☐ **Write your name and PID on the top of EVERY PAGE.**

☐ **Write the solutions to each problem on separate pages. CLEARLY INDICATE on the top of each page the number of the corresponding problem.**

☐ **Remember this exam is graded by a human being. Write your solutions NEATLY AND COHERENTLY, or they risk not receiving full credit.**

☐ **You may assume that all transition probability functions are STATIONARY.**

☐ **You are allowed to use the textbook, lecture notes and your personal notes. You are not allowed to use the electronic devices (except for accessing the online version of the textbook) or outside assistance. Outside assistance includes but is not limited to other people, the internet and unauthorized notes.**

1. (25 points) Let $(X_t)_{t\geq 0}$ be a continuous time Markov chain on the state space $\{0,1,2\}$ with transition probability functions

$$P(t) = \begin{array}{c|c} & \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \end{array} & \begin{Vmatrix} e^{-3t} & \frac{1}{2} - e^{-3t} + \frac{1}{2}e^{-4t} & \frac{1}{2} - \frac{1}{2}e^{-4t} \\ 0 & \frac{1}{2} + \frac{1}{2}e^{-4t} & \frac{1}{2} - \frac{1}{2}e^{-4t} \\ 0 & \frac{1}{2} - \frac{1}{2}e^{-4t} & \frac{1}{2} + \frac{1}{2}e^{-4t} \end{Vmatrix} \end{array}. \tag{1}$$

(a) (7 points) Compute the generator Q of $(X_t)_{t\geq 0}$. [Hint. Recall that $P'(0) = Q$.]

(b) (6 points) Give the jump-and-hold description of $(X_t)_{t\geq 0}$.

(c) (6 points) Draw the rate diagram of $(X_t)_{t\geq 0}$.

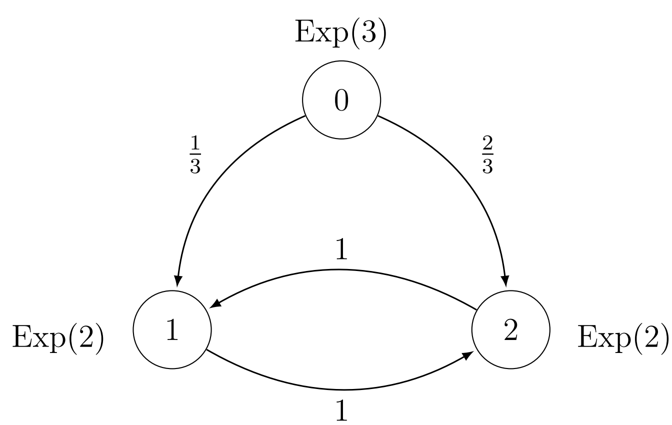
(d) (6 points) Assuming that X_0 is uniformly distributed on $\{0,1,2\}$, compute the probability that $X_1 = 2$.

Solution.

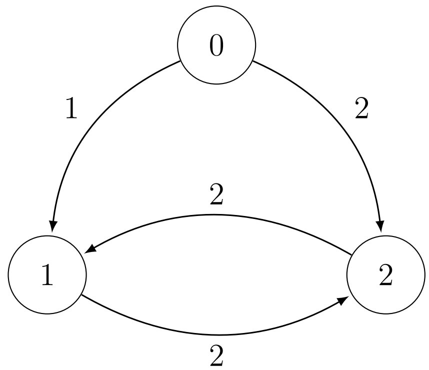
(a) Computing $P'(0)$ gives

$$Q = \begin{array}{c|c} & \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \end{array} & \begin{Vmatrix} -3 & 1 & 2 \\ 0 & -2 & 2 \\ 0 & 2 & -2 \end{Vmatrix} \end{array} \tag{2}$$

(b) The jump-and-hold diagram



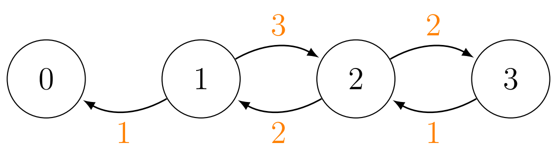
(c) The rate diagram



(d)

$$P(X_1 = 2) = \sum_{i=0}^{2} P(X_1 = 2 \,|\, X_0 = i) P(X_0 = i) \tag{3}$$
$$= \frac{1}{3}\Big(P_{02}(1) + P_{12}(1) + P_{22}(1)\Big) \tag{4}$$
$$= \frac{1}{2} - \frac{1}{6}e^{-4}. \tag{5}$$

2. (25 points) Let $(X_t)_{t\geq 0}$ be a birth and death process on $\{0, 1, 2, 3\}$ described by the following rate diagram



Compute the mean time to absorption starting from state 1 (i.e., given $X_0 = 1$).

Solution.

This chain has three transient states $\{1, 2, 3\}$ and one absorbing state 0. Denote by T the absorption time $T = \min\{t : X_t = 0\}$ and denote by w_i the expected time to absorption starting from state i, i.e.,

$$w_i := E(T \,|\, X_0 = i). \tag{6}$$

In order to solve this problem we have to compute w_1.

Unknown quantities (w_1, w_2, w_3) satisfy the following system of equations:

$$w_1 = \frac{1}{4} + \frac{3}{4}w_2, \tag{7}$$
$$w_2 = \frac{1}{4} + \frac{1}{2}w_1 + \frac{1}{2}w_3, \tag{8}$$
$$w_3 = 1 + w_2. \tag{9}$$

Plugging the first and the third equations into the second equation gives

$$w_2 = \frac{1}{4} + \frac{1}{8} + \frac{3}{8}w_2 + \frac{1}{2} + \frac{1}{2}w_2 = \frac{7}{8} + \frac{7}{8}w_2, \tag{10}$$

from which $w_2 = 7$. Plugging this into the first equation gives the final answer $w_1 = \frac{11}{2}$.

3. (25 points) Let $(X_t)_{t\geq 0}$ be a continuous time Markov chain on the state space $\{0, 1, 2\}$ with generator

$$Q = \begin{array}{c|ccc|} & 0 & 1 & 2 \\ \hline 0 & -2 & 2 & 0 \\ 1 & 3 & -6 & 3 \\ 2 & 0 & 1 & -1 \end{array}. \tag{11}$$

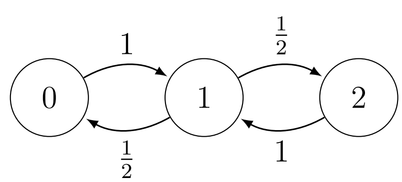
(a) (10 points) Draw the diagram for the jump chain of $(X_t)_{t\geq 0}$ and explain why $(X_t)_{t\geq 0}$ is irreducible.

(b) (10 points) Compute the stationary distribution for $(X_t)_{t\geq 0}$.

(c) (5 points) What is the expected average fraction of time that $(X_t)_{t\geq 0}$ will spend in states 1 and 2 in the long run?

Solution.

(a) The diagram of the jump chain is



All states communicate, therefore the jump chain is irreducible. This implies that $(X_t)_{t\geq 0}$ is also irreducible.

(b) Denote by (π_0, π_1, π_2) the limiting distribution. Then (π_0, π_1, π_2) satisfy the following system

$$-2\pi_0 + 3\pi_1 = 0, \tag{12}$$
$$2\pi_0 - 6\pi_1 + \pi_2 = 0, \tag{13}$$
$$3\pi_1 - \pi_2 = 0, \tag{14}$$
$$\pi_0 + \pi_1 + \pi_2 = 1. \tag{15}$$

The first equation gives $\pi_0 = \frac{3}{2}\pi_1$, the third equation gives $\pi_2 = 3\pi_1$. Plugging this into the last equation gives

$$\pi_1\left(\frac{3}{2} + 1 + 3\right) = \pi_1\frac{11}{2} = 1, \tag{16}$$

so

$$\pi_1 = \frac{2}{11}, \quad \pi_0 = \frac{3}{11}, \quad \pi_2 = \frac{6}{11}. \tag{17}$$

(c) In the long run, the process will spend $\pi_1 + \pi_2 = \frac{8}{11}$ of time in states 1 and 2.

4. (25 points) Certain printing facility has two printers operating on a 24/7 basis and one repairman that takes care of the printers. The amount of time (in hours) that a printer works before breaking down has exponential distribution with mean 2. If a printer is broken, the repairman needs exponentially distributed amount of time with mean 1 (hour) to repair the broken printer. The repairman cannot repair two printers simultaneously. Each printer can produce 100 pages per minute.

   Let X_t denote the number of printers in operating state at time t.

   (a) (10 points) Assuming without proof that $(X_t)_{t\geq 0}$ is a Markov process, determine the generator of $(X_t)_{t\geq 0}$ (you can provide rigorous computations for only one entry of matrix Q.)

   [Hint. If $T \sim \text{Exp}(\gamma)$, then $P(T \leq h) = \gamma h + o(h)$ as $h \to 0$.]

   (b) (10 points) Compute the stationary distribution for $(X_t)_{t\geq 0}$.

   (c) (5 points) In the long run, how many pages does the facility produce on average per minute?

Solution.

(a) The generator of $(X_t)_{t\geq 0}$ is given by

$$Q = \begin{array}{c|c} & \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \end{array} & \left\| \begin{array}{ccc} -1 & 1 & 0 \\ \frac{1}{2} & -\frac{3}{2} & 1 \\ 0 & 1 & -1 \end{array} \right\| \end{array}. \tag{18}$$

Example of computations of q_{ij}.

- $X_0 = 1$ means that one printer is operating and one printer is broken.
  - $X_h = 0$ means that the operating printer stopped working before time h and the broken printed was not repaired before time h, so

$$P(X_h = 0 \,|\, X_0 = 1) \tag{19}$$
$$= P(\text{printer's working time} \leq h) P(\text{printer's repair time} > h) \tag{20}$$
$$= (1 - e^{-\frac{1}{2}h}) e^{-h} = \frac{1}{2}h + o(h) \tag{21}$$

and $q_{10} = \frac{1}{2}$.

- $X_h = 2$ means that the broken printer got repared before time h and the operating printer was is still working at time h, so

$$
\begin{aligned}
& P(X_h = 2 \,|\, X_0 = 1) && (22)\\
&\quad = P(\text{repare time} \le h) P(\text{working time} > h) && (23)\\
&\quad = (1 - e^{-h}) e^{-\frac{1}{2}h} = h + o(h) && (24)
\end{aligned}
$$

and $q_{12} = 1$.

- $X_0 = 2$ means that both printers are working
  - $X_h = 1$ means that one of the two printers stopped working before time h and the other is working at time h (note that there are two choices of which of the two got broken), so

$$
\begin{aligned}
& P(X_h = 1 \,|\, X_0 = 2) && (25)\\
&\quad = 2P(\text{printer's working time} \le h) P(\text{printer's working time} > h) && (26)\\
&\quad = 2(1 - e^{-\frac{1}{2}h}) e^{-\frac{1}{2}h} = 2\Big(\frac{1}{2}h + o(h)\Big) = h + o(h) && (27)
\end{aligned}
$$

and $q_{21} = 1$.

(b) Let (π_0, π_1, π_2) be the stationary distribution. Then (π_0, π_1, π_2) should satisfy the following system

$$
\begin{aligned}
-\pi_0 + \frac{1}{2}\pi_1 &= 0, && (28)\\
\pi_0 - \frac{3}{2}\pi_1 + \pi_2 &= 0, && (29)\\
\pi_1 - \pi_2 &= 0, && (30)\\
\pi_0 + \pi_1 + \pi_2 &= 1. && (31)
\end{aligned}
$$

From the first and the third equations we have that $\pi_0 = \frac{1}{2}\pi_1$ and $\pi_2 = \pi_1$. Plugging this into the last equation gives

$$\pi_1\Big(\frac{1}{2} + 1 + 1\Big) = \frac{5}{2}\pi_1 = 1, \tag{32}$$

from which we get that

$$\pi_0 = \frac{1}{5}, \quad \pi_1 = \frac{2}{5}, \quad \pi_2 = \frac{2}{5}. \tag{33}$$

(c) In the long run, $\frac{1}{5}$ of the time both printers are broken printing 0 pages per minute, $\frac{2}{5}$ of the time only one printer is working producing 100 pages per minute and $\frac{2}{5}$ of the time both printer are working producing 200 pages per minute. Therefore, on average the printing facility produces

$$\frac{1}{5} \cdot 0 + \frac{2}{5} \cdot 100 + \frac{2}{5} \cdot 200 = 120 \tag{34}$$

pages per minute.