Problem 1

1. (25 points) The time intervals between two consecutive rainstorms in San Diego are independent identically distributed random variables with density (in years)

$$f(x) = \begin{cases} 2(1-x), & x \in (0,1) \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

(a) (10 points) Compute the long run expected time between the last rainstorm and the next rainstorm.

(b) (15 points) What is the long run probability that there will be no rainstorms in San Diego in the next 6 months?

Solution.

(a) If $\delta(t)$ is the current life (age) of the renewal process at time t (time from the last rainstorm to time t), and $\gamma(t)$ is the residual life of the renewal process at time t (time until the next rainstorm after time t), then we have to compute

$$\lim_{t\to\infty} E(\delta(t)+\gamma(t)) = \lim_{t\to\infty} E(\beta(t)). \tag{2}$$

Lecture 14-15, page 16:

$$\lim_{t\to\infty} E(\beta(t)) = \frac{\sigma^2+\mu^2}{\mu}, \tag{3}$$

where μ and σ^2 are the mean and variance of the interrenewal times.

$$\mu = \int_0^1 2x(1-x)dx = (x^2 - \frac{2x^3}{3})\Big|_0^1 = \frac{1}{3}, \tag{4}$$

$$\mu^2+\sigma^2 = \int_0^1 2x^2(1-x)dx = \Big(\frac{2x^3}{3} - \frac{x^4}{2}\Big)\Big|_0^1 = \frac{1}{6}, \tag{5}$$

therefore

$$\lim_{t\to\infty} E(\beta(t)) = \frac{1}{2}. \tag{6}$$

(2) In terms of the renewal process, the long run probability that there will be no rainstorm in the next 6 months is given by

$$\lim_{t\to\infty} P(\gamma(t) > 0.5). \tag{7}$$

Lecture 14-15, page 8:

$$\lim_{t\to\infty} P(\gamma(t) > 0.5) = \int_{0.5}^{\infty} \frac{1}{\mu}(1-F(x))dx, \tag{8}$$

where $F(x)$ is the interrenewal distribution. Note, that $F(x) = 1$ for $x \geq 1$. For $x \in (0,1)$

$$F(x) = \int_0^x 2(1-s)ds = -(1-s)^2\Big|_0^x = 1-(1-x)^2. \tag{9}$$

Therefore,

$$\lim_{t\to\infty} P(\gamma(t) > 0.5) = \int_{0.5}^{1} 3(1-x)^2 dx = -(1-x)^3\Big|_{0.5}^{1} = \frac{1}{8}. \tag{10}$$

Problem 2

2. (25 points) Let ξ_i be independent indentically distributed random variables having normal distribution $N(0,2)$ with mean zero and variance 2.
   (a) (15 points) Show that the random variable $(X_n)_{n\geq 0}$, given by

   $$X_0 = 1, \quad X_n = \frac{1}{2^n}\xi_1^2 \cdots \xi_n^2,$$

   defines a nonnegative martingale.

   (b) (10 points) Estimate the probability that $(X_n)_{n\geq 0}$ ever exceeds 100.

Solution.

(a) Check the definition of a martingale:

$$E(|X_n|) = \frac{1}{2^n}E(\xi_1^2\cdots\xi_n^2) = \frac{1}{2^n}(E(\xi^2))^n = 1 < \infty, \tag{11}$$

$$E(X_{n+1}|X_0,\dots,X_n) = E\Big(\frac{\xi_{n+1}^2}{2}X_n|X_0,\dots,X_n\Big) = E\Big(\frac{\xi_{n+1}^2}{2}\Big)X_n = X_n. \tag{12}$$

Since $X_n \geq 0$, $(X_n)_{n\geq 0}$ is a nonnegative martingale.

(b) Using the maximal inequality for nonnegative martingales (Lecure 18, page 7)

$$P(\max_{n\geq 0} X_n \geq 100) \leq \frac{E(X_0)}{100} = \frac{1}{100}. \tag{13}$$

3. (25 points) Let ξ_i be independent indentically distributed random variables having normal distribution $N(0,4)$ with mean zero and variance 4.
   (a) (15 points) Show that the random variable $(X_n)_{n\geq 0}$, given by

   $$X_0 = 1, \quad X_n = \frac{1}{4^n}\xi_1^2 \cdots \xi_n^2,$$

   defines a nonnegative martingale.

   (b) (10 points) Estimate the probability that $(X_n)_{n\geq 0}$ ever exceeds 100.

Solution.

(a) Check the definition of a martingale:

$$E(|X_n|) = \frac{1}{4^n} E(\xi_1^2 \cdots \xi_n^2) = \frac{1}{4^n} (E(\xi^2))^n = 1 < \infty, \tag{14}$$

$$E(X_{n+1}|X_0, \ldots, X_n) = E\Big(\frac{\xi_{n+1}^2}{4} X_n | X_0, \ldots, X_n\Big) = E\Big(\frac{\xi_{n+1}^2}{4}\Big) X_n = X_n. \tag{15}$$

Since $X_n \geq 0$, $(X_n)_{n\geq 0}$ is a nonnegative martingale.

(b) Using the maximal inequality for nonnegative martingales (Lecure 18, page 7)

$$P(\max_{n\geq 0} X_n \geq 100) \leq \frac{E(X_0)}{100} = \frac{1}{100}. \tag{16}$$

Problem 3

4. (25 points) Let $(X_t)_{t\geq 0}$ be a Brownian motion with drift μ and variance parameter σ^2. It is given that $X_0 = 0$, $E(X_1) = 1$ and $\mathrm{Var}(X_1) = 1$.

(a) Determine μ and σ^2.

(b) Suppose that the price fluctuations of a share are modeled by the process $(Z_t)_{t\geq 0}$ given by

$$Z_t = e^{X_t}. \tag{17}$$

Determine the probability that the price of the share doubles before it drops by one half (i.e., probability that the price increases from 1 to 2 before in drops from 1 to 1/2).

Solution.

(a) If $(X_t)_{t\geq 0}$ is a Brownian motion with drift μ and variance σ^2, then $E(X_t) = \mu t$ and $\mathrm{Var}(X_t) = \sigma^2$, therefore we conclude that $\mu = 1$ and $\sigma^2 = 1$.

(b) If $(X_t)_{t\geq 0}$ is a Brownian motion with drift μ and variance σ^2, then the process $(Z_t)_{t\geq 0}$ given by $Z_t = e^{X_t}$ is a geometric Brownian motion with drift α, where

$$\alpha = \mu + \sigma^2/2 = 3/2. \tag{18}$$

Denote $T := \min\{t : Z_t = 2 \text{ or } Z_t = 1/2\}$. Compute

$$1 - \frac{2\alpha}{\sigma^2} = 1 - \frac{3}{1} = -2. \tag{19}$$

Using the "gambler's ruin" theorem for geometric Brownian motion (Lecture 22-23, page 15)

$$P(Z_T = 2) = \frac{1 - (1/2)^{-2}}{2^{-2} - (1/2)^{-2}} = \frac{1-4}{1/4 - 4} = \frac{4}{5}. \tag{20}$$

5. (25 points) Let $(X_t)_{t\geq 0}$ be a Brownian motion with drift μ and variance parameter σ^2. It is given that $X_0 = 0$, $E(X_1) = \frac{1}{2}$ and $\mathrm{Var}(X_1) = 1$.

   (a) Determine μ and σ^2.

   (b) Suppose that the price fluctuations of a share are modeled by the process $(Z_t)_{t\geq 0}$ given by
   $$Z_t = e^{X_t}. \tag{21}$$
   Determine the probability that the price of the share triples before it drops by two thirds (i.e., probability that the price increases from 1 to 3 before in drops from 1 to 1/3).

Solution.

(a) If $(X_t)_{t\geq 0}$ is a Brownian motion with drift μ and variance σ^2, then $E(X_t) = \mu t$ and $\mathrm{Var}(X_t) = \sigma^2$, therefore we conclude that $\mu = 1/2$ and $\sigma^2 = 1$.

(b) If $(X_t)_{t\geq 0}$ is a Brownian motion with drift μ and variance σ^2, then the process $(Z_t)_{t\geq 0}$ given by $Z_t = e^{X_t}$ is a geometric Brownian motion with drift α, where
$$\alpha = \mu + \sigma^2/2 = 1. \tag{22}$$
Denote $T := \min\{t \,:\, Z_t = 3 \text{ or } Z_t = 1/3\}$. Compute
$$1 - \frac{2\alpha}{\sigma^2} = 1 - \frac{2}{1} = -1. \tag{23}$$
Using the "gambler's ruin" theorem for geometric Brownian motion (Lecture 22-23, page 15)
$$P(Z_T = 3) = \frac{1-(1/3)^{-1}}{3^{-1}-(1/3)^{-1}} = \frac{1-3}{1/3-3} = \frac{3}{4}. \tag{24}$$

Problem 4

6. (25 points) Certain device consists of two components. The amount of time that the components work before breaking down has exponential distribution with rate 1. If any of the components fails, the repair time has exponential distribution with mean 2. The two components work independently and are repaired independently of each other.

   The number of components working at time t is given by the process $(X_t)_{t\geq 0}$ which is a continuous time Markov chain.

   (a) (10 points) Determine the genetator Q of $(X_t)_{t\geq 0}$.

   (b) (10 points) Determine the stationary distribution of $(X_t)_{t\geq 0}$.

   (c) (5 points) In the long run, what fraction of time both components work simultaneously?

Solution

(a) Let T_i and R_i denote the working and repair times of the ith component correspondingly for $i \in \{1, 2\}$. Then $T_i \sim \text{Exp}(1)$, $R_i \sim \text{Exp}(1/2)$ and the generator matrix Q of the Markov process $(X_t)_{t\geq 0}$ is given by (similarly as in Problem 4 of Practice Midterm 1)

$$Q = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array}\!\!\begin{array}{c} \begin{array}{ccc} 0 & \quad 1 & \quad 2 \end{array} \\ \begin{Vmatrix} -1 & 1 & 0 \\ 1 & -3/2 & 1/2 \\ 0 & 2 & -2 \end{Vmatrix} \end{array}. \tag{25}$$

For example, the (2,1) entry of Q can be computed from

$$P(X_h = 1 \mid X_0 = 2) = P(\text{one of two working components fails during } (0, h)) + o(h) \tag{26}$$
$$= P(T_1 < h, T_2 > h) + P(T_2 < h, T_1 > h) + o(h) \tag{27}$$
$$= (1 - e^{-h})e^{-h} + (1 - e^{-h})e^{-h} + o(h) \tag{28}$$
$$= 2h + o(h) \tag{29}$$

as $h \to 0$, and we get that $q_{21} = 2$.

(b) Let $\pi = (\pi_0, \pi_1, \pi_2)$ be the stationary distribution of the Markov chain $(X_t)_{t\geq 0}$. Then (see Lecture 9, page 4) π satisfies the following system of equations

$$\pi Q = 0, \qquad \pi_0 + \pi_1 + \pi_2 = 1. \tag{30}$$

Solve this system

$$-\pi_0 + \pi_1 = 0, \tag{31}$$
$$\pi_0 - \frac{3}{2}\pi_1 + 2\pi_2 = 0, \tag{32}$$
$$\frac{1}{2}\pi_1 - 2\pi_2 = 0, \tag{33}$$
$$\pi_0 + \pi_1 + \pi_2 = 0. \tag{34}$$

From the first and the third equations we get that $\pi_0 = \pi_1 = 4\pi_2$, which together with the fourth equation gives

$$\pi_0 = \frac{4}{9}, \quad \pi_1 = \frac{4}{9}, \quad \pi_2 = \frac{1}{9}. \tag{35}$$

(c) The average long run fraction of time spent in state 2, corresponding to both components working simultaneously, is given by $\pi_2 = 4/9$ (see Lecture 9, page 10).

7. (25 points) Certain device consists of two components. The amount of time that the components work before breaking down has exponential distribution with mean 4. If any of the components fails, the repair time has exponential distribution with rate 1. The two components work independently and are repaired independently of each other.

   The number of components working at time t is given by the process $(X_t)_{t\geq 0}$ which is a continuous time Markov chain.

   (a) (10 points) Determine the genetator Q of $(X_t)_{t\geq 0}$.

   (b) (10 points) Determine the stationary distribution of $(X_t)_{t\geq 0}$.

   (c) (5 points) In the long run, what fraction of time the device has at least one component working?

Solution

(a) Let T_i and R_i denote the working and repair times of the ith component correspondingly for $i \in \{1,2\}$. Then $T_i \sim \text{Exp}(1/4)$, $R_i \sim \text{Exp}(1)$ and the generator matrix Q of the Markov process $(X_t)_{t\geq 0}$ is given by (similarly as in Problem 4 of Practice Midterm 1)

$$Q = \begin{array}{c|ccc|} & 0 & 1 & 2 \\ \hline 0 & -2 & 2 & 0 \\ 1 & 1/4 & -5/4 & 1 \\ 2 & 0 & 1/2 & -1/2 \end{array}. \tag{36}$$

For example, the (2,1) entry of Q can be computed from

$$P(X_h = 1 \mid X_0 = 2) = P(\text{one of two working components fails during } (0,h)) + o(h) \tag{37}$$

$$= P(T_1 < h, T_2 > h) + P(T_2 < h, T_1 > h) + o(h) \tag{38}$$

$$= (1 - e^{-h/4})e^{-h/4} + (1 - e^{-h/4})e^{-h/4} + o(h) \tag{39}$$

$$= \frac{1}{2}h + o(h) \tag{40}$$

as $h \to 0$, and we get that $q_{21} = 1/2$.

(b) Let $\pi = (\pi_0, \pi_1, \pi_2)$ be the stationary distribution of the Markov chain $(X_t)_{t\geq 0}$. Then (see Lecture 9, page 4) π satisfies the following system of equations

$$\pi Q = 0, \qquad \pi_0 + \pi_1 + \pi_2 = 1. \tag{41}$$

Solve this system

$$-2\pi_0 + \frac{1}{4}\pi_1 = 0, \tag{42}$$
$$2\pi_0 - \frac{5}{4}\pi_1 + \frac{1}{2}\pi_2 = 0, \tag{43}$$
$$\pi_1 - \frac{1}{2}\pi_2 = 0, \tag{44}$$
$$\pi_0 + \pi_1 + \pi_2 = 0. \tag{45}$$

From the first and the third equations we get that $\pi_1 = 8\pi_0$, $\pi_2 = 2\pi_1$, which together with the fourth equation gives

$$\pi_0 = \frac{1}{25}, \quad \pi_1 = \frac{8}{25}, \quad \pi_2 = \frac{16}{25}. \tag{46}$$

(c) The average long run fraction of time spent in states 1 or 2, corresponding to at least one components working, is given by $\pi_1 + \pi_2 = 24/25$ (see Lecture 9, page 10).