

☐ Write your name and PID on the top of **EVERY PAGE**.

☐ Write the solutions to each problem on separate pages. **CLEARLY INDICATE** on the top of each page the number of the corresponding problem.

☐ Remember this exam is graded by a human being. Write your solutions **NEATLY AND COHERENTLY**, or they risk not receiving full credit.

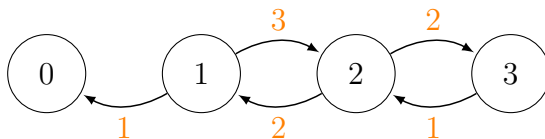
☐ You may assume that all transition probability functions are **STATIONARY**.

☐ You are allowed to use the textbook, lecture notes and your personal notes. You are not allowed to use the electronic devices (except for accessing the online version of the textbook) or outside assistance. Outside assistance includes but is not limited to other people, the internet and unauthorized notes.

1. (25 points) Let $(X_t)_{t \geq 0}$ be a continuous time Markov chain on the state space $\{0, 1, 2\}$ with transition probability functions

$$P(t) = \begin{matrix} & \begin{matrix} 0 & 1 & 2 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \end{matrix} & \left\| \begin{matrix} e^{-3t} & \frac{1}{2} - e^{-3t} + \frac{1}{2}e^{-4t} & \frac{1}{2} - \frac{1}{2}e^{-4t} \\ 0 & \frac{1}{2} + \frac{1}{2}e^{-4t} & \frac{1}{2} - \frac{1}{2}e^{-4t} \\ 0 & \frac{1}{2} - \frac{1}{2}e^{-4t} & \frac{1}{2} + \frac{1}{2}e^{-4t} \end{matrix} \right\| \end{matrix}. \quad (1)$$

- (a) (7 points) Compute the generator Q of $(X_t)_{t \geq 0}$. [Hint. Recall that $P'(0) = Q$.]
 (b) (6 points) Give the jump-and-hold description of $(X_t)_{t \geq 0}$.
 (c) (6 points) Draw the rate diagram of $(X_t)_{t \geq 0}$.
 (d) (6 points) Assuming that X_0 is uniformly distributed on $\{0, 1, 2\}$, compute the probability that $X_1 = 2$.
2. (25 points) Let $(X_t)_{t \geq 0}$ be a birth and death process on $\{0, 1, 2, 3\}$ described by the following rate diagram



Compute the mean time to absorption starting from state 1 (i.e., given $X_0 = 1$).

3. (25 points) Let $(X_t)_{t \geq 0}$ be a continuous time Markov chain on the state space $\{0, 1, 2\}$ with generator

$$Q = \begin{matrix} & \begin{matrix} 0 & 1 & 2 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \end{matrix} & \left\| \begin{matrix} -2 & 2 & 0 \\ 3 & -6 & 3 \\ 0 & 1 & -1 \end{matrix} \right\| \end{matrix}. \quad (2)$$

- (a) (10 points) Draw the diagram for the jump chain of $(X_t)_{t \geq 0}$ and explain why $(X_t)_{t \geq 0}$ is irreducible.
 (b) (10 points) Compute the stationary distribution for $(X_t)_{t \geq 0}$.
 (c) (5 points) What is the expected average fraction of time that $(X_t)_{t \geq 0}$ will spend in states 1 and 2 in the long run?

4. (25 points) Certain printing facility has two printers operating on a 24/7 basis and one repairman that takes care of the printers. The amount of time (in hours) that a printer works before breaking down has exponential distribution with mean 2. If a printer is broken, the repairman needs exponentially distributed amount of time with mean 1 (hour) to repair the broken printer. The repairman cannot repair two printers simultaneously. Each printer can produce 100 pages per minute.

Let X_t denote the number of printers in operating state at time t .

- (a) (10 points) Assuming without proof that $(X_t)_{t \geq 0}$ is a Markov process, determine the generator of $(X_t)_{t \geq 0}$ (you can provide rigorous computations for only one entry of matrix Q .)

[Hint. If $T \sim \text{Exp}(\gamma)$, then $P(T \leq h) = \gamma h + o(h)$ as $h \rightarrow 0$.]

- (b) (10 points) Compute the stationary distribution for $(X_t)_{t \geq 0}$.
- (c) (5 points) In the long run, how many pages does the facility produce on average per minute?