MATH 180A: Introduction to Probability

Lecture A00 (Au)

www.math.ucsd.edu/~bau/w21.180a

Lecture B00 (Nemish)

www.math.ucsd.edu/~ynemish/teaching/180a

Today: Confidence Intervals. Poisson Approximation

Next: ASV 4.5

Video: Prof. Todd Kemp, Fall 2019

Week 6:

- Homework 5 (due Sunday, February 14)

Example

Flip a fair coin n times. How does

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{\#\text{Heads}}{n} \geq 50.01\%\right) = 0$$

$\frac{1}{2} + 0.0001$

behave as $n \to \infty$?

$50.01\% = 0.5001$

$n = 10^{10}$ $\sqrt{n} = 10^5$
$2\sqrt{n}(0.0001) = 200$
$1-\Phi(200) = 0$

$\sqrt{n} = 100$
$n = 10{,}000$
$\varepsilon = 0.0001$

Suppose after 10,000 flips, there are 5,001 Heads. Should we doubt that the coin is really fair?

$1-\Phi(0.02) \geq 40\%$ (49%)

$\sqrt{n} = 1000$

What if, after 1,000,000 flips, there are 500,100 Heads. Now how confident should we be that the coin is really fair?

$\Phi(0.2)$ $1-$ $\geq 40\%$

$S_n = \#\text{Heads} \sim \text{Bin}(n, \frac{1}{2})$

$$\mathbb{P}\left(\frac{S_n}{n} \geq \frac{1}{2} + \varepsilon\right) = \mathbb{P}\left(\frac{S_n - \frac{1}{2}n}{n} \geq \varepsilon\right) = \mathbb{P}\left(\frac{S_n - \frac{1}{2}n}{\sqrt{n}/2} \geq 2\varepsilon\sqrt{n}\right) \approx \mathbb{P}(X \geq 2\sqrt{n}\varepsilon)$$

Normal: $\mathcal{N}(0,1)$

$= 1 - \mathbb{P}(X < 2\varepsilon\sqrt{n})$

$= 1 - \Phi(2\varepsilon\sqrt{n})$

replace n w/ $\sqrt{\text{Var}(S_n)} = \sqrt{n \cdot \frac{1}{2} \cdot \frac{1}{2}} = \frac{\sqrt{n}}{2}$

Confidence

Suppose we have a coin that is biased by some unknown amount;

$$X \sim Ber(p)$$ unknown p !

How can we figure out what p is?

Use the law of large numbers: $p = \lim_{n\to\infty} \frac{S_n}{n}$

We can't actually wait around for $n \to \infty$. Instead, we estimate

$$p \approx \hat{p} := \frac{S_n}{n}$$ for some large n.

The question is: how good an estimate is this for given n? Or, turning it around: how big must you take n to get an estimate of a certain accuracy?

$$|\hat{p} - p| < \varepsilon \ (= 0.01)$$

$$\mathbb{P}(|\hat{p} - p| < \varepsilon) \geq 95\%$$

"$\hat{p}$ is within margin of error ε of p, w probability 95%."

A Maximum Likelihood Estimate

Want to find n large enough that (with $\hat{p} = S_n/n$)

$$\mathbb{P}(|\hat{p}-p| < \varepsilon) \geq \text{(high probability)}$$

$\uparrow$ chosen tolerance

$X \sim N(0,1)$

$$\mathbb{P}(|\hat{p}-p| < \varepsilon) = \mathbb{P}\left(\frac{|S_n - np|}{n} < \varepsilon\right) = \mathbb{P}\left(\frac{|S_n-np|}{\sqrt{np(1-p)}} < \frac{\varepsilon\sqrt{n}}{\sqrt{p(1-p)}}\right) \approx \mathbb{P}\left(|X| < \frac{\varepsilon\sqrt{n}}{\sqrt{p(1-p)}}\right)$$

$\frac{S_n}{n}$ $\qquad \sqrt{np(1-p)}$

$$= \Phi\left(\frac{\varepsilon\sqrt{n}}{\sqrt{p(1-p)}}\right) - \Phi(-\;''\;'')$$

$$\mathbb{P}(|\hat{p}-p| < \varepsilon) \approx 2\Phi\left(\varepsilon\sqrt{n}/\sqrt{p(1-p)}\right) - 1.$$

$0 \leq p \leq 1$

$p(1-p) \leq \frac{1}{4}$

max @ $p = \frac{1}{2}$

$$\frac{1}{\sqrt{p(1-p)}} \geq 2$$

$\Phi \uparrow$

Conclusion: $\mathbb{P}(|\hat{p}-p| < \varepsilon) \underset{(\approx)}{\geq} 2\Phi(2\varepsilon\sqrt{n}) - 1.$

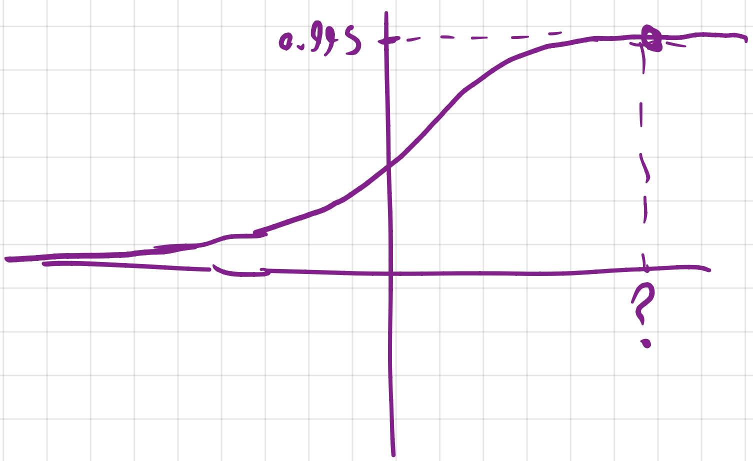
Example (of the Beast): How many times should we flip a coin, biased an unknown amount p, so that the estimate $\hat{p} = S_n/n$ is within a tolerance of 0.05 of the true value p, with probability $\geq 99\%$?

Want n large enough that

$$\mathbb{P}(|\hat{p} - p| < 0.05) \gtrsim 99\%$$

make sure

We know $\mathbb{P}(|\hat{p}-p| < 0.05) \gtrsim (\approx)\ 2\Phi(2(0.05)\sqrt{n}) - 1 \geq 99\%$



$$\Phi(\underbrace{2(0.05)\sqrt{n}}) \geq 0.995$$

$$\therefore\ 2(0.05)\sqrt{n} \geq 2.58$$

$$\sqrt{n} \geq 25.8$$

$$n \geq 665.64$$

666

Confidence Intervals

Turning this around: if we can't control n, we would like to say how accurate the sample mean is as an estimate of the true mean, for a given number n of samples.

Eg. A coin (of unknown bias p) is tossed 1000 times. 450 Heads come up. Within what tolerance can we say we know the true value of p with probability $\geq 95\%$?

Estimate $p \approx \hat{p} = \frac{S_{1000}}{1000} = 0.45$

Want $\mathbb{P}(|p-\hat{p}| < \varepsilon) \geq 95\%$

Know: $\mathbb{P}(|p-\hat{p}| < \varepsilon) \gtrapprox 2\Phi(2\varepsilon\sqrt{1000}) - 1 \geq 0.95$

$\Phi(2\varepsilon\sqrt{1000}) \geq 0.975$

$2\varepsilon\sqrt{1000} \geq 1.96 \rightsquigarrow \varepsilon \geq \frac{1.96}{2\sqrt{1000}} \doteq 0.031$

I.e. $|p - 0.45| < 0.031$ w $P \geq 95\%$

$0.45 - 0.031 < p < 0.45 + 0.031$

$p \in [0.419, 0.481]$ w $P \geq 95\%$

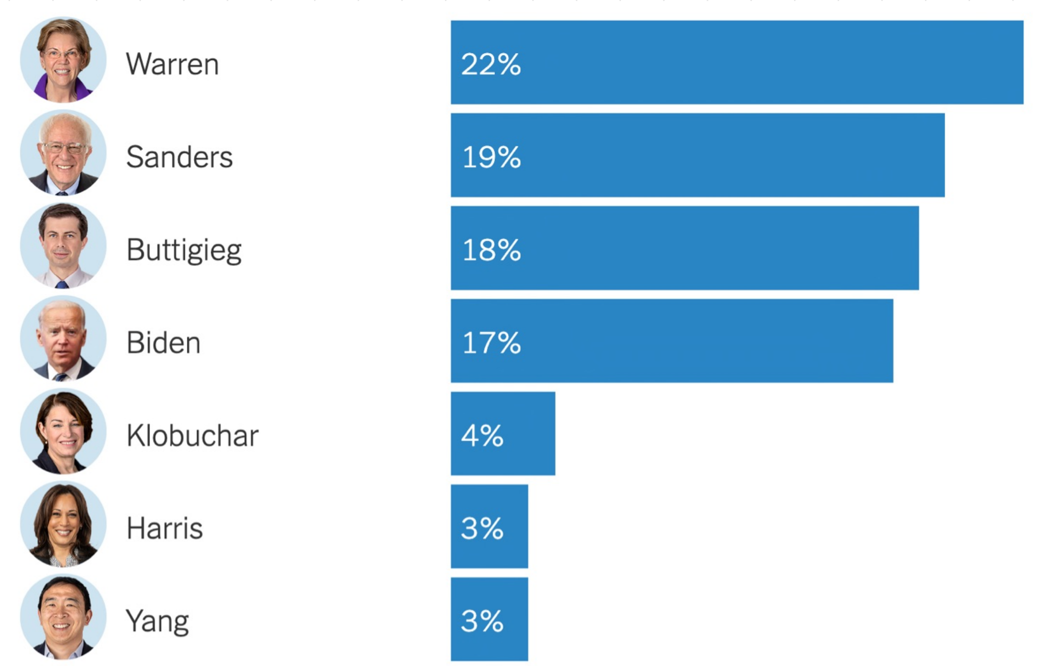
95% confidence interval

If an experiment is repeated in many independent trials, and the preceding (normal approximation) estimates yield

$$\mathbb{P}(|\hat{p}-p| < \varepsilon) \geq 95\%$$

we say $[\hat{p}-\varepsilon, \hat{p}+\varepsilon]$ is the 95% <u>confidence interval</u> for p.

The same statement might be given as "$p = \hat{p}$ with margin of error ε (95 times out of 100)".



Source: New York Times Upshot/Siena College poll conducted Oct. 25-30.

Poll conducted Oct 25-30 of 439 Iowa Democratic caucusgoers.

$$\mathbb{P}(|p-\underset{0.22}{\hat{p}}| < \varepsilon) \geq 2\Phi(2\varepsilon\sqrt{439})-1$$

$$(\approx) \quad \geq 0.95$$

i.e. $2\varepsilon\sqrt{439} \geq 1.96$

$\varepsilon \geq 4.68\%$

Margin of error: 4.7%

Poisson vs. Normal Approximation – Quantitative

4.4

Theorem. Let $S_n \sim \mathrm{Bin}(n,p)$

$X \sim \mathrm{Poisson}(np)$

$Y \sim \mathcal{N}(0,1)$

For any subset $A \subseteq \mathbb{N}$,

$$|\mathbb{P}(S_n \in A) - \mathbb{P}(X \in A)| \le np^2$$

if $p = \frac{\lambda}{n^{0.51}}$ $(\lambda > 0)$

$np^2 = n\left(\frac{\lambda}{n^{0.51}}\right)^2 = \frac{\lambda^2 n}{n^{1.02}}$

$\to 0$ as $n \to \infty$

OTOH, for any $x \in \mathbb{R}$,

Berry–Esseen Thm.

$$\Big|\underbrace{\mathbb{P}\left(\frac{S_n - np}{\sqrt{np(1-p)}} \le x\right)}_{\text{CDF of } \frac{S_n - np}{\sqrt{np(1-p)}}} - \underbrace{\mathbb{P}(Y \le x)}_{\Phi(x)}\Big| \le \frac{3}{\sqrt{np(1-p)}}$$

3 is not optimal

optimal

Upshot: if np^2 is small, use Poisson Approximation.

if $np(1-p)$ is quite large, use Normal Approximation.

Beyond independent trials:

* The normal approximation breaks down qickly if the trials are dependent.
* The Poisson approximation holds up well under "weak dependence"

Example. A factory experiences 3 accidents per month, on average. What is the probability there will be 3 accidents this month?

X = # accidents in a given month.

well modeled by a Poisson.

$X \sim \text{Poisson}(\lambda)$

$3 = \mathbb{E}(X) = \lambda$

$$\mathbb{P}(X=3) = e^{-3}\frac{3^3}{3!} \doteq 22.4\%$$

$$\frac{3^3}{3!} = \frac{3^2\cdot\cancel{3}}{\cancel{3}\cdot 2\cdot 1} = \frac{3^2}{2!}.$$

$$\mathbb{P}(X=2) = e^{-3}\frac{3^2}{2!} \doteq 22.4\%$$