MATH180C: Introduction to Stochastic Processes II

www.math.ucsd.edu/~ynemish/teaching/180c

Today: Introduction. Birth processes

> Q&A: October 5

Next: PK 6.2-6.3

Week 0/1:

- visit course web site
- homework 0 (due Wednesday October 7)
- homework 1 (due Friday October 9)
- join Piazza

Stochastic (random) processes

Def. Let $(\Omega, \mathcal{F}, P)$ be a probability space.
Stochastic process is a collection $(X_t : t \in T)$ of random variables $X_t : \Omega \to S \subset \mathbb{R}$ (all defined on the same probability space)

- often t represents time, but can be 1-D space
- T is called the index set, S is called the state space
- $X : \Omega \times T \to S$ $(X_t(\omega) \in S)$
- for any fixed ω, we get a realization of all random variables $(X_t(\omega) : t \in T)$ ← sample path / trajectory

$$X_\cdot(\omega) : T \to S \quad \leftarrow \text{function in } t$$

- stochastic process = random function

Stochastic processes. Classification

Questions:

- What is T?
- What is S?
- Relations between X_{t_1} and X_{t_2} for $t_1 \neq t_2$?
- Properties of the trajectory?

Discrete time

$T = \mathbb{N}, \mathbb{Z}$, finite set

random vector

Continuous time

$T = \mathbb{R}, [0, +\infty), [0,1]$

Real-valued

$S = \mathbb{R}$

Integer-valued

$S = \mathbb{Z}$

Nonnegative

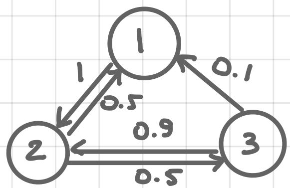
$S \subset [0, +\infty)$

....

Continuous, right-continuous (cadlag) sample path

Examples of stochastic processes

- Gaussian processes: for any $t \in T$, X_t has normal distrib.
- stationary processes: distribution doesn't change in time
- processes with stationary/independent increments (Lévy)
- Poisson process: increments are indep. and Poisson$(\cdot)$
- Markov processes: "distribution in the future depends only on the current state, but does not depend on the past"

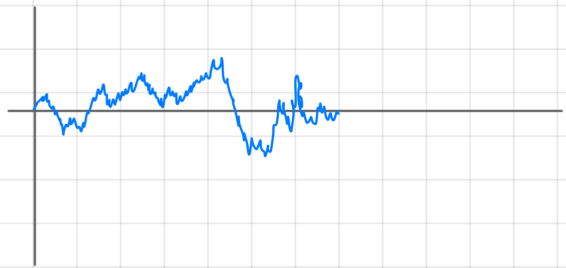


Examples of stochastic processes

- martingales : $E[X_{n+1} \mid X_n, X_{n-1}, \dots, X_0] = X_n$ ('fair game')

- Brownian motion (Wiener process) is continuous-time s.p.
  Gaussian, martingale, has stationary and independent increments, Markov, $Var[W_t] = t$, $Cov(W_t, W_s) = \min\{s, t\}$, its sample path is everywhere continuous and nowhere differentiable

- diffusion processes (stochastic differential equations)



- ......

Continuous time MC

Continuous Time Markov Chains

Def (Discrete-time Markov chain)

Let $(X_n)_{n\geq 0}$ be a discrete time stochastic process taking values in $\mathbb{Z}_+ = \{0,1,2,\dots\}$ (for convenience). $(X_n)_{n\geq 0}$ is called Markov chain if for any $n\in\mathbb{N}$ and $i_0, i_1, \dots, i_{n-1}, i, j \in \mathbb{Z}_+$

$$P(X_{n+1}=j \mid X_0=i_0, X_1=i_1, \dots, X_{n-1}=i_{n-1}, X_n=i) = P(X_{n+1}=j \mid X_n=i)$$

Def (Continuous-time Markov chain)

Let $(X_t)_{t\geq 0} = (X_t : 0\leq t<\infty)$ be a continuous time process taking values in $\mathbb{Z}_+$. $(X_t)_{t\geq 0}$ is called Markov chain if for any $n\in\mathbb{N}$, $0\leq t_0<t_1<\dots<t_{n-1}<s$, $t>0$, $i_0, i_1, \dots, i_{n-1}, i, j \in \mathbb{Z}_+$

$$P(X_{s+t}=j \mid X_{t_0}=i_0, X_{t_1}=i_1, \dots, X_{t_{n-1}}=i_{n-1}, X_s=i) = P(X_{s+t}=j \mid X_s=i)$$

Markov property

Example: Poisson process as MC

Is Poisson process a continuous time MC?

Poisson process:

- ✓ continuous time
- ✓ discrete state
- ✓ Markov property

Let $(X_t)_{t\geq 0}$ be a Poisson process, let $i_0 \leq i_1 \leq \cdots \leq i_{n-1} \leq i \leq j$

$$P(X_{s+t}=j \mid X_{t_0}=i_0, X_{t_1}=i_1, \ldots, X_{t_{n-1}}=i_{n-1}, X_s=i)$$

$$= \frac{P(X_{t_0}=i_0, X_{t_1}-X_{t_0}=i_1-i_0, \ldots, X_s-X_{t_{n-1}}=i-i_{n-1}, X_{s+t}-X_s=j-i)}{P(X_{t_0}=i_0, X_{t_1}-X_{t_0}=i_1-i_0, \ldots, X_s-X_{t_{n-1}}=i-i_{n-1})}$$

$$= P(X_{s+t}-X_s=j-i)$$

$$= P(X_{s+t}-X_s=j-i \mid X_s=i) = P(X_{s+t}=j \mid X_s=i)$$

Transition probability function

One way of describing a continuous time MC is by using the transition probability functions.

Def. Let $(X_t)_{t\geq 0}$ be a MC. We call

$$P(X_{s+t}=j \mid X_s=i), \quad i,j\in\{0,1,\dots\}, \quad s\geq 0, \ t>0$$

the transition probability function for $(X_t)_{t\geq 0}$.

If $P(X_{s+t}=j \mid X_s=i)$ does not depend on s, we say that $(X_t)_{t\geq 0}$ has stationary transition probabilities and we define $P_{ij}(t) := P(X_{s+t}=j \mid X_s=i) \ \big(= P(X_t=j \mid X_0=i)\big)$

[compare with n-step transition probabilities]

Characterization of the Poisson process

Experiment: count events occurring along $[0,+\infty)$ $\left\{\begin{array}{l}\text{time}\\ \text{or 1-D space}\end{array}\right.$

Denote by $N((a,b])$ the number of events that occur on $(a,b]$.

Assumptions:

1. Number of events happening in disjoint intervals are independent.
2. For any $t \geq 0$ and $h > 0$, the distribution of $N((t,t+h])$ does not depend on t (only on h, the length of the interval)
3. There exists $\lambda > 0$ s.t. $P(N((t,t+h]) \geq 1) = \lambda h + o(h)$ as $h \to 0$
   (rare events)
4. Simultaneous events are not possible: $P(N((t,t+h]) \geq 2) = o(h), h \to 0$

Then $X_t := N((0,t])$ is a Poisson process with rate λ.

Transition probabilities of the Poisson process

Let $(X_t)_{t\geq 0}$ be the Poisson process.

Define the transition probability functions

$$P_{ij}(h) := P(X_{t+h}=j \mid X_t = i), \quad i,j \in \{0,1,2,\dots\}, \quad t\geq 0, \; h>0$$

What are the infinitesimal (small h) transition probability functions for $(X_t)_{t\geq 0}$? As $h\to 0$,

$$P_{ii}(h) = P(X_{t+h}=i \mid X_t=i)$$

$$= P(X_{t+h}-X_t=0 \mid X_t=i) = P(X_{t+h}-X_t=0) = 1-\lambda h + o(h)$$

$$P_{i,i+1}(h) = P(X_{t+h}=i+1 \mid X_t=i) = P(X_{t+h}-X_t=1) = \lambda h + o(h)$$

$$\sum_{j\notin\{i,i+1\}} P_{ij}(h) = o(h)$$

Poisson process and transition probabilities

To sum up: $(X_t)_{t\geq 0}$ is a MC with (infinitesimal) transition probabilities satisfying

$$P_{ii}(h) = 1 - \lambda h + o(h)$$

$$P_{i,i+1}(h) = \lambda h + o(h) \qquad \text{as } h \to 0$$

$$\sum_{j \notin \{i,i+1\}} P_{i,j}(h) = o(h)$$

What if we allow $P_{ij}(h)$ depend on i?

↳ birth and death processes

Pure birth processes

Def Let $(\lambda_k)_{k \geq 0}$ be a sequence of positive numbers.
We define a pure birth process as a Markov process $(X_t)_{t \geq 0}$ whose stationary transition probabilities satisfy

1. $P_{k,k+1}(h) = \lambda_k h + o(h)$
2. $P_{k,k}(h) = 1 - \lambda_k h + o(h)$ as $h \to 0_+$
3. $P_{k,j}(h) = 0$ for $j < k$
4. $X_0 = 0$

Related model. Yule process : $\lambda_k = \beta k$ for some $\beta > 0$.
Describes the growth of a population
- birth rate is proportional to the size of the population

Birth processes and related differential equations

Now define $P_n(t) = P(X_t = n)$. For small $h > 0$

$$P_n(t+h) = P(X_{t+h} = n) = \sum_{k=0}^{n} P(X_{t+h} = n \mid X_t = k)\, P(X_t = k)$$

$$= \sum_{k=0}^{n} P_{k,n}(h) \cdot P(X_t = k)$$

$$= P_{n,n}(h) \cdot P_n(t) + P_{n-1,n}(h) \cdot P_{n-1}(t) + \sum_{k=0}^{n-2} P_{k,n}(h) \cdot P(X_t = k)$$

$$= (1 - \lambda_n h) P_n(t) + \lambda_{n-1} h\, P_{n-1}(t) + o(h)$$

$$= P_n(t) - \lambda_n h\, P_n(t) + \lambda_{n-1} h\, P_{n-1}(t) + o(h)$$

$$P_n(t+h) - P_n(t) = -\lambda_n h\, P_n(t) + \lambda_{n-1} h\, P_{n-1}(t) + o(h)$$

$$P_n'(t) = \lim_{h \to 0} \frac{P_n(t+h) - P_n(t)}{h} = -\lambda_n P_n(t) + \lambda_{n-1} P_{n-1}(t)$$

Birth processes and related differential equations

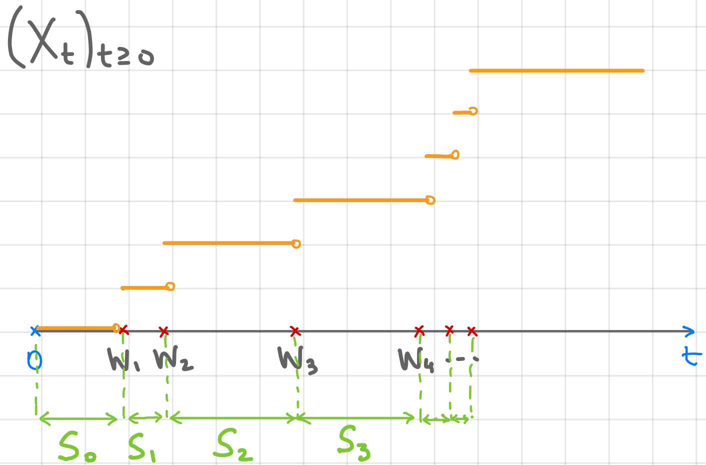
$P_n(t)$ satisfies the following system of differentian eqs. with initial conditions

$$(*)\begin{cases} P_0'(t) = -\lambda_0 P_0(t) & P_0(0) = 1 \\ P_1'(t) = -\lambda_1 P_1(t) + \lambda_0 P_0(t) & P_1(0) = 0 \\ P_2'(t) = -\lambda_2 P_2(t) + \lambda_1 P_1(t) & P_2(0) = 0 \\ \vdots & \vdots \\ P_n'(t) = -\lambda_n P_n(t) + \lambda_{n-1} P_{n-1}(t) & P_n(0) = 0 \\ \vdots & \vdots \end{cases}$$

Solving this system gives the p.m.f. of X_t for any t

$$P(X_t = k) = P_k(t)$$

Description of the birth processes via sojourn times



W_i - i-th "birth time" S_i - "time between (i-1)-th birth and i-th birth"

↳ sojourn times

$$W_i = \sum_{l=0}^{i-1} S_l$$

Alternative way of characterizing $(X_t)_{t\geq 0}$:

- describe the distribution of $(S_i)_{i\geq 0}$
- describe the jumps $X_{W_{i+1}} - X_{W_i}$

Description of the birth processes via sojourn times

Theorem

Let $(\lambda_k)_{k\geq 0}$ be a sequence of positive numbers. Let $(X_t)_{t\geq 0}$ be a non-decreasing right-continuous process, $X_0=0$, taking values in $\{0,1,2\dots\}$. Let $(S_i)_{i\geq 0}$ be the sojourn times associated with $(X_t)_{t\geq 0}$, and define $W_\ell = \sum_{i=0}^{\ell-1} S_i$.

Then conditions

(a) $S_0, S_1, S_2, \dots$ are independent exponential r.v.s of rate $\lambda_0, \lambda_1, \lambda_2 \dots$

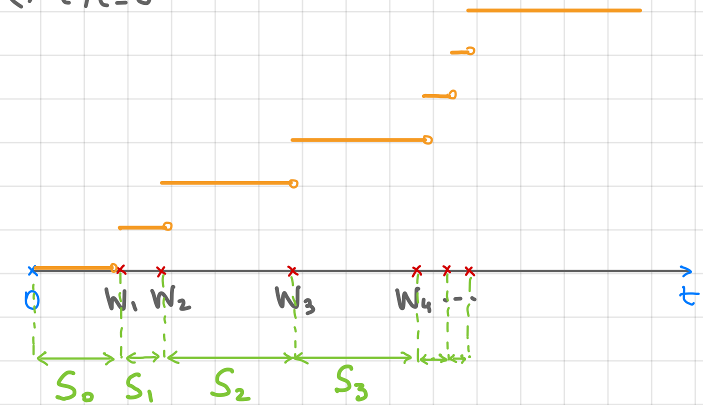
(b) $X_{W_i} = i$ (jumps of magnitude 1)

are equivalent to

(c) $(X_t)_{t\geq 0}$ is a pure birth process with paremeters $(\lambda_k)_{k\geq 0}$.

Explosion

$(X_t)_{t\geq 0}$



population becomes infinite in finite time

Thm. Let $(X_t)_{t\geq 0}$ be a pure birth process of rates $(\lambda_k)_{k\geq 0}$.

Then
- if $\displaystyle\sum_{k=0}^{\infty} \frac{1}{\lambda_k} < \infty$, then $P\big((X_t)_{t\geq 0} \text{ explodes}\big) = 1$
- if $\displaystyle\sum_{k=0}^{\infty} \frac{1}{\lambda_k} = \infty$, then $P\big((X_t)_{t\geq 0} \text{ does not explode}\big) = 1$

Hint. $\displaystyle E\Big(\sum_{k=0}^{\infty} S_k\Big) = \sum_{k=0}^{\infty} \frac{1}{\lambda_k}$

Solving the system of differential equations (*)

$$(*)\begin{cases} P_0'(t) = -\lambda_0 P_0(t), & P_0(0)=1 \\ P_n'(t) = -\lambda_n P_n(t) + \lambda_{n-1} P_{n-1}(t), & P_n(0)=0 \quad \text{for } n \geq 1 \end{cases}$$

$P_0(t)$:

$$P_0'(t) = -\lambda_0 P_0(t)$$

$$\frac{P_0'(t)}{P_0(t)} = -\lambda_0 \qquad \Big(\underbrace{\log(P_0(t))}_{g(t)}\Big)' = \frac{1}{P_0(t)} \cdot P_0'(t) = \frac{P_0'(t)}{P_0(t)}$$

$$g'(t) = -\lambda_0$$

$$g(t) = -\lambda_0 t + K = \log(P_0(t))$$

$$\Rightarrow P_0(t) = e^K e^{-\lambda_0 t} = C e^{-\lambda_0 t}, \quad C > 0$$

$$P_0(0) = C = 1 \Rightarrow C = 1$$

$$\Rightarrow P_0(t) = e^{-\lambda_0 t}$$

Solving the system of differential equations (*)

$P_n(t)$, $n \geq 1$

Consider the function $Q_n(t) = e^{\lambda_n t} P_n(t)$

$$(Q_n(t))' = (e^{\lambda_n t} P_n(t))' = \lambda_n e^{\lambda_n t} P_n(t) + e^{\lambda_n t} (P_n(t))'$$

$$= \lambda_n e^{\lambda_n t} P_n(t) + e^{\lambda_n t}(-\lambda_n P_n(t) + \lambda_{n-1} P_{n-1}(t))$$

$$= \lambda_{n-1} e^{\lambda_n t} P_{n-1}(t)$$

$$Q_n(t) = \int_0^t \lambda_{n-1} e^{\lambda_n s} P_{n-1}(s)\,ds$$

$\hookrightarrow$ $P_n(t) = e^{-\lambda_n t} \int_0^t \lambda_{n-1} e^{\lambda_n s} P_{n-1}(s)\,ds$ $\leftarrow$ apply recursively

(if $\lambda_1 \neq \lambda_0$)

$$P_1(t) = e^{-\lambda_1 t} \int_0^t \lambda_0 e^{\lambda_1 s} e^{-\lambda_0 s}\,ds = e^{-\lambda_1 t} \int_0^t \lambda_0 e^{(\lambda_1 - \lambda_0)s}\,ds$$

$$= e^{-\lambda_1 t} \frac{\lambda_0}{\lambda_1 - \lambda_0} \left(e^{(\lambda_1 - \lambda_0)t} - 1\right) = \frac{\lambda_0}{\lambda_1 - \lambda_0} e^{-\lambda_0 t} - \frac{\lambda_0}{\lambda_1 - \lambda_0} e^{-\lambda_1 t}$$