☐ **Write your name and PID on the top of EVERY PAGE.**

☐ **Write the solutions to each problem on separate pages. CLEARLY INDICATE on the top of each page the number of the corresponding problem. Different parts of the same problem can be written on the same page (for example, part (a) and part (b))**

☐ **Remember this exam is graded by a human being. Write your solutions NEATLY AND COHERENTLY, or they risk not receiving full credit.**

☐ **From the moment you access the midterm problems on Gradescope you have 65 MINUTES to COMPLETE AND UPLOAD your exam to Gradescope. Plan your time accordingly.**

☐ **For combinatorial problems, you can leave the expressions without simplifications (unless the problem specifically asks to simplify).**

☐ **You are allowed to use the textbook, lecture notes and your personal notes. You are not allowed to use the electronic devices (except for accessing the online version of the textbook) or outside assistance. Outside assistance includes but is not limited to other people, the internet and unauthorized notes.**

1. (20 points) A team of two robots plays repeatedly the following game. Each of the two robots chooses uniformly at random and independently of each other a number from the set $\{1, 2, 3, 4, 5\}$. The team wins if both robots choose the same number.

   (a) (5 points) What is the probability that the team wins in one particular game?

   (b) (10 points) Compute the probability that in the first *three* games the team wins *at least once*. (Simplify the answer for a full credit.)

   (c) (5 points) Compute the probability that in the first *four* games the team wins *exactly three times*.

Solutions.

(a) If $S := \{1, 2, 3, 4, 5\}$, then the sample space for one game is $\Omega = S^2$ and $\#\Omega = 5^2 = 25$. If $A = \{\text{team wins}\}$, then $A = \{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5)\}$ and $\#A = 5$. Therefore $P(A) = \#A/\#\Omega = 5/25 = 1/5$.

Correct answer:

$$P(\text{win}) = 1/5$$

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(b) Let S_3 be the number of wins in 3 games. Then $S_3 \sim \text{Bin}(3, 1/5)$. Now $P(S_3 \geq 1) = 1 - P(S_3 = 0)$, and $P(S_3 = 0) = 4^3/5^3$, so

$$P(S_3 \geq 1) = 1 - 4^3/5^3 = (125 - 64)/125 = 61/125.$$

Correct answer:

$$61/125.$$

(c) If S_4 is the number of wins in 4 games, then $S_4 \sim \text{Bin}(4, 1/5)$, therefore

$$P(S_4 = 3) = \binom{4}{3} \frac{1}{5^3} \frac{4}{5}.$$

Correct answer:

$$\binom{4}{3} \frac{1}{5^3} \frac{4}{5}.$$

2. (25 points) Consider the familiar setting of a car insurance with deductibles. Your car is in a minor accident and the damage repair cost is uniformly distributed on the interval $[100, 1500]$ (in USD). Your insurance deductible is 500 USD. This means that you personally pay for 100% of the repairs up to 500 USD, with the insurance company paying the rest.

   Denote by Y the amount of money paid by the *insurance company* for the car repair.

   (a) (15 points) Compute and plot the cumulative distribution function of Y.

(b) (10 points) If Y is discrete, compute its probability mass function; if Y is continuous, compute its probability density function; if neither of the two are possible, explain why.

Solutions.

(a) If X is the cost of repairs, then $X \sim \text{Unif}[100, 1500]$. The first 500 USD are paid by the client, the insurance company pays the remaining part between 0 USD and 1000 USD, i.e., for $t < 0$, $P(Y \le t) = 0$, and for $t \ge 1000$, $P(Y \le t) = 1$.
For $t = 0$ we have

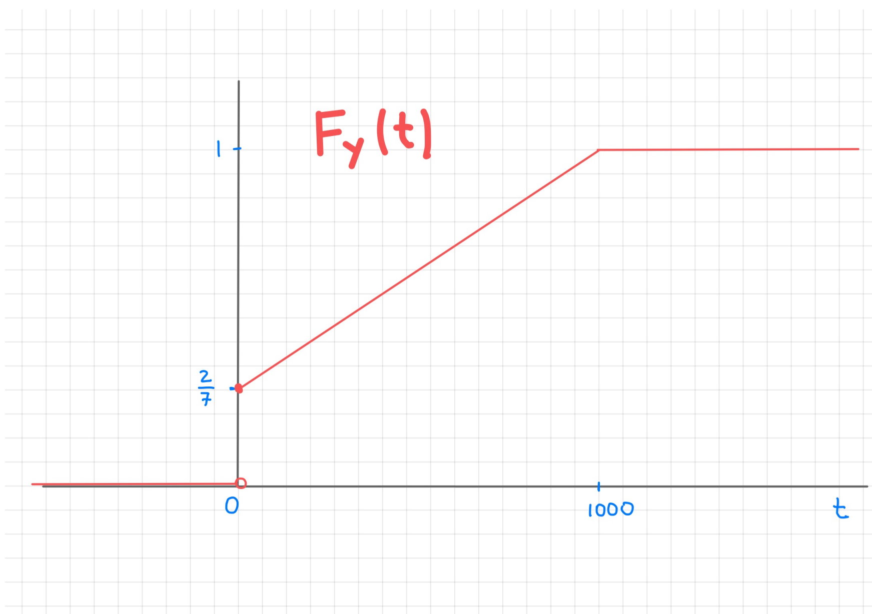
$$P(Y \le 0) = P(Y = 0) = P(X \le 500) = \frac{400}{1400} = 2/7,$$

and for $t \in (0, 1000)$,

$$P(Y \le t) = P(X \le 500 + t) = \frac{400 + t}{1400} = \frac{2}{7} + \frac{t}{1400}.$$

To sum up,

$$F_Y(t) = \begin{cases} 0, & t < 0, \\ 2/7 + t/1400, & 0 \le t < 1000, \\ 1, & t \ge 1000 \end{cases}$$



(b) The CDF is not continuous at zero, therefore Y is not a continuous random variable. At the same time, CDF is not piece-wise constant on $(0, 1000)$, which means that Y is not discrete. We conclude that Y is neither discrete nor continuous.

3. (25 points) Each employee of the Department of Mathematics owns one (and only one) car. 60% of the cars owned by the employees are manufactured in America, 25% in Asia and 15% in Europe. You know that 36% of these cars are electric, in particular, every tenth car owned by the employees is a European electric car, and 20% of Asian cars are electric.

   You overheard that Professor Jamie A. has an electric car. What is the probability that Professor Jamie A. owns an American car?

   Solutions.

   (Suppose you randomly choose a car of an employee, define the events $A = \{\text{American}\}$, $B = \{\text{Asian}\}$, $C = \{\text{European}\}$, $E = \{\text{Electric}\}$. Then we are given the following information:

$$P(A) = 0.6, \quad P(B) = 0.25, \quad P(C) = 0.15, \quad P(E) = 0.36,$$
$$P(E \cap C) = 0.1, \quad P(E|B) = 0.2.$$

   Now we have to compute

$$P(A|E) = \frac{P(A \cap E)}{P(E)}.$$

   Compute

$$P(A \cap E) = P(E) - P(B \cap E) - P(C \cap E) = 0.36 - 0.2 \cdot 0.25 - 0.1 = 0.21,$$

   so that $P(A|E) = 0.21/0.36 = 7/12$.

   Correct answer:

$$P(\text{American}|\text{Electric}) = 7/12.$$

4. (30 points) Mathematicians from three countries (Canada, US and Mexico) participate in a scientific meeting. Each country is represented by 10 participants. There are 5 time slots available for scientific talks, and each participant is ready to give one (and only one) talk. The organizing committee decides to choose the speakers uniformly at random from all participants

   (a) (10 points) What is the probability that the first talk will be given by Michel from Canada, and the last talk will be given by Mireille from Canada?

   (b) (20 points) What is the probability that there will be at least one speaker from each country?

   [Remark: parts (a) and (b) are independent; you do not need to simplify your answer for part (b).]

   Solutions.

(a) Since we distinguish the order of the speakers, we model part (a) as sampling without replacement, order matters. Therefore the size of the sample space is

$$\#\Omega = 30 \cdot 29 \cdot 28 \cdot 27 \cdot 26.$$

If we fix the first and the last speaker, we have to choose 3 more speakers from the remaining 28 participants, which gives

$$\#\{\text{Michel first, Mireille last}\} = 28 \cdot 27 \cdot 26.$$

Correct probability

$$\frac{1}{30 \cdot 29}.$$

(b) In this part we are only interested in the speakers but not the exact order in which they give talks, therefore we model it as sampling without replacement, order does not matter. The size of the sample space is

$$\#\Omega = \binom{30}{5}.$$

The event that each country has at least one speaker can be realized as (2 countries have 2 speakers and 1 country has one speaker) or (1 country has three speakers and 2 countries have 1 speaker).

In the first situation, if we fix the country that has only one speaker, that the number of different combinations of speakers is

$$\binom{10}{2}\binom{10}{2}\binom{10}{1}.$$

There are 3 ways of choosing the one country that has only one speaker, so we end up with

$$3\binom{10}{2}\binom{10}{2}\binom{10}{1}$$

combinations of speakers of the first situation.

In the second situation there are 3 ways of choosing the country that has three speakers, and if we fix this country, we have

$$\binom{10}{1}\binom{10}{1}\binom{10}{1}$$

possible combinations. We end up with

$$3\binom{10}{3}\binom{10}{1}\binom{10}{1}$$

.

The final answer is

$$\frac{3\binom{10}{2}\binom{10}{2}\binom{10}{1} + 3\binom{10}{3}\binom{10}{1}\binom{10}{1}}{\binom{30}{5}}$$