MATH180C: Introduction to Stochastic Processes II

www.math.ucsd.edu/~ynemish/teaching/180c

Today: Reflection principle

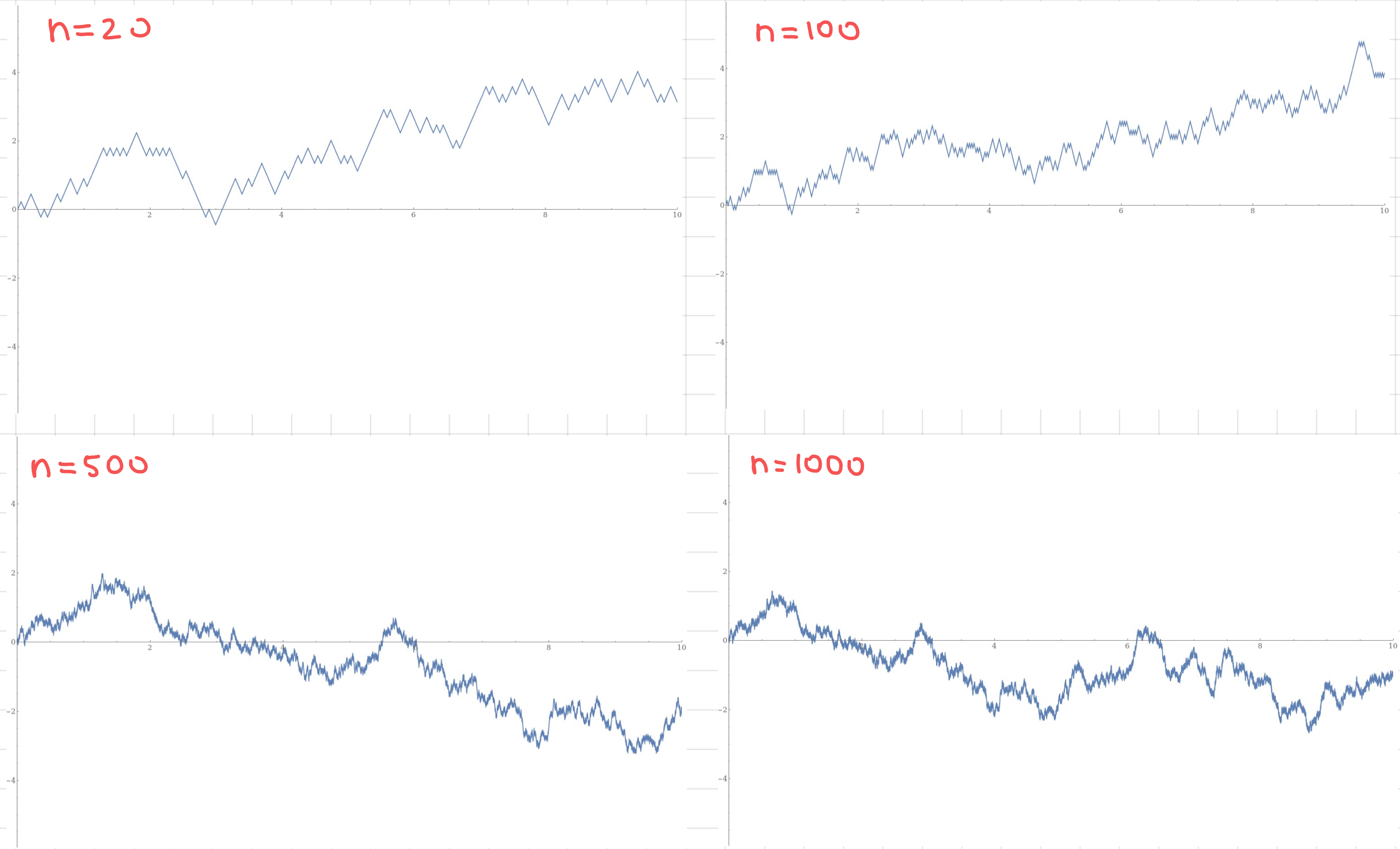
> Q&A: December 4

Next: PK 8.3

This week:

- Homework 7 (due THURSDAY, December 3)
- Homework 8 (due THURSDAY, December 10)
- CAPE at www.cape.ucsd.edu

Approximating a BM with random walks X_t^n



Stopping times and the strong Markov property (lec. 3)

Def (Informal). Let $(X_t)_{t\geq 0}$ be a stochastic process and let $T \geq 0$ be a random variable. We call T a stopping time if the event

$$\{T \leq t\}$$

can be determined from the knowledge of the process up to time t (i.e., from $\{X_s : 0 \leq s \leq t\}$)

Examples: Let $(X_t)_{t\geq 0}$ be right-continuous

1. $\min\{t \geq 0 : X_t = x\}$ is a stopping time

2. $\sup\{t \geq 0 : X_t = x\}$ is not a stopping time

Stopping times and the strong Markov property (lec. 3)

Theorem (no proof)

Let $(X_t)_{t\geq 0}$ be a Markov process, let T be a stopping time of $(X_t)_{t\geq 0}$. Then, conditional on $T<\infty$ and $X_T = x$,

$$(X_{T+t})_{t\geq 0}$$

(i) is independent of $\{X_s,\ 0\leq s\leq T\}$

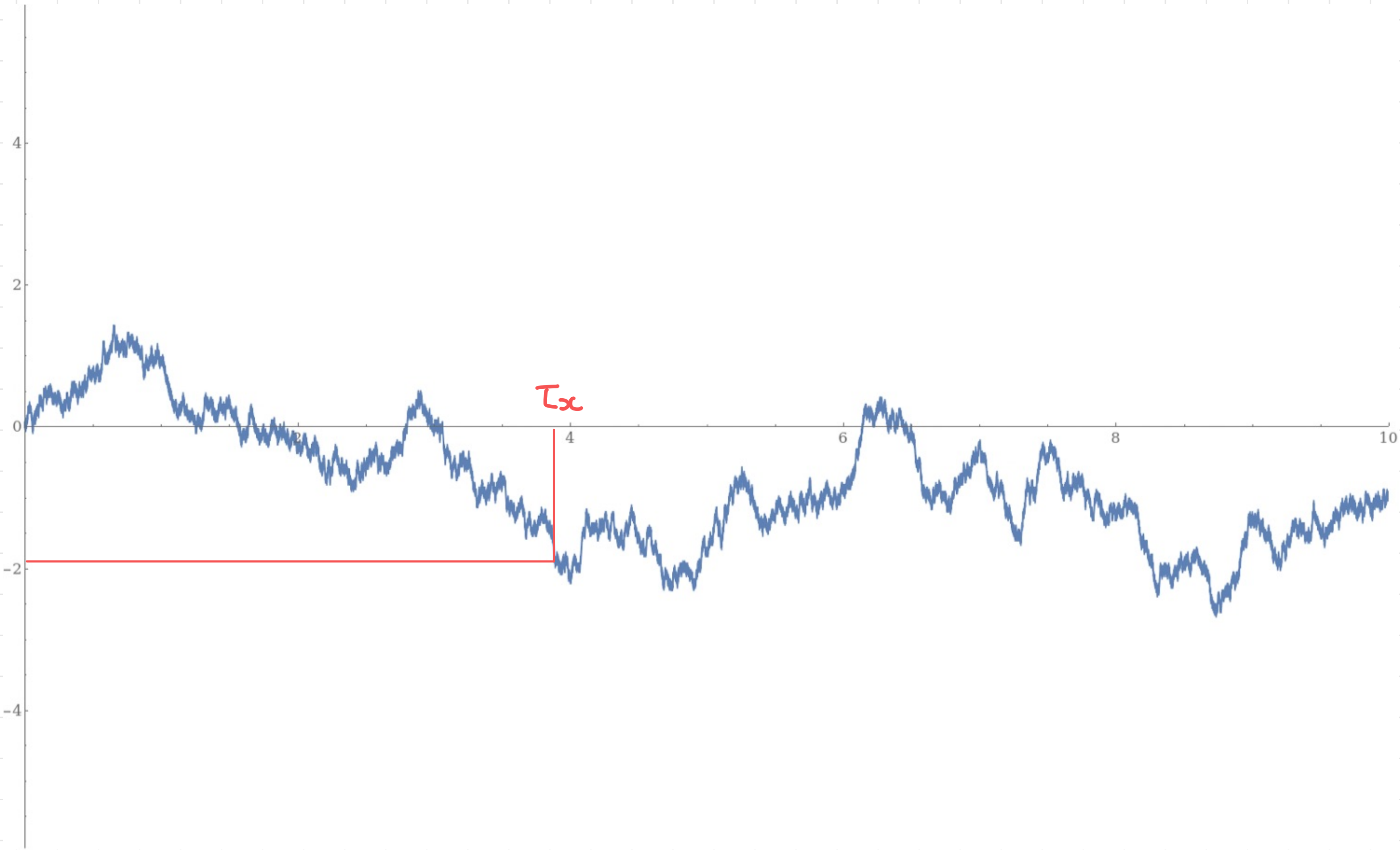
(ii) has the same distribution as $(X_t)_{t\geq 0}$ starting from x

Example $(B_t)_{t\geq 0}$ is Markov. For any $x\in\mathbb{R}$ define

$$\tau_x = \min\{t : B_t = x\}. \text{ Then}$$

- $(B_{t+\tau_x} - B_{\tau_x})_{t\geq 0}$ is a BM starting from x
- $(B_{t+\tau_x} - B_{\tau_x})_{t\geq 0}$ is independent of $\{B_s,\ 0\leq s\leq \tau_x\}$

(independent of what B was doing before it hit x)



4
2
0
-2
-4
4
6
8
10
τ_x

Reflection principle

Thm. Let $(B_t)_{t\geq 0}$ be a standard BM. Then for any $t \geq 0$ and $x > 0$

Proof. Let $\tau_x = \min\{t : B_t = x\}$. Note that τ_x is a stopping time and is uniquely determined by $\{B_u, 0 \leq u \leq \tau_x\}$

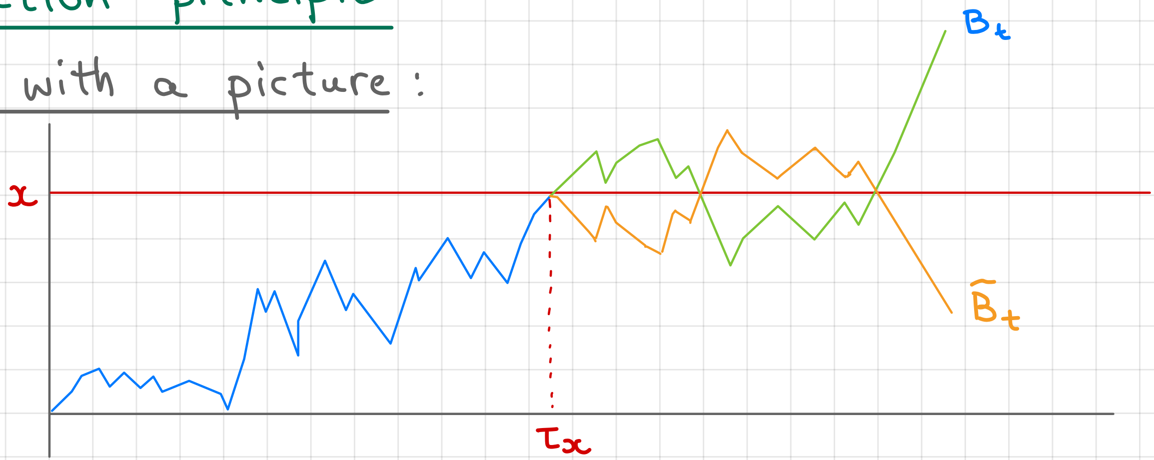
From the definition of τ_x, . Then

$$P\left(\max_{0\leq u\leq t} B_u \geq x,\ B_t < x\right) =$$

Now $P\left(\max_{0\leq u\leq t} B_u \geq x\right) =$

Reflection principle

Proof with a picture:



If $(B_t)_{t\geq 0}$ is a BM, then $(\tilde{B}_t)_{t\geq 0}$ is a BM, where

$$\tilde{B}_t = \begin{cases} B_t , & t \leq \tau_x \\ B_{\tau_x} - (B_t - B_{\tau_x}), & t > \tau_x \end{cases}$$

$\Rightarrow$ to each sample path with $\max\limits_{0\leq u\leq t} B_u > x$ and $B_t > x$ we associate a unique path with $\max\limits_{0\leq u\leq t} \tilde{B}_u > x$ and $\tilde{B}_t < x$, so

$$P(\max_{0\leq u\leq t} B_u \geq x, B_t < x) = P(B_t > x) \Rightarrow P(\max_{0\leq u\leq t} B_u \geq x) = 2P(B_t \geq x) \blacksquare$$

Application of the RP: distribution of the hitting time τ_x

By definition, $\tau_x \le t \iff \max\limits_{0 \le u \le t} B_t \ge x$, so

$$P(\tau_x \le t) =$$

$$=$$

$$=$$

$\Rightarrow$ p.d.f. of τ_x $\quad f_{\tau_x}(t) =$

<u>Thm</u>. $F_{\tau_x}(t) = \sqrt{\frac{2}{\pi}} \int\limits_{x/\sqrt{t}}^{\infty} e^{-\frac{v^2}{2}} dv$,

$$f_{\tau_x}(t) = \frac{x}{\sqrt{2\pi}} t^{-3/2} e^{-\frac{x^2}{2t}}$$

Zeros of BM

Denote by $\theta(t,t+s)$ the probability that $B_u = 0$ on $(t,t+s)$

$$\theta(t,t+s) :=$$

__Thm__. For any $t, s > 0$

$$\theta(t,t+s) =$$

__Proof__ Compute $P(\,B_u = 0$ for some $u \in (t,t+s]\,)$ by conditioning on the value of B_t.

$$\theta(t,t+s) =$$

$(*)$

Define $\tilde{B}_u = B_{t+u} - B_t$. Then

$$P(B_u = 0 \text{ on } (t,t+s] \mid B_t = x) =$$

$(**)$

Zeros of BM

Plugging (**) into (*) gives

$$\Theta(t,t+s) = \int_{-\infty}^{+\infty} P(B_u = x \text{ for some } u \in (0,s]) \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}} dx$$

$$= \int_{0}^{+\infty} P(B_u = x \text{ for some } u \in (0,s]) \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}} dx$$

$$+ \int_{0}^{\infty} P(B_u = -x \text{ for some } u \in (0,s]) \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}} dx$$

$$=$$

Finally, $P(B_u = x > 0 \text{ for some } u \in (0,s]) =$

$$(*) = \int_0^{\infty} \sqrt{\frac{2}{\pi t}} e^{-\frac{x^2}{2t}} \left(\int_0^{s} \frac{x}{\sqrt{2\pi}} y^{-3/2} e^{-\frac{x^2}{2y}} dy \right) dx =$$

Zeros of BM

$$\int_0^\infty x e^{-\frac{x^2}{2}\left(\frac{1}{t}+\frac{1}{y}\right)} dx =$$

$\Rightarrow (*) =$

Now use the change of variable $z=\sqrt{\frac{y}{t}}$, $dy = 2tdz$

$$(*) = \frac{\sqrt{t}}{\pi}\int_0^{\sqrt{s/t}} \frac{1}{t(1+z^2)\sqrt{t}\,z}\cdot 2t\,dz = \frac{2}{\pi}\int_0^{\sqrt{s/t}} \frac{1}{1+z^2}dz = \frac{2}{\pi}\arctan\left(\sqrt{\frac{s}{t}}\right)$$

$$= \frac{2}{\pi}\arccos\left(\sqrt{\frac{t}{s+t}}\right)$$

exercise

$\blacksquare$

<u>Remark</u> Let $T_0 := \inf\{t>0 : B_t = 0\}$. Then $P(T_0 = 0) = 1$

There is a sequense of zeros of $B_t(\omega)$ converging to 0.

To understand the structure of the set of zeros → Cantor set

Behavior of BM as $t \to \infty$

<u>Thm</u>. Let $(B_t)_{t \geq 0}$ be a (standard) BM. Then

$$P\left(\sup_{t \geq 0} B_t = +\infty , \inf_{t \geq 0} B_t = -\infty \right) = 1$$

(BM "oscilates with increasing amplitude")

<u>Proof</u>. Denote $Z = \sup_{t \geq 0} B_t$. Then for any $c > 0$

$$cZ =$$

By property (iii), cB_{t/c^2} is a standard BM, so cZ has the same distribution as $Z \Rightarrow P(Z=0) = p, P(Z=\infty) = 1-p$

$p = P(Z=0)$

$\Rightarrow P(Z=0) = 0, P(Z=\infty) = 1$. Similarly for $\inf_{t \geq 0} B_t$ ∎

Sample paths of $(B_t)_t$ are not differentiable

Thm. $P(\ B_t$ is not differentiable at zero $)=1$

Proof. $P(\sup_{t\geq 0} B_t = \infty,\ \inf_{t\geq 0} B_t = -\infty) = 1.$ $(\star)$

Consider $\tilde{B}_t = t B_{1/t}$. $(\tilde{B}_t)_{t\geq 0}$ is a BM (by property (iv))

By $(\star)$, for any $\varepsilon > 0$ $\exists\ t<\varepsilon,\ s<\varepsilon$ such that

$\tilde{B}_t > 0,\ \tilde{B}_s < 0 \Rightarrow$ only differentiable if $\tilde{B}'_0 = 0$

But if $\tilde{B}'_0 = 0$, then

for some $t>0$ and all $0<s<t$,

which imples that for all $0<s<t$, which

contradicts to $(\star)$ ∎

Thm $P((B_t)_{t\geq 0}$ is nowhere differentiable $)=1$

Reflected BM

Def. Let $(B_t)_{t\geq 0}$ be a standard BM. The stochastic process

$$|B_t| = \begin{cases} \quad , & \text{if } B(t) \geq 0 \\ \quad , & \text{if } B(t) < 0 \end{cases}$$

is called reflected BM.

Think of a movement in the vicinity of a boundary.

Moments: $E(R_t) =$

$$Var(R_t) = E(B_t^2) - (E(|B_t|))^2 =$$

Transition density: $P(R_t \leq y \mid R_0 = x) =$

$=$ $\qquad \Rightarrow p_t(x,y) =$

Thm (Lévy) Let $M_t = \max_{0 \leq u \leq t} B_u$. Then $(M_t - B_t)_{t\geq 0}$ is a reflected BM.

Reflected BM