MATH 180A Introduction to Probability - FINAL

Winter 2021

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1 Final 10 AM

1.1 Problem 1

1. (20 points) Three archers have one arrow each. The probability of hitting the target for the first archer is $\frac{4}{5}$, for the second $\frac{3}{4}$, and for the third $\frac{2}{3}$.

   All three archers shoot simultaneously, and two of them hit the target.

   What is the probability that the archer that missed the target is the third archer?

   (Hint: Consider the events $A_i = \{i\text{-th archer missed}\}$).

   Solution. Consider the events

$$A_i = \{i\text{-th archer missed}\} \quad \text{and} \quad B = \{\text{exactly two hit the target}\}. \tag{1.1}$$

We are asked to compute the conditional probability

$$P(A_3|B). \tag{1.2}$$

By definition of the conditional probability

$$P(A_3|B) = \frac{P(A_3 \cap B)}{P(B)}. \tag{1.3}$$

Events $A_3 \cap B$ and B can be written in terms of events A_i as

$$A_3 \cap B = \overline{A_1} \cap \overline{A_2} \cap A_3, \quad B = \left(A_1 \cap \overline{A_2} \cap \overline{A_3}\right) \cup \left(\overline{A_1} \cap A_2 \cap \overline{A_3}\right) \cup \left(\overline{A_1} \cap \overline{A_2} \cap A_3\right). \tag{1.4}$$

Notice that the unions in (1.4) are disjoint.

Using the properties of the probability measure

$$P(A_1 \cap \overline{A_2} \cap \overline{A_3}) = \frac{1}{5} \cdot \frac{3}{4} \cdot \frac{2}{3} = \frac{6}{60} \tag{1.5}$$

$$P(\overline{A_1} \cap A_2 \cap \overline{A_3}) = \frac{4}{5} \cdot \frac{1}{4} \cdot \frac{2}{3} = \frac{8}{60} \tag{1.6}$$

$$P(\overline{A_1} \cap \overline{A_2} \cap A_3) = \frac{4}{5} \cdot \frac{3}{4} \cdot \frac{1}{3} = \frac{12}{60}, \tag{1.7}$$

therefore

$$P(A_3|B) = \frac{\frac{12}{60}}{\frac{6+8+12}{60}} = \frac{12}{26} = \frac{6}{13}. \tag{1.8}$$

1.2 Problem 2

2. (20 points) Let X be a random variable uniformly distributed on the interval $(0, 1)$. Let $Y = -\log \sqrt{X}$.

   (a) (10 points) Compute the CDF of Y.

(b) (6 points) Compute the density of Y and $E(Y)$.

(c) (4 points) What can you say about the distribution of the random variable $-\log X^\alpha$ for general $\alpha > 0$?

Solution.

(a)

$$F_Y(t) = P(Y \leq t) = P(-\log\sqrt{X} \leq t) = P(\sqrt{X} \geq e^{-t}) = P(X \geq e^{-2t}). \tag{1.9}$$

Since $X \sim \text{Unif}(0,1)$, we get

$$F_Y(t) = \begin{cases} 0, & t < 0, \\ 1 - e^{-2t}, & t \geq 0. \end{cases} \tag{1.10}$$

(b) Using the result of part (a) we have that Y is a continuous random variable taking values on the set $(0, +\infty)$ with density

$$f_Y(x) = \begin{cases} 0, & x < 0, \\ 2e^{-2x}, & x \geq 0. \end{cases} \tag{1.11}$$

We see that $Y \sim \text{Exp}(2)$, and thus $E(Y) = \frac{1}{2}$.

(c) If we now repeat the computations from part (a) with $\sqrt{X} = X^{1/2}$ replaced by X^α, we get that for $Y_\alpha := -\log X^\alpha$ we get that

$$F_{Y_\alpha}(t) = \begin{cases} 0, & t < 0, \\ 1 - e^{-t/\alpha}, & t \geq 0. \end{cases} \tag{1.12}$$

We conclude that $Y_\alpha \sim \text{Exp}(\frac{1}{\alpha})$.

1.3 Problem 3

3. (20 points) Let X be a continuous random variable with density

$$f_X(x) = \begin{cases} xe^{-x}, & x \geq 0, \\ 0, & \text{otherwise}. \end{cases} \tag{1.13}$$

(a) (15 points) Compute the moment generating function of X.

(b) (5 points) Compute $E(X)$, $E(X^2)$, $E(X^3)$.

Solution.

(a) By definition

$$M_X(t) = E(e^X) = \int_0^\infty e^{tx} x e^{-x} dx = \int_0^\infty x e^{(t-1)x} = \begin{cases} +\infty, & t \geq 1, \\ (1-t)^{-2}, & t < 1. \end{cases} \tag{1.14}$$

(b) We can compute the moments of X by computing the derivative of $M_X(t)$ at $t = 0$

$$M'_X(t) = 2(1-t)^{-3}, \quad M'_X(0) = E(X) = 2. \tag{1.15}$$

$$M''_X(t) = 6(1-t)^{-4}, \quad M''_X(0) = E(X^2) = 6. \tag{1.16}$$

$$M^{(3)}_X(t) = 24(1-t)^{-5}, \quad M^{(3)}_X(0) = E(X^3) = 24. \tag{1.17}$$

More generally, one can show that $E(X^n) = (n+1)!$.

1.4 Problem 4

4. (20 points) The joint probability mass function of the discrete random variables X and Y is given by the following table

		X		
		2	4	6
	1	0.2	0	0.2
Y	2	0	0.2	0
	3	0.2	0	0.2

(a) (10 points) Show that X and Y are uncorrelated.

(b) (5 points) Show that X and Y are not independent.

(c) (5 points) Change the *second row* in the joint probability mass function table above to make X and Y independent.

Solution.

(a) First, compute the marginal probability mass functions for X and Y

$$P(X = 2) = 0.4, P(X = 4) = 0.2, P(X = 6) = 0.4, \tag{1.18}$$

$$P(Y = 1) = 0.4, P(Y = 2) = 0.2, P(Y = 3) = 0.4. \tag{1.19}$$

Now, compute $E(X)$ and $E(Y)$

$$E(X) = 2 \cdot 0.4 + 4 \cdot 0.2 + 6 \cdot 0.4 = 0.8 + 0.8 + 2.4 = 4, \tag{1.20}$$

$$E(Y) = 1 \cdot 0.4 + 2 \cdot 0.2 + 3 \cdot 0.4 = 2, \tag{1.21}$$

and $E(XY)$

$$E(XY) = 0.2(2 \cdot 1 + 2 \cdot 3 + 4 \cdot 2 + 6 \cdot 1 + 6 \cdot 3) = 0.2(2 + 6 + 8 + 6 + 18) = 8. \tag{1.22}$$

Using the formula for the covariance of X and Y gives

$$\mathrm{Cov}(X, Y) = E(XY) - E(X)E(Y) = 8 - 4 \cdot 2 = 0, \tag{1.23}$$

which means that X and Y are uncorrelated.

(b) We have that $P(X=2,Y=2)=0\neq P(X=2)P(Y=2)=0.08$, therefore X and Y are not independent.

(c) In order for X and Y to be independent, we need that the following equalities hold for all possible values k and ℓ of X and Y correspondingly

$$P(X=k,Y=\ell)=P(X=k)P(Y=\ell). \tag{1.24}$$

Since we are allowed to change only the second row, the marginal probability mass function of Y will remain unchanged, and since the first row does not change, we new table (for independent X and Y) has to satisfy (1.24) with $\ell=1$, which gives the new marginal PMF for X

$$P(X=2,Y=1)=0.2=P(X=2)\cdot 0.4 \quad\Rightarrow\quad P(X=2)=0.5, \tag{1.25}$$
$$P(X=4,Y=1)=0=P(X=2)\cdot 0.2 \quad\Rightarrow\quad P(X=4)=0, \tag{1.26}$$
$$P(X=6,Y=1)=0.2=P(X=2)\cdot 0.4 \quad\Rightarrow\quad P(X=6)=0.5. \tag{1.27}$$

Applying now (1.24) for $Y=2$ and the new PMF for X gives

$$P(X=2,Y=2)=0.5\cdot 0.2=0.1, \tag{1.28}$$
$$P(X=4,Y=2)=0\cdot 0.2=0, \tag{1.29}$$
$$P(X=6,Y=2)=0.5\cdot 0.2=0.1. \tag{1.30}$$

Therefore, as one can check, changing the second row to $(0.1,0,0.1)$ makes the random variables X and Y independent.

1.5 Problem 5

5. (20 points) Let X be a continuous nonnegative random variable with the following moments

$$E(X)=\frac{4}{3},\quad E(X^2)=2,\quad E(X^3)=4. \tag{1.31}$$

(a) (10 points) Show that

$$P(X\geq 3)\leq\frac{4}{27}. \tag{1.32}$$

(b) (10 points) Estimate $P(X\geq 3)$ using Chebyshev's inequality and compare it with the estimate obtained in part (a).

Solution.

(a) Using Markov's inequality applied to X^3 we have

$$P(X\geq 3)=P(X^3\geq 27)\leq\frac{E(X^3)}{27}=\frac{4}{27}. \tag{1.33}$$

(b) In order to use the Chebyshev's inequality, we first need to compute the variance of X

$$\mathrm{Var}(X) = E(X^2) - (E(X))^2 = 2 - \left(\frac{4}{3}\right)^2 = \frac{2}{9}. \tag{1.34}$$

Now the Chebyshev's inequality gives

$$P(X \geq 3) = P\Big[\left(X - \frac{4}{3}\right)^2 \geq \left(3 - \frac{4}{3}\right)^2\Big] \leq \frac{\frac{2}{9}}{\frac{25}{9}} = \frac{2}{25}. \tag{1.35}$$

The bound obtained in part (b) is better than the bound obtained in part (a).

2 Final 7 PM

2.1 Problem 1

6. (20 points) Urn A has 3 white and 2 black balls. Urn B has 3 white and 1 black balls.

We choose two balls at random from urn A and put them into urn B. After that we choose a ball at random from urn B.

Compute the probability that the ball chosen from urn B is white.

Solution. Consider the events that describe the possible combinations of balls chosen from urn A:

$$A_1 = \{\text{2 white balls}\}, \quad A_2 = \{\text{1 white ball, 1 black ball}\}, \quad A_3 = \{\text{2 black balls}\}, \tag{2.1}$$

and denote

$$B = \{\text{white ball chosen from urn B}\}. \tag{2.2}$$

We are asked to compute the probability of the event B, and for this we use the law of total probability: events A_1, A_2, A_3 give a partition of the sample space, so

$$P(B) = P(B|A_1)P(A_1) + P(B|A_2)P(A_2) + P(B|A_3)P(A_3). \tag{2.3}$$

Now we compute each of the probabilities in (2.3)

$$P(B|A_1) = \frac{5}{6}, \quad P(B|A_2) = \frac{4}{6}, \quad P(B|A_3) = \frac{3}{6}. \tag{2.4}$$

For the probabilities of A_i, we use the formula for sampling without replacement, order doesn't matter

$$P(A_1) = \frac{\binom{3}{2}\binom{2}{0}}{\binom{5}{2}} = \frac{3}{10}, \quad P(A_2) = \frac{\binom{3}{1}\binom{2}{1}}{\binom{5}{2}} = \frac{6}{10}, \quad P(A_3) = \frac{\binom{3}{0}\binom{2}{2}}{\binom{5}{2}} = \frac{1}{10}. \tag{2.5}$$

Plugging this into (2.3) gives

$$P(B) = \frac{5}{6}\cdot\frac{3}{10} + \frac{4}{6}\cdot\frac{6}{10} + \frac{3}{6}\cdot\frac{1}{10} = \frac{15+24+3}{60} = \frac{42}{60} = 0.7. \tag{2.6}$$

7. (20 points) An urn has one ball, which can be with equal probability white or black.

On the first step, we take a white ball and add it to the urn. Now the urn contains two balls.

On the second step, we choose one ball at random from the urn and observe that the chosen ball is white.

What is the probability that the other ball, that remains inside the urn, is also white?

Solution. Consider the events

$$A_1 = \{\text{The first ball was white}\}, \quad A_2 = \{\text{The first ball was black}\}, \tag{2.7}$$

and

$$B = \{\text{Ball chosen on the second step is white}\}. \tag{2.8}$$

With this notation, we are asked to compute the conditional probability $P(A_1|B)$. Events A_1, A_2 for a partition of the sample space, therefore we use the Bayes' formula

$$P(A_1|B) = \frac{P(B|A_1)P(A_1)}{P(B|A_1)P(A_1) + P(B|A_2)P(A_2)}. \tag{2.9}$$

It is given that

$$P(A_1) = P(A_2) = \frac{1}{2}. \tag{2.10}$$

Now

$$P(B|A_1) = 1, \quad P(B|A_2) = \frac{1}{2}. \tag{2.11}$$

Plugging this into (2.9) we get

$$P(A_1|B) = \frac{1 \cdot \frac{1}{2}}{1 \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2}} = \frac{2}{3}. \tag{2.12}$$

2.2 Problem 2

8. (20 points) Let X be a random variable describing the lifetime of a certain component. We know that X is a continuous random variable with PDF (in years)

$$f_X(x) = \begin{cases} \frac{2}{x^3}, & x \in [1, +\infty), \\ 0, & \text{otherwise.} \end{cases} \tag{2.13}$$

The component is replaced either when it fails, or after 4 years of service even if it is still operational.

Denote by Y the time the component is in operation (i.e., from being installed to being replaced). Compute $E(Y)$.

Solution. The time in operation Y is equal to X if X is less than 4 years, or equal to 4 if X is greater than 4 years. Therefore, we can express Y as a function of X

$$Y = g(X), \tag{2.14}$$

where

$$g(x) = \begin{cases} x, & x \le 4, \\ 4, & x > 4. \end{cases} \tag{2.15}$$

We can thus compute $E(Y)$ as $E(g(X))$

$$E(Y) = \int_1^{+\infty} g(x) f_X(x) dx = \int_1^4 x \cdot \frac{2}{x^3} dx + \int_4^{+\infty} 4 \cdot \frac{2}{x^3} dx \tag{2.16}$$

The first integral gives

$$\int_1^4 \frac{2}{x^2} dx = -\frac{2}{x}\Big|_1^4 = 2 - \frac{1}{2} = \frac{3}{2}, \tag{2.17}$$

the second integral gives

$$\int_4^{+\infty} \frac{8}{x^3} dx = -\frac{8}{2x^2}\Big|_4^{\infty} = \frac{1}{4}. \tag{2.18}$$

Therefore,

$$E(Y) = \frac{3}{2} + \frac{1}{4} = \frac{7}{4}. \tag{2.19}$$

9. (20 points) Let X be a random variable describing the lifetime of a certain component. We know that X is a continuous random variable with PDF (in years)

$$f_X(x) = \begin{cases} \frac{3}{x^4}, & x \in [1, +\infty), \\ 0, & \text{otherwise}. \end{cases} \tag{2.20}$$

The component is replaced either when it fails, or after 2 years of service even if it is still operational.

Denote by Y the time the component is in operation (i.e., from being installed to being replaced). Compute $E(Y)$.

Solution. The time in operation Y is equal to X if X is less than 2 years, or equal to 2 if X is greater than 2 years. Therefore, we can express Y as a function of X

$$Y = g(X), \tag{2.21}$$

where

$$g(x) = \begin{cases} x, & x \le 2, \\ 2, & x > 2. \end{cases} \tag{2.22}$$

We can thus compute $E(Y)$ as $E(g(X))$

$$E(Y) = \int_1^{+\infty} g(x) f_X(x) dx = \int_1^2 x \cdot \frac{3}{x^4} dx + \int_2^{+\infty} 2 \cdot \frac{3}{x^4} dx \tag{2.23}$$

The first integral gives

$$\int_1^2 \frac{3}{x^3} dx = -\frac{3}{2x^2}\Big|_1^2 = \frac{3}{2} - \frac{3}{8} = \frac{9}{8}, \tag{2.24}$$

the second integral gives

$$\int_2^{+\infty} \frac{6}{x^4} dx = -\frac{6}{3x^3}\Big|_2^{\infty} = \frac{1}{4}. \tag{2.25}$$

Therefore,

$$E(Y) = \frac{9}{8} + \frac{1}{4} = \frac{11}{8}. \tag{2.26}$$

2.3 Problem 3

10. (20 points) According to the US Department of Treasury, one in every 10,000 US dollar notes is counterfeit.

    A cash-in-transit van operating in San Diego area transports 20,000 US dollar notes from a supermarket to a bank.

    Estimate the probability that there are at least 3 counterfeit notes in this van.

    [Explain your choice of approximation. You may leave the answers in terms of $\Phi(x)$ or e^x. Do not use the continuity correction.]

    Solution. We model the number of the counterfeit notes in the van using a random variable X having binomial distribution with parameters $n = 20000$ and $p = 10^{-4}$, $X \sim \text{Bin}(20000, 10^{-4})$. We are asked to compute the probability that $X \geq 3$. For this, we use the Poisson approximation with $\lambda = np = 20000 \cdot 10^{-4} = 2$, so that for any $k \in \{0, 1, 2, \ldots\}$

    $$P(X = k) \approx \frac{2^k}{k!} e^{-2}. \tag{2.27}$$

    We choose the Poisson approximation since $np^2 = 2 \cdot 10^{-4}$ is much smaller than 1, and

    $$npq = 20000 \cdot 10^{-4}(1 - 10^{-4}) \approx 2, \tag{2.28}$$

    which is not enough to guarantee a good normal approximation.

    In order to estimate $P(X \geq 3)$ we use the complement formula together with the Poisson approximation

    $$P(X \geq 3) = 1 - [P(X = 0) + P(X = 1) + P(X = 2)] \approx 1 - \left[e^{-2} + \frac{2}{1!} e^{-2} + \frac{4}{2!} e^{-2}\right] = 1 - 5e^{-2}. \tag{2.29}$$

11. (20 points) About one in every thousand wild oysters has a pearl inside.

    You have just bought a container of 3,000 wild oysters.

    Estimate the probability that you will find at least 2 pearls in this container of oysters.

    [Explain your choice of approximation. You may leave the answers in terms of $\Phi(x)$ or e^x. Do not use the continuity correction.]

    Solution. We model the number of pearls in the container using a random variable X having binomial distribution with parameters $n = 3000$ and $p = 10^{-3}$, $X \sim \text{Bin}(3000, 10^{-3})$. We are asked to compute the probability that $X \geq 2$. For this, we use the Poisson approximation with $\lambda = np = 3000 \cdot 10^{-3} = 3$, so that for any $k \in \{0, 1, 2, \ldots\}$

    $$P(X = k) \approx \frac{3^k}{k!} e^{-3}. \tag{2.30}$$

We choose the Poisson approximation since $np^2 = 3 \cdot 10^{-3}$ is much smaller than 1, and

$$npq = 3000 \cdot 10^{-3}(1 - 10^{-3}) \approx 3, \tag{2.31}$$

which is not enough to guarantee a good normal approximation.

In order to estimate $P(X \geq 2)$ we use the complement formula together with the Poisson approximation

$$P(X \geq 2) = 1 - [P(X = 0) + P(X = 1)] \approx 1 - \left[e^{-3} + \frac{3}{1!}e^{-3}\right] = 1 - 4e^{-3}. \tag{2.32}$$

2.4 Problem 4

12. (20 points) Let X and Y be independent random variables having the same distribution. We know that for any $n \in \mathbb{N}$

$$E((X+Y)^n) = \begin{cases} 0, & n = 2k+1, \\ \frac{(2k)!}{k!}, & n = 2k, \end{cases} \tag{2.33}$$

with $k \in \mathbb{N}$.

Determine the distribution of X. (You may assume without proof that the moment generating function of X is well defined in an interval around zero.)

Solution. Denote $Z = X + Y$ and let $M_Z(t)$ be the moment generating function of $X + Y$. Then from (2.33)

$$M_Z^{(n)}(0) = \begin{cases} 0, & n = 2k+1, \\ \frac{(2k)!}{k!}, & n = 2k. \end{cases} \tag{2.34}$$

Therefore, $M_Z(t)$ admits the following Taylor series expansion at $t = 0$

$$M_Z(t) = \sum_{n=0}^{\infty} M_Z^{(n)}(0)\frac{t^n}{n!} = \sum_{k=0}^{\infty} M_Z^{(2k)}(0)\frac{t^{2k}}{(2k)!} = \sum_{k=0}^{\infty} \frac{(2k)!}{k!} \cdot \frac{t^{2k}}{(2k)!} = \sum_{k=0}^{\infty} \frac{t^{2k}}{k!} = e^{t^2}. \tag{2.35}$$

Since X and Y are independent and have the same distribution, we have that

$$M_X(t) = M_Y(t), \quad M_{X+Y}(t) = M_X(t)M_Y(t) \quad \Rightarrow \quad M_X(t) = \sqrt{M_{X+Y}(t)}, \tag{2.36}$$

which together with (2.35) implies that

$$M_X(t) = e^{t^2/2}. \tag{2.37}$$

Therefore, $X \sim \mathcal{N}(0,1)$, random variable X has standard normal distribution.

2.5 Problem 5

13. (20 points) Let $X_1, X_2, X_3, \ldots$ be a sequence of independent identically distributed random variables all having Poisson distribution with parameter 1. Denote

$$S_n := X_1 + X_2 + \cdots + X_n. \tag{2.38}$$

(a) (12 points) Compute

$$\lim_{n\to\infty} P(S_n \le n). \tag{2.39}$$

(b) (8 points) Using the result from part (a) compute the following limit

$$\lim_{n\to\infty} e^{-n} \sum_{k=0}^{n} \frac{n^k}{k!}. \tag{2.40}$$

(Hint. Notice that the sum of independent random variables with Poisson distributions is again a random variable with Poisson distribution.)

Solution.

(a) Since $X_i \sim \text{Pois}(1)$, we have that $E(X_i) = 1$, $\text{Var}(X_i) = 1$ and we can apply the (general) Central Limit Theorem to compute the limit

$$\lim_{n\to\infty} P(S_n \le n) = \lim_{n\to\infty} P\Big(\frac{S_n - n}{\sqrt{n}} \le \frac{n-n}{\sqrt{n}}\Big) = P(Z \le 0), \tag{2.41}$$

where $Z \sim \mathcal{N}(0,1)$. Since $P(Z \le 0) = 1/2$ we conclude that

$$\lim_{n\to\infty} P(S_n \le n) = \frac{1}{2}. \tag{2.42}$$

(b) If $Y_1, Y_2, Y_3, \ldots$ are independent random variables, $Y_i \sim \text{Pois}(\lambda_i)$ for some $\lambda_i > 0$, then

$$Y_1 + Y_2 + \cdots + Y_n \sim \text{Pois}(\lambda_1 + \lambda_2 + \cdots + \lambda_n) \tag{2.43}$$

(see Example 7.2 in ASV, or Example 8.19 in ASV, or Example 9.14 in ASV, or Lecture 22). Therefore,

$$S_n \sim \text{Pois}(n), \tag{2.44}$$

and

$$P(S_n \le n) = \sum_{k=0}^{n} \frac{n^k}{k!} e^{-n}. \tag{2.45}$$

Using the result from part (a) we have that

$$\lim_{n\to\infty} e^{-n} \sum_{k=0}^{n} \frac{n^k}{k!} = \lim_{n\to\infty} P(S_n \le n) = \frac{1}{2}. \tag{2.46}$$