

Math 180A: Introduction to Probability

Lecture A00 (Au)

math.ucsd.edu/~bau/w21.180a

Lecture B00 (Nemish)

math.ucsd.edu/~ynemish/teaching/180a

Today: ASV 3.1 (Probability distributions)

ASV 3.2 (Cumulative distribution function)

Video: Prof. Todd Kemp, Fall 2019

Next: ASV 3.2

Week 3: Quiz 2 (Wednesday, Jan 20, on Lectures 3-5)

Homework 2 (due Friday, Jan 22)

Random Variables

3.1

Given a probability space (Ω, \mathcal{F}, P) a random variable is a function

$$X : \Omega \rightarrow \mathbb{R}$$

This is a bad, old-fashioned name. Would be better to call it a random function or random measurement.

E.g. Toss a fair coin 4 times. Let $X = \text{number of tails}$.

E.g. Shoot an arrow at a circular target. $Y = \text{distance from center}$.

E.g. Your car is in a minor accident; the damage repair cost is a random number between \$100 and \$1500. Your insurance deductible is \$500. $Z = \text{your out of pocket expenses}$.

In all these examples, think about what you observe. You can't really see a formula for the function. By repeating the experiment over and over, all you can learn is the probability distribution.

Probability Distribution

3.2

Given a probability space (Ω, \mathcal{F}, P) and a random variable
 $X: \Omega \rightarrow \mathbb{R}$

the probability distribution or law of X is a probability measure,
 μ_X ON \mathbb{R} .

$$A \subseteq \mathbb{R} \rightsquigarrow \mu_X(A) = P(\{X \in A\})$$

[Caution: for this to make sense, we need to have a designated set of allowed "events" in \mathbb{R} ; call this collection $\mathcal{B}(\mathbb{R})$.
Then, we must have

For each $A \in \mathcal{B}(\mathbb{R})$, $\{X \in A\} \in \mathcal{F}$.

This is a condition on X ; we call such functions **measurable**.

We will ignore these technicalities in this course; all our random variables are indeed measurable.

E.g. Toss a fair coin 4 times. Let $X = \text{number of tails}$.

$\underbrace{\quad}_{\Omega = \{(x_1, x_2, x_3, x_4) \in \{H, T\}^4\}}$

$$\Omega = \{(x_1, x_2, x_3, x_4) \in \{H, T\}^4\}$$

P = uniform on Ω ;

$$P\{(x_1, x_2, x_3, x_4)\} = \frac{1}{2^4} = \frac{1}{16}$$

$$\{X=2\} = \{(T, T, H, H), (T, H, T, H), \dots\}$$

$$\#\{X=2\} = \binom{4}{2}.$$

$$P(X=2) = \frac{\binom{4}{2}}{16} = \frac{3}{8}$$

$$P(X=k) = \frac{\binom{4}{k}}{16}$$

$X = \# \text{tails in } n \text{ coin tosses}$,

$$P_X(k) = P(X=k) = \frac{1}{2^n} \binom{n}{k} \quad 0 \leq k \leq n$$

\downarrow

$$X \in \{0, 1, 2, 3, 4\} \subset \mathbb{R}$$

if $A \subseteq \mathbb{R}$ does not contain one of these numbers, $\mu_X(A) = 0$.

By additivity of probability measures, to understand μ_X , just need to know

$$\mu_X(\{k\}) = ? \quad 0 \leq k \leq 4$$

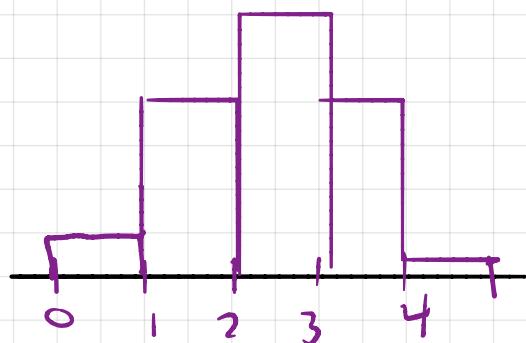
||

$$P(X=k) = P_X(k)$$

k	0	1	2	3	4
$P_X(k)$	$\frac{1}{16}$	$\frac{1}{4}$	$\frac{3}{8}$	$\frac{1}{4}$	$\frac{1}{16}$

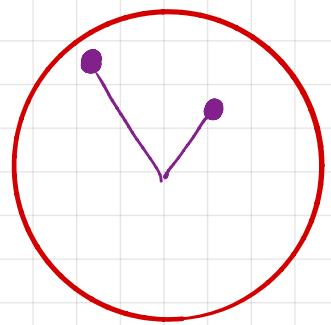
Binomial

$\text{Bin}(n, \frac{1}{2})$



E.g. Shoot an arrow at a circular target of radius 1.

Y = distance from center.



Last lecture, you calculated that:

- If $r \leq 0$, $P(Y \leq r) = 0$
- If $r \geq 1$, $P(Y \leq r) = 1$.
- If $r \in [0, 1]$, $P(Y \leq r) =$

$$\frac{\text{Area}(\{Y \leq r\})}{\text{Area}(\{Y \leq 1\})} = r^2.$$

$P(Y = 0.4) \leq P(Y \in (0.4 - \varepsilon, 0.4]) \rightarrow 0 \text{ as } \varepsilon \rightarrow 0.$

as $\varepsilon \rightarrow 0$,
this $\rightarrow 0$.

$$P(\{Y \in (-\infty, r]\})$$

Can get information about
other sets $A \subseteq \mathbb{R}$ using the
properties of P .

What can we say about $P(Y = 0.4) \neq 0$?

$$(0.3, 0.4]$$

$$P(Y \in (-\infty, 0.4]) = P(Y \in (-\infty, 0.3]) + P(Y \in (0.3, 0.4])$$

$$[0.3 \quad 0.4]$$

$$(-\infty, 0.4] = (-\infty, 0.3] \cup (0.3, 0.4]$$

$$0.4^2 - 0.3^2 = 0.07$$
$$P(Y \in (0.4 - \varepsilon, 0.4]) = 0.4^2 - (0.4 - \varepsilon)^2$$

We will focus mostly on two kinds of random variables:

discrete: There are finitely (or countably) many possible values $\{k_1, k_2, k_3, \dots\}$ for X .

↪ μ_X is described by the probability mass function $p_X(k) = P(X=k)$
 $k \in \{k_1, k_2, k_3, \dots\}$

In this case, by the laws of probability,

$$p_X(k) \geq 0 \text{ for each } k, \quad \sum_{j=1}^n p_X(j) = 1.$$

continuous: For any real number $t \in \mathbb{R}$, $P(X=t) = 0$.

↪ μ_X is captured by understanding $P(X \leq r)$ as a function of r .

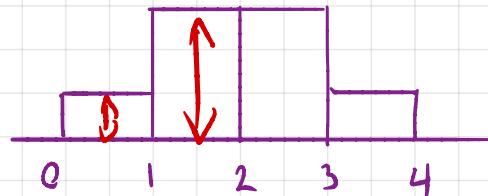
$$\begin{aligned} \text{Eg. } P(X \in [a, b]) &= P(\{X=a\} \cup \{X \in (a, b]\}) \\ &= P(X \neq a) + P(X \in (a, b]) \\ &\quad \swarrow 0 \\ &= P(X \leq b) - P(X \leq a) \end{aligned}$$

Cumulative Distribution Function (CDF)

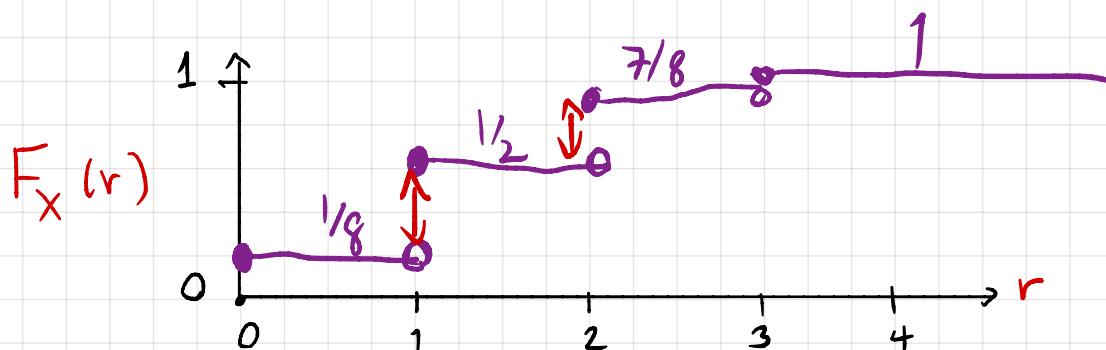
For any random variable X , $F_X(r) = P(X \leq r)$.

E.g. $M_X = \text{Bin}(3, \frac{1}{2})$

k	0	1	2	3
$P_X(k)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$



- $r < 0$, $\{X \leq r\} = \emptyset$ $P(X \leq r) = 0$
- $0 \leq r < 1$, $\{X \leq r\} = \{X=0\}$ $P(X \leq r) = P(X=0) = \frac{1}{8}$
- $1 \leq r < 2$, $\{X \leq r\} = \{X=0, 1\}$ $P(X \leq r) = P(X=0) + P(X=1) = \frac{1}{8} + \frac{3}{8} = \frac{1}{2}$
- $2 \leq r < 3$, $\{X \leq r\} = \dots$
- $r \geq 3$, $\{X \leq r\} = \dots$



Actually works
in all cases -
including discrete.

Properties of the CDF $F_X(r) = P(X \leq r)$

(1) Monotone increasing: $s \leq t \Rightarrow F_X(s) \leq F_X(t)$

(2) $\lim_{r \rightarrow -\infty} F_X(r) = 0$, $\lim_{r \rightarrow +\infty} F_X(r) = 1$.

(3) The function F_X is right-continuous: $\lim_{t \rightarrow r^+} F_X(t) = F_X(r)$.

Corollary: If X is a continuous random variable, F_X is a continuous function.

Densities

Some continuous random variables have probability densities.

This is an infinitesimal version of a probability mass function.

X discrete, $\in \{k_1, k_2, k_3, \dots\}$

$$p_X(k) = P(X=k)$$

$$P(X \in A) = \sum_{k \in A} P(X=k)$$

$$= \sum_{k \in A} p_X(k)$$

X continuous

$$P(X=t) = 0 \text{ for all } t \in \mathbb{R}.$$

$$P(X \in A) = \int f_X(t) dt$$

$$p_X(k) \geq 0, \quad \sum_k p_X(k) = 1.$$

E.g. Shoot an arrow at a circular target of radius 1.

Y = distance from center.

$$\int_{-\infty}^r f(t) dt \stackrel{?}{=} P(Y \in (-\infty, r]) = F_Y(r) = \begin{cases} 0, & r \leq 0 \\ r^2, & 0 \leq r \leq 1 \\ 1, & r \geq 1 \end{cases}$$

