

## Lecture 8: Random Processes, Part II

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EE210: Probability and Introductory Random Processes  
KAIST EE

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## Markov Chain

- (1) Definition, Transition Probability Matrix, State Transition Diagram
- (2)  $n$ -step Transition Probability
- (3) Classification of States
- (4) Steady-state Behaviors and Stationary Distribution
- (5) Transient Behaviors

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- Markov chain
- One of the most popular random processes in engineering!

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 $\mathbb{P}(X_{n+1} = 1 | X_n = 1) = 1 - b, \quad \mathbb{P}(X_{n+1} = 2 | X_n = 1) = b$   
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- What will happen at  $(n + 1)$ -th day depends only on what happens at  $n$ -th day?

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for all  $n \geq 0$ , all  $i, j \in \mathcal{S}$ , and all possible sequences  $i_0, \dots, i_{n-1}$  of earlier states,

$$[ ] = \mathbb{P}(X_{n+1} = j | X_n = i, X_{n-1} = i_{n-1}, \dots, X_0 = i_0)$$

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- **Alternate definition via conditional independence.** For any fixed  $n$ , the future of the process after  $n$  is **independent** of  $\{X_1, \dots, X_{n-1}\}$ , **given**  $X_n$ .

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- Thus, for any  $n \geq 0$ , we introduce a simple notation  $p_{ij}$   
$$p_{ij} \triangleq \mathbb{P}(X_{n+1} = j | X_n = i)$$
- (Q) Any convenient way of describing a MC for intuitive understanding?



- Machine example:  $\mathcal{S} = \{1, 2\}$

$$p_{11} = \mathbb{P}(X_{n+1} = 1 | X_n = 1) = 1 - b,$$

$$p_{21} = \mathbb{P}(X_{n+1} = 1 | X_n = 2) = r,$$

$$p_{12} = \mathbb{P}(X_{n+1} = 2 | X_n = 1) = b$$

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The  $m \times m$  matrix  $\mathbf{P} = [p_{ij}]$ , where  $p_{ij} \triangleq \mathbb{P}(X_{n+1} = j | X_n = i)$

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- Property.

$$\sum_{j=1}^m p_{ij} = 1 \text{ (for each row } i, \text{ the column sum} = 1\text{)}$$

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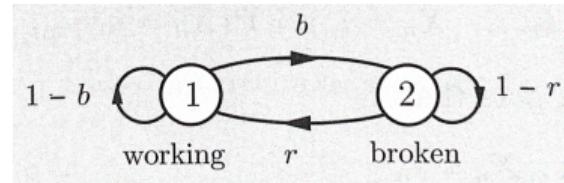
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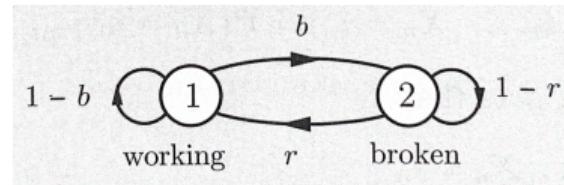
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- Any other way? **State Transition Diagram**



- Transition probability matrix and state transition diagram are the two ways of completely describing a given Markov chain.

- A fly moves along a line in unit increments.

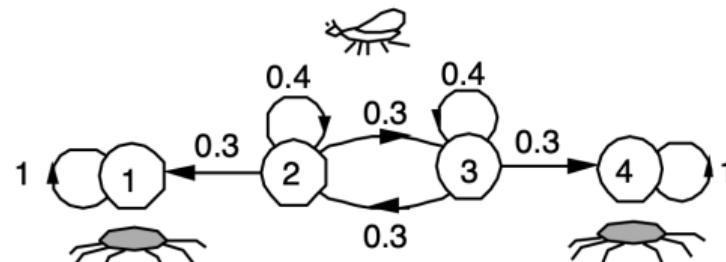
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# Spider-Fly Example (Example 7.2)

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	1	2	3	4
1	1.0	0	0	0
2	0.3	0.4	0.3	0
3	0	0.3	0.4	0.3
4	0	0	0	1.0

$$p_{ij}$$

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VIDEO PAUSE 1

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- $(Y_n : n \geq 0)$  is not a MC.

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  - **Infection model.** If a person becomes infected during a time slot, then he/she will be in an infectious condition (F) during the following time slot, and from then on will be in an purely infected condition (I).

$N \rightarrow F$  (just one slot)  $\rightarrow I$

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- **Contact model.** During every time slot, each of the  $\binom{N}{2}$  pairs of persons are independently in contact w.p.  $p$ 
  - When  $F$  meets with  $N$ , then  $N$  becomes infected, following **infection model**.

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  - Being successful in good modeling depends on the choice of “state” (good modeling sense).
  - Markov chain can be used widely if we choose the state space appropriately.

## Markov Chain

- (1) Definition, Transition Probability Matrix, State Transition Diagram
- (2) *n*-step Transition Probability
- (3) Classification of States
- (4) Steady-state Behaviors and Stationary Distribution
- (5) Transient Behaviors

(Q) What is the probability of a sample path in a Markov chain with the transition probability matrix  $P = [p_{ij}]$ ?



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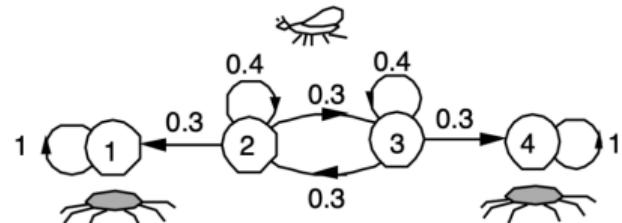
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- Spider-Fly example

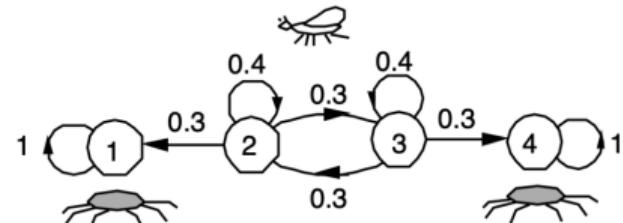


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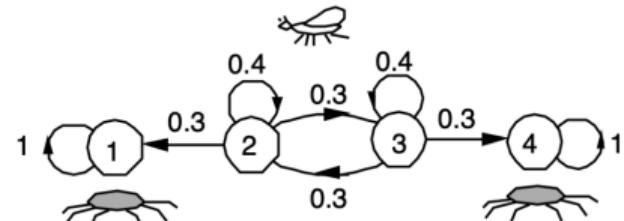


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$$\begin{aligned}\mathbb{P}(X_0 = 2, X_1 = 2, X_2 = 2, X_3 = 3, X_4 = 4) \\ &= \mathbb{P}(X_0 = 2)p_{22}p_{22}p_{23}p_{34} \\ &= \mathbb{P}(X_0 = 2)(0.4)^2(0.3)^2\end{aligned}$$



(Q) What is the probability that my state is  $j$ , after  $n$  steps, starting from  $i$ ?

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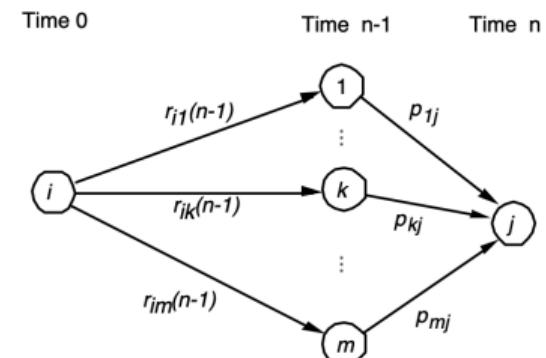
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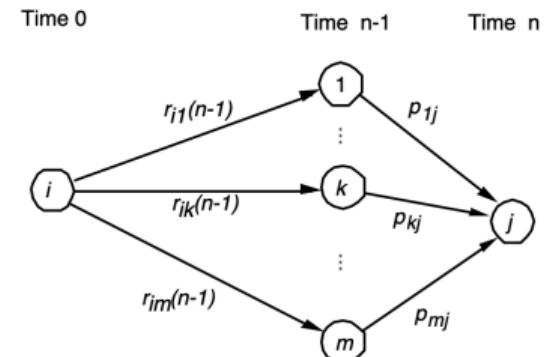

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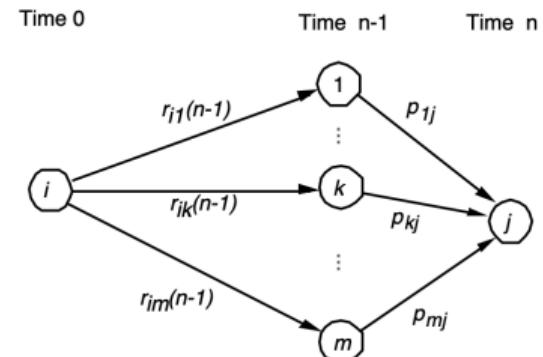
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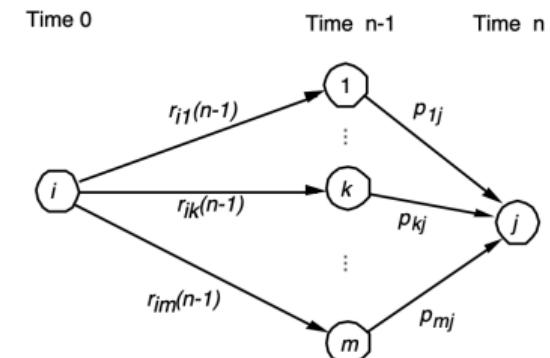
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- Possible to compute  $r_{ij}(n)$  recursively. This is called Chapman-Kolmogorov equation.

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- (Q) What is the relation between  $\mathbf{P}^{(n)}$  and  $\mathbf{P}$ ? Can we express  $\mathbf{P}^{(n)}$  with  $\mathbf{P}$ ?

- $r_{ij}(n+1) = \sum_{k=1}^m r_{ik}(n)r_{kj}(1)$  and  $\mathbf{P}^{(n)} \triangleq [r_{ij}(n)]$

- $r_{ij}(n + l) = \sum_{k=1}^m r_{ik}(n)r_{kj}(l)$  and  $\mathbf{P}^{(n)} \triangleq [r_{ij}(n)]$
- Then, by letting  $n = 1, l = 1$ ,  
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- By letting  $n = 2, l = 1$ ,  
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- In other words,  $n$ -step transition probability matrix is just a  **$n$ -time multiplication** of the transition probability matrix  $\mathbf{P}$ .

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- By computing  $P^4$ , we get  $r_{2,1}(4) = 0.4352$  and  $r_{2,2}(4) = 0.4872$

## Example: Urn with Two Balls

- An urn always contains 2 balls. Ball colors are **red** and **blue**.
- At each stage, a ball is randomly chosen, and then replaced by a new ball, which with probability 0.8 is the same color, and with probability 0.2 is the opposite color, as the ball it replaces.
- If initially both balls are red, find the probability that the **fifth ball** selected is red.
- **Solution.** Let  $X_n$  be the number of red balls after  $n$ -th stage (selection and replacement). Then,  $\mathcal{S} = \{0, 1, 2\}$ .

$$\bullet \quad P = \begin{pmatrix} 0.8 & 0.2 & 0 \\ 0.1 & 0.8 & 0.1 \\ 0 & 0.2 & 0.8 \end{pmatrix}$$

- Let  $A = \{\text{fifth ball is red}\}$ .

$$\begin{aligned} \mathbb{P}(A) &= \sum_{i=0}^2 \mathbb{P}(A|X_4 = i) \mathbb{P}(X_4 = i | X_0 = 2) \\ &= (0)r_{2,0}(4) + (0.5)r_{2,1}(4) + (1)r_{2,2}(4) \end{aligned}$$

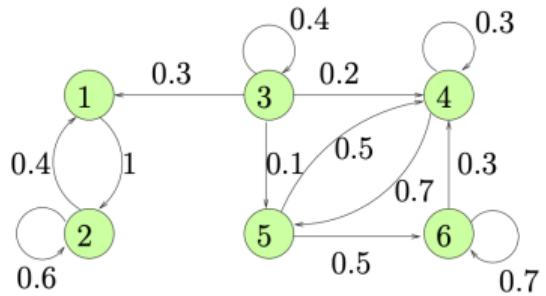
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- Thus,  $\mathbb{P}(A) = 0.7048$

## Markov Chain

- (1) Definition, Transition Probability Matrix, State Transition Diagram
- (2)  $n$ -step Transition Probability
- (3) **Classification of States**
- (4) Steady-state Behaviors and Stationary Distribution
- (5) Transient Behaviors

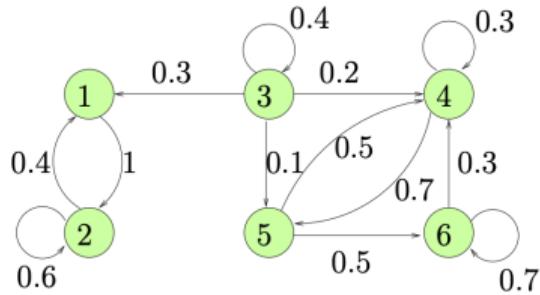
# Different States and Classes?

- Classes



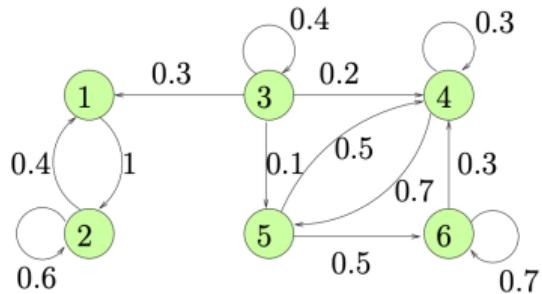
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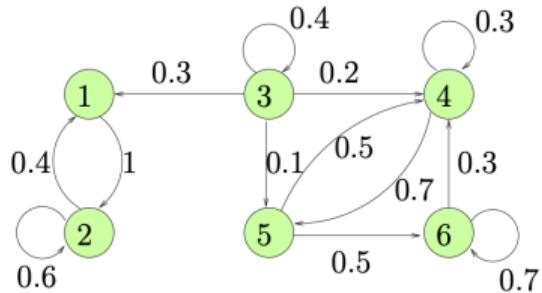
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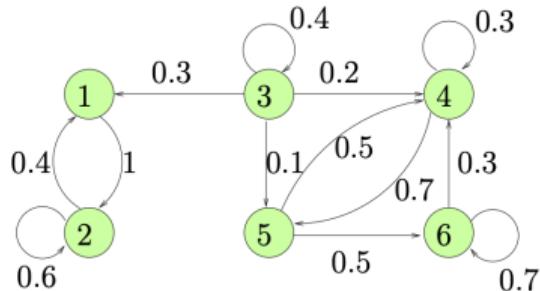
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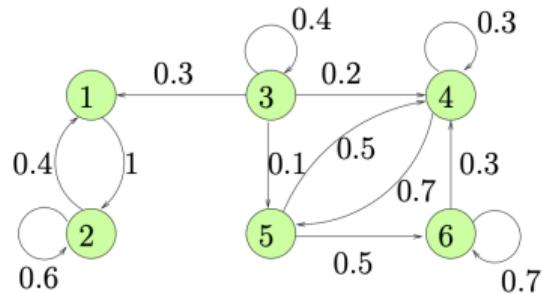
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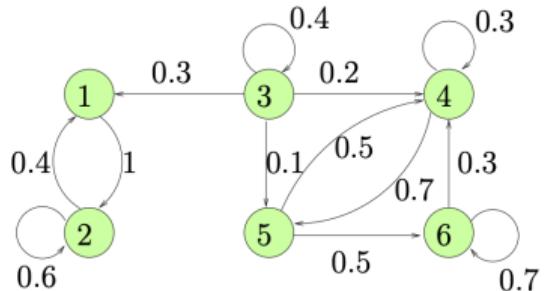
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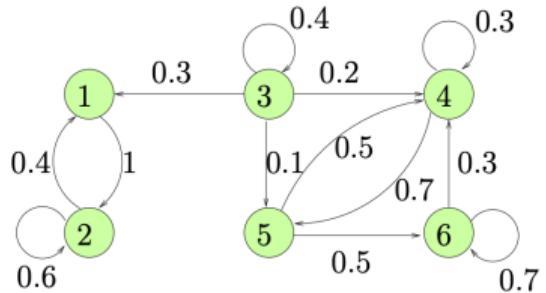
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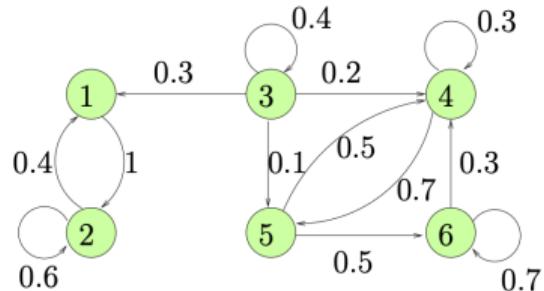
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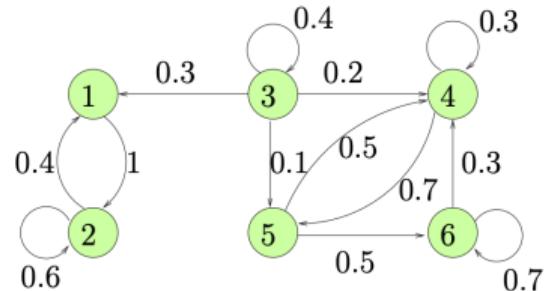
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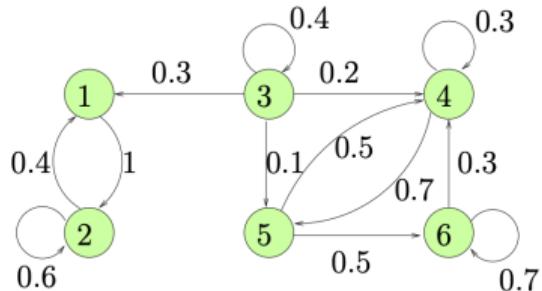
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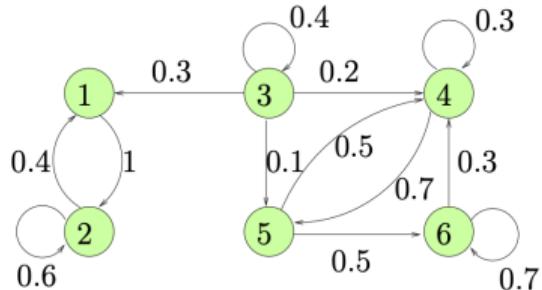
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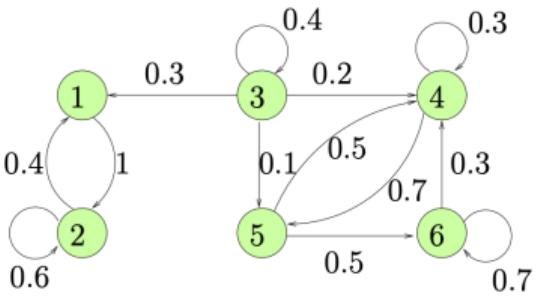


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- **Message 3.** States in the same class share some properties.

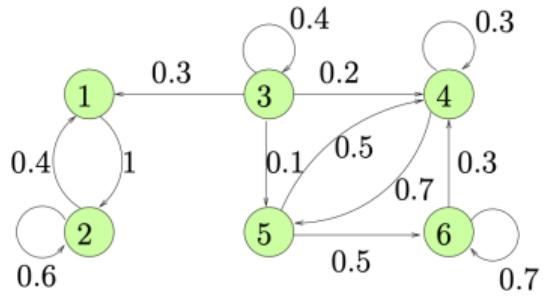


# Classification of States (1)



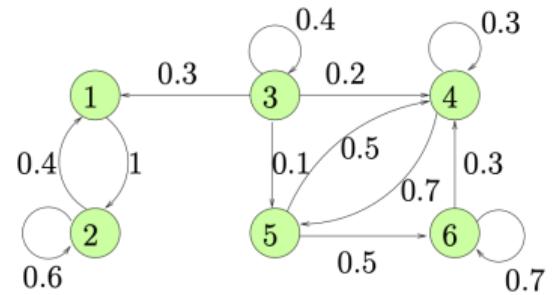
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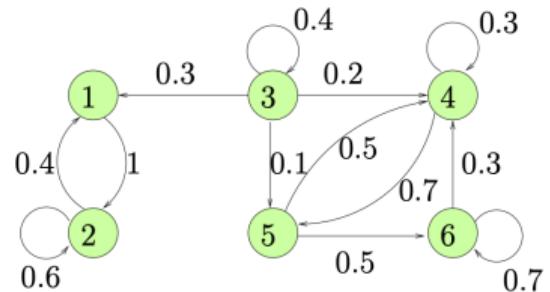
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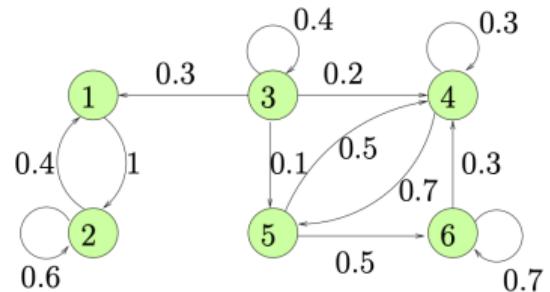
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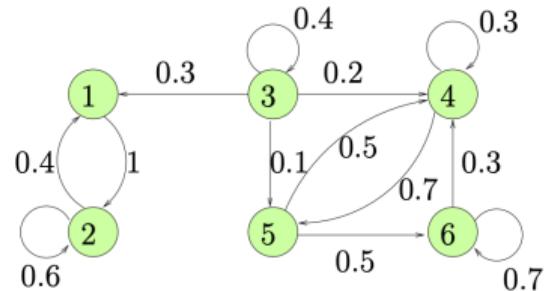
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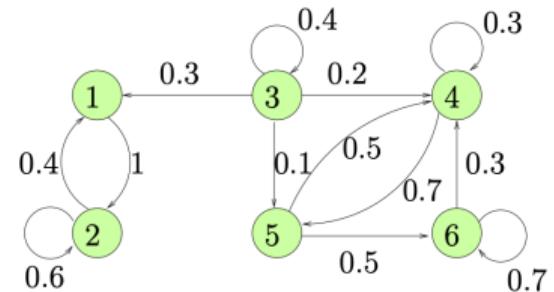
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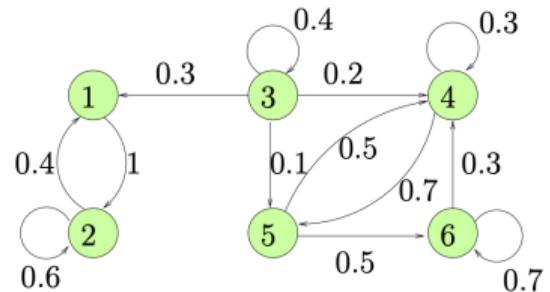
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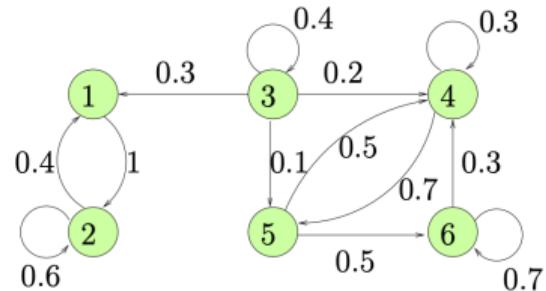
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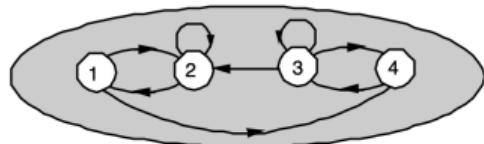
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  - A state that is not recurrent is **transient**.
  - 2 is recurrent? Yes. 3 is recurrent? No.
  - If we start from a recurrent state  $i$ , then there is always some probability of returning to  $i$ . It means that, given enough time, it is certain that it returns to  $i$ .

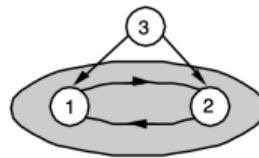


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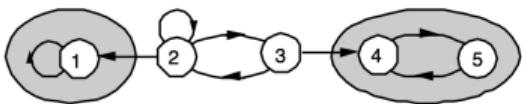
- A set of recurrent states which communicate with each other form a **class**.



Single class of recurrent states



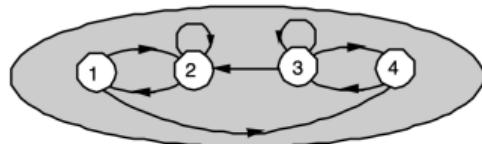
Single class of recurrent states (1 and 2)  
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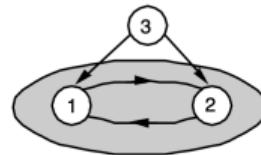
Two classes of recurrent states  
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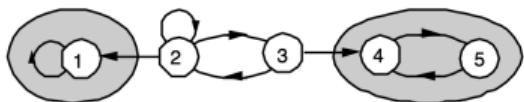
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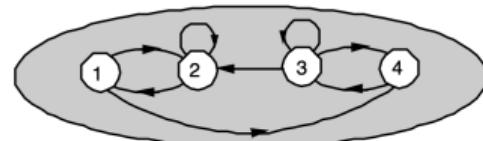
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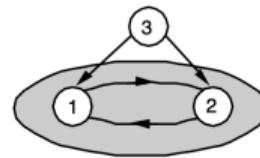
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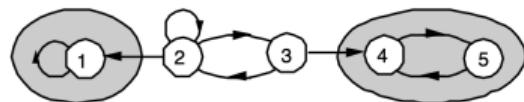
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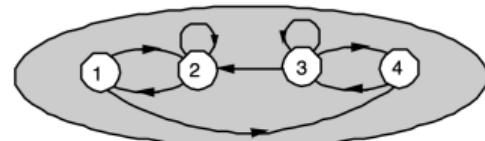
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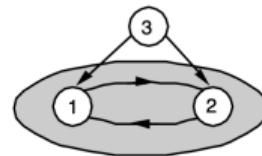
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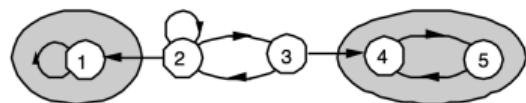
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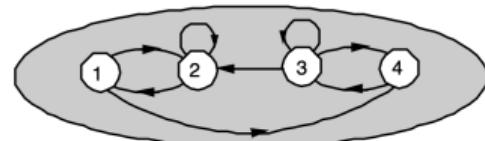
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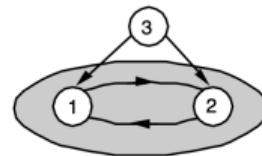
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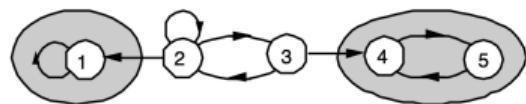
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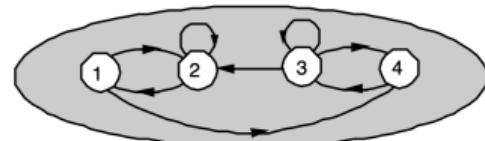
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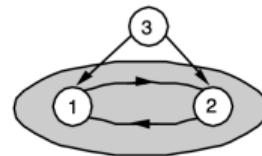
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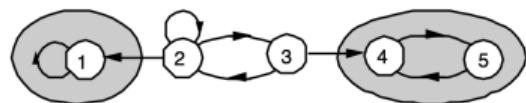
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  - A transient state is not accessible from any recurrent state.
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- The MC with only a single recurrent class is said to be **irreducible** (더이상 분해할 수 없는).



Single class of recurrent states

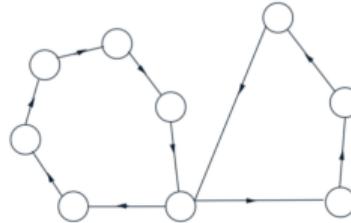
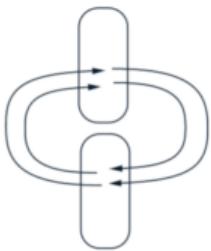
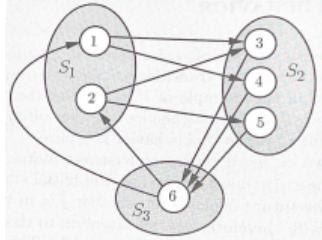


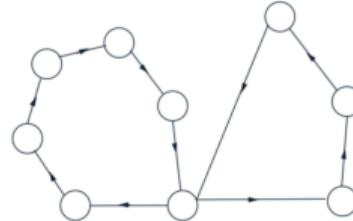
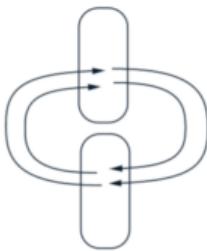
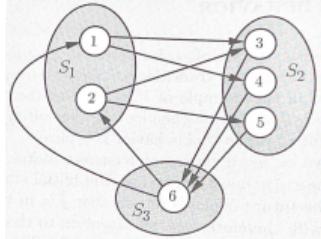
Single class of recurrent states (1 and 2)  
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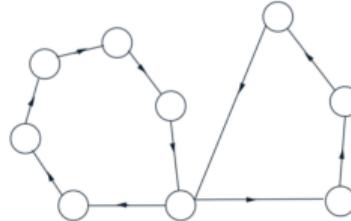
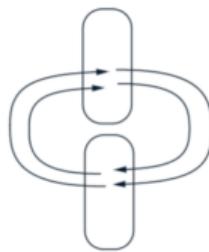
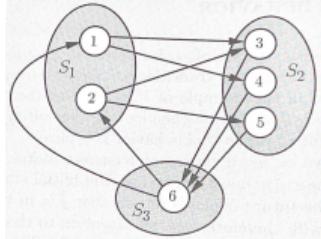
Two classes of recurrent states  
(class of state 1 and class of states 4 and 5)  
and two transient states (2 and 3)

# Periodicity

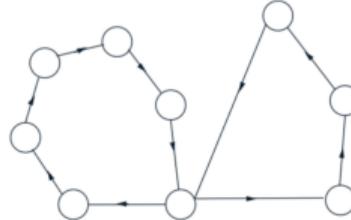
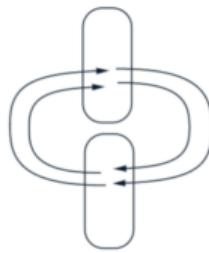
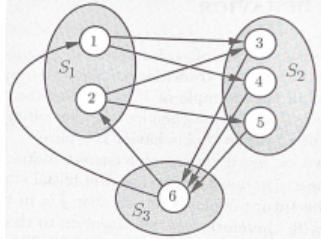




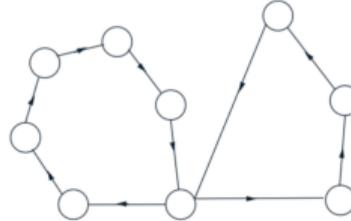
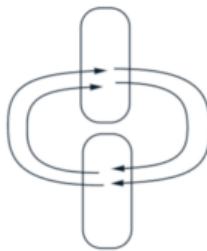
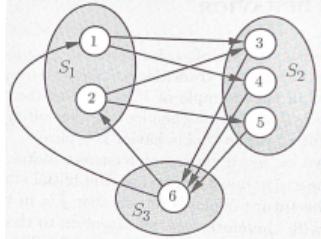
- **Definition.** A recurrent class is said to be **periodic**, if its states can be grouped in  $d > 1$  disjoint subsets  $S_1, \dots, S_d$  so that all transitions from  $S_k$  lead to  $S_{k+1}$  (or to  $S_1$  if  $k = d$ ). We call  $d$  the **period** of the recurrent class.



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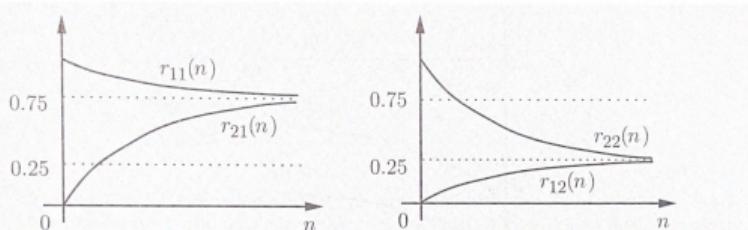


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- Often, it is not easy to see some MC is periodic or not. But, one easy way is to check whether there exists a self-transition or not. **An MC with a self-transition must be aperiodic.**

## Markov Chain

- (1) Definition, Transition Probability Matrix, State Transition Diagram
- (2)  $n$ -step Transition Probability
- (3) Classification of States
- (4) Steady-state Behaviors and Stationary Distribution
- (5) Transient Behaviors

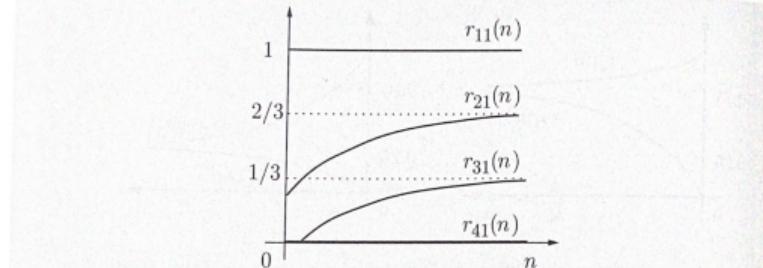
# $n$ -step transition prob.: $r_{ij}(n)$ for large $n$



$n$ -step transition probabilities as a function of the number  $n$  of transitions

	U	B				
U	0.8	0.2	.76	.24	.752	.248
B	0.6	0.4	.72	.28	.744	.256
$r_{ij}(1)$			$r_{ij}(2)$		$r_{ij}(3)$	
			$r_{ij}(4)$		$r_{ij}(5)$	

Sequence of  $n$ -step transition probability matrices



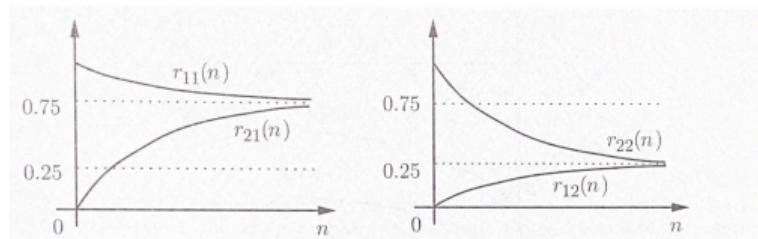
$n$ -step transition probabilities into state 1

	1	2	3	4		1	2	3	4		1	2	3	4		1	2	3	4		1	2	3	4	
1	1.0	0	0	0		1.0	0	0	0		1.0	0	0	0		1.0	0	0	0		1.0	0	0	0	
2	0.3	0.4	0.3	0		.42	.25	.24	.09		.50	.17	.17	.16		.55	.12	.12	.21		2/3	0	0	1/3	
3	0	0.3	0.4	0.3		.09	.24	.25	.42		.16	.17	.17	.50		.21	.12	.12	.55		1/3	0	0	2/3	
4	0	0	0	1.0		0	0	0	1.0		0	0	0	1.0		0	0	0	1.0		0	0	0	1.0	
						$r_{ij}(1)$					$r_{ij}(2)$					$r_{ij}(3)$					$r_{ij}(4)$				

Sequence of transition probability matrices

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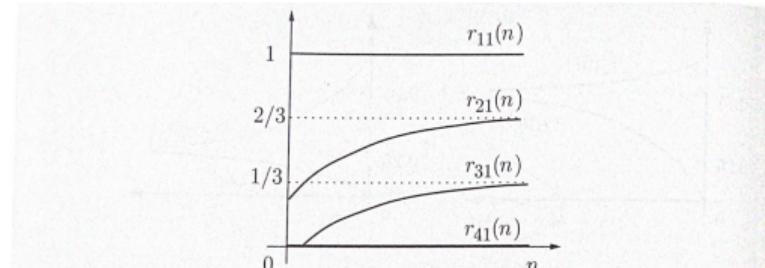
- Convergence **irrespective** of the start state



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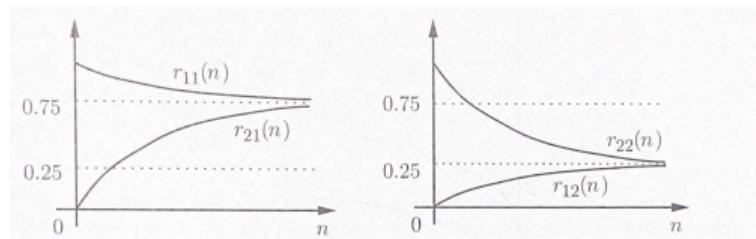


$n$ -step transition probabilities into state 1

	1	2	3	4						
1	1.0	0	0	0	$r_{ij}(1)$	$r_{ij}(2)$	$r_{ij}(3)$	$r_{ij}(4)$	$r_{ij}(\infty)$	
2	0.3	0.4	0.3	0						
3	0	0.3	0.4	0.3						
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Sequence of transition probability matrices

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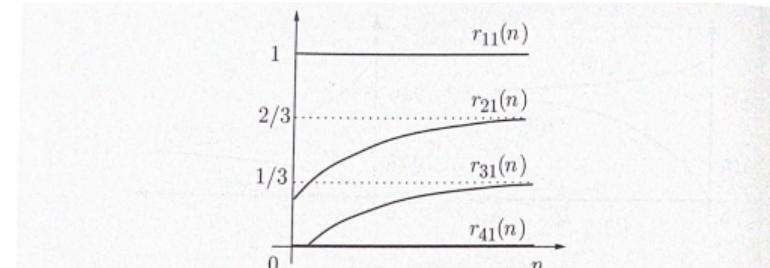


*n*-step transition probabilities as a function of the number  $n$  of transitions

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						$r_{ij}(4)$
						$r_{ij}(5)$

Sequence of *n*-step transition probability matrices

- Convergence **depending on** the start state

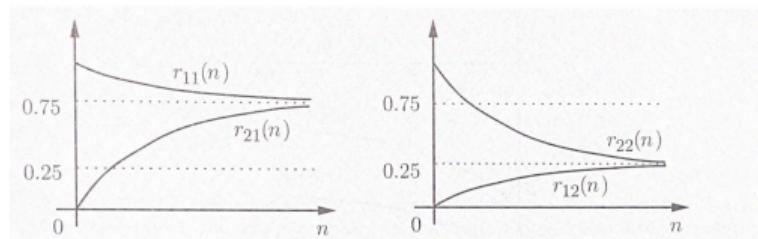


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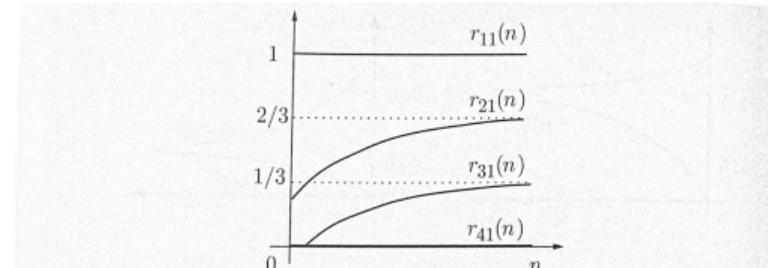


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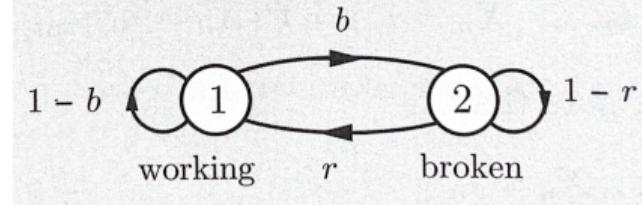
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Sequence of transition probability matrices

(Q) Under what conditions, convergence occurs, **independent of** the start state? If so, how does it depend on the start state and the shape of the MC?

- $r_{ij}(n) \xrightarrow{n \rightarrow \infty} \pi_j$ , for some  $\pi_j \leq 1$ ?

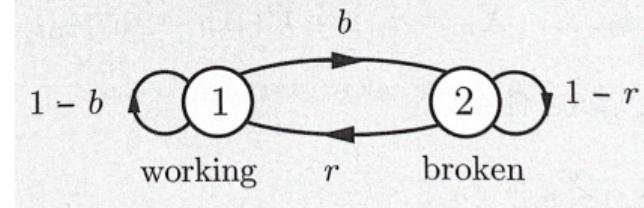


$$\pi_{\text{working}} = \alpha$$

$$\pi_{\text{broken}} = \beta$$

- $r_{ij}(n) \xrightarrow{n \rightarrow \infty} \pi_j$ , for some  $\pi_j \leq 1$ ?
- Interpretation.

$\pi_j \approx \mathbb{P}(X_n = j)$  for large  $n$



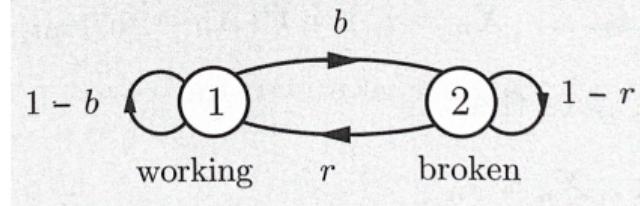
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- After running the MC for a long time, we see how long the MC will stay at which state on average.



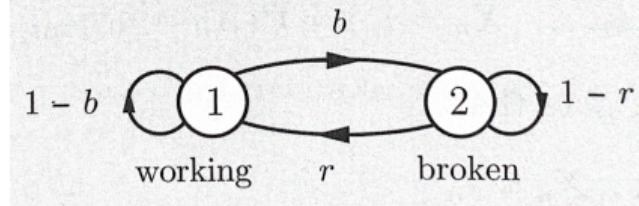
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- After running the MC for a long time, we see how long the MC will stay at which state on average.
- Helps in understanding how this MC behaves.



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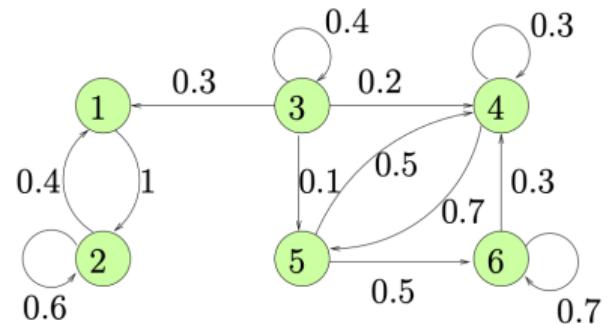
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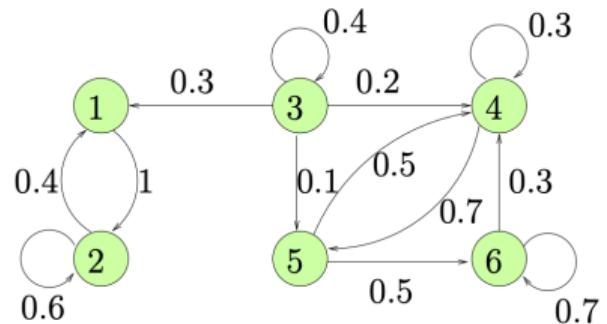
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**C1.** For the case of multiple recurrent classes, one stays at the class including the starting state.



(a) multiple recurrent classes

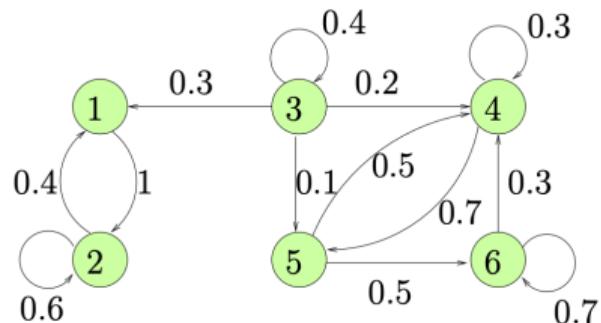
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  - Convergence occurs, **independent of** the starting state, if:
    - C1.** Only a **single recurrent class**
    - C2.** such recurrent class is **aperiodic**
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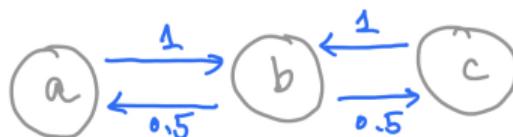
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- C1.** For the case of multiple recurrent classes, one stays at the class including the starting state.
- C2.** Divergent behavior for periodic recurrent classes.



(a) multiple recurrent classes



(b) single recurrent, but periodic class

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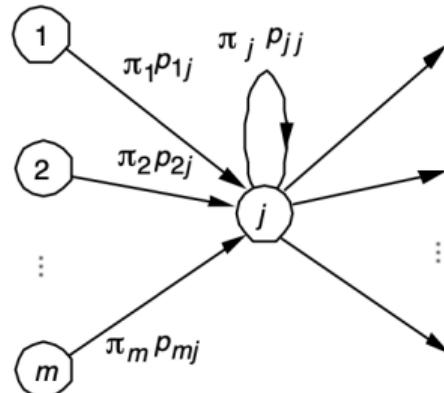
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- $\sum_{i=1}^m \pi_i = 1$ : (Normalization equation)
- Balance eqn. + Normalization eqn.  $\implies$  Finding the steady-state probabilities  $\{\pi_i\}$ .
  - Solving linear equations

- Probability: often interpreted as the **relative frequencies** out of many independent trials
- $\pi_j = \lim_{n \rightarrow \infty} \frac{v_{ij}(n)}{n}$ , where  $v_{ij}(n)$  is the expected number of visits to state  $j$  up to the first  $n$  transitions
- In other words,  $\pi_j$ : long-term **expected fraction of time** that the MC is at the state  $j$ .
- $\pi_j p_{jk}$ : the long-term expected **fraction of transitions** that move the state **from  $j$  to  $k$** .

- Balance equation:  $\sum_{k=1}^m \pi_k p_{kj} = \pi_j$ 
  - The expected frequency of visits to  $j$  = The sum of the expected frequencies of transitions that lead to  $j$ .



- A two-state MC with:  $\begin{pmatrix} 0.8 & 0.2 \\ 0.6 & 0.4 \end{pmatrix}$
- (Balance equation)

$$\pi_1 =$$

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- (Normalization equation)  $\pi_1 + \pi_2 = 1$
- Steady-state probabilities:  $\pi_1 = 0.25$ ,  $\pi_2 = 0.75$ .

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- Similarly, we have  $\mathbb{P}(X_n = j) = \pi_j$ , for all  $n$  and  $j$ .

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$$\mathbb{P}(X_0 = j) = \pi_j \xrightarrow{\text{total prob. theorem}} \mathbb{P}(X_1 = j) = \sum_{k=1}^m \mathbb{P}(X_0 = k) p_{kj} = \sum_{k=1}^m \pi_k p_{kj} = \pi_j$$

- Similarly, we have  $\mathbb{P}(X_n = j) = \pi_j$ , for all  $n$  and  $j$ .
- If the initial state is chosen according to  $\{\pi_j\}$ , the state at any future time will have the same distribution (i.e., the distribution does not change over time).

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- $\{\pi_j\}$  is also called a **stationary distribution**. Why?
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- We say that "the limiting distribution (steady-state distribution) is equal to the stationary distribution"

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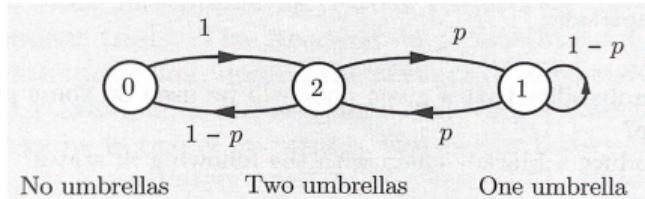
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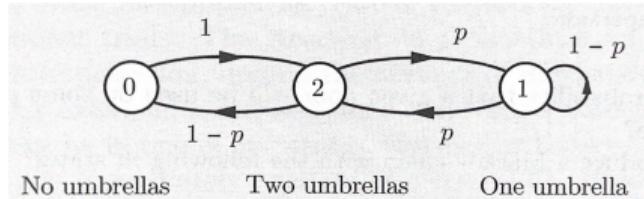


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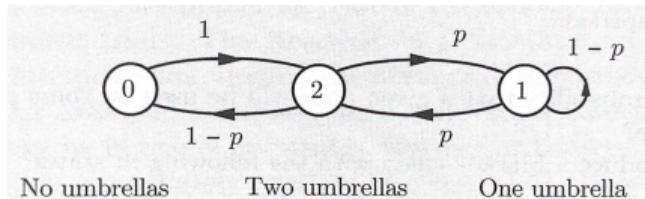
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- Single recurrent class and aperiodic
- Balance and normalization equation

$$\pi_0 = \quad , \quad \pi_1 =$$

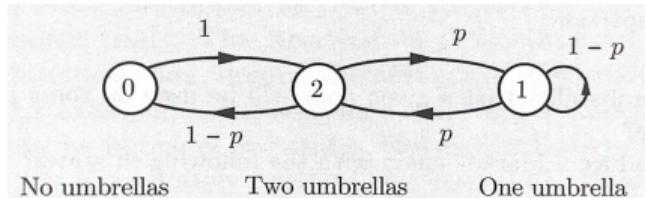
$$\pi_2 = \quad , \quad \pi_0 + \pi_1 + \pi_2 = 1$$

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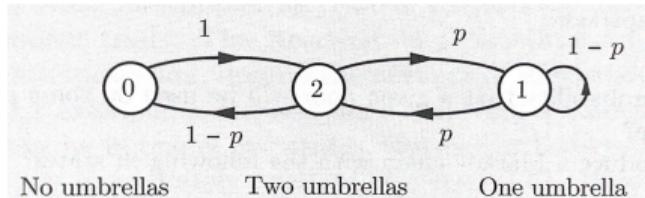
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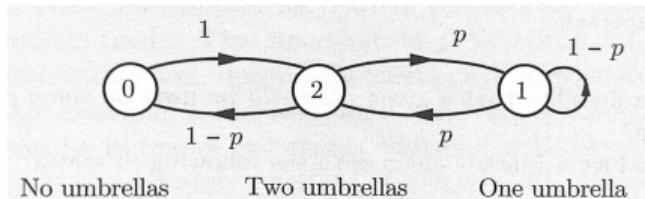
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- $\pi_0 = \frac{1-p}{3-p}$ ,  $\pi_1 = \frac{1}{3-p}$ ,  $\pi_2 = \frac{1}{3-p}$ .
- The answer is  $p \times \pi_0$ .

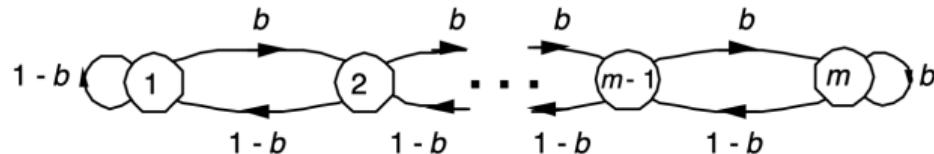
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## Example 3: Random Walk with Reflecting Barriers

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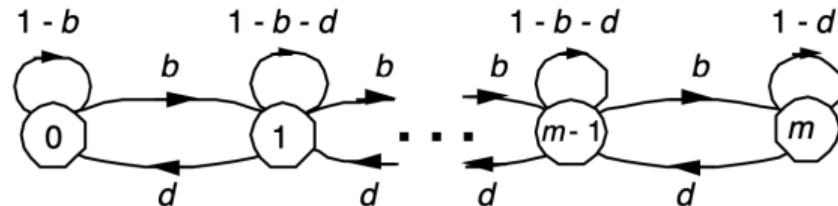
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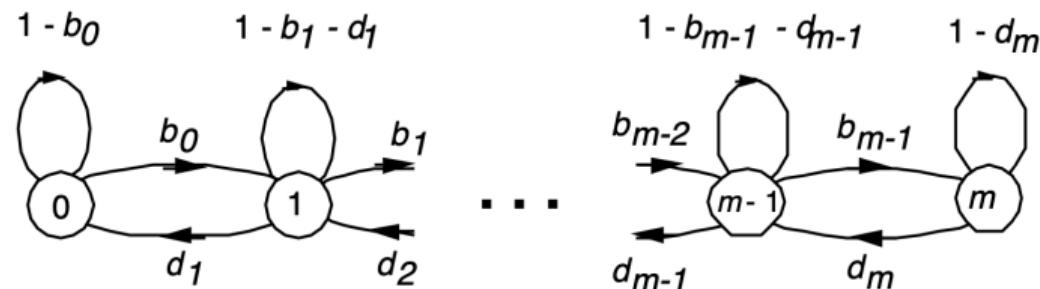


- A special type of Markov chain where the states are **linearly arranged** and transitions can occur only to a **neighboring** state.

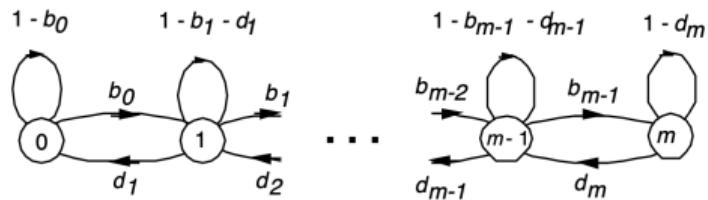
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- Birth and Death

$b_i = \mathbb{P}(X_{n+1} = i + 1 | X_n = i)$ , birth probability at state  $i$

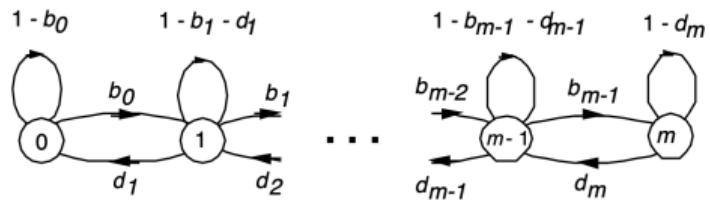
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- State transition diagram



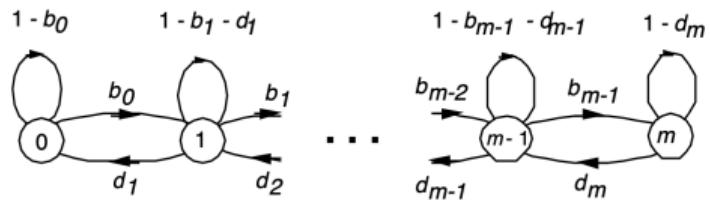
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- Balance eqn at state 0

$$\pi_0(1 - b_0) + \pi_1 d_1 = \pi_0 \leftrightarrow$$

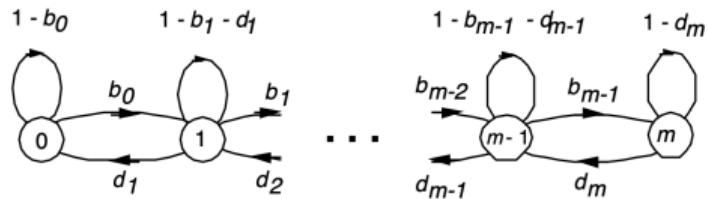
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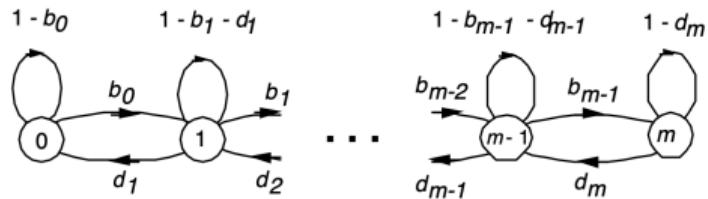
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$$\pi_0 b_0 + \pi_1 (1 - b_1 - d_1) + \pi_2 d_2 = \pi_1$$

$\leftrightarrow$   
 $\leftrightarrow$

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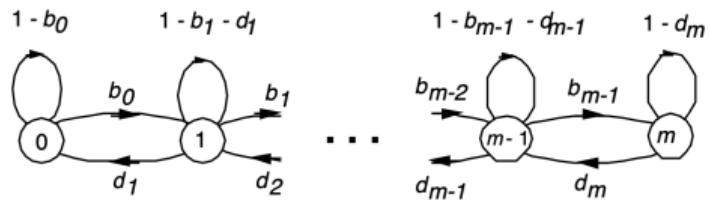
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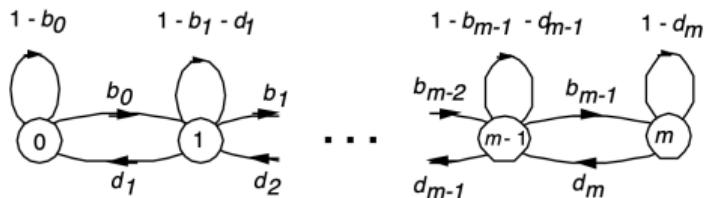
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## Birth-Death Process (2)

- State transition diagram



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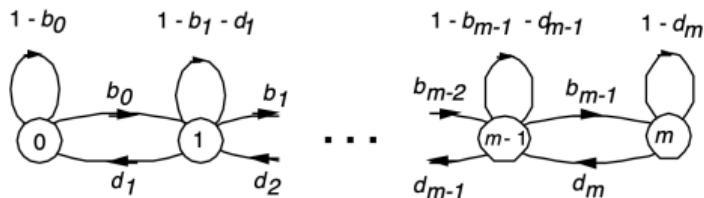
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[ ]:

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- State transition diagram



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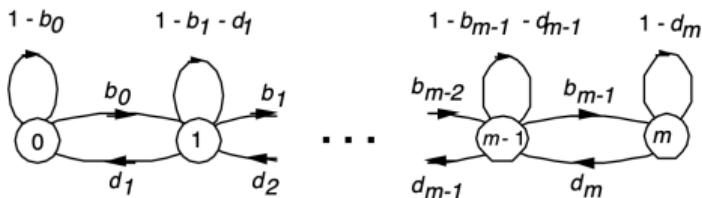
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$$\pi_i b_i = \pi_{i+1} d_{i+1}, i = 0, 1, \dots, m-1$$

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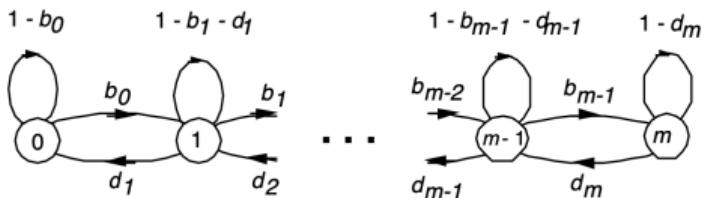
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- Using the above local balance eqn,

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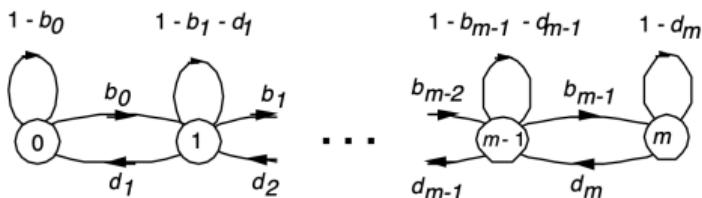
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- Examples 3 and 4 are the special cases of birth-death process. So, please compute the steady-state probabilities for both examples as your homeworks.

## Markov Chain

- (1) Definition, Transition Probability Matrix, State Transition Diagram
- (2)  $n$ -step Transition Probability
- (3) Classification of States
- (4) Steady-state Behaviors and Stationary Distribution
- (5) Transient Behaviors

- In the previous lecture,
- In this lecture,

- In the previous lecture,
  - A MC with a single recurrent, aperiodic class  $\mathcal{R} = \{1, 2, \dots, m\}$
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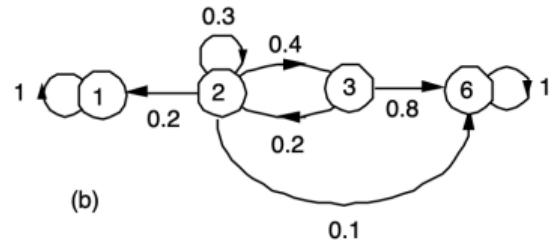
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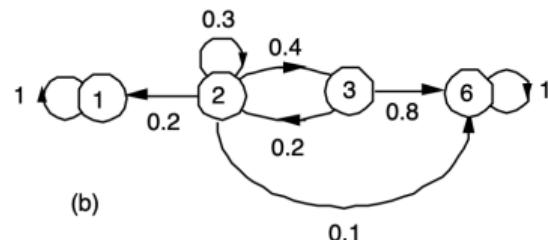
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- (Q) Transient behavior: what is the first recurrent state to be entered as well as the time until this happens?

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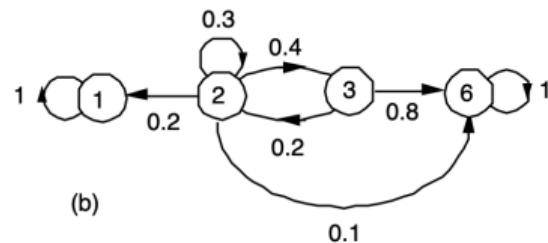
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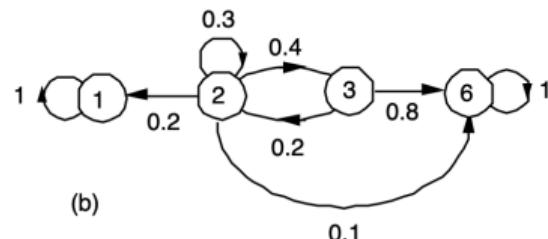
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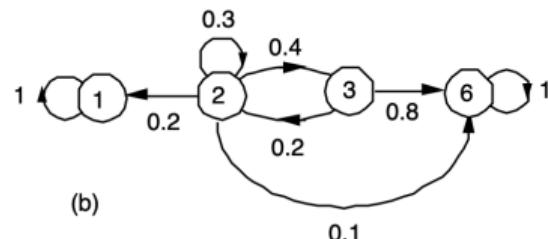
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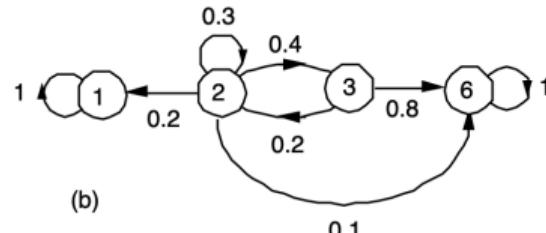
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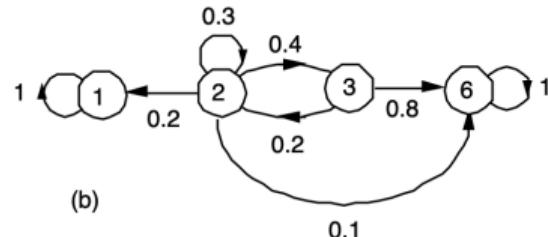
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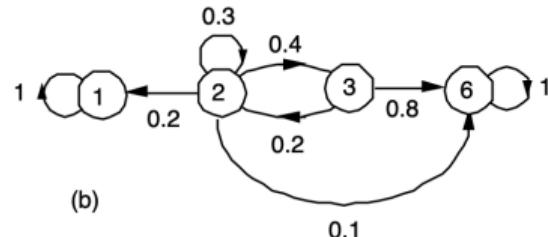
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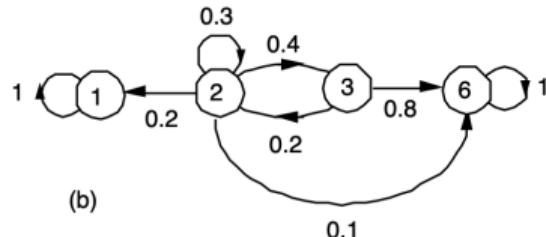
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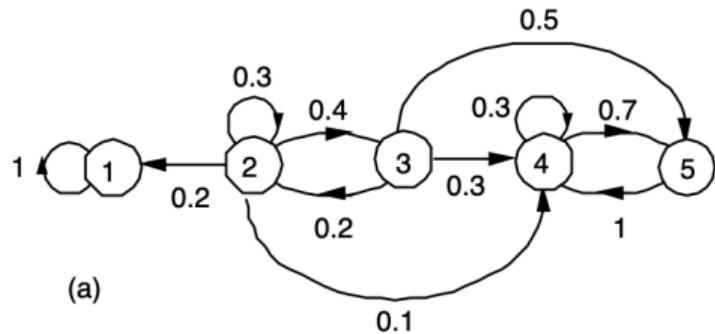
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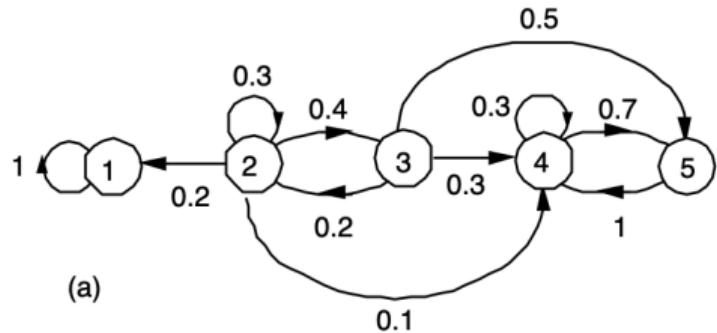


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- $a_2 = 21/31$  and  $a_3 = 29/31$

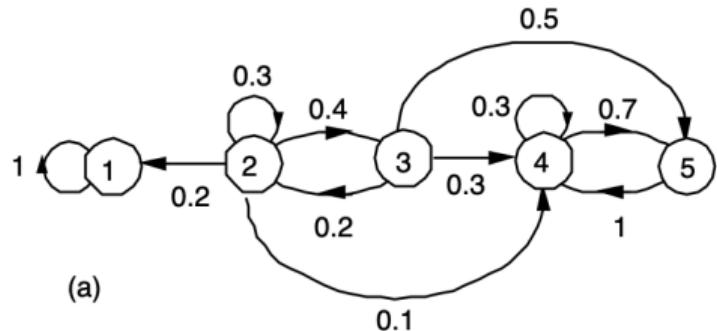
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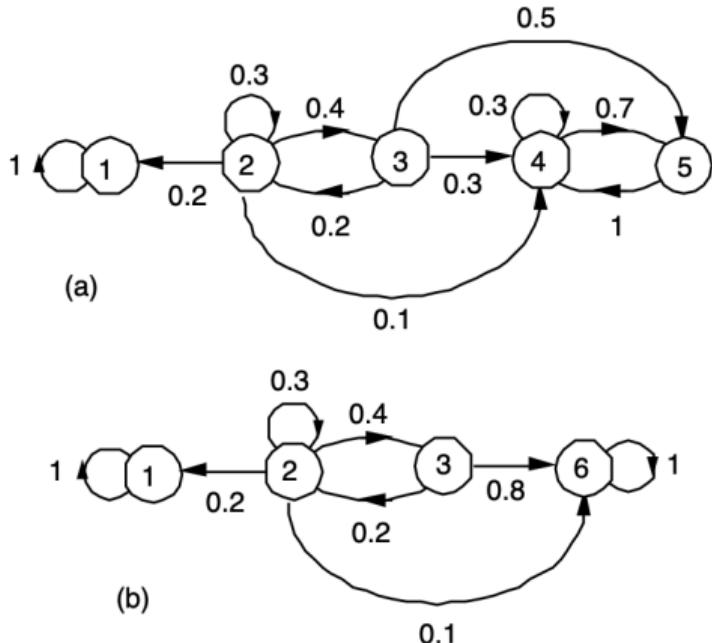
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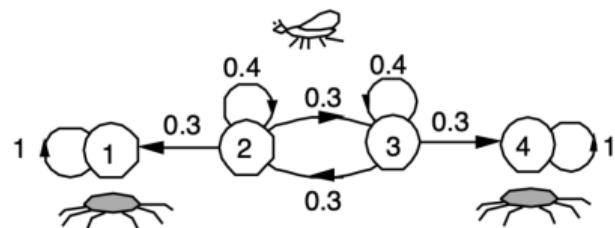
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- Thus, convert it into the one only with absorbing recurrent states ((a)  $\rightarrow$  (b)).

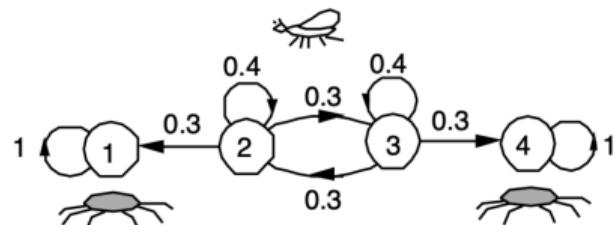


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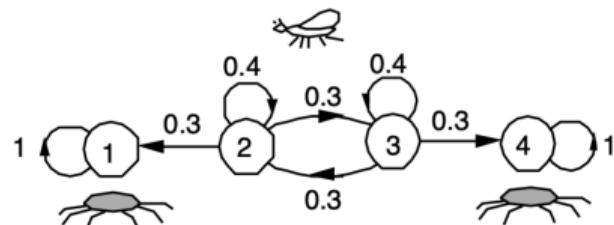


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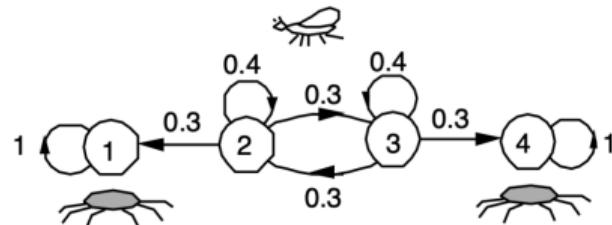


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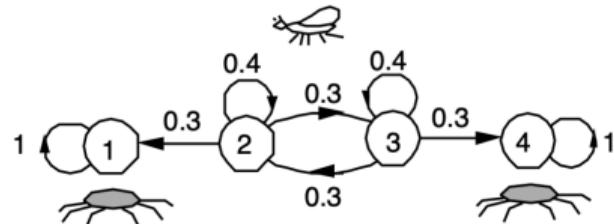


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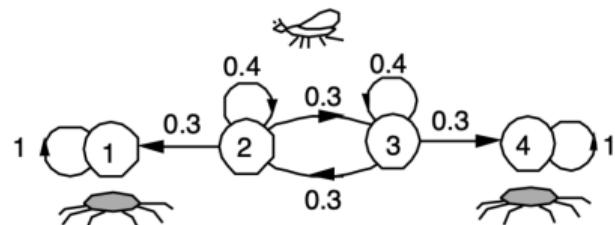


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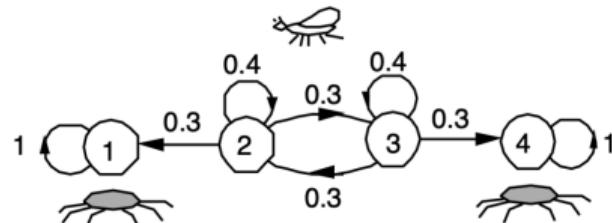
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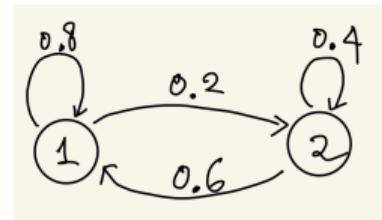
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- Again, for general MCs, convert them into the one with only recurrent states that are absorbing

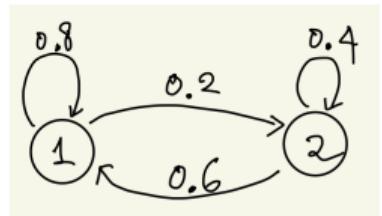


- Assume a single recurrent class for simplicity



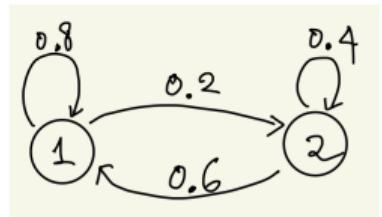
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$$t_2 = 1 + p_{21}t_1 + p_{22}t_2 = 1 + 0.4t_2 \implies t_2 = 5/3$$

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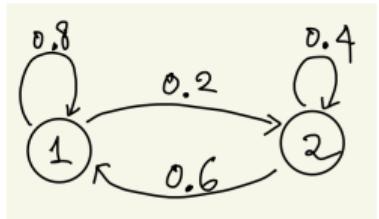
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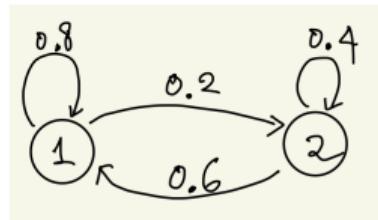
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- Mean first recurrence time from 1 to 1

$$t_1^* = 1 + p_{11}t_1 + p_{12}t_2 = 1 + 0 + 0.2\frac{5}{3} = \frac{4}{3}$$



Questions?

- 1) Why do you think Markov chain (MC) is important?
- 2) What is the Markov property and its meaning? What's the key difference of MC from Bernoulli processes?
- 3) What are the limiting distribution and the stationary distribution of MCs?
- 4) How are you going to compute the stationary distribution, if you are given a transition probability matrix?
- 5) What are recurrent and transient states in MC?