

Math 180A: Introduction to Probability

Lecture A00 (Au)

math.ucsd.edu/~bau/w21.180a

Lecture B00 (Nemish)

math.ucsd.edu/~ynemish/teaching/180a

Today: ASV 2.2 (Bayes' rule)

ASV 2.3 (Independence)

Video: Prof. Todd Kemp, Fall 2019

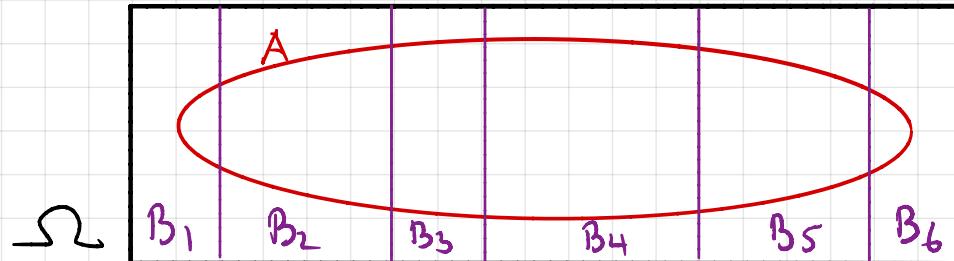
Next: ASV 1.5, 3.1

Week 2: Quiz 1 (on Wednesday, Jan 13, covering Lectures 1 and 2)

Homework 1 (due Friday, Jan 15)

Law of Total Probability

If B_1, B_2, \dots, B_n partition Ω (disjoint, $B_1 \cup \dots \cup B_n = \Omega$, $P(B_j) > 0$)



then for any event A :

$$P(A) = P(AB_1 \cup AB_2 \cup \dots \cup AB_n) = \sum_{j=1}^n P(B_j) P(A|B_j)$$

E.g. 90% of coins are fair, 9% are biased to come up heads 60%.
1% are biased to come up heads 80%.

You find a coin on the street. How likely is it to come up heads?

$$B_1 = \{\text{fair coins}\} \quad P(B_1) = 90\%$$

$$B_2 = \{60\% \text{ heads}\} \quad P(B_2) = 9\%$$

$$B_3 = \{80\% \text{ heads}\} \quad P(B_3) = 1\%$$

$$A = \{\text{heads}\}$$

$$P(A|B_1) = 50\%$$

$$P(A|B_2) = 60\%$$

$$P(A|B_3) = 80\%$$

$$\rightarrow P(A) = 51.2\%$$

Question:

90% of coins are fair. 9% are biased to come up heads 60%.
1% are biased to come up heads 80%.

2.2

You find a coin on the street. You toss it, and it comes up heads.

How likely is it that this coin is heavily biased?

$$P(B_3 | H) \neq P(H | B_3) = 80\%$$

E.g. According to Forbes Magazine, as of April 10, 2019, there are
2208 billionaires in the world.

1964 of them are men.

$$P(M | B) = \frac{1964}{2208} = 89\% \neq P(B|M)$$

Bayes' Rule (A relationship between $P(A|B)$ and $P(B|A)$)

Let B_1, B_2, \dots, B_n partition the sample space. Then for any event A with $P(A) > 0$,

$$\begin{aligned} P(B_k | A) &= P(B_k A) / P(A) \\ &= \frac{P(A | B_k) P(B_k)}{P(A)} = \frac{P(A | B_k) P(B_k)}{\sum_{j=1}^n P(A | B_j) P(B_j)} \end{aligned}$$

E.g. (Coins) $P(C_{80}|H)$

$$\begin{aligned} &= \frac{P(C_{80}H)}{P(H)} = \frac{P(H|C_{80}) P(C_{80})}{P(H)} \quad \downarrow \\ &\xrightarrow{\text{(80\%)(1\%)}} \frac{P(H|C_{80}) P(C_{80})}{P(H|C_{80}) P(C_{80}) + P(H|C_{60}) P(C_{60}) + P(H|C_0) P(C_0)} \\ &\xrightarrow{\text{II}} \frac{0.8 \cdot 0.01}{0.8 \cdot 0.01 + 0.2 \cdot 0.01 + 0} \end{aligned}$$

$\approx 1.56\%$

Epidemiological Confusion

An HIV test is 99% accurate. (1% false positives, 1% false negatives.)
0.33% of US residents have HIV.

If you test positive, what is the probability you have HIV?

(a) 99%

$$T = \{\text{positive test}\}$$

(b) 1%

$$H = \{\text{have HIV}\}$$

(c) 25%

$$\Omega = H \cup H^c$$

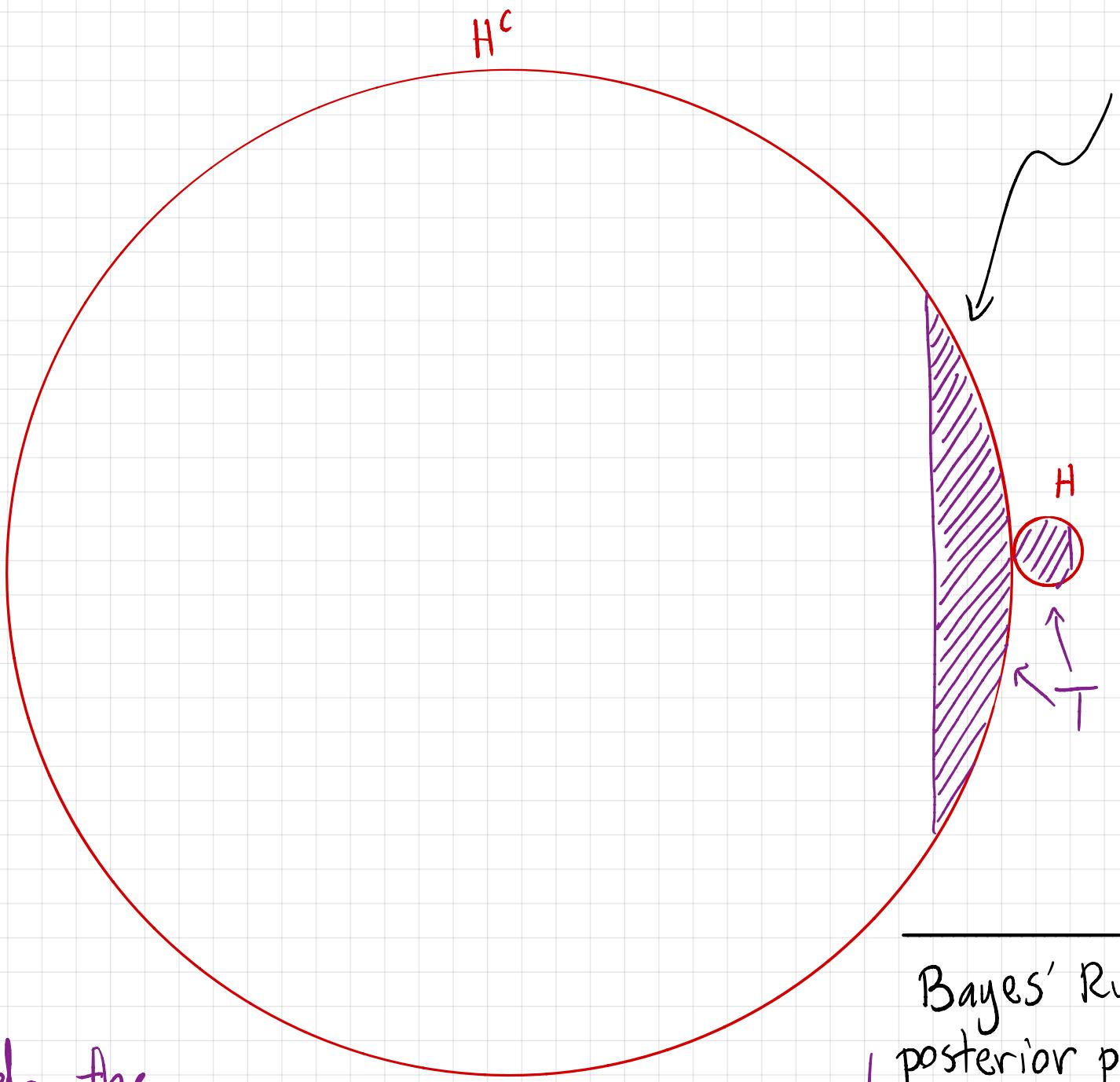
(d) 0.33%

(e) There is not enough information to answer.

$$\begin{aligned} P(H|T) &= \frac{P(HT)}{P(T)} = \frac{P(T|H)P(H)}{P(T|H)P(H) + P(T|H^c)P(H^c)} \\ &= \frac{(0.99)(0.0033)}{(0.99)(0.0033) + (0.01)199.67\%} = 24.69\% \end{aligned}$$

$$P(T|H^c) = 1\% = P(T^c|H)$$

$$P(H) = 0.33\% \quad "1 - P(T|H)"$$



even though this part of T is only 1% of H^c , it is 3 times as big as the part of T in H (which takes up 99% of H).

This is possible because H^c dwarfs H in this example.

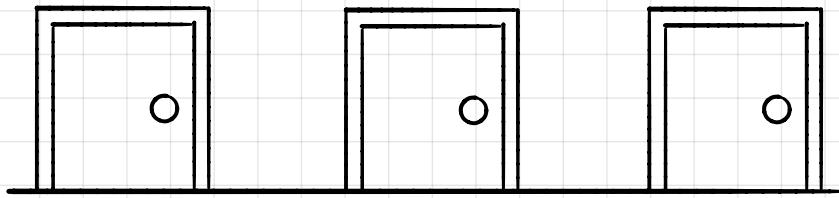
Redo the

$$\text{Calc } w \quad P(H) = 0.03 \\ \Rightarrow P(H|T) = 25.3\%$$

$$P(H) = 0.3 \\ P(H|T) = 97.7\%$$

Bayes' Rule shows that posterior probabilities are **highly sensitive** to prior inputs.

The Monty Hall Problem



At the climax of a gameshow, you are shown 3 doors. The host tells you that, behind one of them is a valuable prize (a car), while the other two hide nothing of value (goats).

You choose one. The host then opens **one of the two doors you did not choose**, revealing a goat. He then asks you if you want to stick with your original choice, or switch to the other closed door.

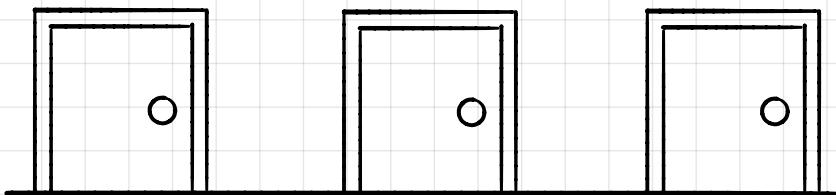
Should you switch??

(a) Yes.

(b) No.

(c) Doesn't matter.

The Monty Hall Problem



Let's decide to call the door you chose originally #1.

∴ Monty will open #2 or #3. We'll focus our analysis on #2.

$$B_i = \{ \text{the car is behind door } \#i \}$$

$$P(B_1) = P(B_2) = P(B_3) = \frac{1}{3}$$

$$A = \{ \text{Monty opens door } \#2 \}$$

$$P(A | B_2) = 0$$

$$\text{We want to know } P(B_3 | A).$$

$$P(A | B_3) = 1$$

$$P(A | B_1) = \frac{1}{2}$$

$$P(B_3 | A) = \frac{P(B_3 A)}{P(A)} = \frac{P(B_3) P(A | B_3)}{P(B_1) P(A | B_1) + P(B_2) P(A | B_2) + P(B_3) P(A | B_3)}$$

$$= \frac{\left(\frac{1}{3}\right) \cdot 1}{\left(\frac{1}{3}\right) 1 + 0 + \left(\frac{1}{3}\right) \left(\frac{1}{2}\right)} = \frac{2}{3}.$$

Suppose two events A and B really have nothing to do with each other. That doesn't mean they're disjoint; it means they have no influence on each other.

2.3

E.g. Flip a coin 3 times. $A = \{\text{the first toss is heads}\}$
 $B = \{\text{the second toss is tails}\}$.

$$A = \{ \text{HHH}, \text{HHT}, \text{HTH}, \text{HTT} \}$$

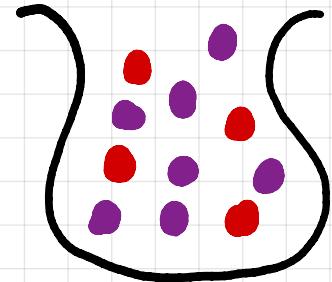
$$B = \{ \text{HTH}, \text{HTT}, \text{TTH}, \text{TTT} \}$$

$$P(AB) = \frac{2}{8} = \frac{1}{4} . \quad P(A) = P(B) = \frac{4}{8} = \frac{1}{2} .$$

$$P(B|A) = \frac{P(AB)}{P(A)} = \frac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{2} = P(B)$$

Def: Two events A, B are (statistically) independent if $P(AB) = P(A)P(B)$

E.g.



An urn has 4 red and 7 blue balls.

Two balls are sampled *with replacement

$A = \{1^{\text{st}} \text{ ball is red}\}$

*without

$B = \{2^{\text{nd}} \text{ ball is blue}\}$

Are A and B independent?

- (a) Yes.
- (b) No.
- (c) Can't tell from the question.