MATH 180A: Introduction to Probability

Lecture A00 (Au)

www.math.ucsd.edu/~bau/w21.180a

Lecture B00 (Nemish)

www.math.ucsd.edu/~ynemish/teaching/180a

Today: Covariance and correlation coefficient. MGF of sums of independent random variables

Next: ASV 9.3

Video: Prof. Todd Kemp, Fall 2019

Week 9:

- Homework 7 (due Friday, March 5)
- Homework 5 regrades (until Friday, March 12)
- CAPE at www.cape.ucsd.edu

Flashback to Math 20C

Given two vectors $\underline{v}, \underline{w}$ in $\mathbb{R}^n$, their dot product is

$$\underline{v} \cdot \underline{w} = \sum_{j=1}^{n} v_j w_j \qquad \underline{v} \cdot \underline{v} = \sum_{j=1}^{n} v_j^2$$

It is a positive bilinear form:

(1) $(a\underline{u} + b\underline{v}) \cdot \underline{w} = a\,\underline{u} \cdot \underline{w} + b\,\underline{v} \cdot \underline{w}$

$(\underline{v} \cdot \underline{w} = \underline{w} \cdot \underline{v})$

(2) $\underline{v} \cdot (a\underline{u} + b\underline{w}) = a\,\underline{v} \cdot \underline{u} + b\,\underline{v} \cdot \underline{w}$

(3) $\underline{v} \cdot \underline{v} > 0$ unless $\underline{v} = \underline{0}$, in which case $\underline{v} \cdot \underline{v} = 0$.

The length^2 of a vector is $\|\underline{v}\|^2 = \underline{v} \cdot \underline{v}$, length $= \|\underline{v}\| = \sqrt{\underline{v} \cdot \underline{v}}$

Two vectors are orthogonal if $\underline{v} \cdot \underline{w} = 0$

Cauchy-Schwarz Inequality: $|\underline{v} \cdot \underline{w}| \leq \|\underline{v}\|\,\|\underline{w}\|$

$$\frac{\underline{v} \cdot \underline{w}}{\|\underline{v}\|\,\|\underline{w}\|} = \cos\theta$$

$$-1 \leq \frac{\underline{v} \cdot \underline{w}}{\|\underline{v}\|\,\|\underline{w}\|} \leq 1$$

$= -1$ iff $\underline{v} = a\underline{w}$ for $a<0$

$= 1$ iff $\underline{v} = a\underline{w}$ for some $a>0$

The Geometry of Random Variables

$$\underline{\mathbb{E}(XY)-\mathbb{E}(X)\mathbb{E}(Y)}$$
$$\|$$

Dot product $\rightsquigarrow$ Covariance $\quad \mathrm{Cov}(X,Y) = \underline{\mathbb{E}\big((X-\mathbb{E}(X))(Y-\mathbb{E}(Y))\big)}$.

$$\Downarrow$$

(Almost) positive bilinear form

$$\left\{\begin{array}{ll} (1) & \mathrm{Cov}(aX_1+bX_2, Y) = a\,\mathrm{Cov}(X_1,Y) + b\,\mathrm{Cov}(X_2,Y) \\ (2) & \mathrm{Cov}(X, aY_1+bY_2) = a\,\mathrm{Cov}(X,Y_1) + b\,\mathrm{Cov}(X,Y_2) \\ (3) & \mathrm{Cov}(X,X) = \mathrm{Var}(X) > 0 \text{ unless } X \equiv \text{const.} \\ & \qquad (\mathrm{Var}(X) = 0) \end{array}\right.$$

Orthogonal $\rightsquigarrow$ $\mathrm{Cov}(X,Y)=0$ uncorrelated ($\Leftarrow$ independent)

Length^2 $\rightsquigarrow$ $\mathrm{Var}(X)$ $\quad$ Length $\rightsquigarrow \sqrt{\mathrm{Var}(X)} = \mathrm{S.D}(X)$

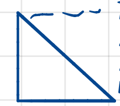
"Angles" $\rightsquigarrow$ $\dfrac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}(X)}\sqrt{\mathrm{Var}(Y)}} =: \mathrm{Corr}(X,Y)$

Theorem: $-1 \leq \mathrm{Corr}(X,Y) \leq 1$

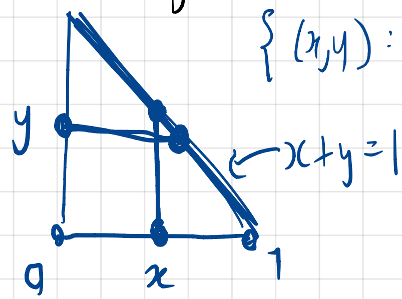
$\mathrm{Corr}(X,Y)=0$ iff (X,Y) are uncorrelated $\not\Rightarrow$ independence

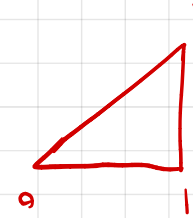
$\mathrm{Corr}(X,Y)=1$ iff $Y = aX+b$ for some $a>0$, $b\in\mathbb{R}$

$\mathrm{Corr}(X,Y)=-1$ iff $Y = aX+b$ for some $a<0$, $b\in\mathbb{R}$.

Eg. (X,Y) uniformly distributed on the triangle  Area $= \frac{1}{2}$

$\{(x,y) : x,y \geq 0,\ x+y \leq 1\}$





$$f_{(X,Y)}(x,y) = 2\,\mathbb{1}_{\Delta}(x,y)$$

$$f_X(x) = \int_0^{1-x} f_{(X,Y)}(x,y)\,dy = \int_0^{1-x} 2\,dy = 2(1-x)\,\mathbb{1}_{[0,1]}(x)$$

$$f_Y(x) = \int_0^{1-y} 2\,dx = 2(1-y)\,\mathbb{1}_{[0,1]}(y)$$

$\Big\}\ X \sim Y$

$$E(X) = E(Y) = \int x f_X(x)\,dx = \int_0^1 x \cdot 2(1-x)\,dx = \frac{1}{3}$$

$$E(X^2) = E(Y^2) = \int x^2 f_X(x)\,dx = \int_0^1 x^2\, 2(1-x)\,dx = \frac{1}{6}.$$

$$\mathrm{Var}(X) = \mathrm{Var}(Y) = \frac{1}{6} - \left(\frac{1}{3}\right)^2 = \frac{1}{18}.$$

$$\mathrm{Cov}(X,Y) = E(XY) - \underbrace{E(X)E(Y)}_{} = \iint_{\Delta} xy\, f_{(X,Y)}(x,y)\,dx\,dy - \left(\frac{1}{3}\right)\left(\frac{1}{3}\right)$$

$$= \int_0^1 dx \int_0^{1-x} dy\ xy\,2 - \frac{1}{9} = \frac{1}{12} - \frac{1}{9} = \frac{-1}{36}.$$

$$\mathrm{Corr}(X,Y) = \frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}X}\sqrt{\mathrm{Var}Y}} = \frac{-1/36}{\sqrt{\frac{1}{18}}\sqrt{\frac{1}{18}}} = \frac{-\frac{1}{36}}{\frac{1}{18}} = \boxed{-\frac{1}{2}}$$

Moment Generating Function Revisited

8.3

Suppose X, Y are independent, and $\boxed{M_X, M_Y < \infty}$ on an interval containing 0. Then

$$M_{X+Y}(t) = \mathbb{E}\left(e^{t(X+Y)}\right) = \mathbb{E}\left(\underbrace{e^{tX}}_{\text{fn of } X}\underbrace{e^{tY}}_{\text{fn of } Y}\right) = \mathbb{E}(e^{tX})\,\mathbb{E}(e^{tY}) = M_X(t) \cdot M_Y(t).$$

$\star$ If $M_Z(t) < \infty$ for all $t \in (-\delta, \delta)$, then M_Z determines the dist. of Z.

$\Rightarrow$ M_{X+Y} determines the dist. of $X+Y$.

E.g. $X \sim \text{Poisson}(\lambda)$, $Y \sim \text{Poisson}(\mu)$ independent

$M_X(t) = e^{\lambda(e^t - 1)}$ $\quad M_Y(t) = e^{\mu(e^t-1)}$

$$M_{X+Y}(t) = e^{\lambda(e^t-1)} e^{\mu(e^t-1)} = e^{\lambda(e^t-1)+\mu(e^t-1)} = e^{(\mu+\lambda)(e^t-1)}$$

the MGF of Poisson$(\lambda+\mu)$

$\Rightarrow X + Y \sim \text{Poisson}(\lambda+\mu)$

Eg. $X \sim N(\mu_1, \sigma_1^2)$ $\quad Y \sim N(\mu_2, \sigma_2^2)$ independent

$Z \sim N(0,1)$, $M_Z(t) = e^{t^2/2}$

$X \sim \sigma_1 Z + \mu_1$ $\quad \therefore M_X(t) = M_{\sigma_1 Z + \mu_1}(t) = E\left(e^{t(\sigma_1 Z + \mu_1)}\right)$

$$= E\left(e^{(\sigma_1 t)Z} e^{\mu_1 t}\right)$$

$$= e^{\mu_1 t} \underbrace{E\left(e^{(\sigma_1 t)Z}\right)}_{M_Z(\sigma_1 t)} = e^{\mu_1 t} e^{(\sigma_1 t)^2/2}$$

$$= e^{\frac{\sigma_1^2}{2} t^2 + \mu_1 t}$$

$$\therefore M_{X+Y}(t) = e^{\frac{\sigma_1^2}{2} t^2 + \mu_1 t} \cdot e^{\frac{\sigma_2^2}{2} t^2 + \mu_2 t}$$

$$= e^{\frac{1}{2}(\sigma_1^2 + \sigma_2^2) t^2 + (\mu_1 + \mu_2) t} \Longleftarrow \text{Normal}$$

$$\simeq \text{MGF of } N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$$