MATH 142A: Introduction to Analysis

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Today: Basic properties of the derivative
> Q&A: February 25
Next: Ross § 29

Week 8:

- Homework 7 (due Sunday, February 27)

Differentiability and derivative

<u>Def</u> Let $f: I \to \mathbb{R}$, I open interval. Let $a \in I$.

We say that f is differentiable at $a \in I$, or that f has a derivative at a, if the limit

exists and is finite. If f is differentiabe $\forall a \in I$, we get a function

<u>Examples</u>   1) Let $f(x) = x$. Then $\forall a \in \mathbb{R}$ $f'(a) = 1$ (so $f'(x) = 1$)

2) Let $f(x) = \sin x$. Then $f'(x) = \cos x$

Important examples (limits of functions)

__IE 13__ $\lim\limits_{x \to 0} \dfrac{\log(1+x)}{x}$

__Proof.__ ① $\dfrac{\log(1+x)}{x}$ is well-defined on $(-1, +\infty) \setminus \{0\}$

②   Write $\dfrac{\log(1+x)}{x} =$

③ $\lim\limits_{x \to 0_+} (1+x)^{\frac{1}{x}} =$: Let (x_n) be a sequence in $(0,1)$

   $\lim x_n = 0$. Define   Then

   and

④ $\lim\limits_{x \to 0_-} (1+x)^{\frac{1}{x}} =$. As in ③

⑤   By Thm 20.10

⑥ $\log$ is continuous on $(0, +\infty) \Rightarrow$

Limits of functions. Examples

<u>Warm up</u>   Last time : $\lim\limits_{x \to \infty} \left(1 + \frac{1}{x}\right)^x = e$

① $\lim\limits_{n \to \infty} \left(1 - \frac{1}{n}\right)^{-n} = \lim\limits_{n \to \infty} \left(\frac{n-1}{n}\right)^{-n} =$

② $\lim\limits_{n \to \infty} \left(1 - \frac{1}{n}\right)^{-(n-1)} = \lim\limits_{n \to \infty} \left(1 - \frac{1}{n-1}\right)^{-n} =$

③ $\forall \ x < 0$ $\left(1 + \frac{1}{\lceil x \rceil}\right)^{\lceil x \rceil + 1} < \left(1 + \frac{1}{x}\right)^x < \left(1 + \frac{1}{\lceil x \rceil + 1}\right)^{\lceil x \rceil}$

④   Fix $\varepsilon > 0$. $\exists N \ \forall n > N$ $\left| \left(1 - \frac{1}{n}\right)^{-(n+1)} - e \right| < \varepsilon, \ \left| \left(1 - \frac{1}{n+1}\right)^{-n} - e \right| < \varepsilon$

    For $x <$

            $< \left(1 + \frac{1}{x}\right)^x - e <$

Important examples (limits of functions)

IE 14

$$\lim_{x \to 0} \frac{e^x - 1}{x} =$$

Proof   Denote $f(x) :=$, so that $x =$

Then $\dfrac{e^x - 1}{x} =$ where

① $f(x)$ is continuous on $\mathbb{R}$,

② g is defined on $(-1, +\infty)$, by

$\Rightarrow$

Examples

3) $(e^x)' = e^x$

For any $x \in \mathbb{R}$

$$\lim_{h \to 0} \frac{e^{x+h} - e^x}{h} =$$

Thm 28.2   f is differentiable at point a $\Rightarrow$ f is continuous at a

Proof.   f differentiable at a $\Rightarrow$ $\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = f'(a)$

Rewrite $f(x) =$

Then $\lim_{x \to a} f(x) =$

Important examples (limits of functions)

IE 15

$$\forall \alpha \in \mathbb{R}$$

$$\lim_{x \to 0} \frac{(1+x)^{\alpha} - 1}{x} =$$

Proof

① Write $\dfrac{(1+x)^{\alpha} - 1}{x} =$

② Denote $f(x) =$

$$g(y) =$$

Then by IE 14 , so

by Thm 20.5

③ By IE 13

Derivatives and arithmetic operations

Thm 28.3 Let f and g be differentiable at a, $c \in \mathbb{R}$. Then $c \cdot f$, $f+g$ and $f \cdot g$ are differentiable at a. If additionally $g(a) \neq 0$, then $\frac{f}{g}$ is differentiable at a. Moreover

$$(cf)'(a) = c \cdot f'(a), \quad (f+g)'(a) = f'(a) + g'(a), \quad (f \cdot g)'(a) = f'(a)g(a) + f(a)g'(a)$$

$$\left(\frac{f}{g}\right)'(a) = \frac{f'(a)g(a) - f(a)g'(a)}{g^2(a)}$$

Proof. $(cf)'$, $(f+g)'$ — exercise.

$$\lim_{x \to a} \frac{f(x)g(x) - f(a)g(a)}{x-a} = \lim_{x \to a} \frac{\big(f(x) - f(a)\big)g(x) + f(a)\big(g(x) - g(a)\big)}{x-a}$$

$$= f'(a) \cdot g(a) + f(a) \cdot g'(a)$$

If $g(a) \neq 0$, then

$$\lim_{x \to a} \frac{\frac{f(x)}{g(x)} - \frac{f(a)}{g(a)}}{x-a} = \lim_{x \to a} \frac{1}{g(x)g(a)} \cdot \frac{f(x)g(a) - f(a)g(x) - f(a)g(a) + f(a)g(a)}{x-a} = \frac{f'(a)g(a) - f(a)g'(a)}{\big(g(a)\big)^2} \quad \blacksquare$$

Derivative of a composition

__Thm 28.4__ If f is differentiable at a, and g is differentiable at $f(a)$, then $g \circ f$ is differentiable at a and

$$(g \circ f)'(a) =$$

__Remark__ $\dfrac{g(f(x)) - g(f(a))}{x - a} =$

Take $f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$, $g(y) = e^y$: $\displaystyle \lim_{x \to 0} \dfrac{e^{x^2 \sin \frac{1}{x}} - e^0}{x^2 \sin\left(\frac{1}{x}\right)}$ is not well defined $\left(x_n = \frac{1}{\pi n}\right)$

__Proof:__ ① g is defined on $(f(a) - c, f(a) + c)$ for some $c > 0$.

  f is cont. at $a \Rightarrow$

Derivative of a composition

Case 1: $\exists\ \eta \leq \delta$ s.t $\forall\ x \in (a-\eta, a+\eta)$

Then $\dfrac{g(f(x)) - g(f(a))}{f(x) - f(a)}$ can be written on $(a-\eta, a+\eta)$ as $\varphi \circ f(x)$

where $\varphi(y) = \begin{cases} & \end{cases}$ is defined on $(f(a)-c, f(a)+c)$

g is differentiable at $f(a) \Rightarrow$

By Thm 20.5

$\Rightarrow \displaystyle\lim_{x \to a} \frac{g(f(x)) - g(f(a))}{x-a} = \lim_{x \to a} \frac{g(f(x)) - g(f(a))}{f(x) - f(a)} \cdot \lim_{x \to a} \frac{f(x) - f(a)}{x-a}$

$=$

<u>Case 2:</u> $\exists (x_n)$, $\lim x_n = a$, $\forall n \; x_n \neq a$, $f(x_n) = f(a)$

T20.2 $\rightarrow$ f is continuous at a, and

$\llcorner\rightarrow$ g is continuous at $f(a)$, $f(x_n) = f(a) \; \forall n$, so

if $\displaystyle\lim_{x \to a} \frac{g \circ f(x) - g \circ f(a)}{x - a} = (g \circ f)'(a)$ exists, then

$$(g \circ f)'(a) =$$

Fix $\varepsilon > 0$.

Then

Then $\forall x \in (a - \delta', a + \delta') \backslash \{a\}$, $f(x) \neq f(a)$