

MATH 180A: Introduction to Probability

Lecture A00 (Au)

www.math.ucsd.edu/~bau/w21.180a

Lecture B00 (Nemish)

www.math.ucsd.edu/~ynemish/teaching/180a

Today: CDF and PDF

Next: ASV 2.4, 2.5, 4.4

Video: Prof. Todd Kemp, Fall 2019

Week 3:

- Homework 2 (due Friday, January 22)
- Midterm 1 (Wednesday, January 27) - Lectures 1-7
- Regrades for HW1: Mon, Jan 25 - Tue, Jan 26 (PST) on Gradescope

Cumulative Distribution Function (CDF)

For any random variable X , $F_X(r) = P(X \leq r)$. $r \in \mathbb{R}$

(1) Monotone increasing: $s < t \Rightarrow F_X(s) \leq F_X(t)$

(2) $\lim_{r \rightarrow -\infty} F_X(r) = 0$, $\lim_{r \rightarrow +\infty} F_X(r) = 1$.

(3) The function F_X is right-continuous: $\lim_{t \rightarrow r^+} F_X(t) = F_X(r)$.

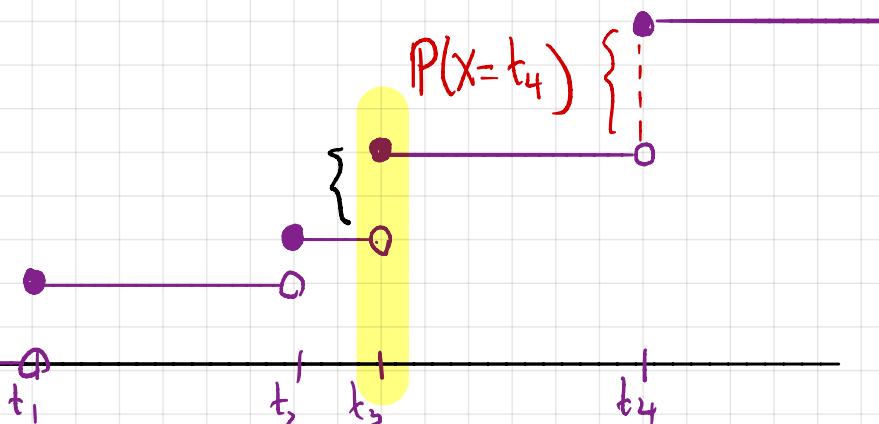
Discrete random variable:

finite or countable set of values

t_1, t_2, t_3, \dots with $P(X=t_j) > 0$

and

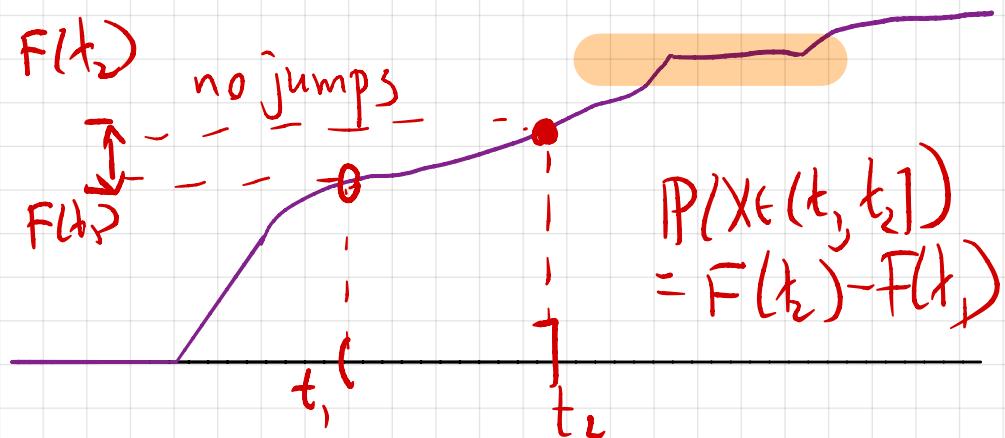
$$\sum_j P(X=t_j) = 1.$$



Continuous random variable

for each real number t , $P(X=t)=0$.

Because of (1) & (3) above, this implies that F_X is continuous.



Densities

Some continuous random variables have probability densities.

This is an infinitesimal version of a probability mass function.

X discrete, $\in \{t_1, t_2, t_3, \dots\}$

$$p_X(t) = P(X=t) \quad \text{probability mass function}$$

$$\underline{P(X \in A)} = \sum_{t \in A} P(X=t)$$

$$= \sum_{t \in A} p_X(t)$$

$$p_X(t) \geq 0,$$

$$\sum_t p_X(t) = 1.$$

X continuous

$$P(X=t) = 0 \text{ for all } t \in \mathbb{R}.$$

BUT

Maybe there is an "infinitesimal" prob. mass function f_X .

$$P(X \in A) = \int_A f_X(t) dt$$

$$\text{i.e. } A = (-\infty, r]$$

$$P(X \leq r) =$$

$$\int_{-\infty}^r f_X(t) dt$$

$$P(X \in [a, b]) =$$

$$\int_a^b f_X(t) dt$$

$$\int_{-\infty}^{\infty} f_X(t) dt = 1,$$

$$f_X(t) \geq 0.$$

E.g. Shoot an arrow at a circular target of radius 1.

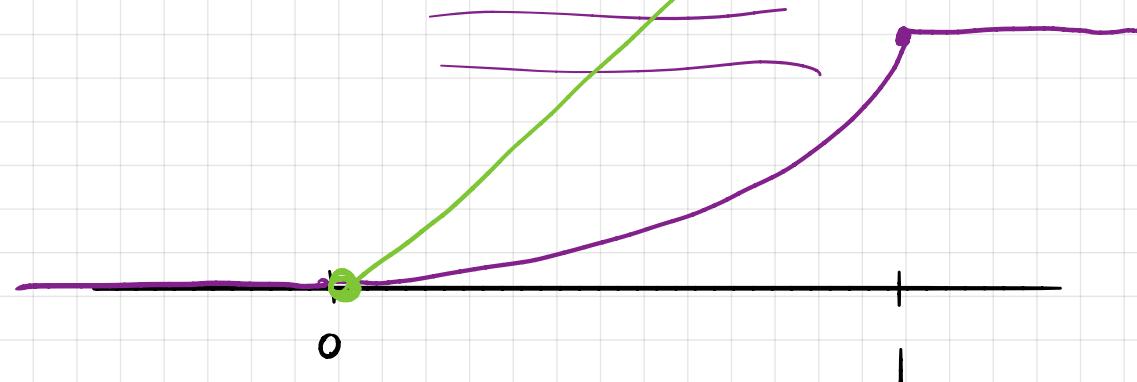
Y = distance from center.

$$\int_{-\infty}^r f(t) dt \stackrel{?}{=} P(Y \in (-\infty, r]) = F_Y(r) = \begin{cases} 0, & r < 0 \\ r^2, & 0 \leq r < 1 \\ 1, & r \geq 1 \end{cases}$$

"Solve for f "

$$\frac{d}{dr} \int_0^r f(t) dt = \frac{d}{dr} \begin{cases} r^2 & 0 \leq r < 1 \\ 1 & r \geq 1 \end{cases}$$

$$FTC \quad f(r) = \begin{cases} 2r & 0 < r \leq 1 \\ 0 & r > 1 \end{cases}$$

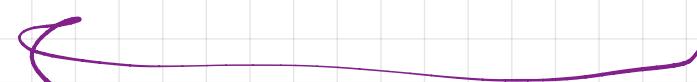


$$f_Y(r) = \begin{cases} 0 & r \leq 0 \\ 2r & 0 < r \leq 1 \\ 0 & r > 1 \end{cases}$$

$$P(Y \in [0.1, 0.2] \cup [0.9, 1]) = \int_{0.1}^{0.2} 2r dr + \int_{0.9}^1 2r dr$$

$$(0.2)^2 - (0.1)^2 + 1^2 - (0.9)^2$$

$$P(Y \in [0.1, 0.2] \cup [0.9, 1])$$



Theorem: If F_X is continuous and piecewise differentiable, then X has a density $f_X = F'_X$.

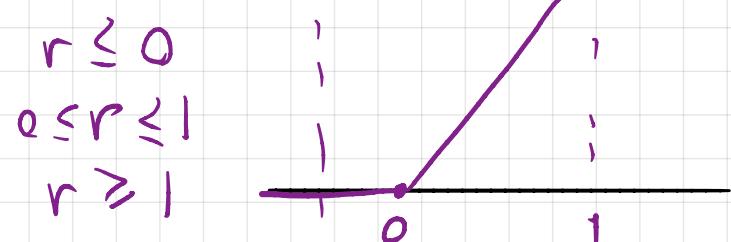
Proof: FTC. \square

Eg. Let X = a uniformly random number in $[0,1]$.

As we discussed in lecture 2, this means

$$\rightarrow P(X \in [s, t]) = t - s \quad \text{if } 0 \leq s < t \leq 1.$$

$$F_X(r) = P(X \leq r) = \begin{cases} 0 & r \leq 0 \\ r & 0 < r \leq 1 \\ 1 & r > 1 \end{cases}$$



$$\therefore f_X(r) = F'_X(r) = \begin{cases} 0 & r < 0 \\ 1 & 0 \leq r \leq 1 \\ 0 & r > 1 \end{cases}$$

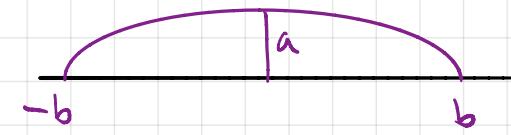
$$X \sim \text{Unif}([0,1])$$

$$Z \sim \text{Unif}([a,b]) \rightarrow$$

$$f_Z(t) = \begin{cases} 0 & t < a \\ \frac{1}{b-a} & a \leq t \leq b \\ 0 & t > b \end{cases}$$



Eg. Let $f(t) = c\sqrt{b^2 - t^2}$ for $|t| \leq b$, 0 otherwise
 (for some positive constants $b, c > 0$).



Is f a probability density?

- $f \geq 0$ ✓

- $\int_{-\infty}^{\infty} f(t) dt = \int_{-b}^b c\sqrt{b^2 - t^2} dt = c \int_{-b}^b \sqrt{b^2 - t^2} dt$

||

$$cb^2 \frac{\pi}{2} = 1$$

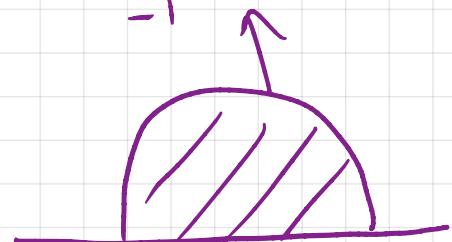
Must have $cb^2 = \frac{2}{\pi}$. ✓

E.g. For any $b, c = \frac{2}{\pi b^2}$. ✓

Subs: $t = bs$.

$$\begin{aligned} & c \int_{-1}^1 \sqrt{b^2 - (bs)^2} b ds \\ &= cb \int_{-1}^1 \sqrt{b^2(1-s^2)} ds \end{aligned}$$

$$= cb^2 \int_{-1}^1 \sqrt{1-s^2} ds$$



E.g. Your car is in a minor accident; the damage repair cost is a random number between \$100 and \$1500. Your insurance deductible is \$500. $Z =$ your out of pocket expenses.

The random variable Z is

(a) continuous

(b) discrete

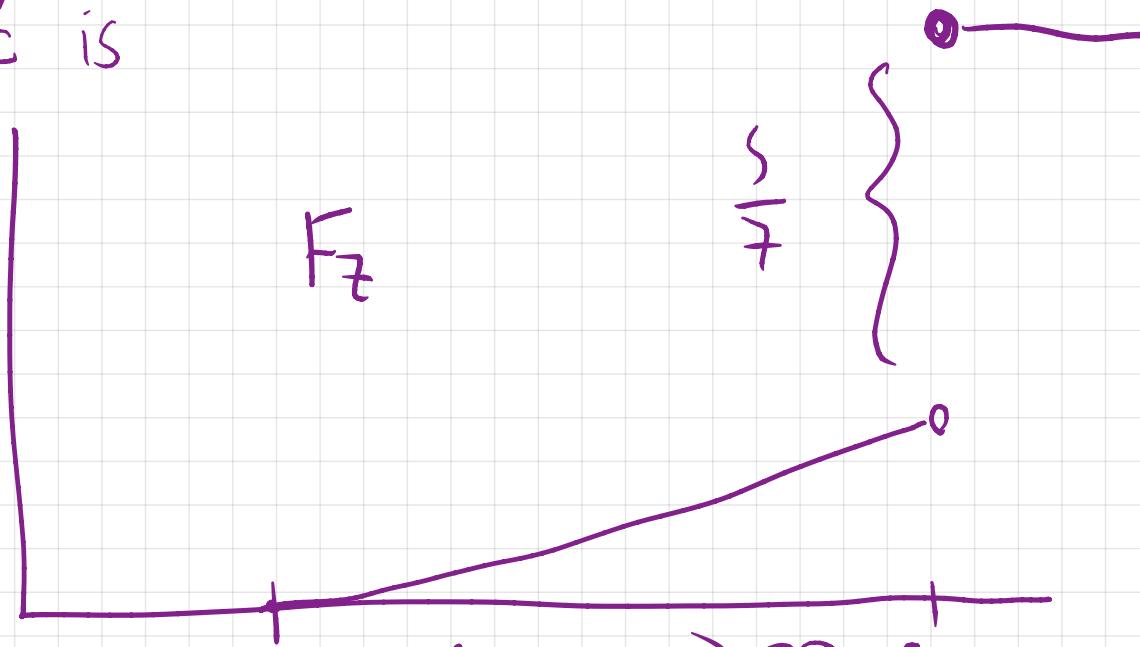
(c) neither

~~(d) both~~

$$X \sim \text{Unif}([100, 1500])$$

$$f_X(t) = \begin{cases} \frac{1}{1500-100} & 100 \leq t \leq 1500 \\ 0 & \text{otherwise} \end{cases}$$

$$\text{But if } r < 500 \quad P(Z=r) = 0.$$



$$Z = \max_{\min}(X, 500) ?? \quad \$00$$

$$P(Z=500) = P(Z \geq 500)$$

Correction
from
lecture

$$= P(X \geq 500) \\ = \int_{500}^{1500} \frac{1}{1400} dt = \frac{5}{7} > 0$$