MATH180C: Introduction to Stochastic Processes II

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Today: General continuous time MC. Q-matrices. Matrix exponentials

> Q&A: October 16

Next: PK 6.3, 6.6, Durrett 4.2

Week 2:

- No homework!

Q-matrices (infinitesimal generators)

Let $S=\{0,1,\dots,N\}$. We call $Q=(q_{ij})_{i,j=0}^{N}$ a Q-matrix if Q satisfies the following conditions:

(a) $0 \le -q_{ii} < \infty$ for all i

(b) $q_{ij} \ge 0$ for all $i \ne j$

(c) $\sum_j q_{ij} = 0$ for all i

Denote $q_i = \sum_{j\ne i} q_{ij}$

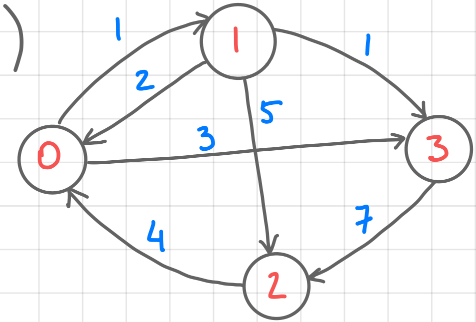
then $q_{ii} = -q_i$

Examples

(a)

$$Q = \begin{pmatrix} -2 & 1 & 1 \\ 2 & -7 & 5 \\ 0 & 2 & -2 \end{pmatrix}$$

(b)



$$\begin{array}{c|c} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{pmatrix} -4 & 1 & 0 & 3 \\ 2 & -8 & 5 & 1 \\ 4 & 0 & -4 & 0 \\ 0 & 0 & 7 & -7 \end{pmatrix} \end{array}$$

Matrix exponentials

Let $Q=(q_{ij})_{i,j=1}^{N}$ be a matrix. Then the series $\sum_{k=0}^{\infty}\frac{Q^k}{k!}$ converges componentwise, and we denote its sum $\sum_{k=0}^{\infty}\frac{Q^k}{k!} =: e^{Q}$, the matrix exponential of Q.

In particular, we can define $e^{tQ}=\sum_{k=0}^{\infty}\frac{Q^k t^k}{k!}$ for $t\geq 0$.

Thm. Define $P(t)=e^{tQ}$. Then

(i) $P(t+s)=P(t)P(s)$ for all s,t

(ii) $(P(t))_{t\geq 0}$ is the unique solution to the equations

$$\begin{cases}\frac{d}{dt}P(t)=P(t)Q\\ P(0)=I\end{cases}, \quad \text{and} \quad \begin{cases}\frac{d}{dt}P(t)=QP(t)\\ P(0)=I\end{cases}$$

Main theorem

Let $P(t)$ be a matrix-valued function $t \geq 0$.

Consider the following properties

(a) $P_{ij}(t) \geq 0$, $\sum_j P_{ij}(t) = 1$ for all $i, j, t \geq 0$

(b) $P(0) = I$

(c) $P(t+s) = P(t)P(s)$ for all $t, s \geq 0$

(d) $\lim_{t \downarrow 0} P(t) = I$ (continuous at 0)

<u>Theorem A.</u> $P(t)$ satisfies (a)–(d)

if and only if

Main theorem. Remarks

This theorem establishes one-to-one correspondance between matrices $P(t)$ satisfying (a)-(d) and the Q-matrices of the same dimension.

Remarks

1. Conditions (a)-(d) imply that $P(t)$ is differentiable

2. If $P(t) = e^{tQ}$, then $P(h) =$ as $h \to 0$

$$P(h) =$$

Q-matrices and Markov chains

Let $(X_t)_{t\geq 0}$ be a continuous time MC, $X_t \in \{0,1,\dots,N\}$
with right-continuous sample paths

Denote $P_{ij}(t) = P(X_t = j \mid X_0 = i)$, $i,j \in \{0,1,\dots,N\}$

stationary

Then

- $P_{ij}(t)$, $\sum_{j=0}^{N} P_{ij}(t)$ $\left(= \sum_{j=0}^{N} P(X_t = j \mid X_0 = i) \right)$

- $P_{ij}(0) =$ $\left(P(X_0 = j \mid X_0 = i) = \begin{cases} 1, & j = i \\ 0, & j \neq i \end{cases} \right)$

- $P_{ij}(t+s) = P(X_{t+s} = j \mid X_0 = i)$

- $\lim_{h \downarrow 0} P(X_h = j \mid X_0 = i) =$

Q-matrices and Markov chains (cont.)

$P(t)$ satisfies properties (a)-(d) from Theorem A.

$\Longrightarrow$ there is a Q-matrix Q such that

$$P(t) =$$

In particular,

$$P(h) =$$

This implies the one-to-one correspondance between Q-matrices and continuous time MC with right-continuous sample paths.

Q is called the infinitesimal generator of $(X_t)_{t\geq 0}$

Infinitesimal description of cont. time MC

Let $Q = (q_{ij})_{i,j=0}^{N}$ be a Q-matrix, let $(X_t)_{t\geq 0}$ be right-continuous stochastic process, $X_t \in \{0, 1, \dots, N\}$.

We call $(X_t)_{t\geq 0}$ a Markov chain with generator Q, if

(i) $(X_t)_{t\geq 0}$ satisfies the Markov property

(ii) $P(X_{t+h} = j \mid X_t = i) =$

Example

Pure death process

- $P_{i,i-1}(h) = \mu_i h + o(h)$
- $P_{ii}(h) = 1 - \mu_i h + o(h)$
- $P_{ij}(h) = o(h)$ for $j \notin \{i-1, i\}$

The corresponding Q-matrix

$$Q = \begin{pmatrix} \\ \\ \\ \end{pmatrix}$$

Sojourn time description

Let $Q = (q_{ij})_{i,j=0}^{N}$ be a Q-matrix. Denote $q_i = \sum_{j \neq i} q_{ij}$

so that

$$Q = \begin{pmatrix} & q_{01} & q_{02} & \cdots \\ q_{10} & & q_{12} & \cdots \\ q_{20} & q_{21} & & \cdots \\ \vdots & \vdots & & \ddots \end{pmatrix}$$

$q_0 = \sum_{i \neq 0} q_{0i}$

$\vdots$

Denote $Y_k := X_{W_k}$ (jump chain).

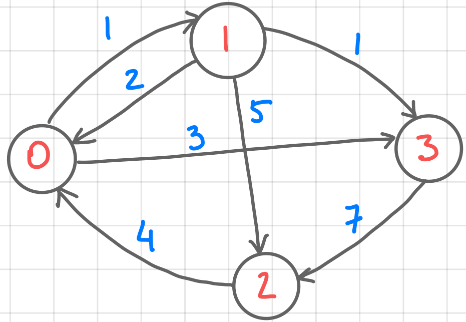
Then the MC with generator matrix Q has the following equivalent jump and hold description

- sojourn times S_k are independent r.v. with $P(S_k > t \mid Y_k = i) =$

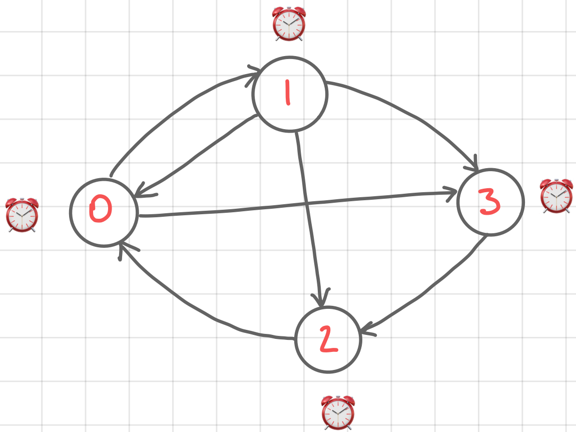
- transition probabilities $P(Y_{k+1} = j \mid Y_k = i) =$

Example

$$
\begin{array}{c|cccc}
 & 0 & 1 & 2 & 3 \\
\hline
0 & -4 & 1 & 0 & 3 \\
1 & 2 & -7 & 5 & 1 \\
2 & 4 & 0 & -4 & 0 \\
3 & 0 & 0 & 7 & -7
\end{array}
$$

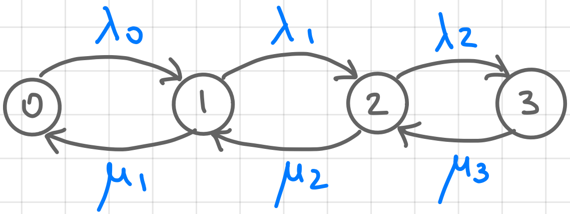


$i \xrightarrow{\alpha} j$ " = "



Example

Birth and death process on $\{0, 1, 2, 3\}$



$$Q = \begin{pmatrix} -\lambda_0 & \lambda_0 & & \\ \mu_1 & -(\lambda_1+\mu_1) & \lambda_1 & \\ & \mu_2 & -(\lambda_2+\mu_2) & \lambda_2 \\ & & \mu_3 & -\mu_3 \end{pmatrix}$$