# MATH180C: Introduction to Stochastic Processes II

www.math.ucsd.edu/~ynemish/teaching/180c

## Today: Birth processes. Yule process

> Q&A: October 7

## Next: PK 6.2-6.3

Week 1:

- visit course web site
- homework 0 (due Wednesday October 7)
- homework 1 (due Friday October 9)
- join Piazza

# Continuous Time Markov Chains. Transition probabilities

<u>Def</u> (Continuous-time Markov chain)

Let $(X_t)_{t\geq 0} = (X_t : 0 \leq t < \infty)$ be a continuous time process taking values in $\mathbb{Z}_+$. $(X_t)_{t\geq 0}$ is called Markov chain if for any $n \in \mathbb{N}$, $0 \leq t_0 < t_1 < \cdots < t_{n-1} < s$, $t > 0$, $i_0, i_1, \ldots, i_{n-1}, i, j \in \mathbb{Z}_+$

$$P(X_{s+t} = j \mid X_{t_0} = i_0, X_{t_1} = i_1, \ldots, X_{t_{n-1}} = i_{n-1}, X_s = i) = P(X_{s+t} = j \mid X_s = i)$$

Markov property

We call $P_{ij}(t) := P(X_{s+t} = j \mid X_s = i) \ \left(= P(X_t = j \mid X_0 = i)\right)$

the stationary transition probability function for $(X_t)_{t\geq 0}$.

# Pure birth processes

<u>Def</u> Let $(\lambda_k)_{k \geq 0}$ be a sequence of positive numbers. We define a pure birth process as a Markov process $(X_t)_{t \geq 0}$ whose stationary transition probabilities satisfy

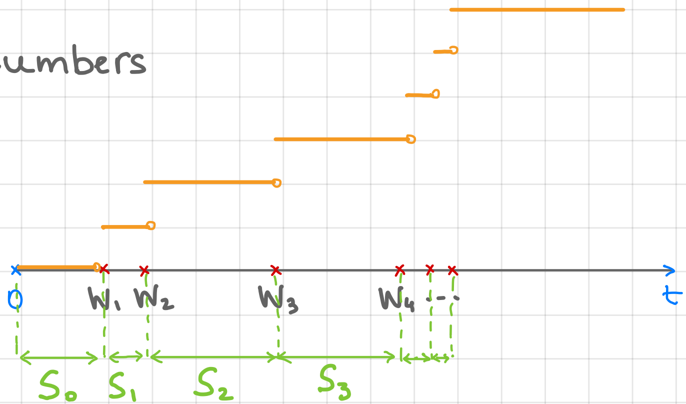
1. $P_{k,k+1}(h) = \lambda_k h + o(h)$
2. $P_{k,k}(h) = 1 - \lambda_k h + o(h)$ as $h \to 0_+$
3. $P_{k,j}(h) = 0$ for $j < k$
4. $X_0 = 0$

# Description of the birth processes via sojourn times

$(\lambda_k)_{k\geq 0}$ sequence of positive numbers

$(X_t)_{t\geq 0}$ : right-continuous

$X_t \in \mathbb{Z}_+$




Then conditions

(a) $S_0, S_1, S_2, \ldots$ are independent exponential r.v.s of rate $\lambda_0, \lambda_1, \lambda_2 \ldots$

(b) $X_{W_i} = i$ (jumps of magnitude 1)

are equivalent to

(c) $(X_t)_{t\geq 0}$ is a pure birth process with paremeters $(\lambda_k)_{k\geq 0}$.

# Birth processes and related differential equations

$P_n(t)$ satisfies the following system of differentian eqs. with initial conditions

$$(*)\quad \begin{cases} P_0'(t) = -\lambda_0 P_0(t) & P_0(0) = 1 \\ P_1'(t) = -\lambda_1 P_1(t) + \lambda_0 P_0(t) & P_1(0) = 0 \\ P_2'(t) = -\lambda_2 P_2(t) + \lambda_1 P_1(t) & P_2(0) = 0 \\ \vdots & \vdots \\ P_n'(t) = -\lambda_n P_n(t) + \lambda_{n-1} P_{n-1}(t) & P_n(0) = 0 \\ \vdots & \vdots \end{cases}$$

Solving this system gives the p.m.f. of $X_t$ for any $t$

$$P(X_t = k) = P_k(t)$$

# General solution to (*)

Assume that $\lambda_i \neq \lambda_j$ for $i \neq j$.

Then for $n \geq 1$

$$P_n(t) = \lambda_0 \cdots \lambda_{n-1} \left( B_{0n} e^{-\lambda_0 t} + \cdots + B_{nn} e^{-\lambda_n t} \right)$$

$$B_{kn} = \prod_{\substack{l=0 \\ l \neq k}}^{n} \frac{1}{\lambda_l - \lambda_k}$$

$$P_1(t) = \lambda_0 \left( \frac{1}{\lambda_1 - \lambda_0} e^{-\lambda_0 t} + \frac{1}{\lambda_0 - \lambda_1} e^{-\lambda_1 t} \right)$$

$$P_2(t) = \lambda_0 \lambda_1 \left( \frac{1}{\lambda_1 - \lambda_0} \frac{1}{\lambda_2 - \lambda_0} e^{-\lambda_0 t} + \frac{1}{\lambda_0 - \lambda_1} \frac{1}{\lambda_2 - \lambda_1} e^{-\lambda_1 t} + \frac{1}{\lambda_0 - \lambda_2} \frac{1}{\lambda_1 - \lambda_2} e^{-\lambda_2 t} \right)$$

$\vdots$

# The Yule process

**Setting:** In a certain population each individual during any (small) time interval of length $h$ gives a birth to one new individual with probability $\beta h + o(h)$, independently of other members of the population. All members of the population live forever. At time $0$ the population consists of one individual.

**Question:** What is the distribution on the size of the population at a given time $t$?

# The Yule process

Let $X_t$, $t \geq 0$, be the size of the population at time $t$.

$X_0 = 1$ (start from one common ancestor).

Compute $\tilde{P}_n(t) = P(X_t = n \mid X_0 = 1)$

If $X_t = n$, then during a time interval of length $h$

(a) $P(X_{t+h} = n+1 \mid X_t = n) = n\beta h + o(h)$

(b) $P(X_{t+h} = n \mid X_t = n) = 1 - n\beta h + o(h)$

(c) $P(X_{t+h} > n+1 \mid X_t = n) = o(h)$

all n indiv. give 0 births

(b) $P(0 \text{ births} \mid X_t = n) = (1 - \beta h + o(h))^n = 1 - n\beta h + o(h)$

(a), (b), (c) $\Rightarrow (X_t)_{t \geq 0}$ is a pure birth process with rates $\lambda_n = n\beta$

$\tilde{P}_n(t)$ satisfies the system of differential equations

# The Yule process

$$(*)\begin{cases} \tilde{P}_1'(t) = -\beta \tilde{P}_1(t) & \tilde{P}_1(0) = 1 \\ \tilde{P}_2'(t) = -2\beta \tilde{P}_2(t) + \beta \tilde{P}_1(t) & \tilde{P}_2(0) = 0 \\ \vdots & \vdots \\ \tilde{P}_n'(t) = -n\beta \tilde{P}_n(t) + (n-1)\beta \tilde{P}_{n-1}(t) & \tilde{P}_n(0) = 0 \\ \vdots & \vdots \end{cases}$$

The same system with shifted indices

$$\tilde{P}_1(t) = P_0(t) \qquad \tilde{P}_n(t) = P_{n-1}(t) \qquad \text{with } \lambda_n = \beta(n+1)$$

$$P_n(t) = \lambda_0 \cdots \lambda_{n-1}\left(B_{0n} e^{-\lambda_0 t} + \dots + B_{nn} e^{-\lambda_n t}\right) \qquad \lambda_0 \cdots \lambda_{n-1} = \beta^n n!$$

$$B_{kn} = \prod_{\substack{\ell=0 \\ \ell\neq k}}^{n} \frac{1}{\lambda_\ell - \lambda_k} \qquad B_{kn} = \prod_{\ell=0}^{k-1} \frac{1}{\lambda_\ell - \lambda_k} \prod_{\ell=k+1}^{n} \frac{1}{\lambda_\ell - \lambda_k} = \frac{1}{\beta^n} \frac{1}{(-1)^k k!} \frac{1}{(n-k)!}$$

# The Yule process

$$P_n(t) = \lambda_0 \cdots \lambda_{n-1} \left( B_{0n} e^{-\lambda_0 t} + \cdots + B_{nn} e^{-\lambda_n t} \right)$$

$$= \sum_{k=0}^{n} \cancel{\beta^n} \, n! \frac{(-1)^k}{\cancel{\beta^n} \, k! \, (n-k)!} e^{-\beta(k+1)t}$$

$$= e^{-\beta t} \sum_{k=0}^{n} \binom{n}{k} \left(-e^{-\beta t}\right)^k = e^{-\beta t} \left(1 - e^{-\beta t}\right)^n$$

$$\tilde{P}_n(t) = P_{n-1}(t) = e^{-\beta t} \left(1 - e^{-\beta t}\right)^{n-1} \leftarrow \text{Geom}\left(e^{-\beta t}\right)$$

$$X_t \sim \text{Geom}\left(e^{-\beta t}\right)$$

# Graphical representation. Exponential sojourn times