Math 180A: Introduction to Probability

Lecture A00 (Au)

math.ucsd.edu/~bau/w21.180a

Lecture B00 (Nemish)

math.ucsd.edu/~ynemish/teaching/180a

Today: ASV 1.5 (Random variables)

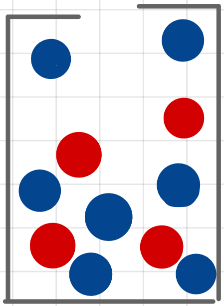
ASV 3.1 (Probability distributions)

Video: Prof. Yuriy Nemish, Fall 2019

Next: ASV 3.2

Week 2: Homework 1 (due Friday, Jan 15)

E.g. (from last lecture)



An urn has 4 red and 7 blue balls. Choose two balls.

$A = \{1^{st} \text{ ball is red}\}$

$B = \{2^{nd} \text{ ball is blue}\}$

1) choose balls with replacement

$$P(A) = \frac{4 \cdot 11}{11 \cdot 11} = \frac{4}{11}$$

$$P(B) = \frac{11 \cdot 7}{11 \cdot 11} = \frac{7}{11}$$

$$P(A \cap B) = \frac{4 \cdot 7}{11 \cdot 11} = P(A)\,P(B)$$

$\boxed{\text{A and B independent}}$

2) choose balls without replacement

$$P(A) = \frac{4 \cdot 10}{11 \cdot 10} = \frac{4}{11}$$

$$P(B) = \frac{10 \cdot 7}{11 \cdot 10} = \frac{7}{11}$$

$$P(A \cap B) = \frac{4 \cdot 7}{11 \cdot 10} \neq \frac{4 \cdot 7}{11 \cdot 11}$$

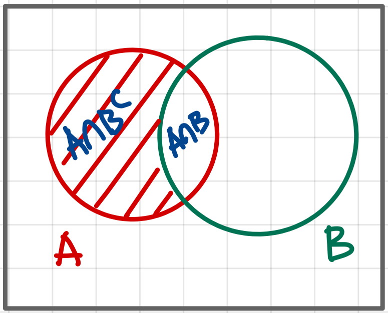
A and B are <u>not</u> independent

A and B independent $\iff$ A and B^c independent

<u>Proof</u>. $(\Rightarrow)$ Suppose that A and B are indep.

$$P(A\cap B^c) = P(A) - P(A\cap B) \overset{\text{indep of } A \,\&\, B}{=} P(A) - P(A)P(B) = P(A)(1-P(B))$$

$$= P(A)P(B^c)$$



$$A = (A\cap B) \underset{\text{disjoint}}{\cup} (A\cap B^c)$$

$$P(A) = P(A\cap B) + P(A\cap B^c)$$

$(\Leftarrow)$

More than two events?

<u>Def.</u> A collection $A_1, \dots, A_n$ of events is mutually independent if

<u>for any subcollection</u> $A_{i_1}, A_{i_2}, \dots, A_{i_k}$

$(1 \le i_1 < i_2 < \cdots < i_k \le n)$

$$P(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}) = P(A_{i_1}) P(A_{i_2}) \cdots P(A_{i_k})$$

<u>E.g.</u> When $n=3$, this means that we must have

$$P(A_1 \cap A_2) = P(A_1) P(A_2)$$

$$P(A_1 \cap A_3) = P(A_1) P(A_3)$$

$$P(A_2 \cap A_3) = P(A_2) P(A_3)$$

$$P(A_1 \cap A_2 \cap A_3) = P(A_1) P(A_2) P(A_3)$$

Important example

Toss a coin three times

$A = \{$there is exactly 1 Tails in the first two$\}$

$B = \{$there is exactly 1 Tails in the last two$\}$

$C = \{$there is exactly 1 Tails in first and last toss$\}$

$A = \{(H,T,*), (T,H,*)\}$ $\quad B = \{(*,H,T), (*,T,H)\}$

$C = \{(H,*,T), (T,*,H)\}$

$P(A) = \frac{4}{8} = \frac{1}{2} = P(B) = P(C)$

$P(A \cap B) = \frac{2}{8} = \frac{1}{4} = P(A \cap C) = P(B \cap C)$

↳ A, B, C are pairwise indep.

$A \cap B \cap C = \emptyset$

$P(A \cap B \cap C) = 0$

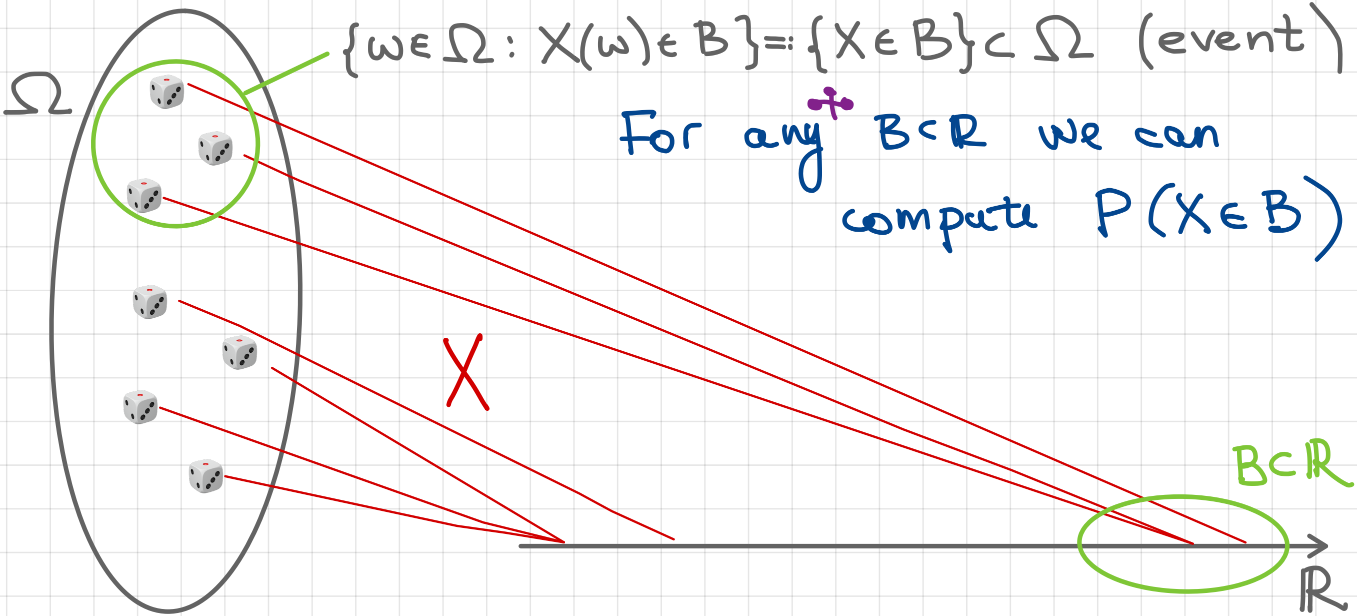
Random variables

$(\Omega, \mathcal{F}, P)$ - probability space

<u>Definition</u>. A (measurable†) function $X : \Omega \to \mathbb{R}$ is called a random variable.

$\{\omega \in \Omega : X(\omega) \in B\} =: \{X \in B\} \subset \Omega$ (event)

For any† $B \subset \mathbb{R}$ we can compute $P(X \in B)$



Def. Let X be a random variable (rv). The **probability distribution of X** is the collection of probabilities $P(X \in B)$ for all $B \subset \mathbb{R}$.

Remark. Strictly speaking, $X : (\Omega, \mathcal{F}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$

Borel sets

Examples 1) Coin toss: $\Omega = \{H, T\}$, $X(H) = 0$, $X(T) = 1$

$P(X=0) = P(\{H\}) = \frac{1}{2} = P(X=1)$ (fair coin)

2) Roll a die: $\Omega = \{1, 2, \dots, 6\}$, $X(\omega) = \omega$

For any $1 \le i \le 6$ $P(X = i) = \frac{1}{6}$

3) Roll a die twice : $\Omega = \{ (i,j) : i,j \in \{1, \dots, 6\} \}$

$X_1((i,j)) = i$ (first number) $X_2((i,j)) = j$ (second number)

for $1 \le i \le 6$ $P(X_1 = i) = \frac{1}{6}$ $P(X_2 = i) = \frac{1}{6}$

$S = X_1 + X_2$

$P(S=2) = \frac{1}{36}$ $P(S=7) = \frac{6}{36}$

$P(S=3) = \frac{2}{36}$ $P(S=8) = \frac{5}{36}$

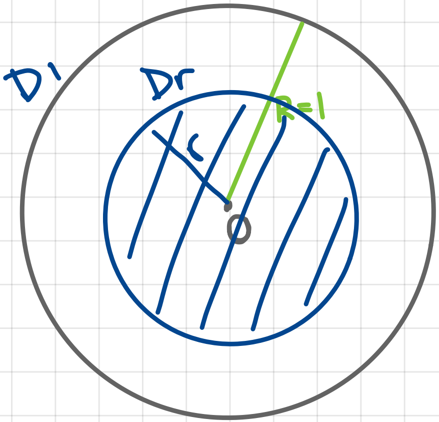
$P(S=4) = \frac{3}{36}$ $P(S=9) = \frac{4}{36}$

$P(S=5) = \frac{4}{36}$ $P(S=10) = \frac{3}{36}$

$P(S=6) = \frac{5}{36}$ $P(S=11) = \frac{2}{36}$

$P(S=12) = \frac{1}{36}$

4) Choosing a point from unit disk unif. at random



$\Omega = \{ \omega \in \mathbb{R}^2 : \text{dist}(\omega, 0) \leq 1 \}$

$X(\omega) = \text{dist}(\omega, 0)$

For any $r < 0$, $P(X \leq r) = 0$

For any $r > 1$, $P(X \leq r) = 1$

For any $r \in [0,1]$, $P(X \leq r) = \dfrac{\text{size } D_r}{\text{size } D_1} = \dfrac{\pi r^2}{\pi} = r^2$

$\{X \leq r\} = \{ X \in (-\infty, r] \}$

missing in class

Def. Random variable X is a discrete rv is there exists a finite or infinite countable collection of points $\{a_1, \dots, \} \subset \mathbb{R}$ such that $\sum_i P(X = a_i) = 1$

Example (lecture 3) Toss a coin until first T.

X = total number of tosses.

(Already computed before) for any $i = 1, 2, \dots$

$$P(X = i) = \frac{1}{2^i}$$

$$\sum_{i=1}^{\infty} P(X = i) = \sum_{i=1}^{\infty} \frac{1}{2^i} = 1 \quad \text{(geometric series)}$$

Discrete rv X is completely described by its **probability mass function (pmf)** p_X

given by
$$p_X(k) = P(X=k)$$
for all possible values of X.

Ex. S = sum of two dice

k	2	3	4	5	6	7	8	9	10	11	12
$p_S(k)$	$\frac{1}{36}$	$\frac{2}{36}$	..	..							

What if for every $x \in \mathbb{R}$ $P(X=x)=0$?

Probability density function

Def. Let X be a rv. If function $f: \mathbb{R} \to \mathbb{R}$ satisfies

$$P(X \leq b) = \int_{-\infty}^{b} f(x)\,dx$$

then f is a probability density function of X

Remark. Definition implies that for $B \subset \mathbb{R}$

$$P(X \in B) = \int_{B} f_X(x)\,dx$$

E.g. Distance to 0 from a random point in a disk

$$\int_{-\infty}^{r} f_X(x)\, dx = P(X \le r) = \begin{cases} 0, & r < 0 \\ r^2, & 0 \le r \le 1 \\ 1, & r > 1 \end{cases},$$

$$f_X(x) =$$