

# MATH 180A: Introduction to Probability

Lecture B00 (Nemish)

[www.math.ucsd.edu/~ynemish/teaching/180a](http://www.math.ucsd.edu/~ynemish/teaching/180a)

Lecture C00 (Au)

[www.math.ucsd.edu/~bau/f20.180a](http://www.math.ucsd.edu/~bau/f20.180a)

## Today: Independent trials

### Next: ASV 3.3

Video: Prof. Todd Kemp, Fall 2019

Week 3:

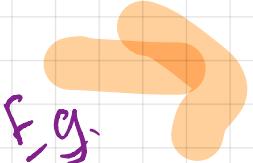
- Homework 3 (due Friday October 23)
- Quiz 2 on Wednesday October 21
- Regrades for HW1: Mon, Oct 19 - Tue, Oct 20 (PST) on Gradescope

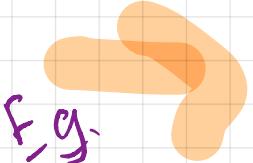
# Independent Random Variables

2.3

A collection  $X_1, X_2, X_3, \dots, X_n$  of random variables defined on the same sample space are independent if

for any  $B_1, B_2, \dots, B_n \subseteq \mathbb{R}$ , the events

  $\{X_1 \in B_1\}, \{X_2 \in B_2\}, \dots, \{X_n \in B_n\}$  are independent.

  $\text{Ex. } P(\{X_1 \in B_1\} \cap \{X_2 \in B_2\} \cap \dots \cap \{X_n \in B_n\}) = P(X_1 \in B_1)P(X_2 \in B_2) \dots P(X_n \in B_n)$

Special Case: if the  $X_j$  are discrete random variables, it suffices to check the simpler condition

for any real numbers  $t_1, t_2, \dots, t_n$

  $P(X_1 = t_1, X_2 = t_2, \dots, X_n = t_n) = P(X_1 = t_1)P(X_2 = t_2) \dots P(X_n = t_n)$

E.g. Let  $X_1, X_2, \dots, X_n$  be fair coin tosses. Denote H ~ 1, T ~ 0.

$$P(X_1 = t_1, X_2 = t_2, \dots, X_n = t_n) = \frac{1}{2^n} = \underbrace{\frac{1}{2} \cdot \frac{1}{2} \cdots \frac{1}{2}}_n = P(X_1 = t_1)P(X_2 = t_2) \dots P(X_n = t_n)$$

# Independent Trials

2.4

Experiments can have numerical observables, but sometimes you only observe whether there is **success** or **failure**

We model this with a random variable **X** taking value **1** with some probability **p**, and value **0** with probability **1-p**.

$$X \sim \text{Ber}(p) \quad (\text{Bernoulli})$$

In practice, we usually repeat the experiment many times, making sure to use the same setup each trial. The previous trials do not influence the future ones.

$X_1, X_2, X_3, \dots, X_n$  independent  $\text{Ber}(p)$  n's.  $\prod_{i=1}^n (1-p)^{1-p}$

$$\begin{aligned} & P(X_1=0, X_2=1, X_3=0, X_4=0, X_5=1, X_6=0) = (1-p)p(1-p)(1-p)p(1-p) \\ & = P(X_1=0)P(X_2=1)P(X_3=0)P(X_4=0)P(X_5=1)P(X_6=0) \end{aligned}$$

## How many successful trials?

Run  $n$  independent trials, each with success probability  $p$ .

$$X_1, X_2, \dots, X_n \sim \text{Ber}(p).$$

Let  $S_n = \# \text{successful trials}$

$$= X_1 + X_2 + X_3 + \dots + X_n.$$

What is the distribution of  $S_n$ ?  $S_n \in \{0, 1, 2, \dots, n\}$

$P(S_n = k) = P(\{\text{exactly } k \text{ of the } n \text{ trials are successful}\})$

$$0 \leq k \leq n$$

$$= \binom{n}{k} p^k (1-p)^{n-k}$$

$$\begin{aligned} p &= \frac{1}{2}, \quad 1-p = \frac{1}{2} \\ p^k (1-p)^{n-k} &= \left(\frac{1}{2}\right)^k \left(\frac{1}{2}\right)^{n-k} = \frac{1}{2^n} \end{aligned}$$

Binomial Distribution  $\overline{\text{Bin}(n, p)}$

$\text{Ber}(p) = \text{Bin}(1, p)$  | # heads in  $n$  <sup>fair</sup> coin tosses  $\approx \text{Bin}(n, \frac{1}{2})$

E.g. Roll a fair die 10 times. What is the probability that  
 (success)  $\rightarrow$  ⑥ comes up at least 3 times?

$$X_1, X_2, \dots, X_{10} \text{ indep. } \text{Ber}\left(\frac{1}{6}\right)$$

$$\therefore S_{10} = \text{Bin}(10, \frac{1}{6})$$

$$P(S_{10} \geq 3) = \sum_{k=3}^{10} P(S_{10} = k) \quad \frac{566299}{2519424} \approx 22.5\%$$

$$\begin{aligned} 1 - P(S_{10} < 3) &= 1 - (P(S_{10} = 0) + P(S_{10} = 1) + P(S_{10} = 2)) \\ &= 1 - \left( \binom{10}{0} \left(\frac{1}{6}\right)^0 \left(\frac{5}{6}\right)^{10} + \binom{10}{1} \left(\frac{1}{6}\right)^1 \left(\frac{5}{6}\right)^9 + \binom{10}{2} \left(\frac{1}{6}\right)^2 \left(\frac{5}{6}\right)^8 \right) \end{aligned}$$

E.g. What is the probability that no 6 is rolled in the 10 rolls?

$$P(S_{10} = 0) = \binom{10}{0} \left(\frac{1}{6}\right)^0 \left(\frac{5}{6}\right)^{10} = \left(\frac{5}{6}\right)^{10} \approx 16\%$$

Now, keep rolling. Let  $N$  denote the first roll where a 6 appears.  $N$  is a random variable. What is its distribution?

## First Success Time

$N$  = first success in repeated independent trials (success rate  $p$ ).

Model trials with (unlimited number) of independent  $\text{Ber}(p)$ 's:

$$X_1, X_2, X_3, X_4, \dots$$

$$N \in \{1, 2, 3, \dots\}$$

$$\{N=k\} = \{X_1=0, X_2=0, X_3=0, \dots, X_{k-1}=0, X_k=1\}$$

$$\begin{aligned}
 P(N=k) &= P(\text{" }) = P(X_1=0) P(X_2=0) \cdots P(X_{k-1}=0) P(X_k=1) \\
 &= (1-p) \cdot (1-p) \cdots (1-p) \cdot p = (1-p)^{k-1} p
 \end{aligned}$$

Geometric Distribution  $\text{Geom}(p)$  on  $\{0, 1, 2, 3, \dots\} = N$ .

$$\sum_{k=1}^{\infty} P(N=k) = \sum_{k=1}^{\infty} (1-p)^{k-1} p = p \sum_{k=1}^{\infty} (1-p)^{k-1} = p \sum_{l=0}^{\infty} (1-p)^l$$

$k-1=l$

geometric series

$$\left[ p \frac{1}{1-(1-p)} \right] = \frac{p}{p} = 1$$

## Rare Events

4.4

If  $S_n = S_{n,p} \sim \text{Bin}(n, p)$ ,  $S_n$  is the number of successes in  $n$  independent trials each with success probability  $p$ .

What if  $p$  is very small, but  $n$  is very large?

One way to handle this mathematically is a **Scaling limit**

↪ For each  $n$ , take  $p \propto \frac{1}{n}$ .  $p = \frac{\lambda}{n}$  for some  $\lambda > 0$ .

$$\begin{aligned} P(S_{n,p} = k) &= \binom{n}{k} p^k (1-p)^{n-k} \\ &= \binom{n}{k} \left(\frac{\lambda}{n}\right)^k \left(1 - \frac{\lambda}{n}\right)^{n-k} \end{aligned}$$

$$\lim_{n \rightarrow \infty} \left( \text{" } \text{" } \text{" } \right) = ?$$

What happens as  $n \rightarrow \infty$ ?

$$P(S_{n,p}=k) = \binom{n}{k} \left(\frac{\lambda}{n}\right)^k \left(1-\frac{\lambda}{n}\right)^{n-k}$$

$$= \frac{n(n-1)(n-2)\dots(n-k+1)}{k!} \frac{\lambda^k}{n^k} \left(1-\frac{\lambda}{n}\right)^n \left(1-\frac{\lambda}{n}\right)^{-k}$$

$n \rightarrow \infty$   
 $k$  is fixed

$$= \frac{n}{n} \cdot \frac{n-1}{n} \cdot \frac{n-2}{n} \cdots \frac{n-k+1}{n} \cdot \frac{\lambda^k}{k!} \cdot \underbrace{\left(1-\frac{\lambda}{n}\right)^n}_{\substack{\downarrow \\ 1}} \cdot \underbrace{\left(1-\frac{\lambda}{n}\right)^{-k}}_{\substack{\downarrow \\ e^{-\lambda}}}$$

$$\left(1 + \frac{\lambda}{n}\right)^n \rightarrow e^\lambda$$

$$\therefore \lim_{n \rightarrow \infty} P(S_{n,\frac{\lambda}{n}} = k) = e^{-\lambda} \frac{\lambda^k}{k!}$$

## Poisson Distribution

A random variable  $X$  has the Poisson( $\lambda$ ) distribution if

$$P(X=k) = e^{-\lambda} \frac{\lambda^k}{k!}, \quad k=0,1,2,\dots$$

Check :  $\sum_{k=0}^{\infty} P(X=k) = 1.$

## Summary

Sampling independent trials, the most important (discrete) probability distributions are:

- $\text{Ber}(p)$ :  $P(X=1)=p, P(X=0)=1-p \quad 0 \leq p \leq 1$   
(single trial with success probability  $p$ )
- $\text{Bin}(n,p)$ :  $P(S_n=k) = \binom{n}{k} p^k (1-p)^{n-k} \quad 0 \leq k \leq n$   
(number of successes in  $n$  independent trials with rate  $p$ )
- $\text{Geom}(p)$   $P(N=k) = (1-p)^{k-1} p \quad k=0, 1, 2, \dots$   
(first successful trial in repeated independent trials with rate  $p$ )
- $\text{Poisson}(\lambda)$   $P(X=k) = e^{-\lambda} \frac{\lambda^k}{k!} \quad k=0, 1, 2, \dots \quad \lambda > 0$ .  
(Approximates  $\text{Bin}(n, \lambda/n)$ ; number of rare events in many trials)