MATH180C: Introduction to Stochastic Processes II

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Today: Asymptotic behaviour of renewal processes. Examples

> Q&A: November 18

Next: PK 2.5, Durrett 5.1-5.2

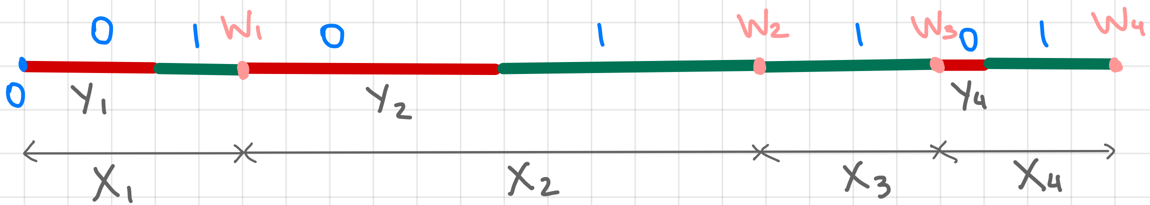
This week:

- Homework 6 (due Saturday, November 21, 11:59 PM)
- Quiz 4 (Wednesday, November 18, lectures 11-15)
- Midterm 2 (Monday, November 23, lectures 10-17)

Two component renewals

Consider the following model:

- $(X_i)_{i=1}^{\infty}$ are interrenewal times
- at each moment of time the system $S(t)$ can be in one of two states: $S(t)=0$ or $S(t)=1$
- random variables Y_i denote the part of X_i during which the system is in state 0, $0 \le Y_i \le X_i$
- collection $((X_i, Y_i))_{i=1}^{\infty}$ is i.i.d.



Q: In the long run (for large t), what is the probability that the system is in state 1 at time t?

Two component renewals

Thm.

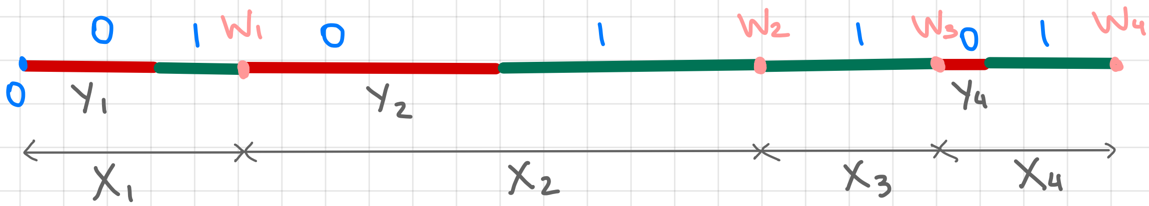
$$\lim_{t\to\infty} P(S(t)=0) =$$

Proof. Denote $g(t) =$. Then

$$g(t) =$$

If $t < x$, then $P(S(t)=0 \mid X_1 = x) =$

If $t \geq x$, then $P(S(t)=0 \mid X_1 = x) =$



Two component renewals

$$g(t) = \int_t^\infty \underbrace{\qquad\qquad\qquad\qquad} + \int_0^t \underbrace{\qquad\qquad\qquad\qquad}$$

Function g satisfies the renewal equation

$$g(t) =$$

Note that $Y_1 \le X_1$, therefore $P(Y_1 > t \mid X_1 = x) =$ for $x < t$,

$$h(t) =$$

$$\int_0^\infty h(t)\,dt =$$

From the Key renewal theorem $\lim\limits_{t \to \infty} g(t) =$

Example: the Peter principle

Setting:
- infinite population of candidates for certain position
- fraction p of the candidates are competent, $q = 1 - p$ are incompetent
- if a competent person is chosen, after time C_i he/she gets promoted
- if an incompetent person is chosen, he/she remains in the job until retirement (r.v. I_j)
- once the position is open again, the process repeats

Question: What fraction of time, denoted f, is the position held by an incompetent person on average in the long run?

Example: the Peter principle

Denote

$$X_i = \begin{cases} \quad , & \text{if occupied by a competent person} \\ \quad , & \text{if occupied by an incompetent person} \end{cases}$$

$$Y_i = \begin{cases} \quad , & \text{if occupied by a competent person} \\ \quad , & \text{if occupied by an incompetent person} \end{cases}$$

KRT for two component renewals can be applied to $((X_i, Y_i))_{i=1}^{\infty}$

If $S(t) = 0$ if the person is incompetent, then

$$\lim_{t\to\infty} P(S(t)=0) = \frac{E(Y_1)}{E(X_1)} \qquad \text{and}$$

$$f := \lim_{t\to\infty} \Big(\qquad\qquad \Big) = \lim_{t\to 0} \frac{1}{t} \int_0^t \qquad du =$$

Finally, if
- $E(C_i) = \mu$
- $E(I_i) = \nu$

, then $f = \dfrac{E(Y_1)}{E(X_1)} = \dfrac{(1-p)\nu}{p\mu + (1-p)\nu}$

Example: the Peter principle (alternative)

Let $X_i = \begin{cases} C_i, & \text{if the } i\text{-th person is competent} \\ I_i, & \text{if the } i\text{-th person is incompetent} \end{cases}$

$Y_i = \begin{cases} 0, & \text{time occupied by a competent person} \\ I_i, & \text{time occupied by an incompetent person} \end{cases}$

and assume that $|X_i| < K$. Then using

$$\leq E\left(\frac{1}{t} \int_0^t \mathbb{1}_{\{S(u)=0\}} \, du \right) \leq$$

Again, if $\bullet\ E(C_i) = \mu$, $\bullet\ E(I_i) = \nu$, then $f = \frac{E(Y_1)}{E(X_1)} = \frac{(1-p)\nu}{p\mu + (1-p)\nu}$

Example: the Peter principle

If we take , then

$$f =$$