

# MATH 142A: Introduction to Analysis

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Today: Series

> Q&A: February 2

Next: Ross § 15

Week 5:

- Homework 4 (due Sunday, February 6)

## Sequences $\left| \frac{S_{n+1}}{S_n} \right|$ and $\sqrt[n]{|S_n|}$

Thm 12.2 Let  $(S_n)$  be a sequence,  $\forall n (S_n \neq 0)$ . Then

$$\liminf_{n \rightarrow \infty} \left| \frac{S_{n+1}}{S_n} \right| \leq \liminf_{n \rightarrow \infty} \sqrt[n]{|S_n|} \leq \limsup_{n \rightarrow \infty} \sqrt[n]{|S_n|} \leq \limsup_{n \rightarrow \infty} \left| \frac{S_{n+1}}{S_n} \right|$$

$\ell$        $\beta$

Proof. If  $\ell = 0$ , then  $\ell \leq \beta$ . Assume that  $\ell > 0$ .

Take any  $0 < \ell_1 < \ell$ . Then by Thm 9.11 (i)  $\exists N$  s.t.

$$\inf \left\{ \left| \frac{S_{n+1}}{S_n} \right| : n \geq N \right\} > \ell_1 \Rightarrow \forall n \geq N \quad \left| \frac{S_{n+1}}{S_n} \right| > \ell_1.$$

$$\begin{aligned} \text{Therefore, } \forall n > N \quad |S_n| &= |S_N| \cdot \frac{|S_{N+1}|}{|S_N|} \cdot \frac{|S_{N+2}|}{|S_{N+1}|} \cdots \frac{|S_n|}{|S_{n-1}|} > |S_N| \cdot \ell_1^{n-N} = \frac{|S_N|}{\ell_1^N} \cdot \ell_1^n \\ \Rightarrow \forall n > N \quad \sqrt[n]{|S_n|} &> \sqrt[n]{\frac{|S_N|}{\ell_1^N} \cdot \ell_1^n} = \ell_1 \cdot \sqrt[n]{\frac{|S_N|}{\ell_1^N}} \stackrel{\text{defn}}{=} \tilde{u}_n \end{aligned}$$

Note that  $(\tilde{u}_n)$  is increasing, so  $\forall K > N \quad \tilde{u}_K \geq \tilde{u}_N \geq \ell_1 \sqrt[n]{\frac{|S_N|}{\ell_1^N}}$

$$\text{Now } \beta = \lim_{k \rightarrow \infty} \tilde{u}_k \stackrel{\text{Cor 9.12}}{\geq} \lim_{n \rightarrow \infty} \left( \ell_1 \sqrt[n]{\frac{|S_N|}{\ell_1^N}} \right) = \ell_1 \lim_{n \rightarrow \infty} \sqrt[n]{\frac{|S_N|}{\ell_1^N}} \stackrel{\text{IE4}}{=} \ell_1 \cdot 1 = \ell_1$$

So  $\forall \ell_1 \in (0, \ell) (\beta \geq \ell_1) \Rightarrow \beta$  is an upper bound for  $(0, \ell) \Rightarrow \beta \geq \ell$ .

## Sequences $\left| \frac{S_{n+1}}{S_n} \right|$ and $\sqrt[n]{|S_n|}$

### Corollary 12.3

If  $\lim_{n \rightarrow \infty} \left| \frac{S_{n+1}}{S_n} \right|$  exists, and  $\lim_{n \rightarrow \infty} \left| \frac{S_{n+1}}{S_n} \right| = L$ , then  $\lim_{n \rightarrow \infty} \sqrt[n]{|S_n|} = L$

### Example

Let  $(a_n)$  be a sequence such that  $\forall n \in \mathbb{N} \quad a_n > 0$ .

Suppose that  $(a_n)$  converges,  $\lim_{n \rightarrow \infty} a_n = a$ . Then

$$\lim_{n \rightarrow \infty} \sqrt[n]{a_1 \cdot a_2 \cdots a_n} = a$$

Proof. Denote  $S_n := a_1 \cdots a_n$ . Then  $\lim_{n \rightarrow \infty} \left| \frac{S_{n+1}}{S_n} \right| = \lim_{n \rightarrow \infty} a_{n+1} = a$

By Corollary 12.3  $\lim_{n \rightarrow \infty} \sqrt[n]{|S_n|} = a = \lim_{n \rightarrow \infty} \sqrt[n]{a_1 \cdots a_n}$  ■

## Series

Let  $(a_n)_{n=1}^{\infty}$  be a sequence of real numbers.

For  $p, q \in \mathbb{N}$ ,  $p < q$  we denote  $a_p + a_{p+1} + \dots + a_q$  by  $\sum_{n=p}^q a_n$

Def 14.1 (Infinite series) We call the expression

$$a_1 + a_2 + a_3 + \dots + a_n + \dots =: \sum_{n=1}^{\infty} a_n$$

$a_n$  (infinite) series.  $a_n$  is called the  $n$ -th term of the series.

Def 14.2 (Convergent series)

We call the sum  $S_n = \sum_{k=1}^n a_k$  the ( $n$ -th) partial sum of the series.

If the sequence  $(S_n)$  of partial sums converges, we say that

the series  $\sum_{n=1}^{\infty} a_n$  is convergent

If  $\lim_{n \rightarrow \infty} S_n = s$ , then we call  $s$  the sum of the series  $\sum_{n=1}^{\infty} a_n$ , and

write it as  $\sum_{n=1}^{\infty} a_n = s$

## Series

If  $\lim_{n \rightarrow \infty} s_n = +\infty (-\infty)$ , we say that  $\sum_{n=1}^{\infty} a_n$  diverges to  $+\infty (-\infty)$   
and we write  $\sum_{n=1}^{\infty} a_n = +\infty (\text{or } -\infty)$

We say that  $\sum_{n=1}^{\infty} a_n$  converges absolutely (is absolutely convergent)  
if the series  $\sum_{n=1}^{\infty} |a_n|$  converges

Remark An infinite series can be viewed as a particular type of a sequence,  $s_n = a_1 + a_2 + \dots + a_n$

so we can use all the relevant results.

For example, if  $\forall n a_n \geq 0$ , then  $s_n$  is increasing.

Partial sums of  $\sum_{n=1}^{\infty} |a_n|$  form an increasing sequence.

Use the criteria on convergence for partial sums etc.

## Important examples

8. Let  $a, r \in \mathbb{R}$ . Then  $\sum_{n=0}^{\infty} ar^n$  is called the **geometric series**.

If  $|r| < 1$ , then  $\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$

Proof Denote  $S_k = \sum_{n=0}^k ar^n = a + ar + \dots + ar^k = a(1 + r + r^2 + \dots + r^k)$

Note that  $r(1 + r + \dots + r^k) = r + r^2 + \dots + r^k + r^{k+1}$ , so

$$\begin{aligned} (1-r)(1+r+\dots+r^k) &= 1+r+\dots+r^k - r(1+r+\dots+r^k) = 1-r^{k+1} \Rightarrow 1+r+\dots+r^k = \frac{1-r^{k+1}}{1-r} \\ \Rightarrow \forall k \quad S_k &= \frac{a(1-r^{k+1})}{1-r}. \text{ By 1.E2 } \lim_{k \rightarrow \infty} r^{k+1} = 0 \Rightarrow \lim_{k \rightarrow \infty} S_k = \frac{a}{1-r} \end{aligned}$$

9. Let  $p > 0$ . Then  $\sum_{n=1}^{\infty} \frac{1}{n^p}$  converges iff  $p > 1$

Proof ( $p=2$ ).  $S_k := \sum_{n=1}^k \frac{1}{n^2}$ . ①  $(S_k)$  is increasing ( $S_{k+1} - S_k = \frac{1}{(k+1)^2} > 0$ )

②  $(S_k)$  is bounded  $S_k = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots + \frac{1}{k^2} \leq 1 + \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \dots + \frac{1}{(k-1)k}$

For any  $n \geq 2$   $\frac{1}{(n-1)n} = \frac{1}{n-1} - \frac{1}{n}$ , so  $S_k \leq 1 + \frac{1}{1} - \cancel{\frac{1}{2}} + \cancel{\frac{1}{2}} - \cancel{\frac{1}{3}} + \cancel{\frac{1}{3}} + \dots + \cancel{\frac{1}{k-1}} - \cancel{\frac{1}{k}} = 2 - \frac{1}{k} < 2$

① + ② + Thm 10.2

## Cauchy criterion

Def 14.3 We say that  $\sum_{n=1}^{\infty} a_n$  satisfies the Cauchy criterion if its sequence of partial sums  $(S_n)$  is a Cauchy sequence, i.e.

$$\forall \varepsilon > 0 \exists N \forall m, n > N |S_n - S_m| < \varepsilon$$

$$\forall \varepsilon > 0 \exists N \forall n > m > N \left| \sum_{k=m+1}^n a_k \right| < \varepsilon$$

Thm 14.4  $\sum a_n$  converges  $\Leftrightarrow$   $\sum a_n$  satisfies the Cauchy criterion

Proof. Follows from Thm 10.11

Corollary 14.5 (Necessary condition for convergence).

$$\sum a_n \text{ converges} \Rightarrow \lim_{n \rightarrow \infty} a_n = 0$$

Proof.  $\sum a_n \text{ converges} \stackrel{\text{Thm 14.4}}{\Leftrightarrow} \forall \varepsilon > 0 \exists N \forall n > m > N \left| \sum_{k=m+1}^n a_k \right| < \varepsilon$  (take  $n = m + 1$ )  
 $\Rightarrow \forall \varepsilon > 0 \exists N \forall n > N + 1 (|a_n| < \varepsilon) \Leftrightarrow \lim a_n = 0 \blacksquare$

## Example

- $\sum_{k=1}^{\infty} \frac{1}{k \cdot 2^k}$  satisfies the Cauchy criterion

Proof.  $\forall k \in \mathbb{N} \quad \frac{1}{k \cdot 2^k} \leq \frac{1}{2^k}, \text{ so } \forall n > m \geq 1$

$$\sum_{k=m+1}^n \frac{1}{k \cdot 2^k} \leq \sum_{k=m+1}^n \frac{1}{2^k} = \frac{1}{2^{m+1}} \sum_{l=0}^{n-m} \frac{1}{2^l} < \frac{1}{2^{m+1}} \sum_{l=0}^{\infty} \frac{1}{2^l} = \frac{1}{2^{m+1}} \cdot \frac{1}{1-\frac{1}{2}} = \frac{1}{2^m}$$

Fix  $\varepsilon > 0$ . By I.E.-2  $\exists N \in \mathbb{N} \quad (\frac{1}{2^m} < \varepsilon)$  Therefore

$\forall n > m > N \quad \sum_{k=m+1}^n \frac{1}{k \cdot 2^k} \leq \frac{1}{2^m} < \varepsilon \Rightarrow \sum_{k=1}^{\infty} \frac{1}{k \cdot 2^k}$  satisfies the Cauchy criterion

In particular,

$$\lim_{k \rightarrow \infty} \frac{1}{k \cdot 2^k} = 0$$

■

- If  $|r| \geq 1$ , then the sequence  $(r^n)$  does not converge to 0 (L8)

$$\Rightarrow \sum_{n=0}^{\infty} r^n \text{ does not converge}$$

- Consider  $\sum_{n=1}^{\infty} \frac{1}{n}$ :  $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$ , but the series diverges (L8)

## Comparison test

Thm 14.6 Let  $(a_n)$  and  $(b_n)$  be two sequences,  $\forall n \quad a_n \geq 0$

Then

$$(i) \left( \sum_{n=1}^{\infty} a_n \text{ converges} \wedge \forall n \quad (|b_n| \leq a_n) \right) \Rightarrow \sum_{n=1}^{\infty} b_n \text{ converges}$$

$$(ii) \left( \sum_{n=1}^{\infty} a_n = +\infty \wedge \forall n \quad (b_n \geq a_n) \right) \Rightarrow \sum_{n=1}^{\infty} b_n = +\infty$$

Proof. (i) Use the Cauchy criterion  $[\forall \varepsilon > 0 \exists N \forall n > m > N \quad |\sum_{k=m+1}^n b_k| < \varepsilon]$

$$\left| \sum_{k=M+1}^n b_k \right| \stackrel{\text{Tr. Ineq.}}{\leq} \sum_{k=M+1}^n |b_k| \leq \sum_{k=M+1}^n a_k$$

Fix  $\varepsilon > 0$ . By Thm 14.4  $\exists N \forall n > m > N \quad \sum_{k=m+1}^n a_k < \varepsilon$ . Then

$$\forall n > m > N \quad \left| \sum_{k=m+1}^n b_k \right| < \sum_{k=m+1}^n a_k < \varepsilon. \text{ By Thm 14.4 } \sum_{n=1}^{\infty} b_n \text{ converges}$$

(ii) Denote  $s_n = \sum_{k=1}^n a_k$ ,  $t_n = \sum_{k=1}^n b_k$ . Then  $\forall n \quad (t_n \geq s_n)$

$$\sum_{n=1}^{\infty} a_n = +\infty \Leftrightarrow \lim_{n \rightarrow \infty} s_n = +\infty \Rightarrow \lim_{n \rightarrow \infty} t_n = +\infty \Leftrightarrow \sum_{n=1}^{\infty} b_n = +\infty$$