

Math 180A: Introduction to Probability

Lecture A00 (Au)

math.ucsd.edu/~bau/w21.180a

Lecture B00 (Nemish)

math.ucsd.edu/~ynemish/teaching/180a

Today: ASV 3.4 (Variance)

ASV 3.5 (Gaussian distribution)

Video: Prof. Todd Kemp, Fall 2019

Next: ASV 3.5, 4.1

Week 5: Quiz 3 (Wednesday, Feb 3, covering Lectures 8-10)

Homework 4 (due Sunday, Feb 7)

Regrades for Homework 2 (Feb 3-5)

Variance

3.4

Definition: The variance of a random variable X is

$$\text{Var}(X) = \mathbb{E}[(X - \mathbb{E}(X))^2]$$

I.e., first compute $\mu = \mathbb{E}(X)$. Then apply the function $g(x) = (x - \mu)^2$ to X , and compute $\mathbb{E}(g(X))$.

* If X is discrete, $\text{Var}(X) = \sum_{t=-\infty}^{\infty} (t - \mu)^2 P(X=t)$

* If X is continuous, $\text{Var}(X) = \underline{\int_{-\infty}^{\infty} (t - \mu)^2 f_X(t) dt}$

In any case, $\text{Var}(X) \geq 0$. Its square root is

$$\sigma(X) = \sqrt{\text{Var}(X)}$$
 is called standard deviation.

E.g. If $X \sim \text{Ber}(p)$, $\mathbb{E}(X) = p$, \therefore

$$\begin{aligned} \text{Var}(X) &= \mathbb{E}((X-p)^2) = \sum_t (t-p)^2 P(X=t) \\ &= (1-p)p[1-p+p] \\ &= \boxed{(1-p)p} \end{aligned}$$

$p = \frac{1}{2}$
 $\text{Var} = \frac{1}{4}$
 $S = \frac{1}{2}$

$= (1-p)^2 p + p^2 (1-p)$
 $= (1-p)^2 p + (1-p)^2 p$

E.g. If $U \sim \text{Unif}([a,b])$, $\mathbb{E}(U) = \frac{a+b}{2}$, \therefore

$$\text{Var}(U) = \int_{-\infty}^{\infty} \left(t - \frac{a+b}{2}\right)^2 f_U(t) dt = \int_a^b \left(t - \frac{a+b}{2}\right)^2 \cdot \frac{1}{b-a} dt$$

$$\begin{aligned} \frac{(b-a)^3}{12(b-a)} &= \frac{(b-a)^2}{12} \\ (\text{St. Dev.}) &= \frac{|b-a|}{\sqrt{12}} \end{aligned}$$

$\frac{1}{b-a}$ on $[a,b]$
otherwise

$$\begin{aligned} &= \frac{1}{3} \left(t - \frac{a+b}{2}\right)^3 \Big|_{t=a}^{t=b} \cdot \frac{1}{b-a} \\ &= \frac{1}{3} \cdot \frac{1}{b-a} \left[\left(b - \frac{a+b}{2}\right)^3 - \left(a - \frac{a+b}{2}\right)^3 \right] \\ &= \frac{1}{3} \cdot \frac{1}{b-a} \left[\left(\frac{b-a}{2}\right)^3 - \left(\frac{a-b}{2}\right)^3 \right] \\ &= \frac{1}{3} \cdot \frac{1}{b-a} \cdot 2(b-a)^3 \end{aligned}$$

Variance is a measure of how "spread out from the mean" the distribution is. For example:

Theorem: Let X be a random variable with finite expectation $E(X) = \mu$. Then

$$\text{Var}(X) = 0 \quad \text{iff} \quad P(X = \mu) = 1 .$$

(i.e. $X \equiv \mu$.)

Pf. (\Leftarrow) If $P(X = \mu) = 1$. So X is discrete.

$$\therefore \text{Var}(X) = E((X - \mu)^2) = (\mu - \mu)^2 P(X = \mu) = 0 .$$

(\Rightarrow) [Here we will assume X is discrete, for now.]

$$0 = \text{Var}(X) = \sum_t \underbrace{(t - \mu)^2}_{\geq 0} \underbrace{P(X = t)}_{\geq 0}$$

\therefore For each t , either $(t - \mu)^2 = 0$ or $P(X = t) = 0$.

If $t \neq \mu \Rightarrow P(X = t) = 0$.

⇒ for each t , $(t - \mu)^2 \cdot P(X = t) = 0$. //

Chebyshov's Inequality

If X has finite mean $E(X) = \mu$ and finite variance $\text{Var}(X) = \sigma^2$, then

$$P(|X - \mu| \geq k\sigma) \leq \frac{1}{k^2}$$

Two ideas in the proof:

(1) Indicators. $\mathbb{1}_A = \begin{cases} 1 & \text{if } A \text{ happens} \\ 0 & \text{if } A \text{ does not} \end{cases}$

$$\therefore E(\mathbb{1}_A) = P(A)$$

$\mathbb{1}_A$ is a 0-1 valued r.v.
It is a $\text{Ber}(p)$ with
 $p = P(A)$

(2) Monotonicity. FACT: if $X \leq Y$ then $E(X) \leq E(Y)$

(Intuitive; for proof, we must wait until week 8.)

Proof of Chebyshov: $P(|X - \mu| \geq k\sigma) = E(\mathbb{1}_{\{|X - \mu| \geq k\sigma\}})$

$$\Leftrightarrow |X - \mu| \geq k\sigma \quad \left| \begin{array}{l} \text{if } Y \geq 1 \\ \left(\frac{X - \mu}{k\sigma} \right)^2 \geq 1 \end{array} \right. \quad \text{then } Y \geq \mathbb{1}_{\{Y \geq 1\}}$$

$$\begin{aligned} &= E(\mathbb{1}_{\left(\frac{(X-\mu)}{k\sigma} \right)^2 \geq 1}) \\ &\leq E \left[\left(\frac{(X-\mu)}{k\sigma} \right)^2 \right] = \frac{1}{k^2\sigma^2} E[(X-\mu)^2] \end{aligned}$$

Normal (Gaussian) Distribution

3.5

The standard normal distribution $\mathcal{N}(0, 1)$
is given by the density

$$f(t) = \frac{1}{\sqrt{2\pi}} e^{-t^2/2}$$

Check:

$$I = \int_{-\infty}^{\infty} e^{-t^2/2} dt = \sqrt{2\pi}$$

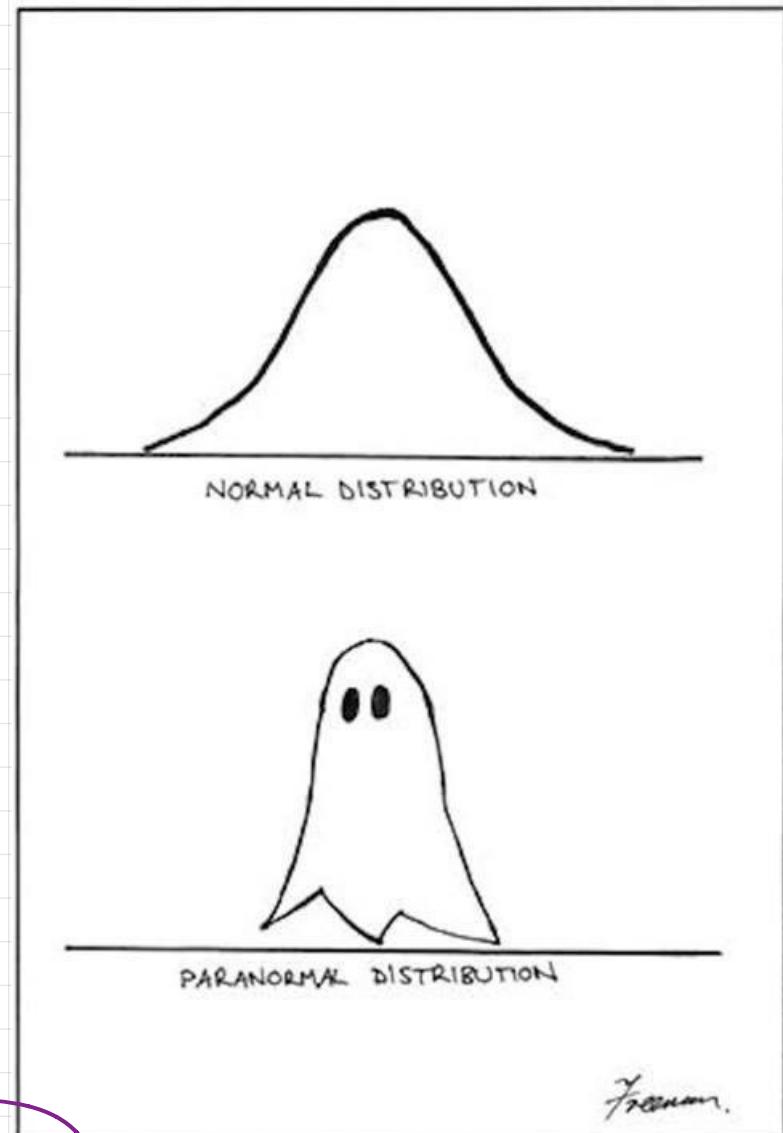
Polar Coords:

$$(x, y) \rightarrow (r, \theta)$$

$$dx dy \rightarrow r dr d\theta$$

$$x^2 + y^2 = r^2$$

$$\begin{aligned} I^2 &= \int_{-\infty}^{\infty} e^{-x^2/2} dx \int_{-\infty}^{\infty} e^{-y^2/2} dy \\ &= \iint e^{-x^2/2} e^{-y^2/2} dx dy \\ &\stackrel{IR^2}{=} \int_0^{2\pi} \int_0^{\infty} r e^{-r^2/2} dr d\theta \\ &= 2\pi \cdot \left[-e^{-r^2/2} \right]_0^{\infty} = 2\pi \cdot 1 \end{aligned}$$



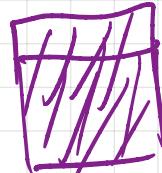
CDF of $\mathcal{N}(0,1)$

Suppose $X \sim \mathcal{N}(0,1)$. What is $P(|X| \leq 1)$?

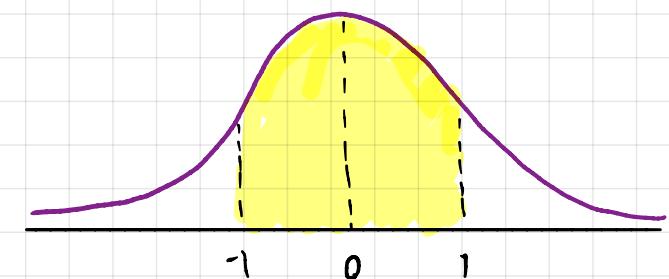
$$P(-1 \leq X \leq 1)$$

$$= \int_{-1}^1 f_X(t) dt = \frac{1}{\sqrt{2\pi}} \int_{-1}^1 e^{-t^2/2} dt$$

$$\int_{-1}^1 \int_{-1}^1 \cdots$$



← not good for polar coords.



$$\Phi(x) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-t^2/2} dt$$