

# MATH180C: Introduction to Stochastic Processes II

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Today: Limiting behavior  
> Q&A: October 23  
Next: Review

This week:

- Midterm 1 on Wednesday, October 28 (lectures 1-9)
- Homework 2 (due Friday, October 23, 11:59 PM)

# Long run behavior of discrete time MC. Summary

Let  $(X_n)_{n \geq 0}$  be a discrete time MC on  $\{0, \dots, N\}$  with stationary transition probability matrix  $P = (P_{ij})_{i,j=0}^N$ .

- $P$  is called **regular** if there exists  $k$  such that  $[P^k]_{ij} > 0$  for all  $i, j$ . **[ $P$  is regular iff  $(X_n)$  is irreducible and aperiodic]**

Thm. If  $P$  is **regular**, then there exist  $\pi_0, \dots, \pi_N \in \mathbb{R}$  s.t.

$$1) \pi_i > 0 \quad \forall i$$

$$2) \sum_{i=0}^N \pi_i = 1$$

$$3) \forall j \quad \lim_{n \rightarrow \infty} P_{ij}^{(n)} = \pi_j$$

$(\pi_0, \dots, \pi_N)$  is called limiting

(stationary) distribution of  $(X_n)$

$(\pi_0, \dots, \pi_N)$  is uniquely defined by the system of equations

$$\begin{cases} \pi_j = \sum_{i=0}^N \pi_i P_{ij}, \\ \sum_{i=0}^N \pi_i = 1 \end{cases}$$

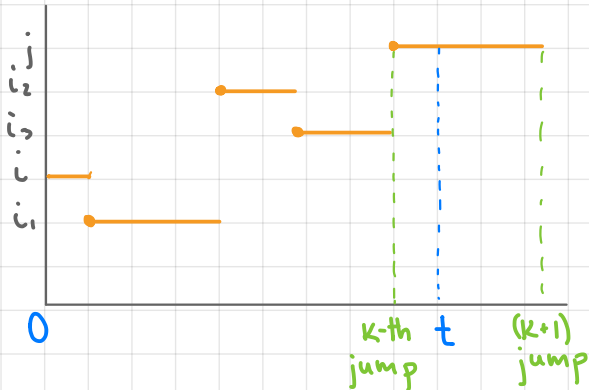
$$(\pi_0, \pi_1, \dots, \pi_N) = (\pi_0, \dots, \pi_N) P$$

## Long run behavior of continuous time MC.

Let  $(X_t)_{t \geq 0}$  be a continuous time MC,  $X_t \in \{0, \dots, N\}$  and let  $(Y_n)_{n \geq 0}$  be the embedded jump chain.

Def.  $(X_t)_{t \geq 0}$  is called irreducible if its jump chain  $(Y_n)_{n \geq 0}$  is irreducible (consisting of one communicating class)

Thm. If  $(X_t)_{t \geq 0}$  is irreducible, then



Idea of the proof:

- $Y_n$  is irreducible  $\Rightarrow \exists i_1, \dots, i_{k-1}$  s.t.

$$P(Y_k = j, Y_{k-1} = i_{k-1}, \dots, Y_1 = i_1 \mid Y_0 = i) > 0$$

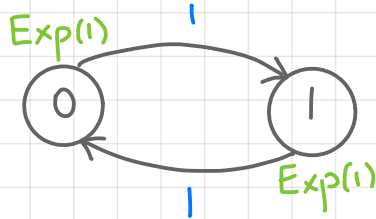
- $P(k\text{-th jump} \leq t < (k+1)\text{-th jump}) > 0 \quad \forall t > 0$

# Long run behavior of continuous time MC

Remarks: Continuous time MCs are "aperiodic"

All irreducible continuous time MCs are "regular"

Example.



Thm. If  $(X_t)_{t \geq 0}$  is irreducible, then there exists  $\pi_0, \dots, \pi_N$

1)  $\pi_i > 0, \sum_{i=0}^N \pi_i = 1$

2)  $\lim_{t \rightarrow \infty} P_{ij}(t) = \pi_j$  for all  $i$

3)  $\pi = (\pi_0, \dots, \pi_N)$  is uniquely determined by

$\pi$  is called limiting/stationary/equilibrium distribution of  $(X_t)$

## Long run behavior of continuous time MC

Remark about 3):  $\pi Q = 0$  is equivalent to  $\forall t$

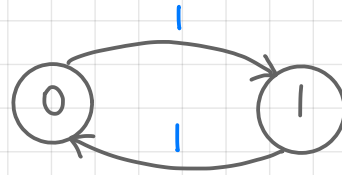
( $\Rightarrow$ ) If  $\pi Q = 0$ , then using Kolmogorov backward equation  
 $(\pi P(t))' =$

so  $\pi P(t)$  is independent of  $t$ . Since  $P(0) = I$ , we get

$$\forall t \quad \pi P(t) =$$

( $\Leftarrow$ ) If  $\pi P(t) = \pi$ , then  $(\pi P(t))' = 0$ . Using Kolmogorov forward equation

## Example: Two-state MC



From Lecture 5: if  $Q = \begin{pmatrix} -\alpha & \alpha \\ \beta & -\beta \end{pmatrix}$ , then

$$P(t) = I + \frac{1}{\alpha + \beta} Q - \frac{1}{\alpha + \beta} e^{-(\alpha + \beta)t} Q \quad . \text{ If } \alpha = \beta = 1$$

$$\lim_{t \rightarrow \infty} P(t) =$$

Note, that the jump process  $(Y_n)$  does not have limiting distribution!

$$\tilde{P}^{Y_n} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

## Long run behavior of discrete time MC. Summary (2)

Let  $(X_n)_{n \geq 0}$  be a discrete time MC on  $\{0, 1, \dots\}$  with stationary transition probability matrix  $P = (P_{ij})_{i,j=0}^{\infty}$

Define  $R_i = \min\{n: X_n = i\}$ ,  $m_i = E(R_i | X_0 = i)$  mean duration between visits

Thm. If  $(X_n)_{n \geq 0}$  is recurrent irreducible aperiodic, then

$$\lim_{n \rightarrow \infty} P_{ij}^{(n)} = \frac{1}{m_j} \quad \forall j$$

If  $\lim_{n \rightarrow \infty} P_{ij}^{(n)} > 0$  for some (all)  $j$ , then MC is positive recurrent

$\lim_{n \rightarrow \infty} P_{ij}^{(n)} = 0$  for some (all)  $j$ , then MC is null recurrent.

If  $(X_n)$  is positive recurrent,  $(\pi_j)_{j=0}^{\infty}$ ,  $\pi_j = \lim_{n \rightarrow \infty} P_{ij}^{(n)}$  is called stationary distribution, uniquely determined by

$$\pi_j = \sum_{i=0}^{\infty} \pi_i P_{ij} \quad \forall j, \quad \sum_{i=0}^{\infty} \pi_i = 1, \quad \pi_i > 0$$

## Long run behavior of continuous time MC (2)

Let  $(X_t)_{t \geq 0}$  be a continuous time MC,  $X_t \in \{0, 1, \dots\}$   
and let  $(Y_n)_{n \geq 0}$  be the embedded jump chain.

Define  $R_i = \min \{t > S_0 : X_t = i\}$ ,

$m_i = E(R_i \mid X_0 = i)$  — mean return time from  $i$  to  $i$

If  $m_i < \infty$ , then  $i$  is positive recurrent (class property).

Thm 1) If  $(X_t)_{t \geq 0}$  is irreducible, then

$$\lim_{t \rightarrow \infty} P_{ij}(t) =$$

2)  $(X_t)_{t \geq 0}$  is positive recurrent iff there exists a (unique)  
solution  $(\pi'_j)_{j=0}^{\infty}$  to

in which case  $\pi_j = \pi'_j$  and  $(\pi_j)_{j=0}^{\infty}$  is called  
limiting/stationary distribution.



## Remarks

- 1) Until now we discussed only the transition probabilities. But in order to describe completely MC  $(X_t)$  we need also the initial / starting distribution

$$\nu = (\nu_0, \nu_1, \dots), \quad \nu_i = P(X_0 = i)$$

$$(X_t) \longleftrightarrow (\nu, Q)$$

- 2) Distribution of  $X_{t_1}$  is given by  $\nu P(t_1)$

$$P(X_{t_1} = i) =$$

More generally

$$P(X_0 = i_0, X_{t_1} = i_1, \dots, X_{t_n} = i_n) =$$

- 3) Stationary distribution remains unchanged in time

$$\pi P(t) = \pi \Rightarrow$$

## Remarks

4) Similarly as in the discrete case,  $\pi_j$  gives the fraction of time spent in state  $j$  in long run

(compare with  $\lim_{m \rightarrow \infty} E \left[ \frac{1}{m} \sum_{n=0}^{m-1} \mathbb{1}_{\{X_n=j\}} \mid X_0=i \right] = \pi_j$  for discrete time MC)

5) If we can find  $(\pi_i)_{i=0}^{\infty}$  such that

then  $(\pi_i)_{i=0}^{\infty}$  satisfies  $\pi Q = 0$

Indeed,  $\sum_{j=0}^{\infty} \pi_i q_{ij} = \pi_i \sum_{j=0}^{\infty} q_{ij} = 0 = \sum_{j=0}^{\infty} \pi_j q_{ji} = (\pi Q)_i$

## Example: Birth and death processes

If we consider the birth and death process, the equation  $\pi Q = 0$  takes the following form

$$\text{where } \theta_i = \frac{\lambda_{i-1}}{\mu_i} \cdot \frac{\lambda_{i-2}}{\mu_{i-1}} \cdots \frac{\lambda_0}{\mu_1}, \quad \theta_0 = 1.$$

Then,  $\sum_{i=0}^{\infty} \pi_i = 1$  implies that

If  $\sum_{i=0}^{\infty} \theta_i < \infty$ , then  $(X_t)$  is positive recurrent and  $\pi_j =$

If  $\sum_{i=0}^{\infty} \theta_i = \infty$ , then  $\pi_j = 0 \quad \forall j$ .

## Example. Linear growth with immigration

Birth and death process,  $\lambda_j = \lambda_j + a$ ,  $\mu_j = \mu_j$  (\*)

Using Kolmogorov's equations we showed (lecture 5) that  $E(X_t) \rightarrow \frac{a}{\mu - \lambda}$ ,  $t \rightarrow \infty$ , if  $\mu > \lambda$ .

What is the limiting distribution of  $X_t$ ?

From the previous slide,  $\pi_j = \frac{\theta_j}{\sum_{i=0}^{\infty} \theta_i}$ ,  $\theta_j = \frac{\lambda_{j-1} \cdots \lambda_0}{\mu_j \cdots \mu_1}$

If we replace  $\lambda_j, \mu_j$  by (\*), we get

$$\pi_j = \left(\frac{\lambda}{\mu}\right)^j \left(1 - \frac{\lambda}{\mu}\right)^{\frac{a}{\lambda}} \frac{\frac{a}{\lambda} \left(\frac{a}{\lambda} + 1\right) \cdots \left(\frac{a}{\lambda} + j - 1\right)}{j!}, \quad j > 1$$
$$\pi_0 = \left(1 - \frac{\lambda}{\mu}\right)^{-\frac{a}{\lambda}}$$

## What you should know for midterm 1 (minimum):

- definition of continuous time MC, Markov property, transition probabilities, generator
- representations of MC: infinitesimal (generator), jump-and-hold, transition probabilities, rate diagram and relations between them (in particular  $Q$  and  $P(t)$ )
- computing absorption probabilities and mean time to absorption
- computing stationary distributions for finite and infinite state MCs and interpretation of  $(\pi_i)_{i=0}^{\infty}$
- basic properties of birth and death processes