MATH180C: Introduction to Stochastic Processes II

www.math.ucsd.edu/~ynemish/teaching/180c

Today: Renewal processes. Examples

> Q&A: November 6

Next: PK 7.4-7.5, Durrett 3.1

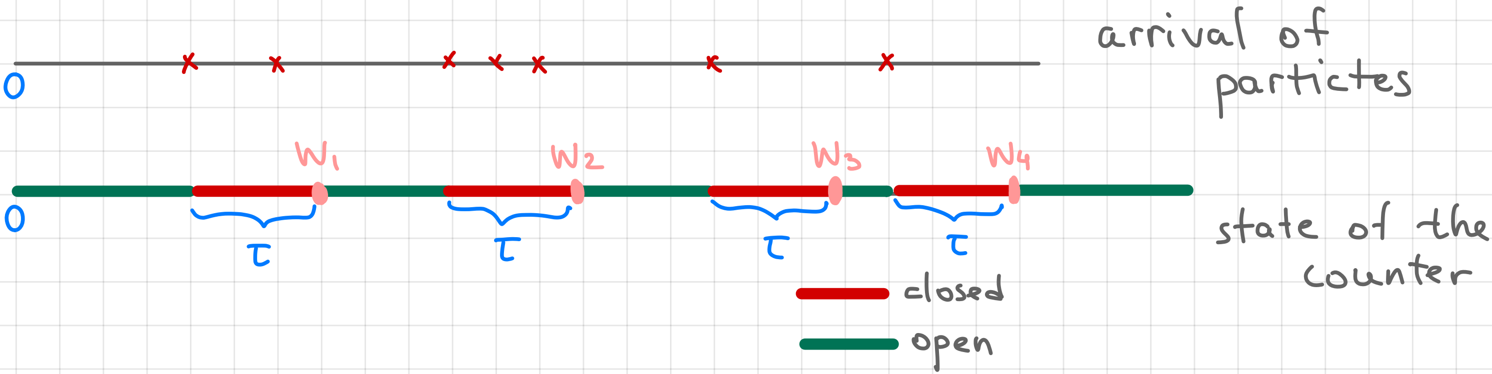
This week:

- Homework 4 (due Friday, November 6, 11:59 PM)

Other renewal processes

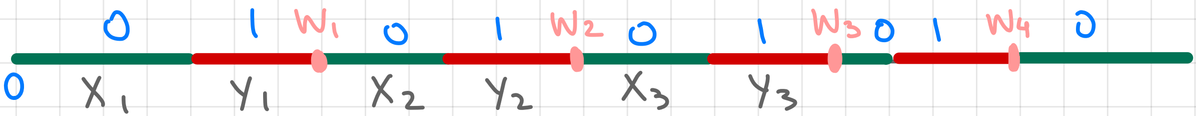
- traffic flow : distances between successive cars are assumed to be i.i.d. random variables

- counter process: particles / signals arrive on a device and lock it for time τ ; particles arrive according to a PP; times at which the counter unlocks form a renewal process



Other renewal processes

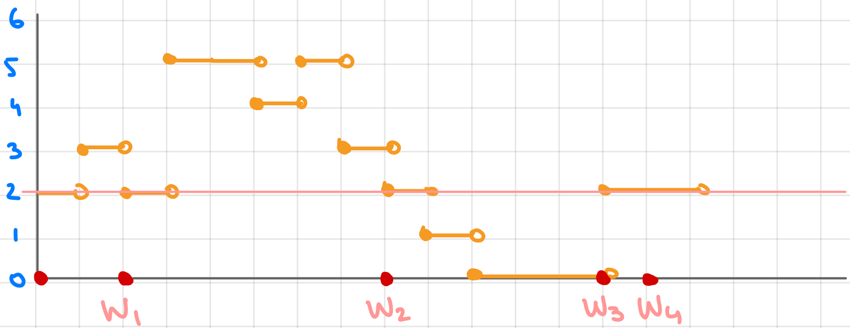
- more generally, if a component has two states (0/1, operating/non-operating etc), switches between them, times spent in 0 are X_i, times spent in 1 are Y_i, $(X_i)_{i=1}^{\infty}$ i.i.d., $(Y_i)_{i=1}^{\infty}$ i.i.d., then the times of switching from 0 to 1 form a renewal process with interrenewal times $X_i + Y_i$



Other renewal processes

- Markov chains: if $(Y_n)_{n\geq 0}$, $Y_n \in \{0,1,\dots\}$ is a recurrent MC starting from $Y_0 = k$, then the times of returns to state k form a renewal process. More precisely define

$$W_1 = \min\{n>0 : Y_n = k\}$$
$$W_p = \min\{n > W_{p-1} : Y_n = k\}$$



Example with $k=2$

Similarly for continuous time MCs.

Strong Markov property!

Other renewal processes

- Queues. Consider a single-server queueing process

(i) if customer arrival times form a renewal process then the times of the starts of successive idle periods generate a second renewal time

(ii) if customes arrive according to a Poisson process, then the times when the server passes from busy to free form a renewal process

Asymptotic behavior

Asymptotic behavior of renewal processes

Let $N(t)$ be a renewal process with interrenewal times X_i, $X_i \in (0,\infty)$.

Thm.

$$P\left(\lim_{t\to\infty} N(t) = \infty\right) = 1$$

Proof. $N(t)$ is nondecreasing, therefore $\exists \lim_{t\to\infty} N(t) =: N_\infty$

N_∞ is the total number of events ever happened.

$N_\infty \leq k$ if and only if $W_{k+1} = \infty$

if and only if $X_i = \infty$ for some $i \leq k+1$

$$P(N_\infty < \infty) = P(X_i = \infty \text{ for some } i) \leq \sum_{i=1}^{\infty} P(X_i = \overset{=0}{\infty}) = 0$$ ∎

Thm (Pointwise renewal thm).

$$P\left(\lim_{t\to\infty} \frac{N(t)}{t} = \frac{1}{\mu}\right) = 1 \qquad (\mu = E(X_1))$$

Elementary Renewal Theorem

Thm. If $M(t) = E(N(t))$ and $E(X_1) = \mu$, then

$$\lim_{t \to \infty} \frac{M(t)}{t} = \frac{1}{\mu}$$

Proof. (Only for bounded X_i: $\exists K$ s.t. $P(X_1 \le K) = 1$)

First note that $W_{N(t)+1} = t + \gamma_t$

In lecture 11 we showed that $E(W_{N(t)+1}) = \mu(M(t)+1)$

so
$$M(t) = \frac{t + E(\gamma_t)}{\mu} - 1$$

$$\frac{M(t)}{t} = \frac{1}{\mu} + \frac{1}{t}\left(\frac{E(\gamma_t)}{\mu} - 1\right) \longrightarrow \frac{1}{\mu} \text{ as } t \to \infty$$

If $X_i \le K$, then $\gamma_t \le K \Rightarrow E(\gamma_t) \le K$

Exercise: $(X_n)_{n \ge 0}$: 1) $P(\lim_{n\to\infty} X_n = 0) = 1$ 2) $\lim_{n \to \infty} E(X_n) \ge c > 0$.

Asymptotic distribution of $N(t)$

Thm. Let $N(t)$ be a renewal process with $E(X_1) = \mu$ and $Var(X_1) = \sigma^2$, then

1) $$\lim_{t\to\infty} \frac{Var(N(t))}{t} = \frac{\sigma^2}{\mu^3}$$

2) $$\lim_{t\to\infty} P\left(\frac{N(t) - E(N(t))}{\sqrt{Var(N(t))}} \le x \right)$$

$$= \lim_{t\to\infty} P\left(\frac{N(t) - \frac{t}{\mu}}{\sqrt{\frac{\sigma^2}{\mu^3} t}} \le x \right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2}\, dy$$

$N(t) \approx \frac{t}{\mu} + \sqrt{\frac{\sigma^2}{\mu^3} t}\, Z$, where $Z \sim N(0,1)$ for large t

No proof.