☐ **Write your name and PID on the top of EVERY PAGE.**

☐ **Write the solutions to each problem on separate pages. CLEARLY INDICATE on the top of each page the number of the corresponding problem. Different parts of the same problem can be written on the same page (for example, part (a) and part (b)).**

☐ **Remember this exam is graded by a human being. Write your solutions NEATLY AND COHERENTLY, or they risk not receiving full credit.**

☐ **This exam is designed to be written in 50 minutes. In order for your exam to be graded, you have to upload it to Gradescope before November 23, 11:59 PM.**

☐ **You may assume that all transition probability functions are STATIONARY.**

☐ **You are allowed to use the textbook, lecture notes and your personal notes. You are not allowed to use the electronic devices (except for accessing the online version of the textbook) or outside assistance. Outside assistance includes but is not limited to other people, the internet and unauthorized notes.**

1. (40 points) Let $Y \sim \text{Exp}(\lambda)$ for some $\lambda > 0$ and let $X \sim \text{Exp}(\frac{1}{Y})$.
   (a) (20 points) Compute the expectation $E(X)$ of the random variable X.
   (b) (20 points) Compute the variance $\text{Var}(X)$ of the random variable X.

Solution.

(a) Condition of the value of Y and use $E(X \mid Y = y) = y$

$$E(X) = \int_0^\infty E(X \mid Y = y)\lambda e^{-\lambda y} dy = \int_0^\infty y\lambda e^{-\lambda y} dy = \frac{1}{\lambda}. \tag{1}$$

(b) Now compute $E(X^2)$ using $E(X^2 \mid Y = y) = 2y^2$

$$E(X^2) = \int_0^\infty 2y^2\lambda e^{-\lambda y} dy = \frac{4}{\lambda^2}. \tag{2}$$

We can now compute the variance

$$\text{Var}(X) = \frac{4}{\lambda^2} - \frac{1}{\lambda^2} = \frac{3}{\lambda^2}. \tag{3}$$

2. (40 points) The economic history of a certain county is characterized by alternating periods of long economic growth and periods of long recession. Suppose that the lengths of all periods are independent and have uniform distribution on $[0, 1]$ (in years), both for growth and recession. At the beginning of our observation (time $t = 0$) a new recession starts.
   (a) (15 points) Let X and Y be independent random variables having uniform distributions on [0,1]. Compute

$$P(X + Y \leq t) = \begin{cases} & t \leq 0, \\ & 0 < t \leq 1, \\ & 1 < t \leq 2, \\ & t > 2. \end{cases}$$

   [Hint. Draw a unit square.]

   (b) (25 points) What is the long-run probability that there will be no new recession starting within next year [Hint. Formulate using the excess life.]

Solution.

(a)

$$P(X + Y \leq t) = \begin{cases} 0, & t \leq 0, \\ t^2/2, & 0 < t \leq 1, \\ 1 - (2-t)^2/2, & 1 < t \leq 2, \\ 1, & t > 2. \end{cases} \tag{4}$$

(b) Let $N(t)$ count the number of times the economy switches from growth to recession on the time interval $[0, t]$. Then $N(t)$ is a renewal process with interrennewal times being the sum of two independent uniformly distributed on $[0, 1]$ random variables (periods of recession and growth). The interrenewal distribution $F(t)$ is then equal to function (4) computed in part (a)

$$F(t) = \begin{cases} 0, & t \leq 0, \\ t^2/2, & 0 < t \leq 1, \\ 1 - (2-t)^2/2, & 1 < t \leq 2, \\ 1, & t > 2. \end{cases} \tag{5}$$

Each renewal denotes the start of a new recession. For any fixed time t, the event that there will be no new recession starting during the next year means in terms of the renewal process that the excess life at time t is greater than 1, i.e., $\gamma_t > 1$. If we observe the situation in the country at some time t with $t \gg 1$ large, then from the theorem about the limiting distribution of the excess life (Lecture 14-15)

$$P(\gamma_t > 1) = \frac{1}{\mu} \int_1^\infty (1 - F(x))dx, \tag{6}$$

where μ is the mean interrenewal time. Plugging (4) and $\mu = 1$ into the above formula, we get

$$P(\gamma_t > 1) = \frac{1}{\mu} \int_1^\infty (1 - F(x))dx = \int_1^2 \frac{(2-x)^2}{2} dx = \frac{1}{6}. \tag{7}$$

3. (20 points) Suppose that certain company is using age replacement policy for replacing lightbulbs in its offices: a lightbulb is replaced either upon its failure, or after reaching age $T > 0$, whichever comes first. Suppose that each bulb replacement costs 1 dollar, but if it happens due to a failure, then it incurs **additional** costs of 4 dollars per replacement. It is given that the lifetime of a lightbulb has a uniform distribution on the interval [0,2].

Determine the optimal replacement age T (that minimizes the long run mean cost of the replacement) and compute the long run mean replacement cost per unit of time for this choice of T. Compare it to the costs of replacement upon failure.

Solution. Use age replacement strategy from Lecture 17. If the cost of one replacement is K dollars, each replacement due to a failure costs additional c dollars, T is the replacement age and the interrenewal distribution is given by F, then the long run replacement cost (per unit) is given by

$$C(T) = \frac{K + cF(T)}{\int_0^T (1 - F(x))dx}. \tag{8}$$

In our particular case, $K = 1$, $c = 4$ and

$$F(t) = \begin{cases} 0, & t \leq 0, \\ t/2, & 0 < t \leq 2, \\ 1, & t > 2, \end{cases} \tag{9}$$

so

$$\int_0^T (1 - F(x))dx = T - \frac{T^2}{4} \tag{10}$$

for $0 \le T \le 2$. Therefore,

$$C(T) = \frac{1 + 2T}{T - T^2/4}. \tag{11}$$

Find the minimum

$$C'(T) = \frac{2(T - T^2/4) - (1 + 2T)(1 - T/2)}{(T - T^2/4)^2} = \frac{T^2/2 + T/2 - 1}{(T - T^2/4)^2} = 0. \tag{12}$$

Multiplying the numerator by 2, we get that the equation

$$T^2 + T - 2 = 0, \tag{13}$$

has two solutions, $T = -2$ and $T = 1$. Point $T = 1$ is the point of minimum, therefore, the optimal long run replacement cost is equal to

$$C(1) = \frac{1 + 2}{1 - 1/4} = 4. \tag{14}$$

The cost of replacement upon failure is $K + c = 1 + 4 = 5$. The age replacement policy with the replacement age $T = 1$ will save the company $5 - 4 = 1$ dollar per each replacement in the long run.