Joint density

of a linear transformation of a random vector (change of variable in a multivariate integral, no proof).

<u>Lemma</u>. Let U be an n-dimensional random vector, $B \in \mathbb{R}^{n \times n}$, $\det B \neq 0$, $\vec{c} \in \mathbb{R}^n$. Let $f_U(u_1, \dots, u_n)$ be the joint density of U. Define $V := BU + \vec{c}$. Then

$$f_V(v_1, \dots, v_n) = f_U\left(B^{-1}(\vec{v} - \vec{c})\right) \frac{1}{|\det B|} \qquad (*)$$

<u>Example</u> Suppose we have r.v.s X_1, X_2 with known joint density $f_{X_1,X_2}(x_1, x_2)$ Define $Y_1 = X_1 + X_2$, $Y_2 = X_1 - X_2$ $\rightarrow$
Then we can compute the joint density of Y_1 and Y_2 using $(*)$ with $U = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$, $V = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = BU$ and $B = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

Example cont.

To apply (*) we need (i) det B and (ii) B^{-1}

(i) $\det B = -2$

(ii) $B^{-1} = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{pmatrix}$, so $B^{-1}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}y_1 + \frac{1}{2}y_2 \\ \frac{1}{2}y_1 - \frac{1}{2}y_2 \end{pmatrix}$

The from (*) we have that the joint density of Y_1 and Y_2 is given by

$$f_{Y_1,Y_2}(y_1,y_2) = f_{X_1,X_2}\left(\frac{1}{2}y_1 + \frac{1}{2}y_2, \frac{1}{2}y_1 - \frac{1}{2}y_2\right) \cdot \frac{1}{|-2|} =$$

$$= \frac{1}{2} f_{X_1,X_2}\left(\frac{1}{2}y_1 + \frac{1}{2}y_2, \frac{1}{2}y_1 - \frac{1}{2}y_2\right)$$