MATH180C: Introduction to Stochastic Processes II

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Today: Martingales

> Q&A: November 25

Next: PK 8.1

This week:

- Thanksgiving
- Next homework deadline: December 2 (HW 7)

Martingales

we get that

Definition. A stochastic process (Xn, n > 0) is a martingale if for n = 0,1,...

(a)

(a)
(b)

After taking the expectation of both sides of (b)

(Xn)n=0 is a martingale =>

- · submartingale:
- · supermartingale:

Examples of martingales (i) Let X, X2, ... be independent RV's with E(IXxI) <0 and E(Xx)=0. Define Sn=X1+···+Xn, So=0. Then => (ii) Let X1, X2,... be independent RV with Xx20, E (1Xx1) 200 and E(Xx)=1. Define Mn = X, X2 -- Xn, Mo=1. Then

=)

Example

Stock prices in a perfect market

Let Xn be the closing price at the end of day n of a certain publicly traded security such as a share or stock. Many scholars believe that in a perfect market these price sequences should be martingales.

(see PK page 73 for more details).

History and gambling Let (Xn) n20 be a stochastic process describing your total winnings in n games with unit stake. Think of Xn-Xn-1 as your net winnings per unit stake in game n, n ≥ 1, in a series of games, played at times n=1,2,... In the martingale case Some early works of martingales was motivated by gambling. Note that there exists a betting strategy called the "martingale system" - doubling bets after losses

Some basic properties

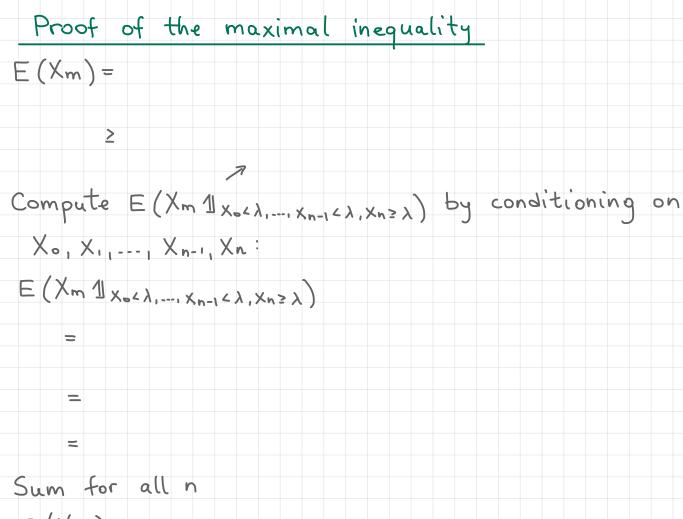
Let (Xn)_{n≥0} be a martingale.

Proof

=>

· Markov inequality: If Xn20 Vn, then for any 1>0

Maximal inequality for nonegative martingales Thm. Let (Xn)n≥0 be a martingale with nonnegative values. For any 1>0 and me N and (2) Proof. We prove (1), (2) follows by taking the limit m+00. Take the vector (Xo, X.,..., Xm) and partition the sample space wrt the index of the first r.v. rising above & using the above partition Compute



E(Xm) ≥

Example A gambler begins with a unit amount of money and faces a series of independent fair games. In each game the gamblers bets fraction p of his current fortune, wins with probability & loses with probability &. Estimate the probability that the gambler ever doubles the initial fortune. Denote by Zn, n > 0, the gambler's fortune after n-th game. Denote Then

Martingale transform In the previous example the stake in n-th game is P Zn-1. What if we choose another strategy? Def Let (Xn)nzo be a nonnegative martingale, and (et (Cn)nzo be a stochastic process with Cn = fn (Xo,..., Xn-1). Then the stochastic process is called the Think of • Xx-Xx-1 as the winning per unit stake in x-th game · Ck as your stake in K-th game decision is made based on the previous history . (C.X), as total winnings up to time n

Martingale transform

Prop. Let $Z_n = X_0 + (C \cdot X)_n$. Let $C_k > 0$ bounded if $Z_{k-1} > 0$ and $C_k = 0$ if $Z_{k-1} = 0$. Then $(Z_n)_{n \ge 0}$ is a martingale $Proof: F(Z_{n+1}, Z_{n+1}, Z_$

Note that

If Zn>o, then C1>0,..., Cn>o,

$$E(Z_{n+1}|Z_{0},...,Z_{n})=$$

If Zn=0, then Cn+1=0 and E(Zn1,120,..., Zn)=0=2n

Gambling example:

Start from the initial fortune Xo=1. Define

Zn=

Then (Zn)n>o is a nonnegative martingale, E(Zo)=1

Convergence of nonnegative martingales Thm If (Xn)nzo is a nonnegative (super) martingale, then with probability 1 and Example An urn initially contains one red ball and one green ball. Choose a ball and return it to the urn

together with another ball of the same color. Repeat.

Denote by Xn the fraction of red ball after n iterations.

Example (cont.)

(i)
$$(X_n)_{n\geq 0}$$
 is a martingale

Denote by Rn the number of red balls after n-th iteration

 $R_n =$

Then

 $E(X_{n+1}|X_{0},...,X_{n}) =$
 $=$

(ii) X_n is nonnegative $=$)

(iii) Compute the distribution of X_{∞}
 $P(X_n = \frac{K}{n+2}) = \frac{1}{n+1}$ for $K \in \{1,2,...,n+1\}$
 $P(X_{\infty} \leq x) = x$, $x \in \{0,1\} \Rightarrow X_{\infty} \sim U_{ni} \in \{0,1\}$