MATH180C: Introduction to Stochastic Processes II

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Today: Conditioning on continuous random variables

> Q&A: October 30

Next: PK 7.1, Durrett 3.1

This week:

- Homework 3 (due Saturday, October 31, 11:59 PM)

Conditioning on continuous r.v.

Def. Let X and Y be jointly distributed continuous random variables with joint probability density function $f_{X,Y}(x,y)$. We call the function

$$f_{X|Y}(x|y) := \frac{f_{X,Y}(x,y)}{f_Y(y)} \quad \text{if } f_Y(y) > 0$$

the conditional probability density function of X given $Y=y$.

The function $$F_{X|Y}(x|y) = \int_{-\infty}^{x} f_{X|Y}(s|y)\,ds$$

is called conditional distribution of X given $Y=y$

Conditional expectation

Def. Let X and Y be jointly distributed continuous random variables, let $f_{X|Y}(x|y)$ be a conditional distribution of X given $Y=y$ and let $g:\mathbb{R}\to\mathbb{R}$ be a function for which $E(|g(X)|)<\infty$.

Then we call

$$E(g(X)\,|\,Y=y):=\int_{-\infty}^{+\infty} g(x)\,f_{X|Y}(x|y)\,dx \quad \text{if } f_Y(y)>0$$

the conditional expectation of $g(X)$ given $Y=y$.

In particular, if $g(x)=\mathbb{1}_A(x)$ indicator of set A, then

$$E\left(\mathbb{1}_A(X)\,|\,Y=y\right)=P\left(X\in A\,|\,Y=y\right)=\int_A g(x)\,f_{X|Y}(x|y)\,dx$$

Remark

If Y is a continuous random variable, then

$$P(Y=y)=0 \quad \text{for all } y\in\mathbb{R}$$

Therefore, we cannot define $P(X\in A \mid Y=y)$ as

$$P(X\in A \mid Y=y) = \xcancel{\frac{P(X\in A,\ Y=y)}{P(Y=y)}}$$

On the other hand consider example:

X, Y i.i.d. r.v., $X, Y \sim \text{Unif}\,[0,1]$. Define $Z=X-Y$

If $Y=\frac{1}{2}$, $Z\sim \text{Unif}\,[-\frac{1}{2},\frac{1}{2}]$ makes perfect sense

Intuitive explanation / derivation

$$P(X\in[x,x+\Delta x],Y\in[y,y+\Delta y])$$

$$= f_{X,Y}(x,y)\,\Delta x\,\Delta y + o(\Delta x\,\Delta y)$$

Using the multiplication rule $(f_Y(y)>0 \text{ on } [y,y+\Delta y])$

$$P(X\in[x,x+\Delta x],Y\in[y,y+\Delta y])$$

$$= P(X\in[x,x+\Delta x]\mid Y\in[y,y+\Delta x])\,P(Y\in[y,y+\Delta y])$$

$$\frac{P(X\in[x,x+\Delta x]\mid Y\in[y,y+\Delta y])}{\Delta x} = \frac{P(X\in[x,x+\Delta x],Y\in[y,y+\Delta y])}{\frac{P(Y\in[y,y+\Delta y])}{\Delta y}\;\Delta x\,\Delta y}$$

Left side:

$$\Big\downarrow\ \Delta x\to 0$$

$$\text{"}f_X(x\mid Y\in[y,y+\Delta y])\text{"}$$

$$\Big\downarrow\ \Delta y\to 0$$

$$\text{"}f_X(x\mid Y=y)\text{"} = f_{X\mid Y}(x\mid y)$$

Right side:

$$\Big\downarrow\ \begin{matrix}\Delta x\to 0\\ \Delta y\to 0\end{matrix}$$

$$\frac{f_{X,Y}(x,y)}{f_Y(y)}$$

Properties of conditional probability/expectation

1) $$P(a<X<b,\ c<Y<d) = \int_{c}^{d} \left(\int_{a}^{b} f_{X|Y}(x|y)\,dx \right) f_Y(y)\,dy$$

$$= \int_{c}^{d} P(X\in(a,b) \mid Y=y) f_Y(y)\,dy$$

2) $$P(a<X<b) = \int_{-\infty}^{+\infty} \left(\int_{a}^{b} f_{X|Y}(x|y)\,dx \right) f_Y(y)\,dy$$

$$= \int_{-\infty}^{+\infty} P(X\in(a,b) \mid Y=y) f_Y(y)\,dy$$

3) $$E(g(X)) = \int_{-\infty}^{+\infty} E(g(X) \mid Y=y) f_Y(y)\,dy$$

Further properties of conditional expectation (PK, p.50)

4) $E(c_1 g_1(X_1) + c_2 g_2(X_2) | Y=y) = c_1 E(g_1(X_1) | Y=y) + c_2 E(g_2(X_2)) Y=y)$

5) $E(\nu(X,Y) | Y=y) = E(\nu(X,y) | Y=y)$

In particular, $E(\nu(X,Y)) = \int_{-\infty}^{+\infty} E(\nu(X,y) | Y=y) f_Y(y)\, dy$

6) $E(g(X)h(Y)) = \int_{-\infty}^{+\infty} h(y) E(g(X) | Y=y) f_Y(y)\, dy$

$= E(h(Y) E(g(X) | Y))$

7) $E(g(X) | Y=y) = E(g(X))$ if X and Y are independent

Example 1

Let (X,Y) be jointly continuous r.v.s with density

$$f_{X,Y}(x,y) = \frac{1}{y} e^{-\frac{x}{y}-y} \quad , \ x,y>0$$

Compute the conditional density of X given $Y=y$.

1) Compute the marginal density of Y

$$f_Y(y) = \int_0^\infty \frac{1}{y} e^{-\frac{x}{y}-y} dx = e^{-y} \int_0^\infty \frac{1}{y} e^{-\frac{x}{y}} dx = e^{-y} \quad (Y \sim Exp(1))$$

2) Compute the conditional density

$$f_{X|Y}(x|y) = \frac{\frac{1}{y} e^{-\frac{x}{y}-y}}{e^{-y}} = \frac{1}{y} e^{-\frac{x}{y}} \quad \Bigg| \Rightarrow \ \text{given } Y=y \quad X \sim Exp\left(\tfrac{1}{y}\right)$$

Example 1 (cont.)

Suppose that $Y \sim Exp(1)$, and X has exponential distribution with parameter $\frac{1}{Y}$. Compute $E(X)$

First, $E(X \mid Y=y) = y$, and using property 3)

$$E(X) = \int_0^\infty E(X \mid Y=y)\, f_y(y)\, dy$$

$$= \int_0^\infty y\, e^{-y}\, dy = 1$$

$$E(X) = \int_0^\infty \int_0^\infty x \frac{1}{y} e^{-\frac{x}{y} - y}\, dx\, dy = \int_0^\infty \Big(\overbrace{\int_0^\infty \frac{x}{y} e^{-\frac{x}{y}}\, dx}^{= E(X \mid Y=y)} \Big) e^{-y}\, dy$$

$$= \int_0^\infty y e^{-y}\, dy = 1$$

Example 2 : continuous and discrete r.v.s

Let $N \in \mathbb{N}$, $P \sim \text{Unif}[0,1]$, $X \sim \text{Bin}(N, P)$

What is the distribution of X ?

$$P(X=k) = \int_0^1 P(X=k \mid P=s) f_P(s)\, ds$$

$$= \int_0^1 P(X=k \mid P=s)\, ds$$

$$= \int_0^1 \frac{N!}{k!\,(N-k)!} \cdot s^k (1-s)^{N-k}\, ds$$

$$= \frac{N!}{k!\,(N-k)!} \cdot \frac{k!\,(N-k)!}{(N+1)!} = \frac{1}{N+1}$$

$\Rightarrow$ X is uniformly distributed $\{0, 1, \dots, N\}$

Example 3

Let X and Y be i.i.d. $\mathrm{Exp}(\lambda)$ r.v.

Define $Z = \frac{X}{Y}$. Compute the density of Z.

- If $X \sim \mathrm{Exp}(\lambda)$, then for $\alpha > 0$ $\alpha X \sim \mathrm{Exp}\left(\frac{\lambda}{\alpha}\right)$

$$P(\alpha X > t) = P\left(X > \frac{t}{\alpha}\right) = e^{-\lambda \frac{t}{\alpha}} \Rightarrow \alpha X \sim \mathrm{Exp}\left(\frac{\lambda}{\alpha}\right)$$

- $$P(Z > t) = \int_0^\infty P(Z > t \mid Y = y) f_Y(y)\, dy$$

$$= \int_0^\infty P\left(\frac{1}{y} X > t\right) \lambda e^{-\lambda y}\, dy$$

$$= \int_0^\infty \lambda e^{-\lambda y t - \lambda y}\, dy = \lambda \int_0^\infty e^{-\lambda (t+1) y}\, dy = \frac{1}{1+t}$$

$$f_Z(t) = \frac{1}{(1+t)^2}$$