

MATH 180A: Introduction to Probability

Lecture A00 (Au)

www.math.ucsd.edu/~bau/w21.180a

Lecture B00 (Nemish)

www.math.ucsd.edu/~ynemish/teaching/180a

Today: Definition of Probability.
Random sampling.

Next: ASV 1.2

Video: Prof. Todd Kemp, Fall 2019

Week 1:

- Homework 0 (due Friday, January 8)
- Join Piazza

The world around us is fundamentally **random**. 1.1

- * 1600s - games of chance
- * 1900s - quantum theory
- * 1950s - finance / insurance
- * 1980s - chemical reactions inside our cells
- * 2000s - complex networks; machine learning

The modern rigorous foundation of probability theory goes back to **1933** Kolmogorov

Ingredients

Sample Space

Ω = the set of possible outcomes in an experiment. $\{\text{HH}, \text{HT}, \text{TH}, \text{TT}\}$

Events

\mathcal{F} = collections of outcomes. $E = \{\text{at least 1 H}\}$
 $= \{\text{HH}, \text{HT}, \text{TH}\}$

Probability Measure $P : \mathcal{F} \rightarrow [0, 1]$

Kolmogorov's Axioms

$$P(E \cup T) = \frac{2}{3}$$

$$\begin{aligned} P(E) + P(T) &= \frac{1}{2} + \frac{1}{3} \\ &= \frac{5}{6} \end{aligned}$$

(i) For any event $A \subseteq \mathcal{F}$, $0 \leq P(A) \leq 1$.

(ii) $P(\Omega) = 1$ $P(\emptyset) = 0$

(iii) If A_1, A_2, A_3, \dots are disjoint events, then

$$P(A_1 \cup A_2 \cup A_3 \cup \dots) = P(A_1) + P(A_2) + P(A_3) + \dots$$

E.g. A fair die. $\Omega = \{1, 2, 3, 4, 5, 6\}$

$$P(\{1\}) = P(\{2\}) = \dots = P(\{6\}) = \frac{1}{6}$$

$E = \text{even}$, $O = \text{odd}$

$$E = \{2, 4, 6\} \quad O = \{1, 3, 5\}$$

$T = \text{divisible by 3}$

$$T = \{3, 6\}$$

$$P(E \cup T)$$

$$= P\{2, 3, 4, 6\}$$

$$= P\{2\} + P\{3\} + P\{4\} + P\{6\}$$

$$= 4 \cdot \frac{1}{6} = \frac{2}{3}.$$

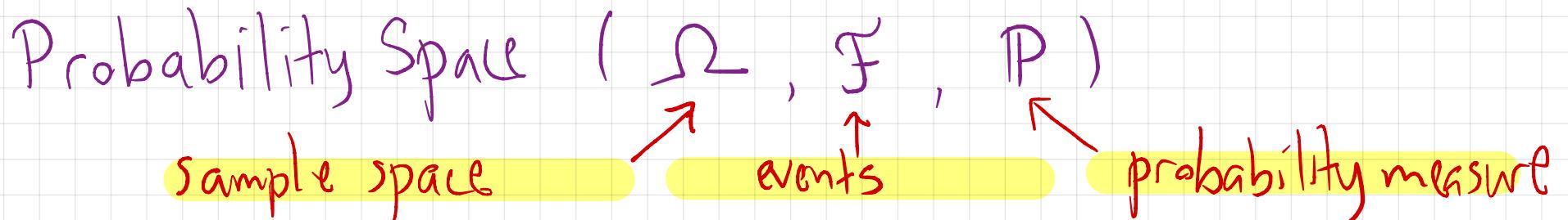
$$E \cup O = \Omega$$

$$E \cup T = \{2, 3, 4, 6\}$$

$$P(E) = \frac{3}{6} = \frac{1}{2}$$

$$P(T) = \frac{2}{6} = \frac{1}{3}.$$

↑



Determined by the experiment being modeled

Key property: if A_1, A_2, A_3, \dots are disjoint ($A_i \cap A_j = \emptyset$)
 $\underbrace{A_1, A_2, A_3, \dots}_{\in \mathcal{F}}$
 $i \neq j$

$$\text{then } P(A_1 \cup A_2 \cup A_3 \cup \dots) = \sum_j P(A_j)$$

Important special case: if $\#\Omega < \infty$ then

$$\Omega = \{\omega_1, \omega_2, \omega_3, \dots, \omega_n\}$$

the singleton sets $\{\omega_1\}, \{\omega_2\}, \dots, \{\omega_n\}$ are disjoint

$$\therefore P(A) = \{a_1\} \cup \{a_2\} \cup \dots \cup \{a_r\} \quad P(A) = \sum_{j=1}^r P(\{a_j\})$$

$A = \{a_1, a_2, \dots, a_r\}$

Uniform Probability Measure (Sampling)

1.2

If Ω is finite, the uniform probability measure is defined by:

$$\text{For each } w \in \Omega, P(\{w\}) = \frac{1}{\#\Omega}$$

$$\Rightarrow \text{For any event } A, P(A) = \frac{\#A}{\#\Omega}$$

This means calculating probabilities in such models is tantamount to Counting.

E.g. A fair die is cast 2 times. What is the probability that the sum is 4?

$$\Omega = \{(i,j) : 1 \leq i, j \leq 6\} \quad \#\Omega = 36 \quad P(A) = \frac{3}{36}$$

$$A = \{(1,3), (2,2), (3,1)\} \quad \#A = 3 \quad = \frac{1}{12}$$

(sum is 4)

$\approx 8.3\%$

E.g. A fair coin is tossed 3 times.

$$A = \{\text{at least two tails}\}$$

$$B = \{\text{exactly two tails}\}$$

$$\begin{aligned}\Omega &= \{HHH, HHT, \dots, TTT\} \\ &= \{(i,j,k) : i, j, k \in \{T, H\}\}\end{aligned}$$

$$A = \{TTH, THT, HTT, TTT\}$$

$$B = \{TTH, THT, HTT\}$$

$$P(A) = \frac{\#A}{8} = \frac{4}{8} = \frac{1}{2}$$

$$P(B) = \frac{3}{8}.$$

THINK PAIR SHARE

There are 10 people on a committee.

How many different ways are there to select a subcommittee of 4 people?

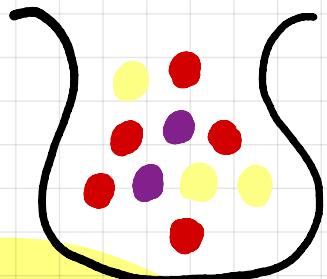
(a) $10 \cdot 10 \cdot 10 \cdot 10 = 10^4 = 10,000$

(b) $10 \cdot 9 \cdot 8 \cdot 7 = 5,040$

(c) $\binom{10}{4} = \frac{10 \cdot 9 \cdot 8 \cdot 7}{4!} = 210$

(d) $\frac{10!}{4!} = 151,200$

Combinatorics



A collection of labeled balls $\{1, 2, \dots, n\}$ are in an urn. k are taken out one by one

- * with replacement, or
 - * without replacement ($k \leq n$)
- They are lined up
- $b_i \neq b_j$ if $i \neq j$
- * in the order they came out, or
 - * disregarding order.
- $$\Omega = \left\{ \{b_1, \dots, b_k\} : \dots \right\}$$

How many ways?

* with replacement : n^k

* without : { & ordered : $n \cdot (n-1) \cdot (n-2) \cdots (n-k+1)$

not :
$$\frac{n(n-1)(n-2)\cdots(n-k+1)}{k!} = \frac{n!}{k!(n-k)!} = \binom{n}{k}$$

when $k=n$
$n!$

Sampling with Replacement

Toss a fair coin n times; record a statistic observing
 $\# H$ vs. $\# T$.

E.g. $n = 10$, $P\{\text{odd rolls are all } H\}$.

$$\Omega = \{(c_1, c_2, c_3, \dots, c_{10}) : \forall j \quad c_j \in \{H, T\}\}$$

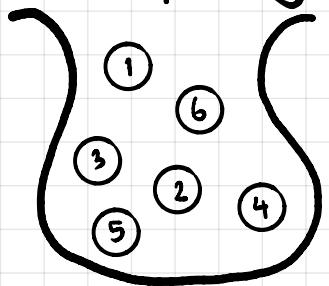
$$\#\Omega = 2^{10}$$

$$\begin{aligned} A &= \{c_1 = c_3 = c_5 = c_7 = c_9 = H\} \\ &= \{(H, *, H, *, H, *, H, *, H, *)\} \end{aligned}$$

$$\#A = 2^5$$

$$P(A) = \frac{2^5}{2^{10}} = \frac{1}{2^5} = \frac{1}{32}.$$

Sampling without Replacement (order matters)



There are 6 labeled balls in an urn.
3 are removed in sequence (without replacement), and lined up in order.

What is the probability that the first two are (3, 6)?

$$\Omega = \{(b_1, b_2, b_3) : 1 \leq b_j \leq 6, b_1 \neq b_2, b_2 \neq b_3, b_1 \neq b_3\}$$

$$\#\Omega = 6 \cdot 5 \cdot 4 = 120$$

$$A = \{(3, 6, *)\} \quad \underbrace{\in \{1, 2, 4, 5\}}_{\#\Lambda = 4}$$

$$\begin{aligned} P(A) &= \frac{4}{120} \\ &= \frac{1}{30}. \end{aligned}$$