MATH180C: Introduction to Stochastic Processes II

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Today: Asymptotic behaviour of renewal processes

> Q&A: November 9, 13

Next: PK 7.5, Durrett 3.1, 3.3

This week:

- Homework 5 (due Friday, November 13, 11:59 PM)

Last time

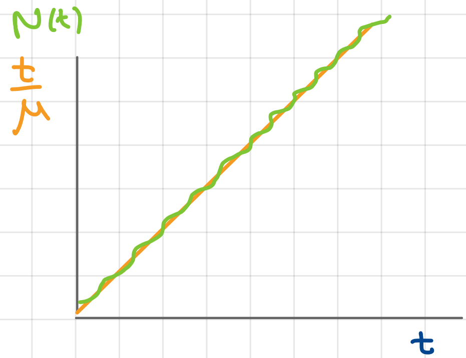
Let $N(t)$ be a renewal process with interrenewal times X_i, $E(X_i)=\mu$.

Thm.

$$P\left(\lim_{t\to\infty} N(t)=\infty\right)=1$$

Thm (Pointwise renewal thm).

$$P\left(\lim_{t\to\infty}\frac{N(t)}{t}=\frac{1}{\mu}\right)=1$$



Thm (CLT for renewal processes)

If $Var(X_i)=\sigma^2$, then

$$\lim_{t\to\infty} P\left(\frac{N(t)-\frac{t}{\mu}}{\sqrt{\frac{\sigma^2}{\mu^3}t}}\le x\right)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{y^2}{2}}\,dy$$

Last time (cont.)

Thm. (Elementary renewal thm)

If $M(t) = E(N(t))$ and $E(X_1) = \mu$, then

$$\lim_{t\to\infty} \frac{M(t)}{t} = \frac{1}{\mu} \qquad \left(M(t) \approx \frac{t}{\mu} \text{ for large } t \right)$$

Thm. If $Var(X_1) = \sigma^2$, then

$$\lim_{t\to\infty} \frac{Var(N(t))}{t} = \frac{\sigma^2}{\mu^3}$$

Elementary renewal theorem and continuous X_i's

Two more results (without proofs) about the limiting behaviour of $M(t)$ for models with continuous interrenewal times.

<u>Thm</u>. Let $E(X_1)=\mu$ and let $m(t)=\frac{d}{dt}M(t)$ be the renewal density. Then

$$\lim_{t\to\infty} m(t) = \lim_{t\to\infty} \frac{dM(t)}{dt} = \frac{1}{\mu}$$

<u>Remark</u> $\lim_{t\to\infty} \frac{f(t)}{t} = \alpha$ does not imply in general $\lim_{t\to\infty} f'(t) = \alpha$

(E.g., take $f(t) = t + \sin(t)$)

<u>Thm</u>. If additionally $Var(X_1)=\sigma^2$, then

$$\lim_{t\to\infty}\left(M(t) - \frac{t}{\mu}\right) = \frac{\sigma^2-\mu^2}{2\mu^2}$$

Example: $X_i \sim \text{Gamma}(2,1)$

Let $N(t)$ be a renewal process with interrenewal times X_i having Gamma distribution with parameters $(2,1)$ i.e., $f_{X_1}(t) = t\,e^{-t}$. Then from the properties of the Gamma distribution (or from direct computations)

$$X_1 + \cdots + X_n \sim \text{Gamma}(2n, 1), \text{ so}$$

$$f^{*n}(t) = \frac{t^{2n-1}}{(2n-1)!} e^{-t}, \text{ for } t > 0$$

We can compute the renewal density

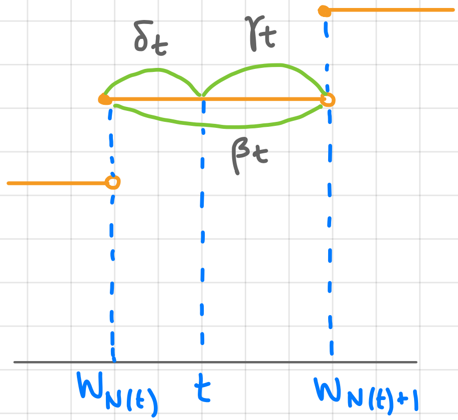
$$m(t) = \sum_{n=1}^{\infty} f^{*n}(t) = \sum_{n=1}^{\infty} \frac{t^{2n-1}}{(2n-1)!} e^{-t} = \left(\frac{e^{t} - e^{-t}}{2} \right) e^{-t} = \frac{1 - e^{-2t}}{2}$$

so that $M(t) = \int_0^t m(x)\,dx = \frac{t}{2} - \frac{1}{4}\left(1 - e^{-2t}\right)$

Finally, $E(X_1) = \mu = 2$, $\text{Var}(X_1) = \sigma^2 = 2$, so $\frac{\sigma^2 - \mu^2}{2\mu^2} = -\frac{2}{2 \cdot 4} = -\frac{1}{4}$

Joint distribution of age and excess life

From the definition of γ_t and δ_t



$P(\delta_t \geq x, \gamma_t > y)$ $\quad (x \leq t)$

$$= P(W_{N(t)} \leq t - x, W_{N(t)+1} > t + y)$$

- Partition wrt the values of $N(t)$

$$= \sum_{k=0}^{\infty} P(W_k \leq t - x, W_{k+1} > t + y)$$

condition on the value of W_k (c.d.f. of W_k is $F^{*k}(t)$)

$$= 1 - F(t+y) + \sum_{k=1}^{\infty} \int_0^{\infty} P(W_k \leq t - x, W_k + X_{k+1} > t + y \mid W_k = u) \, dF^{*k}(u)$$

$$= 1 - F(t+y) + \sum_{k=1}^{\infty} \int_0^{t-x} P(X_{k+1} > t + y - u) \, dF^{*k}(u)$$

$$= 1 - F(t+y) + \sum_{k=1}^{\infty} \int_0^{t-x} (1 - F(t+y-u)) \, dF^{*k}(u)$$

Limiting distribution of age and excess life

Assume that X_i are continuous. Then

$$P(\delta_t \geq x, \gamma_t > y) = 1 - F(t+y) + \sum_{k=1}^{\infty} \int_0^{t-x} (1 - F(t+y-u))\, dF^{*k}(u)$$

$$= 1 - F(t+y) + \int_0^{t-x} (1 - F(t+y-u))\, d\underbrace{\sum_{k=1}^{\infty} F^{*k}(u)}_{= M(u)}$$

$$= 1 - F(t+y) + \int_0^{t-x} (1 - F(\underbrace{t+y-u}_{= w}))\, m(u)\, du$$

$$= 1 - F(t+y) + \int_{y+x}^{y+t} (1 - F(w))\, m(t+y-w)\, dw$$

Recall that $\varepsilon(s) := m(s) - \frac{1}{\mu} \to 0$ as $s \to \infty$ $(\mu = E(X_1))$. Then

$$\lim_{t\to\infty} P(\delta_t \geq x, \gamma_t > y) = \lim_{t\to\infty} \left[1 - F(t+y) + \int_{y+x}^{y+t} (1 - F(w)) \left\{ \frac{1}{\mu} + \varepsilon(t+y-w) \right\} dw \right]$$

$$= \int_{y+x}^{\infty} (1 - F(w)) \frac{1}{\mu}\, dw + \lim_{t\to\infty} \int_{y+x}^{y+t} (1 - F(w))\, \varepsilon(t+y-w)\, dw$$

0 Exercise

Joint/ limiting distribution of (γ_t, δ_t)

Thm. Let $F(t)$ be the c.d.f. of the interrenewal times. Then

(a) $$P(\gamma_t > y, \delta_t \geq x) = 1 - F(t+y) + \sum_{k=1}^{\infty} \int_0^{t-x} (1 - F(t+y-u))\, dF^{*k}(u)$$

$$= 1 - F(t+y) + \int_0^{t-x} (1 - F(t+y-u))\, dM(u)$$

(b) if additionally the interrenewal times are continuous,

$$\lim_{t \to \infty} P(\gamma_t > y, \delta_t \geq x) = \frac{1}{\mu} \int_{x+y}^{\infty} (1 - F(w))\, dw \qquad (*)$$

If we denote by $(\gamma_\infty, \delta_\infty)$ a pair of r.v.s with distribution $(*)$ then γ_∞ and δ_∞ are continuous r.v.s with densities

$$f_{\gamma_\infty}(x) = f_{\delta_\infty}(x) = \frac{1}{\mu} (1 - F(x))$$

Example

Renewal process (counting earthquakes in California) has interrenewal times uniformly distributed on [0,1] (years).

(a) What is the long-run probability that an earthquake will hit California within 6 months?

$$\lim_{t \to \infty} P(\gamma_t \leq 0.5) = \int_0^{0.5} 2 \cdot (1 - x)\, dx = 1 - x^2 \Big|_0^{0.5} = 0.75$$

(b) What is the long-run probability that it has been at most 6 months since the last earthquake?

$$\lim_{t \to \infty} P(\delta_t \leq 0.5) = \int_0^{0.5} 2 \cdot (1 - x)\, dx = 0.75$$

Key renewal theorem

Suppose $H(t)$ is an unknown function that satisfies

$$H(t) = h(t) + H * F(t) \quad (*)$$

renewal equation

E.g.: $M(t) = F(t) + M * F(t)$,

$m(t) = f(t) + m * F(t) = f(t) + m * f(t)$

Remark about notation

- Convolution with c.d.f.: $g * F(t) = \int_{-\infty}^{+\infty} g(t-x)\, dF(x)$
- Convolution with p.d.f.: $g * f(t) = \int_{-\infty}^{+\infty} g(t-x) f(x)\, dx$

Def. Function h is called locally bounded if $\max\limits_{0 \le x \le t} |h(x)| < \infty \ \forall t$

Def. Function h is absolutely integrable if

$$\int_{-\infty}^{+\infty} |h(x)|\, dx < \infty$$

Key renewal theorem

Thm (Key renewal theorem) Let h be locally bounded.

(a) If H satisfies $H = h + h * M$, then H is locally bounded and

$$H = h + H * F \quad (*)$$

(b) Conversely, if H is a locally bounded solution to $(*)$, then

$$H = h + h * M \quad (**)$$

[convolution in the Riemann-Stieltjes sense]

(c) If h is absolutely integrable, then

$$\lim_{t \to \infty} H(t) = \frac{\int_0^\infty h(x)\,dx}{\mu}$$

No proof.

Remark. Key renewal theorem says that if h is locally bounded, then there exists a unique locally bounded solution to $(*)$ given by $(**)$

Examples

- Renewal function: $M(t)$ satisfies $M = F + M * F$

  and $M = F + F * M$

  $F(t)$ is nondecreasing, so (c) does not apply to the renewal equation for $M(t)$

- Renewal density: $m(t)$ satisfies $m = f + m * f$

  and $m = f + f * m$

  $= f + f * M$ (in the Riemann-Stieltjes sense)

  f is absolutely integrable, $\int_0^\infty f(x)\,dx = 1$, so

$$\lim_{t \to \infty} m(t) = \frac{\int_0^\infty f(x)\,dx}{\mu} = \frac{1}{\mu}$$

Important remark

Let $W = (W_1, W_2, \dots)$ be arrival times of a renewal process, and denote $W' = (W_1', W_2', \dots)$ with

$$W_i' = W_{i+1} - W_1 = X_2 + X_3 + \dots + X_{i+1},$$

shifted arrival times.

Then:

- W' is independent of $W_1 = X_1$, and
- W' has the same distribution as W

Example

Example. Compute $\lim_{t\to\infty} E(\gamma_t)$. Take $H(t) = E(\gamma_t)$

If $X_1 > t$, then $\gamma_t = X_1 - t$; if $X_1 \le t$ condition on $X_1 = s$

$$E(\gamma_t) = E\left((X_1 - t)\mathbb{1}_{X_1 > t}\right) + E\left(\gamma_t \mathbb{1}_{X_1 \le t}\right)$$

$$E\left(\gamma_t \mathbb{1}_{X_1 \le t}\right) = \int_0^\infty P\left((W_{N(t)+1} - t)\mathbb{1}_{X_1 \le t} > \omega\right) dw$$

$$= \int_0^\infty \sum_{k=1}^\infty P\left((W_k - t)\mathbb{1}_{X_1 \le t} > w,\ N(t) = k-1\right) dw$$

$$= \int_0^\infty \sum_{k=2}^\infty P\left(\left(X_1 + \sum_{j=2}^k X_j - t\right)\mathbb{1}_{X_1 \le t} > w,\ N(t) = k-1\right) dw$$

$$= \int_0^\infty \left[\sum_{k=2}^\infty \int_0^t P\left(\overbrace{\sum_{j=2}^k X_j}^{=W'_{k-1}} - (t-s) > w,\ \overbrace{N(t) = k-1}^{\Leftrightarrow N'(t-s) = k-2}\right) dF(s)\right] dw$$

$$= \int_0^t \left[\int_0^\infty \underbrace{\sum_{l=1}^\infty P\left(W'_l - (t-s) > w,\ N'(t-s) = l-1\right)}_{= P(\gamma'_{t-s} > w)} dw\right] dF(s) = \underbrace{\int_0^t E(\gamma_{t-s})\, dF(s)}_{= H * F(t)}$$

Example (cont)

Assume that $E(X_1)=\mu$, $Var(X_1)=\sigma^2$

$$E\left((X_1-t)\mathbb{1}_{X_1>t}\right) = \int_t^\infty (x-t)\,dF(x) = \int_t^\infty (t-x)\,d(1-F(x))$$

$$= (t-x)(1-F(x))\Big|_t^\infty + \int_t^\infty (1-F(x))\,dx$$

0

Since we assume that $Var(X_1)=\sigma^2$, $2\int_0 x(1-F(x))\,dx = E(X_1^2)<\infty$

and $x(1-F(x))\to 0$, as $x\to\infty$

Finally, we have that

$$H(t) = \int_t^\infty (1-F(x))dx + H*F(t)$$

therefore $H(t) = h(t) + h*M(t)$

with $h(t) = \int_t^\infty (1-F(x))\,dx$

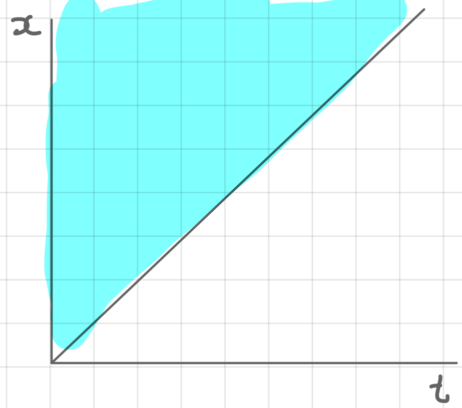
Example (cont)

In particular,

$$\int_0^\infty \int_t^\infty (1-F(x))\,dx\,dt = \int_0^\infty \left(\int_0^x (1-F(x))\,dt \right) dx$$

$$= \int_0^\infty (1-F(x))\,x\,dx = \frac{1}{2} E(X_1^2)$$

$$= \frac{1}{2}(\sigma^2+\mu^2) \Rightarrow h(t) \text{ is absolutely integrable}$$



$\Rightarrow$ by part (c) of the key renewal theorem

$$\lim_{t\to\infty} E(\gamma_t) = \frac{\sigma^2+\mu^2}{2\mu}$$

Similarly $\lim_{t\to\infty} E(\delta_t) = \dfrac{\sigma^2+\mu^2}{2\mu}$, $\lim_{t\to\infty} E(\beta_t) = \dfrac{\sigma^2+\mu^2}{\mu} > \mu$

Example

What is the expected time to the next earthquake in the long run?

For $X_1 \sim \text{Unif}[0,1]$

$$\int_0^1 x^2\,dx = \frac{1}{3} = \sigma^2 + \mu^2$$

therefore, $\lim_{t\to\infty} E(\gamma_t) = \dfrac{\frac{1}{3}}{2 \cdot \frac{1}{2}} = \frac{1}{3}$

And the long run expected time between two consecutive earthquakes is $\frac{2}{3} > \frac{1}{2} = E(X_1)$