MATH 180A - INTRODUCTION TO PROBABILITY
PRACTICE MIDTERM 2

FALL 2020

☐ **Write your name and PID on the top of EVERY PAGE.**

☐ **Write the solutions to each problem on separate pages. CLEARLY INDICATE on the top of each page the number of the corresponding problem. Different parts of the same problem can be written on the same page (for example, part (a) and part (b))**

☐ **Remember this exam is graded by a human being. Write your solutions NEATLY AND COHERENTLY, or they risk not receiving full credit.**

☐ **From the moment you access the midterm problems on Gradescope you have 70 MINUTES to COMPLETE AND UPLOAD your exam to Gradescope. Plan your time accordingly.**

☐ **You are allowed to use the textbook, lecture notes and your personal notes. You are not allowed to use the electronic devices (except for accessing the online version of the textbook) or outside assistance. Outside assistance includes but is not limited to other people, the internet and unauthorized notes.**

1. Suppose that the time it takes for you to complete your probability homework is distributed according to an exponential random variable with mean 1 hour. You start your homework at 8:00 PM. Your bedtime is 10:00 PM. If you finish your homework before your bedtime, you watch TV until your bedtime and then go to sleep. If you do not finish by your bedtime, you go to sleep anyway, and so you do not watch TV at all. Let Y be the random variable that measures the amount of time in hours that you spend watching TV.

(a) Calculate the CDF of Y.

Proof. Let $X \sim \mathrm{Exp}(1)$. Then

$$Y = \begin{cases} 2 - X & \text{if } X \in [0, 2], \\ 0 & \text{if } X \in (2, \infty). \end{cases}$$

So,

$$F_Y(t) = \mathbb{P}(Y \le t) = \begin{cases} 0 & \text{if } t < 0, \\ \mathbb{P}(X \in [2, \infty)) = e^{-2} & \text{if } t = 0, \\ \mathbb{P}(X \in [2 - t, \infty)) = e^{-(2-t)} & \text{if } t \in (0, 2], \\ 1 & \text{if } t > 2, \end{cases}$$

where we have used the tail probability formula for the exponential distribution. You can simplify this to

$$F_Y(t) = \mathbb{P}(Y \le t) = \begin{cases} 0 & \text{if } t < 0, \\ e^{t-2} & \text{if } t \in [0, 2], \\ 1 & \text{if } t > 2, \end{cases}$$

but this is not necessary on the actual exam. $\square$

(b) Calculate the expected value $\mathbb{E}[Y]$.

Proof. Based on our work in part (a), we see that Y is neither continuous nor discrete. So, we calculate the expectation using the fact that Y is a function of X (this reasoning was implicit in the calcuation of the CDF of Y). In particular, $Y = g(X)$, where $g : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is the function

$$g(x) = \begin{cases} 2 - x & \text{if } x \in [0, 2], \\ 0 & \text{if } x \in (2, \infty). \end{cases}$$

We can then compute

$$\mathbb{E}[Y] = \mathbb{E}[g(X)] = \int_0^\infty g(x)e^{-x}\,dx = \int_0^2 (2-x)e^{-x}\,dx + \int_2^\infty 0 \cdot e^{-x}\,dx.$$

Since

$$\int_2^\infty 0 \cdot e^{-x}\,dx = 0,$$

we only need to calculate

$$\int_0^2 (2-x)e^{-x}\,dx.$$

Integration by parts tells us that the indefinite integral

$$\int (2-x)e^{-x}\,dx = -(2-x)e^{-x} - \int e^{-x}\,dx = -(2-x)e^{-x} + e^{-x} + C.$$

So,

$$\int_0^2 (2-x)e^{-x}\,dx = \left[-(2-x)e^{-x} + e^{-x}\right]\Big|_0^2 = e^{-2} + 1.$$

□

2. Let $X \sim \text{Poisson}(\lambda)$. Compute

$$\mathbb{E}\left[\frac{1}{1+X}\right].$$

Proof. This is a direct calculation:

$$\begin{aligned}
\mathbb{E}\left[\frac{1}{1+X}\right] &= \sum_{k=0}^{\infty}\left(\frac{1}{1+k}\right)\mathbb{P}(X=k) \\
&= \sum_{k=0}^{\infty}\left(\frac{1}{1+k}\right)\left(e^{-\lambda}\frac{\lambda^k}{k!}\right) \\
&= e^{-\lambda}\sum_{k=0}^{\infty}\frac{1}{1+k}\frac{\lambda^k}{k!} \\
&= e^{-\lambda}\sum_{k=0}^{\infty}\frac{\lambda^k}{(k+1)!} \\
&= e^{-\lambda}\sum_{k=0}^{\infty}\frac{\lambda^{k+1}}{\lambda(k+1)!} \\
&= \frac{e^{-\lambda}}{\lambda}\sum_{k=0}^{\infty}\frac{\lambda^{k+1}}{(k+1)!} \\
&= \frac{e^{-\lambda}}{\lambda}\sum_{k=1}^{\infty}\frac{\lambda^{k}}{k!} \\
&= \frac{e^{-\lambda}}{\lambda}\left(\left(\sum_{k=0}^{\infty}\frac{\lambda^{k}}{k!}\right)-1\right) \\
&= \frac{e^{-\lambda}}{\lambda}\left(e^{\lambda}-1\right) \\
&= \frac{1-e^{-\lambda}}{\lambda},
\end{aligned}$$

though you did not need to simplify to the last line. The penultimate line would also receive full credit. $\square$

3. Suppose that we plan to interview n randomly chosen individuals to estimate the unknown fraction $p \in (0,1)$ of the population that likes ice cream. Let $\hat{p} = \frac{S_n}{n}$ be the random variable that records the proportion of the individuals who say they do like ice cream. How many people must we interview to have at least a 95% chance of capturing the true fraction p with a margin of error .01? You may leave your answer in terms of the inverse Φ^{-1} of the CDF of the standard normal.

Proof. We start with the formula

$$2\Phi(2\varepsilon\sqrt{n}) - 1 \geq .95.$$

We are told $\varepsilon = .01$. So, this simplifies to

$$\Phi(.02\sqrt{n}) \geq .975.$$

Since the density φ of the standard normal is positive everywhere, its CDF is strictly increasing and hence invertible. This allows us to conclude that

$$.02\sqrt{n} \geq \Phi^{-1}(.975).$$

In other words, we need

$$n \geq (50\Phi^{-1}(.975))^2 = 2500[\Phi^{-1}(.975)]^2.$$

$\square$

4. Suppose that the random variable X has p.d.f.

$$f(x) = \frac{\lambda}{2} e^{-\lambda|x|},$$

where $\lambda > 0$.

(a) Compute the moment generating function $M_X(t)$ of X.

Proof. This is another direct computation

$$\begin{aligned}
M_X(t) &= \mathbb{E}[e^{tX}] \\
&= \int_{-\infty}^{\infty} e^{tx} \frac{\lambda}{2} e^{-\lambda|x|}\, dx \\
&= \int_{-\infty}^{0} e^{tx} \frac{\lambda}{2} e^{-\lambda|x|}\, dx + \int_{0}^{\infty} \frac{\lambda}{2} e^{-\lambda|x|}\, dx \\
&= \int_{-\infty}^{0} e^{tx} \frac{\lambda}{2} e^{\lambda x}\, dx + \int_{0}^{\infty} e^{tx} \frac{\lambda}{2} e^{-\lambda x}\, dx \\
&= \int_{-\infty}^{0} \frac{\lambda}{2} e^{(t+\lambda)x}\, dx + \int_{0}^{\infty} \frac{\lambda}{2} e^{(t-\lambda)x}\, dx
\end{aligned}$$

Note that

$$\int_{-\infty}^{0} \frac{\lambda}{2} e^{(t+\lambda)x}\, dx = \infty \quad \text{if} \quad t + \lambda \leq 0.$$

Similarly,

$$\int_{0}^{\infty} \frac{\lambda}{2} e^{(t-\lambda)x}\, dx = \infty \quad \text{if} \quad t - \lambda \geq 0.$$

If $t \in (-\lambda, \lambda)$, then we can compute

$$\int_{-\infty}^{0} \frac{\lambda}{2} e^{(t+\lambda)x}\, dx = \left[\frac{\lambda}{2(t+\lambda)} e^{(t+\lambda)x}\right]\Bigg|_{-\infty}^{0} = \frac{\lambda}{2(t+\lambda)}.$$

Similarly, if $t \in (-\lambda, \lambda)$, then

$$\int_{0}^{\infty} \frac{\lambda}{2} e^{(t-\lambda)x}\, dx = \left[\frac{\lambda}{2(t-\lambda)} e^{(t-\lambda)x}\right]\Bigg|_{0}^{\infty} = \frac{-\lambda}{2(t-\lambda)}.$$

So,

$$M_X(t) = \begin{cases} \frac{\lambda}{2(t+\lambda)} + \frac{-\lambda}{2(t-\lambda)} = \frac{\lambda^2}{\lambda^2 - t^2} & \text{if } |t| < \lambda, \\ \infty & \text{if } |t| \geq \lambda. \end{cases}$$

Again, you do not need to simplify the function to $\frac{\lambda^2}{\lambda^2 - t^2}$ in the first line to get full credit. □

(b) Use the moment generating function to compute the nth moment of X.

Proof. We find the Taylor series of the moment generating function in part (a) in the region $(-\lambda, \lambda)$:

$$\begin{aligned} M_X(t) &= \frac{\lambda^2}{\lambda^2 - t^2} \\ &= \frac{1}{1 - \frac{t^2}{\lambda^2}} \\ &= \sum_{k=0}^{\infty} \left(\frac{t^2}{\lambda^2}\right)^k \\ &= \sum_{k=0}^{\infty} \frac{t^{2k}}{\lambda^{2k}} \\ &= \sum_{k=0}^{\infty} \frac{(2k)!}{\lambda^{2k}} \frac{t^{2k}}{(2k)!}. \end{aligned}$$

In particular, the coefficients of the odd power terms t^{2k+1} are all 0, so the odd moments of X are all zero. The even moments m_{2k} are then the coefficients of the Taylor series:

$$m_{2k} = \frac{(2k)!}{\lambda^{2k}} \quad \text{for} \quad k \in \mathbb{N}.$$

□