☐ **Write your name and PID on the top of EVERY PAGE.**

☐ **Write the solutions to each problem on separate pages. CLEARLY INDICATE on the top of each page the number of the corresponding problem. Different parts of the same problem can be written on the same page (for example, part (a) and part (b))**

☐ **Remember this exam is graded by a human being. Write your solutions NEATLY AND COHERENTLY, or they risk not receiving full credit.**

☐ **From the moment you access the midterm problems on Gradescope you have 70 MINUTES to COMPLETE AND UPLOAD your exam to Gradescope. Plan your time accordingly.**

☐ **For combinatorial problems, you can leave the expressions without simplifications (unless the problem specifically asks to simplify).**

☐ **You are allowed to use the textbook, lecture notes and your personal notes. You are not allowed to use the electronic devices (except for accessing the online version of the textbook) or outside assistance. Outside assistance includes but is not limited to other people, the internet and unauthorized notes.**

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1. (20 points) In a survey conducted by the City of San Diego, 48 of 400 interviewed San Diegans confirmed they had at least one encounter with coyotes in the past six months.

   Using the above data, construct a 90% confidence interval for the unknown fraction of the city population who encountered coyotes in the past six months. Clearly state (in words) your conclusion.

   [You may leave your answer in terms of Φ]

   Solution. Denote by p the fraction of San Diegans who encountered at least one coyote in the past six months. In order to construct the confidence interval for p, take the estimate $\hat{p} = 48/400 = 0.12$ and find the smallest $\varepsilon > 0$ for which

$$P(|p - \hat{p}| < \varepsilon) \geq 0.9. \tag{1}$$

   Using the normal approximation of the binomial distribution from Lecture 14 we estimate the above probability as

$$P(|p - \hat{p}| < \varepsilon) \geq 2\Phi(2\varepsilon\sqrt{n}) - 1, \tag{2}$$

   where n is the number of interviewed San Diegans. The 90% confidence interval is obtained by taking ε such that

$$2\Phi(2\varepsilon\sqrt{n}) - 1 = 0.9. \tag{3}$$

   Since the CDF of the standard normal distribution Φ is strictly increasing, we can solve the above equation with respect to ε

$$\varepsilon = \frac{\Phi^{-1}(0.95)}{2 \cdot 20}. \tag{4}$$

   Conclusion. With probability at least 90% the fraction of the San Diego population that encountered coyotes lies in the interval

$$\Big(0.12 - \frac{\Phi^{-1}(0.95)}{40}, 0.12 + \frac{\Phi^{-1}(0.95)}{40}\Big). \tag{5}$$

2. (25 points) Let X be a continuous random variable with the probability density function given by

$$f_X(x) = \begin{cases} c \cdot (x+1)^2, & x \in (0,1), \\ 0, & \text{otherwise,} \end{cases}$$

   with $c > 0$ a constant to be determined later. Let $Y = \frac{1}{X+1}$.

   (a) (10 points) Compute $E(Y)$.

   (b) (10 points) Compute $\mathrm{Var}(Y)$.

   (c) (5 points) Determine the value of c.

[You may leave the answers to (a) and (b) without writing explicitly the value of c.]

Solution.

(a) Using the formula for the expectation of a function of a random variable, we get

$$E\Big(\frac{1}{X+1}\Big) = \int_{-\infty}^{\infty} \frac{1}{x+1} f_X(x)dx = \int_0^1 \frac{1}{x+1} c(1+x)^2 dx = c\int_0^1 (1+x)dx = \frac{3c}{2}.$$

(b) We use the formula $\mathrm{Var}(Y) = E(Y^2) - (E(Y))^2$.

$$E\Big(\frac{1}{(1+X)^2}\Big) = \int_{-\infty}^{+\infty} \frac{1}{(1+x)^2} f_X(x)dx = \int_0^1 \frac{1}{(1+x)^2} c(1+x)^2 dx = \int_0^1 c dx = c,$$

which together with part (a) gives

$$\mathrm{Var}(Y) = c - \frac{9}{4}c^2 = c\Big(1 - \frac{9c}{4}\Big). \tag{6}$$

(c) Finally, from the properties of the density function $\int_{\mathbb{R}} f_X(x)dx = 1$ we get that

$$\int_0^1 c(1+x)^2 dx = c\frac{(1+x)^3}{3}\Big|_0^1 = c \cdot \frac{7}{3} = 1 \quad \Rightarrow \quad c = \frac{3}{7}. \tag{7}$$

3. (25 points) The final MATH 180B exam consists of 192 multiple choice question. Each question has four answer options, only one of which is correct. In order to pass the exam, students have to answer correctly to at least 64 questions.

Student A decides not to study at all, choose the answers at random and rely completely on pure luck. Estimate the probability that student A passes the exam.

[Explain your choice of approximation. You may leave the answers in terms of $\Phi(x)$ or e^x. Do not use the continuity correction.]

Solution. Denote by X the number of correct answers given by student A. The probability of successfully answering each question is $1/4$, and the outcomes for different questions are independent (the student has no prior knowledge of MATH 180B), so we may assume that X has binomial distribution with parameters $n = 192$ and $p = 1/4$, $X \sim \mathrm{Bin}(192, 1/4)$. Then we have to estimate the probability

$$P(\text{student A passes the exam}) = P(X \geq 64). \tag{8}$$

In order to choose the best approximation, we compare $np(1-p)$ and np^2

$$np(1-p) = 36, \quad np^2 = 12. \tag{9}$$

$np(1-p) > 10$, p is not very close to 0 or 1, therefore the normal approximation should give a good approximation of the probability (8). At the same time, $np^2 = 12 > 1$, which means that the Poisson approximation is most likely useless in this case.

Using the normal approximation (without applying the continuity correction), we get

$$P(X \geq 64) = P\Big(\frac{X - 192/4}{\sqrt{36}} \geq \frac{64-48}{6}\Big) \approx P\Big(Z \geq \frac{8}{3}\Big) = 1 - \Phi\Big(\frac{8}{3}\Big), \tag{10}$$

where $Z \sim \mathcal{N}(0,1)$.

We conclude that student A passes the exam with probability around $1-\Phi(\frac{8}{3}) \approx 0.0038$.

4. (30 points) Let X be a random variable that describes the lifetime (in hours) of a certain component of an analog clock. The average lifetime of the component is 10 hours. When the component fails, the clock pointers (hands) stop rotating. A new component has been installed at noon.

   Compute the probability, that when this component fails, the minute pointer (hand) will indicate a number between 0 and 30. [Hint. Start by describing this event in terms of X]

   For the full credit the answer should be written in the closed form (not involving infinite sums).

Solution. Denote by X the lifetime (in hours) of the component. Since X describes the lifetime, we assume that X has exponential distribution, $X \sim \text{Exp}(\lambda)$. In order to determine $\lambda > 0$, note that $E(X) = 10$, therefore

$$E(X) = \frac{1}{\lambda} = 10 \quad \Rightarrow \quad \lambda = \frac{1}{10}. \tag{11}$$

Denote by A the event that when the component fails, the minute hand is indicating a number between 0 and 30. Since the new component has been installed at noon, when the minute hand is at 0, the even A can be rewritten as

$$A = \cup_{n=0}^{\infty}\{X \in (n, n+1/2)\} = \{X \in \cup_{n=0}^{\infty}(n, n+1/2)\}. \tag{12}$$

Events $\{X \in (n, n+1/2)\}$ are disjoint for different n, and

$$P(X \in (n, n+1/2)) = \int_n^{n+1/2} \frac{1}{10} e^{-x/10} dx = e^{-n/10} - e^{-(n+1/2)/10}. \tag{13}$$

Therefore,

$$P(A) = \sum_{n=0}^{\infty} \left(e^{-n/10} - e^{-(n+1/2)/10}\right) \tag{14}$$

$$= \left(1 - e^{-1/20}\right) \sum_{n=0}^{\infty} e^{-n/10} \tag{15}$$

$$= \frac{1 - e^{-1/20}}{1 - e^{-1/10}} \tag{16}$$

$$= \frac{1}{1 + e^{-1/20}} \tag{17}$$