☐ **Write your name and PID on the top of EVERY PAGE.**

☐ **Write the solutions to each problem on separate pages. CLEARLY INDICATE on the top of each page the number of the corresponding problem.**

☐ **Remember this exam is graded by a human being. Write your solutions NEATLY AND COHERENTLY, or they risk not receiving full credit.**

☐ **You may assume that all transition probability functions are STATIONARY.**

☐ **You are allowed to use the textbook, lecture notes and your personal notes. You are not allowed to use the electronic devices (except for accessing the online version of the textbook) or outside assistance. Outside assistance includes but is not limited to other people, the internet and unauthorized notes.**

1. (25 points) Certain solar power plant has three operating modes: low intensity, medium intensity and high intensity. The transitions from one operating mode to another form a continuous time Markov chain on the states $\{L, M, H\}$ with generator $Q = \left(\begin{array}{cccc} & L & M & H \\ L & -3 & 3 & 0 \\ M & 3 & -4 & 1 \\ H & 0 & 2 & -2 \end{array}\right)$. Denote this Markov chain by $(X_t)_{t\geq 0}$

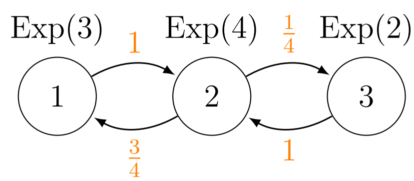
   (a) (10 points) Draw the diagram for the jump chain of $(X_t)_{t\geq 0}$ 0 and explain why $(X_t)_{t\geq 0}$ is irreducible.

   (b) (10 points) Compute the stationary distribution for $(X_t)_{t\geq 0}$.

   (c) (5 points) What is the expected average fraction of time that the plant spends in the low intensity mode in the long run?

Solution.

(a) The parameters of the jump-and-hold diagram can be read off from the generator matrix Q



(b) The stationary distribution $\pi = (\pi_L, \pi_M, \pi_H)$ is determined from the equations $\pi Q = 0$ and $\pi_L + \pi_M + \pi_H = 1$.

$$-3\pi_L + 3\pi_M = 0, \tag{1}$$
$$3\pi_L - 4\pi_M + 2\pi_H = 0, \tag{2}$$
$$\pi_M - 2\pi_H = 0, \tag{3}$$
$$\pi_L + \pi_M + \pi_H = 1. \tag{4}$$

The first and third equations give $\pi_L = \pi_M$ and $\pi_M = 2\pi_H$. Plugging this into the last equation gives

$$\pi_H = 0.2, \quad \pi_M = 0.4, \quad \pi_L = 0.4. \tag{5}$$

(c) The average fraction of time spent in the low intensity state in the long run is given by (see lecture 9, page 10) $\pi_L = 0.4$.

2. (25 points) Let $(N_t)_{t\geq 0}$ be a renewal process with the interrenewal times uniformly distributed on the interval $[0, 12]$. Compute the asymptotic expression (linear and constant terms) of the expected number of renewals up to time t for $t \gg 1$.

Solution. Denote the interrenewal times by X_i and the expected number of renewals up to time t by $M(t) := E(N(t))$ (renewal function).

The asymptotic behavior of $M(t)$ is given by (lecture 14-15, page 3)

$$\lim_{t\to\infty} \frac{M(t)}{t} = \frac{1}{\mu}, \tag{6}$$

i.e., $M(t) \approx \frac{t}{\mu}$ for large t, where $\mu = E(X_i) = 6$.

Since the interrenewal times have finite variance $\sigma^2 = 12$, we can use the theorem on page 4 of the same lecture

$$\lim_{t\to\infty} \left(M(t) - \frac{t}{6}\right) = \frac{\sigma^2 - \mu^2}{2\mu^2} = \frac{12 - 36}{2 \cdot 36} = -\frac{1}{3}. \tag{7}$$

We conclude that for large t

$$M(t) \approx \frac{t}{6} - \frac{1}{3}. \tag{8}$$

3. (25 points) Let τ_1 be the smallest zero of a standard Brownian motion that exceeds $b > 0$. Compute $P(\tau_1 < t)$ for $t > b$.

Solution. Let $(B_t)_{t\geq 0}$ be a standard Brownian motion and let $t > b$. The event $\{\tau_1 < t\}$ means that B has a zero on the interval (b, t). Using the theorem about the distribution of zero of Brownian motion (lecture 21 page 9) we get that

$$P(\tau_1 < t) = P(B \text{ has zero on } (b,t)) = \frac{2}{\pi} \arccos \sqrt{b/t}. \tag{9}$$

4. (25 points) Let X and Y be two random variables. Suppose that Y has exponential distribution with rate $\lambda > 0$, and suppose that given $Y = y$, $y > 0$, the random variable X has normal distribution with mean y and variance 1.

   (a) (15 points) Compute $E(X)$.

   (b) (10 points) Compute $P(X > Y)$.

Solution.

(a) Compute $E(X)$ by conditioning on the value of Y:

$$E(X) = \int_0^\infty E(X \mid Y = y) \lambda e^{-\lambda y} dy = \int_0^\infty y \lambda e^{-\lambda y} dy = \frac{1}{\lambda}. \tag{10}$$

(b) Compute $P(X > Y)$ by conditioning on the value of Y:

$$P(X > Y) = \int_0^\infty P(X > Y \mid Y = y) \lambda e^{-\lambda y} dy = \int_0^\infty \frac{1}{2} \lambda e^{-\lambda y} dy = \frac{1}{2}. \tag{11}$$