MATH 180A - INTRODUCTION TO PROBABILITY
PRACTICE MIDTERM #1

WINTER 2021

Name (Last, First): ____________________

Student ID: ____________________

REMEMBER THIS EXAM IS GRADED BY A HUMAN BEING. WRITE YOUR SOLUTIONS NEATLY AND COHERENTLY, OR THEY RISK NOT RECEIVING FULL CREDIT.

1. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space.

(a) Suppose that $A, B \in \mathcal{F}$ satisfy

$$\mathbb{P}(A) + \mathbb{P}(B) > 1.$$

Making no further assumptions on A and B, prove that $A \cap B \neq \emptyset$.

Solution. First method (proof by contradiction). Assume that $A \cap B = \emptyset$. Then $P(A \cup B) = P(A) + P(B) > 1$, contradiction. Therefore, $A \cap B \neq 0$.

Second method.

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) \leq 1,$$

therefore $P(A \cap B) \geq P(A) + P(B) - 1 > 0$, and we conclude that $A \cap B \neq \emptyset$.

(b) Prove that A is independent from itself if and only if $\mathbb{P}(A) \in \{0, 1\}$.

Solution. By definition, events A and B are independent if $P(A \cap B) = P(A)P(B)$. Take $A = B$, so that A being independent of A is equivalent to $P(A \cap A) = P(A)P(A)$. Since $A \cap A = A$, we have that $P(A)$ satisfies

$$(P(A))^2 = P(A).$$

The only two numbers that satisfy the above equation are numbers 0 and 1.

2. Suppose we have an urn with 10 red balls, 15 blue balls, and 20 green balls. We draw three balls without replacement uniformly at random.

(a) What is the probability that we have more red balls than non-red balls?

Solution. Define the events $A = \{\text{more red balls than non-red balls}\}$ and for $i \in \{0, 1, 2, 3\}$ $R_i = \{\text{exactly } i \text{ red balls}\}$. Then $A = R_2 \cup R_3$, $\#A = \#R_2 + \#R_3$. Since the balls are chosen without replacement and order does not matter, we have that

$$\#\Omega = \binom{45}{3}, \quad \#R_i = \binom{10}{i}\binom{35}{3-i}, \tag{1}$$

therefore

$$P(A) = \frac{\binom{10}{2}\binom{35}{1} + \binom{10}{3}}{\binom{45}{3}}. \tag{2}$$

(b) What is the probability that we have more red balls than green balls?

Solution. Define the events $B = \{\text{ more red balls than green balls}\}$, $D(i, j, k) = \{i \text{ red balls, } j \text{ blue balls, } k \text{ green balls}\}$. Then $B = D(1, 2, 0) \cup D(2, 1, 0) \cup D(3, 0, 0) \cup D(2, 0, 1)$, and thus

$$P(B) = \frac{\binom{10}{1}\binom{15}{2} + \binom{10}{2}\binom{15}{1} + \binom{10}{3} + \binom{10}{2}\binom{20}{1}}{\binom{45}{3}}. \tag{3}$$

(c) What is the probability that at least two of the three balls have the same color?

Solution. Define the event $C = \{$ at least two have the same color$\}$. Then the complement of this event is given by $C^{\complement} = \{$ all three balls of different colors$\}$, which means that there is one red, one blue and one green. Therefore

$$P(C^{\complement}) = \frac{\binom{10}{1}\binom{15}{1}\binom{20}{1}}{\binom{45}{3}}, \quad P(C) = 1 - \frac{\binom{10}{1}\binom{15}{1}\binom{20}{1}}{\binom{45}{3}}. \tag{4}$$

3. A box contains 3 coins, two of which are fair and the third has probability 3/4 of coming up heads. A coin is chosen randomly from the box and tossed 3 times.

(a) What is the probability that all 3 tosses are heads?

Solution. Define the following events:

$$\begin{aligned} A &= \{\text{all 3 tosses are heads}\} \\ B_1 &= \{\text{chosen coin is fair}\} \\ B_2 &= \{\text{chosen coin is biased}\}. \end{aligned}$$

Then B_1 and B_2 form a partition of the sample space with

$$P(B_1) = \frac{2}{3}, \qquad P(B_2) = \frac{1}{3},$$

and depending on which coin was chosen from the box, we have the conditional probabilities of observing heads on all three tosses

$$P(A|B_1) = \left(\frac{1}{2}\right)^3, \qquad P(A|B_2) = \left(\frac{3}{4}\right)^3.$$

Then we can compute $P(A)$ using the law of total probability

$$\begin{aligned} P(A) &= P(A|B_1)P(B_1) + P(A|B_2)P(B_2) \\ &= \frac{1}{8} \cdot \frac{2}{3} + \left(\frac{3}{4}\right)^3 \frac{1}{3} \\ &= \frac{43}{192}. \end{aligned}$$

(b) Given that all three tosses are heads, what is the probability that the biased coin was chosen?

Solution. Using the Bayes' rule, we have

$$P(B_2|A) = \frac{P(A|B_2)P(B_2)}{P(A)} = \frac{27}{43}.$$

4. Let X be a discrete random variable taking the values $\{1, 2, \ldots, n\}$ all with equal probability. Let Y be another discrete random variable taking values in $\{1, 2, \ldots, n\}$. Assume that X and Y are independent. Show that $\mathbb{P}(X = Y) = \frac{1}{n}$. (Hint: you do not need to know the distribution of Y to calculate this.)

Solution. Since both random variables X and Y take values in the set $\{1, 2, \ldots, n\}$, the event $\{X = Y\}$ can be written as a disjoint union

$$\{X = Y\} = \cup_{k=1}^{n} \{X = k, Y = k\},$$

therefore the probability of $\{X = Y\}$ is equal to

$$P(X = Y) = \sum_{k=1}^{n} P(X = k, Y = k).$$

From the independence of X and Y and the fact that $P(X = k) = 1/n$ for all $k \in \{1, \ldots, n\}$ we have that

$$\begin{aligned}\sum_{k=1}^{n} P(X = k, Y = k) &= \sum_{k=1}^{n} P(X = k)(Y = k).\\ &= \frac{1}{n} \sum_{k=1}^{n} P(Y = k).\end{aligned}$$

Since Y is distributed on $\{1, \ldots, n\}$, $\sum_{k=1}^{n} P(Y = k) = 1$ and we conclude that $P(X = Y) = 1/n$.

5. Consider a point $P = (X, Y)$ chosen uniformly at random inside of the triangle in $\mathbb{R}^2$ that has vertices $(1, 0)$, $(0, 1)$, and $(0, 0)$. Let $Z = \max(X, Y)$ be the random variable defined as the maximum of the two coordinates of the point. For example, if $P = (\frac{1}{2}, \frac{1}{3})$, then $Z = \max(X, Y) = \frac{1}{2}$. Determine the cumulative distribution function of Z. Determine if Z is a continuous random variable, a discrete random variable, or neither. If continuous, determine the probability density function of Z. If discrete, determine the probability mass function of Z. If neither, explain why.

(Hint: Draw a picture.)

Solution.

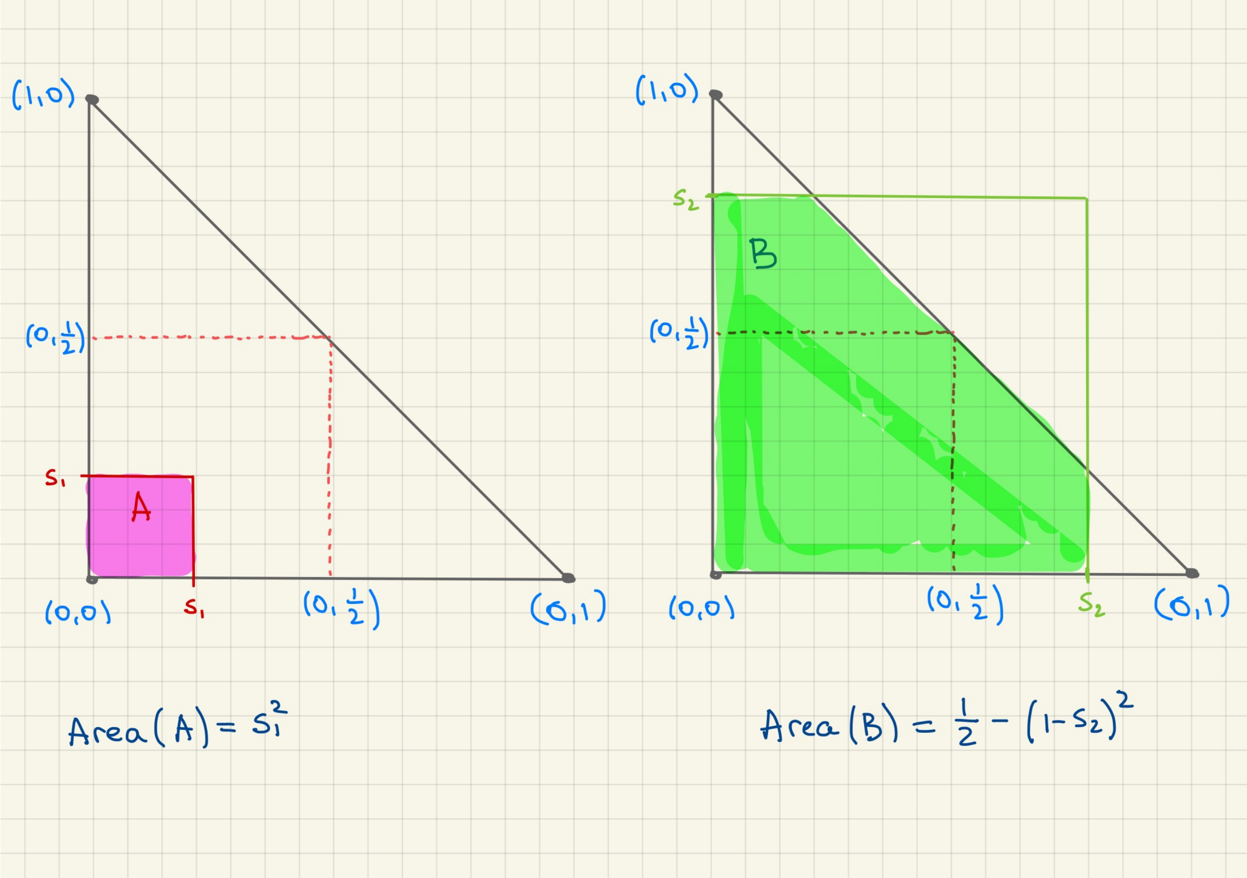
First note, that since $Z = \max\{X, Y\}$, the event $\{Z \leq s\}$ can be rewritten as

$$\{\max\{X, Y\} \leq s\} = \{X \leq s, Y \leq s\}.$$

So in order to compute the CDF of Z, we have to compute the probability that $X \leq s, Y \leq s$ for all $s \in \mathbb{R}$, where (X, Y) is uniformly chosen from the triangle with vertices $(0, 0)$, $(0, 1)$ and $(1, 0)$.

It is clear that $F_Z(s) = P(Z \leq s) = 0$ if $s \leq 0$ and $F_Z(s) = 1$ if $s \geq 1$.

For $s \in (0, 1)$ two situations are possible (see the picture below)



If $0 \le s \le 1/2$, then

$$P(X \le s, Y \le s) = \frac{\text{Area}(A)}{\text{Area of the triagle}} = \frac{s^2}{\frac{1}{2}} = 2s^2.$$

If $1/2 \le s \le 1$, then

$$P(X \le s, Y \le s) = \frac{\text{Area}(B)}{\text{Area of the triagle}} = \frac{\frac{1}{2} - (1-s)^2}{\frac{1}{2}} = 1 - 2(1-s)^2.$$

We finally get that

$$F_Z(s) = \begin{cases} 0, & s \le 0, \\ 2s^2, & 0 \le s \le 1/2, \\ 1 - 2(1-s)^2, & 1/2 \le s \le 1, \\ 1, & s \ge 0. \end{cases}$$

It is clear that the function is continuous at points $s = 0$ and $s = 1$, and we can check that it is also continuous at $s = 1/2$

$$2\left(\frac{1}{2}\right)^2 = \frac{1}{2} = 1 - 2\left(1 - \frac{1}{2}\right)^2.$$

Since the CDF of Z is continuous, the random variable Z is a continuous random variable. In order to compute its probability density function, differentiate the CDF

$$f_Z(s) = \begin{cases} 0, & s \leq 0, \\ 4s, & 0 \leq s \leq 1/2, \\ 4 - 4s, & 1/2 \leq s \leq 1, \\ 0, & s \geq 0. \end{cases}$$