☐ **Write your name and PID on the top of EVERY PAGE.**

☐ **Write the solutions to each problem on separate pages. CLEARLY INDICATE on the top of each page the number of the corresponding problem.**

☐ **Remember this exam is graded by a human being. Write your solutions NEATLY AND COHERENTLY, or they risk not receiving full credit.**

☐ **You may assume that all transition probability functions are STATIONARY.**

☐ **You are allowed to use the textbook, lecture notes and your personal notes. You are not allowed to use the electronic devices (except for accessing the online version of the textbook) or outside assistance. Outside assistance includes but is not limited to other people, the internet and unauthorized notes.**

1. (30 points) Let $Y > 0$ be a random variable having Gamma distribution with parameters 2 and λ, i.e., the p.d.f. of Y is given by

$$f_Y(y) = \lambda^2 y e^{-\lambda y}, \qquad y > 0, \tag{1}$$

and let $X \sim \text{Unif}[0, Y]$ be a random variable uniformly distributed on $[0, Y]$.

It is given that $E(X) = 1$.

(a) (10 points) Determine the unknown parameter λ. [Hint. Compute $E(X)$ with unknown parameter λ.]

(b) (10 points) Determine

$$P(X \le t \,|\, Y = y) = \begin{cases} \quad\;\;, & t < 0, \\ \quad\;\;, & 0 \le t < y, \\ \quad\;\;, & t \ge y. \end{cases} \tag{2}$$

for $y > 0$.

(c) (10 points) Using the results from (a) and (b), compute $P(X \le t)$ and determine the marginal distribution of X.

Solution.

(a) Compute $E(X)$ by conditioning on the value of Y

$$E(X) = \int_0^\infty E(X \,|\, Y = y) f_Y(y) dy. \tag{3}$$

For any $y > 0$, $E(X \,|\, Y = y) = \frac{y}{2}$, therefore, using the integration by parts,

$$E(X) = \int_0^\infty \frac{y}{2} \lambda^2 y e^{-\lambda y} dy = -\frac{\lambda}{2} \int_0^\infty y^2 d e^{-\lambda y} = \int_0^\infty y \lambda e^{-\lambda y} dy = \frac{1}{\lambda}. \tag{4}$$

We conclude that $\lambda = 1$.

(b) Given $Y = y$, X has uniform distribution on $[0, y]$, so

$$P(X \le t \,|\, Y = y) = \begin{cases} 0, & t < 0, \\ \frac{t}{y}, & 0 \le t < y, \\ 1, & t \ge y. \end{cases} \tag{5}$$

(c) Using the results from parts (a) and (b), we compute $P(X \le t)$ for $t > 0$ by conditioning on the value of Y

$$P(X \le t) = \int_0^\infty P(X \le t \,|\, Y = y) f_Y(y) dy \tag{6}$$

$$= \int_0^t f_Y(y) dy + \int_t^\infty \frac{t}{y} f_Y(y) dy \tag{7}$$

$$= \int_0^t y e^{-y} dy + \int_t^\infty \frac{t}{y} y e^{-y} dy \tag{8}$$

$$= (-te^{-t} + 1 - e^{-t}) + te^{-t} \tag{9}$$

$$= 1 - e^{-t}. \tag{10}$$

We see that X has exponential distribution with rate 1.

2. (35 points) Certain device consists of two components, A and B. Whenever one of the components fails, the whole device is immideately replaced by a new one. Components A and B are the only components that can fail.

   Suppose that the lifetimes of components A and B (in days) are independent random variables both having exponential distributions with rate λ. Let $N(t)$ be the renewal process counting the number of the replacements of the device on the time interval $[0, t]$.

   (a) (10 points) Express the interrenewal times in terms of the lifetimes of components A and B (hint: this is **not** a sum) and compute the distribution of the interrenewal times.

   (b) (15 points) Determine an asymptotic expression for the mean age of the device at time t in the long run.

   (c) (10 points) What is the long run probability that the device will fail within next 24 hours?

   Solution. Denote by X_i and Y_i, $i \ge 1$, the random variables describing the lifetimes of the components A and B correspondingly. Then all X_i and Y_i, $i \ge 1$, are i.i.d. with exponential distribution with rate λ.

   (a) Denote by Z_i the lifetime of the i-th device. Then $Z_i = \min\{X_i, Y_i\}$ is the interrenewal time of the process that counts the number of the replacement of the device. Since X_i and Y_i are independent exponentially distributed random variables, $Z_i \sim \mathrm{Exp}(2\lambda)$.

   (b) Let δ_t denote the age of the device at time t. Then, using the limit theorem for the average age, we have

$$\lim_{t\to\infty} E(\delta_t) = \frac{\mathrm{Var}(Z_1) + (E(Z_1))^2}{2E(Z_1)} = \frac{(2\lambda)^{-2} + (2\lambda)^{-2}}{2\lambda^{-1}} = \frac{1}{2\lambda}. \tag{11}$$

(c) We have to compute the probability that $\delta_t \leq 1$ in the limit as $t \to \infty$. The limiting density of δ_t is given by

$$f(x) = 2\lambda e^{-2\lambda x}, \tag{12}$$

therefore

$$\lim_{t\to\infty} P(\delta_t \leq 1) = \int_0^1 2\lambda e^{-2\lambda x} dx = 1 - e^{-2\lambda}. \tag{13}$$

3. (35 points) The climate of a certain tropical country is characterized by the alternating periods of rain and (sunny) periods without precipitations. Let $(X_i)_{i\geq 0}$ and $(Y_i)_{i\geq 1}$ be the random variables describing the lengths of the consecutive rainy and sunny periods of time correspondingly, and assume that $(X_i)_{i\geq 0}$ and $(Y_i)_{i\geq 0}$ are two independent families of *i.i.d.* continuous random variables.

   We start the observation at the beginning of one of the rainy periods and count the number of times the weather changes from sunny to rainy. Suppose that

$$E(X_1) = \alpha, \quad \mathrm{Var}(X_1) = 2\alpha^2, \quad E(Y_1) = \beta, \quad \mathrm{Var}(Y_1) = 2\beta^2 \tag{14}$$

   for some $\alpha > 0, \beta > 0$.

   (a) (20 points) Determine an asymptotic expression (linear and constant terms) of the expected number of times the weather changes from sunny to rainy on the interval $[0, t]$ for $t \gg 1$.

   (b) (15 points) What is the long run average fraction of time that the weather in this country is sunny?

Solution.

(a) Denote by $N(t)$ the renewal process that counts the number of times the weather changes from sunny to rainy. Then the interrenewal times are given by $X_i + Y_i$. Denote

$$\mu = E(X_i + Y_i) = \alpha + \beta, \qquad \sigma^2 = \mathrm{Var}(X_i + Y_i) = 2(\alpha^2 + \beta^2). \tag{15}$$

Then the asymptotic behavior of $E(N(t))$ (see (7.17) in the textbook) yields that for $t \gg 1$

$$E(N(t)) \approx \frac{t}{\alpha+\beta} + \frac{\sigma^2 - \mu^2}{2\mu^2} = \frac{t}{\alpha+\beta} + \frac{(\alpha-\beta)^2}{2(\alpha+\beta)^2}. \tag{16}$$

(b) From the theorem for two component renewals, similarly as in the Peter principle, the long run fraction of sunny days is

$$\frac{E(Y_1)}{E(X_1+Y_1)} = \frac{\beta}{\alpha+\beta}. \tag{17}$$