

MATH 180A: Introduction to Probability

Lecture A00 (Au)

www.math.ucsd.edu/~bau/w21.180a

Lecture B00 (Nemish)

www.math.ucsd.edu/~ynemish/teaching/180a

Today: Conditional probability

Next: ASV 2.2 - 2.3

Video: Prof. Todd Kemp, Fall 2019

Week 2:

- Homework 1 (due Friday, January 15)
- Quiz 1 (Wednesday, January 13) - Lectures 1-2

Conditional Probability

2.1

E.g. Your friend rolls two fair dice, and asks you what is the probability the sum is 10.

$$[\Omega = \{(i,j) : 1 \leq i, j \leq 6\} \quad A = \{i+j=10\} = \{(4,6), (5,5), (6,4)\} \therefore P(A) = \frac{3}{36} = \frac{1}{12}]$$

Before you answer, however, she reveals that the actual sum that came up was a two digit number. In light of this information, was your probability calculation correct?

"Updated" $\tilde{\Omega} = \{\text{sum has 2 digits}\}$

$$\begin{aligned} &= \{i+j=10\} \cup \{i+j=11\} = \{i+j=10, 11\} \\ &= \{(4,6), (5,5), (6,4), (1,5), (5,6), (6,6)\} \end{aligned}$$

$$\tilde{P}(A) = \frac{\#\tilde{A}}{\#\tilde{\Omega}} = \frac{3}{6} = \frac{1}{2}$$

$$\tilde{A} = A \cap \tilde{\Omega}$$

Conditional Probability

Moral: given information (ie that an event B is known to have happened), we **condition** on B ; we make B the new sample space.

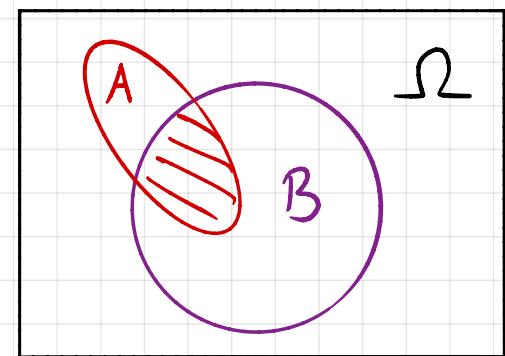
We must modify events afterward so they're "in" B :

$$\Omega \rightarrow B = \tilde{\Omega}$$

$$\mathcal{F} \rightarrow \mathcal{F}_B = \{AB : A \in \mathcal{F}\}$$

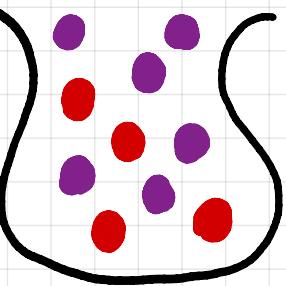
Problem: $P(\tilde{\Omega}) = P(B) < 1$

$$P(A|B) \stackrel{?}{=} \tilde{P} = \frac{P}{P(B)} \quad (\text{caveat: } P(B) \neq 0)$$



Def: Given an event B with $P(B) > 0$, we define the **conditional probability** of an event A given B as

$$P(A|B) = P(AB)/P(B).$$

E.g.  An urn contains 4 red balls and 6 blue balls. 3 are sampled, without replacement.

What is the probability that exactly two are red?

$$\left[\Omega = \{ \{b_1, b_2, b_3\} : b_i \neq b_j \text{ if } i \neq j \} \quad A = \{2 \text{ red, } 1 \text{ blue}\} \quad P(A) = \frac{6 \cdot 6}{120} = 30\% \right]$$

$$\# \Omega = \binom{10}{3}$$

$$\# A = \binom{4}{2} \binom{6}{1}$$

Suppose we somehow know a priori that at least one red is sampled. What is the conditional probability that exactly two red balls are sampled?

$$A = \{\text{exactly 2 red}\} \quad B = \{\text{at least one red}\}$$

$$P(A|B) = \frac{P(B|A)}{P(B)} = \frac{P(A)}{\cancel{P(B)}} = \frac{3/10}{\cancel{5/6}} = \frac{9}{25}$$

$$AB = A$$

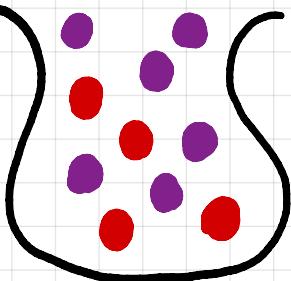
$$P(B^c) = \frac{\binom{6}{3}}{\binom{10}{3}} = \frac{20}{120} = \frac{1}{6}, \quad P(B) = 1 - P(B^c) = \frac{5}{6}$$

Recovering P from $P(\cdot | B)$

By definition, $P(B|A) = \frac{P(AB)}{P(A)}$; $\Rightarrow P(AB) = P(A)P(B|A)$
"multiplication rule"

Can generalize: $P(ABC) = P(AB)P(C|AB) = P(A)P(B|A)P(C|AB)$

E.g.



An urn contains 4 red balls and 6 blue balls.
2 are sampled, without replacement.

What is the probability that both are red?

$$R_1 = \{1^{\text{st}} \text{ red}\}$$

$$R_2 = \{2^{\text{nd}} \text{ red}\}$$

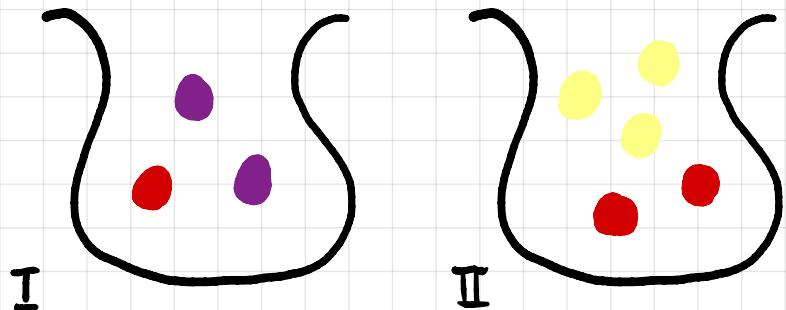
$$\begin{aligned} P(R_1, R_2) &= P(R_1)P(R_2|R_1) \\ &= (0.4)\left(\frac{3}{9}\right) = \frac{2}{15} \end{aligned}$$

(old way: $\frac{\binom{4}{2}\binom{6}{2}}{\binom{10}{2}}$)

Two-Stage Experiments

- * perform an experiment, measure a random outcome.
- * perform a second experiment whose setup depends on the outcome of the first!

E.g.



- * First, choose an urn at random.
- * Then, sample a ball at random from the chosen urn.

What is the probability it is red?

$$\begin{aligned} P(R) &= P((R \cap I) \cup (R \cap II)) \\ &= P(R|I) + P(R|II) \\ &= P(I)P(R|I) + P(II)P(R|II) \\ &= \frac{1}{2}\left(\frac{1}{3}\right) + \frac{1}{2}\left(\frac{2}{3}\right) = \frac{1}{6} + \frac{1}{3} = \frac{11}{30} \end{aligned}$$

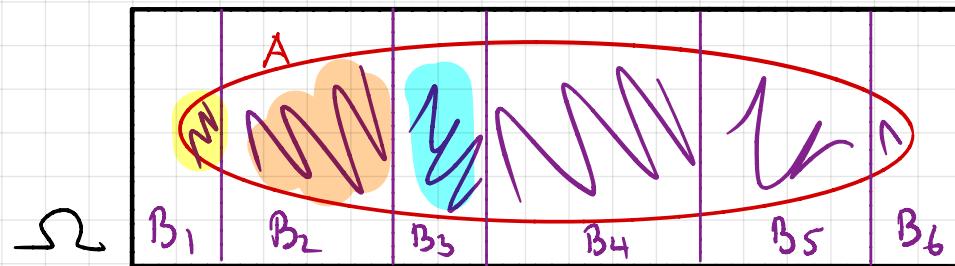
Law of Total Probability

If B_1, B_2, \dots, B_n partition Ω (disjoint, $B_1 \cup \dots \cup B_n = \Omega$, $P(B_j) > 0$)

then for any event A :

$$P(A) = P(AB_1 \cup AB_2 \cup \dots \cup AB_n) = \sum_{j=1}^n P(AB_j)$$

$$= \sum_{j=1}^n P(B_j) P(A|B_j)$$



E.g. 90% of coins are fair, B_1 . 9% are biased to come up heads B_2 . 1% are biased to come up heads B_3 .

You find a coin on the street. How likely is it to come up heads?

$$\begin{aligned}
 P(H) &= P(B_1) P(H|B_1) + P(B_2) P(H|B_2) + P(B_3) P(H|B_3) \\
 &= (0.9)(50\%) + (0.09)(60\%) + (0.01)(80\%) = 0.512
 \end{aligned}$$

Subtler question:

90% of coins are fair. 9% are biased to come up heads 60%.

1% are biased to come up heads 80%.

2.2

You find a coin on the street. You toss it, and it comes up heads.

How likely is it that this coin is heavily biased?

E.g. According to Forbes Magazine, as of April 10, 2019, there are
2208 billionaires in the world.
1964 of them are men.