MATH180C: Introduction to Stochastic Processes II

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Today: Introduction to renewal processes

> Q&A: November 2

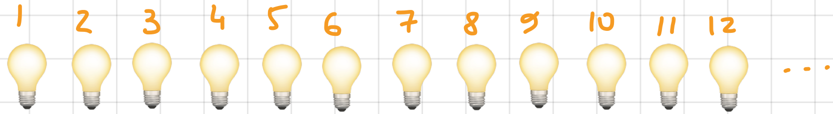
Next: PK 7.2-7.3, Durrett 3.1

This week:

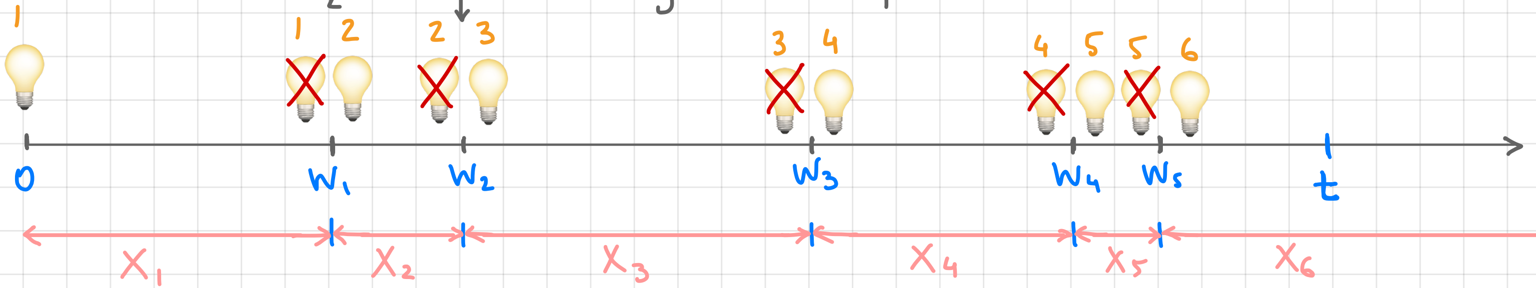
- Quiz 3 (November 4)
- Homework 4 (due Friday, November 6, 11:59 PM)

Renewal process

Imagine lightbulbs



"renewal" = lightbulb replacement



X_i – lifetime of the lightbulb #i, W_i = time of i-th "renewal"

Lightbulbs are identical $\Rightarrow$ X_i are i.i.d.

Let $N(t)$ denote the number of renewals up to time t

- What are the properties of $(N(t))_{t\geq 0}$?
- How they depend on the distribution of X_i?

Renewal process. Definition

Def. Let $\{X_i\}_{i\geq 1}$ be i.i.d. r.v.s, $X_i > 0$.

Denote $W_n := X_1 + \dots + X_n$, $n \geq 1$, and $W_0 := 0$.

We call the counting process

$$N(t) = \#\{k : W_k \leq t\} = \max\{n : W_n \leq t\}$$

the renewal process.

Remarks. 1) W_n are called the waiting/renewall times

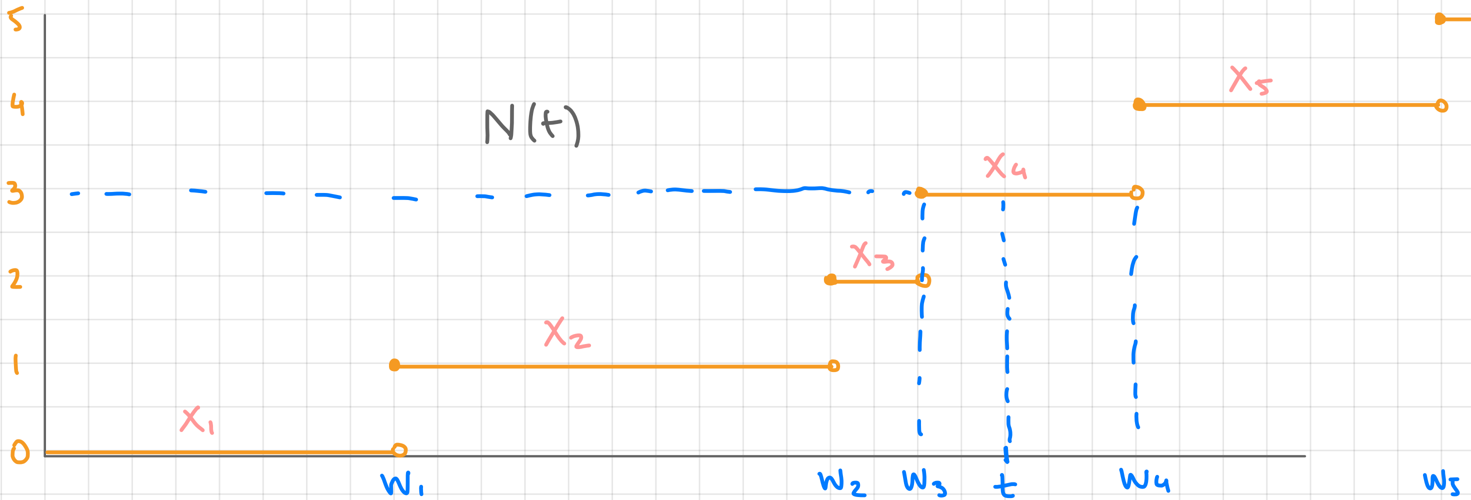
X_i are called the interrenewal times

2) $N(t)$ is characterised by the distribution of $X_i > 0$

3) More generally, we can define for $0 \leq a < b < \infty$

$$N((a,b]) = \#\{k : a < W_k \leq b\}$$

Renewal process. Definition



$N(t) \geq k$ if and only $W_k \leq t$ $\quad (*)$

Remarks 1) $(*)$ implies that $(N(t))_{t\geq 0}$ is determined by $(W_k)_{k\geq 0}$, so sometimes $(W_k)_{k\geq 0}$ is called renewal process

2) For any t, $\quad W_{N(t)} \leq t < W_{N(t)+1}$

Convolutions of c.d.f.s

Suppose that X and Y are independent r.v.s

$F: \mathbb{R} \to [0,1]$ is the c.d.f. of X (i.e. $P(X \leq t) = F(t)$).

$G: \mathbb{R} \to [0,1]$ is the c.d.f. of Y

- if Y is discrete, then

$$F_{X+Y}(t) = P(X+Y \leq t) = \sum_k P(X+Y \leq t \mid Y=k)\,P(Y=k)$$

$$= \sum_k P(X+k \leq t)\,P(Y=k) = \sum_k P(X \leq t-k)\,P(Y=k)$$

$$= \sum_k F(t-k)\,P(Y=k) = \int_{-\infty}^{+\infty} F(t-x)\,dG(x) = F * G(t)$$

- if Y is continuous, then

$$F_{X+Y}(t) = P(X+Y \leq t) = \int_{-\infty}^{+\infty} P(X+y \leq t)\,f_Y(y)\,dy$$

$$= \int_{-\infty}^{+\infty} F(t-y)\,f_Y(y)\,dy = \int_{-\infty}^{+\infty} F(t-x)\,dG(x) = F * G(t)$$

Distribution of W_k

Let $X_1, X_2, \ldots$ be i.i.d. r.v.s, $X_i > 0$, and let $F : \mathbb{R} \to [0,1]$ be the c.d.f. of X_i (we call F the interoccurrence or interrenewal distribution). Then

- $F_1(t) := F_{W_1}(t) = P(W_1 \leq t) = P(X_1 \leq t) = F(t)$
- $F_2(t) := F_{W_2}(t) = F_{X_1+X_2}(t) = F * F(t)$
- $F_3(t) := F_{W_3}(t) = F_{(X_1+X_2)+X_3}(t) = (F*F)*F =: F^{*3}(t)$
- More generally,

$$F_n(t) := F_{W_n}(t) = P(W_n \leq t) = F^{*n}(t) \leftarrow \text{n-fold convolution of } F$$

<u>Remark</u>: $$F^{*(n+1)}(t) = \int_0^t F^{*n}(t-x)\, dF(x) = \int_0^t F(t-x)\, dF^{*n}(x)$$

Renewal function

<u>Def</u>. Let $(N(t))_{t\geq 0}$ be a renewal process with interrenewal distribution F. We call

$$M(t) = E(N(t))$$

the **renewal function**.

<u>Proposition</u> 1. $M(t) = \sum_{k=1}^{\infty} F_k(t) = \sum_{k=1}^{\infty} F^{*k}(t)$

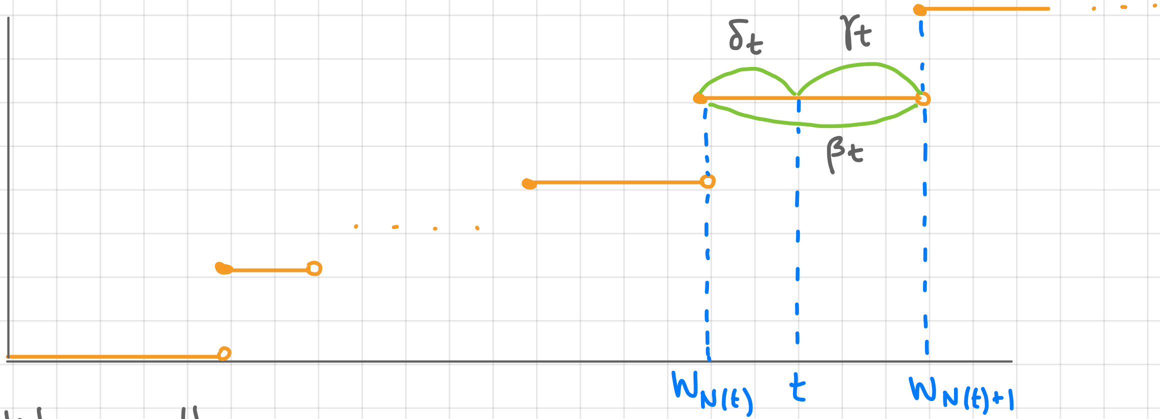
<u>Proof</u>. $M(t) = E(N(t)) = \sum_{k=1}^{\infty} P(N(t) \geq k)$

$$= \sum_{k=1}^{\infty} P(W_k \leq t)$$

$$= \sum_{k=1}^{\infty} F_k(t) = \sum_{k=1}^{\infty} F^{*k}(t).$$

Related quantities

Let $N(t)$ be a renewal process.



Def. We call

- $\gamma_t := W_{N(t)+1} - t$ the excess (or residual) lifetime
- $\delta_t := t - W_{N(t)}$ the current life (or age)
- $\beta_t := \gamma_t + \delta_t$ the total life

Remarks

1) $\gamma_t > h \geq 0$ iff $N(t+h) = N(t)$

2) $t \geq h$ and $\delta_t \geq h$ iff $N(t-h) = N(t)$

Expectation of W_n

Proposition 2. Let $N(t)$ be a renewal process with interrenewal times $X_1, X_2, \dots$ and renewal times $(W_n)_{n\geq 1}$. Then

$$E(W_{N(t)+1}) = E(X_1)\;E(N(t)+1)$$
$$= \mu(M(t)+1)$$

where $\mu = E(X_1)$.

Proof. $E(W_{N(t)+1}) = E(X_1 + X_2 + \dots + X_{N(t)+1}) = \underbrace{E(X_1)}_{=\mu} + E(X_2 + \dots + X_{N(t)+1})$

$$E(X_2 + \dots + X_{N(t)+1}) = E(X_2 \mid N(t)=1)P(N(t)=1)$$
$$+ E(X_2 + X_3 \mid N(t)=2)\;P(N(t)=2)$$
$$+ E(X_2 + X_3 + X_4 \mid N(t)=3)\;P(N(t)=3)$$
$$\vdots$$
$$+ E\left(\sum_{k=2}^{n+1} X_k \mid N(t)=n\right)P(N(t)=n) + \cdots$$

$$= \sum_{n=1}^{\infty} E(X_2 \mid N(t)=n)\,P(N(t)=n) + \sum_{n=2}^{\infty} E(X_3 \mid N(t)=n)\,P(N(t)=n) +$$

Expectation of W_n

$$E\left(\sum_{j=2}^{N(t)+1} X_j\right) = \sum_{j=2}^{\infty} \sum_{n=j-1}^{\infty} E\left(X_j \mid N(t)=n\right) P(N(t)=n)$$

$$= \sum_{j=2}^{\infty} E\left(X_j \mid N(t) \geq j-1\right) P(N(t) \geq j-1)$$

Since $\quad N(t) \geq j-1 \iff W_{j-1} \leq t \iff X_1 + X_2 + \cdots + X_{j-1} \leq t$

$$= \sum_{j=2}^{\infty} E\left(X_j \mid X_1 + \cdots + X_{j-1} \leq t\right) P(N(t) \geq j-1)$$

independent

$$= \sum_{j=2}^{\infty} E(X_j)\, P(N(t) \geq \overset{=\ell}{j-1}) = \mu \sum_{\ell=1}^{\infty} P(N(t) \geq \ell)$$

$$= \mu\, E(N(t)) = \mu M(t) \qquad \blacksquare$$

Remark For proof in PK take $1 = \sum_{i=1}^{\infty} \mathbb{1}_{\{N(t)=i\}}$.

Renewal equation

<u>Proposition 3.</u> Let $(N(t))_{t\geq 0}$ be a renewal process with interrenewal distribution F. Then $M(t) = E(N(t))$ satisfies

$$M(t) = F(t) + M * F(t) = F(t) + \int_0^t M(t-x)\,dF(x)$$

renewal equation

<u>Proof.</u> We showed in Proposition 1 that

$$M = \sum_{n=1}^{\infty} F^{*n}.$$

Then

$$M * F = \left(\sum_{n=1}^{\infty} F^{*n}\right) * F = \sum_{n=2}^{\infty} F^{*n} = \sum_{n=1}^{\infty} F^{*n} - F$$

$$= M - F \qquad \blacksquare$$