☐ **Write your name and PID on the top of EVERY PAGE.**

☐ **Write the solutions to each problem on separate pages. CLEARLY INDICATE on the top of each page the number of the corresponding problem. Different parts of the same problem can be written on the same page (for example, part (a) and part (b))**

☐ **Remember this exam is graded by a human being. Write your solutions NEATLY AND COHERENTLY, or they risk not receiving full credit.**

☐ **From the moment you access the midterm problems on Gradescope you have 65 MINUTES to COMPLETE AND UPLOAD your exam to Gradescope. Plan your time accordingly.**

☐ **For combinatorial problems, you can leave the expressions without simplifications (unless the problem specifically asks to simplify).**

☐ **You are allowed to use the textbook, lecture notes and your personal notes. You are not allowed to use the electronic devices (except for accessing the online version of the textbook) or outside assistance. Outside assistance includes but is not limited to other people, the internet and unauthorized notes.**

1. (20 points) Let $(\Omega, \mathcal{F}, P)$ be a probability space.
   (a) (10 points) Suppose that $A, B \in \mathcal{F}$ and $P(A) = 0.6$, $P(B) = 0.9$. Show that
   $$0.5 \leq P(A \cap B) \leq 0.6.$$
   (b) (10 points) Show that for any $F, G \in \mathcal{F}$
   $$P(F \cap G) \geq 1 - P(F^C) - P(G^C).$$

Solution.

(a) By the inclusion-exclusion formula for two events, for any events $A, B \in \mathcal{F}$
$$P(A \cup B) = P(A) + P(B) - P(A \cap B). \tag{1}$$
Therefore, using also that $P(A \cup B) \leq$, we get
$$P(A \cap B) = P(A) + P(B) - P(A \cup B) \geq P(A) + P(B) - 1. \tag{2}$$
On the other had, by monotonicity of the probability measure,
$$A \cap B \subset A \Rightarrow P(A \cap B) \leq P(A), \quad A \cap B \subset B \Rightarrow P(A \cap B) \leq P(B). \tag{3}$$
From (2) and (3) with $P(A) = 0.6$ and $P(B) = 0.9$, we get
$$0.5 = 0.6 + 0.9 - 1 \leq P(A \cap B) \leq \min\{0.6, 0.9\} = 0.6. \tag{4}$$

(b) Using (b) we get
$$\begin{aligned} 1 - P(F^C) - P(G^C) &= 1 - (1 - P(F)) - (1 - P(G)) && (5)\\ &= P(F) + P(G) - 1 && (6)\\ &\leq P(F \cap G). && (7) \end{aligned}$$

2. (25 points) An urn contains 2 white balls and 4 black balls. You remove the balls one by one from the urn (without replacement).
   (a) (15 points) What is the probability that the first two balls removed from the urn are black?
   (b) (10 points) What is the probability that the last removed ball is white?

Solution.

(a) Suppose that we choose two balls from the urn, and let A be the event that two balls are black. Since the balls are chosen without replacement, order does not matter, we have

$$P(A) = \frac{\binom{4}{2}}{\binom{6}{2}} = \frac{2}{5}. \tag{8}$$

(b) Suppose that we remove all balls from the urn, and let B be the event that the last ball is white. Then

$$\#\Omega = 6!, \quad \#B = 5! \cdot 2, \quad P(B) = \frac{5! \cdot 2}{6!} = \frac{1}{3}. \tag{9}$$

3. (25 points) Let X be a point chosen uniformly at random from the interval $[1, 3]$. Let Y be a random variable defined by $Y = \frac{1}{2}(X - 1)$.
   (a) (15 points) Compute the CDF of Y.
   (b) (10 points) If Y is continuous, compute its PDF. Otherwise explain why Y is not a continuous random variable.

Solution.

(a) For any $r \in \mathbb{R}$

$$F_Y(r) = P(Y \le r) = P\Big(\frac{1}{2}(X-1) \le r\Big) = P\Big(X \le 1 + 2r\Big). \tag{10}$$

Since $X \sim \text{Unif}[1, 3]$,

$$F_Y(r) = P\Big(X \le 1 + 2r\Big) = \begin{cases} 0, & r < 0, \\ r, & 0 \le r < 1, \\ 1, & r \ge 1. \end{cases} \tag{11}$$

(b) The CDF of Y is continuous, therefore Y is a continuous random variable. The PDF of Y is given by

$$f_Y(r) = F_Y'(r) = \begin{cases} 0, & r < 0, \\ 1, & 0 \le r < 1, \\ 0, & r \ge 1. \end{cases} \tag{12}$$

$Y \sim \text{Unif}[0, 1]$.

4. (30 points) You roll three fair *four-sided* dice.
   (a) (15 points) Compute the probability that there will be at least one four, *given* that all three dice give different numbers.

(b) (15 points) Compute the (unconditional) probability that there will be at least one four. [**Hint.** Use complement]

Solution. Denote by A the event that there will be at least one four and by B the event that all three dice give different numbers. If Ω is the sample space, then $\#\Omega = 4^3$.

(a)
$$P(A|B) = \frac{P(A \cap B)}{P(B)}. \tag{13}$$

$A \cap B$ is the event that all three numbers are different, and one of the numbers is four. Then $\#(A \cap B) = 3 \cdot 3 \cdot 2$ and $\#B = 4 \cdot 3 \cdot 2$. Therefore

$$P(A|B) = \frac{3 \cdot 3 \cdot 2}{4 \cdot 3 \cdot 2} = \frac{3}{4}. \tag{14}$$

(b) Without conditioning on B, $P(A) = 1 - P(A^C)$, where A^C is the event that none of the dice gives four. Therefore, $\#A^C = 3^3$, and thus

$$P(A) = 1 - P(A^C) = 1 - \frac{3^3}{4^3} = \frac{64 - 27}{64} = \frac{37}{64}. \tag{15}$$

Note that $P(A|B) > P(A)$.