Lecture 8: Random Processes, Part II

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EE210: Probability and Introductory Random Processes
KAIST EE

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Roadmap

Markov Chain

(1) Definition, Transition Probability Matrix, State Transition Diagram

(2) n-step Transition Probability

(3) Classification of States

(4) Steady-state Behaviors and Stationary Distribution

(5) Transient Behaviors

Roadmap

Markov Chain

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Recap and Markov Chain

- Assume discrete times $n = 1, 2, \ldots$

- Random process: A sequence of $X_1, X_2, X_3, \cdots$
- "Simplest" random process
  - Process without memory

$$\mathbb{P}(X_n = i_n \mid X_{n-1} = i_{n-1}, X_{n-2} = i_{n-2}, X_{n-3} = i_{n-3}, \ldots, X_1 = i_1) = \mathbb{P}(X_n = i_n)$$

  - Bernoulli process
- A random process that is just a little more general than the above?
  - Process that depends only on "yesterday", not the entire history

$$\mathbb{P}(X_n = i_n \mid X_{n-1} = i_{n-1}, X_{n-2} = i_{n-2}, X_{n-3} = i_{n-3}, \ldots, X_1 = i_1) = \mathbb{P}(X_n = i_n \mid X_{n-1} = i_{n-1})$$

  - Markov chain
  - One of the most popular random processes in engineering!

Example: Machine Failure, Repair, and Replacement

- A machine: working or broken down on a given day.
  - If working, break down in the next day w.p. b, and continue working w.p. $1-b$.
  - If broken down, it will be repaired and be working in the next day w.p. r, and continue to be broken down w.p. $1-r$.
- $X_n \in \{1,2\}$: status of the machine, 1: working and 2: broken down
- $(X_n)_{n=1}^{\infty}$: A random process satisfying: for any $n \geq 1$,

  $\mathbb{P}(X_{n+1}=1|X_n=1)=1-b, \quad \mathbb{P}(X_{n+1}=2|X_n=1)=b$

  $\mathbb{P}(X_{n+1}=1|X_n=2)=r, \quad \mathbb{P}(X_{n+1}=2|X_n=2)=1-r$
- What will happen at $(n+1)$-th day depends only on what happens at n-th day?

Markov Chain: Definition (1)

- Definition. Let $X_1, \ldots, X_n, \ldots$ be a sequence of random variables taking values in some finite space $\mathcal{S} = \{1,2,\ldots,m\}$, such that for all $i,j \in \mathcal{S}$, $n \geq 0$, the following Markov property is satisfied:
  for all $n \geq 0$, all $i,j \in \mathcal{S}$, and all possible sequences $i_0, \ldots, i_{n-1}$ of earlier states,

$$\boxed{\mathbb{P}(X_{n+1}=j|X_n=i)} = \mathbb{P}(X_{n+1}=j|X_n=i, X_{n-1}=i_{n-1}, \ldots, X_0=i_0)$$

- Alternate definition via conditional independence. For any fixed n, the future of the process after n is independent of $\{X_1, \ldots, X_{n-1}\}$, given X_n.

[0]A Markov chain can be also defined for the infinte $\mathcal{S} = \{1,2,\ldots\}$, but we just focus on the finite state space in this lecture.

Markov Chain: Definition (2)

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- The value that X_n can take is called *state* (e.g., working or broken down in the previous example). Thus, the space $\mathcal{S} = \{1, \dots, m\}$ is called *state space*.
- We will focus on the MC of time homogeneity. The probability $\mathbb{P}(X_{n+1} = j | X_n = i)$ does NOT depends on n.
  - In the machine failure example, $\mathbb{P}(X_{100} = 1 | X_{99} = 1) = \mathbb{P}(X_{200} = 1 | X_{199} = 1) = 1 - b$.
- Thus, for any $n \geq 0$, we introduce a simple notation $\boxed{p_{ij}}$

$$\boxed{p_{ij}} \triangleq \mathbb{P}(X_{n+1} = j | X_n = i)$$

- (Q) Any convenient way of describing a MC for intuitive understanding?

Transition Probability Matrix

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- Machine example: $\mathcal{S} = \{1, 2\}$

$$p_{11} = \mathbb{P}(X_{n+1} = 1 | X_n = 1) = 1 - b, \qquad p_{12} = \mathbb{P}(X_{n+1} = 2 | X_n = 1) = b$$
$$p_{21} = \mathbb{P}(X_{n+1} = 1 | X_n = 2) = r, \qquad p_{22} = \mathbb{P}(X_{n+1} = 2 | X_n = 2) = 1 - r$$

$$\boldsymbol{P} = \begin{pmatrix} 1-b & b \\ r & 1-r \end{pmatrix}$$

- Transition Probability Matrix

  The $m \times m$ matrix $\boldsymbol{P} = [p_{ij}]$, where $p_{ij} \triangleq \mathbb{P}(X_{n+1} = j | X_n = i)$

- Property.

  $\sum_{j=1}^{m} p_{ij} = 1$ (for each row i, the column sum $= 1$)

State Transition Diagram

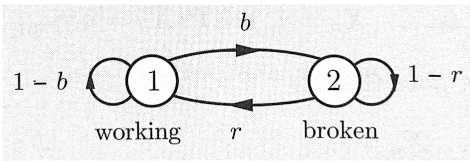
- Machine example.

$$p_{11} = \mathbb{P}(X_{n+1} = 1 | X_n = 1) = 1 - b, \qquad p_{12} = \mathbb{P}(X_{n+1} = 2 | X_n = 1) = b$$
$$p_{21} = \mathbb{P}(X_{n+1} = 1 | X_n = 2) = r, \qquad p_{22} = \mathbb{P}(X_{n+1} = 2 | X_n = 2) = 1 - r$$

- Transition probability matrix

$$\boldsymbol{P} = \begin{pmatrix} 1-b & b \\ r & 1-r \end{pmatrix}$$

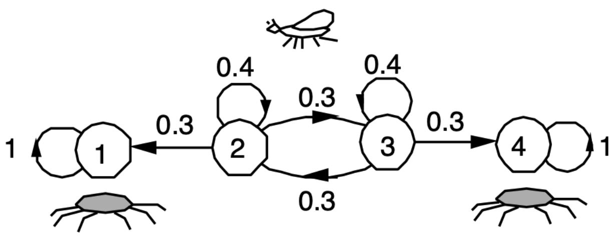
- Any other way? State Transition Diagram



- Transition probability matrix and state transition diagram are the two ways of completely describing a given Markov chain.

Spider-Fly Example (Example 7.2)

- A fly moves along a line in unit increments.
- At each time, it moves one unit (i) left w.p. 0.3, (ii) right w.p. 0.3 and (iii) stays in place w.p. 0.4, independent of the past history of movements.
- Two spiders lurk at positions 1 and 4: if the fly lands there, it is captured by the spider, and the process terminates. Assume that the fly starts in a position between 1 and 4.
- X_n: position of the fly. Please draw the state transition diagram and find the transition probability matrix.



	1	2	3	4
1	1.0	0	0	0
2	0.3	0.4	0.3	0
3	0	0.3	0.4	0.3
4	0	0	0	1.0

p_{ij}

Modified Spider-Fly (Prob. 3, pp 380)

- Assume that the process starts at any of the four positions with equal probability 1/4.
- Let $Y_n = 1$ whenever the MC is at position 1 or 2, and $Y_n = 2$ whenever the MC is at position 3 or 4.
- Is $(Y_n : n \geq 0)$ a Markov chain? VIDEO PAUSE 1
- The key is the Markov property. In other words, given Y_1, $Y_2 \perp\!\!\!\perp Y_0$ or not?
- For example, compare $\mathbb{P}(Y_2 = 2 | Y_1 = 2, Y_0 = 1)$ and $\mathbb{P}(Y_2 = 2 | Y_1 = 2, Y_0 = 2)$.
- Given $Y_1 = 2$ (i.e., at time 1, I am at poisition 3 or 4), the event that I am still at position 3 or 4 at time 2 depends on where I was at time 0 or not? VIDEO PAUSE 2
- $\mathbb{P}(Y_2 = 2 | Y_1 = 2, Y_0 = 1) = \mathbb{P}(X_2 \in \{3, 4\} | X_1 = 3) = 0.7$ (because $Y_0 = 1$ implies that X_1 has to be 3).
- $\mathbb{P}(Y_2 = 2 | Y_1 = 2, Y_0 = 2) > 0.7$, because $X_1 = 3$ or $X_1 = 4$.
- $(Y_n : n \geq 0)$ is not a MC.

COVID-19 Infection Example (1)

- Discrete time slots. N persons. Each person is in one of the three conditions: (F) infectious: infected and infectious, (I) purely infected: infected, but not infectious or (N) noninfected
  - Infection model. If a person becomes infected during a time slot, then he/she will be in an infectious condition (F) during the following time slot, and from then on will be in an purely infected condition (I).

    N $\rightarrow$ F (just one slot) $\rightarrow$ I

- Contact model. During every time slot, each of the $\binom{N}{2}$ pairs of persons are independently in contact w.p. p
  - When F meets with N, then N becomes infected, following infection model.

COVID-19 Infection Example (2)

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- X_n: number of infectious (F) persons at the beginning of time slot n.
- Y_n: number of noninfected (N) persons at the beginning of time slot n.

Q1. Is $(X_n : n \geq 0)$ a MC?

- X_n only depends on X_{n-1}?
- X_n also depends on the number of noninfected persons at $n-1$ time slot. Thus, No.

Q2. Is $(Y_n : n \geq 0)$ a MC?

- Y_n only depends on Y_{n-1}?
- Y_n also depends on the number of infections persons at $n-1$ time slot. Thus, No.

COVID-19 Infection Example (3)

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Q3. Is $\Big((X_n, Y_n) : n \geq 0\Big)$ a MC?

- (X_n, Y_n) only depends on (X_{n-1}, Y_{n-1})?
- Yes.

- Messages
  - Being successful in good modeling depends on the choice of "state" (good modeling sense).
  - Markov chain can be used widely if we choose the state space appropriately.

Markov Chain

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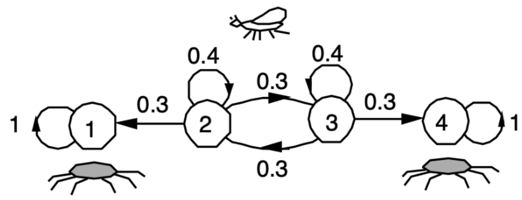
(Q) What is the probability of a sample path in a Markov chain with the transition probability matrix $\boldsymbol{P} = [p_{ij}]$?

$$\begin{aligned}
&\mathbb{P}\left(X_0 = i_0, X_1 = i_1, X_2 = i_2, \ldots, X_n = i_n\right) \\
&= \mathbb{P}\left(X_n = i_n | X_0 = i_0, X_1 = i_1, \ldots, X_{n-1} = i_{n-1}\right) \cdot \mathbb{P}\left(X_0 = i_0, X_1 = i_1, \ldots, X_{n-1} = i_{n-1}\right) \\
&= p_{i_{n-1} i_n} \cdot \mathbb{P}\left(X_0 = i_0, X_1 = i_1, \ldots, X_{n-1} = i_{n-1}\right) = \mathbb{P}(X_0 = i_0) \cdot p_{i_0 i_1} \cdot p_{i_1 i_2} \cdots p_{i_{n-1} i_n}
\end{aligned}$$

- Spider-Fly example

$$\begin{aligned}
&\mathbb{P}(X_0 = 2, X_1 = 2, X_2 = 2, X_3 = 3, X_4 = 4) \\
&= \mathbb{P}(X_0 = 2) p_{22} p_{22} p_{23} p_{34} \\
&= \mathbb{P}(X_0 = 2)(0.4)^2(0.3)^2
\end{aligned}$$



Probability after n Steps

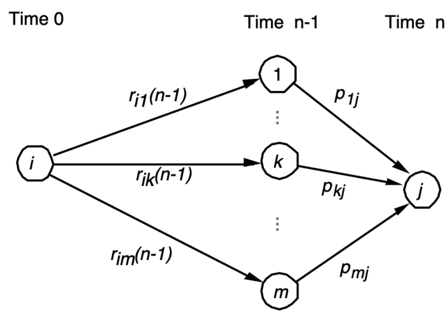
(Q) What is the probability that my state is j, after n steps, starting from i?

- n-step transition probability: $r_{ij}(n) \triangleq \mathbb{P}(X_n = j \mid X_0 = i)$
- Recursive formula, starting with $r_{ij}(1) = p_{ij}$,

$$r_{ij}(n) = \mathbb{P}(X_n = j | X_0 = i) = \sum_{k=1}^{m} \mathbb{P}(X_n = j, X_{n-1} = k | X_0 = i)$$

$$\sum_{k=1}^{m} \mathbb{P}(X_{n-1} = k | X_0 = i)\mathbb{P}(X_n = j | X_{n-1} = k, X_0 = i)$$

$$= \sum_{k=1}^{m} r_{ik}(n-1)p_{kj}$$



- Possible to compute $r_{ij}(n)$ recursively. This is called Chapman-Kolmogorov equation.

[0] $\mathbb{P}(A, C | B) = \mathbb{P}(C | B)\mathbb{P}(A | C, B)$

More Generalized Chapman-Kolmogorov Equation (1)

- $r_{ij}(n+l) = \sum_{k=1}^{m} r_{ik}(n) r_{kj}(l)$

$$r_{ij}(n+l) = \mathbb{P}(X_{n+l} = j \mid X_0 = i) = \sum_{k=1}^{m} \mathbb{P}(X_{n+l} = j, X_n = k \mid X_0 = i)$$

$$= \sum_{k=0}^{m} \mathbb{P}(X_{n+l} = j \mid X_n = k, X_0 = i)\mathbb{P}(X_n = k \mid X_0 = i) = \sum_{k=1}^{m} r_{ik}(n) r_{kj}(l)$$

- Let $\boldsymbol{P}^{(n)}$ be the matrix of n-step transition probability, i.e., $\boldsymbol{P}^{(n)} \triangleq [r_{ij}(n)]$
- (Q) What is the relation between $\boldsymbol{P}^{(n)}$ and $\boldsymbol{P}$? Can we express $\boldsymbol{P}^{(n)}$ with $\boldsymbol{P}$?

More Generalized Chapman-Kolmogorov Equation (2)

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- $r_{ij}(n+l) = \sum_{k=1}^{m} r_{ik}(n) r_{kj}(l)$ and $\boldsymbol{P}^{(n)} \triangleq [r_{ij}(n)]$
- Then, by letting $n=1, l=1$,
  $\boldsymbol{P}^{(2)} = \left[\sum_{k=1}^{m} r_{ik}(1) r_{kj}(1)\right] = \left[\sum_{k=1}^{m} p_{ik} p_{kj}\right] = \boldsymbol{P} \times \boldsymbol{P} = \boldsymbol{P}^2.$
- By letting $n=2, l=1$,
  $\boldsymbol{P}^{(3)} = \left[\sum_{k=1}^{m} r_{ik}(2) r_{kj}(1)\right] = \left[\sum_{k=1}^{m} r_{ik}(2) p_{kj}\right] = \boldsymbol{P}^{(2)} \times \boldsymbol{P} = \boldsymbol{P}^3$
- Then, by induction, $\boldsymbol{P}^{(n)} = \boldsymbol{P}^n$
- In other words, n-step transition probability matrix is just a n-time multiplication of the transition probability matrix $\boldsymbol{P}$.

Example: Urn with Two Balls

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- An urn always contains 2 balls. Ball colors are red and blue.
- At each stage, a ball is randomly chosen, and then replaced by a new ball, which with probability 0.8 is the same color, and with probability 0.2 is the opposite color, as the ball it replaces.
- If initially both balls are red, find the probability that the fifth ball selected is red.
- Solution. Let X_n be the number of red balls after n-th stage (selection and replacement). Then, $\mathcal{S} = \{0, 1, 2\}$.

- $\boldsymbol{P} = \begin{pmatrix} 0.8 & 0.2 & 0 \\ 0.1 & 0.8 & 0.1 \\ 0 & 0.2 & 0.8 \end{pmatrix}$
- Let $A = \{\text{fifth ball is red}\}$.

$$\begin{aligned}\mathbb{P}(A) &= \sum_{i=0}^{2} \mathbb{P}(A|X_4 = i)\mathbb{P}(X_4 = i|X_0 = 2) \\ &= (0) r_{2,0}(4) + (0.5) r_{2,1}(4) + (1) r_{2,2}(4)\end{aligned}$$

- By computing $\boldsymbol{P}^4$, we get $r_{2,1}(4) = 0.4352$ and $r_{2,2}(4) = 0.4872$
- Thus, $\mathbb{P}(A) = 0.7048$

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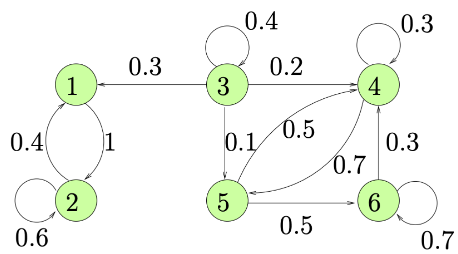
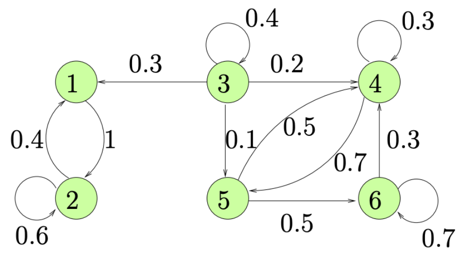
Different States and Classes?



- Classes
  - 3 can only be reached from 3
  - 1 and 2 can reach each other but no other state
  - 4, 5, and 6 all reach each other.
  - Divide into three classes: $\{3\}, \{1, 2\}, \{4, 5, 6\}$
  - Message 1. Multiple classes may exist.
- Difference between 1 and 3
  - 1: If I start from 1, visit 1 infinite times.
  - 3: If I start from 3, visit 3 only finite times (move to other classes and don't return).
  - Message 2. Some states are visited infinite times, but some states are not.
- State 2 will share the above properties with 1 (similarly, $\{4, 5, 6\}$)
- Message 3. States in the same class share some properties.

Classification of States (1)

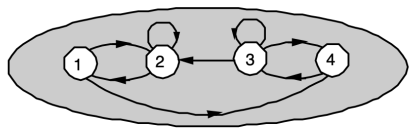
- Definition. State j is accessible from state i, if for some n $r_{ij}(n) > 0$. denoted by $i - -- > j$
  - 6 is accessible from 3, but not the other way around.
- Definition. If i is accessible from j and j is accessible from i, we say that i communicates with j, denoted by $i \leftrightarrow j$.
  - $1 \leftrightarrow 2$, but 3 does not communicate with 5.



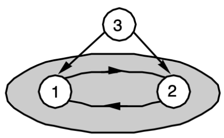
- Definition. Let $A(i) = \{\text{states accessible from } i\}$. State i is recurrent, if $\forall j \in A(i)$, i is also accessible from j. In other words, "I communicate with all of my (direct/indirect) friends!"
  - A state that is not recurrent is transient.
  - 2 is recurrent? Yes. 3 is recurrent? No.
  - If we start from a recurrent state i, then there is always some probability of returning to i. It means that, given enough time, it is certain that it returns to i.

Classification of States (2)

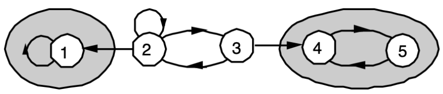
- A set of recurrent states which communicate with each other form a class.
- Markov chain decomposition
  - A MC can be decomposed into one or more recurrent classes, plus possibly some transient states.
  - A recurrent state is accessible from all states in its class, but it not accessible from recurrent states in other classes.
  - A transient state is not accessible from any recurrent state.
  - At least one, possibly more, recurrent states are accessible from a given transient state.
- The MC with only a single recurrent class is said to be irreducible (더이상 분해할 수 없는).



Single class of recurrent states

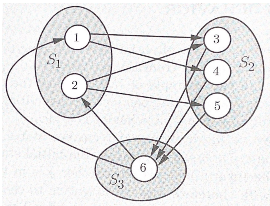


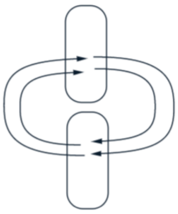
Single class of recurrent states (1 and 2)
and one transient state (3)

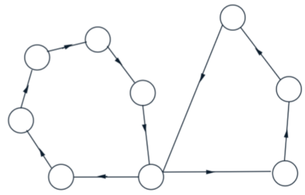


Two classes of recurrent states
(class of state1 and class of states 4 and 5)
and two transient states (2 and 3)

Periodicity







- Definition. A recurrent class is said to be periodic, if its states can be grouped in $d > 1$ disjoint subsets $S_1, \ldots, S_d$ so that all transitions from S_k lead to S_{k+1} (or to S_1 if $k = d$). We call d the period of the recurrent class.
- A recurrent class that is not periodic (i.e., period $d = 1$) is said to be aperiodic.
- For any state i in the d-period recurrent class, $r_{ii}(n) = 0$, whenever n is not divisible by d, where d is the greatest integer with this property.
- Often, it is not easy to see some MC is periodic or not. But, one easy way is to check whether there exists a self-transition or not. An MC with a self-transition must be aperiodic.

Roadmap

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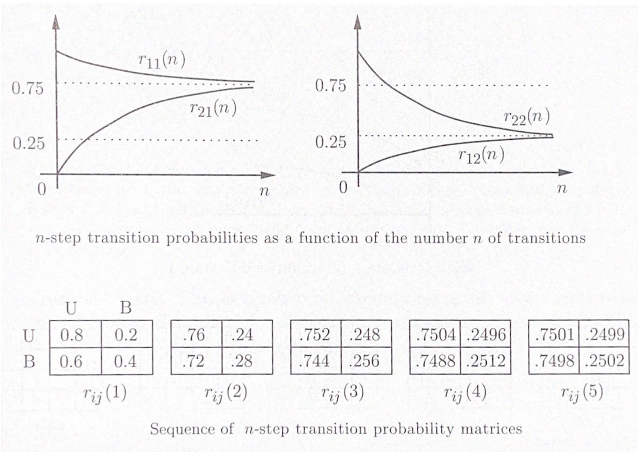
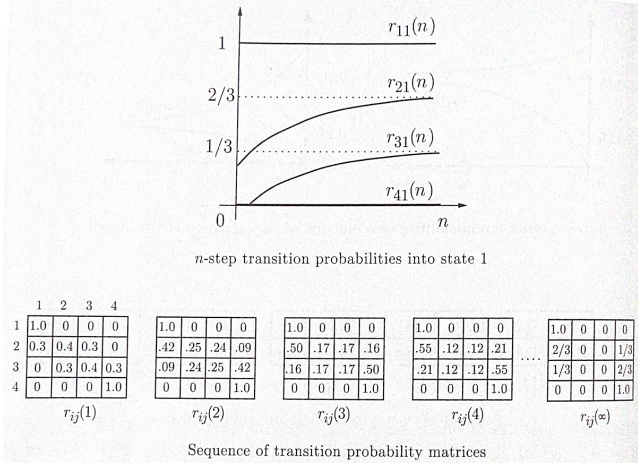
- Convergence irrespective of the start state



- Convergence depending on the start state



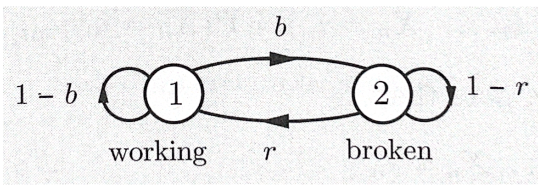
(Q) Under what conditions, convergence occurs, independent of the start state? If so, how does it depend on the start state and the shape of the MC?

Steady-state behavior: Why Impotant?

- $r_{ij}(n) \xrightarrow{n\to\infty} \pi_j$, for some $\pi_j \le 1$?
- Interpretation.

$$\boxed{\pi_j \approx \mathbb{P}(X_n = j) \text{ for large } n}$$

- After running the MC for a long time, we see how long the MC will stay at which state on average.
- Helps in understanding how this MC behaves.



$$\pi_{\text{working}} = \alpha$$

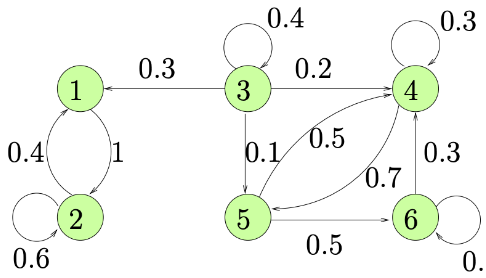
$$\pi_{\text{broken}} = \beta$$

Steady-state behavior: Convergence Condition

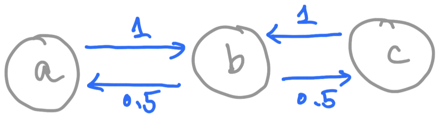
- $r_{ij}(n) \xrightarrow{n\to\infty} \pi_j$, for some $\pi_j \le 1$?
- Convergence occurs, independent of the starting state, if:
  - **C1.** Only a single recurrent class
  - **C2.** such recurrent class is aperiodic

C1. For the case of multiple recurrent classes, one stays at the class including the starting state.

C2. Divergent behavior for periodic recurrent classes.



(a) multiple recurrent classes



(b) single recurrent, but periodic class

Steady-state Convergence Theorem

(Q) How to easily compute $(\pi_1, \pi_2, \ldots, \pi_m)$ rather than taking the limit?

- If $r_{ij}(n) \xrightarrow{n\to\infty} \pi_j$, for some $\pi_j \le 1$, from Chapman-Kolmogorov equation,

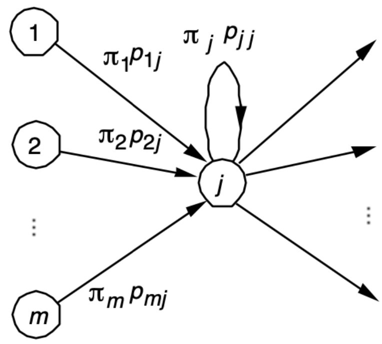
$$r_{ij}(n) = \sum_{k=1}^{m} r_{ik}(n-1)p_{kj} \Longrightarrow \pi_j = \sum_{k=1}^{m} \pi_k p_{kj} \qquad \text{(Balance equation)}$$

- $\sum_{i=1}^{m} \pi_i = 1$: (Normalization equation)
- Balance eqn. + Normalization eqn. $\Longrightarrow$ Finding the steady-state probabilities $\{\pi_i\}$.
  - Solving linear equations

Long-term Frequency Interpretation

- Probability: often interpreted as the relative frequencies out of many independent trials
- $\pi_j = \lim_{n\to\infty} \frac{v_{ij}(n)}{n}$, where $v_{ij}(n)$ is the expected number of visits to state j up to the first n transitions
- In other words, π_j: long-term expected fraction of time that the MC is at the state j.
- $\pi_j p_{jk}$: the long-term expected fraction of transitions that move the state from j to k.

- Balance equation: $\sum_{k=1}^{m} \pi_k p_{kj} = \pi_j$
    - The expected frequency of visits to j = The sum of the expected frequencies of transitions that lead to j.



Example 1

- A two-state MC with: $\begin{pmatrix} 0.8 & 0.2 \\ 0.6 & 0.4 \end{pmatrix}$
- (Balance equation)

$$\pi_1 = \pi_1 p_{11} + \pi_2 p_{21} = 0.8\pi_1 + 0.6\pi_2,$$
$$\pi_2 = \pi_2 p_{22} + \pi_1 p_{12} = 0.4\pi_2 + 0.2\pi_1$$

- (Normalization equation) $\pi_1 + \pi_2 = 1$
- Steady-state probabilities: $\pi_1 = 0.25$, $\pi_2 = 0.75$.

Stationary Distribution

- $\{\pi_j\}$ is also called a stationary distribution. Why?
- Distribution, because $\sum_{j=1}^{m} \pi_j = 1$.
- Stationary, because, if you choose the initial state according to $\{\pi_j\}$, then for any $j \in \{1, \ldots, m\}$

$$\mathbb{P}(X_0 = j) = \pi_j \xrightarrow{\text{total prob. theorem}} \mathbb{P}(X_1 = j) = \sum_{k=1}^{m} \mathbb{P}(X_0 = k) p_{kj} = \sum_{k=1}^{m} \pi_k p_{kj} = \pi_j$$

  - Similarly, we have $\mathbb{P}(X_n = j) = \pi_j$, for all n and j.
  - If the initial state is chosen according to $\{\pi_j\}$, the state at any future time will have the same distribution (i.e., the distribution does not change over time).
- We say that "the limiting distribution (steady-state distribution) is equal to to the stationary distribution"

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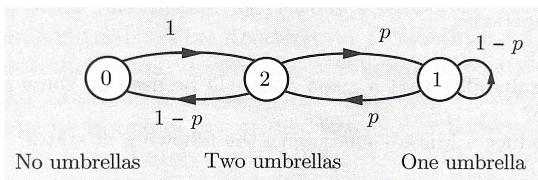
[0] stationary: not moving or not intended to be moved.

Example 2

- An absent-minded professor: two umbrellas from home to office and back.
- If it rains and an umbrella is available, she takes it. If it is not raining, she always forgets to take an umbrella.
- Suppose that it rains w.p. $0 < p < 1$ each time when she commutes, independent of other times.
- (Q) What is the steady-state probability that she gets wet during a commute?
- (Hint) If you think that this can be modeled by a MC, think about what should be chosen as states. What is changing over time? VIDEO PAUSE

- state $i \in \{0, 1, 2\}$: i umbrellas available in her location.
- Transition diagram



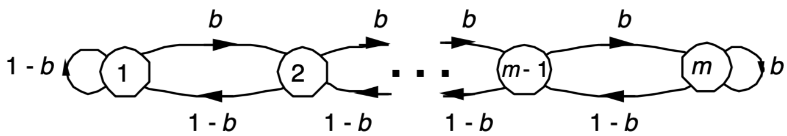
- Single recurrent class and aperiodic
- Balance and normalization equation

$$\pi_0 = (1-p)\pi_2, \quad \pi_1 = (1-p)\pi_1 + p\pi_2$$
$$\pi_2 = \pi_0 + p\pi_1, \quad \pi_0 + \pi_1 + \pi_2 = 1$$

- $\pi_0 = \frac{1-p}{3-p}$, $\pi_1 = \frac{1}{3-p}$, $\pi_2 = \frac{1}{3-p}$.
- The answer is $p \times \pi_0$.

Example 3: Random Walk with Reflecting Barriers

- A person walks along a straght line, and at each time, moves right w.p. b and moves left w.p. $1 - b$.
- Starts in one of the positions $1, 2, \ldots, m$.
- If he reaches position 0 (or position $m + 1$), his step is instantly reflected back to position 1 (or position m, respectively).



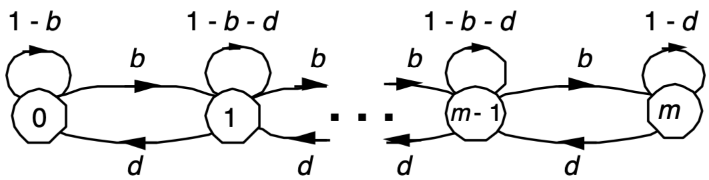
Example 4: Queueing

- Customers arrive at the supermarket counter. If there are some customers at the counter, then new customers should wait in a line whose capacity is m.
- If there are m customers, then new customer cannot wait in the line, and is discarded.
- We assume discrete time slots. We assume that at each time slot, exactly one of the followings (a), (b), and (c) occurs

(a) A new cutomer arrives w.p. $b > 0$

(b) One existing customer at the counter leaves w.p. $d > 0$. If there are no customers, nothing happens.

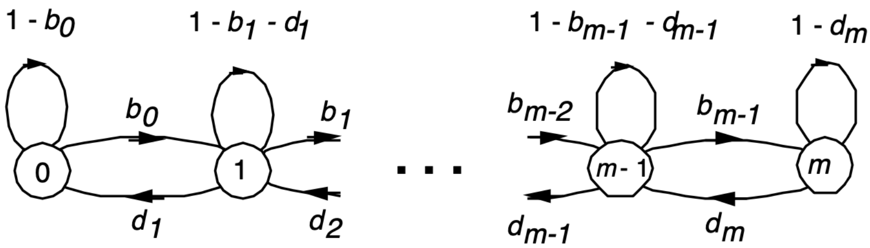
(c) No new customer and no existing customer leaves w.p. $1 - b - d$, if there is at least one customer at the counter and w.p. $1 - b$ otherwise.



Birth-Death Process (1)

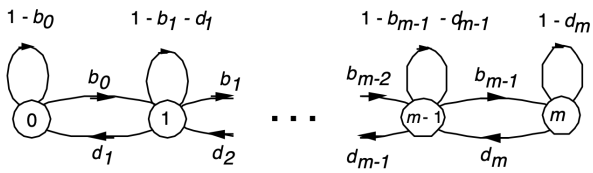
- A special type of Markov chain where the states are linearly arranged and transitions can occur only to a neighboring state.
- Birth and Deadth

$$b_i = \mathbb{P}(X_{n+1} = i+1 | X_n = i), \quad \text{birth probability at state } i$$
$$d_i = \mathbb{P}(X_{n+1} = i-1 | X_n = i), \quad \text{death probability at state } i$$



Birth-Death Process (2)

- State transition diagram



- Balance eqn at state 0

$$\pi_0(1-b_0) + \pi_1 d_1 = \pi_0 \leftrightarrow \pi_0 b_0 = \pi_1 d_1$$

- Balance eqn at state 1

$$\pi_0 b_0 + \pi_1(1-b_1-d_1) + \pi_2 d_2 = \pi_1$$
$$\leftrightarrow \pi_1 d_1 + \pi_1(1-b_1-d_1) + \pi_2 d_2 = \pi_1$$
$$\leftrightarrow \pi_1 b_1 = \pi_2 d_2$$

- By induction, we have the following: called local balance equation :

$$\boxed{\pi_i b_i = \pi_{i+1} d_{i+1}, \; i = 0, 1, \ldots, m-1}$$

- Using the above local balance eqn,

$$\pi_i = \pi_0 \frac{b_0 b_1 \cdots b_{i-1}}{d_1 d_2 \cdots d_i}, \quad i = 1, \ldots, m$$

- Using the above and $\sum \pi_i = 1$, we can easily compute the $[\pi_i]$.
- Examples 3 and 4 are the special cases of birth-death process. So, please compute the steady-state probabilities for both examples as your homeworks.

Roadmap

KAIST EE

Markov Chain

(1) Definition, Transition Probability Matrix, State Transition Diagram

(2) n-step Transition Probability

(3) Classification of States

(4) Steady-state Behaviors and Stationary Distribution

(5) Transient Behaviors

Motivating Questions

KAIST EE

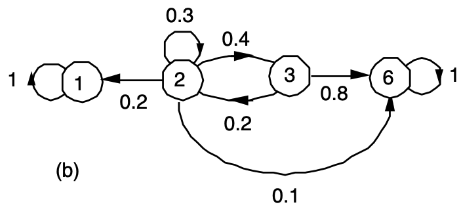
- In the previous lecture,
  - A MC with a single recurrent, aperiodic class $\mathcal{R} = \{1, 2, \dots m\}$
  - Every state will be visited an infinite number of times

  (Q) Steady-state behavior: what are the long-term average frequencies of states?
- In this lecture,
  - A MC with multiple recurrent classes, say, $\mathcal{R}_1, \dots, \mathcal{R}_k$ and a set of transient states $\mathcal{T}$
  - Assume that we start from a state $i \in \mathcal{T}$.
  - Transient states will be visited a finite number of times. Then, the MC will enter a recurrent class whose states are visited infinite number of times, but the states in other recurrent classes will not be visited.

  (Q) Transient behavior: what is the first recurrent state to be entered as well as the time until this happens?

Special Case: Every Recurrent State is Absorbing

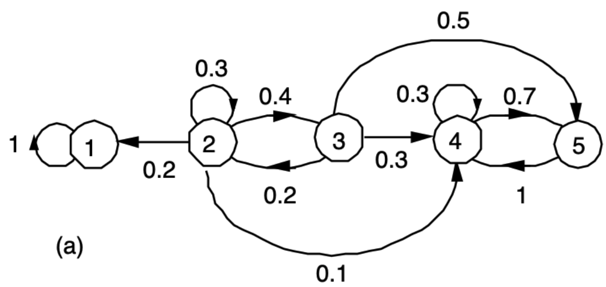
- Rather than dealing with a general MC, let's focus on the Markov chain that every recurrent state is absorbing.
- Definition. A state k is absorbing, if $p_{kk} = 1$, and $p_{kj} = 0$ for all $j \neq k$.
  - states 1 and 6 are absorbing
- For a given absorbing state s, the probability $a_i = a_i(s)$ of reaching s, starting from a state i?
- Fix $s = 6$:
  $a_1 = 0, \quad a_6 = 1, \quad a_2 = 0.2a_1 + 0.3a_2 + 0.4a_3 + 0.1a_6, \quad a_3 = 0.2a_2 + 0.8a_6$

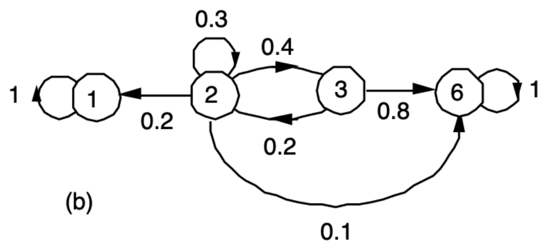


- Our interest: a_2 and a_3
- $a_2 = 21/31$ and $a_3 = 29/31$

For General MCs

- Recurrent classes: $\{1\}$ and $\{4, 5\}$
- (Q) Probability that the state eventually enters the recurrent class $\{4, 5\}$?
- Possible transitions within the class $\{4, 5\}$ are NOT important. Why?
- Thus, convert it into the one only with absorbing recurrent states ((a) $\rightarrow$ (b)).





Expected Time to Any Recurrent State

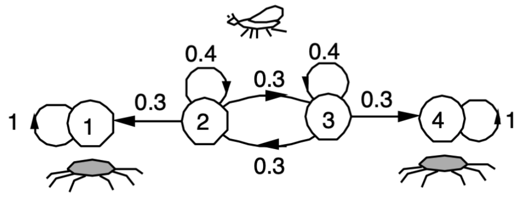
(Q) Starting from a transient state i, what is the expected number of steps until a recurrent state is entered (which we call absorption)?

- Special case when all recurrent states are absorbing
- μ_i: expected number of transitions until absorption, starting from i
- Spider-fly example

$$\mu_1 = \mu_4 = 0 \quad \text{(for recurrent states)}$$

$$\mu_2 = 1 + 0.4\mu_2 + 0.3\mu_3, \quad \mu_3 = 1 + 0.3\mu_2 + 0.4\mu_3 \quad \text{(for transient states)}$$

- Again, for general MCs, convert them into the one with only recurrent states that are absorbing

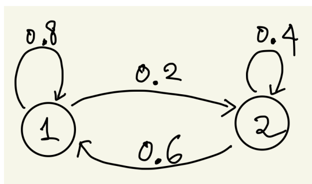


Expected Time to a Particular Recurrent State s

- Assume a single recurrent class for simplicity

(Q) Mean first passage time. Starting from i, expected number of transitions t_i to reach s for the first time?

(Q) Mean first recurrence time. Starting from s, expected number of transitions $t_s^\star$ to reach s for the first time?



- Mean first passage time from 2 to 1: $t_i = t_i(1)$

$$t_1 = 0$$

$$t_2 = 1 + p_{21}t_1 + p_{22}t_2 = 1 + 0.4t_2 \implies t_2 = 5/3$$

- Mean first recurrence time from 1 to 1

$$t_1^\star = 1 + p_{11}t_1 + p_{12}t_2 = 1 + 0 + 0.2\frac{5}{3} = \frac{4}{3}$$

Questions?

Review Questions (1)

1) What is MC? Explain its definition and also its relation to BP and PP. Why do you think Markov chain (MC) is important?
2) What is the Markov property and its meaning? What's the key difference of MC from Bernoulli processes?
3) What are two representation tools to completely describe a given MC?
4) How are you going to compute the steady-state probability, if you are given a transition probability matrix? What is the meaning of the balance equation?
5) What is n-step transition probability? What is its relation to $\boldsymbol{P}^n$, where $\boldsymbol{P}$ is the state transition matrix? What is the limiting distribution?

Review Questions (2)

6) What are recurrent and transient states in MC? How can we talk about periodicity of a given MC?

7) What are the limiting distribution (steady-state distribution) and the stationary distribution of MCs? When are they equal? What is the steady-state convergence theorem and its meaning?

8) What is birth-death process? What is the local balance equation?

9) What are the main questions to investigate the transient behavior of MCs?