MATH 180C HOMEWORK 9. SOLUTIONS

FALL 2020

1. *Pinsky and Karlin, Exercise 8.2.3.* Suppose that net inflows to a reservoir are described by a standard Brownian motion. If at time 0, the reservoir has $x = 3.29$ units of water, what is the probability that the reservoir never becomes empty in the first $t = 4$ units of time?

Solution. Let X_t denote the amount of water in the reservoir at time t. We have to compute

$$P(\min_{0\le t\le 4} X_t > 0). \tag{1}$$

Let $(B_t)_{t\ge 0}$ be a standard Brownian motion starting from 0 such that $X_t = 3.29 + B_t$. Then

$$P(\min_{0\le t\le 4} X_t > 0) = P(\min_{0\le t\le 4}(3.29 + B_t) > 0) = P(\min_{0\le t\le 4} B_t > -3.29). \tag{2}$$

Using the reflection symmetry of the Brownian motion at zero (lecture 20, page 4),

$$P(\min_{0\le t\le 4} B_t > -3.29) = P(\max_{0\le t\le 4} B_t < 3.29). \tag{3}$$

Finally, we can compute the last quantity using the reflection principle (lecture 21, page 6)

$$P(\max_{0\le t\le 4} B_t < 3.29) = P(|B_4| < 3.29) = P\Big(|B_1| < \frac{3.29}{2}\Big) \approx 0.9. \tag{4}$$

2. *Pinsky and Karlin, Exercise 8.2.5.* Let τ_0 be the largest zero of a standard Brownian motion not exceeding $a > 0$. That is $\tau_0 = \max\{u \ge 0;\ B(u) = 0 \text{ and } u \le a\}$. Show that

$$P(\tau_0 < t) = \frac{2}{\pi}\arcsin\sqrt{t/a}. \tag{5}$$

Solution. Firstly, note that for any $t < a$

$$P(\tau_0 < t) = P\big(\forall\, u \in (t, a]\,, B(u) \ne 0\big) = 1 - \theta(t, a), \tag{6}$$

where $\theta(t, a)$ is the probability that there exists a standard Brownian motion has zero on the interval $(t, a]$ (see lecture 21, page 9). From the same lecture we know that

$$\theta(t, a) = \frac{2}{\pi}\arccos\sqrt{t/a}. \tag{7}$$

We conclude that

$$P(\tau_0 < t) = 1 - \theta(t, a) = \frac{2}{\pi}\Big(\frac{\pi}{2} - \arccos\sqrt{t/a}\Big) = \frac{2}{\pi}\arcsin\sqrt{t/a}. \tag{8}$$

3. *Pinsky and Karlin, Exericse 8.3.3.* The net inflow to a reservoir is well described by a Brownian motion. Because a reservoir cannot contain a negative amount of water, we

suppose that the water level $R(t)$ at time t is a reflected Brownian motion. What is the probability that the reservoir contains more than 10 units of water at time $t = 25$? Assume that the reservoir has unlimited capacity and that $R(0) = 5$.

Solution. Let $(B_t)_{t\geq 0}$ be a standard Brownian motion such that $R(t)$ is given by

$$R(t) = |5 + B_t|, \tag{9}$$

i.e., the amount of water is modeled by a Brownian motion starting from $R(0) = 5$ and reflected at zero (taking absolute value). Then

$$P(R(25) > 10) = P(5 + B_{25} < -10) + P(5 + B_{25} > 10) \tag{10}$$
$$= P(B_{25} < -15) + P(B_{25} > 5) \tag{11}$$
$$= P(B_1 < -3) + P(B_1 > 1) \tag{12}$$
$$\approx 0.16. \tag{13}$$

4. *Pinsky and Karlin, Problem 8.3.2* Let B_t be a standard Brownian motion process. Determine the conditional mean and variance of B_t, $0 < t < 1$, given that $B_1 = b$.

Solution. Firstly, note that (B_t, B_1) is a Gaussian vector with zero mean and covariance matrix $\begin{pmatrix} t & t \\ t & 1 \end{pmatrix}$ and the corresponding joint density. Next, the marginal density of B_1 at point b is given by

$$f_{B_1}(b) = \frac{1}{\sqrt{2\pi}} e^{-\frac{b^2}{2}}. \tag{14}$$

Compute the conditional density of B_t given $B_1 = b$

$$f_{B_t\,|\,B_1}(x\,|\,b) = \frac{\sqrt{2\pi}}{2\pi\sqrt{t(1-t)}} e^{-\frac{1}{2t(1-t)}(x^2 - 2tbx + tb^2) + \frac{b^2}{2}} \tag{15}$$
$$= \frac{1}{\sqrt{2\pi t(1-t)}} e^{-\frac{1}{2t(1-t)}(x-tb)^2}. \tag{16}$$

We conclude, that given $B_1 = b$, B_t has normal distribution with mean bt and variance $t(1-t)$.

5. *Pinsky and Karlin, Exercise 8.4.2.* A Brownian motion $(X_t)_{t\geq 0}$ has parameters $\mu = 0.1$ and $\sigma = 2$. Evaluate the probability of exiting the interval $(a, b]$ at the point b starting from $X_0 = 0$ for $b = 1$, 10 and 100 and $a = -b$. Why do the probabilities change when a/b is the same in all cases?

Solution. Denote by $u_0^{(x)}$ the probability that the process X exits the interval $(-x, x]$ at point x. Compute

$$\frac{2\mu}{\sigma^2} = \frac{2 \cdot 0.1}{4} = 0.05. \tag{17}$$

Using the formula for the gambler's ruin probability for the Brownian motion with drift (lecture 22-23, page 9), we have that

$$u_0^{(1)} = \frac{1 - e^{0.05}}{e^{-0.05} - e^{0.05}} \approx 0.51. \tag{18}$$

Similarly,

$$u_0^{(10)} \approx 0.62, \quad u_0^{(100)} \approx 0.99. \tag{19}$$

Intuitive explanation: the larger is b, the longer it takes to reach either b or $-b$, the stronger is the influence of the drift.

6. *Pinsky and Karlin, Exercise 8.4.3.* A Brownian motion (X_t) has parameters $\mu = 0.1$ and $\sigma = 2$. Evaluate the mean time to exit the interval $(a, b]$ from $X_0 = 0$ for $b = 1$, 10 and 100 and $a = -b$. Can you guess how this mean time varies with b for b large?

Solution. Denote by $T^{(x)}$ the mean time to exit the interval $(-x, x)$. Similarly as in the previous problem, using the formula for the mean time in the gambler's ruin problem (lecture 22-23, page 9), we have that

$$T^{(1)} = \frac{1}{0.1}(u_0^{(1)}2 - 1) \approx 0.25, \tag{20}$$

$$T^{(10)} \approx 24.5, \quad T^{(100)} \approx 986. \tag{21}$$

Intuitive explanation: the larger is the value b, the longer it takes to reach either b or $-b$, and thus the stronger is the role of the deterministic drift (linear in t) compared to the random fluctuations (of order $\sqrt{t}$). So for $b \gg 1$, the mean time behaves as $\frac{b}{\mu} = 10b$.