MATH180C: Introduction to Stochastic Processes II

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Today: Asymptotic behaviour of renewal processes. Examples

> Q&A: November 16

Next: PK 7.5, Durrett 3.1, 3.3

This week:

- Homework 6 (due Saturday, November 21, 11:59 PM)
- Quiz 4 (Wednesday, November 18, lectures 11-15)

Remark: moments of nonnegative r.v.s

Proposition. Let X be a nonnegative random variable. Then

$$E(X^n) = n\int_0^\infty x^{n-1} P(X > x)\,dx$$

$$= n\int_0^\infty x^{n-1} (1 - F_X(x))\,dx$$

Proof. $X \geq 0 \Rightarrow X^n \geq 0$. Using the "tail" formula for the expectation of nonnegative random variables

$$E(X^n) = \int_0^\infty P(X^n > t)\,dt = \int_0^\infty P(X > t^{1/n})\,dt$$

After the change of variable $x = t^{1/n}$ we get

$$E(X^n) = n\int_0^\infty x^{n-1} P(X > x)\,dx = n\int_0^\infty x^{n-1} (1 - F_X(x))\,dx$$

$\blacksquare$

Remark. $M(t)$ is finite for all t

Proposition. Let $N(t)$ be a renewal process with interrenewal times X_i having distribution F. If there exist $c>0$ and $\alpha\in(0,1)$ such that $P(X_1>c)>\alpha$, then $M(t)=E(N(t))<\infty \quad \forall t$

Proof: Recall that $M(t)=\sum_{k=1}^{\infty} P(W_k\le t)=\sum_{k=1}^{\infty} P\left(\sum_{j=1}^{k} X_j\le t\right)$ $(*)$

Fix $t>0$, take $L\in\mathbb{N}$ such that $c\cdot L>t$. Then

$$P\left(\sum_{j=1}^{L} X_j>t\right)\ge P(X_1>c, X_2>c,\dots,X_L>c)>\alpha^L$$

$$P\left(\sum_{j=1}^{L} X_j\le t\right)\le 1-\alpha^L.$$ Thus, for any $n\in\mathbb{N}$

$P(W_{nL}\le t)=P\left(\sum_{j=1}^{nL} X_j\le t\right)\le (1-\alpha^L)^n$, from which we conclude (exercise) that $\sum_{k=1}^{\infty} P(W_k\le t)<\infty$ ∎

Example: Age replacement policies (PK, p. 363)

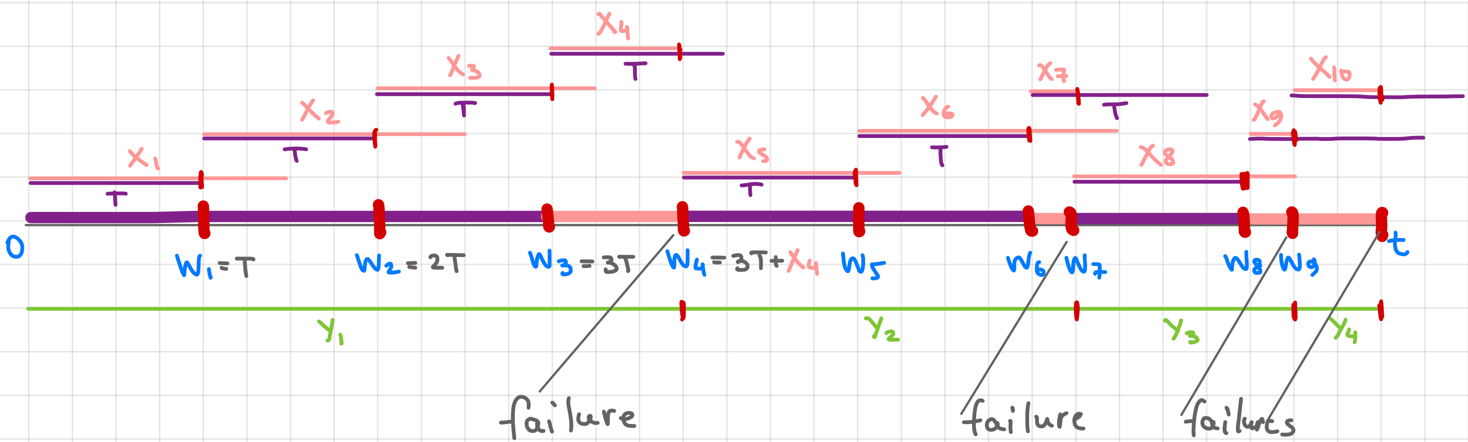
Setting:
- component's lifetime has distribution function F
- component is replaced
  (A) either when it fails,
  (B) or after reaching age T (fixed)
  whichever occurs first
- replacements (A) and (B) have different costs:
  replacement of a failed component (A) is more expensive than the planned replacement (B)

Question: How does the long-run cost of replacement depend on the cost of (A), (B) and age T?
What is the optimal T that minimizes the long-run cost of replacement?

Example: Age replacement policies (PK, p. 363)

Notation: X_i - lifetime of i-th component, $F_{X_i}(t) = F(t)$

Y_i - times between failures



Here we have two renewal processes

(1) renewal process $N(t)$ generated by renewal times $(W_i)_{i=1}^{\infty}$

(2) renewal process $Q(t)$ generated by interrenewal times $(Y_i)_{i=1}^{\infty}$

$N(t) = \#$ replacements on $[0, t]$, $Q(t) = \#$ failure replacements on $[0, t]$

Example: Age replacement policies (PK, p. 363)

Compute the distribution of the interrenewal times for $N(t)$

$$W_i - W_{i-1} = \begin{cases} X_i, & \text{if } X_i \le T \\ T, & \text{if } X_i > T \end{cases}, \text{ so}$$

$$F_T(x) := P(W_i - W_{i-1} \le x) = \begin{cases} F(x), & x < T \\ 1, & x \ge T \end{cases}$$

In particular,

$$E(W_i - W_{i-1}) = \int_0^T (1 - F(x))\,dx =: \mu_T \le \mu = E(X_i)$$

Using the elementary renewal theorem for $N(t)$, the total number of replacements has a long-run rate

$$\frac{E(N(t))}{t} \approx \frac{1}{\mu_T} \quad \text{for large } t$$

Example: Age replacement policies (PK, p. 363)

Compute the distribution of the interrenewal times for $Q(t)$.

$$Y_1 = \begin{cases} X_1 & \text{if } X_1 \le T \\ T + X_2 & \text{if } X_1 > T,\ X_2 \le T \\ \vdots & \\ nT + X_{n+1}, & \text{if } X_1 > T, \dots, X_n > T,\ X_{n+1} \le T \\ \vdots & \end{cases}$$

so $Y_1 = LT + Z$, where $P(L \ge n) = (1 - F(T))^n$, $Z \in [0, T]$

and for $z \in [0, T]$

$$P(Z \le z) = P(X_1 \le z, \cancel{X_1 \le T}) + P(X_2 \le z, X_1 > T, \cancel{X_2 \le T})$$

$$+ \cdots + P(X_{n+1} \le z, X_1 > T, \dots, X_n > T, \cancel{X_{n+1} \le T}) + \cdots$$

$$= P(X_1 \le z) + P(X_2 \le z) P(X_1 > T) + \cdots + P(X_{n+1} \le z) P(X_1 > T) \cdots P(X_n > T) + \cdots$$

$$= F(z)\left(1 + (1 - F(T)) + (1 - F(T))^2 + \cdots + \cdots\right) = \frac{F(z)}{F(T)}$$

Example: Age replacement policies (PK, p. 363)

Now we can compute the long-run rate of the replacements due to failures

$$E(Y_1) = T\,E(L) + E(Z)$$

$$E(L) = \sum_{n=1}^{\infty} P(L \geq n) = \sum_{n=1}^{\infty} (1-F(T))^n = \frac{1-F(T)}{F(T)}$$

$$E(Z) = \frac{\int_0^{\infty} F(T)-F(x)\,dx}{F(T)} \quad , \quad \text{so}$$

$$E(Y_1) = \frac{1}{F(T)}\left(T(1-\cancel{F(T)}) + \int_0^T (\cancel{F(T)}-F(x))\,dx\right) = \frac{\mu_T}{F(T)}$$

Applying the elementary renewal theorem to $Q(t)$

$$\frac{E(Q(t))}{t} \approx \frac{F(T)}{\mu_T} \quad \text{for large } t$$

Example: Age replacement policies (PK, p. 363)

Suppose that the cost of one replacement is K, and each replacement due to a failure costs additional c. Then, in the long run the total amount spent on the replacements of the component per unit of time is given by

$$C(T) \approx K \cdot \frac{1}{\mu_T} + c \cdot \frac{F(T)}{\mu_T} = \frac{K + c\,F(T)}{\int_0^T (1 - F(x))\,dx}$$

If we are given c, K and the distribution of the component's lifetime F, we can try to minimize the overall costs by choosing the optimal value of T.

Example: Age replacement policies (PK, p. 363)

For example, if $K=1$, $c=4$ and $X_1 \sim \mathrm{Unif}[0,1]$ $\left(F(x) = x \mathbb{1}_{[0,1]} \right)$

For $T \in [0,1]$, $\mu_T = \int_0^T (1-x)\,dx = T\left(1 - \frac{T}{2}\right)$ and

the average (per unit of time) long-run costs are

$$C(T) = \frac{1+4T}{T\left(1-\frac{T}{2}\right)}$$

$$\frac{d}{dT} C(T) = \frac{4T\left(1-\frac{T}{2}\right) - (1+4T)(1-T)}{\left(T\left(1-\frac{T}{2}\right)\right)^2} = \frac{2T^2+T-1}{\left(T\left(1-\frac{T}{2}\right)\right)^2} = 0$$

$2T^2 + T - 1 = 0$

$T_{min} = \frac{1}{2}$

$C(T_{min}) = 8$

$C(1) = 10 > 8$