MATH180C: Introduction to Stochastic Processes II

www.math.ucsd.edu/~ynemish/teaching/180c

Today: Martingales

> Q&A: November 25

Next: PK 8.1

This week:

- Thanksgiving
- Next homework deadline: December 2 (HW 7)

Martingales

<u>Definition</u>. A stochastic process $(X_n, n \geq 0)$ is a martingale if for $n = 0, 1, \ldots$

(a)

(b)

After taking the expectation of both sides of (b) we get that

$(X_n)_{n \geq 0}$ is a martingale $\Rightarrow$

- submartingale :
- supermartingale:

Examples of martingales

(i) Let $X_1, X_2, \dots$ be independent RV's with $E(|X_k|) < \infty$ and $E(X_k) = 0$. Define $S_n = X_1 + \dots + X_n$, $S_0 = 0$.

Then

$\Rightarrow$

(ii) Let $X_1, X_2, \dots$ be independent RV with $X_k \geq 0$, $E(|X_k|) < \infty$ and $E(X_k) = 1$. Define $M_n = X_1 X_2 \cdots X_n$, $M_0 = 1$.

Then

$\Rightarrow$

Example

Stock prices in a perfect market

Let X_n be the closing price at the end of day n of a certain publicly traded security such as a share or stock. Many scholars believe that in a perfect market these price sequences should be martingales. (see PK page 73 for more details).

History and gambling

Let $(X_n)_{n\geq 0}$ be a stochastic process describing your total winnings in n games with unit stake.

Think of $X_n - X_{n-1}$ as your net winnings per unit stake in game n, $n \geq 1$, in a series of games, played at times $n = 1, 2, \dots$.

In the martingale case

Some early works of martingales was motivated by gambling. Note that there exists a betting strategy called the "martingale system" ← doubling bets after losses

Some basic properties

Let $(X_n)_{n\geq 0}$ be a martingale.

•

Proof

Exercise :

• Markov inequality : If $X_n \geq 0$ $\forall n$, then for any $\lambda > 0$

$\Rightarrow$

Maximal inequality for nonnegative martingales

Thm. Let $(X_n)_{n\geq 0}$ be a martingale with nonnegative values.
For any $\lambda>0$ and $m\in\mathbb{N}$

(1)

and

(2)

Proof. We prove (1), (2) follows by taking the limit $m\to\infty$.
Take the vector $(X_0, X_1, \dots, X_m)$ and partition the sample space wrt the index of the first r.v. rising above λ

Compute using the above partition

Proof of the maximal inequality

$E(X_m) =$

$\geq$

Compute $E(X_m \mathbb{1}_{X_0<\lambda,\dots,X_{n-1}<\lambda,X_n\geq\lambda})$ by conditioning on

$X_0, X_1, \dots, X_{n-1}, X_n$:

$E(X_m \mathbb{1}_{X_0<\lambda,\dots,X_{n-1}<\lambda,X_n\geq\lambda})$

$=$

$=$

$=$

Sum for all n

$E(X_m) \geq$

Example

A gambler begins with a unit amount of money and faces a series of independent fair games. In each game the gamblers bets fraction p of his current fortune, wins with probability $\frac{1}{2}$, loses with probability $\frac{1}{2}$. Estimate the probability that the gambler ever doubles the initial fortune.

Denote by $Z_n, n \geq 0$, the gambler's fortune after n-th game.

Denote

Then

Martingale transform

In the previous example the stake in n-th game is pZ_{n-1}. What if we choose another strategy?

<u>Def</u> Let $(X_n)_{n\geq 0}$ be a nonnegative martingale, and let $(C_n)_{n\geq 0}$ be a stochastic process with $C_n = f_n(X_0, \dots, X_{n-1})$. Then the stochastic process

is called the

Think of
- $X_k - X_{k-1}$ as the winning per unit stake in k-th game
- C_k as your stake in k-th game
  decision is made based on the previous history
- $(C \bullet X)_n$ as total winnings up to time n

Martingale transform

Prop. Let $Z_n = X_0 + (C \cdot X)_n$. Let $C_k > 0$ bounded if $Z_{k-1} > 0$ and $C_k = 0$ if $Z_{k-1} = 0$. Then $(Z_n)_{n \geq 0}$ is a martingale

Proof: $E(Z_{n+1} \mid Z_0, \dots, Z_n) =$

$=$

Note that

If $Z_n > 0$, then $C_1 > 0, \dots, C_n > 0$,

and

$E(Z_{n+1} \mid Z_0, \dots, Z_n) =$

$=$

If $Z_n = 0$, then $C_{n+1} = 0$ and $E(Z_{n+1} \mid Z_0, \dots, Z_n) = 0 = Z_n$

Gambling example:

Start from the initial fortune $X_0 = 1$. Define

$$Z_n =$$

fortune after n-th game with strategy C

Then $(Z_n)_{n\geq 0}$ is a nonnegative martingale, $E(Z_0)=1$

$$\Rightarrow$$

Convergence of nonnegative martingales

Thm.

If $(X_n)_{n\geq 0}$ is a nonnegative (super)martingale, then with probability 1

and

Example

An urn initially contains one red ball and one green ball. Choose a ball and return it to the urn together with another ball of the same color. Repeat.

Denote by X_n the fraction of red ball after n iterations.

Example (cont.)

(i) $(X_n)_{n\geq 0}$ is a martingale

Denote by R_n the number of red balls after n-th iteration

$$R_n =$$

Then

$$E(X_{n+1} \mid X_0, \dots, X_n) =$$

$$=$$

(ii) X_n is nonnegative $\Rightarrow$

(iii) Compute the distribution of X_∞

$$P\left(X_n = \frac{k}{n+2}\right) = \frac{1}{n+1} \quad \text{for } k \in \{1, 2, \dots, n+1\}$$

$$P(X_\infty \leq x) = x, \quad x \in (0,1) \Rightarrow X_\infty \sim \mathrm{Unif}[0,1)$$