MATH180C: Introduction to Stochastic Processes II

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Today: Kolmogorov's equations

> Q&A: October 21

Next: PK 6.4, 6.6, Durrett 4.3

This week:

- Quiz 2 on Wednesday, October 21 (lectures 4-6)
- Homework 2 (due Friday, October 23, 11:59 PM)

Kolmogorov equations

Jump and hold description is very intuitive, gives a very clear picture of the process, but does not answer to some very basic questions, e.g., computing $P_{ij}(t) := P(X_t = j \mid X_0 = i)$.

For computing the transition probabilities the differential equation approach is more appropriate.

In order to derive the system of differential equations for $P_{ij}(t)$ from the infinitesimal description, we start from the familiar relation:

Chapman-Kolmogorov equation (semigroup property)

Chapman-Kolmogorov equation

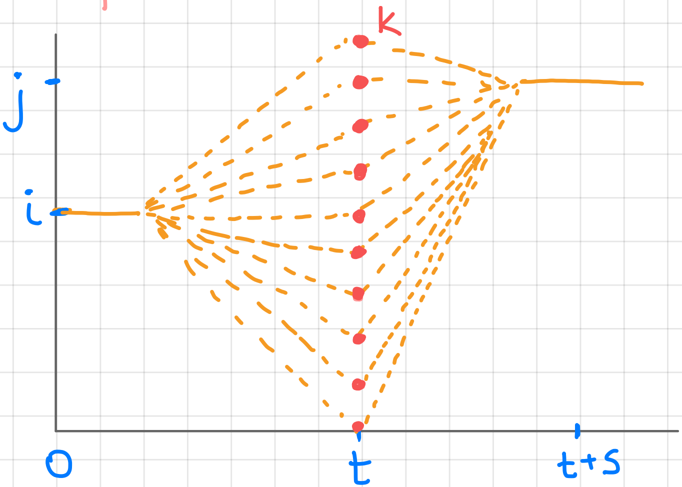
$$P_{ij}(t+s) = P(X_{t+s}=j \mid X_0=i)$$

condition on the value of X_t

$$= \sum_{k=0}^{N} P(X_{t+s}=j \mid X_t=k, X_0=i)\, P(X_t=k \mid X_0=i)$$

Markov $= \sum_{k=0}^{N} P(X_{t+s}=j \mid X_t=k)\, P(X_t=k \mid X_0=i)$

stationary trans. prob. $= \sum_{k=0}^{N} P(X_s=j \mid X_0=k)\, P(X_t=k \mid X_0=i) = \sum_{k=0}^{N} P_{kj}(s)\, P_{ik}(t)$



Or in matrix form

$$P(t+s) = P(t)\,P(s$$

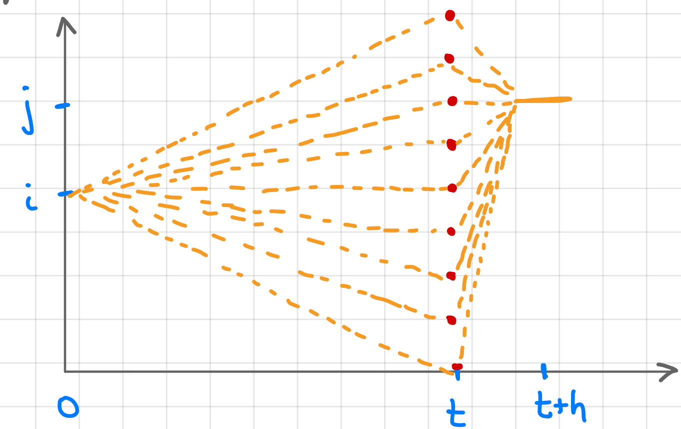
Kolmogorov forward equations

$Q = (q_{ij})_{i,j=0}^{N}$

Apply Chapman-Kolmogorov equations to compute

$P_{ij}(t+h)$:

$$P_{ij}(t+h) = \sum_{k=0}^{N} P_{ik}(t)\, P_{kj}(h) \quad (*)$$



Use infinitesimal description:

$$P_{kj}(h) = \begin{cases} q_{kj}\, h + o(h), & k \neq j \\ 1 + q_{jj}\, h + o(h), & k = j \end{cases}$$

$$(*) = P_{ij}(t)\left(1 + q_{jj} h + o(h)\right) + \sum_{\substack{k=0 \\ k \neq j}}^{N} P_{ik}(t)\left(q_{kj} h + o(h)\right)$$

$$= P_{ij}(t) + \underbrace{\sum_{k=0}^{N} P_{ik}(t)\, q_{kj}}_{[P(t)\,Q]_{ij}} h + o(h)$$

$\Rightarrow$ $P(t+h) = P(t) + P(t)Qh + o(h)$

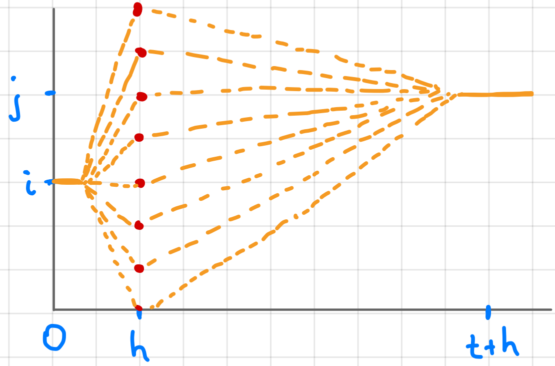
$$\boxed{\frac{d}{dt} P(t) = P(t)\, Q}$$

Kolmogorov backward equations

$$P_{ij}(t+h) = \sum_{k=0}^{N} P_{ik}(h) P_{kj}(t)$$

$$= (1 + q_{ii} h + o(h)) P_{ij}(t)$$

$$+ \sum_{\substack{k=0 \\ k \neq i}}^{N} (q_{ik} h + o(h)) P_{kj}$$



$$= P_{ij}(t) + \sum_{k=0}^{N} q_{ik} P_{kj}(t) h + o(h)$$

$\hookrightarrow$ $$\boxed{\frac{d}{dt} P(t) = Q P(t)}$$

$$P(0) = I$$

Kolmogorov equations. Remarks

1. e^{tQ} satisfies both (forward and backward) equations.

Indeed, omitting technical details, differentiate term-by-term

$$\frac{d}{dt} e^{tQ} = \frac{d}{dt}\left(\sum_{k=0}^{\infty} \frac{Q^k t^k}{k!} \right) = \sum_{k=0}^{\infty} \frac{Q^k}{k!} \frac{d}{dt}(t^k) = \sum_{k=1}^{\infty} \frac{Q^k}{(k-1)!} t^{k-1}$$

Now $\displaystyle \sum_{k=1}^{\infty} \frac{Q^k}{(k-1)!} t^{k-1} \overset{l=k-1}{=} \sum_{l=0}^{\infty} \frac{Q^{l+1}}{l!} t^l = Q \sum_{l=0}^{\infty} \frac{Q^l t^l}{l!} = \sum_{l=0}^{\infty} \frac{Q^l t^l}{l!} Q$

2. Redundancy is related to the stationarity of transition probabilities. If transition probabilities $P_{ij}(s,t) = P(X_t = j \mid X_s = i)$ are not stationary, then

$\frac{\partial}{\partial t} P_{ij}(s,t) \to$ forward equation, $\frac{\partial}{\partial s} P_{ij}(s,t) \to$ backward equation

Example

Two-state MC

$$Q = \begin{pmatrix} -\alpha & \alpha \\ \beta & -\beta \end{pmatrix}$$

$$Q^2 = \begin{pmatrix} -\alpha & \alpha \\ \beta & -\beta \end{pmatrix}\begin{pmatrix} -\alpha & \alpha \\ \beta & -\beta \end{pmatrix} = \begin{pmatrix} \alpha(\alpha+\beta) & -\alpha(\alpha+\beta) \\ -\beta(\alpha+\beta) & \beta(\alpha+\beta) \end{pmatrix} = -(\alpha+\beta)Q$$

$$\hookrightarrow Q^k = (-(\alpha+\beta))^{k-1} Q$$

$$e^{tQ} = \sum_{k=0}^{\infty} \frac{Q^k t^k}{k!} = I + \sum_{k=1}^{\infty} \frac{(-(\alpha+\beta))^{k-1} t^k}{k!} Q$$

$$= I + \frac{1}{-(\alpha+\beta)} \sum_{k=1}^{\infty} \frac{(-(\alpha+\beta))^{k} t^k}{k!} Q$$

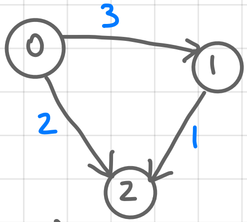
$$= I - \frac{1}{\alpha+\beta}\left(e^{-(\alpha+\beta)t} - 1\right) Q$$

$$= I + \frac{1}{\alpha+\beta} Q - \frac{1}{\alpha+\beta} e^{-(\alpha+\beta)t} Q$$

Example

Let $(X_t)_{t\geq 0}$ be a MC with generator Q

$$Q = \begin{pmatrix} -5 & 3 & 2 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$



Compute $P_{01}(t)$

For any k, $Q^k = \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix}$, $\Rightarrow P_{10}(t) = P_{20}(t) = P_{21}(t) = 0$

$$P'(t) = \begin{pmatrix} P_{00} & P_{01} & P_{02} \\ 0 & P_{11} & P_{12} \\ 0 & 0 & P_{22} \end{pmatrix} \begin{pmatrix} -5 & 3 & 2 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$P_{00}'(t) = -5P_{00}(t)$, $P_{00}(0) = 1 \Rightarrow P_{00}(t) = e^{-5t}$

$P_{11}'(t) = -P_{11}(t)$, $P_{11}(0) = 1 \Rightarrow P_{11}(t) = e^{-t}$

$P_{22}'(t) = 0$, $P_{22}(0) = 1 \Rightarrow P_{22}(t) = 1$

$$\begin{cases} P_{01}'(t) = 3P_{00}(t) - P_{01}(t) \\ P_{01}(0) = 0 \end{cases}$$

$P_{01}'(t) = -P_{01}(t) + 3e^{-5t}$

$P_{01}(t) = e^{-t}\cdot C - \frac{3}{4}e^{-5t} \Rightarrow C = \frac{3}{4}$

$P_{01}(t) = \frac{3}{5}\cdot\frac{5}{4}\left(e^{-t} - e^{-5t}\right)$

Forward and backward equations for B&D processes

Forward equation:

$$P_{ij}(t+h) = \sum_{k=0}^{\infty} P_{ik}(t) P_{kj}(h)$$

$$= P_{ij}(t)\left(1-(\lambda_j+\mu_j)h+o(h)\right)$$

$$+ P_{i,j-1}(t)\left(\lambda_{j-1}h+o(h)\right) + P_{i,j+1}(t)\left(\mu_{j+1}h+o(h)\right)$$

$$+ \underbrace{\sum_{\substack{k=0 \\ |k-j|>1}}^{\infty} P_{ik}(t)\left[o(h)_{kj}\right]}_{\theta_{ij}(h)}$$

If $\theta_{ij} = o(h)$ (requires additional technical assumptions)

$$\begin{cases} P_{ij}'(t) = \lambda_{j-1} P_{i,j-1}(t) - (\lambda_j+\mu_j) P_{ij}(t) + \mu_{j+1} P_{i,j+1}(t) \\ P_{i0}'(t) = -\lambda_0 P_{i0}(t) + \mu_1 P_{i1}(t) \end{cases}, \quad \text{with } P_{ij}(0) = \delta_{ij}$$

Forward and backward equations for B & D processes

Similarly, we derive the backward equations

$$\begin{cases} P'_{ij}(t) = \mu_i P_{i-1,j}(t) - (\lambda_i + \mu_i) P_{ij}(t) + \lambda_i P_{i+1,j}(t) \\ P_{0j}(t) = -\lambda_0 P_{0j}(t) - \lambda_0 P_{1j}(t) \end{cases}, \quad \text{with } P_{ij}(0) = \delta_{ij}$$

<u>Example</u> Linear growth with immigration.

Recall $\lambda_k = \lambda \cdot k + a$ immigration

linear birth rate

$\mu_k = \mu \cdot k$

linear death rate

Example: Linear growth with immigration.

Use forward equations to compute $E(X_t \mid X_0 = i)$

$$\begin{cases} P_{ij}'(t) = \lambda_{j-1} P_{i,j-1}(t) - (\lambda_j + \mu_j) P_{ij}(t) + \mu_{j+1} P_{i,j+1}(t) \\ P_{i0}'(t) = -\lambda_0 P_{i0}(t) + \mu_1 P_{i1}(t) \end{cases}$$

$$M'(t) = \sum_{j=0}^{\infty} j P_{ij}'(t)$$

$$E(X_t \mid X_0 = i) = \sum_{j=0}^{\infty} j \cdot P(X_t = j \mid X_0 = i) = \sum_{j=0}^{\infty} j \cdot P_{ij}(t) =: M(t)$$

$$P_{ij}'(t) = (\lambda(j-1) + a) P_{i,j-1}(t) - ((\lambda+\mu)j + a) P_{ij}(t) + \mu(j+1) P_{i,j+1}(t)$$

$$j P_{ij}'(t) = j(\lambda(j-1) + a) P_{i,j-1}(t) - j((\lambda+\mu)j + a) P_{ij}(t) + j\mu(j+1) P_{i,j+1}(t)$$

$$P_{ik}(t)\left[(\cancel{k}+1)(\lambda k + a) - k((\cancel{\lambda} + \cancel{\mu})k + \cancel{a}) + (\cancel{k}-1)\mu k\right]$$

$$= P_{ik}(t)\left[\lambda k + a - \mu k\right] = \underbrace{(k(\lambda - \mu) + a)}_{\text{coefficient of } P_{ik}(t) \text{ after summing over all } j} P_{ik}(t)$$

Example: Linear growth with immigration.

$$M'(t) = \sum_{j=0}^{\infty} j\, P_{ij}'(t)$$

$$= \sum_{k=0}^{\infty} \left(k(\lambda-\mu)+\alpha\right) P_{ik}(t)$$

$$= (\lambda-\mu) \underbrace{\sum_{k=0}^{\infty} k\, P_{ik}(t)}_{\substack{\shortparallel \\ M(t)}} + \alpha \underbrace{\sum_{k=0}^{\infty} P_{ik}(t)}_{1}$$

$$\begin{cases} M'(t) = (\lambda-\mu) M(t) + a, \\ M(0) = i \end{cases}$$

$$M(t) = i + at \quad \text{if } \lambda = \mu$$

$$M(t) = \frac{a}{\lambda-\mu}\left(e^{(\lambda-\mu)t} - 1\right) + i\, e^{(\lambda-\mu)t} \quad \text{if } \lambda \neq \mu$$