

MATH 180A: Introduction to Probability

Lecture A00 (Au)

www.math.ucsd.edu/~bau/w21.180a

Lecture B00 (Nemish)

www.math.ucsd.edu/~ynemish/teaching/180a

Today: Joint distributions.
Independence of random
variables

Next: ASV 8.1-8.3

Video: Prof. Todd Kemp, Fall 2019

Week 8:

- Homework 7 (due Friday, March 5)

Jointly Continuous Random Vectors

A random vector $\underline{X} = (X_1, \dots, X_n)$ has a pdf $f_{\underline{X}}: \mathbb{R}^n \rightarrow \mathbb{R}_+$ if, for "nice" subsets $B \subseteq \mathbb{R}^n$

$$P(\underline{X} \in B) = \int_B f_{\underline{X}}(x_1, \dots, x_n) dx_1 \dots dx_n.$$

(We say X_1, X_2, \dots, X_n are jointly continuous.)

Properties: $f_{\underline{X}}(\underline{x}) \geq 0$ for all $\underline{x} \in \mathbb{R}^n$

$$\int_{\mathbb{R}^n} f(\underline{x}) d\underline{x} = 1.$$

$$\|\underline{x}\|^2 = x_1^2 + \dots + x_n^2$$

E.g. Standard Multivariate Normal

$$f(\underline{x}) = (2\pi)^{-\frac{n}{2}} e^{-\|\underline{x}\|^2/2}$$

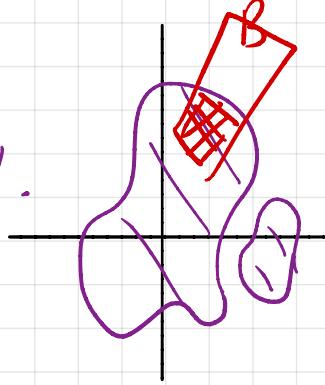
$$(n=2) \quad f(x,y) = \frac{1}{2\pi} e^{-(x^2+y^2)/2} = \frac{1}{2\pi} e^{-x^2/2} e^{-y^2/2} > 0$$

$$\therefore \iint_{\mathbb{R}^2} f(x,y) dx dy = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{2\pi} e^{-x^2/2} e^{-y^2/2} dx dy = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-y^2/2} \left(\int_{-\infty}^{\infty} e^{-x^2/2} dx \right) dy = \frac{(\sqrt{2\pi})^2}{2\pi} = 1.$$

E.g. Uniform Probability

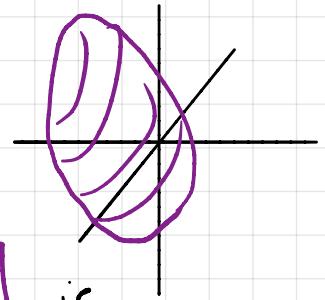
Let A be a bounded region in \mathbb{R}^2 with area α .

$$P((X_1, X_2) \in B) = \frac{\text{Area}(B \cap A)}{\text{Area}(A)}$$



Let V be a bounded region in \mathbb{R}^3 with volume ϑ

:



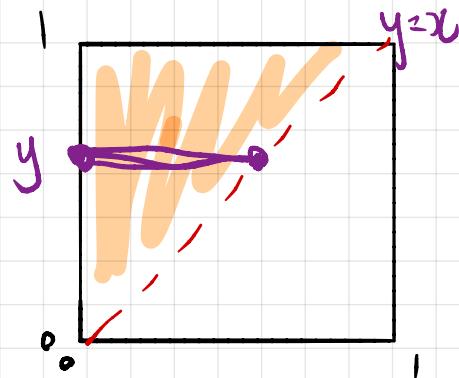
A random vector $X = (X_1, X_2)$ [resp. $X = (X_1, X_2, X_3)$] is uniformly distributed in A [resp. V] if it is jointly continuous and has density

$$f_A(x_1, x_2) = \begin{cases} \frac{1}{\alpha} \mathbb{1}_{\{(x_1, x_2) \in A\}} & \text{resp. } f_V(x_1, x_2, x_3) = \frac{1}{\vartheta} \mathbb{1}_{\{(x_1, x_2, x_3) \in V\}} \\ 0 & \text{if } (x_1, x_2) \notin A \end{cases}$$

$$\text{Area}(A \cap B) / \alpha.$$

$\Rightarrow P((X_1, X_2) \in B) = \iint_B f_A(x_1, x_2) dx_1 dx_2 = \iint_B \frac{1}{\alpha} \mathbb{1}_{\{(x_1, x_2) \in A\}} dx_1 dx_2 = \iint_{A \cap B} \frac{1}{\alpha} dx_1 dx_2$

E.g. Suppose (X, Y) has joint density $f(x, y) = \frac{3}{2}(xy^2 + y) \mathbb{1}_{(0 \leq x, y \leq 1)}$



$$\begin{aligned} \iint_{\mathbb{R}^2} f(x, y) dx dy &= \int_0^1 \int_0^1 \frac{3}{2}(xy^2 + y) dx dy \\ &= \frac{3}{2} \int_0^1 dy \left(y^2 \cdot \frac{x^2}{2} + yx \right) \Big|_{x=0}^1 = \frac{3}{2} \int_0^1 \left(\frac{1}{2}y^2 + y \right) dy \\ &= \frac{3}{2} \left(\frac{1}{6}y^3 + \frac{1}{2}y^2 \right) \Big|_{y=0}^1 \\ &= \frac{3}{2} \left(\frac{1}{6} + \frac{1}{2} \right) = 1. \end{aligned}$$

Compute $P(X < Y)$.

$$T = \{(x, y) : x < y\}$$

$$P((X, Y) \in T) = \iint_T \frac{3}{2}(xy^2 + y) dx dy$$

$$\begin{aligned} &= \frac{3}{2} \int_0^1 dy \int_0^y dx (xy^2 + y) = \frac{3}{2} \int_0^1 dy \left(y^2 \cdot \frac{x^2}{2} + yx \right) \Big|_{x=0}^y \\ &= \frac{3}{2} \int_0^1 dy \left(y^2 \cdot \frac{y^2}{2} + y^2 \right) \\ &= \frac{3}{2} \left(\frac{1}{10}y^5 + \frac{1}{3}y^3 \right) \Big|_0^1 = \frac{3}{2} \left(\frac{1}{10} + \frac{1}{3} \right) \\ &= \frac{13}{20}. \end{aligned}$$

Marginals

Let $\underline{X} = (X_1, X_2, \dots, X_n)$ be a jointly continuous random vector, with joint density $f_{\underline{X}}$. The density of X_j is

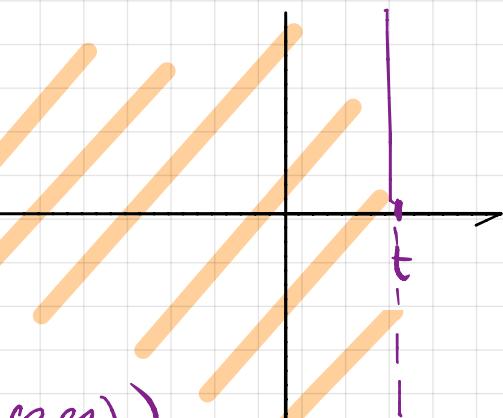
$$f_{X_j}(t) = \int_{\mathbb{R}^{n-1}} f_{\underline{X}}(x_1, x_2, \dots, x_{j-1}, t, x_{j+1}, \dots, x_n) dx_1 \dots dx_{j-1} dx_{j+1} \dots dx_n.$$

(Integrate out all but the j^{th} variable)

Proof ($n=2, j=1$) $\underline{X} = (X, Y)$

$$P(X \leq t) = P(X \leq t, Y \in \mathbb{R}) = P((X, Y) \in (-\infty, t] \times (-\infty, \infty))$$

$$F_X(t) = \int_{-\infty}^t dx \int_{-\infty}^{\infty} dy f_{(X,Y)}(x, y) = \int_{-\infty}^t g(x) dx$$



$$g(x) = \int_{-\infty}^x f_{(X,Y)}(z, y) dy$$

$$\therefore f_X(t) = \frac{d}{dt} F_X(t) = \frac{d}{dt} \int_0^t g(x) dx = g(t)$$

FTC

E.g. Suppose \underline{X} is uniformly distributed on the disk of radius 2.

Find the marginal density of X_1 ($\underline{X} = (X_1, X_2)$)

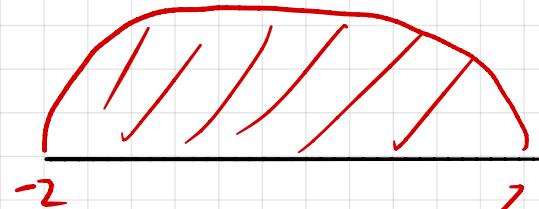
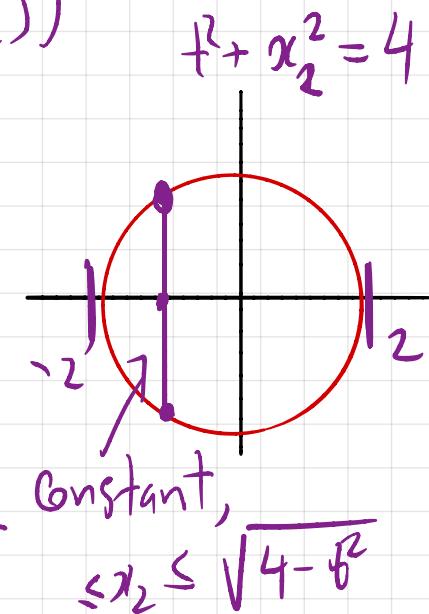
$$f_{\underline{X}}(x_1, x_2) = \frac{1}{4\pi} \mathbb{1}_{D_2}(x_1, x_2)$$

$$f_{X_1}(t) = \int_{-\infty}^{\infty} f_{\underline{X}}(t, x_2) dx_2 = \frac{1}{4\pi} \int_{-\infty}^{\infty} \mathbb{1}_{D_2}(t, x_2) dx_2$$

$$\begin{aligned} & \text{for } -2 \leq t \leq 2 \\ &= \frac{1}{4\pi} \int_{-\sqrt{4-t^2}}^{\sqrt{4-t^2}} dx_2 = \frac{1}{4\pi} (2\sqrt{4-t^2}) \end{aligned}$$

$$= \begin{cases} \frac{\sqrt{4-t^2}}{2\pi} & -2 \leq t \leq 2 \\ 0 & |t| > 2 \end{cases}$$

"Semicircle Law"



Expectation

If \underline{X} is a random vector in \mathbb{R}^n with joint density $f_{\underline{X}}: \mathbb{R}^n \rightarrow \mathbb{R}_+$, and $g: \mathbb{R}^n \rightarrow \mathbb{R}$, then

$$\mathbb{E}(g(\underline{X})) = \int_{\mathbb{R}^n} g(\underline{x}) f_{\underline{X}}(\underline{x}) d\underline{x}$$

E.g. $\underline{X} = (X, Y)$, $f_{\underline{X}}(x, y) = \frac{3}{2}(xy^2 + y) \mathbb{1}_{(0 \leq x, y \leq 1)}$. Find $\mathbb{E}(X^2Y)$.

$$\mathbb{E}(X^2Y) = \iint_{\mathbb{R}^2} x^2y f_{\underline{X}}(x, y) dx dy = \int_0^1 \int_0^1 x^2y \cdot \frac{3}{2}(xy^2 + y) dx dy$$

Exercise

$$= \frac{25}{96}.$$

CAUTION!

If X_1, X_2, \dots, X_n are discrete random variables, then

$\underline{X} = (X_1, \dots, X_n)$ is a (jointly) discrete random vector.

BUT

Just because X_1, \dots, X_n are (separately) continuous random variables does not necessarily imply that $\underline{X} = (X_1, \dots, X_n)$ has a joint density!

$$\text{E.g. } X \sim N(0, 1), \quad Y = -X \sim N(0, 1) \quad P(Y \leq t) = P(-X \leq t)$$

$$= P(X \geq -t)$$

$$= 1 - P(X < -t)$$

$$= 1 - \Phi(-t)$$

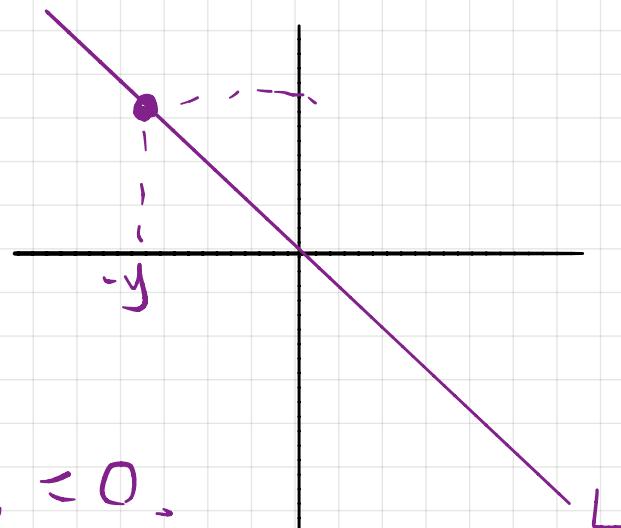
$$= \Phi(t)$$

Must have $P(X = -Y) = 1$.

If f_{XY} existed,

$$P(X = -Y) = \iint f_{XY}(x, y) dx dy$$

$$\int_{-\infty}^{\infty} dy \int_{-y}^{\infty} f_{XY}(x, y) dx = 0.$$



Joint Distributions & Independence

6.3

Suppose $\underline{X} = (X_1, \dots, X_n)$ is jointly continuous. Then

X_1, \dots, X_n are independent iff $f_{X_1, \dots, X_n}(x_1, \dots, x_n) = f_{X_1}(x_1) f_{X_2}(x_2) \cdots f_{X_n}(x_n)$.

$$(h=2) \quad f_{(X,Y)}(x,y) = f_X(x) f_Y(y).$$

$$\text{Pf. } (\Leftarrow) \quad \overbrace{\Pr(X \in A, Y \in B)} = \Pr((X, Y) \in A \times B)$$

$$= \iint_{A \times B} f_X(x) f_Y(y) dx dy$$

$$= \int_A f_X(x) dx \int_B f_Y(y) dy = P(X \in A) P(Y \in B)$$

$\therefore X, Y$ independent.

(=>) NEXT TIME.