MATH180C: Introduction to Stochastic Processes II

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Today: Poisson process as a renewal process

> Q&A: November 4

Next: PK 7.2-7.3, Durrett 3.1

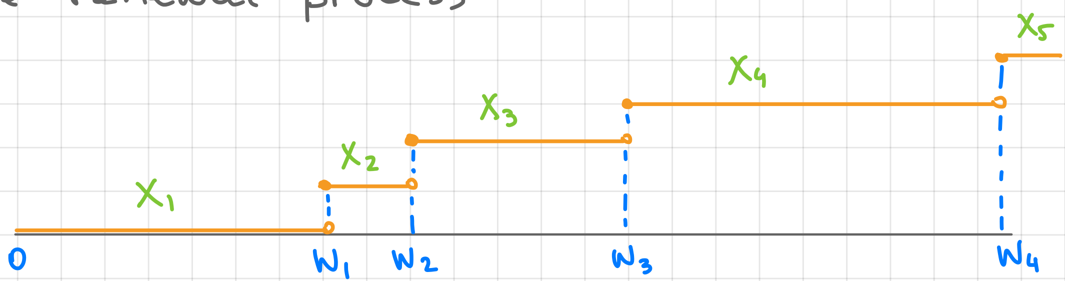
This week:

- Quiz 3 (November 4)
- Homework 4 (due Friday, November 6, 11:59 PM)

Poisson process as a renewal process

The Poisson process $N(t)$ with rate $\lambda > 0$ is a renewal process with $F(x) = 1 - e^{-\lambda x}$.

- sojourn times S_i are i.i.d., $S_i \sim Exp(\lambda)$
- S_i represent intervals between two consecutive events (arrivals of customers)
- $W_n = \sum_{i=0}^{n-1} S_i$
- we can take $X_i = S_{i-1}$ in the definition of the renewal process



Poisson process as a renewal process

We know that $N(t) \sim \text{Pois}(\lambda t)$, so in particular

$$E(N(t)) = \lambda t$$

Example Compute $M(t) = \sum_{n=1}^{\infty} F^{*n}(t)$ for PP

$$F_2(t) = \int_0^t \left(1 - e^{-\lambda(t-x)}\right) \lambda e^{-\lambda x}\, dx = 1 - e^{-\lambda t} - \lambda \int_0^t e^{-\lambda t}\, \overset{dx}{\cancel{dt}} = F(t) - \lambda t e^{-\lambda t}$$

Denote $\varphi_k(t) = \frac{(\lambda t)^k}{k!} e^{-\lambda t}$:

$$\varphi_k * F(t) = \int_0^t \frac{\lambda^k (t-x)^k}{k!} e^{-\lambda(t-x)} \lambda e^{-\lambda x}\, dx = \varphi_{k+1}(t)$$

$$F * F(t) = F(t) - \varphi_1(t)$$

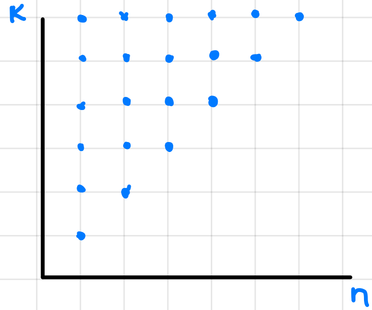
$$F^{*3}(t) = (F - \varphi_1) * F(t) = F(t) - \varphi_1(t) - \varphi_2(t)$$

$$\vdots$$

$$F^{*n}(t) = F(t) - \varphi_1(t) - \cdots - \varphi_{n-1}(t)$$

Poisson process as a renewal process (cont.)

$$\sum_{n=1}^{\infty} F^{*n}(t) = \sum_{n=1}^{\infty}\left[1-\sum_{k=0}^{n-1}\frac{(\lambda t)^k}{k!}e^{-\lambda t}\right] = e^{-\lambda t}\sum_{n=1}^{\infty}\sum_{k=n}^{\infty}\frac{(\lambda t)^k}{k!}$$



$$= e^{-\lambda t}\sum_{k=1}^{\infty}\sum_{n=1}^{k}\frac{(\lambda t)^k}{k!} = e^{-\lambda t}\sum_{k=1}^{\infty}\frac{(\lambda t)^k}{(k-1)!}$$

$$= \lambda t$$

$$M(t) = \lambda t$$

Renewal density

Proposition Let $N(t)$ be a renewal process with continuous interrenewal times X_i having density $f(x)$. Denote $m(t) = \sum_{n=1}^{\infty} f^{*n}(t)$. Then $M(t) = \int_0^t m(x)\,dx$

and $m(t) = f(t) + m * f(t)$ $\quad (*)$

$\uparrow$ renewal density

Proof: $\frac{d}{dt} F^{*n}(t) = \left(\frac{d}{dt} F^{*(n-1)}\right) * f(t) = f^{*n}(t)$ ■

Example: Compute the renewal density for PP using $(*)$.

$f(x) = \lambda e^{-\lambda x}$, so $(*)$ becomes

$$m(t) = \lambda e^{-\lambda t} + \int_0^t m(t-x)\,\lambda e^{-\lambda x}\,dx = \lambda e^{-\lambda t} + \int_0^t m(x)\,\lambda e^{-\lambda(t-x)}\,dx$$

$$= \lambda e^{-\lambda t}\left(1 + \int_0^t m(x)\, e^{\lambda x}\,dx\right)$$

(cont.)

$$e^{\lambda t} m(t) = \lambda \left(1 + \int_0^t e^{\lambda x} m(x)\, dx\right) \leftarrow \text{differentiate}$$

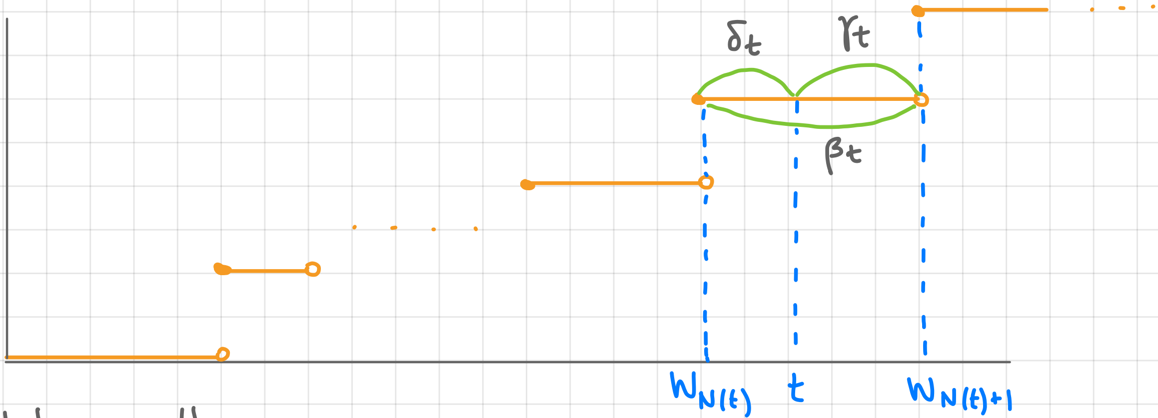
$$\begin{cases} \frac{d}{dt}\left(e^{\lambda t} m(t)\right) = \lambda \left(e^{\lambda t} m(t)\right) \\ m(0) = \lambda \end{cases} \Rightarrow e^{\lambda t} m(t) = \lambda e^{\lambda t}$$

$$m(t) = \lambda$$

Indeed, $M(t) = \lambda t = \int_0^t \lambda \, dt$

Excess life and current life of PP (summary)

Recall: Let $N(t)$ be a renewal process.



Def. We call

- $\gamma_t := W_{N(t)+1} - t$ the excess (or residual) lifetime
- $\delta_t := t - W_{N(t)}$ the current life (or age)
- $\beta_t := \gamma_t + \delta_t$ the total life

Remarks

1) $\gamma_t > h \geq 0$ iff $N(t+h) = N(t)$

2) $t \geq h$ and $\delta_t \geq h$ iff $N(t-h) = N(t)$

Excess life and current life of PP

Let $N(t)$ be a PP. Then

- excess life $\gamma \sim Exp(\lambda)$

$$P(\gamma_t > x) = P(N(t+x) - N(t) = 0) = P(N(x) = 0) = e^{-\lambda x}$$

- current life δ_t

$$P(\delta_t > x) = \begin{cases} 0, & \text{if } x \geq t \\ P(N(t) - N(t-x) = 0) = e^{-\lambda x}, & x < t \end{cases}$$

- total life $\beta_t = \gamma_t + \delta_t$

$$E(\gamma_t + \delta_t) = \frac{1}{\lambda} + E(\delta_t) = \frac{1}{\lambda} + \int_0^\infty P(\delta_t > x)\,dx$$

$$= \frac{1}{\lambda} + \int_0^t e^{-\lambda x}\,dx = \frac{1}{\lambda} + \frac{1}{\lambda}\left(1 - e^{-\lambda t}\right) \longrightarrow \frac{2}{\lambda}, \quad t \to \infty$$

Excess life and current life of PP (cont.)

- Joint distribution of (γ_t, δ_t)

$$P(\gamma_t > x, \delta_t > y) = \begin{cases} 0 & \text{if } y > t \\ P\big(N(t+x) - N(t-y) = 0\big) = e^{-\lambda(x+y)}, & y \le t \end{cases}$$

$\Rightarrow$ γ_t and δ_t are independent (for PP)