

## Lecture 8: Random Processes, Part II

Yi, Yung (이용)

EE210: Probability and Introductory Random Processes  
KAIST EE

June 12, 2021

### Markov Chain

- (1) Definition, Transition Probability Matrix, State Transition Diagram
- (2)  $n$ -step Transition Probability
- (3) Classification of States
- (4) Steady-state Behaviors and Stationary Distribution
- (5) Transient Behaviors

June 12, 2021 1 / 1

June 12, 2021 2 / 1

## Roadmap

### Recap and Markov Chain

#### Markov Chain

- (1) Definition, Transition Probability Matrix, State Transition Diagram
- (2)  $n$ -step Transition Probability
- (3) Classification of States
- (4) Steady-state Behaviors and Stationary Distribution
- (5) Transient Behaviors

- Assume discrete times  $n = 1, 2, \dots$
- Random process: A sequence of  $X_1, X_2, X_3, \dots$
- “Simplest” random process
  - Process without memory
  - Bernoulli process
- A random process that is just a little more general than the above?
  - Process that depends only on “yesterday”, not the entire history
  - $\mathbb{P}(X_n = i_n | X_{n-1} = i_{n-1}, X_{n-2} = i_{n-2}, X_{n-3} = i_{n-3}, \dots, X_1 = i_1) = \mathbb{P}(X_n = i_n)$
  - Markov chain
  - One of the most popular random processes in engineering!

- A machine: working or broken down on a given day.
  - If working, break down in the next day w.p.  $b$ , and continue working w.p.  $1 - b$ .
  - If broken down, it will be repaired and be working in the next day w.p.  $r$ , and continue to be broken down w.p.  $1 - r$ .
- $X_n \in \{1, 2\}$ : status of the machine, 1: working and 2: broken down
- $(X_n)_{n=1}^{\infty}$ : A random process satisfying: for any  $n \geq 1$ ,
 
$$\begin{aligned}\mathbb{P}(X_{n+1} = 1 | X_n = 1) &= 1 - b, & \mathbb{P}(X_{n+1} = 2 | X_n = 1) &= b \\ \mathbb{P}(X_{n+1} = 1 | X_n = 2) &= r, & \mathbb{P}(X_{n+1} = 2 | X_n = 2) &= 1 - r\end{aligned}$$
- What will happen at  $(n + 1)$ -th day depends only on what happens at  $n$ -th day?

- **Definition.** Let  $X_1, \dots, X_n, \dots$  be a sequence of random variables taking values in some finite space  $\mathcal{S} = \{1, 2, \dots, m\}$ , such that for all  $i, j \in \mathcal{S}$ ,  $n \geq 0$ , the following **Markov property** is satisfied:  
for all  $n \geq 0$ , all  $i, j \in \mathcal{S}$ , and all possible sequences  $i_0, \dots, i_{n-1}$  of earlier states,

$$\boxed{\mathbb{P}(X_{n+1} = j | X_n = i)} = \mathbb{P}(X_{n+1} = j | X_n = i, X_{n-1} = i_{n-1}, \dots, X_0 = i_0)$$

- **Alternate definition via conditional independence.** For any fixed  $n$ , the future of the process after  $n$  is **independent** of  $\{X_1, \dots, X_{n-1}\}$ , **given**  $X_n$ .

<sup>0</sup>A Markov chain can be also defined for the infinite  $\mathcal{S} = \{1, 2, \dots\}$ , but we just focus on the finite state space in this lecture.

- The value that  $X_n$  can take is called **state** (e.g., working or broken down in the previous example). Thus, the space  $\mathcal{S} = \{1, \dots, m\}$  is called **state space**.
- We will focus on the MC of **time homogeneity**. The probability  $\mathbb{P}(X_{n+1} = j | X_n = i)$  does NOT depends on  $n$ .
  - In the machine failure example,  $\mathbb{P}(X_{100} = 1 | X_{99} = 1) = \mathbb{P}(X_{200} = 1 | X_{199} = 1) = 1 - b$ .
- Thus, for any  $n \geq 0$ , we introduce a simple notation  $\boxed{p_{ij}}$
- **(Q)** Any convenient way of describing a MC for intuitive understanding?

$$\boxed{p_{ij}} \triangleq \mathbb{P}(X_{n+1} = j | X_n = i)$$

- Machine example:  $\mathcal{S} = \{1, 2\}$

$$\begin{aligned}p_{11} &= \mathbb{P}(X_{n+1} = 1 | X_n = 1) = 1 - b, & p_{12} &= \mathbb{P}(X_{n+1} = 2 | X_n = 1) = b \\ p_{21} &= \mathbb{P}(X_{n+1} = 1 | X_n = 2) = r, & p_{22} &= \mathbb{P}(X_{n+1} = 2 | X_n = 2) = 1 - r\end{aligned}$$

$$\boldsymbol{P} = \begin{pmatrix} 1 - b & b \\ r & 1 - r \end{pmatrix}$$

- **Transition Probability Matrix**

The  $m \times m$  matrix  $\boldsymbol{P} = [p_{ij}]$ , where  $p_{ij} \triangleq \mathbb{P}(X_{n+1} = j | X_n = i)$

- **Property.**

$$\sum_{j=1}^m p_{ij} = 1 \text{ (for each row } i, \text{ the column sum} = 1\text{)}$$

- Machine example.

$$p_{11} = \mathbb{P}(X_{n+1} = 1 | X_n = 1) = 1 - b,$$

$$p_{12} = \mathbb{P}(X_{n+1} = 2 | X_n = 1) = b$$

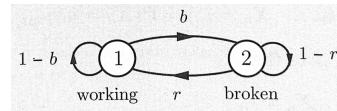
$$p_{21} = \mathbb{P}(X_{n+1} = 1 | X_n = 2) = r,$$

$$p_{22} = \mathbb{P}(X_{n+1} = 2 | X_n = 2) = 1 - r$$

- Transition probability matrix

$$\mathbf{P} = \begin{pmatrix} 1 - b & b \\ r & 1 - r \end{pmatrix}$$

- Any other way? State Transition Diagram



- Transition probability matrix and state transition diagram are the two ways of completely describing a given Markov chain.

L8(1)

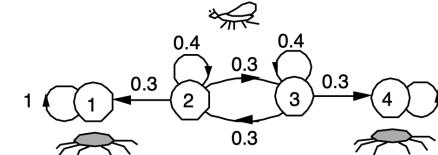
June 12, 2021 9 / 1

- A fly moves along a line in unit increments.

- At each time, it moves one unit (i) left w.p. 0.3, (ii) right w.p. 0.3 and (iii) stays in place w.p. 0.4, independent of the past history of movements.

- Two spiders lurk at positions 1 and 4: if the fly lands there, it is captured by the spider, and the process terminates. Assume that the fly starts in a position between 1 and 4.

- $X_n$ : position of the fly. Please draw the state transition diagram and find the transition probability matrix.



	1	2	3	4
1	1.0	0	0	0
2	0.3	0.4	0.3	0
3	0	0.3	0.4	0.3
4	0	0	0	1.0

$p_{ij}$

## Modified Spider-Fly (Prob. 3, pp 380)

- Assume that the process starts at any of the four positions with equal probability 1/4.
- Let  $Y_n = 1$  whenever the MC is at position 1 or 2, and  $Y_n = 2$  whenever the MC is at position 3 or 4.
- Is  $(Y_n : n \geq 0)$  a Markov chain?
- The key is the Markov property. In other words, given  $Y_1, Y_2 \perp\!\!\!\perp Y_0$  or not?
- For example, compare  $\mathbb{P}(Y_2 = 2 | Y_1 = 2, Y_0 = 1)$  and  $\mathbb{P}(Y_2 = 2 | Y_1 = 2, Y_0 = 2)$ .
- Given  $Y_1 = 2$  (i.e., at time 1, I am at position 3 or 4), the event that I am still at position 3 or 4 at time 2 depends on where I was at time 0 or not?
- $\mathbb{P}(Y_2 = 2 | Y_1 = 2, Y_0 = 1) = \mathbb{P}(X_2 \in \{3, 4\} | X_1 = 3) = 0.7$  (because  $Y_0 = 1$  implies that  $X_1$  has to be 3).
- $\mathbb{P}(Y_2 = 2 | Y_1 = 2, Y_0 = 2) > 0.7$ , because  $X_1 = 3$  or  $X_1 = 4$ .
- $(Y_n : n \geq 0)$  is not a MC.

VIDEO PAUSE 1

VIDEO PAUSE 2

N → F (just one slot) → I

- Discrete time slots.  $N$  persons. Each person is in one of the three conditions: (F) infectious: infected and infectious, (I) purely infected: infected, but not infectious or (N) noninfected

- Infection model. If a person becomes infected during a time slot, then he/she will be in an infectious condition (F) during the following time slot, and from then on will be in an purely infected condition (I).

- Contact model. During every time slot, each of the  $\binom{N}{2}$  pairs of persons are independently in contact w.p.  $p$ 
  - When  $F$  meets with  $N$ , then  $N$  becomes infected, following infection model.

L8(1)

June 12, 2021 11 / 1

L8(1)

June 12, 2021 12 / 1

- $X_n$ : number of infectious (F) persons at the beginning of time slot  $n$ .
  - $Y_n$ : number of noninfected (N) persons at the beginning of time slot  $n$ .
- Q1. Is  $(X_n : n \geq 0)$  a MC?
- $X_n$  only depends on  $X_{n-1}$ ?
  - $X_n$  also depends on the number of noninfected persons at  $n - 1$  time slot. Thus, No.
- Q2. Is  $(Y_n : n \geq 0)$  a MC?
- $Y_n$  only depends on  $Y_{n-1}$ ?
  - $Y_n$  also depends on the number of infections persons at  $n - 1$  time slot. Thus, No.

L8(1)

June 12, 2021 13 / 1

- Q3. Is  $((X_n, Y_n) : n \geq 0)$  a MC?
- $(X_n, Y_n)$  only depends on  $(X_{n-1}, Y_{n-1})$ ?
  - Yes.

- **Messages**
  - Being successful in good modeling depends on the choice of "state" (good modeling sense).
  - Markov chain can be used widely if we choose the state space appropriately.

June 12, 2021 14 / 1

## Roadmap

## Markov Chain

- (1) Definition, Transition Probability Matrix, State Transition Diagram
- (2) ***n*-step Transition Probability**
- (3) Classification of States
- (4) Steady-state Behaviors and Stationary Distribution
- (5) Transient Behaviors

L8(2)

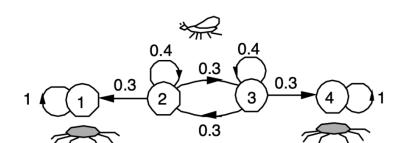
June 12, 2021 15 / 1

- (Q) What is the probability of a sample path in a Markov chain with the transition probability matrix  $\mathbf{P} = [p_{ij}]$ ?

$$\begin{aligned} & \mathbb{P}(X_0 = i_0, X_1 = i_1, X_2 = i_2, \dots, X_n = i_n) \\ &= \mathbb{P}(X_n = i_n | X_0 = i_0, X_1 = i_1, \dots, X_{n-1} = i_{n-1}) \cdot \mathbb{P}(X_0 = i_0, X_1 = i_1, \dots, X_{n-1} = i_{n-1}) \\ &= p_{i_{n-1}i_n} \cdot \mathbb{P}(X_0 = i_0, X_1 = i_1, \dots, X_{n-1} = i_{n-1}) = \mathbb{P}(X_0 = i_0) \cdot p_{i_0i_1} \cdot p_{i_1i_2} \cdots p_{i_{n-1}i_n} \end{aligned}$$

- Spider-Fly example

$$\begin{aligned} & \mathbb{P}(X_0 = 2, X_1 = 2, X_2 = 2, X_3 = 3, X_4 = 4) \\ &= \mathbb{P}(X_0 = 2)p_{22}p_{22}p_{23}p_{34} \\ &= \mathbb{P}(X_0 = 2)(0.4)^2(0.3)^2 \end{aligned}$$



L8(2)

June 12, 2021 16 / 1

(Q) What is the probability that my state is  $j$ , after  $n$  steps, starting from  $i$ ?

- $n$ -step transition probability:  $r_{ij}(n) \triangleq \mathbb{P}(X_n = j \mid X_0 = i)$

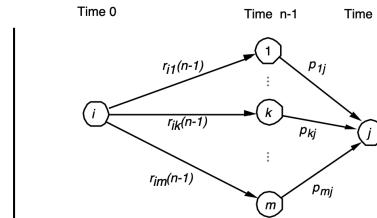
- Recursive formula, starting with  $r_{ij}(1) = p_{ij}$ ,

$$r_{ij}(n) = \mathbb{P}(X_n = j \mid X_0 = i) = \sum_{k=1}^m \mathbb{P}(X_n = j, X_{n-1} = k \mid X_0 = i)$$

$$\begin{aligned} & \sum_{k=1}^m \mathbb{P}(X_{n-1} = k \mid X_0 = i) \mathbb{P}(X_n = j \mid X_{n-1} = k, X_0 = i) \\ &= \sum_{k=1}^m r_{ik}(n-1) p_{kj} \end{aligned}$$

- Possible to compute  $r_{ij}(n)$  recursively. This is called **Chapman-Kolmogorov equation**.

$${}^0\mathbb{P}(A, C|B) = \mathbb{P}(C|B)\mathbb{P}(A|C, B)$$



- $r_{ij}(n+l) = \sum_{k=1}^m r_{ik}(n)r_{kj}(l)$

$$\begin{aligned} r_{ij}(n+l) &= \mathbb{P}(X_{n+l} = j \mid X_0 = i) = \sum_{k=0}^m \mathbb{P}(X_{n+l} = j, X_n = k \mid X_0 = i) \\ &= \sum_{k=0}^m \mathbb{P}(X_{n+l} = j \mid X_n = k, X_0 = i) \mathbb{P}(X_n = k \mid X_0 = i) = \sum_{k=1}^m r_{ik}(n)r_{kj}(l) \end{aligned}$$

- Let  $\mathbf{P}^{(n)}$  be the matrix of  $n$ -step transition probability, i.e.,  $\mathbf{P}^{(n)} \triangleq [r_{ij}(n)]$

- (Q) What is the relation between  $\mathbf{P}^{(n)}$  and  $\mathbf{P}$ ? Can we express  $\mathbf{P}^{(n)}$  with  $\mathbf{P}$ ?

$${}^0\mathbb{P}(A, C|B) = \mathbb{P}(C|B)\mathbb{P}(A|C, B)$$

June 12, 2021 17 / 1

L8(2)

June 12, 2021 18 / 1

- $r_{ij}(n+l) = \sum_{k=1}^m r_{ik}(n)r_{kj}(l)$  and  $\mathbf{P}^{(n)} \triangleq [r_{ij}(n)]$
- Then, by letting  $n = 1, l = 1$ ,  
 $\mathbf{P}^{(2)} = \left[ \sum_{k=1}^m r_{ik}(1)r_{kj}(1) \right] = \left[ \sum_{k=1}^m p_{ik}p_{kj} \right] = \mathbf{P} \times \mathbf{P} = \mathbf{P}^2$ .
- By letting  $n = 2, l = 1$ ,  
 $\mathbf{P}^{(3)} = \left[ \sum_{k=1}^m r_{ik}(2)r_{kj}(1) \right] = \left[ \sum_{k=1}^m r_{ik}(2)p_{kj} \right] = \mathbf{P}^{(2)} \times \mathbf{P} = \mathbf{P}^3$
- Then, by induction,  $\mathbf{P}^{(n)} = \mathbf{P}^n$
- In other words,  $n$ -step transition probability matrix is just a  **$n$ -time multiplication of the transition probability matrix  $\mathbf{P}$** .

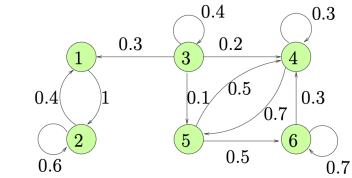
- An urn always contains 2 balls. Ball colors are **red** and **blue**.
- At each stage, a ball is randomly chosen, and then replaced by a new ball, which with probability 0.8 is the same color, and with probability 0.2 is the opposite color, as the ball it replaces.
- If initially both balls are red, find the probability that the **fifth ball** selected is red.
- Solution.** Let  $X_n$  be the number of red balls after  $n$ -th stage (selection and replacement). Then,  $\mathcal{S} = \{0, 1, 2\}$ .

$$\begin{aligned} \bullet \quad & \mathbf{P} = \begin{pmatrix} 0.8 & 0.2 & 0 \\ 0.1 & 0.8 & 0.1 \\ 0 & 0.2 & 0.8 \end{pmatrix} \\ \bullet \quad & \text{Let } A = \{\text{fifth ball is red}\}. \\ \mathbb{P}(A) &= \sum_{i=0}^2 \mathbb{P}(A|X_4 = i)\mathbb{P}(X_4 = i|X_0 = 2) \\ &= (0)r_{2,0}(4) + (0.5)r_{2,1}(4) + (1)r_{2,2}(4) \\ \bullet \quad & \text{By computing } \mathbf{P}^4, \text{ we get } r_{2,1}(4) = 0.4352 \text{ and } r_{2,2}(4) = 0.4872 \\ \bullet \quad & \text{Thus, } \mathbb{P}(A) = 0.7048 \end{aligned}$$

## Markov Chain

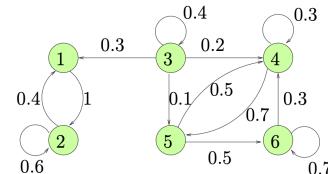
- (1) Definition, Transition Probability Matrix, State Transition Diagram
- (2)  $n$ -step Transition Probability
- (3) **Classification of States**
- (4) Steady-state Behaviors and Stationary Distribution
- (5) Transient Behaviors

- Classes
  - 3 can only be reached from 3
  - 1 and 2 can reach each other but no other state
  - 4, 5, and 6 all reach each other.
  - Divide into three classes: {3}, {1, 2}, {4, 5, 6}
  - **Message 1. Multiple classes may exist.**
- Difference between 1 and 3
  - 1: If I start from 1, visit 1 infinite times.
  - 3: If I start from 3, visit 3 only finite times (move to other classes and don't return).
  - **Message 2. Some states are visited infinite times, but some states are not.**
- State 2 will share the above properties with 1 (similarly, {4, 5, 6})
- **Message 3. States in the same class share some properties.**



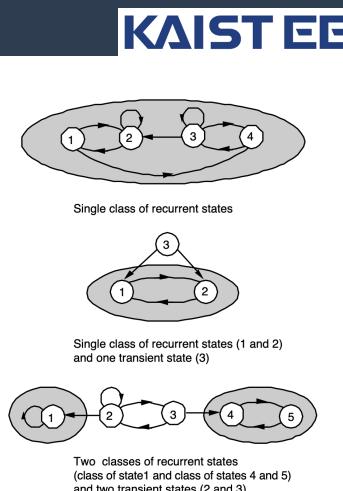
## Classification of States (1)

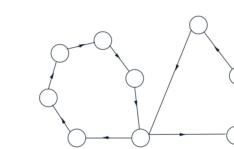
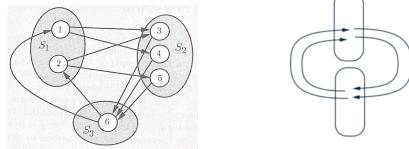
- **Definition.** State  $j$  is **accessible** from state  $i$ , if for some  $n$   $r_{ij}(n) > 0$ . denoted by  $i \dashrightarrow j$ 
  - 6 is accessible from 3, but not the other way around.
- **Definition.** If  $i$  is accessible from  $j$  and  $j$  is accessible from  $i$ , we say that  $i$  **communicates** with  $j$ , denoted by  $i \leftrightarrow j$ .
  - 1  $\leftrightarrow$  2, but 3 does not communicate with 5.
- **Definition.** Let  $A(i) = \{\text{states accessible from } i\}$ . State  $i$  is **recurrent**, if  $\forall j \in A(i)$ ,  $i$  is also accessible from  $j$ . In other words, "I communicate with all of my (direct/indirect) friends!"
  - A state that is not recurrent is **transient**.
  - 2 is recurrent? Yes. 3 is recurrent? No.
  - If we start from a recurrent state  $i$ , then there is always some probability of returning to  $i$ . It means that, given enough time, it is certain that it returns to  $i$ .



## Classification of States (2)

- A set of recurrent states which communicate with each other form a **class**.
- **Markov chain decomposition**
  - A MC can be decomposed into **one or more recurrent classes**, plus possibly **some transient states**.
  - A recurrent state is accessible from all states in its class, but it is not accessible from recurrent states in other classes.
  - A transient state is not accessible from any recurrent state.
  - At least one, possibly more, recurrent states are accessible from a given transient state.
- The MC with only a single recurrent class is said to be **irreducible** (더이상 분해할 수 없는).





- Definition.** A recurrent class is said to be **periodic**, if its states can be grouped in  $d > 1$  disjoint subsets  $S_1, \dots, S_d$  so that all transitions from  $S_k$  lead to  $S_{k+1}$  (or to  $S_1$  if  $k = d$ ). We call  $d$  the **period** of the recurrent class.
- A recurrent class that is not periodic (i.e., period  $d = 1$ ) is said to be **aperiodic**.
- For any state  $i$  in the  $d$ -period recurrent class,  $r_{ii}(n) = 0$ , whenever  $n$  is not divisible by  $d$ , where  $d$  is the greatest integer with this property.
- Often, it is not easy to see some MC is periodic or not. But, one easy way is to check whether there exists a self-transition or not. **An MC with a self-transition must be aperiodic.**

L8(3)

June 12, 2021 25 / 1

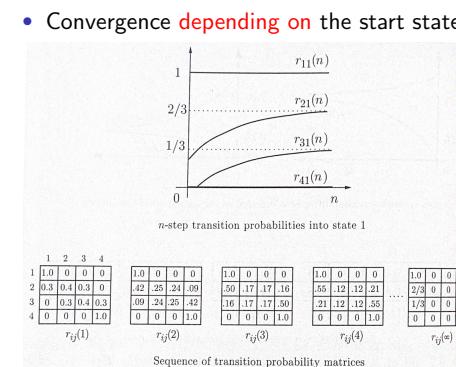
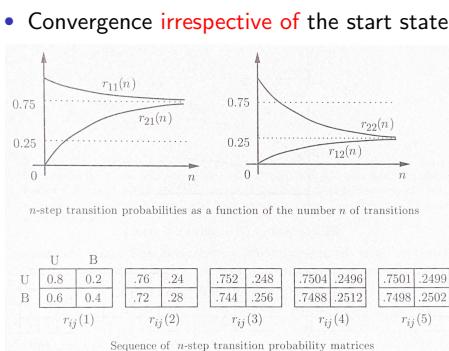
L8(4)

June 12, 2021 26 / 1

## Markov Chain

- (1) Definition, Transition Probability Matrix, State Transition Diagram
- (2)  $n$ -step Transition Probability
- (3) Classification of States
- (4) Steady-state Behaviors and Stationary Distribution
- (5) Transient Behaviors

$n$ -step transition prob.:  $r_{ij}(n)$  for large  $n$



(Q) Under what conditions, convergence occurs, **independent** of the start state? If so, how does it depend on the start state and the shape of the MC?

L8(4)

June 12, 2021 27 / 1

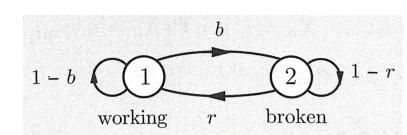
L8(4)

June 12, 2021 28 / 1

Steady-state behavior: Why Important?

- $r_{ij}(n) \xrightarrow{n \rightarrow \infty} \pi_j$ , for some  $\pi_j \leq 1$ ?
- Interpretation.

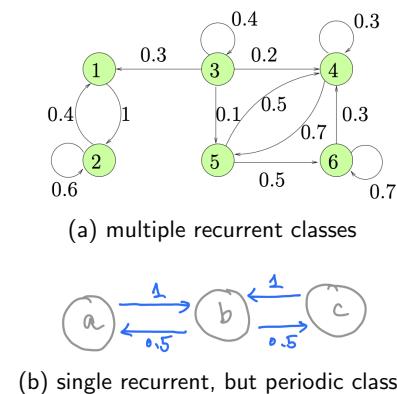
$$\pi_j \approx \mathbb{P}(X_n = j) \text{ for large } n$$



$$\begin{aligned}\pi_{\text{working}} &= \alpha \\ \pi_{\text{broken}} &= \beta\end{aligned}$$

- After running the MC for a long time, we see how long the MC will stay at which state on average.
- Helps in understanding how this MC behaves.

- $r_{ij}(n) \xrightarrow{n \rightarrow \infty} \pi_j$ , for some  $\pi_j \leq 1$ ?
- Convergence occurs, **independent of** the starting state, if:
  - Only a **single recurrent class**
  - such recurrent class is **aperiodic**
- For the case of multiple recurrent classes, one stays at the class including the starting state.
- Divergent behavior for periodic recurrent classes.



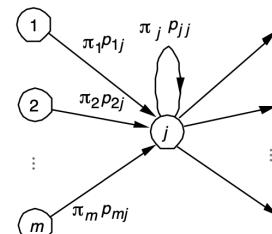
(Q) How to easily compute  $(\pi_1, \pi_2, \dots, \pi_m)$  rather than taking the limit?

- If  $r_{ij}(n) \xrightarrow{n \rightarrow \infty} \pi_j$ , for some  $\pi_j \leq 1$ , from Chapman-Kolmogorov equation,
$$r_{ij}(n) = \sum_{k=1}^m r_{ik}(n-1)p_{kj} \implies \pi_j = \sum_{k=1}^m \pi_k p_{kj} \quad (\text{Balance equation})$$
- $\sum_{i=1}^m \pi_i = 1$ : (Normalization equation)
- Balance eqn. + Normalization eqn.  $\implies$  Finding the steady-state probabilities  $\{\pi_i\}$ .
  - Solving linear equations

## Long-term Frequency Interpretation

- Probability: often interpreted as the **relative frequencies** out of many independent trials
- $\pi_j = \lim_{n \rightarrow \infty} \frac{v_{ij}(n)}{n}$ , where  $v_{ij}(n)$  is the expected number of visits to state  $j$  up to the first  $n$  transitions
- In other words,  $\pi_j$ : long-term **expected fraction of time** that the MC is at the state  $j$ .
- $\pi_j p_{jk}$ : the long-term expected **fraction of transitions** that move the state **from  $j$  to  $k$** .

- Balance equation:  $\sum_{k=1}^m \pi_k p_{kj} = \pi_j$ 
  - The **expected frequency of visits to  $j$**  = The sum of the expected frequencies of transitions that lead to  $j$ .



- A two-state MC with:  $\begin{pmatrix} 0.8 & 0.2 \\ 0.6 & 0.4 \end{pmatrix}$

- (Balance equation)

$$\pi_1 = \pi_1 p_{11} + \pi_2 p_{21} = 0.8\pi_1 + 0.6\pi_2,$$

$$\pi_2 = \pi_2 p_{22} + \pi_1 p_{12} = 0.4\pi_2 + 0.2\pi_1$$

- (Normalization equation)  $\pi_1 + \pi_2 = 1$

- Steady-state probabilities:  $\pi_1 = 0.25$ ,  $\pi_2 = 0.75$ .

- $\{\pi_j\}$  is also called a **stationary distribution**. Why?
- **Distribution**, because  $\sum_{j=1}^m \pi_j = 1$ .
- **Stationary**, because, if you choose the initial state according to  $\{\pi_j\}$ , then for any  $j \in \{1, \dots, m\}$

$$\mathbb{P}(X_0 = j) = \pi_j \xrightarrow{\text{total prob. theorem}} \mathbb{P}(X_1 = j) = \sum_{k=1}^m \mathbb{P}(X_0 = k) p_{kj} = \sum_{k=1}^m \pi_k p_{kj} = \pi_j$$

- Similarly, we have  $\mathbb{P}(X_n = j) = \pi_j$ , for all  $n$  and  $j$ .
- If the initial state is chosen according to  $\{\pi_j\}$ , the state at any future time will have the same distribution (i.e., the distribution does not change over time).
- We say that "the limiting distribution (steady-state distribution) is equal to the stationary distribution"

<sup>0</sup>stationary: not moving or not intended to be moved.

L8(4)

June 12, 2021 33 / 1

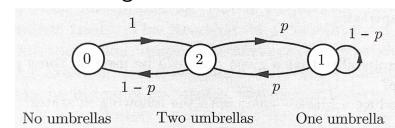
L8(4)

June 12, 2021 34 / 1

- An absent-minded professor: two umbrellas from home to office and back.
- If it rains and an umbrella is available, she takes it. If it is not raining, she always forgets to take an umbrella.
- Suppose that it rains w.p.  $0 < p < 1$  each time when she commutes, independent of other times.
- (Q) What is the steady-state probability that she gets wet during a commute?
- (Hint) If you think that this can be modeled by a MC, think about what should be chosen as states. What is changing over time?

VIDEO PAUSE

- state  $i \in \{0, 1, 2\}$ :  $i$  umbrellas available in her location.
- Transition diagram



No umbrellas      Two umbrellas      One umbrella

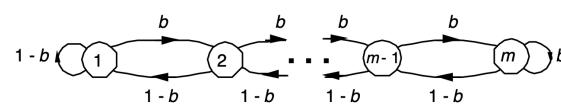
- Single recurrent class and aperiodic
- Balance and normalization equation

$$\begin{aligned}\pi_0 &= (1-p)\pi_2, & \pi_1 &= (1-p)\pi_1 + p\pi_2 \\ \pi_2 &= \pi_0 + p\pi_1, & \pi_0 + \pi_1 + \pi_2 &= 1\end{aligned}$$

- $\pi_0 = \frac{1-p}{3-p}$ ,  $\pi_1 = \frac{1}{3-p}$ ,  $\pi_2 = \frac{1}{3-p}$ .
- The answer is  $p \times \pi_0$ .

### Example 3: Random Walk with Reflecting Barriers

- A person walks along a straight line, and at each time, moves right w.p.  $b$  and moves left w.p.  $1 - b$ .
- Starts in one of the positions  $1, 2, \dots, m$ .
- If he reaches position 0 (or position  $m + 1$ ), his step is instantly reflected back to position 1 (or position  $m$ , respectively).



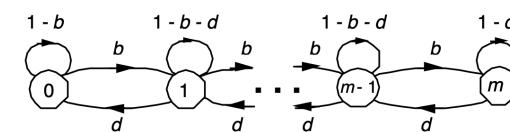
L8(4)

June 12, 2021 35 / 1

### Example 4: Queueing

- Customers arrive at the supermarket counter. If there are some customers at the counter, then new customers should wait in a line whose capacity is  $m$ .
- If there are  $m$  customers, then new customer cannot wait in the line, and is discarded.
- We assume discrete time slots. We assume that at each time slot, exactly one of the followings (a), (b), and (c) occurs

- (a) A new customer arrives w.p.  $b > 0$
- (b) One existing customer at the counter leaves w.p.  $d > 0$ . If there are no customers, nothing happens.
- (c) No new customer and no existing customer leaves w.p.  $1 - b - d$ , if there is at least one customer at the counter and w.p.  $1 - b$  otherwise.



L8(4)

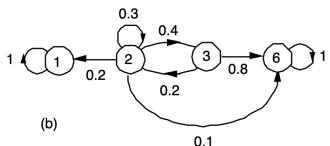
June 12, 2021 36 / 1



- Rather than dealing with a general MC, let's focus on the Markov chain that **every recurrent state is absorbing**.
- Definition.** A state  $k$  is **absorbing**, if  $p_{kk} = 1$ , and  $p_{kj} = 0$  for all  $j \neq k$ .  
states 1 and 6 are absorbing
- For a given absorbing state  $s$ , the probability  $a_i = a_i(s)$  of reaching  $s$ , starting from a state  $i$ ?

• Fix  $s = 6$ :

$$a_1 = 0, \quad a_6 = 1, \quad a_2 = 0.2a_1 + 0.3a_2 + 0.4a_3 + 0.1a_6, \quad a_3 = 0.2a_2 + 0.8a_6$$

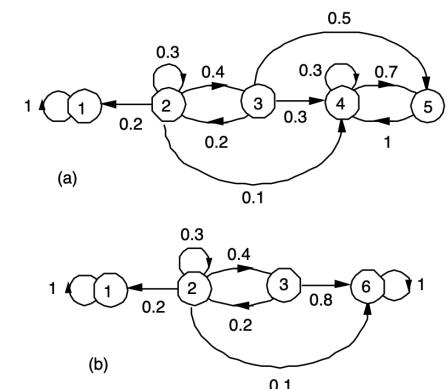


- Our interest:  $a_2$  and  $a_3$
- $a_2 = 21/31$  and  $a_3 = 29/31$

L8(5)

June 12, 2021 41 / 1

- Recurrent classes:  $\{1\}$  and  $\{4, 5\}$
- (Q) Probability that the state eventually enters the recurrent class  $\{4, 5\}$ ?
- Possible transitions within the class  $\{4, 5\}$  are NOT important. Why?
- Thus, convert it into the one only with absorbing recurrent states ((a)  $\rightarrow$  (b)).



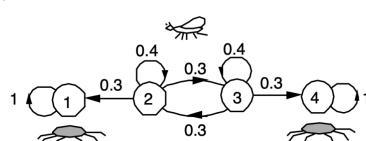
L8(5)

June 12, 2021 42 / 1

## Expected Time to Any Recurrent State

Expected Time to a Particular Recurrent State  $s$ 

- (Q) Starting from a transient state  $i$ , what is the **expected number of steps until a recurrent state is entered** (which we call **absorption**)?
- Special case when all recurrent states are absorbing
  - $\mu_i$ : expected number of transitions until absorption, starting from  $i$
  - Spider-fly example
- $\mu_1 = \mu_4 = 0$  (for **recurrent** states)
- $\mu_2 = 1 + 0.4\mu_2 + 0.3\mu_3, \quad \mu_3 = 1 + 0.3\mu_2 + 0.4\mu_3$  (for **transient** states)
- Again, for general MCs, convert them into the one with only recurrent states that are absorbing



L8(5)

June 12, 2021 43 / 1

- Assume a single recurrent class for simplicity

(Q) **Mean first passage time.** Starting from  $i$ , expected number of transitions  $t_i$  to reach  $s$  for the first time?

(Q) **Mean first recurrence time.** Starting from  $s$ , expected number of transitions  $t_s^*$  to reach  $s$  for the first time?

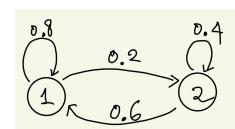
- Mean first passage time from 2 to 1:  $t_2 = t_2(1)$

$$t_1 = 0$$

$$t_2 = 1 + p_{21}t_1 + p_{22}t_2 = 1 + 0.4t_2 \implies t_2 = 5/3$$

- Mean first recurrence time from 1 to 1

$$t_1^* = 1 + p_{11}t_1 + p_{12}t_2 = 1 + 0 + 0.2 \frac{5}{3} = \frac{4}{3}$$



L8(5)

June 12, 2021 44 / 1

## Questions?

- 1) Why do you think Markov chain (MC) is important?
- 2) What is the Markov property and its meaning? What's the key difference of MC from Bernoulli processes?
- 3) What are the limiting distribution and the stationary distribution of MCs?
- 4) How are you going to compute the stationary distribution, if you are given a transition probability matrix?
- 5) What are recurrent and transient states in MC?