MATH180C: Introduction to Stochastic Processes II

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Today: Limiting behavior

> Q&A: October 23

Next: Review

This week:

- Midterm 1 on Wednesday, October 28 (lectures 1-9)
- Homework 2 (due Friday, October 23, 11:59 PM)

Long run behavior of discrete time MC. Summary

Let $(X_n)_{n\geq 0}$ be a disrete time MC on $\{0,\dots,N\}$ with stationary transition probability matrix $P=(P_{ij})_{i,j=0}^{N}$.

- P is called regular if there exists k such that $[P^k]_{ij}>0$ for all i,j. [P is regular iff (X_n) is irreducible and aperiodic]

__Thm.__ If P is regular, then there exist $\pi_0,\dots,\pi_N\in\mathbb{R}$ s.t.

1) $\pi_i>0 \quad \forall i$
2) $\sum_{i=0}^{N}\pi_i=1$
3) $\forall j \quad \lim_{n\to\infty} P_{ij}^{(n)}=\pi_j$

$(\pi_0,\dots,\pi_n)$ is called limiting (stationary) distribution of (X_n)

$(\pi_0,\dots,\pi_n)$ is uniquely defined by the system of equations

$$\begin{cases}\pi_j=\sum_{i=0}^{N}\pi_i P_{ij}, \\ \sum_{i=0}^{N}\pi_i=1\end{cases}$$

$$(\pi_0,\pi_1,\dots,\pi_N)=(\pi_0,\dots,\pi_N)\,P$$

$$P=\begin{pmatrix}0&1\\1&0\end{pmatrix}$$

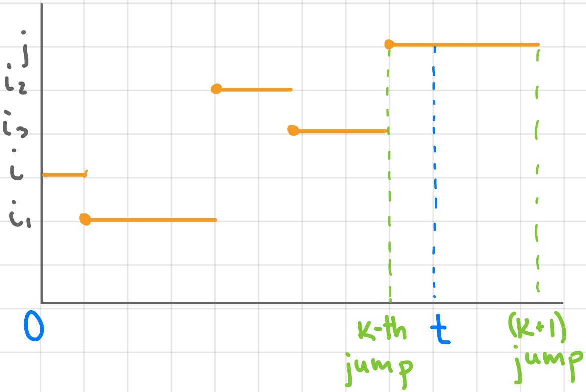
Long run behavior of continuous time MC.

Let $(X_t)_{t\geq 0}$ be a continuous time MC, $X_t \in \{0, \dots, N\}$ and let $(Y_n)_{n\geq 0}$ be the embedded jump chain.

<u>Def</u>. $(X_t)_{t\geq 0}$ is called irreducible if its jump chain $(Y_n)_{n\geq 0}$ is irreducible (consisting of one communicating class)

<u>Thm</u>. If $(X_t)_{t\geq 0}$ is irreducible, then

$$P_{ij}(t) > 0 \text{ for all } i,j \text{ and for all } t > 0$$



Idea of the proof:

- Y_n is irreducible $\Rightarrow \exists\, i_1, \dots, i_{k-1}$ s.t.

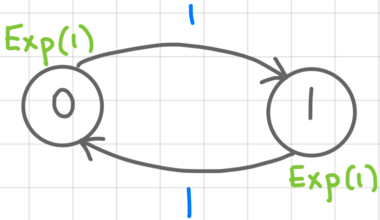
  $P(Y_k = j, Y_{k-1} = i_{k-1}, \dots, Y_1 = i_1 \mid Y_0 = i) > 0$
- $P(k\text{-th jump} \leq t < (k+1)\text{-th jump}) > 0 \quad \forall t > 0$

Long run behavior of continuous time MC

Remarks: Continuous time MCs are "aperiodic"

All irreducible continuous time MCs are "regular"

Example.



$(Y_n)_{n\geq 0}$ has period 2

$P(X_t = 0 \mid X_0 = 0) \geq P(S_0 > t) = e^{-t} > 0$

Thm. If $(X_t)_{t\geq 0}$ is irreducible, then there exists $\pi_0, \dots, \pi_N$

1) $\pi_i > 0$, $\sum_{i=0}^{N} \pi_i = 1$

2) $\lim_{t\to\infty} P_{ij}(t) = \pi_j$ for all i

3) $\pi = (\pi_0, \dots, \pi_N)$ is uniquely determined by $\pi Q = 0$ and 1)

π is called limiting/stationary/equilibrium distribution of (X_t)

Long run behavior of continuous time MC

Remark about 3): $\pi Q = 0$ is equivalent to $\pi P(t) = \pi \quad \forall t$

($\Rightarrow$) If $\pi Q = 0$, then using Kolmogorov backward equation

$$(\pi P(t))' = \pi P'(t) = \pi Q P(t) = 0$$

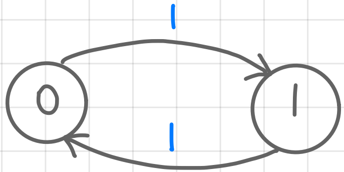
so $\pi P(t)$ is independent of t. Since $P(0) = I$, we get

$$\forall t \qquad \pi P(t) = \pi P(0) = \pi$$

($\Leftarrow$) If $\pi P(t) = \pi$, then $(\pi P(t))' = 0$. Using Kolmogorov forward equation

$$0 = (\pi P(t))' = \pi P'(t) = \pi P(t) Q = \pi Q$$

Example: Two-state MC



$$(\pi_0 \ \pi_1)\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} = (0 \ 0) \quad \Bigg| \Rightarrow \quad \begin{aligned} -\pi_0 + \pi_1 &= 0 \\ \pi_0 - \pi_1 &= 0 \\ \pi_0 + \pi_1 &= 1 \end{aligned} \quad \Bigg| \Rightarrow \pi_0 = \pi_1 = \frac{1}{2}$$

From Lecture 5: if $Q = \begin{pmatrix} -\alpha & \alpha \\ \beta & -\beta \end{pmatrix}$, then

$$P(t) = I + \frac{1}{\alpha+\beta} Q - \frac{1}{\alpha+\beta} e^{-(\alpha+\beta)t} Q \xrightarrow[t\to\infty]{} I + \frac{1}{\alpha+\beta} Q$$

. If $\alpha = \beta = 1$

$$\lim_{t\to\infty} P(t) = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

Note, that the jump process (Y_n) does not have limiting distribution! $\tilde{P}^{Y_n} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

Long run behavior of discrete time MC. Summary (2)

Let $(X_n)_{n\geq 0}$ be a disrete time MC on $\{0,1,\dots\}$ with stationary transition probability matrix $P=(P_{ij})_{i,j=0}^{\infty}$

Define $R_i = \min\{n: X_n = i\}$, $m_i = E(R_i \mid X_0 = i)$ mean duration between visits

Thm. If $(X_n)_{n\geq 0}$ is recurrent irreducible aperiodic, then

$$\lim_{n\to\infty} P_{ij}^{(n)} = \frac{1}{m_j} \quad \forall j$$

If $\lim_{n\to\infty} P_{ij}^{(n)} > 0$ for some (all) j, then MC is positive recurrent

$\lim_{n\to\infty} P_{ij}^{(n)} = 0$ for some (all) j, then MC is null recurrent.

If (X_n) is positive recurrent, $(\pi_j)_{j=0}^{N}$, $\pi_j = \lim_{n\to\infty} P_{ij}^{(N)}$ is called stationary distribution, uniquely determined by

$$\pi_j = \sum_{i=0}^{\infty} \pi_i P_{ij} \quad \forall j, \quad \sum_{i=0}^{\infty} \pi_i = 1, \quad \pi_i > 0$$

Long run behavior of continuous time MC (2)

Let $(X_t)_{t\geq 0}$ be a continuous time MC, $X_t \in \{0, 1, \dots\}$

and let $(Y_n)_{n\geq 0}$ be the embedded jump chain.

Define $R_i = \min\{t > S_0 : X_t = i\}$,

$m_i = E(R_i \mid X_0 = i)$ — mean return time from i to i

If $m_i < \infty$, then i is positive recurrent (class property).

<u>Thm</u> 1) If $(X_t)_{t\geq 0}$ is irreducible, then

$$\lim_{t\to\infty} P_{ij}(t) = \frac{1}{q_j m_j} =: \pi_j \geq 0$$

2) $(X_t)_{t\geq 0}$ is positive recurrent iff there exists a (unique)

solution $(\pi_j')_{j=0}^{\infty}$ to $\sum_{i=0}^{\infty} \pi_i' q_{ij} = 0$, $\sum_{i=0}^{\infty} \pi_i' = 1$, $\pi_i' > 0$

in which case $\pi_j = \pi_j'$ and $(\pi_j)_{j=0}^{\infty}$ is called

limiting/stationary distribution.

Remarks

1) Until now we discussed only the transition probabilities. But in order to describe completely MC (X_t) we need also the initial / starting distribution

$$\nu = (\nu_0, \nu_1, \dots), \quad \nu_i = P(X_0 = i)$$

$$(X_t) \longleftrightarrow (\nu, Q)$$

2) Distribution of X_{t_1} is given by $\nu P(t_1)$

$$P(X_{t_1} = i) = [\nu P(t_1)]_i$$

More generally

$$P(X_0 = i_0, X_{t_1} = i_1, \dots, X_{t_n} = i_n) = P_{i_{n-1}, i_n}(t_n - t_{n-1}) \cdots P_{i_0, i_1}(t_1)\, \nu_{i_0}$$

3) Stationary distribution remains unchaged in time

$$\pi P(t) = \pi \implies \text{if } X_0 \sim \pi \text{ then } X_t \sim \pi$$

Remarks

4) Similarly as in the discrete case, π_j gives the fraction of time spent in state j in long run

$$\lim_{T\to\infty} E\left[\frac{1}{T}\int_0^T \mathbb{1}_{\{X_t=j\}}\,dt \,\middle|\, X_0=i\right]=\pi_j$$

(compare with $\lim_{m\to\infty} E\left[\frac{1}{m}\sum_{n=0}^{m-1}\mathbb{1}_{\{X_n=j\}} \,\middle|\, X_0=i\right]=\pi_j$ for discrete time MC)

5) If we can find $(\pi_i)_{i=0}^{\infty}$ such that $\underbrace{\pi_i q_{ij}=\pi_j q_{ji},\ i\neq j}_{\text{detailed balance eq.}}$

then $(\pi_i)_{i=0}^{\infty}$ satisfies $\pi Q=0$

Indeed, $\sum_{j=0}^{\infty}\pi_i q_{ij}=\pi_i\sum_{j=0}^{\infty}q_{ij}=0=\sum_{j=0}^{\infty}\pi_j q_{ji}=(\pi Q)_i$

Example: Birth and death processes

If we consider the birth and death process, the equation $\pi Q = 0$ takes the following form

$$-\pi_0 \lambda_0 + \pi_1 \mu_1 = 0 \qquad \pi_1 = \frac{\lambda_0}{\mu_1} \pi_0 = \theta_1 \pi_0$$

$$\pi_0 \lambda_0 - \pi_1 (\lambda_1 + \mu_1) + \pi_2 \mu_2 = 0 \qquad \pi_2 = \frac{\lambda_1}{\mu_2} \pi_1 = \frac{\lambda_1}{\mu_2} \cdot \frac{\lambda_0}{\mu_1} \pi_0 = \theta_2 \pi_0$$

$$\pi_{i-1} \lambda_{i-1} - \pi_i (\lambda_i + \mu_i) + \pi_{i+1} \mu_{i+1} = 0 \qquad \pi_{i+1} = \frac{\lambda_i}{\mu_{i+1}} \pi_i = \theta_{i+1} \pi_0$$

where $\theta_i = \frac{\lambda_{i-1}}{\mu_i} \cdot \frac{\lambda_{i-2}}{\mu_{i-1}} \cdot \dots \cdot \frac{\lambda_0}{\mu_1}$, $\theta_0 = 1$.

Then, $\sum_{i=0}^{\infty} \pi_i = 1$ implies that $\left(\sum_{i=0}^{\infty} \theta_i \right) \pi_0 = 1$

If $\sum_{i=0}^{\infty} \theta_i < \infty$, then (X_t) is positive recurrent and $\pi_j = \frac{\theta_j}{\sum_{i=0}^{\infty} \theta_i}$

If $\sum_{i=0}^{\infty} \theta_i = \infty$, then $\pi_j = 0 \ \forall j$.

Example. Linear growth with immigration

Birth and death process, $\lambda_j = \lambda j + a$, $\mu_j = \mu j$ $(*)$

Using Kolmogorov's equations we showed (lecture 5) that $E(X_t) \to \frac{a}{\mu - \lambda}$, $t \to \infty$, if $\mu > \lambda$.

What is the limiting distribution of X_t?

From the previous slide, $\pi_j = \dfrac{\theta_j}{\sum_{i=0}^{\infty} \theta_i}$, $\theta_j = \dfrac{\lambda_{j-1} \cdots \lambda_0}{\mu_j \cdots \mu_1}$

If we replace λ_j, μ_j by $(*)$, we get

$$\pi_j = \left(\frac{\lambda}{\mu}\right)^j \left(1 - \frac{\lambda}{\mu}\right)^{a/\lambda} \frac{\frac{a}{\lambda}\left(\frac{a}{\lambda}+1\right) \cdots \left(\frac{a}{\lambda}+j-1\right)}{j!}, \quad j \geq 1$$

$$\pi_0 = \left(1 - \frac{\lambda}{\mu}\right)^{-\frac{a}{\lambda}}$$

What you should know for midterm 1 (minimum):

- definition of continuous time MC, Markov property, transition probabilities, generator
- representations of MC: infinitesimal (generator), jump-and-hold, transition probabilities, rate diagram and relations between them (in particular Q and $P(t)$)
- computing absorption probabilities and mean time to absorption
- computing stationary distributions for finite and infinite state MCs and interpretation of $(\pi_i)_{i=0}^{\infty}$
- basic properties of birth and death processes