

MATH 180A: Introduction to Probability

Lecture A00 (Au)

www.math.ucsd.edu/~bau/w21.180a

Lecture B00 (Nemish)

www.math.ucsd.edu/~ynemish/teaching/180a

Today: Expectation

Next: ASV 3.4

Video: Prof. Todd Kemp, Fall 2019

Week 4:

- Homework 3 (due Sunday, January 31)

Poisson Distribution

A random variable X has the Poisson(λ) distribution if

$$\lim_{n \rightarrow \infty} P(S_{n, \frac{\lambda}{n}} = k) = P(X = k) = e^{-\lambda} \frac{\lambda^k}{k!}, \quad k = 0, 1, 2, \dots$$

$$\sum_{k=0}^{\infty} e^{-\lambda} \frac{\lambda^k}{k!} = e^{-\lambda} \underbrace{\sum_{k=0}^{\infty} \frac{\lambda^k}{k!}}_{\geq 0} = e^{-\lambda} e^{\lambda} = 1 \quad \checkmark$$

E.g. A 100 year storm is a storm magnitude expected to occur in any given year with probability $1/100$.

Over the course of a century, how likely is it to see at least 4 100 year storms?

$$P(S_{100, \frac{1}{100}} \geq 4) = \sum_{k=4}^{100} P(S_{100, \frac{1}{100}} = k) \approx \sum_{k=4}^{100} e^{-1} \frac{(1)^k}{k!}$$

$$\begin{aligned} \sum_{k=4}^{100} \binom{100}{k} \left(\frac{1}{100}\right)^k \left(1 - \frac{1}{100}\right)^{100-k} &\doteq 1.8374 \underbrace{e^{-1}}_{= 0.018374} = \sum_{k=4}^{\infty} e^{-1} \frac{1}{k!} \\ &= 1 - \sum_{k=0}^3 e^{-1} \frac{1}{k!} \\ &= 1.8988\% \end{aligned}$$

Summary

Sampling independent trials, the most important (discrete) probability distributions are:

- $\text{Ber}(p)$: $P(X=1)=p, P(X=0)=1-p \quad 0 \leq p \leq 1$
(single trial with success probability p)
- $\text{Bin}(n,p)$: $P(S_n=k) = \binom{n}{k} p^k (1-p)^{n-k} \quad 0 \leq k \leq n$
(number of successes in n independent trials with rate p)
- $\text{Geom}(p)$ $P(N=k) = (1-p)^{k-1} p \quad k=1, 2, \dots$
(first successful trial in repeated independent trials with rate p)
- $\text{Poisson}(\lambda)$ $P(X=k) = e^{-\lambda} \frac{\lambda^k}{k!} \quad k=0, 1, 2, \dots \quad \lambda > 0$.
(Approximates $\text{Bin}(n, \lambda/n)$; number of rare events in many trials)

Expectation

3.3

Toss a fair coin 1000 times, and record the sequence of outcomes.

110100011011001011...

Average them. $\frac{1}{1000}(1+1+0+1+0+0+0+1+1+0+1+1\dots)$

What size do you **expect** this number to have?

[About ~~X~~ of the outcomes are 1, about ~~X~~ are 0.]
 $\Rightarrow p \quad 1-p$

$\therefore \text{Average} = \frac{1+0}{2} = \frac{1}{2}$
 $= p(1) + (1-p) \cdot 0$

What if the coin is biased $P(X_j=1)=p$, $P(X_j=0)=1-p$.

Definition: Let X be a discrete random variable with possible values t_1, t_2, t_3, \dots . The **expectation** or **expected value** of X is

$$E(X) := \sum_j t_j P(X=t_j)$$

weighted average
"balance point"

Question: Is the expectation $E(X)$ the value X is equal to most often?

- (a) Yes, always.
- (b) No, not generally.

E.g. Let X be the number rolled on a fair die. $X \in \{1, 2, 3, 4, 5, 6\}$

$$E(X) = \sum_{k=1}^6 k \cdot \frac{1}{6} = \frac{1}{6}(1+2+3+4+5+6) = \frac{21}{6} = \frac{7}{2}.$$

E.g. Let Y be $\text{Ber}(p)$. $E(Y) = p \cdot 1 + (1-p) \cdot 0 = p$

E.g. You toss a biased coin (Y) repeatedly until the first heads.
How long do you expect it to take?

N = the time the 1st heads comes up. $N \sim \text{Geom}(p)$

$$\begin{aligned} E(N) &= \sum_{k=1}^{\infty} k \cdot P(N=k) = p \sum_{k=1}^{\infty} k(1-p)^{k-1} \\ &\quad \underbrace{(1-p)^{k-1}}_{p} = p \cdot \frac{t}{(1-(1-p))^2} = \boxed{p \frac{1}{p} \frac{d}{dx} \sum_{k=0}^{\infty} kx^{k-1}} = \frac{1}{(1-x)^2} \end{aligned}$$

E.g. $S_n \sim \text{Bin}(n, p)$ ($S_n = X_1 + X_2 + \dots + X_n$ for X_j independent $\text{Ber}(p)$)

$$\mathbb{E}(S_n) = \sum_{k=0}^n k \cdot P(S_n=k) = \sum_{k=0}^n k \cdot \binom{n}{k} p^k (1-p)^{n-k} = \boxed{np}$$

$$S_n = X_1 + X_2 + \dots + X_n$$

$$\mathbb{E}(X_j) = p$$

$$\begin{aligned} & \mathbb{E}(X_1 + \dots + X_n) \\ &= \mathbb{E}(X_1) + \mathbb{E}(X_2) + \dots + \mathbb{E}(X_n) \end{aligned}$$

E.g. $X \sim \text{Poisson}(\lambda)$

$$\begin{aligned} \mathbb{E}(X) &= \sum_{k=0}^{\infty} k \cdot e^{-\lambda} \frac{\lambda^k}{k!} = e^{-\lambda} \sum_{k=1}^{\infty} \frac{\lambda^k}{(k-1)!} = e^{-\lambda} \sum_{j=0}^{\infty} \frac{\lambda^{j+1}}{j!} = \lambda e^{-\lambda} \sum_{j=0}^{\infty} \frac{\lambda^j}{j!} \\ &= \boxed{\lambda} \end{aligned}$$

↳ E.g. A factory has, on average, 3 accidents per month.

Estimate the probability that there will be exactly 2 accidents this month.

$$X = \#\text{accidents/month} \quad \left. \begin{array}{l} \mathbb{E}(X) = 3 = \lambda \\ P(X=2) = e^{-3} \frac{3^2}{2!} = 22.4\% \end{array} \right\}$$

$X \sim \text{Poisson}(3)$

E.g. Toss a fair coin until tails comes up. If this is on the first toss, you win \$2 and stop. If heads comes up, the pot doubles, and you continue. That is, if the first tails is on the k^{th} toss, you win 2^k dollars.

What is your expected winnings?

$$W = \left\{ 2^k \text{ if the first tails is on the } k^{\text{th}} \text{ toss} \right\}$$

$$E(W) = \sum_{k=1}^{\infty} 2^k \cdot P(W=2^k) = \underbrace{\left(\frac{1}{2}\right)^{k-1} \cdot \frac{1}{2}}_{\text{(1)}} \cdot \sum_{k=1}^{\infty} 2^k \cdot \frac{1}{2^k} = \infty.$$