MATH180C: Introduction to Stochastic Processes II

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Today: Processes generated by BM

> Q&A: December 7,9

Next: Review

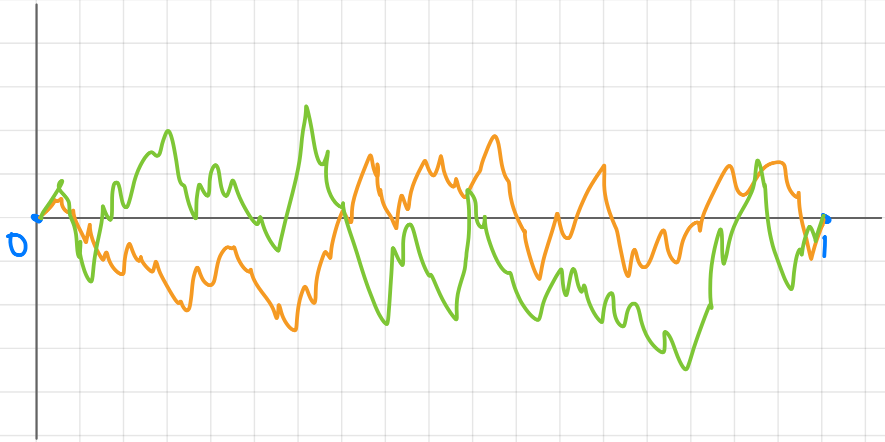
This week:

- Homework 8 (due THURSDAY, December 10)
- Homework 9 (do not submit, practice problems)
- Quiz 5 on Wednesday, December 9 (lectures 18-20)

Brownian bridge

Brownian bridge is constructed from a BM by conditioning on the event $\{B(0)=0, B(1)=0\}$.

$(B^\circ_t)_{t\geq 0}$



<u>Thm</u> 1. Brownian bridge is a continuous Gaussian process on $[0,1]$ with mean 0 and covariance function

$$\Gamma(s,t) = \min\{s,t\} - st$$

Conditioned multivariate normal distribution

Lemma Let (X,Y) be a random vector with multivariate normal distribution $N(0,\Sigma)$ with $\Sigma = \begin{pmatrix} \sigma_x^2 & \sigma_{xy} \\ \sigma_{xy} & 1 \end{pmatrix}$.

Then
$$f_{X|Y}(x|0) = \frac{1}{\sqrt{2\pi(\sigma_x^2 - \sigma_{xy}^2)}} e^{-\frac{1}{2}\frac{x^2}{\sigma_x^2 - \sigma_{xy}^2}}$$

i.e., given $Y=0$, X is Gaussian with mean 0 and variance $\sigma_x^2 - \sigma_{xy}^2$

Proof. By definition of the joint normal distribution,
$$f_{X,Y}(x,0) = \frac{1}{2\pi(\det\Sigma)^{1/2}} \exp\left(-\frac{1}{2}(x,0)\,\Sigma^{-1}\begin{pmatrix} x \\ 0 \end{pmatrix}\right)$$

Then
$$f_{X|Y}(x|0) = \frac{f_{X,Y}(x,0)}{f_Y(0)} = \frac{\sqrt{2\pi}}{2\pi(\det\Sigma)^{1/2}} \exp\left(-\frac{1}{2}(x,0)\,\Sigma^{-1}\begin{pmatrix} x \\ 0 \end{pmatrix}\right)$$

Now $(x,0)\,\Sigma^{-1}\begin{pmatrix} x \\ 0 \end{pmatrix} = x \cdot \frac{1}{\det\Sigma}$, $\det\Sigma = \sigma_x^2 - \sigma_{xy}^2$, therefore

$$f_{X|Y}(x|0) = \frac{1}{\sqrt{2\pi(\sigma_x^2 - \sigma_{xy}^2)}} e^{-\frac{1}{2}\frac{x^2}{\sigma_x^2 - \sigma_{xy}^2}} \quad \blacksquare$$

Proof of Theorem 1 (1)

Let $(B_t)_{t\geq 0}$ be a standard BM. Denote by $(B^\circ_t)_{t\in[0,1]}$ the part of B on $[0,1]$ conditioned on the event $B_1=0$.

1) B° is continuous on $[0,1]$

2) In order to show that B° is Gaussian, we need to show that $\forall\, \alpha_i \in \mathbb{R}$ and $0\leq t_1<t_2<\cdots<t_n\leq 1$

$$\sum_{i=1}^{n} \alpha_i B^\circ_{t_i} \text{ is Gaussian}$$

$$\Longleftrightarrow \text{ given } B_1=0,\ \sum_{i=1}^{n}\alpha_i B_{t_i} \text{ is Gaussian}$$

B is Gaussian $\Rightarrow \forall\, \beta_1,\beta_2\in\mathbb{R}$ $\beta_1 B_1 + \beta_2 \sum_{i=1}^{n}\alpha_i B_{t_i}$ is Gaussian

$$\Rightarrow \Big(\overset{X}{\sum_{i=1}^{n}\alpha_i B_{t_i}},\ \overset{Y}{B_1}\Big) \text{ are jointly normal}$$

$$\overset{\text{Lemma}}{\Longrightarrow} \text{ given } B_1=0,\ \sum_{i=1}^{n}\alpha_i B_{t_i} \text{ is Gaussian}$$

Proof of Theorem 1 (2)

3) From Lemma we also know that $E(B_t^\circ)=0$.

To compute the covariance function, note that for $0<s<t<1$

$$f_{B_s,B_t,B_1}(x,y,0)=(2\pi)^{-3/2}(\det\Sigma)^{-1/2}\exp\left(-\tfrac{1}{2}(x,y,0)\Sigma^{-1}\begin{pmatrix}x\\y\\0\end{pmatrix}\right),$$

where $\Sigma=\begin{pmatrix}s&s&s\\s&t&t\\s&t&1\end{pmatrix}$ Also note that $f_{B_1}(0)=(2\pi)^{-1/2}$

If $\Sigma^{-1}=\begin{pmatrix}T & \begin{matrix}*\\ *\end{matrix}\\ *\ * & *\end{pmatrix}$, then $(x,y,0)\Sigma^{-1}\begin{pmatrix}x\\y\\0\end{pmatrix}=(x,y)\,T\begin{pmatrix}x\\y\end{pmatrix}$

Now, $\det\Sigma=s(t-s)(1-t)$ and $T=\frac{1}{s(t-s)(1-t)}\begin{pmatrix}t(1-t)&-s(1-t)\\-s(1-t)&s(1-s)\end{pmatrix}$

$$\det T=\frac{1}{s(t-s)(1-t)}\quad\text{and}\quad T^{-1}=\begin{pmatrix}s(1-s)&s(1-t)\\s(1-t)&t(1-t)\end{pmatrix}=:\tilde{\Sigma}$$

Finally, $$\frac{f_{B_s,B_t,B_1}(x,y,0)}{f_{B_1}(0)}=(2\pi)^{-1}(\det\tilde{\Sigma})^{-1/2}e^{-\frac{1}{2}(x,y)\tilde{\Sigma}^{-1}\begin{pmatrix}x\\y\end{pmatrix}}$$

$\min\{s,t\}-st$

$s-st$

$\Rightarrow \mathrm{Cov}(B_s^\circ,B_t^\circ)=s(1-t)$ ∎

Brownian bridge. Remark

Remark. Let $(B_t)_{t\geq 0}$ be a BM. Then the process

$$(X_t)_{t\in[0,1]}, \quad X_t = B_t - tB_1 \quad \text{for } t\in[0,1]$$

is a Brownian bridge.

Indeed: 1) $(X_t)_{t\in[0,1]}$ is continuous, $X_0 = 0$

2) $(X_t)_{t\in[0,1]}$ is Gaussian: $\forall\, \alpha_i,\ 0\leq t_1 < \dots < t_n \leq 1$

$$\sum_{i=1}^{n} \alpha_i X_{t_i} = \sum_{i=1}^{n} \alpha_i \left(B_{t_i} - t_i B_1\right) = \sum_{i=1}^{n} \alpha_i B_{t_i} - \left(\sum_{i=1}^{n} \alpha_i t_i\right) B_1$$

which is Gaussian since $(B_t)_{t\geq 0}$ is a Gaussian process

3) $\mathrm{Cov}(X_s, X_t) = \mathrm{Cov}(B_s - sB_1, B_t - tB_1) = \min\{s,t\} - st - st + st\cdot 1$

$$= \min\{s,t\} - st$$

Brownian motion with drift

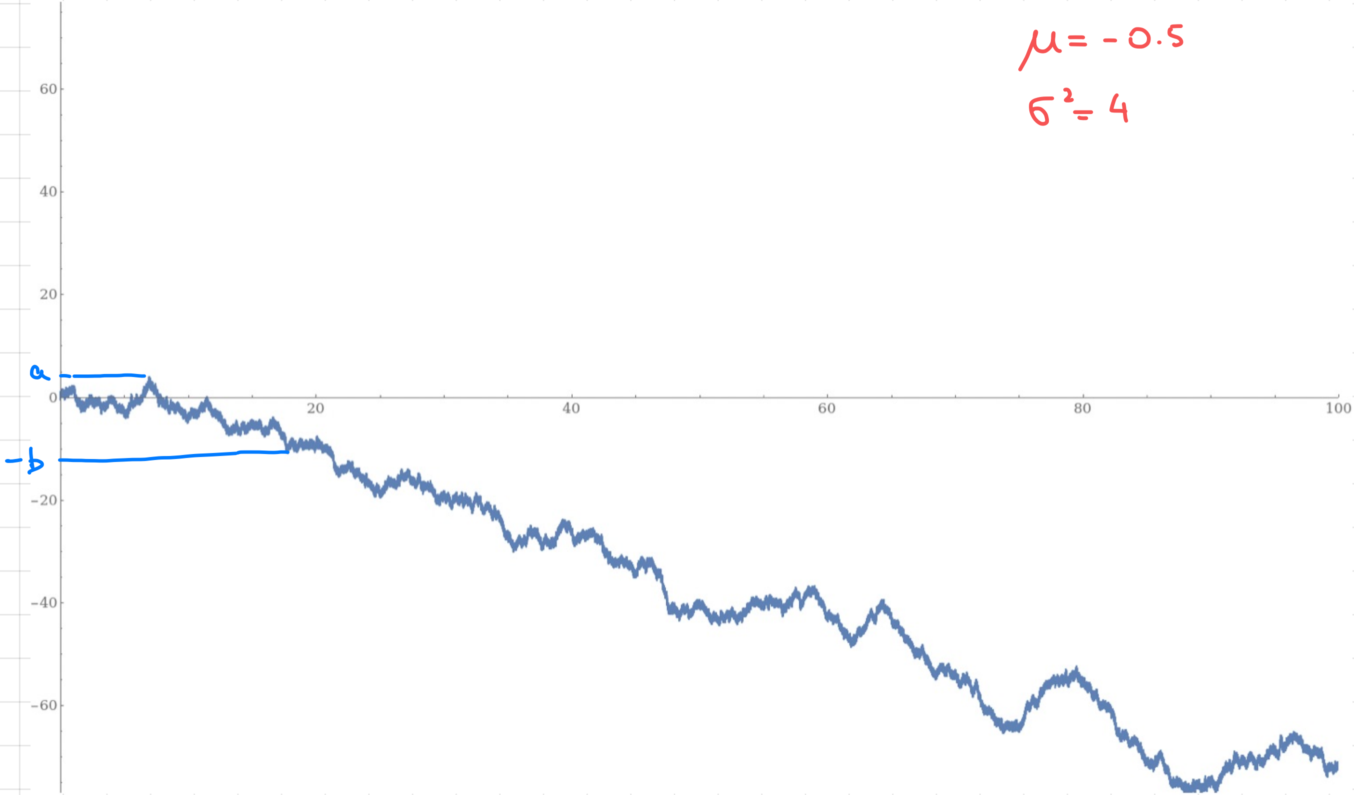
Def Let $(B_t)_{t\geq 0}$ be a standard BM. Then for $\mu \in \mathbb{R}$ and $\sigma > 0$ the process $(X_t)_{t\geq 0}$ with $X_t = \mu t + \sigma B_t$, $t \geq 0$ is called the Brownian motion with drift μ and variance paremeter σ^2.

Remark BM with drift μ and variance paremeter σ is a stochastic process $(X_t)_{t\geq 0}$ satisfying

1) $X_0 = 0$, $(X_t)_{t\geq 0}$ has continuous sample paths
2) $(X_t)_{t\geq 0}$ has independent increments
3) For $t > s$ $\quad X_t - X_s \sim N(\mu(t-s), \sigma^2(t-s))$

In particular, $X_t \sim N(\mu t, \sigma^2 t) \Rightarrow X_t$ is not centered, not symmetric w.r.t. the origin

Brownian motion with drift



Gambler's ruin problem for BM with drift

Let $(X_t)_{t\geq 0}$ be a BM with drift $\mu \in \mathbb{R}$ and variance parameter $\sigma^2 > 0$. Fix $a < x < b$ and denote

$$T = T_{ab} = \min\{t \geq 0 : X_t = a \text{ or } X_t = b\}, \text{ and}$$

$$u(x) = P(X_T = b \mid X_0 = x).$$

Theorem.

(i) $$u(x) = \frac{\exp(-2\mu x/\sigma^2) - \exp(-2\mu a/\sigma^2)}{\exp(-2\mu b/\sigma^2) - \exp(-2\mu a/\sigma^2)}$$

(ii) $$E(T_{ab} \mid X_0 = x) = \frac{1}{\mu}\Big(u(x)(b-a) - (x-a)\Big)$$

No proof

$$\left(u(x) = \frac{b-x}{b-a}\right)$$

SBM

Example

Fluctuations of the price of a certain share is modeled by the BM with drift $\mu = 1/10$ and variance $\sigma^2 = 4$. You buy a share at 100 \$ and plan to sell it if its price increases to 110 \$ or drops to 95 \$.

(a) What is the probability that you will sell at profit?

(b) What is the expected time until you sell the share?

Denote by $(X_t)_{t\geq 0}$ a BM with drift $\frac{1}{10}$ and variance 4, $x = 100$, $b = 110$, $a = 95$. Then $2\mu/\sigma^2 = \frac{2 \cdot 0.1}{4} = \frac{1}{20}$ and

(a) $P(X_T = 110 \mid X_0 = 100) = \dfrac{e^{-\frac{1}{20}\cdot 100} - e^{-\frac{1}{20}\cdot 95}}{e^{-\frac{1}{20}\cdot 110} - e^{-\frac{1}{20}\cdot 95}} \approx 0.419$

(b) $E(T \mid X_0 = 100) = \frac{1}{0.1}\Big(0.419\,(110 - 95) - (100 - 95)\Big) \approx 12.88$

Maximum of a BM with negative drift

Thm Let $(X_t)_{t\geq 0}$ be a BM with drift $\mu<0$, variance σ^2 and $X_0=0$. Denote $M=\max_{t\geq 0} X_t$. Then

$$M \sim \mathrm{Exp}(-2\mu/\sigma^2)$$

Proof. $X_0=0$, therefore $M\geq 0$. For any $b>0$

$$P(M>b) = P\Big(\bigcup_{n\geq 1} \{X \text{ hits } b \text{ before } -n\}\Big)$$

$$= \lim_{n\to\infty} P(X \text{ hits } b \text{ before } -n)$$

$$= \lim_{n\to\infty} \frac{1-e^{2n\mu/\sigma^2}}{e^{-2b\mu/\sigma^2}-e^{2n\mu/\sigma^2}} = \frac{1}{e^{-2b\mu/\sigma^2}} = e^{-2b\mu/\sigma^2}$$

$$P(M>b) = e^{-(-2\mu/\sigma^2)b} \Rightarrow M \sim \mathrm{Exp}(-2\mu/\sigma^2)$$

$\blacksquare$

Geometric BM

Def. Stochastic process $(Z_t)_{t\geq 0}$ is called a geometric Brownian motion with drift parameter α and variance σ^2 if $X_t = \log Z_t$ is a BM with drift $\mu = \alpha - \frac{1}{2}\sigma^2$ and variance σ^2.

In other words, $Z_t = z e^{(\alpha - \frac{1}{2}\sigma^2)t + \sigma B_t}$, where $(B_t)_{t\geq 0}$ is a standard BM and $z > 0$ is the starting point $Z_0 = z$.

If $0 \leq t_1 < t_2 < \cdots < t_n$, then $\frac{Z_{t_i}}{Z_{t_{i-1}}} = e^{(\alpha - \frac{1}{2}\sigma^2)(t_i - t_{i-1}) + \sigma(B_{t_i} - B_{t_{i-1}})}$

Since B has independent increments

$$\frac{Z_{t_1}}{Z_{t_0}}, \frac{Z_{t_2}}{Z_{t_1}}, \dots, \frac{Z_{t_n}}{Z_{t_{n-1}}}$$ are independent and

$$\frac{Z_{t_n}}{Z_{t_0}} = \frac{Z_{t_1}}{Z_{t_0}} \cdot \frac{Z_{t_2}}{Z_{t_1}} \cdots \frac{Z_{t_n}}{Z_{t_{n-1}}}$$ $\leftarrow$ "relative change of price = product of independentent relative changes"

Expectation of Geometric BM

Let $(Z_t)_{t\geq 0}$ be geometric BM with paremeters α and σ.

Then

$$E(Z_t \mid Z_0 = z) = E\left(z e^{(\alpha - \frac{1}{2}\sigma^2)t + \sigma B_t}\right) = z e^{(\alpha - \frac{1}{2}\sigma^2)t} E\left(e^{\sigma B_t}\right)$$

$$E(e^{\sigma B_t}) = e^{\frac{t\sigma^2}{2}}$$

$$\Rightarrow E(Z_t \mid Z_0 = z) = z e^{(\alpha - \frac{1}{2}\sigma^2)t} e^{t\frac{\sigma^2}{2}} = z e^{\alpha t}$$

Remark

It can be shown that for $0 < \alpha < \frac{1}{2}\sigma^2$ $Z_t \to 0$ as $t \to \infty$

At the same time, for $\alpha > 0$ $E(Z_t) \to \infty$.

Variance of geometric BM

$$E(Z_t^2 \mid Z_0 = z) = E\left(z^2 e^{2X_t}\right) = E\left(z^2 e^{(2\alpha-\sigma^2)t} e^{2\sigma B_t}\right)$$

$$= z^2 e^{(2\alpha-\sigma^2)t} e^{2\sigma^2 t} = z^2 e^{2\alpha t + \sigma^2 t}$$

$$\mathrm{Var}(Z_t \mid Z_0 = z) = z^2 e^{2\alpha t + \sigma^2 t} - z^2 e^{2\alpha t} = z^2 e^{2\alpha t}\left(e^{\sigma^2 t} - 1\right)$$

Theorem.

Let $(Z_t)_{t\geq 0}$ be geometric BM with paremeters α and σ^2.

Then

(i) $E(Z_t \mid Z_0 = z) = z e^{\alpha t}$

(ii) $\mathrm{Var}(Z_t \mid Z_0 = z) = z^2 e^{2\alpha t}\left(e^{\sigma^2 t} - 1\right)$

Gambler's ruin for geometric BM

Let $(Z_t)_{t\geq 0}$ be geometric BM with paremeters α and σ^2.

Let $A<1<B$, and denote $T=\min\{t: \frac{Z_t}{Z_0}=A \text{ or } \frac{Z_t}{Z_0}=B\}$.

Theorem

$$P\left(\frac{Z_T}{Z_0}=B\right)=\frac{1-A^{1-2\alpha/\sigma^2}}{B^{1-2\alpha/\sigma^2}-A^{1-2\alpha/\sigma^2}}$$

Example Fluctuations of the price are modeled by a geometric BM with drift $\alpha=0.1$ and variance $\sigma^2=4$. You buy a share at 100\$ and plan to sell it if its price increases to 110\$ or drops to 95\$.

Take $A=0.95$, $B=1.1$, $2\alpha/\sigma^2=\frac{1}{20}$, $1-2\alpha/\sigma^2=\frac{19}{20}=0.95$

$$P(X_T=110\mid X_0=100)=\frac{1-0.95^{0.95}}{1.1^{0.95}-0.95^{0.95}}\simeq 0.334$$

Black-Scholes option pricing formula

Call option gives the buyer the right (not obligation) to buy a block of shares at a specific price ← striking price at any time during a certain period. How much should you pay for it?

<u>Example</u>: For the premium of 6\$ the call allows you to buy 60\$ of shares during the period of one month. If at some point during this period the actual price of the shares becomes $x > 66\$$, you can buy the shares using the call option, then immediately sell it gaining $(x-66)\$$. Or you may opt not to buy the shares at all → lose 6\$.

Let z be the current value of the share and τ be the length of the time period. Denote $F(z,\tau)$ the value of the call.

Black-Scholes option pricing formula

Then $F(z,\tau) = e^{-r\tau} E\left((Z_\tau - a)^+ \mid Z_0 = z \right)$ [BS], where

- a is the striking price
- r is the return rate for riskless investments
- $(Z_t)_{t\geq 0}$ is a geometric BM with parameters r and σ^2
- σ^2 is the volatility (variance parameter) of the share price

Computing the conditional expectation gives

$$F(z,\tau) = z\,\Phi\left(\frac{\log\frac{z}{a} + (r + \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}} \right) - a e^{-r\tau}\,\Phi\left(\frac{\log\frac{z}{a} + (r - \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}} \right)$$

Black-Scholes formula