

MATH 142A: Introduction to Analysis

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Today: Subsequential limits
> Q&A: February 1

Next: Ross § 14

Week 5:

- Homework 4 (due Sunday, February 7)
- Quiz 3 (Wednesday, February 3) - Lectures 8-9
- Midterm 1 regardes (Monday, February 1 - Tuesday, February 2)
- Homework 2 regardes (Monday, February 1 - Tuesday, February 2)

Subsequential limits

Def 11.1 Let (s_n) be a sequence of real numbers and let

$1 \leq n_1 < n_2 < \dots < n_k < \dots$ be an increasing sequence of natural numbers.

Then $(s_{n_k})_{k=1}^{\infty} = (s_{n_1}, s_{n_2}, s_{n_3}, \dots)$ is called a **subsequence** of $(s_n)_{n=1}^{\infty}$.

$$\left(\underset{1}{\textcolor{yellow}{1}}, \underset{2}{\textcolor{yellow}{\frac{1}{2}}}, \underset{3}{\textcolor{yellow}{\frac{1}{3}}}, \underset{4}{\frac{1}{4}}, \underset{5}{\textcolor{yellow}{\frac{1}{5}}}, \underset{6}{\frac{1}{6}}, \underset{7}{\textcolor{yellow}{\frac{1}{7}}}, \underset{8}{\frac{1}{8}}, \underset{9}{\frac{1}{9}}, \underset{10}{\frac{1}{10}}, \underset{11}{\textcolor{yellow}{\frac{1}{11}}}, \underset{12}{\frac{1}{12}}, \underset{13}{\textcolor{yellow}{\frac{1}{13}}}, \underset{14}{\frac{1}{14}}, \dots \right)$$

$$\left(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{11}, \frac{1}{13}, \dots \right)$$

Def 11.6 Let (s_n) be a sequence in \mathbb{R} . A subsequential limit is any real number or symbol $+\infty$ or $-\infty$ that is the limit of some subsequence of (s_n) .

Example • $a_n = (-1)^n$, $(-1, 1, -1, 1, \dots)$ $\lim_{k \rightarrow \infty} a_{2k-1} = -1$, $\lim_{k \rightarrow \infty} a_{2k} = 1$

Example • $b_n = 2^{n+1}$, $\left(\frac{1}{2}, 2^2, \frac{1}{2^3}, 2^4, \dots\right)$ $\lim_{k \rightarrow \infty} b_{2k-1} = \lim_{k \rightarrow \infty} \frac{1}{2^{2k-1}} = 0$

$\lim_{k \rightarrow \infty} b_{2k} = \lim_{k \rightarrow \infty} 2^{2k} = +\infty$

Subsequential limits and liminf / limsup

Thm 11.7 Let (s_n) be a sequence. Then there exist

- (i) a monotonic subsequence of (s_n) that converges to $\limsup s_n$
- (ii) a monotonic subsequence of (s_n) that converges to $\liminf s_n$

Proof. If (s_n) is not bounded above, then $\limsup s_n = +\infty$. And by

Thm 11.2(ii) there exist a subsequence of (s_n) that diverges to $+\infty$.

Suppose (s_n) is bounded above, $\limsup_{n \rightarrow \infty} s_n = t \in \mathbb{R}$

By Thm 11.2(i) there exists a monotonic subsequence of (s_n) that converges to t iff $\forall \varepsilon > 0$ the set $\{n : |s_n - t| < \varepsilon\}$ is infinite. Fix $\varepsilon > 0$

$$\limsup_{n \rightarrow \infty} s_n = t \Rightarrow \exists N \quad \forall k > N \quad \sup\{s_n : n > k\} < t + \varepsilon \Rightarrow \forall n > N+1 \quad (s_n < t + \varepsilon)$$

Suppose that $\{n : t - \varepsilon < s_n < t + \varepsilon\}$ is finite. Then $\exists N_1 > N$ s.t.

$$\forall n > N_1 \quad (s_n \leq t - \varepsilon) \Rightarrow \sup\{s_n : n > N_1\} \leq t - \varepsilon \Rightarrow \limsup_{n \rightarrow \infty} s_n \leq t - \varepsilon$$

contradiction

Subsequential limits and convergence

Thm 11.8. Let (s_n) be a sequence. Denote by S the set of all subsequential limits of (s_n) . Then

- (i) S is nonempty (follows Thm 11.7)
- (ii) $\sup S = \limsup s_n, \inf S = \liminf s_n$
- (iii) $\lim s_n$ exists iff S has only one element, $S = \{\lim s_n\}$

Proof. (iii) follows from (ii) and Thm 10.7 ($\lim s_n = t \Leftrightarrow \limsup s_n = \liminf s_n = t$)

(ii) Suppose $t \in S \Leftrightarrow \exists$ subsequence (s_{n_r}) s.t. $\lim_{r \rightarrow \infty} s_{n_r} = t$

Then by Thm 10.7 $\liminf_{r \rightarrow \infty} s_{n_r} = \limsup_{r \rightarrow \infty} s_{n_r} = t$

Note that $\forall k (n_r \geq k)$, therefore $\forall N \{s_n : n > N\} \supset \{s_{n_r} : k > N\}$

and $\forall N \inf \{s_n : n > N\} \leq \inf \{s_{n_r} : k > N\} \leq \sup \{s_{n_r} : k > N\} \leq \sup \{s_n : n > N\}$

$$\begin{array}{ccccccc} N \rightarrow \infty & \downarrow & 9.12 & \downarrow & \downarrow & 9.12 & \downarrow \\ \text{liminf } s_n & \in S & \leq & \text{liminf } s_{n_r} = t = \limsup s_{n_r} & \leq & \limsup s_n \\ & & & & & & \text{Thm 11.7} \end{array}$$

Examples

For each sequence below let S denote the set of subsequential limits.

- $a_n = (-1)^n$,

① $S = \{-1, 1\}$

$$\lim_{k \rightarrow \infty} a_{2k-1} = -1, \quad \lim_{k \rightarrow \infty} a_{2k} = 1 \Rightarrow \{-1, 1\} \subset S$$

If $t \notin \{-1, 1\}$, then $\forall n \in \mathbb{N} \quad |s_n - t| \geq \min\{|t-1|, |t-(-1)|\} > 0 = \epsilon \in S$
 $\Rightarrow S \subset \{-1, 1\}$

② $\limsup a_n = 1 \quad \liminf a_n = -1$

- $b_n = 2^{n(-1)^n}$

① $S = \{0, +\infty\}$

$$\lim_{k \rightarrow \infty} b_{2k-1} = 0, \quad \lim_{k \rightarrow \infty} b_{2k} = +\infty \Rightarrow \{0, +\infty\} \subset S$$

If $t \in \mathbb{R}, t \neq 0$, then $\exists N_1 \quad \forall k > N_1 \quad b_{2k-1} < \frac{t}{2}$ $\left. \exists N_2 \quad \forall k > N_2 \quad b_{2k} > t+1 \right| \Rightarrow |b_n - t| > \min\{\frac{t}{2}, 1\} > 0$

② $(\limsup b_n = +\infty, \liminf b_n = 0)$

$$S \subset \{0, +\infty\}$$

The set of subsequential limits is closed

Thm 11.9 Let (s_n) be a sequence. Denote by S the set of all subsequential limits of (s_n) . Then

Let (t_n) be a sequence in $S \cap \mathbb{R}$, i.e. $\forall n \in \mathbb{N} : t_n \in S$.

If (t_n) has a limit, then $\lim_{n \rightarrow \infty} t_n \in S$.

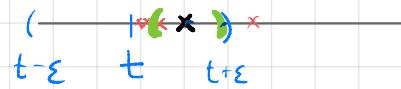
Proof. Suppose $\lim_{n \rightarrow \infty} t_n = t \in \mathbb{R}$. Then

$$x_n = \frac{1}{2^n}, \forall n \in \mathbb{N}, \lim x_n = 0 \quad 0 \notin (0,1)$$

$t \in S \Leftrightarrow \exists$ subsequence of (s_n) that converges to $t \Leftrightarrow \forall \varepsilon > 0 \{ n \in \mathbb{N} : |s_n - t| < \varepsilon \}$ infinite

Fix $\varepsilon > 0$. Then $\exists n_0$ s.t. $t - \varepsilon < t_{n_0} < t + \varepsilon$

Since $t + \varepsilon - t_{n_0} > 0$, $t_{n_0} - (t - \varepsilon) > 0$



$\delta := \min\{t + \varepsilon - t_{n_0}, t_{n_0} - (t - \varepsilon)\} > 0$ and $(t_{n_0} - \delta, t_{n_0} + \delta) \subset (t - \varepsilon, t + \varepsilon)$

$t_{n_0} \in S$ (subsequential limit) $\stackrel{\text{Thm 11.2}}{\Rightarrow} \{n : |s_n - t_{n_0}| < \delta\}$ is infinite $\Rightarrow \{n : |s_n - t| < \varepsilon\}$ is infinite ■

limsup's and liminf's

Thm 12.1 Let (s_n) and (t_n) be two sequences. Then

$$\left((s_n) \text{ converges} \wedge \lim_{n \rightarrow \infty} s_n = s > 0 \right) \Rightarrow \limsup(s_n \cdot t_n) = s \cdot \limsup t_n$$

Convention: For any $s \in \mathbb{R}$, $s > 0$, $s \cdot (+\infty) = +\infty$, $s \cdot (-\infty) = -\infty$.

Proof

①: $\limsup(s_n t_n) \geq s \cdot t$ (only for $\limsup t_n = t \in \mathbb{R}$)

$$\text{Thm. 11.7} \Rightarrow \exists (t_{n_k}) \text{ such that } \lim_{k \rightarrow \infty} t_{n_k} = t. \quad \begin{array}{l} \text{Thm 9.4} \\ \Rightarrow \end{array}$$

$$\text{Thm 11.3} \Rightarrow \lim_{k \rightarrow \infty} s_{n_k} = s.$$

$$\Rightarrow s \cdot t \text{ is a subsequential limit for } (s_n t_n) \Rightarrow \limsup(s_n t_n) \geq s \cdot t \quad \text{Thm 11.8}$$

②: $\limsup(s_n t_n) \leq s \cdot t$ (only for $s_n > 0 \ \forall n$). Thm 9.5 $\Rightarrow \lim_{n \rightarrow \infty} \frac{1}{s_n} = \frac{1}{s} > 0$

$$\text{Then } t = \limsup t_n = \limsup \frac{1}{s_n} (s_n \cdot t_n) \stackrel{\textcircled{1}}{\geq} \frac{1}{s} \limsup(s_n \cdot t_n) \Rightarrow$$

$$\stackrel{\textcircled{1} \textcircled{2}}{\Rightarrow} \limsup(s_n t_n) = \limsup(t_n) \cdot s \quad st \geq \limsup(s_n t_n)$$

Remark

If (s_n) and (t_n) are two sequences, and $\lim_{n \rightarrow \infty} s_n = 0$, then there is nothing we can say in general about $\limsup(s_n t_n)$.

- $s_n = \frac{1}{n}$, $t_n = n \Rightarrow \limsup \frac{1}{n} \cdot n = 1$
- $s_n = \frac{1}{n^2}$, $t_n = n \Rightarrow \limsup \frac{1}{n^2} \cdot n = 0$
- $s_n = \frac{1}{n}$, $t_n = n^2 \Rightarrow \limsup \frac{1}{n} \cdot n^2 = +\infty$

Also it is important that one sequence converges.

- $s_n = (0, 1, 0, 1, 0, 1, \dots)$ $\limsup s_n = 1$ but $\forall n s_n \cdot t_n = 0$
- $t_n = (1, 0, 1, 0, 1, 0, \dots)$ $\limsup t_n = 1$
- $s_n = (-1)^n$, $t_n = (-1)^{n+1}$ $\limsup s_n = \limsup t_n = 1$, but $\forall n s_n \cdot t_n = (-1)^{2n+1} = -1$