MATH180C: Introduction to Stochastic Processes II

www.math.ucsd.edu/~ynemish/teaching/180c

Today: FSA for general MC

> Q&A: October 19

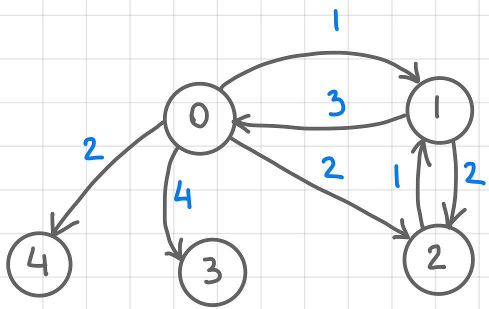
Next: PK 6.3, 6.6, Durrett 4.2

This week:

- Quiz 2 on Wednesday, October 21 (lectures 4-6)
- Homework 2 (due Friday, October 23, 11:59 PM)

General continuous time finite state MCs

Rate diagram



Generator

$$Q = \begin{array}{c|c} & \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{pmatrix} -9 & 1 & 2 & 4 & 2 \\ 3 & -5 & 2 & & \\ & 1 & -1 & & \\ & & & 0 & \\ & & & & 0 \end{pmatrix} \end{array}$$

Infinitesimal description

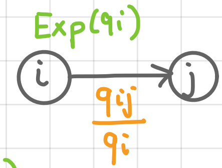
$$P_{ij}(h) = q_{ij}h + o(h) \quad , i \neq j$$

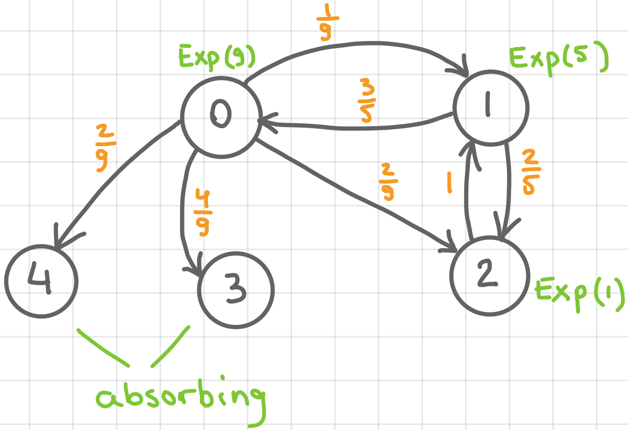
$$P_{ii}(h) = 1 - q_i h + o(h)$$

$$P_{02}(h) = 2h + o(h)$$

$$P_{00}(h) = 1 - 9h + o(h)$$

Jump and hold





Absorption probabilities for finite state chains

By considering the jump chain $(Y_n)_{n\geq 0}$ with $Y_n = X_{W_n}$ and its transition probabilities $P(Y_{n+1}=j \mid Y_n = i) = \frac{q_{ij}}{q_i}$ we can apply the first step analysis to compute, e.g., the absorption probabilities (similarly as for B&D)

If state i is absorbing, then $q_{ij} = 0$ for all $j \neq i$ (no jumps from state i), so $q_i = q_{ii} = 0$. Let Q be given by

$$Q = \begin{array}{cc} & \begin{array}{ccc|ccc} 0 & \cdots & K-1 & K & \cdots & N \end{array} \\ \begin{array}{c} 0 \\ \vdots \\ K-1 \\ K \\ \vdots \\ N \end{array} & \left(\begin{array}{ccc|ccc} -q_0 & & & \cdots & q_{ij} & \\ \vdots & & & & \vdots & \\ q_{ij} & \cdots & -q_{K-1} & & & \\ \hline & & & 0 & & 0 \\ & 0 & & & \ddots & \\ & & & & & 0 \end{array}\right) \end{array}$$

with $\{0,\dots,K-1\}$ transient,
$\{K,\dots,N\}$ absorbing

Absorption probabilities for finite state chains

$$Q = \begin{array}{c|c} & \begin{array}{ccc|ccc} 0 & \cdots & K-1 & K & \cdots & N \end{array} \\ \hline \begin{array}{c} 0 \\ \vdots \\ K-1 \\ K \\ \vdots \\ N \end{array} & \left(\begin{array}{ccc|ccc} -q_0 & & & \cdots & q_{ij} & \\ \vdots & & & & \vdots & \\ q_{ij} & \cdots & -q_{K-1} & & & \\ \hline & & & 0 & & 0 \\ & 0 & & & \ddots & \\ & & & & & 0 \end{array}\right) \end{array}$$

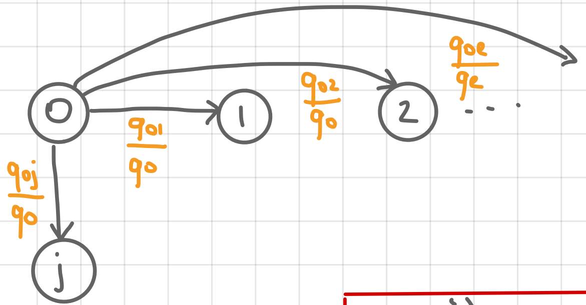
Let $i \in \{0, \dots, K-1\}$, $j \in \{K, \dots, N\}$.

Let $\mathbf{M} = \min\{n : Y_n \in \{K, \dots, N\}\}$

Denote $u_i^{(j)} = P(Y_{\mathbf{M}} = j \mid X_0 = i)$.

Then FSA leads to the system

Jump chain



$$u_i^{(j)} = P(Y_{\mathbf{M}} = j \mid Y_0 = i)$$

$$= \sum_{\substack{\ell=0 \\ \ell \neq i}}^{N} P(Y_{\mathbf{M}} = j \mid Y_0 = i, Y_1 = \ell)\, P(Y_1 = \ell \mid Y_0 = i)$$

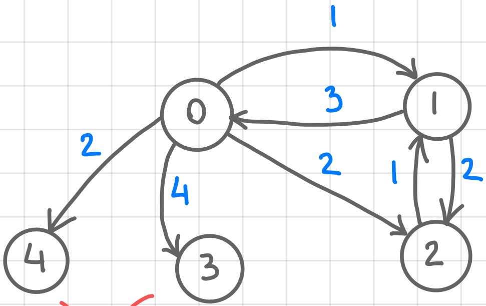
$$= \sum_{\substack{\ell=0 \\ \ell \neq i}}^{K-1} \underbrace{P(Y_{\mathbf{M}} = j \mid Y_1 = \ell)}_{\text{MP} \rightarrow \, = \, u_\ell^{(j)}} \frac{q_{i\ell}}{q_i} + \underbrace{P(Y_{\mathbf{M}} = j \mid Y_1 = j)}_{=1} \frac{q_{ij}}{q_i}$$

$$\boxed{u_i^{(j)} = \frac{q_{ij}}{q_i} + \sum_{\substack{\ell=0 \\ \ell \neq i}}^{K-1} \frac{q_{i\ell}}{q_i} u_\ell^{(j)}}$$

$P(Y_{n+1} = j \mid Y_n = i)$ $\qquad$ $P(Y_{n+1} = \ell \mid Y_n = i)$

Example

Rate diagram



absorbing

Generator

$$Q = \begin{array}{c|ccccc} & 0 & 1 & 2 & 3 & 4 \\ \hline 0 & -9 & 1 & 2 & 4 & 2 \\ 1 & 3 & -5 & 2 & & \\ 2 & & 1 & -1 & & \\ 3 & & & & 0 & \\ 4 & & & & & 0 \end{array}$$

Compute $P(Y_M = 3)$ if $P(X_0 = i) = p_i$ for $i = 0,1,2$

$$\sum_{i=0}^{2} p_i = 1$$

Denote $u_i = P(Y_M = 3 \mid Y_0 = i)$.

$$\begin{cases} u_0 = \frac{1}{9} u_1 + \frac{2}{9} u_2 + \frac{4}{9} \\ u_1 = \frac{3}{5} u_0 + \frac{2}{5} u_2 \\ u_2 = u_1 \end{cases}$$

$$\begin{cases} 9 u_0 = 4 + 3 u_0 \\ 3 u_1 = 3 u_0 \\ u_2 = u_1 \end{cases}$$

$$u_0 = u_1 = u_2 = \frac{2}{3}$$

$$P(Y_M = 3) = \sum_{i=0}^{2} u_i p_i = \frac{2}{3}$$

Mean time to absorption

Similar analysis as was applied to B&D processes can be used to compute the mean time to absorption: before each jump from step i to state j the process sojourns $\frac{1}{q_i}$ on average in state i.

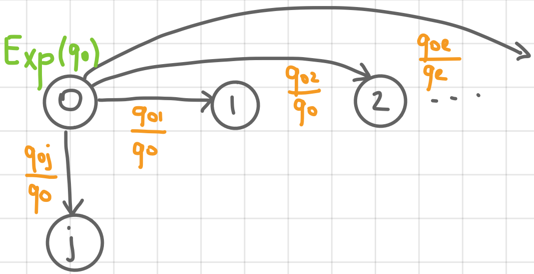
$$
Q = \begin{array}{c} \\ 0 \\ \vdots \\ K\text{-}1 \\ K \\ \vdots \\ N \end{array}
\begin{array}{c}
\begin{array}{cccc|ccc} 0 & \cdots & & K\text{-}1 & K & \cdots & N \end{array} \\
\left(\begin{array}{ccc|ccc}
-q_0 & & & \cdots & q_{ij} & \\
\vdots & & & & \vdots & \\
q_{ij} & \cdots & -q_{K-1} & & & \\
\hline
 & & & 0 & & 0 \\
 & 0 & & & \ddots & \\
 & & & & & 0
\end{array}\right)
\end{array}
$$

Let $T = \min\{t : X_t \in \{K, \dots, N\}\}$

$M = \min\{n : Y_n \in \{K, \dots, N\}\}$

Denote $w_i = E(T \mid X_0 = i)$

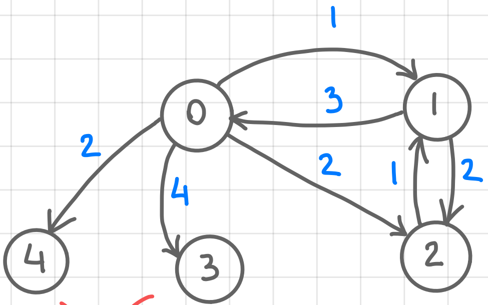
Then FSA gives



$$
\boxed{w_i = \frac{1}{q_i} + \sum_{\substack{\ell=0 \\ \ell \neq i}}^{K-1} \frac{q_{i\ell}}{q_i} w_\ell}
$$

Example

Rate diagram



absorbing

Generator

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$$T = \min\{t : X_t \in \{3,4\}\}$$

$$w_i = E(T \mid X_0 = i)$$

$$\begin{cases} w_0 = \frac{1}{9} + \frac{1}{9} w_1 + \frac{2}{9} w_2 \\ w_1 = \frac{1}{5} + \frac{3}{5} w_0 + \frac{2}{5} w_2 \\ w_2 = 1 + 1 \cdot w_1 \end{cases}$$

$$\begin{cases} w_0 = \frac{3}{9} + \frac{3}{9} w_1 \\ \frac{3}{5} w_1 = \frac{3}{5} + \frac{3}{5} w_0 \\ w_2 = 1 + w_1 \end{cases}$$

$$w_1 = 1 + w_0$$

$$3w_0 = 1 + 1 + w_0$$

$$w_0 = 1, \quad w_2 = 2$$

$$w_3 = 3$$