

MATH 142A: Introduction to Analysis

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Today: Limit theorems for sequences
> Q&A: January 19

Next: Ross § 10

Week 3:

- homework 2 (due Friday, January 21)

Inequalities

• Cauchy-Schwarz-Bunyakovsky inequality

Let $n \in \mathbb{N}$ and $\{a_1, \dots, a_n, b_1, \dots, b_n\} \subset \mathbb{R}$. Then

$$\left(\sum_{k=1}^n a_k b_k \right)^2 \leq \sum_{k=1}^n a_k^2 \cdot \sum_{k=1}^n b_k^2 \quad (*)$$

Proof: Denote $\sum_{k=1}^n a_k^2 = A$, $\sum_{k=1}^n b_k^2 = B$, $\sum_{k=1}^n a_k b_k = C$.

① $A = 0 \Rightarrow \forall k \ a_k = 0 \Rightarrow C = 0 \quad 0 \leq 0 \cdot B$, (*) holds

② $A > 0$. Consider the function $p(x) := Ax^2 + 2Cx + B$

$$p(x) = \sum_{k=1}^n a_k^2 x^2 + 2a_k b_k x + b_k^2 = \sum_{k=1}^n (a_k x + b_k)^2 \geq 0 \Rightarrow 4C^2 - 4AB \leq 0 \\ \Rightarrow AB \geq C^2$$

Exercise

$$\left(\sum_{k=1}^n a_k b_k \right)^2 = \sum_{k=1}^n a_k^2 \cdot \sum_{k=1}^n b_k^2$$

$$\Leftrightarrow \exists \lambda, \mu \in \mathbb{R}, |\lambda| + |\mu| \neq 0 \quad \forall k \quad \lambda a_k = \mu b_k.$$

Inequalities

AM-GM inequality:

Let $n \in \mathbb{N}$, $\{a_1, a_2, \dots, a_n\} \subset [0, +\infty)$. Then

$$G_n := \sqrt[n]{a_1 a_2 \cdots a_n} \leq \frac{a_1 + a_2 + \cdots + a_n}{n} =: A_n$$

Proof ① If $a_1 a_2 \cdots a_n = 0$, then $G_n = 0$, $A_n \geq 0$ ✓

② If $n=1$, then $a_1 \leq a_1$ ✓

③ Suppose $n > 1$ and $\forall k \quad a_k > 0$. Then $\forall k \quad A_k > 0$ and

$\forall k \in \{2, 3, \dots, n\} \quad \frac{A_k}{A_{k-1}} > 0$, which is equivalent to $\frac{A_k}{A_{k-1}} - 1 > -1 \quad \forall k \in \{2, \dots, n\}$

Using Bernoulli's inequality: $\left(\frac{A_k}{A_{k-1}}\right)^k \geq 1 + k \left(\frac{A_k}{A_{k-1}} - 1\right) = \frac{A_{k-1} + k A_k - k A_{k-1}}{A_{k-1}} = \frac{A_k}{A_{k-1}}$

$\Rightarrow A_k \geq a_k A_{k-1}^{k-1}$. Apply for $k = n, n-1, n-2, \dots, 2$

$$A_n^n \geq a_n A_{n-1}^{n-1} \geq a_n a_{n-1} A_{n-2}^{n-2} \geq \cdots \geq a_n a_{n-1} \cdots a_2 a_1$$

Inequalities

- Bernoulli's inequality (L1):

$$\forall a \geq -1 \quad \forall n \in \mathbb{N} \quad (1+a)^n \geq 1+na$$

- Triangle inequality (L2):

$$\forall a, b \in \mathbb{R} \quad |a+b| \leq |a| + |b|$$

- Cauchy - Bunyakovsky - Schwarz inequality

Let $n \in \mathbb{N}$ and $\{a_1, \dots, a_n, b_1, \dots, b_n\} \subset \mathbb{R}$. Then

$$\left(\sum_{k=1}^n a_k b_k \right)^2 \leq \sum_{k=1}^n a_k^2 \cdot \sum_{k=1}^n b_k^2$$

- AM-GM inequality

Let $n \in \mathbb{N}$, $\{a_1, a_2, \dots, a_n\} \subset [0, +\infty)$. Then

$$G_n := \sqrt[n]{a_1 a_2 \cdots a_n} \leq \frac{a_1 + a_2 + \cdots + a_n}{n} =: A_n$$

Limits and inequalities

Thm 9.11(i) Let (a_n) and (b_n) be two convergent sequences, $\lim_{n \rightarrow \infty} a_n = A$, $\lim_{n \rightarrow \infty} b_n = B$.

Then $A < B \Rightarrow \exists N \forall n > N \quad a_n < b_n$

(ii) Let $(a_n), (b_n), (c_n)$ be three sequences such that $\exists N_0 \forall n > N_0 \quad a_n \leq b_n \leq c_n$.

Suppose that (a_n) and (c_n) are convergent, $\lim_{n \rightarrow \infty} a_n = A$, $\lim_{n \rightarrow \infty} c_n = C$

Then $A = C \Rightarrow \lim_{n \rightarrow \infty} b_n = \lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} c_n$

Proof (i). Choose $A < K < B$ Then

$$\lim_{n \rightarrow \infty} a_n = A \Rightarrow \exists N_1 \in \mathbb{N} \quad \forall n > N_1 \quad |a_n - A| < K - A > 0 \quad \left| \Rightarrow \forall n > \max\{N_1, N_2\} \quad a_n < K - A + A = K - B + B < b_n \right.$$

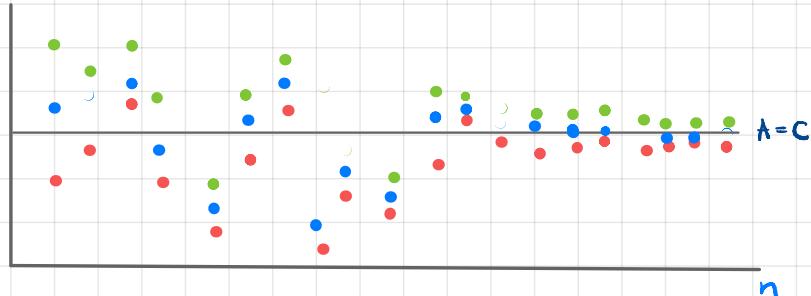
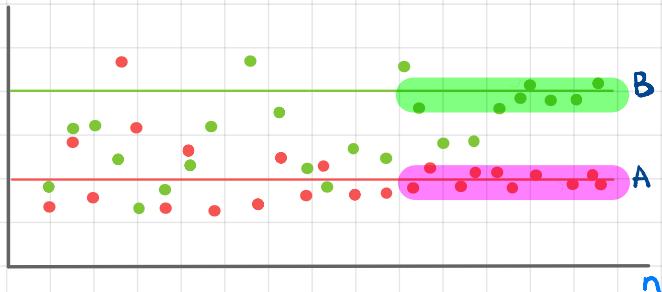
$$\lim_{n \rightarrow \infty} b_n = B \Rightarrow \exists N_2 \in \mathbb{N} \quad \forall n > N_2 \quad |b_n - B| < B - K > 0 \quad \left| \Rightarrow a_n < K - A + A = K - B + B < b_n \right.$$

(ii) Fix $\varepsilon > 0$.

$$\lim_{n \rightarrow \infty} a_n = A \Rightarrow \exists N_1 \quad \forall n > N_1 \quad |a_n - A| < \varepsilon \quad \left| \Rightarrow \forall n > N := \max\{N_0, N_1, N_2\} \quad |b_n - A| < \varepsilon \right.$$

$$\lim_{n \rightarrow \infty} c_n = C \Rightarrow \exists N_2 \quad \forall n > N_2 \quad |c_n - C| < \varepsilon \quad \left| \Rightarrow A - \varepsilon < a_n \leq b_n \leq c_n < C + \varepsilon \quad |b_n - A| < \varepsilon \right.$$

Limits and inequalities



Corollary 9.12 Suppose that $\lim_{n \rightarrow \infty} a_n = A$, $\lim_{n \rightarrow \infty} b_n = B$.

- (i) $\exists N \forall n > N a_n > b_n \Rightarrow A \geq B$ (e.g. $\forall n \frac{1}{n} > 0$, $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$)
- (ii) $\exists N \forall n > N a_n \geq b_n \Rightarrow A \geq B$
- (iii) $\exists N \forall n > N a_n > B \Rightarrow A \geq B$
- (iv) $\exists N \forall n > N a_n \geq B \Rightarrow A \geq B$

Proof: Exercise (for (i) and (ii) use proof by contradiction).

Divergence to $\pm\infty$

Last time: $\lim_{n \rightarrow \infty} \frac{5n^5 - n - 10}{7n^4 - n^2} = \lim_{n \rightarrow \infty} n \frac{\frac{5}{n^4} - \frac{1}{n^3} - \frac{10}{n^5}}{\frac{7}{n^4} - \frac{1}{n^2}} = ?$

Def 9.8. Let (s_n) be a sequence. We say that (s_n) diverges to $+\infty$ ($-\infty$)

$$\lim_{n \rightarrow \infty} s_n = +\infty \text{ if } \forall M > 0 \ \exists N \in \mathbb{N} \ \forall n > N \ s_n > M$$

$$\lim_{n \rightarrow \infty} s_n = -\infty \text{ if } \forall M < 0 \ \exists N \in \mathbb{N} \ \forall n > N \ s_n < M$$

We say that (s_n) has a limit, if it converges, or diverges to $+\infty$ or $-\infty$.

Example

$$\lim_{n \rightarrow \infty} \frac{5n^5 - n - 10}{7n^4 - n^2} = +\infty$$

Proof.

Fix $M > 0$. If $n > 1000$, then $\frac{5n^5 - n - 10}{7n^4 - n^2} > \frac{4n^5}{8n^4} = \frac{n}{2} > M$

Take $N = \max\{1000, [2M]\}$. Then $\forall n > N$

$$\frac{5n^5 - n - 10}{7n^4 - n^2} > \frac{n}{2} > M$$

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Divergence to $\pm\infty$ and arithmetic operations

Thm 9.12 Let (s_n) be a sequence

$$(i) \lim_{n \rightarrow \infty} s_n = +\infty, k > 0 \Rightarrow \lim_{n \rightarrow \infty} (k \cdot s_n) = +\infty$$

$$(ii) \lim_{n \rightarrow \infty} s_n = +\infty \Leftrightarrow \lim_{n \rightarrow \infty} (-s_n) = -\infty$$

$$(iii) \lim_{n \rightarrow \infty} s_n = +\infty, k < 0 \Rightarrow \lim_{n \rightarrow \infty} (k \cdot s_n) = -\infty$$

Proof: Exercise

Thm 9.13 Let (s_n) and (t_n) be two sequences.

If $\lim_{n \rightarrow \infty} s_n = +\infty$ and $\inf\{t_n : n \in \mathbb{N}\} > -\infty$, then $\lim_{n \rightarrow \infty} (s_n + t_n) = +\infty$

Proof. Fix $M > 0$ and denote $m = \inf\{t_n : n \in \mathbb{N}\}$.

$\lim_{n \rightarrow \infty} s_n = +\infty \Rightarrow \exists N \forall n > N \ s_n > M-m$. Then $\forall n > N \ s_n + t_n > s_n + m > M-m+m = M$

Examples

- $\lim_{n \rightarrow \infty} \left(n + \frac{1}{n}\right) = +\infty$
- $\lim_{n \rightarrow \infty} \left(n^2 - n\right) = +\infty$
- $\lim_{n \rightarrow \infty} \left(n + \frac{1}{n}\right)^2 - n^2 = 2$
- $\lim_{n \rightarrow \infty} \left(n - n^2\right) = -\infty$

Divergence to ∞ and arithmetic operations

Thm 9.9 Let (s_n) and (t_n) be sequences such that

$$\lim_{n \rightarrow \infty} s_n = +\infty \text{ and } \left(\lim_{n \rightarrow \infty} t_n = t > 0 \text{ or } \lim_{n \rightarrow \infty} t_n = +\infty \right)$$

Then $\lim_{n \rightarrow \infty} (s_n t_n) = +\infty$

Proof (For $\lim_{n \rightarrow \infty} t_n = t > 0$) Fix $M > 0$. By Thm 9.11 $\exists N_1 \forall n > N_1, t_n > \frac{t}{2} > 0$

$\lim_{n \rightarrow \infty} s_n = +\infty \Rightarrow \exists N_2 \forall n > N_2 \quad s_n > M \cdot \frac{2}{t}$. Then

$$\forall n > \max\{N_1, N_2\} \quad s_n t_n > s_n \cdot \frac{t}{2} > M \cdot \frac{2}{t} \cdot \frac{t}{2} = M \quad \blacksquare$$

Thm 9.10 Let (s_n) be a sequence such that $\forall n \quad s_n > 0$. Then

$$\lim_{n \rightarrow \infty} s_n = +\infty \Leftrightarrow \lim_{n \rightarrow \infty} \left(\frac{1}{s_n} \right) = 0$$

Proof. (\Rightarrow) Suppose $\lim_{n \rightarrow \infty} s_n = +\infty$. Fix $\varepsilon > 0$. $\exists N \forall n > N \quad \frac{1}{s_n} < \varepsilon \quad (\Rightarrow s_n > \frac{1}{\varepsilon})$

$\lim_{n \rightarrow \infty} s_n = +\infty \Rightarrow \exists N \forall n > N \quad s_n > \frac{1}{\varepsilon}$. Then

$$\forall n > N \quad \left| \frac{1}{s_n} - 0 \right| = \frac{1}{s_n} < \varepsilon$$

(\Leftarrow) Exercise.

Important examples

1. If $q \in \mathbb{N}$, then $\lim_{n \rightarrow \infty} \frac{1}{n^q} = 0$ (L4)

2. If $|a| < 1$, then $\lim_{n \rightarrow \infty} a^n = 0$

Proof. ① If $a = 0$, then $a^n = 0$, $\lim_{n \rightarrow \infty} 0 = 0$

② Let $a \neq 0$. Fix $\varepsilon > 0$. [$\exists N \forall n > N |a^n - 0| = |a^n| = |a|^n < \varepsilon$]

Denote $b = \frac{1}{|a|} - 1 > 0$ (so that $|a| = \frac{1}{1+b}$)

By Bernoulli's inequality

$$\left(\frac{1}{|a|}\right)^n = (1+b)^n \geq 1+nb > nb$$

and thus $|a^n - 0| = |a^n| < \frac{1}{nb}$ [enough $\frac{1}{nb} < \varepsilon \Leftrightarrow n > \frac{1}{\varepsilon b}$]

Take $N = \lceil \frac{1}{\varepsilon b} \rceil$. Then $\forall n > N |a^n - 0| < \frac{1}{nb} < \varepsilon$ ■

Important examples

$$3. \lim_{n \rightarrow \infty} \sqrt[n]{n} = 1$$

Proof. ① $\forall n \sqrt[n]{n} \geq 1$

② Write $n = \sqrt{n} \cdot \sqrt{n} \cdot \underbrace{1 \cdot 1 \cdots 1}_{n-2}$. Then by AM-GM inequality

$$\sqrt[n]{n} = \sqrt[n]{\sqrt{n} \cdot \sqrt{n} \cdot \underbrace{1 \cdots 1}_{n-2}} \leq \frac{\sqrt{n} + \sqrt{n} + \underbrace{1 + \cdots + 1}_{n-2}}{n} = \frac{2\sqrt{n} + n - 2}{n} = 1 + \frac{2}{n} - \frac{2}{n}$$

③ $\lim_{n \rightarrow \infty} \sqrt[n]{n} = +\infty$ [By AP $\forall M > 0 \exists N \forall n > N n > M^2 \Rightarrow \sqrt[n]{n} > M$]

④ By Thm 9.10, $\lim_{n \rightarrow \infty} \frac{1}{\sqrt[n]{n}} = 0 \Rightarrow \lim_{n \rightarrow \infty} \left(1 + \frac{2}{\sqrt[n]{n}} - \frac{2}{n} \right) = 1$

⑤ $1 \leq \sqrt[n]{n} \leq 1 + \frac{2}{\sqrt[n]{n}} - \frac{2}{n}$, by Thm 9.11 and ④ $\lim_{n \rightarrow \infty} \sqrt[n]{n} = 1$.

Important examples

$$4. \forall a > 0 \quad \lim_{n \rightarrow \infty} \sqrt[n]{a} = 1$$

Proof. If $a=1$, then $\lim_{n \rightarrow \infty} 1 = 1$

$$\text{If } a > 1, \text{ then } \textcircled{1} \quad \forall n \quad \sqrt[n]{a} \geq 1$$

$$\textcircled{2} \quad \forall n > a \quad \sqrt[n]{a} \leq \sqrt[n]{n}$$

$$\textcircled{3} \quad \text{By Thm 9.11} \quad \lim_{n \rightarrow \infty} \sqrt[n]{a} = 1$$

If $a < 1$, denote $b = \frac{1}{a} > 1$. Then

$$\textcircled{1} \quad \sqrt[n]{a} = \sqrt[n]{\frac{1}{b}} \quad \left| \begin{array}{l} \text{Thm 9.5} \\ \Rightarrow \end{array} \right.$$

$$\textcircled{2} \quad \lim_{n \rightarrow \infty} \sqrt[n]{b} = 1$$

$$\lim_{n \rightarrow \infty} \sqrt[n]{a} = \lim_{n \rightarrow \infty} \sqrt[n]{\frac{1}{b}} = \frac{1}{\lim_{n \rightarrow \infty} \sqrt[n]{b}} = \frac{1}{1} = 1$$

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