MATH180C: Introduction to Stochastic Processes II

www.math.ucsd.edu/~ynemish/teaching/180c

Today: General continuous time MC. Q-matrices. Matrix exponentials

> Q&A: October 14

Next: PK 6.6, Durrett 4.1

Week 2:

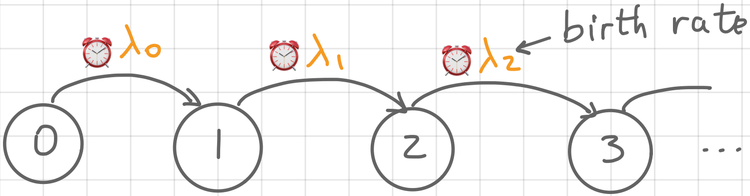
- No homework!
- Quiz 1 on Wednesday, October 14

Birth and death processes

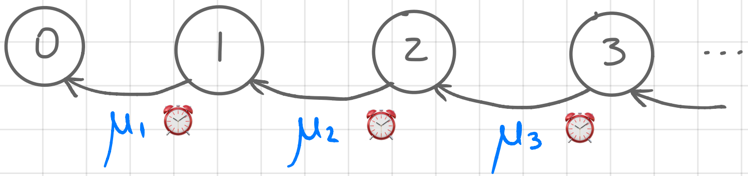
! Only jumps of magnitude 1

$i \to i-1$ or $i \to i+1$

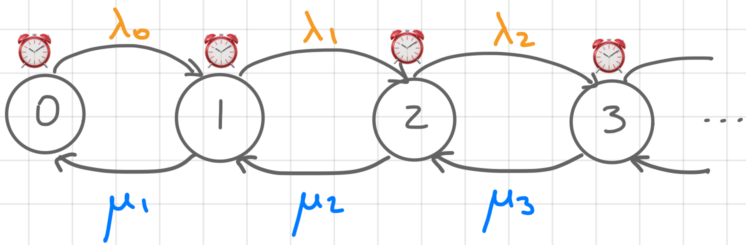
- Pure birth / death processes



$\rightsquigarrow$ Poisson process



- Birth and death processes



$\rightsquigarrow$ random walk

Birth and death processes. Results

- infinitesimal transition probability description
- sojourn time description (jump and hold)

  sojourn times are independent exponential r.v.s

  $$P(i \to i+1) = \frac{\lambda_i}{\lambda_i + \mu_i}, \quad P(i \to i-1) = \frac{\mu_i}{\lambda_i + \mu_i}$$

- system of differential equations for pure birth/death

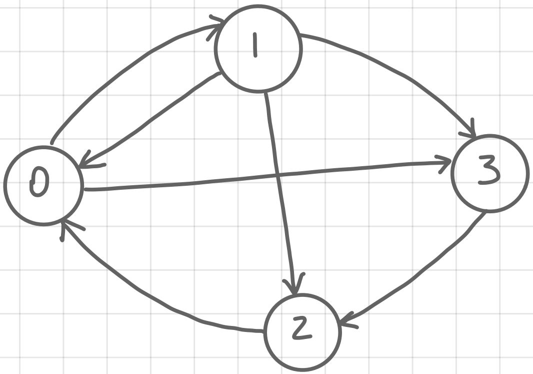
  e.g. $$P_i'(t) = -\lambda_i P_i(t) + \lambda_{i-1} P_{i-1}(t)$$

- distributions of X_t for linear birth (geometric) and linear death (binomial) processes
- first step analysis giving absorption probabilities and mean time to absorption
- explosion, Strong Markov property etc.

General continuous time MC

Assume for simplicity that the state space is finite

birth and death process



general MC

How to define? How to analyze?

Q-matrices (infinitesimal generators)

Let $S = \{0, 1, \dots, N\}$. We call $Q = (q_{ij})_{i,j=0}^{N}$ a Q-matrix if Q satisfies the following conditions:

(a)

(b)

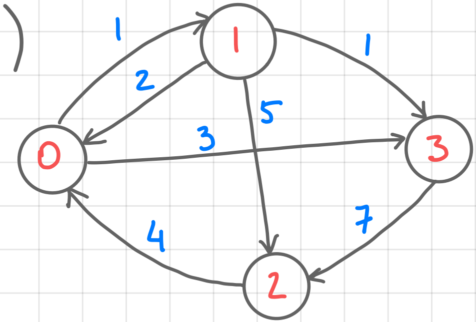
(c)

Examples

(a)

$$Q = \begin{pmatrix} \\ \\ \\ \end{pmatrix}$$

(b)



$$\begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \left(\begin{array}{cccc} & & & \\ & & & \\ & & & \\ & & & \end{array}\right) \end{array}$$

Matrix exponentials

Let $Q = (q_{ij})_{i,j=1}^{N}$ be a matrix. Then the series converges componentwise, and we denote its sum $\sum_{k=0}^{\infty} \frac{Q^k}{k!} =:$, the matrix exponential of Q.

In particular, we can define for $t \geq 0$.

Thm. Define $P(t) = e^{tQ}$. Then

(i) for all s, t

(ii) $(P(t))_{t \geq 0}$ is the unique solution to the equations

$$\begin{cases} \frac{d}{dt} P(t) = \\ P(0) = \end{cases}, \quad \text{and} \quad \begin{cases} \frac{d}{dt} P(t) = \\ P(0) = \end{cases}$$

Matrix exponentials

Properties are easy to remember $\longrightarrow$ scalar exponential

(i) $e^{(t+s)Q} = e^{tQ} e^{sQ} = e^{sQ} e^{tQ}$ $\quad \left(e^{(t+s)\alpha} = e^{t\alpha} e^{s\alpha} \right)$

(note that in general $AB \neq BA$ for matrices A, B)

(ii) $\frac{d}{dt} e^{tQ} = Q e^{tQ} = e^{tQ} Q$ $\quad \left(\frac{d}{dt} e^{t\alpha} = \alpha e^{t\alpha} \right)$

$e^{0 \cdot Q} = I$ $\quad (e^0 = 1)$

Example

(a) $Q_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$

(b) $Q_2 = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$

Matrix exponentials

Results on the previous slide hold for any matrix Q.

<u>Thm</u>. Matrix Q is a Q-matrix

iff $P(t) = e^{tQ}$ is a stochastic matrix $\forall t$

<u>Remarks</u> The semigroup property gives entrywise

$$P_{ij}(t+s) = [P(t)P(s)]_{ij}$$

(if you think about MC $\longrightarrow$

Chapman-Kolmogorov)