MATH 142A: Introduction to Analysis

www.math.ucsd.edu/~ynemish/teaching/142a

Today: Monotone sequences
> Q&A: January 21

Next: Ross § 10

Week 3:

- Homework 2 (due Friday, January 21)

- Midterm 1 on Wednesday, January 26 (lectures 1-7)

Monotone sequences

__Def 10.1__   A sequence (s_n) is called

an **increasing sequence** if

a **decreasing sequence** if

a **monotone/monotonic sequence** if it is increasing or decreasing

Examples

$a_n = 0$

$b_n = n$

$c_n = -n$

$d_n = \frac{1}{n}$

$e_n = \frac{(-1)^n}{n}$

$f_n = \frac{n}{1+n}$

$g_n = n^2 - 4n$

$h_n = \left(1 + \frac{1}{n}\right)^n$

Bounded monotone sequences converge

Thm 10.2 All bounded monotone sequences converge.

Proof Let (s_n) be a bounded increasing sequence.
Denote $S := \{ s_n : n \in \mathbb{N} \}$. Then

(s_n) bounded $\Rightarrow$

Fix $\varepsilon > 0$. Then

①

②

Important example : the number e

Sequence $\quad a_n = \left(1 + \frac{1}{n}\right)^n$ is convergent.

Proof. ① Sequence

② Sequence b_n is bounded below :

③ ① + ② + Thm. 10.2 $\Rightarrow$ sequence $(b_n)_{n=1}^{\infty}$

④ $\lim\limits_{n \to \infty} \left(1 + \frac{1}{n}\right)^n =$

Consider the sequence $(a_n)_{n=1}^{\infty}$ given by $a_1 = \sqrt{5}$, $a_{n+1} = \sqrt{5 + a_n}$

Is (a_n) convergent? If yes, what is the limit?

① (a_n) is monotone

② (a_n) is bounded above

①+②+ Thm. 10.2 $\Rightarrow$

Unbounded monotone sequences

Thm 10.4 (i) If (s_n) is unbounded and increasing, then

(ii) If (s_n) is unbounded and decreasing, then

Proof (i) Fix $M > 0$. (s_n) unbounded $\Rightarrow$

(s_n) increasing $\Rightarrow$

Corollary 10.5 If (s_n) is a monotone sequence, then it has a limit

i.e., (s_n) converges or diverges to $+\infty$ or diverges to $-\infty$.

limsup and liminf

Let (S_n) be convergent, $\lim\limits_{n \to \infty} S_n = S$. Then $\forall \varepsilon > 0 \; \exists N$

$\forall n > N \; |S_n - S| < \varepsilon$

$\lim\limits_{n \to \infty} S_n = S$ iff $\forall \varepsilon > 0 \; \exists N$

$(u_n)_{n=1}^{\infty}$ is

$(v_n)_{n=1}^{\infty}$ is

$\Bigg| \Rightarrow \left(\lim\limits_{n \to \infty} S_n = S \Rightarrow \right.$

Def 10.6   Let (S_n) be a sequence. We define

$$\limsup_{n \to \infty} = \overline{\lim_{n \to \infty}} :=$$

$$\liminf_{n \to \infty} = \underline{\lim_{n \to \infty}} :=$$

If $\sup\{S_n : n \in \mathbb{N}\} = +\infty$, $\limsup\limits_{n \to \infty} S_n =$; if $\inf\{S_n : n \in \mathbb{N}\} = -\infty$, $\liminf\limits_{n \to \infty} S_n =$

limsup and liminf

Examples

1) $a_n = n$, $\forall N$ $\sup \{ a_n : n > N \}$ $\Rightarrow \limsup_{n \to \infty} n$

            $\forall N$ $\inf \{ a_n : n > N \}$ $\Rightarrow \liminf_{n \to \infty} n$

2) $b_n = \frac{1}{n}$, $\forall N$ $\sup \{ b_n : n > N \}$ $\Rightarrow \limsup_{n \to \infty} \frac{1}{n}$

            $\forall N$ $\inf \{ b_n : n > N \}$ $\Rightarrow \liminf_{n \to \infty} \frac{1}{n}$

3) $c_n = \frac{(-1)^n}{n}$ $\forall N$ $\sup \{ c_n : n > N \}$ $\Rightarrow \limsup_{n \to \infty} \frac{(-1)^n}{n}$

            $\forall N$ $\inf \{ c_n : n > N \}$ $\Rightarrow \liminf_{n \to \infty} \frac{(-1)^n}{n}$

4) $d_n = (-1)^n$ $\forall N$ $\sup \{ d_n : n > N \}$ $\Rightarrow \limsup_{n \to \infty} (-1)^n$

            $\forall N$ $\inf \{ d_n : n > N \}$ $\Rightarrow \liminf_{n \to \infty} (-1)^n$

5) $e_n = n^{(-1)^n}$ $\forall N$ $\sup \{ e_n : n > N \}$ $\Rightarrow \limsup_{n \to \infty} n^{(-1)^n}$

            $\forall N$ $\inf \{ e_n : n > N \}$ $\Rightarrow \liminf_{n \to \infty} n^{(-1)^n}$

Convergence and limsup/liminf

<u>Thm. 10.7</u> Let (s_n) be a sequence in $\mathbb{R}$, $s \in \mathbb{R}$ or $s \in \{+\infty, -\infty\}$.

Then 　　　　 (i) $\lim_{n \to \infty} s_n = s \Rightarrow$

　　　　　　 (ii) $\limsup_{n \to \infty} s_n = \liminf_{n \to \infty} s_n = s \Rightarrow$

<u>Proof</u> Denote $u_N = \inf\{s_n : n > N\}$, $v_N = \sup\{s_n : n > N\}$, $u = \liminf_{n \to \infty} s_n$, $v = \limsup_{n \to \infty} s_n$

(i) Three cases: $s = +\infty$, $s = -\infty$, $s \in \mathbb{R}$

　$\boxed{s = +\infty}$ Fix $M > 0$. Then $\exists N \ \forall n > N$ 　　　　, and this implies

　　　　 that

　　　　 On the other hand, $\lim_{n \to \infty} s_n = +\infty \Rightarrow$

　$\boxed{s \in \mathbb{R}}$ Fix $\varepsilon > 0$. Then $\exists N \ \forall n > N \ s - \varepsilon < s_n < s + \varepsilon$. Then

　　(a) 　　　　　　　　　　　　　　　　　(b)

　　(c)

Convergence and limsup/liminf

(ii) Three cases: $S = +\infty$, $S = -\infty$, $S \in \mathbb{R}$.

$\boxed{S = +\infty}$ $\quad \liminf_{n \to \infty} S_n = +\infty \implies \forall M \; \exists N \; \forall n > N \quad u_n > M$

Then

$\boxed{S \in \mathbb{R}}$ $\quad \liminf_{n \to \infty} S_n = \lim_{n \to \infty} u_n = c, \quad \limsup_{n \to \infty} S_n = \lim_{n \to \infty} v_n = c$