MATH180C: Introduction to Stochastic Processes II

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Today: Asymptotic behaviour of renewal processes. Examples

> Q&A: November 18

Next: PK 2.5, Durrett 5.1-5.2

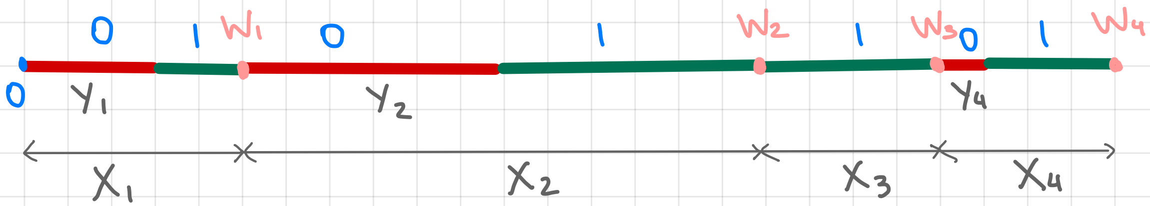
This week:

- Homework 6 (due Saturday, November 21, 11:59 PM)
- Quiz 4 (Wednesday, November 18, lectures 11-15)
- Midterm 2 (Monday, November 23, lectures 10-17)

Two component renewals

Consider the following model:

- $(X_i)_{i=1}^{\infty}$ are interrenewal times
- at each moment of time the system $S(t)$ can be in one of two states: $S(t)=0$ or $S(t)=1$
- random variables Y_i denote the part of X_i during which the system is in state 0, $0 \le Y_i \le X_i$
- collection $((X_i, Y_i))_{i=1}^{\infty}$ is i.i.d.



Q: In the long run (for large t), what is the probability that the system is in state 1 at time t?

Two component renewals

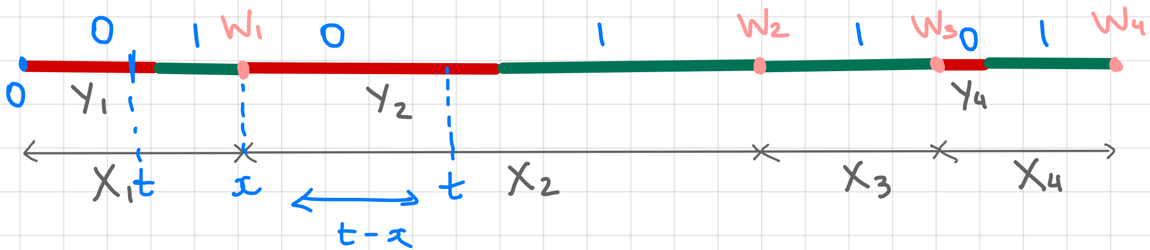
<u>Thm</u>. If $E(X_1) < \infty$, then $\lim_{t\to\infty} P(S(t)=0) = \dfrac{E(Y_1)}{E(X_1)}$

<u>Proof</u>. Denote $g(t) = P(S(t)=0)$. Then

$$g(t) = \int_0^\infty P(S(t)=0 \mid X_1 = x)\, dF(x)$$

If $t < x$, then $P(S(t)=0 \mid X_1 = x) = P(Y_1 > t \mid X_1 = x)$

If $t \geq x$, then $P(S(t)=0 \mid X_1 = x) = P(S(t-x)=0) = g(t-x)$



Two component renewals

$$g(t) = \underbrace{\int_t^\infty P(Y_1 > t \mid X_1 = x)\, dF(x)}_{\stackrel{!!}{=} h(t)} + \underbrace{\int_0^t g(t-x)\, dF(x)}_{= g * F(t)}$$

Function g satisfies the renewal equation

$$g(t) = h(t) + g * F(t)$$

Note that $Y_1 \le X_1$, therefore $P(Y_1 > t \mid X_1 = x) = 0$ for $x < t$,

$$h(t) = \int_0^\infty P(Y_1 > t \mid X_1 = x)\, dF(x) = P(Y_1 > t) \Rightarrow \left[\begin{array}{c}\text{abs. integr.} \\ \text{of } h(t)\end{array}\right]$$

$$\int_0^\infty h(t)\,dt = \int_0^\infty P(Y_1 > t)\, dt = E(Y_1) \le E(X_1) < \infty$$

From the Key renewal theorem $\lim_{t\to\infty} g(t) = \dfrac{E(Y_1)}{E(X_1)}$ ■

Example: the Peter principle

Setting:
- infinite population of candidates for certain position
- fraction p of the candidates are competent, $q = 1 - p$ are incompetent
- if a competent person is chosen, after time C_i he/she gets promoted
- if an incompetent person is chosen, he/she remains in the job until retirement (r.v. I_j)
- once the position is open again, the process repeats

Question: What fraction of time, denoted f, is the position held by an incompetent person on average in the long run?

Example: the Peter principle

Denote $X_i = \begin{cases} C_i, & \text{if } i\text{-th employee is competent} \\ I_i, & \text{if } i\text{-th employee is incompetent} \end{cases}$

$Y_i = \begin{cases} 0, & \text{if } i\text{-th employee is competent} \\ I_i, & \text{if } i\text{-th employee is incompetent.} \end{cases}$

KRT for two component renewals can be applied to $((X_i, Y_i))_{i=1}^{\infty}$

If $S(t) = 0$ if the person is incompetent, then

$$\lim_{t \to \infty} P(S(t) = 0) = \frac{E(Y_1)}{E(X_1)} \qquad \text{and}$$

exercise

$$f := \lim_{t \to \infty} E\left(\frac{1}{t} \int_0^t \mathbb{1}_{\{S(u)=0\}} dt \right) = \lim_{t \to 0} \frac{1}{t} \int_0^t P(S(u) = 0)\, du = \frac{E(Y_1)}{E(X_1)}$$

Finally, if
- $E(C_i) = \mu$
- $E(I_i) = \nu$,

then $f = \dfrac{E(Y_1)}{E(X_1)} = \dfrac{(1-p)\nu}{p\mu + (1-p)\nu}$

Example: the Peter principle (alternative)

Let $X_i = \begin{cases} C_i, & \text{if the } i\text{-th person is competent} \\ I_i, & \text{if the } i\text{-th person is incompetent} \end{cases}$

$Y_i = \begin{cases} 0, & \text{time occupied by a competent person} \\ I_i, & \text{time occupied by an incompetent person} \end{cases}$

and assume that $|X_i| < K$. Then using

$$E\left(\frac{1}{t}\sum_{i=1}^{N(t)-1} Y_i\right) \leq E\left(\frac{1}{t}\int_0^t \mathbb{1}_{\{S(u)=0\}}\,du\right) \leq E\left(\frac{1}{t}\sum_{i=1}^{N(t)} Y_i\right)$$

elementary renewal theorem

exercise

$t \to \infty$: both bounds tend to $\dfrac{E(Y_1)}{E(X_1)}$

Hint: $E\left(\sum_{i=1}^{N(t)+1} Y_i\right) = E(Y_i)\left(1 + E(N(t))\right)$

Again, if
- $E(C_i) = \mu$
- $E(I_i) = \nu$,

then $f = \dfrac{E(Y_1)}{E(X_1)} = \dfrac{(1-p)\nu}{p\mu + (1-p)\nu}$

Example: the Peter principle

If we take $p = \frac{1}{2}$, $\mu = 1$, $\nu = 10$, then

$$f = \frac{10}{11} \approx 0.909$$