1 Exam A

1.1 Problem 1

1. (20 points) You have an urn that initially contains 6 red balls, 2 black balls and 1 green ball. On the first step, you choose one ball uniformly at random from the urn, look at its color, and then return it back to the urn together with one more ball of the same color (e.g., if you pick a red ball, then you put it back to the urn together with another red ball). Then on the second step you choose a ball uniformly at random from the urn (note that on the second step the urn contains the additional ball).

   What is the probability that on the second step you choose a red ball?

2. (20 points) You have an urn that initially contains 5 red balls, 2 black balls and 2 green ball. On the first step, you choose one ball uniformly at random from the urn, look at its color, and then return it back to the urn together with one more ball of the same color (e.g., if you pick a red ball, then you put it back to the urn together with another red ball). Then on the second step you choose a ball uniformly at random from the urn (note that on the second step the urn contains the additional ball).

   What is the probability that on the second step you choose a green ball?

1.2 Problem 2

3. (20 points) Let X and Y be independent random variables uniformly distributed on the interval $[0,1]$, i.e., $X \sim \mathcal{U}[0,1]$, $Y \sim \mathcal{U}[0,1]$.
   (a) (8 points) Compute the moment generating function of the sum $X+Y$.
   (b) (4 points) Show that for any $t \in \mathbb{R}$

   $$(e^t - 1)^2 = e^{2t} - 2e^t + 1 = \sum_{k=2}^{\infty} \frac{2^k}{k!} t^k - \sum_{k=2}^{\infty} \frac{2}{k!} t^k. \tag{1}$$

   (c) (8 points) Use the results of (a) and (b) to compute $E((X+Y)^n)$, moments of the sum, for any $n \in \mathbb{N}$.

1.3 Problem 3

4. (20 points) Let X and Y be a pair of jointly continuous random variables with joint density

   $$f_{X,Y}(x,y) = \begin{cases} c(x+3y), & 0 \le x, y \le 1, \\ 0, & \text{otherwise}, \end{cases} \tag{2}$$

   with an unknown parameter $c > 0$.

(a) (5 points) Determine the value of $c > 0$.

(b) (10 points) Compute the marginal densities of X and Y.

(c) (5 points) Determine if random variables X and Y are independent.

5. (20 points) Let X and Y be a pair of jointly continuous random variables with joint density

$$f_{X,Y}(x,y) = \begin{cases} c(5x+y), & 0 \le x, y \le 1, \\ 0, & \text{otherwise,} \end{cases} \tag{3}$$

with an unknown parameter $c > 0$.

(a) (5 points) Determine the value of $c > 0$.

(b) (10 points) Compute the marginal densities of X and Y.

(c) (5 points) Determine if random variables X and Y are independent.

1.4 Problem 4

6. (20 points) Let X and Y be independent random variables. Suppose that X is Gaussian, $X \sim N(1,1)$ and that Y has Poisson distribution $Y \sim \text{Pois}(2)$.

(a) (10 points) Compute $E(3X - 2Y - 5)$ and $\text{Var}(3X - 2Y - 5)$.

(b) (10 points) Compute $\text{Var}(XY)$.

7. (20 points) Let X and Y be independent random variables. Suppose that X has Poisson distribution, $X \sim \text{Pois}(2)$ and that Y is exponential, $Y \sim \text{Exp}(1)$.

(a) (10 points) Compute $E(3X - 6Y - 1)$ and $\text{Var}(3X - 6Y - 1)$.

(b) (10 points) Compute $\text{Var}(XY)$.

1.5 Problem 5

8. (20 points) Let $(\xi_i)_{i=1}^{\infty}$ be a sequence of independent identically distributed random variables each having Bernoulli distribution with parameter $1/6$, $\xi \sim \text{Ber}(1/6)$, and let $S_n := \xi_1 + \cdots + \xi_n$.

(a) (5 points) Compute $E(4^{\xi_i})$.

(b) (5 points) Compute $E(4^{S_n})$.

(c) (10 points) Using the result of part (b) and Markov's inequality, show that

$$P\Big(\frac{S_n}{n} \ge \frac{1}{2}\Big) \le \Big(\frac{3}{4}\Big)^n. \tag{4}$$

2 Exam B

2.1 Problem 1

9. (20 points) You have two urns. The first urn has 3 red balls, 2 blue balls and 2 green balls. The second urn has 2 red balls and 4 blue balls. You choose one of the urns at random (with equal probability), and then sample one ball from that urn. The ball that you picked is blue.

   What is the probability that the ball was picked from the first urn?

10. (20 points) You have three urns. The first urn has 3 red balls, 2 blue balls and 3 green balls. The second urn has 2 red balls and 4 blue balls. The third urn has 6 green balls. You choose one of the urns at random (with equal probability), and then sample one ball from that urn. The ball that you picked is green.

    What is the probability that the ball was picked from the first urn?

2.2 Problem 2

11. (20 points) Let $X \sim N(0,1)$ be a random variable with standard normal distribution, and let $Y = X^2$.

    Compute the moment generating function of Y and use it to compute $E(Y)$ and $E(Y^2)$.

2.3 Problem 3

12. (20 points) Let X and Y be independent random variables with (marginal) densities

$$f_X(x) = \begin{cases} \frac{1}{2}, & x \in (-1,1), \\ 0, & \text{otherwise}, \end{cases} \tag{5}$$

$$f_Y(y) = \begin{cases} 2y, & y \in (0,1), \\ 0, & \text{otherwise}. \end{cases} \tag{6}$$

   (a) (5 points) Compute the variance of the product $\mathrm{Var}(XY)$.

   (b) (5 points) Compute the probability $P(X > Y)$.

13. (20 points) Let X and Y be independent random variables with (marginal) densities

$$f_X(x) = \begin{cases} \frac{1}{2}, & x \in (-1,1), \\ 0, & \text{otherwise}, \end{cases} \tag{7}$$

$$f_Y(y) = \begin{cases} 3y^2, & y \in (0,1), \\ 0, & \text{otherwise}. \end{cases} \tag{8}$$

(a) (5 points) Compute the variance of the product $\mathrm{Var}(XY)$.

(b) (5 points) Compute the probability $P(X > Y)$.

2.4 Problem 4

14. (20 points) You roll a fair die n times. Each time when the number you get is different from the number obtained on the previous roll, you win 2 dollars. For example, the sequence $(1, 2, 1, 3, 3, 5)$ results in winning 8 dollars (2 dollars on 2nd, 3rd, 4th and 6th rolls).

    Compute your expected winnings after n rolls.

15. (20 points) You roll a fair die n times. Each time when the number you get is the same as the number obtained on the previous roll, you win 3 dollars. For example, the sequence $(2, 2, 1, 3, 3, 2)$ results in winning 6 dollars (3 dollars on 2nd and 5th rolls).

    Question: Compute your expected winnings after n rolls.

2.5 Problem 5

16. (20 points) UCSD Bookstore sells on average 10 books per day. After some research, UCSD Bookstore concluded that the variance of the random variable describing the number of books sold daily is 3.

    (a) (points) Estimate (provide an upper bound) the probability that on a given day UCSD Bookstore sells more than 12 books.

    (b) (points) Assuming that the numbers of books sold on different days are independent and identically distributed, estimate (provide an upper bound) the probability that during the first ten days of December the UCSD Bookstore sells more than 120 books (without using the Central Limit Theorem).