MATH180C: Introduction to Stochastic Processes II

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Today: Brownian Motion

> Q&A: December 2

Next: PK 8.2

This week:

- Homework 7 (due THURSDAY, December 3)
- HW6 regrades (until Wednesday, December 2, 11 PM)

BM as a Gaussian process

<u>Def</u>. Stochastic process $(X_t)_{t \geq 0}$ is called a Gaussian process if for any $0 \leq t_1 < t_2 < \cdots < t_n$

$(X_{t_1}, \ldots, X_{t_n})$ is a Gaussian vector, or equivalently for any $c_1, \ldots, c_n \in \mathbb{R}$

is a Gaussian r.v.

Recall that the distribution of a Gaussian vector is uniquelly defined by its mean and covariance matrix.

Similarly, each Gaussian process is uniquely described by

$$\mu(t) = E(X_t) \quad \text{and} \quad \Gamma(s,t) = \mathrm{Cov}(X_s, X_t) \geq 0$$

↑ covariance function

BM as a Gaussian process

Proposition BM is a Gaussian process with

and

Proof. For any $0 \le t_1 < t_2 < \cdots < t_n$, $B_{t_j} - B_{t_{j-1}}$ are indep. Gaussian, thus

$$\sum_{i=1}^{n} c_i B_{t_i} =$$

is also Gaussian.

By definition . Let $s<t$.

Then $\Gamma(s,t) =$

$=$

$=$

$=$

Some properties of BM

Proposition. Let $(B_t)_{t\geq 0}$ be a standard BM. Then

(i) For any $s>0$, the process is a BM independent of $(B_u,\ 0\leq u\leq s)$.

(ii) The process is a BM

(iii) For any $c>0$, the process is a BM

(iv) The process $(X_t)_{t\geq 0}$ defined by for $t>0$ is a BM.

Proof (i) Define $X_t = B_{t+s} - B_s$. Then

$\Rightarrow$ independent Gaussian increments,

$(X_t)_{t\geq 0}$ has continuous paths $\Rightarrow$

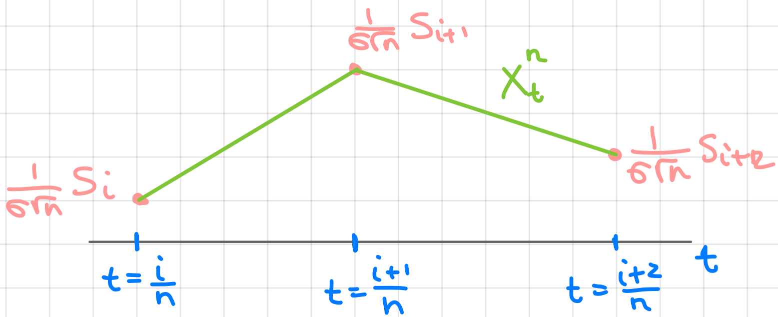
(iv) X_t is Gaussian, for $s<t$

Proof of $\lim_{t\to 0} X_t = 0$ is more technical, thus omitted.

Construction of BM

BM can be constructed as a limit of properly rescaled random walks.

Let $\{\xi_k\}_{k=1}^{\infty}$ be a sequence of i.i.d. r.v.s, $E(\xi_i)=0$, $Var(\xi_i)=\sigma^2<\infty$. Denote and define



Theorem (Donsker)

Applying Donsker's theorem

Example Let $(\xi_i)_{i=1}^{\infty}$ be i.i.d. r.v. $P(\xi_i = 1) = P(\xi_i = -1) = 0.5$

$E(\xi_i) = 0$, $Var(\xi_i) = 1$.

Denote $(S_m)_{m \geq 0}$ is a Markov chain.

From the first step analysis of MC we know that for any $-a < 0 < b$

If X_t^n is the process interpolating S_m, then $\forall n$

$$P(X^n \text{ hits } -a \text{ before } b) =$$

$$\Rightarrow P(B \text{ hits } -a \text{ before } b) =$$

$$\Rightarrow (\tilde{\xi}_i)_{i=1}^{\infty}, E(\tilde{\xi}_i) = 0, Var(\tilde{\xi}_i) = 1, P(\tilde{S} \text{ hits } -a \text{ before } b) \approx \frac{b}{a+b}$$

BM as a martingale

Let $(X_t)_{t\geq 0}$ be a continuous time stochastic process. We say that $(X_t)_{t\geq 0}$ is a martingale if $E(|X_t|) < \infty$ $\forall t \geq 0$ and

Proposition Let $(B_t)_{t\geq 0}$ be a standard BM. Then

(i)

(ii)

"Proof": $E(B_t \mid \{B_u, 0 \leq u \leq s\}) =$

$E(B_t^2 - t \mid \{B_u, 0 \leq u \leq s\}) =$

$=$

Thm (Lévy) Let $(X_t)_{t\geq 0}$ be a continuous martingale such that $(X_t^2 - t)_{t\geq 0}$ is a martingale.