MATH180C: Introduction to Stochastic Processes II

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Today: Brownian Motion

> Q&A: December 2

Next: PK 8.2

This week:

- Homework 7 (due THURSDAY, December 3)
- HW6 regrades (until Wednesday, December 2, 11 PM)

BM as a Gaussian process

<u>Def</u>. Stochastic process $(X_t)_{t\geq 0}$ is called a Gaussian process if for any $0 \leq t_1 < t_2 < \cdots < t_n$

$(X_{t_1}, \dots, X_{t_n})$ is a Gaussian vector, or equivalently for any $c_1, \dots, c_n \in \mathbb{R}$

$c_1 X_{t_1} + c_2 X_{t_2} + \dots + c_n X_{t_n}$ is a Gaussian r.v.

Recall that the distribution of a Gaussian vector is uniquelly defined by its mean and covariance matrix.

Similarly, each Gaussian process is uniquely described by

$\mu(t) = E(X_t)$ and $\Gamma(s,t) = \mathrm{Cov}(X_s, X_t) \geq 0$

covariance function

BM as a Gaussian process

Proposition BM $(B_t)_{t\geq 0}$ is a Gaussian process with
$$\mu(t) = 0 \quad \text{and} \quad \Gamma(s,t) = \min\{s,t\}$$

Proof. For any $0 \leq t_1 < t_2 < \cdots < t_n$, $B_{t_j} - B_{t_{j-1}}$ are indep. Gaussian, thus
$$\sum_{i=1}^{n} c_i B_{t_i} = \sum_{i=1}^{n} c_i \sum_{j=1}^{i} (B_{t_j} - B_{t_{j-1}})$$
is also Gaussian.

By definition $\mu(t) = E(B_t) = 0$. Let $s < t$.

Then
$$\begin{aligned}
\Gamma(s,t) &= \mathrm{Cov}(B_s, B_t) \\
&= \mathrm{Cov}(B_s, B_s + (B_t - B_s)) \\
&= \mathrm{Cov}(B_s, B_s) + \mathrm{Cov}(B_s, B_t - B_s) \\
&= s + 0 = s = \min\{s,t\}
\end{aligned}$$
$\blacksquare$

Some properties of BM

Proposition. Let $(B_t)_{t\geq 0}$ be a standard BM. Then

(i) For any $s>0$, the process $(B_{t+s}-B_s,\ t\geq 0)$ is a BM independent of $(B_u,\ 0\leq u\leq s)$.

(ii) The process $(-B_t,\ t\geq 0)$ is a BM

(iii) For any $c>0$, the process $(cB_{t/c^2},\ t\geq 0)$ is a BM

(iv) The process $(X_t)_{t\geq 0}$ defined by $X_0=0$, $X_t=tB_{1/t}$ for $t>0$ is a BM.

Proof (i) Define $X_t=B_{t+s}-B_s$. Then $X_0=0$ and $X_{t_2}-X_{t_1}=B_{t_2+s}-B_{t_1+s}$

$\Rightarrow$ independent Gaussian increments, $E(X_{t_2}-X_{t_1})=0$, $Var(X_{t_2}-X_{t_1})=t_2-t_1$

$(X_t)_{t\geq 0}$ has continuous paths $\Rightarrow (X_t)_{t\geq 0}$ is a BM

(iv) X_t is Gaussian, for $s<t$ $Cov(sB_{1/s},tB_{1/t})=st\min\{\frac{1}{s},\frac{1}{t}\}=s$

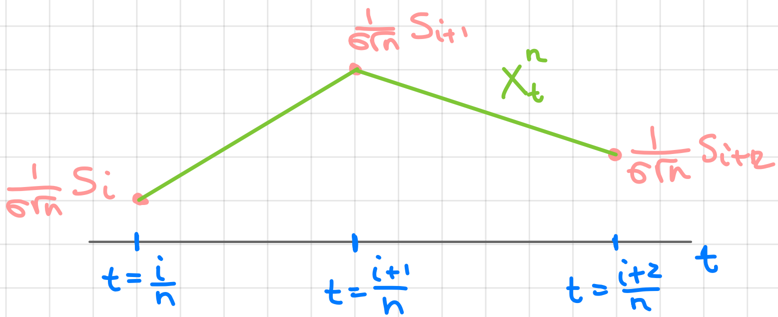
Proof of $\lim_{t\to 0}X_t=0$ is more technical, thus omitted.

Construction of BM

BM can be constructed as a limit of properly rescaled random walks.

Let $\{\xi_k\}_{k=1}^{\infty}$ be a sequence of i.i.d. r.v.s, $E(\xi_i)=0$, $Var(\xi_i)=\sigma^2<\infty$. Denote $S_m=\sum_{k=1}^{m}\xi_k$ and define

$$X_t^n=\frac{1}{\sigma\sqrt{n}}\left(S_{[nt]}+(nt-[nt])\xi_{[nt]+1}\right)$$



Theorem (Donsker) $(X_t^n)_{t\geq 0}$ converges in distribution to the standard BM.

Applying Donsker's theorem

Example Let $(\xi_i)_{i=1}^{\infty}$ be i.i.d. r.v. $P(\xi_i = 1) = P(\xi_i = -1) = 0.5$

$$E(\xi_i) = 0, \quad Var(\xi_i) = 1.$$

Denote $S_m := \sum_{i=1}^{m} \xi_i$, $S_0 = 0$. $(S_m)_{m \geq 0}$ is a Markov chain.

From the first step analysis of MC we know that for any $-a < 0 < b$ $P(S \text{ reaches } -a \text{ before } b) = \frac{b}{a+b}$

If X_t^n is the process interpolating S_m, then $\forall n$

$$P(X^n \text{ hits } -a \text{ before } b) = P(S \text{ hits } -\sqrt{n}a \text{ before } \sqrt{n}b)$$

$$= \frac{\sqrt{n}b}{\sqrt{n}a + \sqrt{n}b} = \frac{b}{a+b}$$

$\Rightarrow P(B \text{ hits } -a \text{ before } b) = \frac{b}{a+b}$

$\Rightarrow (\tilde{\xi}_i)_{i=1}^{\infty}$, $E(\tilde{\xi}_i) = 0$, $Var(\tilde{\xi}_i) = 1$, $P(\tilde{S} \text{ hits } -a \text{ before } b) \approx \frac{b}{a+b}$

BM as a martingale

Let $(X_t)_{t\geq 0}$ be a continuous time stochastic process. We say that $(X_t)_{t\geq 0}$ is a martingale if $E(|X_t|)<\infty$ $\forall t\geq 0$ and

$$E(X_t \mid \{X_u, 0\leq u\leq s\}) = X_s \quad \text{for all } s<t.$$

Proposition Let $(B_t)_{t\geq 0}$ be a standard BM. Then

(i) $(B_t)_{t\geq 0}$ is a martingale

(ii) $(B_t^2 - t)_{t\geq 0}$ is a martingale (w.r.t. $(B_t)_{t\geq 0}$)

"Proof": $E(B_t \mid \{B_u, 0\leq u\leq s\}) = E(B_s + B_t - B_s \mid \{B_u, 0\leq u\leq s\}) = B_s + 0 = B_s$

$$E(B_t^2 - t \mid \{B_u, 0\leq u\leq s\}) = E(B_s^2 + 2B_s(B_t - B_s) + (B_t - B_s)^2 - t \mid \{B_u, 0\leq u\leq s\})$$

$$= B_s^2 + 0 + t - s - t = B_s^2 - s$$

Thm (Lévy) Let $(X_t)_{t\geq 0}$ be a continuous martingale such that $(X_t^2 - t)_{t\geq 0}$ is a martingale. Then $(X_t)_{t\geq 0}$ is BM.