

MATH 180A: Introduction to Probability

Lecture A00 (Au)

www.math.ucsd.edu/~bau/w21.180a

Lecture B00 (Nemish)

www.math.ucsd.edu/~ynemish/teaching/180a

Today: Independence of random variables. Convolutions

Next: ASV 8.2-8.3

Video: Prof. Todd Kemp, Fall 2019

Week 9:

- Homework 7 (due Friday, March 5)

Joint Distributions & Independence

6.3

Suppose $\underline{X} = (X_1, \dots, X_n)$ is jointly continuous. Then

X_1, \dots, X_n are independent iff $f_{X_1, \dots, X_n}(x_1, \dots, x_n) = f_{X_1}(x_1) \cdots f_{X_n}(x_n)$.

(\Rightarrow) Special Case $n=2$)

Suppose $\downarrow X, Y$ are independent. f_X, f_Y .

$$P(X \leq s, Y \leq t) = P(X \leq s) P(Y \leq t)$$

$$\text{P}((X, Y) \in (-\infty, s] \times (-\infty, t]) = \int_{-\infty}^s \int_{-\infty}^t f_{X,Y}(x, y) dx dy$$

↑ equal for all $s, t.$

Now take $\frac{\partial^2}{\partial s^2} t$ of both sides.

E.g. Multivariate standard normal $\frac{1}{(2\pi)^{n/2}} e^{-\|x\|^2/2}$

$$= \left(\frac{1}{\sqrt{2\pi}} e^{-x_1^2/2}\right) \cdots \left(\frac{1}{\sqrt{2\pi}} e^{-x_n^2/2}\right)$$

Since the joint density is a product of single-variable densities, the components of the random vector are independent, with those densities as marginals:

I.e. if \underline{X} is a multivariate standard normal, $\underline{X} = (X_1, \dots, X_n)$
then X_1, \dots, X_n are independent $N(0, 1)$'s.

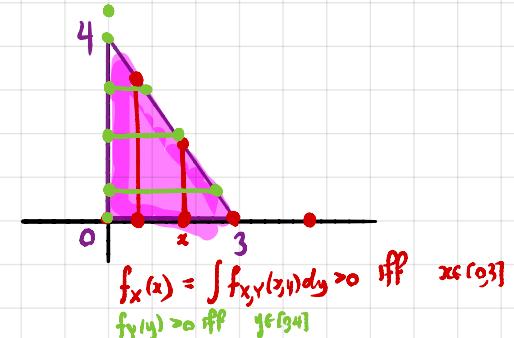
Q: Suppose \underline{X} is uniform on the triangle

(X, Y) Are X, Y independent?

No!

$$f_{(X,Y)}(x,y) = f_X(x)f_Y(y)$$

$\begin{cases} 1/6 & \text{on } \Delta \\ 0 & \text{off } \Delta \end{cases}$



$$\text{Support}(f_X) = \{x : f_X(x) \neq 0\} = [0, 3]$$

$$\text{Support}(f_Y) = \{y : f_Y(y) \neq 0\} = [0, 4]$$

$$\therefore \text{Support } f_{X,Y} = [0, 3] \times [0, 4]$$

Example. Suppose $X \sim \text{Exp}(\lambda)$, $Y \sim \text{Exp}(\mu)$, and X, Y independent.
 Find the distribution of $\min(X, Y)$.

$$F_{\min(X,Y)}(t) = P(\min(X, Y) \leq t) = 1 - P(\min(X, Y) > t)$$



$$\begin{aligned}\therefore &= 1 - e^{-(\lambda+\mu)t} \\ &= F_Z(t)\end{aligned}$$

where $Z \sim \text{Exp}(\lambda+\mu)$

$$\underbrace{\{ \min(X, Y) > t \}}$$

$$= \{ X > t, Y > t \}$$

$$\therefore P(X > t, Y > t)$$

$$= P(X > t)P(Y > t)$$

$$= e^{-\lambda t} e^{-\mu t}$$

$$= e^{-(\lambda+\mu)t}$$

Theorem. Suppose $\underline{X} = (X_1, \dots, X_n)$ is a random vector, with X_1, \dots, X_n independent.

Eg. X, Y indep.

$$\mathbb{E}(X^2Y)$$

$$= \mathbb{E}(X^2) \mathbb{E}(Y)$$

For any functions $g_1, g_2, \dots, g_n : \mathbb{R} \rightarrow \mathbb{R}$,

$$\mathbb{E}(g_1(X_1)g_2(X_2) \cdots g_n(X_n)) = \mathbb{E}(g_1(X_1)) \cdot \mathbb{E}(g_2(X_2)) \cdots \mathbb{E}(g_n(X_n))$$

Pf. ($n=2$, jointly continuous)

$$\mathbb{E}(g(x)h(y)) = \iint_{\mathbb{R}^2} g(x) h(y) f_{(X,Y)}(x,y) dx dy$$

$$= f_X(x) f_Y(y)$$

$$= \int_{-\infty}^{\infty} g(x) f_X(x) dx \cdot \int_{-\infty}^{\infty} h(y) f_Y(y) dy = \mathbb{E}(g(X)) \cdot \mathbb{E}(h(Y)).$$

In particular, $\mathbb{E}(X_1 X_2 \cdots X_n) = \mathbb{E}(X_1) \mathbb{E}(X_2) \cdots \mathbb{E}(X_n)$.

Actually, the last theorem is iff.

↪ Suppose we know

$$\mathbb{E}(g_1(X_1)g_2(X_2)\cdots g_n(X_n)) = \mathbb{E}(g_1(X_1))\mathbb{E}(g_2(X_2))\cdots \mathbb{E}(g_n(X_n))$$

for all functions g_1, g_2, \dots, g_n . Fix $A_1, \dots, A_n \subset \mathbb{R}$.

Just take $g_1 = \mathbb{1}_{A_1}, \dots, g_n = \mathbb{1}_{A_n}$

$$\mathbb{E}(g_1(X_1)) = \mathbb{E}(\mathbb{1}_{A_1}(X_1)) = P(X_1 \in A_1).$$

↪ Ber(p) where $p = P(X_1 \in A_1)$

$$g_1(X_1)g_2(X_2) = \mathbb{1}_{A_1}(X_1)\mathbb{1}_{A_2}(X_2)$$

$$= \underbrace{\{0 \text{ if } X_1 \notin A_1\}}_{1 \text{ if } X_1 \in A_1} \cdot \underbrace{\{0 \text{ if } X_2 \notin A_2\}}_{1 \text{ if } X_2 \in A_2} = \begin{cases} 0 & \text{otherwise} \\ 1 & X_1 \in A_1 \text{ and } X_2 \in A_2 \end{cases}$$

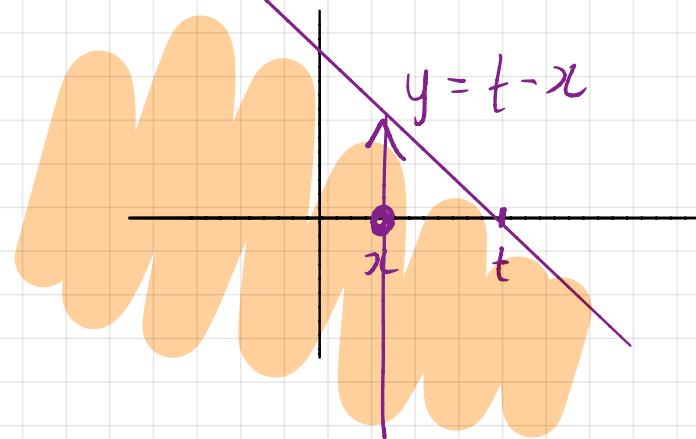
$$\therefore \mathbb{E}(\cdots) = P(X_1 \in A_1, X_2 \in A_2)$$

Convolution

Let X, Y be independent random variables. What can we say about the distribution of $X+Y$?

$$\begin{aligned} F_{X+Y}(t) &= P(X+Y \leq t) \\ &= P((X, Y) \in T_t) \\ &= \iint f_{(X,Y)}(x, y) dx dy \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{t-x} f_{(X,Y)}(x, y) dy dx \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{t-x} f_X(x) f_Y(y) dy dx \end{aligned}$$

$$T_t = \{(x, y) : x+y \leq t\}$$



$$= \int_{-\infty}^{\infty} f_X(x) \int_{-\infty}^{t-x} f_Y(y) dy$$

$$\therefore f_{X+Y}(t) = \frac{d}{dt} \left(\int_{-\infty}^{\infty} f_X(x) f_Y(t-x) dx \right)$$

The Convolution of two prob. densities is $(f * g)(t) = \int_{-\infty}^{\infty} f_X(x) f_Y(t-x) dx$

E.g. $X, Y \sim \text{Exp}(\lambda)$, independent. Find f_{X+Y} .

$$f_{X+Y}(t) = f_X * f_Y(t) = \int_{-\infty}^{\infty} f_X(x) f_Y(t-x) dt$$

$$\begin{aligned} &= \int_{-\infty}^{\infty} \lambda e^{-\lambda x} \underbrace{\lambda e^{-\lambda(t-x)} \mathbb{1}_{x \geq 0}}_{0 \leq x \leq t} dx \\ &\quad \downarrow \quad \quad \quad \downarrow \\ &= \int_0^t \lambda e^{-\lambda x} \cdot \lambda e^{-\lambda(t-x)} dx \end{aligned}$$

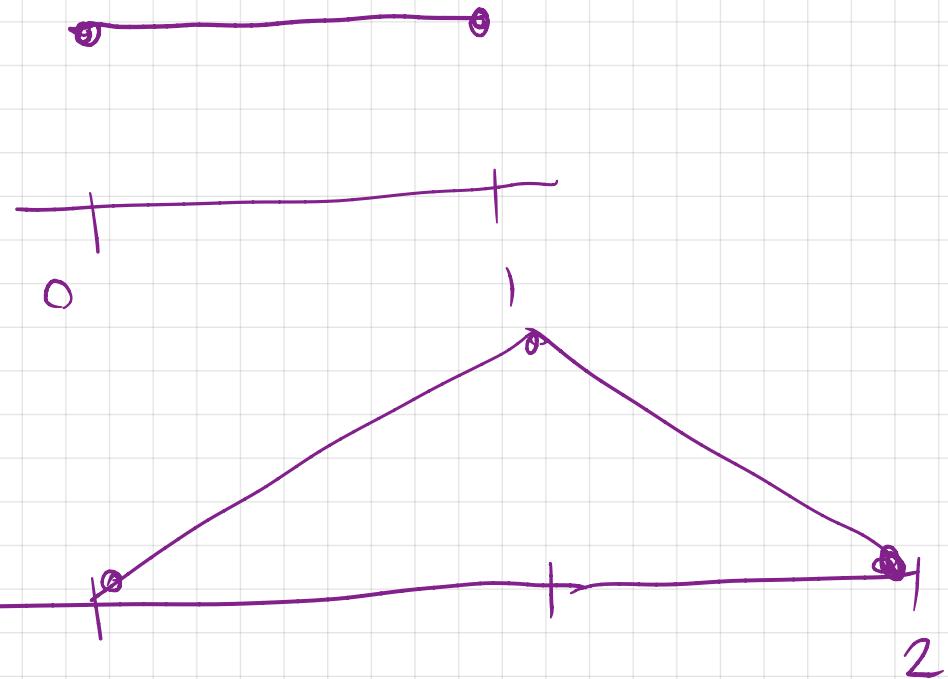
$$= \int_0^t \cancel{\lambda e^{-\lambda x} \cdot \lambda e^{-\lambda(t-x)}} dx$$

$$= \int_0^t \lambda^2 e^{-\lambda t} dx = \lambda^2 t e^{-\lambda t} \Big|_{t=0}^{t=t} = \lambda^2 t e^{-\lambda t} \mathbb{1}_{t \geq 0}$$

E.g. $X, Y \sim \text{Unif}([0,1])$, independent.

$$f_X, f_Y$$

$$f_{X+Y}$$



$$\mathbb{1}_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$$

E.g. $X \sim \text{Unif}[0,1]$, $f_X(x) = \mathbb{1}_{[0,1]}(x)$