Math 180A: Introduction to Probability

Lecture A00 (Au)

math.ucsd.edu/~bau/w21.180a

Lecture B00 (Nemish)

math.ucsd.edu/~ynemish/teaching/180a

Today: ASV 5.1, 5.2, 6.1

Video: Prof. Todd Kemp, Fall 2019

Next: ASV 6.1, 6.2

Week 7:

- Homework 6 (due Wednesday, Feb 24)
- Regrades for Homework 4 (Feb 17-19)
- Midterm 2 (Wednesday, Feb 24) covers up to and including MGF
- Practice midterm posted (solutions will be posted on Monday, Feb 22)

Why Should I Care About $M_X(t)$? $M_X(t) = \mathbb{E}(e^{tX})$

Theorem: Suppose $M_X(t) < \infty$ for all t in some neighborhood of 0 $(-\varepsilon, \varepsilon)$.

Then the function M_X uniquely determines the distribution of X.

If X, Y s.t. $M_X(t) = M_Y(t)$ for $-\varepsilon < t < \varepsilon$ then $X \stackrel{d}{=} Y$

$F_X = F_Y$

(I.e. you can recover F_X from M_X.)

(There is no nice formula $F_X \rightsquigarrow M_X$)

$$M_X(t) = \sum_{k=0}^{\infty} \frac{\mathbb{E}(X^k)}{k!} t^k$$

E.g. Suppose I tell you $M_X(t) = e^{2t^2} = \mathbb{E}(e^{tX})$

If $Y \sim N(0,1)$, $\mathbb{E}(e^{tY}) = e^{t^2/2}$; $\therefore \mathbb{E}(e^{t \cdot 2Y}) = M_Y(2t) = e^{(2t)^2/2} = e^{2t^2}$

$\therefore X \sim 2Y \sim N(0,4)$

F_X, f_X, p_X, M_X different tools to use in different contexts.

5.2

Let $X \sim \mathcal{N}(0,1)$.

Question: what is the distribution of X^2? $g(x)$ $g(t)=t^2$

↳ To begin: which tool should we use? F_{X^2}? f_{X^2}? $\cancel{P_{X^2}}$? M_{X^2}? $\cancel{\mathbb{E}(e^{tX^2})}$

$$F_{X^2}(t) = \mathbb{P}(X^2 \le t) \leftarrow \;=0 \text{ if } t<0$$
$$= \mathbb{P}(|X| \le \sqrt{t})$$
$$= \mathbb{P}(-\sqrt{t} \le X \le \sqrt{t}) = \Phi(\sqrt{t}) - \Phi(-\sqrt{t}) = \underbrace{2\Phi(\sqrt{t})-1}_{\text{only for } t\ge 0}.$$

$$\therefore\; f_{X^2}(t) = \frac{d}{dt}\left(2\Phi(\sqrt{t})-1\right) = 2\Phi'(\sqrt{t})\cdot\frac{1}{2\sqrt{t}}$$

$$\Phi'(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$$

$$= \frac{1}{\sqrt{t}}\cdot\frac{1}{\sqrt{2\pi}}e^{-(\sqrt{t})^2/2} = \begin{cases} \dfrac{1}{\sqrt{2\pi t}}e^{-t/2} & t\ge 0 \\ 0 & t<0. \end{cases}$$

"chi-squared (w/ 1-deg of freedom)"

Eg. Toss a fair die, yielding $X \in \{1,2,3,4,5,6\}$.
What is the probability distribution of $Y = |X-3|$?

each w/ $\mathbb{P} = \frac{1}{6}$

X	$\lvert X-3 \rvert$
1	2
2	1
3	0
4	1
5	2
6	3

k	$p_Y(k) = \mathbb{P}(\lvert X-3 \rvert = k)$
0	$1/6$
1	$1/6 + 1/6 = 1/3$
2	$1/6 + 1/6 = 1/3$
3	$1/6$

$\{|X-3|=3\} = \{X=6\}$

In general: if X is discrete, so is $g(X)$, and

$$\mathbb{P}(g(X)=t) = \mathbb{P}\Big(\bigcup_{k:\, g(k)=t} \{X=k\}\Big) = \sum_{k:\, g(k)=t} \mathbb{P}(X=k)$$

$$p_{g(X)}(t) = \sum_{k \in g^{-1}\{t\}} p_X(k) \qquad g^{-1}\{t\} = \{k : g(k)=t\}$$

Important Example

$F_X(s) \in [0,1]$

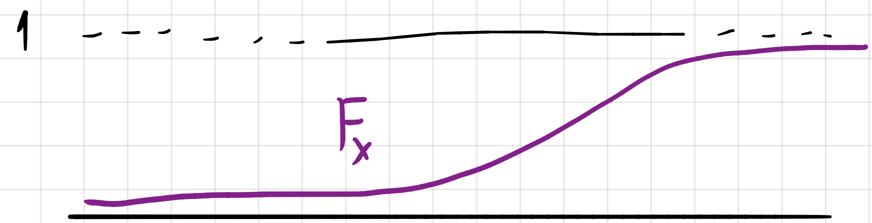
Let X be a random variable.
What is the distribution of $Y = F_X(X)$?

(Assume X continuous. F_X is a strictly increasing funct. $(f_X(t) > 0)$)

$$F_Y(t) = \mathbb{P}(Y \le t) = \mathbb{P}(F_X(X) \le t)$$

$= 0$ if $t < 0$.

$= 1$ if $t > 1$.



$$= \mathbb{P}(F_X(X) \le t) \quad (g = F_X)$$

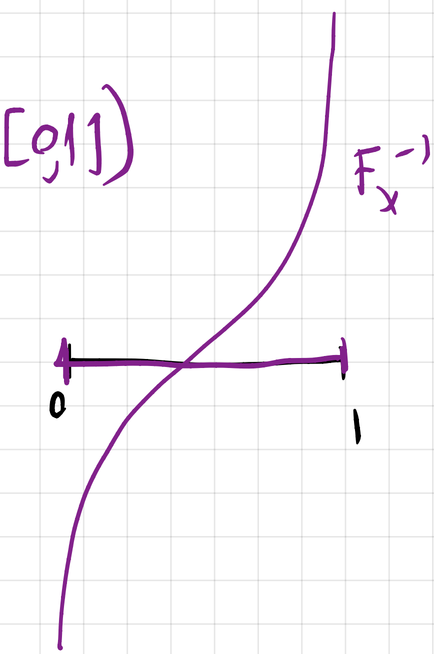
$$= \mathbb{P}(g(X) \le t)$$

$$= \mathbb{P}(X \le g^{-1}(t))$$

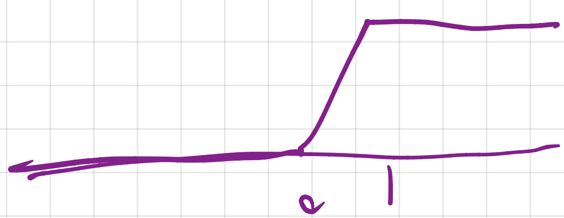
$$= F_X(g^{-1}(t))$$

$$= F_X(F_X^{-1}(t)) = t.$$

$\Big\}$ $Y \sim \mathrm{Unif}([0,1])$



$$F_Y(t) = \begin{cases} 0 & t < 0 \\ t & 0 \le t \le 1 \\ 1 & t > 1 \end{cases}$$



<u>Question</u>: How does a computer generate a $\mathcal{N}(0,1)$ random variable?

To begin: we assume there is a way to produce a $U \sim \text{Unif}([0,1])$ random sample.

Let F be any (strictly increasing) CDF.

Then $F^{-1} : (0,1) \to \mathbb{R}$.

Define $X = F^{-1}(U)$ strictly increasing

$$\mathbb{P}(X \le t) = \mathbb{P}(F^{-1}(U) \le t) = \mathbb{P}(U \le F(t))$$

$$F_X(t) \overset{!!}{=} F(t) \longleftarrow = \begin{cases} 0 & \text{if } \cancel{F(t) < 0} \\ F(t) & \\ 1 & \text{if } \cancel{F(t) > 1} \end{cases}$$

Eg. Sample $\mathcal{N}(0,1)$; just sample $\Phi^{-1}(U)$

Question

6.1

Suppose X and Y are both $\mathrm{Ber}(p)$ random variables. What is $\mathbb{P}(X=Y)$?

(a) $p \cdot p + (1-p)\cdot(1-p)$

(b) $p\cdot(1-p) + (1-p)\cdot p$

(c) 0

(d) 1

(e) Not enough information.

E.g. X, Y independent

$$\{X=Y\} = \{X=1, Y=1\} \cup \{X, Y=0\}$$

$$\mathbb{P}(X=Y) = \mathbb{P}(X=1)\mathbb{P}(X=1) + \mathbb{P}(X=0)\mathbb{P}(X=0) = p\cdot p + (1-p)(1-p)$$

E.g. $X=Y$?

$$\mathbb{P}(X=Y) = 1.$$