MATH180C: Introduction to Stochastic Processes II

www.math.ucsd.edu/~ynemish/teaching/180c

Today: Pure death processes.
Birth and death processes
> Q&A: October 9

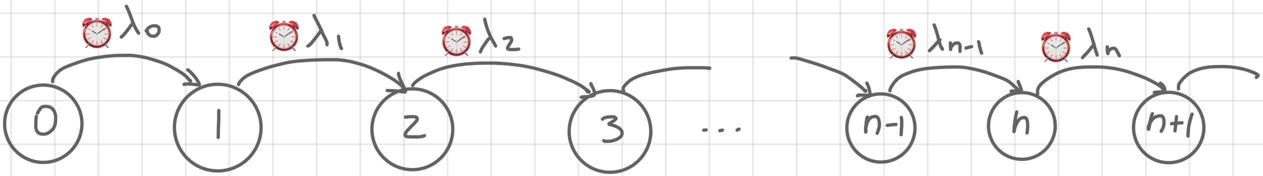
Next: PK 6.5

Week 1:

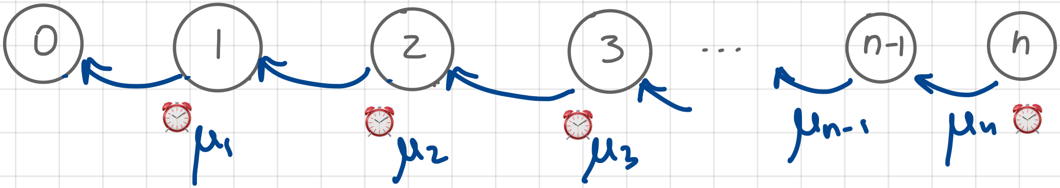
- homework 1 (due Friday October 9)
- join Piazza

Pure death processes

Pure birth process



What if the chain moves in the opposite direction?



Pure death process:

- exponential sojourn times with rates μ_i
- only negative jumps of magnitude 1 allowed

Pure death processes

Infinitesimal description:

Pure death process $(X_t)_{t\geq 0}$ of rates $(\mu_k)_{k=1}^{N}$ is a continuous time MC taking values in $\{0, 1, 2, \dots, N-1, N\}$ (state 0 is absorbing) with stationary infinitesimal transition probability functions

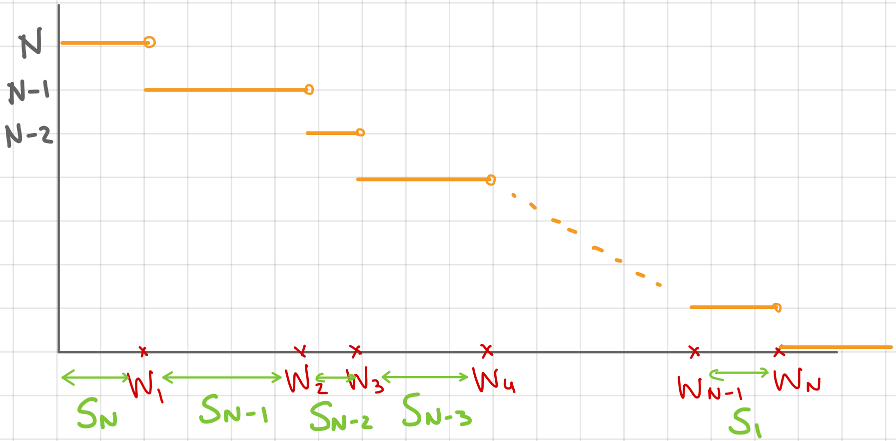
(a) $P_{k,k-1}(h) = \mu_k h + o(h)$ $k = 1, \dots, N$

(b) $P_{kk}(h) = 1 - \mu_k h + o(h)$, $k = 1, \dots, N$

(c) $P_{kj}(h) = 0$ for $j > k$.

State 0 is absorbing $(\mu_0 = 0)$

Pure death process



$S_k \sim Exp(\mu_k)$

Sojourn time / jump description:

Pure death process of rates $(\mu_k)_{k=1}^N$ is a nonincreasing right-continuous process taking values in $\{0, 1, \dots, N\}$

- with sojourn times $S_1, S_2, S_3, \dots, S_N$ being independent exponential r.v.s of rates $\mu_1, \mu_2, \dots, \mu_N$ and
- jumps $X_{W_{i+1}} - X_{W_i} = -1$ of magnitude 1

Differential equations for pure birth processes

Define $P_n(t) = P(X_t = n \mid X_0 = N)$ distribution of X_t

$\uparrow$ starting in state N

(a), (b), (c) implies (check)

$$\begin{cases} P_n'(t) = -\mu_n P_n(t) + \mu_{n+1} P_{n+1}(t) & \text{for } n = 0 \dots N-1 \\ & (\text{note that } \mu_0 = 0) \\ P_N'(t) = -\mu_N P_N(t) \end{cases}$$

Initial conditions: $P_N(0) = 1$, $P_n(0) = 0$ for $n = 0 \dots N-1$

Solve recursively: $P_N(t) = e^{-\mu_N t} \to P_{N-1}(t) \to \cdots \to P_0(t)$

General solution (assume $\mu_i \neq \mu_j$)

$$P_n(t) = \mu_{n+1} \cdots \mu_N \left(A_{nn} e^{-\mu_n t} + \cdots + A_{N,n} e^{-\mu_N t} \right), \quad A_{kn} = \prod_{\substack{l=n \\ l \neq k}}^{N} \frac{1}{\mu_l - \mu_k}$$

Linear death process

Similar to Yule process:

death rate is proportional to the size of the population

$\mu_k = k\alpha$ (linear dependence on k)

Compute $P_n(t)$: • $\mu_{n+1} \cdots \mu_N = \alpha^{N-n} \frac{N!}{n!}$

• $A_{kn} = \prod_{\substack{\ell=n \\ \ell\neq k}}^{N} \frac{1}{\mu_\ell - \mu_k} = \frac{1}{\alpha^{N-n}(-1)^{n-k}(k-n)!(N-k)!}$ $\quad \{ \mu_\ell - \mu_k = \alpha(\ell - k)$

• $P_n(t) = \alpha^{N-n} \frac{N!}{n!} \cdot \frac{1}{\alpha^{N-n}} \sum_{k=n}^{N} \frac{1}{(-1)^{n-k}(k-n)!(N-k)!} \cdot e^{-k\alpha t}$ $\quad \begin{cases} j = k-n \\ k = j+n \end{cases}$

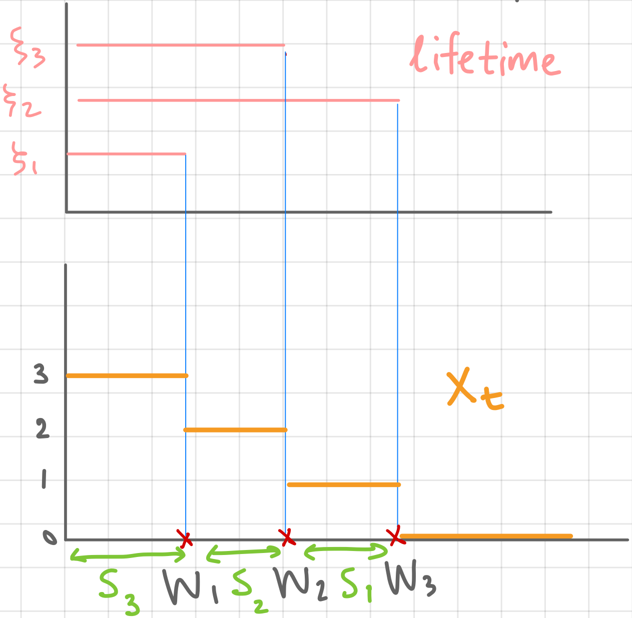
$$= \frac{N!}{n!} \sum_{j=0}^{N-n} \frac{(-1)^j e^{-(j+n)\alpha t}}{j!(N-n-j)!}$$

$$= \frac{N!}{n!} e^{-n\alpha t} \sum_{j=0}^{N-n} \frac{1}{j!(N-n-j)!} \left(-e^{-\alpha t}\right)^j = \frac{N!}{n!(N-n)!} e^{-n\alpha t} \left(1-e^{-\alpha t}\right)^{N-n}$$

$X_t \sim \mathrm{Bin}(N, e^{-\alpha t})$

Interpretation of $X_t \sim Bin(n, e^{-\alpha t})$

Consider the following process: Let ξ_i, $i = 1 \dots N$, be i.i.d. r.v.s, $\xi_i \sim Exp(\alpha)$. Denote by X_t the number of ξ_i's that are bigger than t (ξ_i is the lifetime of an individual, X_t = size of the population at t). $X_0 = N$.



Then: $S_k \sim Exp(\alpha k)$, independent

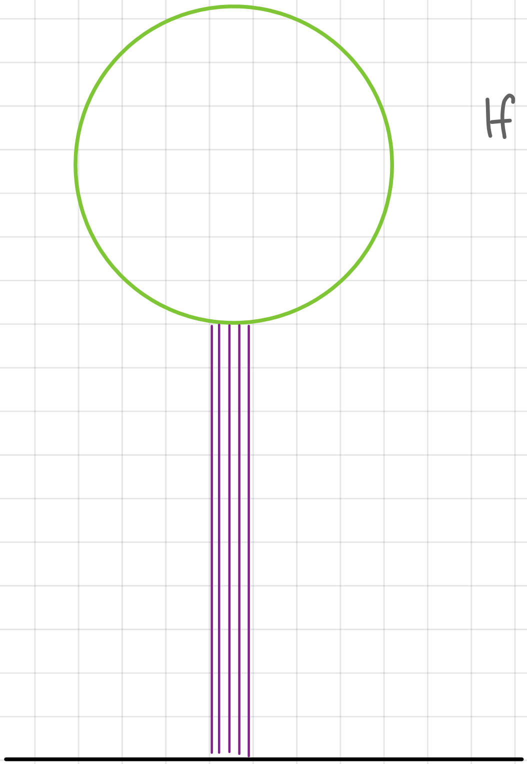
$\hookrightarrow (X_t)_{t \geq 0}$ is a pure death process

Probability that an individual survives to time t is $e^{-\alpha t}$

Probability that exactly n individuals survive to time t is

$$\binom{N}{n} e^{-\alpha t n} (1 - e^{-\alpha t})^{N-n} = P(X_t = n)$$

Example. Cable

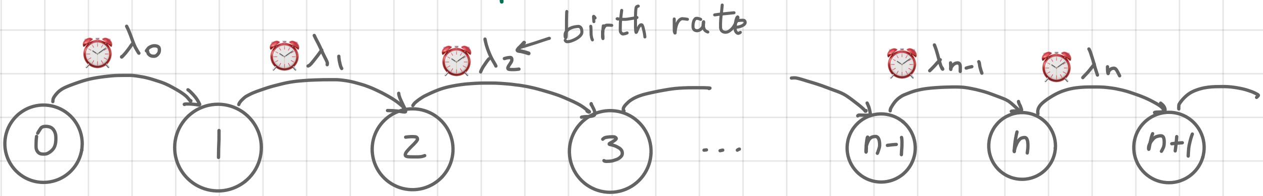


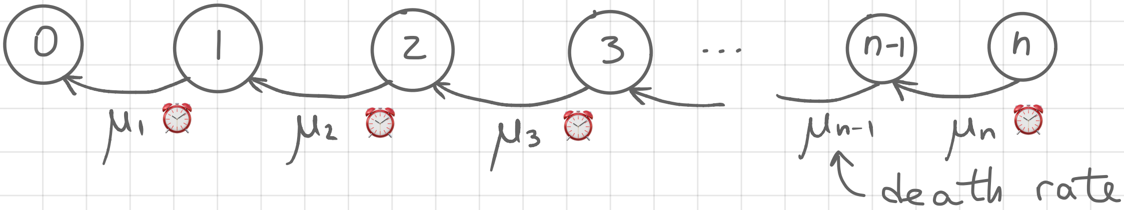
X_t = number of fibers in the cable

If a fiber fails, then this increases the load on the remaining fibers, which results in a shorter lifetime.

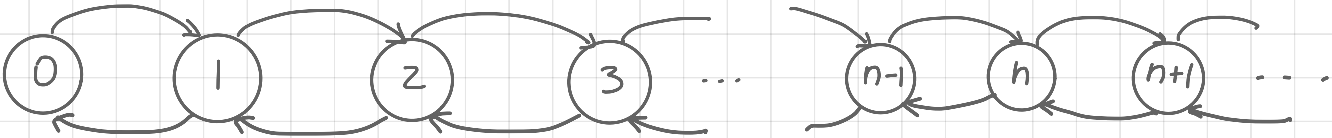
↳ pure death process

Birth and death processes





Combine both



Birth and death processes

Infinitesimal definition

<u>Def</u>. Let $(X_t)_{t\geq 0}$ be a continuous time MC, $X_t \in \{0,1,2,\dots\}$ with stationary transition probabilities. Then $(X_t)_{t\geq 0}$ is called a birth and death process with birth rates (λ_k) and death rates (μ_k) if

1. $P_{i,i+1}(h) = \lambda_i h + o(h)$
2. $P_{i,i-1}(h) = \mu_i h + o(h)$
3. $P_{i,i}(h) = 1 - (\lambda_i + \mu_i) h + o(h)$
4. $P_{ij}(0) = \delta_{ij}$ $\quad \left(P(X_0 = j \mid X_0 = i) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \right)$
5. $\mu_0 = 0$, $\lambda_0 > 0$, $\lambda_i, \mu_i > 0$

Example: Linear growth with immigration

Dynamics of a certain population is described by the following principles:

during any small period of time of length h

- each individual gives birth to one new member with probability $\beta h + o(h)$ independently of other members;
- each individual dies with probability $\alpha h + o(h)$ independently of other members;
- one external member joins the population with probability $a h + o(h)$

Can be modeled as a Markov process

Example: Linear growth with immigration

Let $(X_t)_{t\geq 0}$ denote the size of the population.

Using a similar argument as for the Yule/pure death models:

- $P_{n,n+1}(h) = \underbrace{n\beta h}_{\text{pure birth growth}} + \underbrace{ah}_{\text{immigration growth}} + o(h)$

- $P_{n,n-1}(h) = n\alpha h + o(h)$

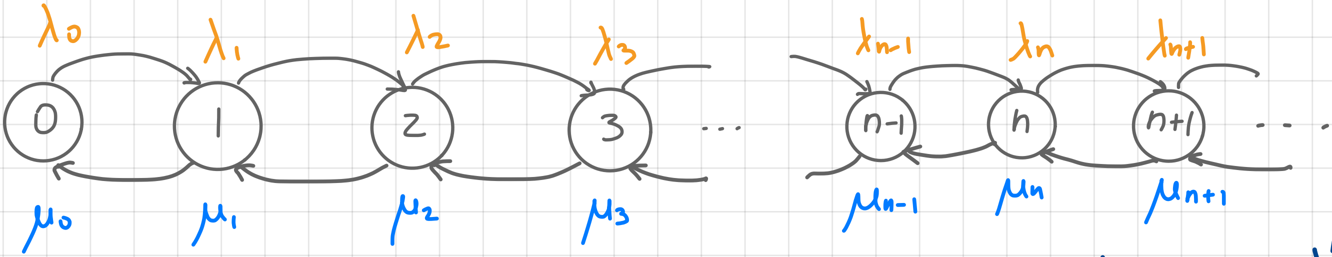
- $P_{n,n}(h) = 1 - (n\beta h + ah + n\alpha h) + o(h)$

↳ birth and death process with

$$\lambda_n = n\beta + a$$

$$\mu_n = n\alpha$$

Alternative (jump and hold) characterization



$\lambda = \mu = 1 \qquad \lambda' = \mu' = 2$

$$Z = \frac{\lambda}{\lambda+\mu} = \frac{\mu}{\lambda+\mu} = \frac{\lambda'}{\lambda'+\mu'} = \frac{\mu'}{\lambda'+\mu'}$$

$\text{Exp}(2) \qquad \text{Exp}(4)$

Sojourn times S_k are independent,

Each transition has two parts

- wait in state i for time $\sim \text{Exp}(\lambda_i + \mu_i)$
- then choose where to go:

go $\to (i+1)$ with probability $\dfrac{\lambda_i}{\lambda_i + \mu_i}$

go $(i-1) \leftarrow$ with pobability $\dfrac{\mu_i}{\lambda_i + \mu_i}$