B_u \geq x) = P(B_t \geq x) + P(\max_{0 \leq u \leq t} B_u \geq x, B_t < x)$$

$$\Rightarrow P(\max_{0 \leq u \leq t} B_u \geq x) = 2 P(B_t \geq x) = P(|B_t| \geq x)$$

Reflection principle

Proof with a picture:



If $(B_t)_{t \geq 0}$ is a BM, then $(\tilde{B}_t)_{t \geq 0}$ is a BM, where

$$\tilde{B}_t = \begin{cases} B_t, & t \leq \tau_x \\ B_{\tau_x} - (B_t - B_{\tau_x}), & t > \tau_x \end{cases}$$

\Rightarrow to each sample path with $\max_{0 \leq u \leq t} B_u > x$ and $B_t > x$ we associate a unique path with $\max_{0 \leq u \leq t} \tilde{B}_u > x$ and $\tilde{B}_t < x$, so

$$P(\max_{0 \leq u \leq t} B_u \geq x, B_t < x) = P(B_t \geq x) \Rightarrow P(\max_{0 \leq u \leq t} B_u \geq x) = 2 P(B_t \geq x)$$

Application of the RP: distribution of the hitting time τ_x

By definition, $\tau_x \leq t \iff \max_{0 \leq u \leq t} B_u \geq x$, so

$$P(\tau_x \leq t) = P\left(\max_{0 \leq u \leq t} B_u \geq x\right) = 2 P(B_t \geq x)$$

$$= 2 \cdot \frac{1}{\sqrt{2\pi t}} \int_x^\infty e^{-\frac{u^2}{2t}} du \quad \left\{ u = v\sqrt{t}, \quad du = \sqrt{t} dv \right.$$

$$= \sqrt{\frac{2}{\pi}} \int_{x/\sqrt{t}}^\infty e^{-\frac{v^2}{2}} dv$$

$$\Rightarrow \text{p.d.f. of } \tau_x \quad f_{\tau_x}(t) = \sqrt{\frac{2}{\pi}} e^{-\frac{x^2}{2t}} \cdot \frac{1}{2} t^{-3/2} = \frac{1}{\sqrt{2\pi}} t^{-3/2} e^{-\frac{x^2}{2t}}$$

Thm. $F_{\tau_x}(t) = \sqrt{\frac{2}{\pi}} \int_{x/\sqrt{t}}^\infty e^{-\frac{v^2}{2}} dv$,

$$f_{\tau_x}(t) = \frac{x}{\sqrt{2\pi}} t^{-3/2} e^{-\frac{x^2}{2t}}$$

Zeros of BM

Denote by $\theta(t, t+s)$ the probability that $B_u = 0$ on $(t, t+s)$

$$\theta(t, t+s) := P(B_u = 0 \text{ for some } u \in (t, t+s))$$

Thm. For any $t, s > 0$

$$\theta(t, t+s) = \frac{2}{\pi} \arccos \sqrt{\frac{t}{t+s}}$$

Proof Compute $P(B_u = 0 \text{ for some } u \in (t, t+s])$ by conditioning on the value of B_t .

$$\theta(t, t+s) = \int_{-\infty}^{+\infty} P(B_u = 0 \text{ for some } u \in (t, t+s] \mid B_t = x) \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}} dx \quad (*)$$

Define $\tilde{B}_u = B_{t+u} - B_t$. Then

$$P(B_u = 0 \text{ on } (t, t+s] \mid B_t = x) = P(\tilde{B}_u = -x \text{ on } (0, s] \mid B_t = x)$$

$$\stackrel{\text{MP}}{=} P(\tilde{B}_u = -x \text{ on } (0, s]) \stackrel{\text{symmetry}}{=} P(\tilde{B}_u = x \text{ on } (0, s]) \quad (**)$$

Zeros of BM

Plugging $(**)$ into $(*)$ gives

$$\begin{aligned}\Theta(t, t+s) &= \int_{-\infty}^{+\infty} P(B_u = x \text{ for some } u \in (0, s]) \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}} dx \\&= \int_0^{+\infty} P(B_u = x \text{ for some } u \in (0, s]) \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}} dx \\&\quad + \int_0^{\infty} P(B_u = -x \text{ for some } u \in (0, s]) \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}} dx \\&= \sqrt{\frac{2}{\pi t}} \int_0^{\infty} P(B_u = x \text{ for some } u \in (0, s]) e^{-\frac{x^2}{2t}} dx\end{aligned}$$

Finally, $P(B_u = x > 0 \text{ for some } u \in (0, s]) = P(\max_{0 \leq u \leq s} B_u \geq x) = P(\tau_x \leq s)$

$$(*) = \int_0^{\infty} \sqrt{\frac{2}{\pi t}} e^{-\frac{x^2}{2t}} \left(\int_0^s \frac{x}{\sqrt{2\pi}} y^{-3/2} e^{-\frac{x^2}{2y}} dy \right) dx = \frac{1}{\pi \sqrt{t}} \int_0^s \left(\int_0^{\infty} x e^{-\frac{x^2}{2} \left(\frac{1}{t} + \frac{1}{y} \right)} dx \right) y^{-3/2} dy$$

Zeros of BM

$$\int_0^{\infty} x e^{-\frac{x^2}{2}(\frac{1}{t} + \frac{1}{y})} dx = \frac{1}{\frac{1}{t} + \frac{1}{y}} = \frac{ty}{t+y}$$

$$\Rightarrow (*) = \frac{1}{\pi\sqrt{t}} \int_0^s \frac{ty}{t+y} y^{-3/2} dy = \sqrt{\frac{t}{\pi}} \int_0^s \frac{1}{(t+y)\sqrt{y}} dy$$

Now use the change of variable $z = \sqrt{\frac{y}{t}}$, $dy = 2t dz$

$$\begin{aligned} (*) &= \frac{\sqrt{t}}{\pi} \int_0^{\sqrt{s/t}} \frac{1}{t(1+z^2)\sqrt{t}z} \cdot 2t dz = \frac{2}{\pi} \int_0^{\sqrt{s/t}} \frac{1}{1+z^2} dz = \frac{2}{\pi} \arctan\left(\sqrt{\frac{s}{t}}\right) \\ &= \frac{2}{\pi} \arccos\left(\sqrt{\frac{t}{s+t}}\right) \end{aligned}$$

↑ exercise

Remark Let $T_0 := \inf\{t > 0 : B_t = 0\}$. Then $P(T_0 = 0) = 1$

There is a sequence of zeros of $B_t(\omega)$ converging to 0.

To understand the structure of the set of zeros \rightarrow Cantor set

Behavior of BM as $t \rightarrow \infty$

Thm. Let $(B_t)_{t \geq 0}$ be a (standard) BM. Then

$$P\left(\sup_{t \geq 0} B_t = +\infty, \inf_{t \geq 0} B_t = -\infty\right) = 1$$


(BM "oscillates with increasing amplitude")

Proof. Denote $Z = \sup_{t \geq 0} B_t$. Then for any $c > 0$

$$cZ = \sup_{t \geq 0} cB_t = \sup_{t \geq 0} cB_{t/c^2}$$

By property (iii), cB_{t/c^2} is a standard BM, so cZ has the same distribution as $Z \Rightarrow P(Z=0)=p, P(Z=\infty)=1-p$

$$p = P(Z=0) \leq P(B_1 \leq 0 \text{ and } \sup_{t \geq 0} (B_{t+1} - B_1) = 0) = \frac{1}{2} P(Z=0) = \frac{1}{2} p$$

$\Rightarrow P(Z=0)=0, P(Z=\infty)=1$. Similarly for $\inf_{t \geq 0} B_t$ 

Sample paths of $(B_t)_t$ are not differentiable

Thm. $P(B_t \text{ is not differentiable at zero}) = 1$

Proof. $P(\sup_{t \geq 0} B_t = \infty, \inf_{t \geq 0} B_t = -\infty) = 1. \quad (\star)$

Consider $\tilde{B}_t = t B_{1/t}$. $(\tilde{B}_t)_{t \geq 0}$ is a BM (by property (iv))

By (\star) , for any $\varepsilon > 0 \exists t < \varepsilon, s < \varepsilon$ such that

$\tilde{B}_t > 0, \tilde{B}_s < 0 \Rightarrow$ only differentiable if $\tilde{B}'_0 = 0$

But if $\tilde{B}'_0 = 0$, then

$|s B_{1/s}| = |\tilde{B}_s| \leq s$ for some $t > 0$ and all $0 < s < t$,

which implies that $|B_{1/s}| \leq 1$ for all $0 < s < t$, which contradicts to (\star) \blacksquare

Thm $P((B_t)_{t \geq 0} \text{ is nowhere differentiable}) = 1$

Reflected BM

Def. Let $(B_t)_{t \geq 0}$ be a standard BM. The stochastic process

$$R_t = |B_t| = \begin{cases} B_t, & \text{if } B_t \geq 0 \\ -B_t, & \text{if } B_t < 0 \end{cases}$$

is called reflected BM.

Think of a movement in the vicinity of a boundary.

Moments: $E(R_t) = \int_{-\infty}^{\infty} |x| \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}} dx \stackrel{\text{symmetry}}{=} 2 \int_0^{\infty} x \frac{e^{-\frac{x^2}{2t}}}{\sqrt{2\pi t}} dx = \sqrt{\frac{2t}{\pi}}$

$$\text{Var}(R_t) = E(B_t^2) - (E(|B_t|))^2 = t - \frac{2t}{\pi} = \left(1 - \frac{2}{\pi}\right)t$$

Transition density: $P(R_t \leq y \mid R_0 = x) = P(-y \leq B_t \leq y \mid B_0 = x)$

$$= \int_{-y}^y \frac{1}{\sqrt{2\pi t}} e^{-\frac{(x-s)^2}{2t}} ds \Rightarrow p_t(x, y) = \frac{1}{\sqrt{2\pi t}} \left(e^{-\frac{(x-y)^2}{2t}} + e^{-\frac{(x+y)^2}{2t}} \right)$$

Thm (Lévy) Let $M_t = \max_{0 \leq u \leq t} B_u$. Then $(M_t - B_t)_{t \geq 0}$ is a reflected BM.

Reflected BM

