MATH180C: Introduction to Stochastic Processes II

www.math.ucsd.edu/~ynemish/teaching/180c

Today: General continuous time MC. Q-matrices. Matrix exponentials

> Q&A: October 14

Next: PK 6.6, Durrett 4.1

Week 2:

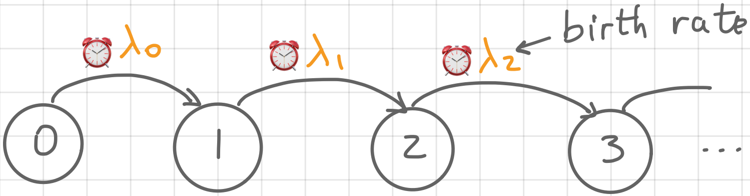
- No homework!
- Quiz 1 on Wednesday, October 14

Birth and death processes

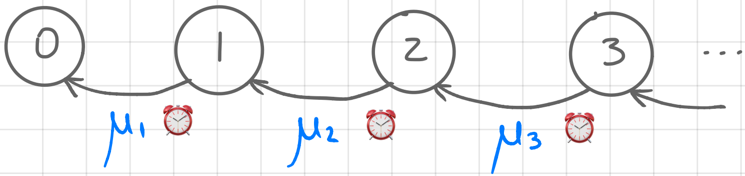
- Pure birth / death processes

! Only jumps of magnitude 1

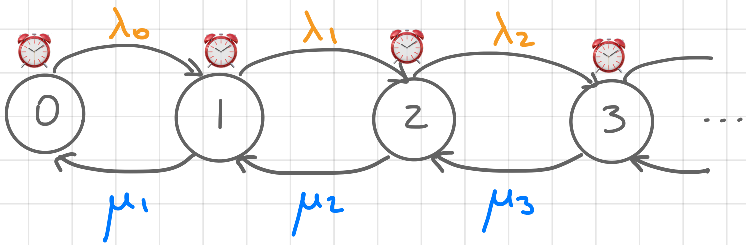
$i \to i-1$ or $i \to i+1$



$\xrightarrow{\lambda_k = \lambda}$ Poisson process



- Birth and death processes



$\rightsquigarrow$ random walk

Birth and death processes. Results

+ infinitesimal transition probability description
+ sojourn time description (jump and hold)

  sojourn times are independent exponential r.v.s

  $$P(i \to i+1) = \frac{\lambda_i}{\lambda_i + \mu_i}, \quad P(i \to i-1) = \frac{\mu_i}{\lambda_i + \mu_i}$$

+ system of differential equations for pure birth/death

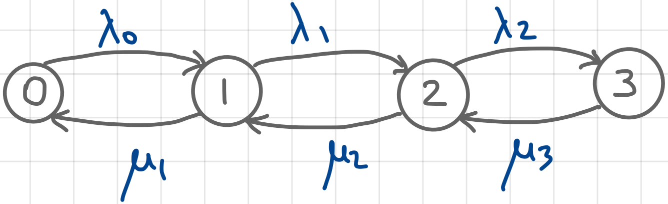
  e.g. $$P_i'(t) = -\lambda_i P_i(t) + \lambda_{i-1} P_{i-1}(t)$$

+ distributions of X_t for linear birth (geometric) and linear death (binomial) processes
+ first step analysis giving absorption probabilities and mean time to absorption
+ explosion, Strong Markov property etc.

General continuous time MC

Assume for simplicity that the state space is finite

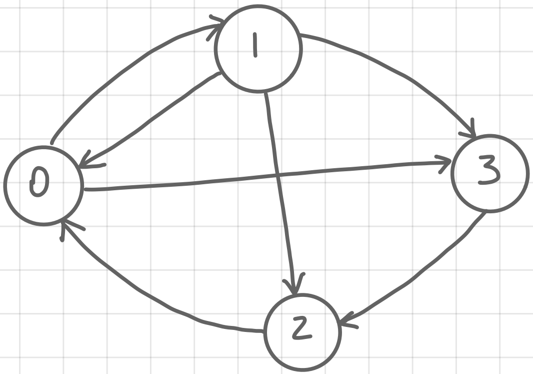
$(X_t)_{t\geq 0}$ is right-continuous



birth and death process

$P_{i,i+1}(h) = \lambda_i h + o(h)$

$P_{i,i}(h) = \ldots$



general MC

$P(X_{t+s}=j \mid X_t = i) = P(X_s = j \mid X_0 = i)$

$P_{ij}(h) = q_{ij} h + o(h)$?

How to define? How to analyze?

Q-matrices (infinitesimal generators)

Let $S=\{0,1,\dots,N\}$. We call $Q=(q_{ij})_{i,j=0}^{N}$ a Q-matrix if Q satisfies the following conditions:

(a) $0 \le -q_{ii} < \infty$ for all i

(b) $q_{ij} \ge 0$ for all $i \ne j$

(c) $\sum_j q_{ij} = 0$ for all i

Denote $q_i = \sum_{j\ne i} q_{ij}$

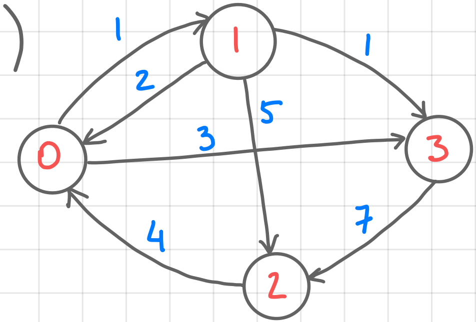
then $q_{ii} = -q_i$

Examples

(a)

$$Q = \begin{pmatrix} -2 & 1 & 1 \\ 2 & -7 & 5 \\ 0 & 2 & -2 \end{pmatrix}$$

(b)



$$\begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & & & & \\ 1 & & & & \\ 2 & & & & \\ 3 & & & & \end{array}\quad\begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array}\begin{pmatrix} -4 & 1 & 0 & 3 \\ 2 & -8 & 5 & 1 \\ 4 & 0 & -4 & 0 \\ 0 & 0 & 7 & -7 \end{pmatrix}$$

Matrix exponentials

Let $Q = (q_{ij})_{i,j=1}^{N}$ be a matrix. Then the series $\sum_{k=0}^{\infty} \frac{Q^k}{k!}$ converges componentwise, and we denote

its sum $\sum_{k=0}^{\infty} \frac{Q^k}{k!} =: e^{Q}$, the matrix exponential of Q.

In particular, we can define $e^{tQ} = \sum_{k=0}^{\infty} \frac{Q^k t^k}{k!}$ for $t \geq 0$.

Thm. Define $P(t) = e^{tQ}$. Then

(i) $P(t+s) = P(t)\,P(s)$ for all s, t

(ii) $(P(t))_{t\geq 0}$ is the unique solution to the equations

$$\begin{cases} \frac{d}{dt} P(t) = P(t)Q \\ P(0) = I \end{cases}, \quad \text{and} \quad \begin{cases} \frac{d}{dt} P(t) = Q\,P(t) \\ P(0) = I \end{cases}$$

Matrix exponentials

Properties are easy to remember $\longrightarrow$ scalar exponential

(i) $e^{(t+s)Q} = e^{tQ} e^{sQ} = e^{sQ} e^{tQ}$ $\qquad \left(e^{(t+s)\alpha} = e^{t\alpha} e^{s\alpha} \right)$

(note that in general $AB \neq BA$ for matrices A, B)

(ii) $\frac{d}{dt} e^{tQ} = Q e^{tQ} = e^{tQ} Q$ $\qquad \left(\frac{d}{dt} e^{t\alpha} = \alpha e^{t\alpha} \right)$

$e^{0 \cdot Q} = I$ $\qquad (e^0 = 1)$

Example

(a) $Q_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $Q_1^2 = 0 \implies e^{tQ_1} = I + tQ_1 + 0 = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$

(b) $Q_2 = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$, $e^{tQ_2} = \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix}$

Matrix exponentials

Results on the previous slide hold for any matrix Q.

Thm. Matrix Q is a Q-matrix

iff $P(t) = e^{tQ}$ is a stochastic matrix $\forall t$

$$\left(\sum_j P_{ij}(t) = 1 \text{ for all } i \right)$$

Remarks The semigroup property gives entrywise

$$P_{ij}(t+s) = [P(t)P(s)]_{ij}$$

$$= \sum_{k=0}^{N} P_{ik}(t) P_{kj}(s)$$

(if you think about MC → Chapman-Kolmogorov)