MATH180C: Introduction to Stochastic Processes II

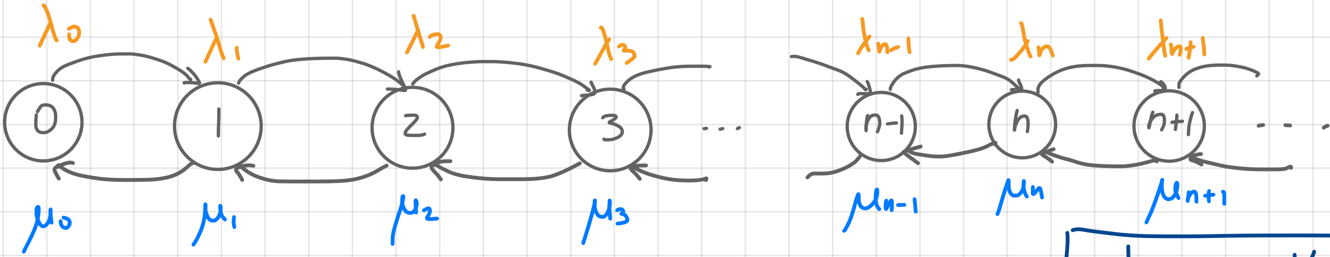
www.math.ucsd.edu/~ynemish/teaching/180c

Today: Strong Markov property.
Hitting probabilities
> Q&A: October 12
Next: PK 6.6, Durrett 4.1

Week 2:

- No homework!
- Quiz 1 on Wednesday, October 14

Alternative (jump and hold) characterization



$\lambda=\mu=1$, $\lambda'=\mu'=2$

$$\frac{\lambda}{\lambda+\mu}=\frac{\mu}{\lambda+\mu}=\frac{1}{2}=\frac{\lambda'}{\lambda'+\mu'}=\frac{\mu'}{\lambda'+\mu'}$$

Exp(2) Exp(4)

Sojourn times S_k are independent,

Each transition has two parts

- wait in state i for time $\sim \text{Exp}(\lambda_i+\mu_i)$
- then choose where to go:

go $\rightarrow$ $i+1$ with probability $\dfrac{\lambda_i}{\lambda_i+\mu_i}$

go $i-1 \leftarrow$ with pobability $\dfrac{\mu_i}{\lambda_i+\mu_i}$

Stopping times

Def (Informal). Let $(X_t)_{t\geq 0}$ be a stochastic process and let $T \geq 0$ be a random variable. We call T a **stopping time** if the event

$$\{T \leq t\}$$

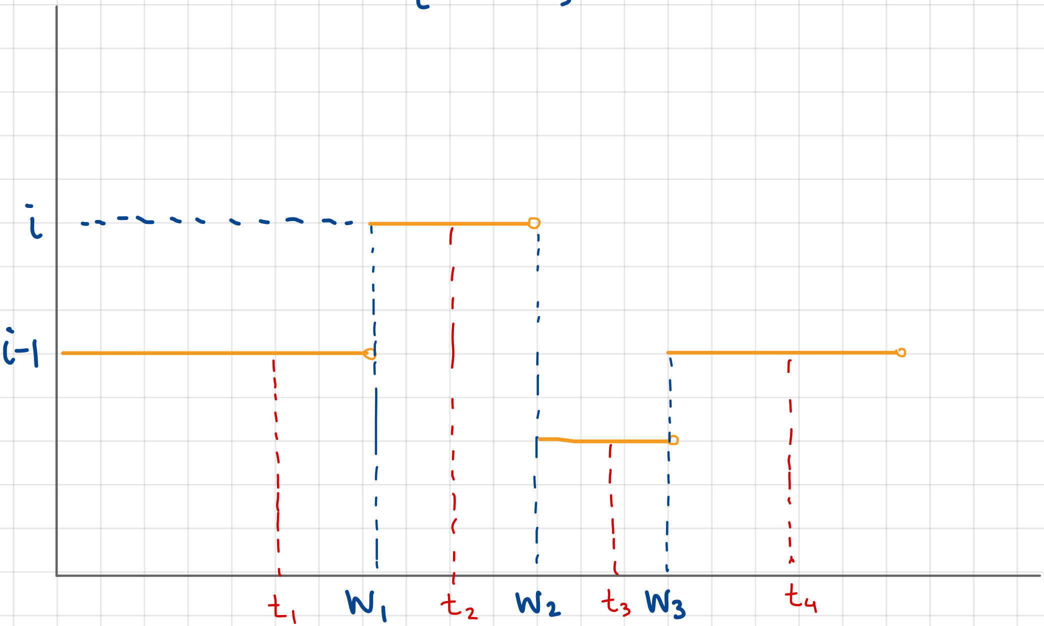
can be determined from the knowledge of the process up to time t (i.e., from $\{X_s : 0 \leq s \leq t\}$)

Examples: Let $(X_t)_{t\geq 0}$ be right-continuous

1. $\min\{t \geq 0 : X_t = i\}$ is a stopping time
2. W_k is a stopping time
3. $\sup\{t \geq 0 : X_t = i\}$ is not a stopping time

Stopping times

$$\{T \leq t\}$$



Strong Markov property

Theorem (no proof)

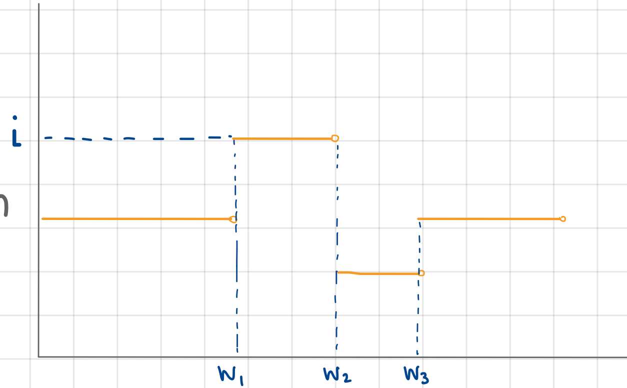
Let $(X_t)_{t\geq 0}$ be a MC, let T be a stopping time of $(X_t)_{t\geq 0}$. Then, conditional on $T<\infty$ and $X_T=i$,

$$(X_{T+t})_{t\geq 0}$$

(i) is independent of $\{X_s,\ 0\leq s\leq T\}$

(ii) has the same distribution as $(X_t)_{t\geq 0}$ starting from i.

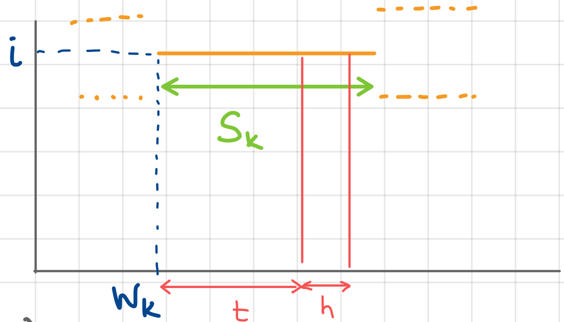
Example



$(X_{W_1+t})_{t\geq 0}$ has the same distribution as $(X_t)_{t\geq 0}$ conditioned on $X_0=i$ and is indep. of what happened before

Alternative (jump and hold) characterization

"Proof"



Denote $G_i(t) := P(S_k > t \mid X_{W_k} = i)$

$$G_i(t+h) = P(S_k > t+h \mid X_{W_k} = i)$$

$$\overset{\text{SMarkov}}{=} P(\text{no jumps on } [0, t+h] \mid X_0 = i)$$

↑ stopping time

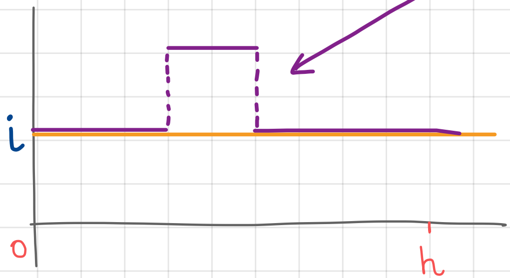
$$\overset{\text{Markov}}{=} P(\text{no jumps on } [0,t] \mid X_0 = i)\, P(\text{no jumps on } [0,h] \mid X_0 = i)$$

$$= P(S_0 > t \mid X_0 = i)\, P(S_0 > h \mid X_0 = i) = G_i(t)\Big(\overbrace{1 - (\lambda_i + \mu_i)h + o(h)}^{P_{ii}(h)}\Big)$$

$$= G_i(t) - (\lambda_i + \mu_i) G_i(t) h + G_i(t) o(h)$$

↳

$$G_i'(t) = -(\lambda_i + \mu_i) G_i(t), \quad G_i(0) = 1$$



Alternative (jump and hold) characterization

"Proof" cont.

$$G_i'(t) = -(\lambda_i+\mu_i)\, G_i(t), \quad G_i(0)=1$$

$\hookrightarrow G_i(t) = e^{-(\lambda_i+\mu_i)t} = P(S_k > t \mid X_{W_k}=i)$

✓ $\hookrightarrow S_k \sim \mathrm{Exp}(\lambda_i+\mu_i)$ (given that the process sojourns in i)

Suppose the process waits $\mathrm{Exp}(\lambda_i+\mu_i)$, then

jumps to $i+1$ with probability $\lambda_i/(\lambda_i+\mu_i)$

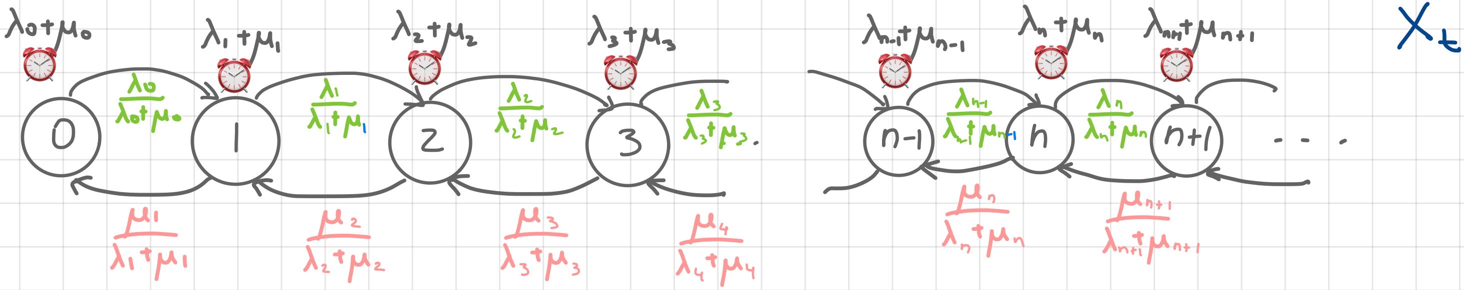
to $i-1$ with probability $\mu_i/(\lambda_i+\mu_i)$

$$P_{i,i+1}(h) \text{ "="} P(S_k \le h \mid X_{W_k}=i)\, P(\text{jump to } i+1)$$ ✓

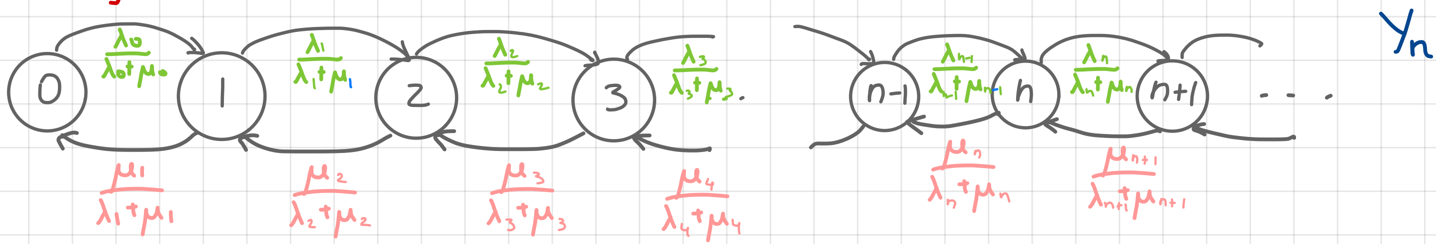
$$= \left(1-e^{-(\lambda_i+\mu_i)h}\right)\frac{\lambda_i}{\lambda_i+\mu_i} = \left((\lambda_i+\mu_i)h+o(h)\right)\frac{\lambda_i}{\lambda_i+\mu_i} = \lambda_i h + o(h)$$

$$P_{i,i-1}(h) \text{ "="} P(S_k \le h \mid X_{W_k}=i)\, P(\text{jump to } i-1) = \left((\lambda_i+\mu_i)h+o(h)\right)\frac{\mu_i}{\lambda_i+\mu_i} = \mu_i h + o(h)$$

Related discrete time MC.



Def. Let $(X_t)_{t\geq 0}$ be a continuous time MC, let $W_n, n\geq 0$, be the corresponding waiting (arrival, jump) times. Then we call $(Y_n)_{n\geq 0}$ defined by $Y_n = X_{W_n}$, $Y_0 = X_0$, $n\geq 1$ the jump chain of $(X_t)_{t\geq 0}$.



random walk

Related discrete time MC.

$(X_t)_{t\geq 0}$ and its jump chain $(Y_n)_{n\geq 0}$ execute the same transitions.

Let $(X_t)_{t\geq 0}$ be a birth and death process. Then the transition probability matrix of the random walk $(Y_n)_{n\geq 0}$ is given by

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ \vdots \end{array} \begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & \cdots \end{array} \\ \begin{pmatrix} & \frac{\lambda_0}{\lambda_0+\mu_0} & & & & \cdots \\ \frac{\mu_1}{\lambda_1+\mu_1} & & \frac{\lambda_1}{\lambda_1+\mu_1} & & & \cdots \\ & \frac{\mu_2}{\lambda_2+\mu_2} & & \frac{\lambda_2}{\lambda_2+\mu_2} & & \cdots \\ & & \frac{\mu_3}{\lambda_3+\mu_3} & & \frac{\lambda_3}{\lambda_3+\mu_3} & \cdots \\ & \vdots & \vdots & & & \end{pmatrix} \end{array}$$

Absorption probabilities for B&D processes

Let $(X_t)_{t\geq 0}$ be a birth and death process, and assume that the state 0 is absorbing, $\lambda_0 = 0$. Then

$P((X_t)_{t\geq 0}$ gets absorbed in $0 \mid X_0 = i)$

$= P((Y_n)_{n\geq 0}$ gets absorbed in $0 \mid Y_0 = i)$

↳ use the first step analysis to compute the absorption probabilities for $(Y_n)_{n\geq 0}$ (and for $(X_t)_{t\geq 0}$)

Denote $u_i = P(Y_n$ is absorded in $0 \mid Y_0 = i)$

Then $\quad u_0 = 1, \quad u_n = \frac{\mu_n}{\lambda_n + \mu_0} u_{n-1} + \frac{\lambda_n}{\lambda_n + \mu_n} u_{n+1}$

Absorption probabilities for B&D processes

$$u_0 = 1 \,, \qquad u_n = \frac{\mu_n}{\lambda_n+\mu_n} u_{n-1} + \frac{\lambda_n}{\lambda_n+\mu_n} u_{n+1}$$

Rewrite
$$(\lambda_n + \mu_n) u_n = \mu_n u_{n-1} + \lambda_n u_{n+1}$$
$$\lambda_n (u_{n+1} - u_n) = \mu_n (u_n - u_{n-1})$$
$$u_{n+1} - u_n = \frac{\mu_n}{\lambda_n}(u_n - u_{n-1})$$
$$= \underbrace{\frac{\mu_n}{\lambda_n} \cdot \frac{\mu_{n-1}}{\lambda_{n-1}} \cdot \cdots \cdot \frac{\mu_1}{\lambda_1}}_{\rho_n} (u_1 - \overset{=1}{u_0})$$

$$(*) \quad u_{n+1} - u_n = \rho_n (u_1 - 1)$$

Note that $\sum_{k=1}^{n-1} (u_{k+1} - u_k) = u_n - u_1 = (u_1 - 1) \sum_{n=1}^{n-1} \rho_n$

If $\sum_{n=1}^{\infty} \rho_n = \infty$, then $u_1 = 1$ and from $(*)$ $u_n = 1 \;\; \forall\, n \geq 0$.

Absorption probabilities for B&D processes

Let $\sum_{k=1}^{\infty} \rho_k < \infty$. If we assume that $u_n \to 0$, $n \to \infty$, then by

taking $n \to \infty$

$$\cancelto{0}{u_n} - u_1 = (u_1 - 1) \sum_{k=1}^{n-1} \rho_k$$

$$u_1 = \frac{\sum_{k=1}^{\infty} \rho_k}{1 + \sum_{k=1}^{\infty} \rho_k}$$

and $$u_n = u_1 + (u_1 - 1) \sum_{k=1}^{n-1} \rho_k = \frac{\sum_{k=1}^{\infty} \rho_k + \left(\sum_{k=1}^{\infty} \rho_k + 1 - \sum_{k=1}^{\infty} \rho_k \right) \sum_{k=1}^{n-1} \rho_k}{1 + \sum_{k=1}^{\infty} \rho_k}$$

$$= \frac{\sum_{k=1}^{\infty} \rho_k - \sum_{k=1}^{n-1} \rho_k}{1 + \sum_{k=1}^{\infty} \rho_k} = \frac{\sum_{k=n}^{\infty} \rho_k}{1 + \sum_{k=1}^{\infty} \rho_k}$$

Mean time until absorption

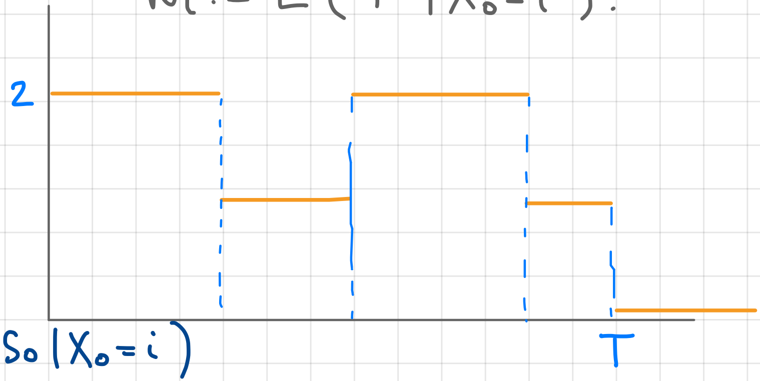
Let $(X_t)_{t\geq 0}$ be a birth and death process. Denote

$T = \min\{t \geq 0 : X_t = 0\}$ absorption time and

$w_i := E(T \mid X_0 = i)$.

Let $(Y_n)_{n\geq 0}$ be the jump chain for $(X_t)_{t\geq 0}$.

$$N := \min\{n \geq 0 : Y_n = 0\}$$



Then $T = \sum_{k=0}^{N-1} S_k$

$$w_i = E\Big(\sum_{k=0}^{N-1} S_k \,\Big|\, X_0 = i\Big) = \overbrace{\frac{1}{\lambda_i + \mu_i}}^{E(S_0 \mid X_0 = i)} + E\Big(\sum_{k=1}^{N-1} S_k \,\Big|\, X_0 = i\Big)$$

$$= \frac{1}{\lambda_i + \mu_i} + \underbrace{E\Big(\sum_{k=1}^{N-1} S_k \,\Big|\, X_0 = i, Y_1 = i+1\Big)}_{\substack{\text{SMP} \,\|\\ w_{i+1}}} P(Y_1 = i+1 \mid Y_0 = i)$$

$$+ \overbrace{E\Big(\sum_{k=1}^{N-1} S_k \,\Big|\, X_0 = i, Y_1 = i-1\Big)}^{\substack{w_{i-1}\\ \| \,\text{SMP}}} P(Y_1 = i-1 \mid Y_0 = i)$$

Mean time until absorption

$$\begin{cases} w_i = \dfrac{1}{\lambda_i+\mu_i} + \dfrac{\lambda_i}{\lambda_i+\mu_i} w_{i+1} + \dfrac{\mu_i}{\lambda_i+\mu_i} w_{i-1}, \\ w_0 = 0 \end{cases}$$

Alternatively, $S_k \sim Exp(\lambda_{Y_k} + \mu_{Y_k})$ and one can show that

$$E(T \mid X_0 = i) = E\left(\sum_{k=0}^{N-1} \frac{1}{\lambda_{Y_k} + \mu_{Y_k}} \,\middle|\, Y_0 = i \right)$$

Now apply the first step analysis for the general MC

$$w_i = E\left(\sum_{k=0}^{N-1} g(Y_k) \,\middle|\, Y_0 = i \right),$$

which leads to (the same) system of equations

$$w_i = g(i) + \sum_{j=1}^{\infty} P_{ij} w_j$$

First step analysis for birth and death processes

Summary:

Let $(X_t)_{t\geq 0}$ be a birth and death process of rates $((\lambda_i, \mu_i))_{i\geq 0}$ with $\lambda_0 = 0$ (state 0 absorbing).

Denote $T = \min\{t : X_t = 0\}$, $u_i := P(X_t \text{ gets absorbed in } 0 \mid X_0 = i)$

$w_i = E(T \mid X_0 = i)$ and $\rho_j = \dfrac{\mu_1 \mu_2 \cdots \mu_j}{\lambda_1 \lambda_2 \cdots \lambda_j}$. Then

$$u_i = \begin{cases} \dfrac{\sum_{j=i}^{\infty} \rho_j}{1 + \sum_{j=1}^{\infty} \rho_j}, & \text{if } \sum_{j=1}^{\infty} \rho_j < \infty \\ 1, & \text{if } \sum_{j=1}^{\infty} \rho_j = \infty \end{cases}$$

$$w_i = \begin{cases} \sum_{j=1}^{\infty} \dfrac{1}{\lambda_j \rho_j} + \sum_{k=1}^{i-1} \rho_k \sum_{j=k+1}^{\infty} \dfrac{1}{\lambda_j \rho_j}, & \text{if } \sum_{j=1}^{\infty} \dfrac{1}{\lambda_j \rho_j} < \infty \\ \infty, & \text{if } \sum_{j=1}^{\infty} \dfrac{1}{\lambda_j \rho_j} = \infty \end{cases}$$