

## Lecture 8: Random Processes, Part II

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## Markov Chain

- (1) Definition, Transition Probability Matrix, State Transition Diagram
- (2)  $n$ -step Transition Probability
- (3) Classification of States
- (4) Steady-state Behaviors and Stationary Distribution
- (5) Transient Behaviors

## Markov Chain

- (1) Definition, Transition Probability Matrix, State Transition Diagram
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- Assume discrete times  $n = 1, 2, \dots$ 
  - Random process: A sequence of  $X_1, X_2, X_3, \dots$
  - “Simplest” random process
    - Process without memory

$$\mathbb{P}(X_n = i_n \mid X_{n-1} = i_{n-1}, X_{n-2} = i_{n-2}, X_{n-3} = i_{n-3}, \dots, X_1 = i_1) = \mathbb{P}(X_n = i_n)$$

- Bernoulli process
- A random process that is just a little more general than the above?
  - Process that depends only on “yesterday”, not the entire history

$$\mathbb{P}(X_n = i_n \mid X_{n-1} = i_{n-1}, X_{n-2} = i_{n-2}, X_{n-3} = i_{n-3}, \dots, X_1 = i_1) = \mathbb{P}(X_n = i_n \mid X_{n-1} = i_{n-1})$$

- Markov chain
  - One of the most popular random processes in engineering!

- A machine: working or broken down on a given day.
  - If working, break down in the next day w.p.  $b$ , and continue working w.p.  $1 - b$ .
  - If broken down, it will be repaired and be working in the next day w.p.  $r$ , and continue to be broken down w.p.  $1 - r$ .
- $X_n \in \{1, 2\}$ : status of the machine, 1: working and 2: broken down
- $(X_n)_{n=1}^{\infty}$ : A random process satisfying: for any  $n \geq 1$ ,  
 $\mathbb{P}(X_{n+1} = 1 | X_n = 1) = 1 - b, \quad \mathbb{P}(X_{n+1} = 2 | X_n = 1) = b$   
 $\mathbb{P}(X_{n+1} = 1 | X_n = 2) = r, \quad \mathbb{P}(X_{n+1} = 2 | X_n = 2) = 1 - r$
- What will happen at  $(n + 1)$ -th day depends only on what happens at  $n$ -th day?

- **Definition.** Let  $X_1, \dots, X_n, \dots$  be a sequence of random variables taking values in some finite space  $\mathcal{S} = \{1, 2, \dots, m\}$ , such that for all  $i, j \in \mathcal{S}$ ,  $n \geq 0$ , the following **Markov property** is satisfied:  
for all  $n \geq 0$ , all  $i, j \in \mathcal{S}$ , and all possible sequences  $i_0, \dots, i_{n-1}$  of earlier states,

$$\mathbb{P}(X_{n+1} = j | X_n = i) = \mathbb{P}(X_{n+1} = j | X_n = i, X_{n-1} = i_{n-1}, \dots, X_0 = i_0)$$

- **Alternate definition via conditional independence.** For any fixed  $n$ , the future of the process after  $n$  is **independent** of  $\{X_1, \dots, X_{n-1}\}$ , **given**  $X_n$ .

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<sup>0</sup>A Markov chain can be also defined for the infinite  $\mathcal{S} = \{1, 2, \dots\}$ , but we just focus on the finite state space in this lecture.

- The value that  $X_n$  can take is called *state* (e.g., working or broken down in the previous example). Thus, the space  $\mathcal{S} = \{1, \dots, m\}$  is called *state space*.
- We will focus on the MC of *time homogeneity*. The probability  $\mathbb{P}(X_{n+1} = j | X_n = i)$  does NOT depends on  $n$ .
  - In the machine failure example,  $\mathbb{P}(X_{100} = 1 | X_{99} = 1) = \mathbb{P}(X_{200} = 1 | X_{199} = 1) = 1 - b$ .
- Thus, for any  $n \geq 0$ , we introduce a simple notation  $p_{ij}$   
$$p_{ij} \triangleq \mathbb{P}(X_{n+1} = j | X_n = i)$$
- (Q) Any convenient way of describing a MC for intuitive understanding?

- Machine example:  $\mathcal{S} = \{1, 2\}$

$$p_{11} = \mathbb{P}(X_{n+1} = 1 | X_n = 1) = 1 - b, \quad p_{12} = \mathbb{P}(X_{n+1} = 2 | X_n = 1) = b$$

$$p_{21} = \mathbb{P}(X_{n+1} = 1 | X_n = 2) = r, \quad p_{22} = \mathbb{P}(X_{n+1} = 2 | X_n = 2) = 1 - r$$

$$\mathbf{P} = \begin{pmatrix} 1 - b & b \\ r & 1 - r \end{pmatrix}$$

- Transition Probability Matrix

The  $m \times m$  matrix  $\mathbf{P} = [p_{ij}]$ , where  $p_{ij} \triangleq \mathbb{P}(X_{n+1} = j | X_n = i)$

- Property.

$$\sum_{j=1}^m p_{ij} = 1 \text{ (for each row } i, \text{ the column sum} = 1)$$

- Machine example.

$$p_{11} = \mathbb{P}(X_{n+1} = 1 | X_n = 1) = 1 - b,$$

$$p_{12} = \mathbb{P}(X_{n+1} = 2 | X_n = 1) = b$$

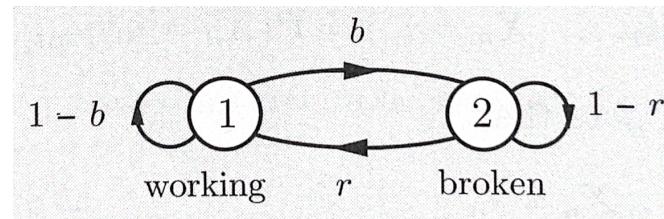
$$p_{21} = \mathbb{P}(X_{n+1} = 1 | X_n = 2) = r,$$

$$p_{22} = \mathbb{P}(X_{n+1} = 2 | X_n = 2) = 1 - r$$

- Transition probability matrix

$$\mathbf{P} = \begin{pmatrix} 1 - b & b \\ r & 1 - r \end{pmatrix}$$

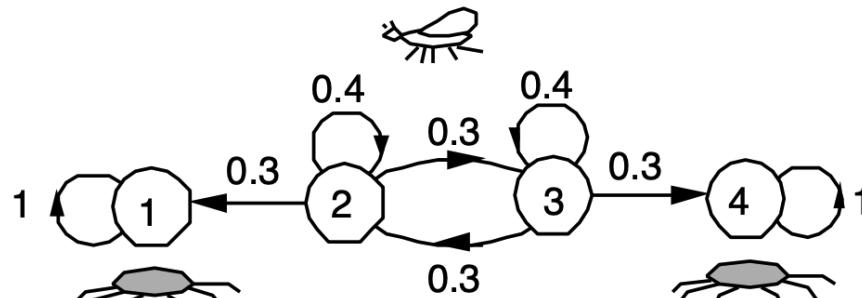
- Any other way? **State Transition Diagram**



- Transition probability matrix and state transition diagram are the two ways of completely describing a given Markov chain.

# Spider-Fly Example (Example 7.2)

- A fly moves along a line in unit increments.
- At each time, it moves one unit (i) left w.p. 0.3, (ii) right w.p. 0.3 and (iii) stays in place w.p. 0.4, independent of the past history of movements.
- Two spiders lurk at positions 1 and 4: if the fly lands there, it is captured by the spider, and the process terminates. Assume that the fly starts in a position between 1 and 4.
- $X_n$ : position of the fly. Please draw the state transition diagram and find the transition probability matrix.



	1	2	3	4
1	1.0	0	0	0
2	0.3	0.4	0.3	0
3	0	0.3	0.4	0.3
4	0	0	0	1.0

$p_{ij}$

- Assume that the process starts at any of the four positions with equal probability 1/4.
- Let  $Y_n = 1$  whenever the MC is at position 1 or 2, and  $Y_n = 2$  whenever the MC is at position 3 or 4.
- Is  $(Y_n : n \geq 0)$  a Markov chain? VIDEO PAUSE 1
- The key is the Markov property. In other words, given  $Y_1, Y_2 \perp\!\!\!\perp Y_0$  or not?
- For example, compare  $\mathbb{P}(Y_2 = 2 | Y_1 = 2, Y_0 = 1)$  and  $\mathbb{P}(Y_2 = 2 | Y_1 = 2, Y_0 = 2)$ .
- Given  $Y_1 = 2$  (i.e., at time 1, I am at position 3 or 4), the event that I am still at position 3 or 4 at time 2 depends on where I was at time 0 or not? VIDEO PAUSE 2
- $\mathbb{P}(Y_2 = 2 | Y_1 = 2, Y_0 = 1) = \mathbb{P}(X_2 \in \{3, 4\} | X_1 = 3) = 0.7$  (because  $Y_0 = 1$  implies that  $X_1$  has to be 3).
- $\mathbb{P}(Y_2 = 2 | Y_1 = 2, Y_0 = 2) > 0.7$ , because  $X_1 = 3$  or  $X_1 = 4$ .
- $(Y_n : n \geq 0)$  is not a MC.

- Discrete time slots.  $N$  persons. Each person is in one of the three conditions: (F) **infectious**: infected and infectious, (I) **purely infected**: infected, but not infectious or (N) **noninfected**
  - **Infection model.** If a person becomes infected during a time slot, then he/she will be in an infectious condition (F) during the following time slot, and from then on will be in an purely infected condition (I).

$$N \rightarrow F \text{ (just one slot)} \rightarrow I$$

- **Contact model.** During every time slot, each of the  $\binom{N}{2}$  pairs of persons are independently in contact w.p.  $p$ 
  - When  $F$  meets with  $N$ , then  $N$  becomes infected, following **infection model**.

- $X_n$ : number of infectious (F) persons at the beginning of time slot  $n$ .
- $Y_n$ : number of noninfected (N) persons at the beginning of time slot  $n$ .

Q1. Is  $(X_n : n \geq 0)$  a MC?

- $X_n$  only depends on  $X_{n-1}$ ?
- $X_n$  also depends on the number of noninfected persons at  $n - 1$  time slot. Thus, No.

Q2. Is  $(Y_n : n \geq 0)$  a MC?

- $Y_n$  only depends on  $Y_{n-1}$ ?
- $Y_n$  also depends on the number of infections persons at  $n - 1$  time slot. Thus, No.

Q3. Is  $((X_n, Y_n) : n \geq 0)$  a MC?

- $(X_n, Y_n)$  only depends on  $(X_{n-1}, Y_{n-1})$ ?
- Yes.

- **Messages**

- Being successful in good modeling depends on the choice of “state” (good modeling sense).
- Markov chain can be used widely if we choose the state space appropriately.

## Markov Chain

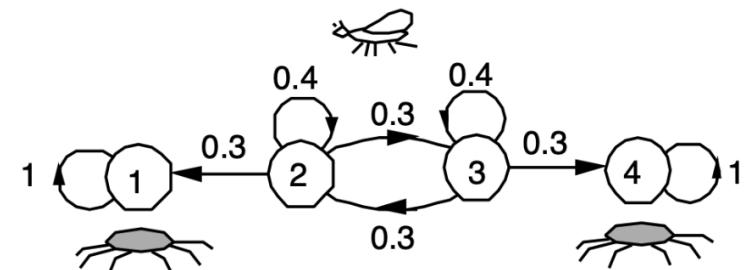
- (1) Definition, Transition Probability Matrix, State Transition Diagram
- (2) *n*-step Transition Probability
- (3) Classification of States
- (4) Steady-state Behaviors and Stationary Distribution
- (5) Transient Behaviors

(Q) What is the probability of a sample path in a Markov chain with the transition probability matrix  $\mathbf{P} = [p_{ij}]$ ?

$$\begin{aligned}\mathbb{P}(X_0 = i_0, X_1 = i_1, X_2 = i_2, \dots, X_n = i_n) \\ &= \mathbb{P}(X_n = i_n | X_0 = i_0, X_1 = i_1, \dots, X_{n-1} = i_{n-1}) \cdot \mathbb{P}(X_0 = i_0, X_1 = i_1, \dots, X_{n-1} = i_{n-1}) \\ &= p_{i_{n-1}i_n} \cdot \mathbb{P}(X_0 = i_0, X_1 = i_1, \dots, X_{n-1} = i_{n-1}) = \mathbb{P}(X_0 = i_0) \cdot p_{i_0i_1} \cdot p_{i_1i_2} \cdots p_{i_{n-1}i_n}\end{aligned}$$

- Spider-Fly example

$$\begin{aligned}\mathbb{P}(X_0 = 2, X_1 = 2, X_2 = 2, X_3 = 3, X_4 = 4) \\ &= \mathbb{P}(X_0 = 2)p_{22}p_{22}p_{23}p_{34} \\ &= \mathbb{P}(X_0 = 2)(0.4)^2(0.3)^2\end{aligned}$$



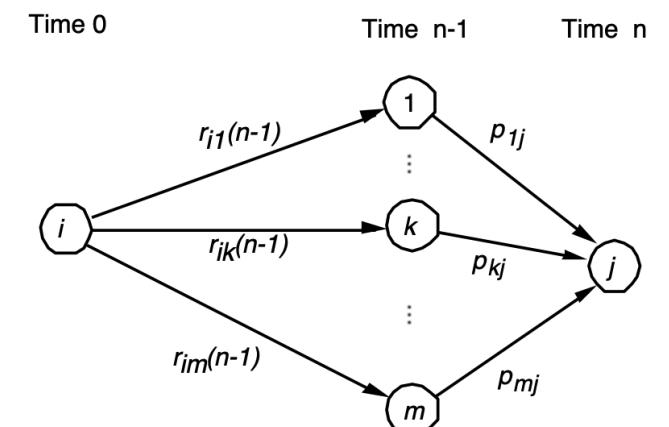
(Q) What is the probability that my state is  $j$ , after  $n$  steps, starting from  $i$ ?

- $n$ -step transition probability:  $r_{ij}(n) \triangleq \mathbb{P}(X_n = j | X_0 = i)$
- Recursive formula, starting with  $r_{ij}(1) = p_{ij}$ ,

$$r_{ij}(n) = \mathbb{P}(X_n = j | X_0 = i) = \sum_{k=1}^m \mathbb{P}(X_n = j, X_{n-1} = k | X_0 = i)$$

$$\sum_{k=1}^m \mathbb{P}(X_{n-1} = k | X_0 = i) \mathbb{P}(X_n = j | X_{n-1} = k, X_0 = i)$$

$$= \sum_{k=1}^m r_{ik}(n-1) p_{kj}$$



- Possible to compute  $r_{ij}(n)$  recursively. This is called Chapman-Kolmogorov equation.

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$${}^0\mathbb{P}(A, C | B) = \mathbb{P}(C | B) \mathbb{P}(A | C, B)$$

L8(2)

$$\bullet \ r_{ij}(n+1) = \sum_{k=1}^m r_{ik}(n)r_{kj}(1)$$

$$r_{ij}(n+1) = \mathbb{P}(X_{n+1} = j \mid X_0 = i) = \sum_{k=1}^m \mathbb{P}(X_{n+1} = j, X_n = k \mid X_0 = i)$$

$$= \sum_{k=0}^m \mathbb{P}(X_{n+1} = j \mid X_n = k, X_0 = i) \mathbb{P}(X_n = k \mid X_0 = i) = \sum_{k=1}^m r_{ik}(n)r_{kj}(1)$$

- Let  $\mathbf{P}^{(n)}$  be the matrix of  $n$ -step transition probability, i.e.,  $\mathbf{P}^{(n)} \triangleq [r_{ij}(n)]$
- (Q) What is the relation between  $\mathbf{P}^{(n)}$  and  $\mathbf{P}$ ? Can we express  $\mathbf{P}^{(n)}$  with  $\mathbf{P}$ ?

- $r_{ij}(n+l) = \sum_{k=1}^m r_{ik}(n)r_{kj}(l)$  and  $\mathbf{P}^{(n)} \triangleq [r_{ij}(n)]$
- Then, by letting  $n = 1, l = 1$ ,  
$$\mathbf{P}^{(2)} = \left[ \sum_{k=1}^m r_{ik}(1)r_{kj}(1) \right] = \left[ \sum_{k=1}^m p_{ik}p_{kj} \right] = \mathbf{P} \times \mathbf{P} = \mathbf{P}^2.$$
- By letting  $n = 2, l = 1$ ,  
$$\mathbf{P}^{(3)} = \left[ \sum_{k=1}^m r_{ik}(2)r_{kj}(1) \right] = \left[ \sum_{k=1}^m r_{ik}(2)p_{kj} \right] = \mathbf{P}^{(2)} \times \mathbf{P} = \mathbf{P}^3$$
- Then, by induction,  $\mathbf{P}^{(n)} = \mathbf{P}^n$
- In other words,  $n$ -step transition probability matrix is just a  **$n$ -time multiplication** of the transition probability matrix  $\mathbf{P}$ .

## Example: Urn with Two Balls

- An urn always contains 2 balls. Ball colors are **red** and **blue**.
- At each stage, a ball is randomly chosen, and then replaced by a new ball, which with probability 0.8 is the same color, and with probability 0.2 is the opposite color, as the ball it replaces.
- If initially both balls are red, find the probability that the **fifth ball** selected is red.
- **Solution.** Let  $X_n$  be the number of red balls after  $n$ -th stage (selection and replacement). Then,  $\mathcal{S} = \{0, 1, 2\}$ .

- $P = \begin{pmatrix} 0.8 & 0.2 & 0 \\ 0.1 & 0.8 & 0.1 \\ 0 & 0.2 & 0.8 \end{pmatrix}$
- Let  $A = \{\text{fifth ball is red}\}$ .

$$\begin{aligned}\mathbb{P}(A) &= \sum_{i=0}^2 \mathbb{P}(A|X_4 = i) \mathbb{P}(X_4 = i|X_0 = 2) \\ &= (0)r_{2,0}(4) + (0.5)r_{2,1}(4) + (1)r_{2,2}(4)\end{aligned}$$

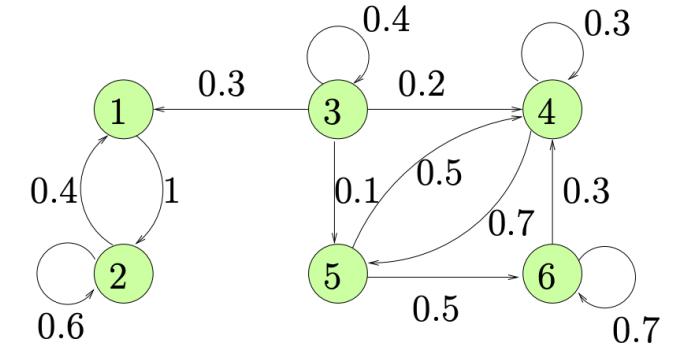
- By computing  $P^4$ , we get  $r_{2,1}(4) = 0.4352$  and  $r_{2,2}(4) = 0.4872$
- Thus,  $\mathbb{P}(A) = 0.7048$

## Markov Chain

- (1) Definition, Transition Probability Matrix, State Transition Diagram
- (2)  $n$ -step Transition Probability
- (3) **Classification of States**
- (4) Steady-state Behaviors and Stationary Distribution
- (5) Transient Behaviors

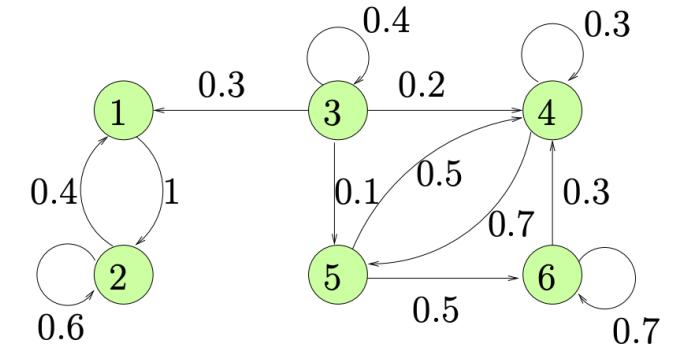
# Different States and Classes?

- Classes
  - 3 can only be reached from 3
  - 1 and 2 can reach each other but no other state
  - 4, 5, and 6 all reach each other.
  - Divide into three classes:  $\{3\}$ ,  $\{1, 2\}$ ,  $\{4, 5, 6\}$
  - **Message 1.** Multiple classes may exist.
- Difference between 1 and 3
  - 1: If I start from 1, visit 1 infinite times.
  - 3: If I start from 3, visit 3 only finite times (move to other classes and don't return).
  - **Message 2.** Some states are visited infinite times, but some states are not.
- State 2 will share the above properties with 1 (similarly,  $\{4, 5, 6\}$ )
- **Message 3.** States in the same class share some properties.



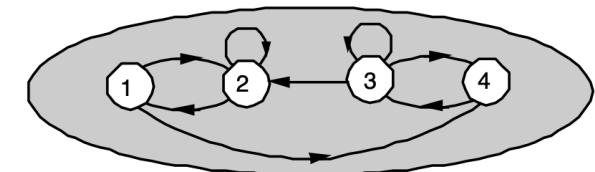
# Classification of States (1)

- **Definition.** State  $j$  is **accessible** from state  $i$ , if for some  $n$   $r_{ij}(n) > 0$ . denoted by  $i \dashrightarrow j$ 
  - 6 is accessible from 3, but not the other way around.
- **Definition.** If  $i$  is accessible from  $j$  and  $j$  is accessible from  $i$ , we say that  $i$  **communicates** with  $j$ , denoted by  $i \leftrightarrow j$ .
  - $1 \leftrightarrow 2$ , but 3 does not communicate with 5.
- **Definition.** Let  $A(i) = \{\text{states accessible from } i\}$ . State  $i$  is **recurrent**, if  $\forall j \in A(i)$ ,  $i$  is also accessible from  $j$ . In other words, “I communicate with all of my (direct/indirect) friends!”
  - A state that is not recurrent is **transient**.
  - 2 is recurrent? Yes. 3 is recurrent? No.
  - If we start from a recurrent state  $i$ , then there is always some probability of returning to  $i$ . It means that, given enough time, it is certain that it returns to  $i$ .

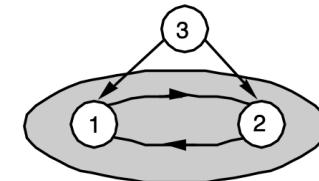


## Classification of States (2)

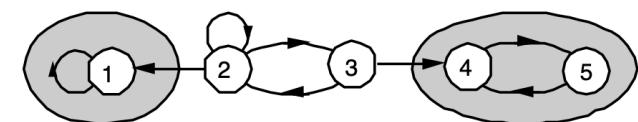
- A set of recurrent states which communicate with each other form a **class**.
- Markov chain decomposition
  - A MC can be decomposed into **one or more recurrent classes**, plus possibly **some transient states**.
  - A recurrent state is accessible from all states in its class, but it not accessible from recurrent states in other classes.
  - A transient state is not accessible from any recurrent state.
  - At least one, possibly more, recurrent states are accessible from a given transient state.
- The MC with only a single recurrent class is said to be **irreducible** (더이상 분해할 수 없는).



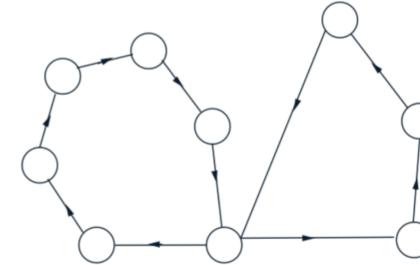
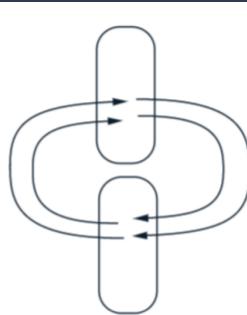
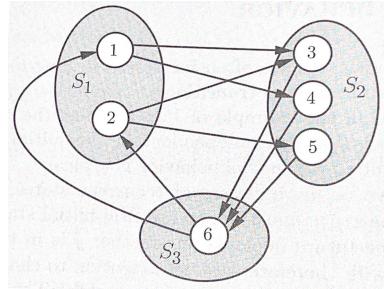
Single class of recurrent states



Single class of recurrent states (1 and 2)  
and one transient state (3)



Two classes of recurrent states  
(class of state1 and class of states 4 and 5)  
and two transient states (2 and 3)



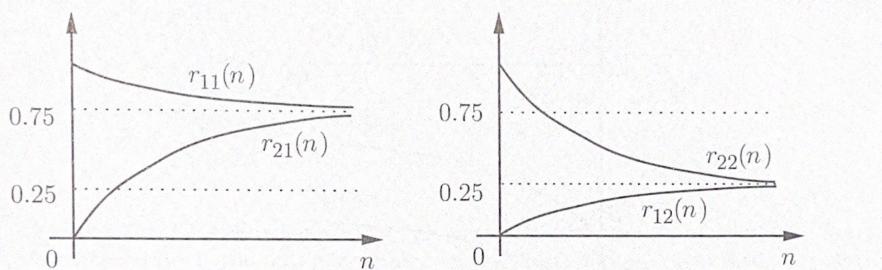
- **Definition.** A recurrent class is said to be **periodic**, if its states can be grouped in  $d > 1$  disjoint subsets  $S_1, \dots, S_d$  so that all transitions from  $S_k$  lead to  $S_{k+1}$  (or to  $S_1$  if  $k = d$ ). We call  $d$  the **period** of the recurrent class.
- A recurrent class that is not periodic (i.e., period  $d = 1$ ) is said to be **aperiodic**.
- For any state  $i$  in the  $d$ -period recurrent class,  $r_{ii}(n) = 0$ , whenever  $n$  is not divisible by  $d$ , where  $d$  is the greatest integer with this property.
- Often, it is not easy to see some MC is periodic or not. But, one easy way is to check whether there exists a self-transition or not. **An MC with a self-transition must be aperiodic.**

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# $n$ -step transition prob.: $r_{ij}(n)$ for large $n$

- Convergence **irrespective of** the start state

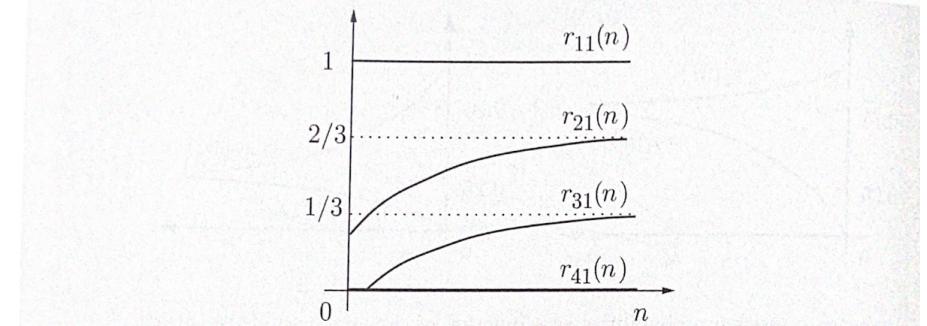


$n$ -step transition probabilities as a function of the number  $n$  of transitions

	U	B				
U	0.8	0.2	.76	.24	.752	.248
B	0.6	0.4	.72	.28	.744	.256
	$r_{ij}(1)$		$r_{ij}(2)$		$r_{ij}(3)$	
						$r_{ij}(4)$
						$r_{ij}(5)$

Sequence of  $n$ -step transition probability matrices

- Convergence **depending on** the start state



$n$ -step transition probabilities into state 1

	1	2	3	4						
1	1.0	0	0	0	1.0	0	0	0	1.0	0
2	0.3	0.4	0.3	0	.42	.25	.24	.09	.50	.17
3	0	0.3	0.4	0.3	.09	.24	.25	.42	.16	.17
4	0	0	0	1.0	0	0	0	1.0	0	0
	$r_{ij}(1)$				$r_{ij}(2)$				$r_{ij}(3)$	
										$r_{ij}(4)$
										$r_{ij}(\infty)$

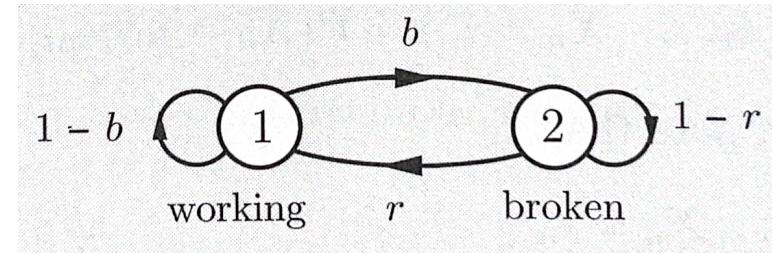
Sequence of transition probability matrices

(Q) Under what conditions, convergence occurs, **independent of the start state**? If so, how does it depend on the start state and the shape of the MC?

- $r_{ij}(n) \xrightarrow{n \rightarrow \infty} \pi_j$ , for some  $\pi_j \leq 1$ ?
- Interpretation.

$$\pi_j \approx \mathbb{P}(X_n = j) \text{ for large } n$$

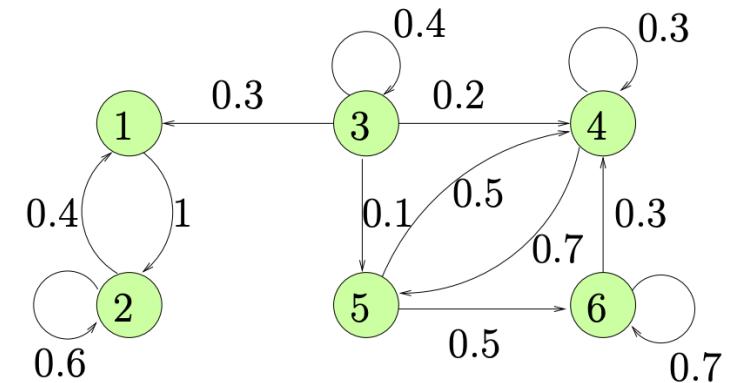
- After running the MC for a long time, we see how long the MC will stay at which state on average.
- Helps in understanding how this MC behaves.



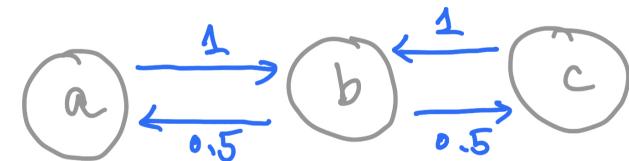
$$\pi_{\text{working}} = \alpha$$

$$\pi_{\text{broken}} = \beta$$

- $r_{ij}(n) \xrightarrow{n \rightarrow \infty} \pi_j$ , for some  $\pi_j \leq 1$ ?
- Convergence occurs, **independent of** the starting state, if:
  - C1.** Only a **single recurrent class**
  - C2.** such recurrent class is **aperiodic**
- C1.** For the case of multiple recurrent classes, one stays at the class including the starting state.
- C2.** Divergent behavior for periodic recurrent classes.



(a) multiple recurrent classes



(b) single recurrent, but periodic class

(Q) How to easily compute  $(\pi_1, \pi_2, \dots, \pi_m)$  rather than taking the limit?

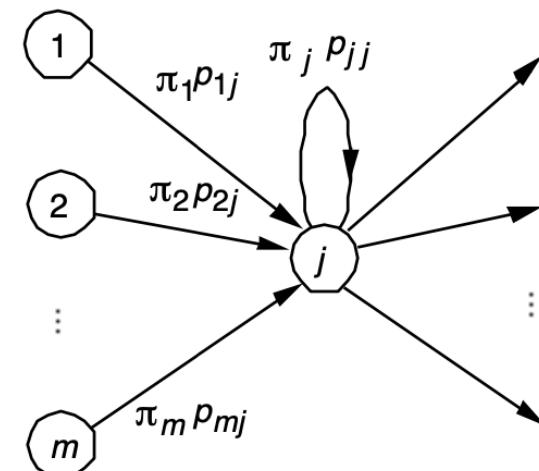
- If  $r_{ij}(n) \xrightarrow{n \rightarrow \infty} \pi_j$ , for some  $\pi_j \leq 1$ , from Chapman-Kolmogorov equation,

$$r_{ij}(n) = \sum_{k=1}^m r_{ik}(n-1)p_{kj} \implies \pi_j = \sum_{k=1}^m \pi_k p_{kj} \quad (\text{Balance equation})$$

- $\sum_{i=1}^m \pi_i = 1$ : (Normalization equation)
- Balance eqn. + Normalization eqn.  $\implies$  Finding the steady-state probabilities  $\{\pi_i\}$ .
  - Solving linear equations

- Probability: often interpreted as the **relative frequencies** out of many independent trials
- $\pi_j = \lim_{n \rightarrow \infty} \frac{v_{ij}(n)}{n}$ , where  $v_{ij}(n)$  is the expected number of visits to state  $j$  up to the first  $n$  transitions
- In other words,  $\pi_j$ : long-term **expected fraction of time** that the MC is at the state  $j$ .
- $\pi_j p_{jk}$ : the long-term expected **fraction of transitions** that move the state from  $j$  to  $k$ .

- Balance equation:  $\sum_{k=1}^m \pi_k p_{kj} = \pi_j$ 
  - **The expected frequency of visits to  $j$  = The sum of the expected frequencies of transitions that lead to  $j$ .**



- A two-state MC with:  $\begin{pmatrix} 0.8 & 0.2 \\ 0.6 & 0.4 \end{pmatrix}$
- (Balance equation)

$$\pi_1 = \pi_1 p_{11} + \pi_2 p_{21} = 0.8\pi_1 + 0.6\pi_2,$$

$$\pi_2 = \pi_2 p_{22} + \pi_1 p_{12} = 0.4\pi_2 + 0.2\pi_1$$

- (Normalization equation)  $\pi_1 + \pi_2 = 1$
- Steady-state probabilities:  $\pi_1 = 0.25$ ,  $\pi_2 = 0.75$ .

- $\{\pi_j\}$  is also called a **stationary distribution**. Why?
- **Distribution**, because  $\sum_{j=1}^m \pi_j = 1$ .
- **Stationary**, because, if you choose the initial state according to  $\{\pi_j\}$ , then for any  $j \in \{1, \dots, m\}$

$$\mathbb{P}(X_0 = j) = \pi_j \xrightarrow{\text{total prob. theorem}} \mathbb{P}(X_1 = j) = \sum_{k=1}^m \mathbb{P}(X_0 = k)p_{kj} = \sum_{k=1}^m \pi_k p_{kj} = \pi_j$$

- Similarly, we have  $\mathbb{P}(X_n = j) = \pi_j$ , for all  $n$  and  $j$ .
- If the initial state is chosen according to  $\{\pi_j\}$ , the state at any future time will have the same distribution (i.e., the distribution does not change over time).
- We say that "the limiting distribution (steady-state distribution) is equal to the stationary distribution"

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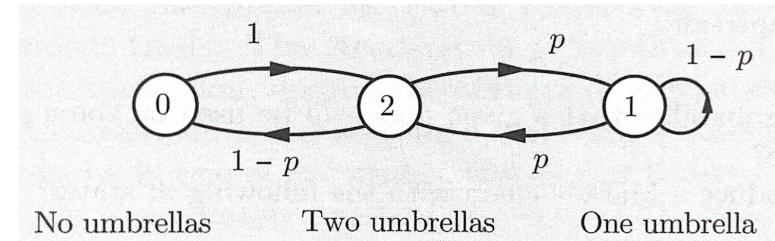
<sup>0</sup>stationary: not moving or not intended to be moved.

## Example 2

- An absent-minded professor: two umbrellas from home to office and back.
- If it rains and an umbrella is available, she takes it. If it is not raining, she always forgets to take an umbrella.
- Suppose that it rains w.p.  $0 < p < 1$  each time when she commutes, independent of other times.
- (Q) What is the steady-state probability that she gets wet during a commute?
- (Hint) If you think that this can be modeled by a MC, think about what should be chosen as states. What is changing over time?

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- state  $i \in \{0, 1, 2\}$ :  $i$  umbrellas available in her location.
- Transition diagram



- Single recurrent class and aperiodic
- Balance and normalization equation

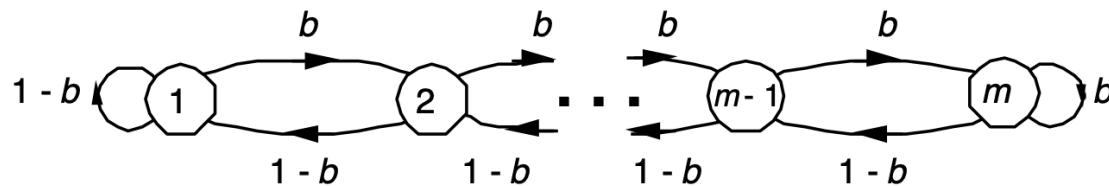
$$\pi_0 = (1 - p)\pi_2, \quad \pi_1 = (1 - p)\pi_1 + p\pi_2$$

$$\pi_2 = \pi_0 + p\pi_1, \quad \pi_0 + \pi_1 + \pi_2 = 1$$

- $\pi_0 = \frac{1-p}{3-p}$ ,  $\pi_1 = \frac{1}{3-p}$ ,  $\pi_2 = \frac{1}{3-p}$ .
- The answer is  $p \times \pi_0$ .

## Example 3: Random Walk with Reflecting Barriers

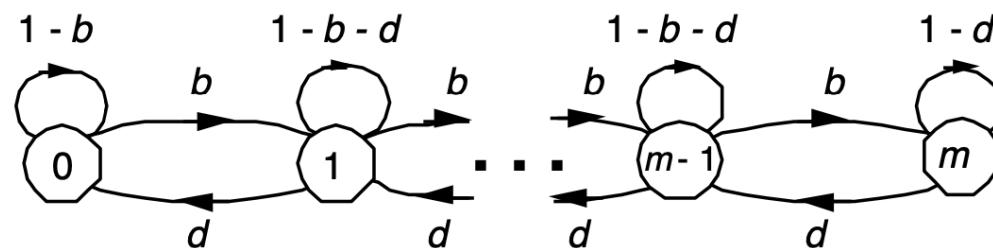
- A person walks along a straight line, and at each time, moves right w.p.  $b$  and moves left w.p.  $1 - b$ .
- Starts in one of the positions  $1, 2, \dots, m$ .
- If he reaches position 0 (or position  $m + 1$ ), his step is instantly reflected back to position 1 (or position  $m$ , respectively).



## Example 4: Queueing

- Customers arrive at the supermarket counter. If there are some customers at the counter, then new customers should wait in a line whose capacity is  $m$ .
- If there are  $m$  customers, then new customer cannot wait in the line, and is discarded.
- We assume discrete time slots. We assume that at each time slot, exactly one of the followings (a), (b), and (c) occurs

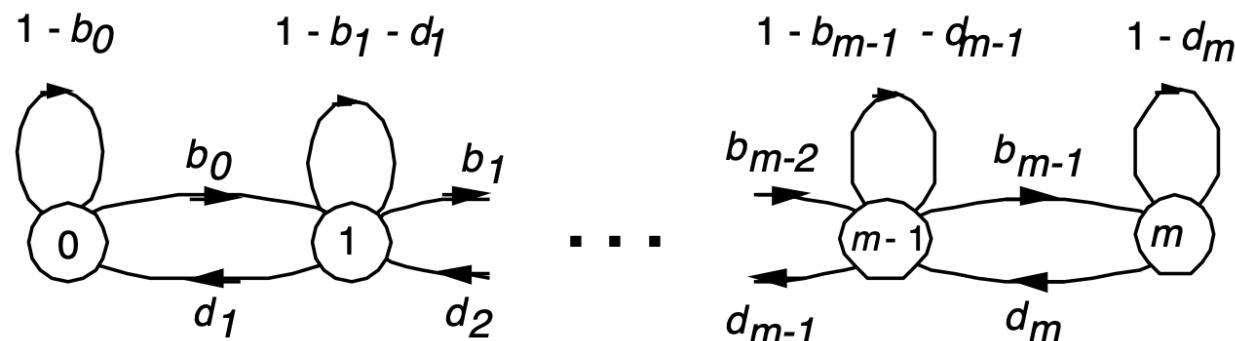
- (a) A new customer arrives w.p.  $b > 0$
- (b) One existing customer at the counter leaves w.p.  $d > 0$ . If there are no customers, nothing happens.
- (c) No new customer and no existing customer leaves w.p.  $1 - b - d$ , if there is at least one customer at the counter and w.p.  $1 - b$  otherwise.



- A special type of Markov chain where the states are **linearly arranged** and transitions can occur only to a **neighboring** state.
- Birth and Death

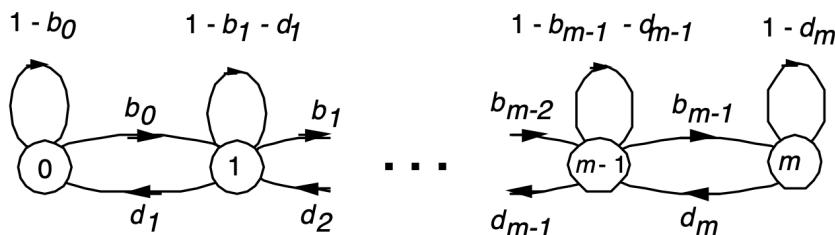
$b_i = \mathbb{P}(X_{n+1} = i + 1 | X_n = i)$ , **birth** probability at state  $i$

$d_i = \mathbb{P}(X_{n+1} = i - 1 | X_n = i)$ , **death** probability at state  $i$



## Birth-Death Process (2)

- State transition diagram



- Balance eqn at state 0

$$\pi_0(1 - b_0) + \pi_1 d_1 = \pi_0 \leftrightarrow \pi_0 b_0 = \pi_1 d_1$$

- Balance eqn at state 1

$$\begin{aligned}\pi_0 b_0 + \pi_1 (1 - b_1 - d_1) + \pi_2 d_2 &= \pi_1 \\ \leftrightarrow \pi_1 d_1 + \pi_1 (1 - b_1 - d_1) + \pi_2 d_2 &= \pi_1 \\ \leftrightarrow \pi_1 b_1 &= \pi_2 d_2\end{aligned}$$

- By induction, we have the following: called **local balance equation**:

$$\pi_i b_i = \pi_{i+1} d_{i+1}, i = 0, 1, \dots, m-1$$

- Using the above local balance eqn,

$$\pi_i = \pi_0 \frac{b_0 b_1 \cdots b_{i-1}}{d_1 d_2 \cdots d_i}, \quad i = 1, \dots, m$$

- Using the above and  $\sum \pi_i = 1$ , we can easily compute the  $[\pi_i]$ .

- Examples 3 and 4 are the special cases of birth-death process. So, please compute the steady-state probabilities for both examples as your homeworks.

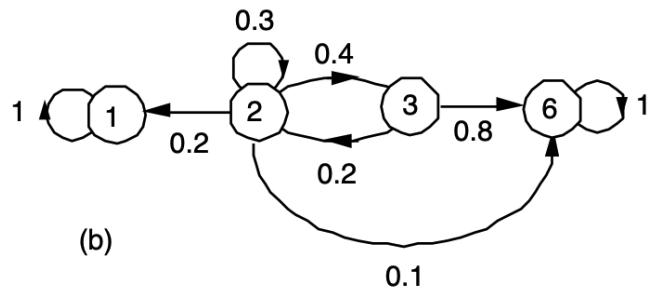
## Markov Chain

- (1) Definition, Transition Probability Matrix, State Transition Diagram
- (2)  $n$ -step Transition Probability
- (3) Classification of States
- (4) Steady-state Behaviors and Stationary Distribution
- (5) Transient Behaviors

- In the previous lecture,
  - A MC with a single recurrent, aperiodic class  $\mathcal{R} = \{1, 2, \dots, m\}$
  - Every state will be visited an infinite number of times
- (Q) Steady-state behavior: what are the long-term average frequencies of states?
- In this lecture,
  - A MC with multiple recurrent classes, say,  $\mathcal{R}_1, \dots, \mathcal{R}_k$  and a set of transient states  $\mathcal{T}$
  - Assume that we start from a state  $i \in \mathcal{T}$ .
  - Transient states will be visited a finite number of times. Then, the MC will enter a recurrent class whose states are visited infinite number of times, but the states in other recurrent classes will not be visited.
- (Q) Transient behavior: what is the first recurrent state to be entered as well as the time until this happens?

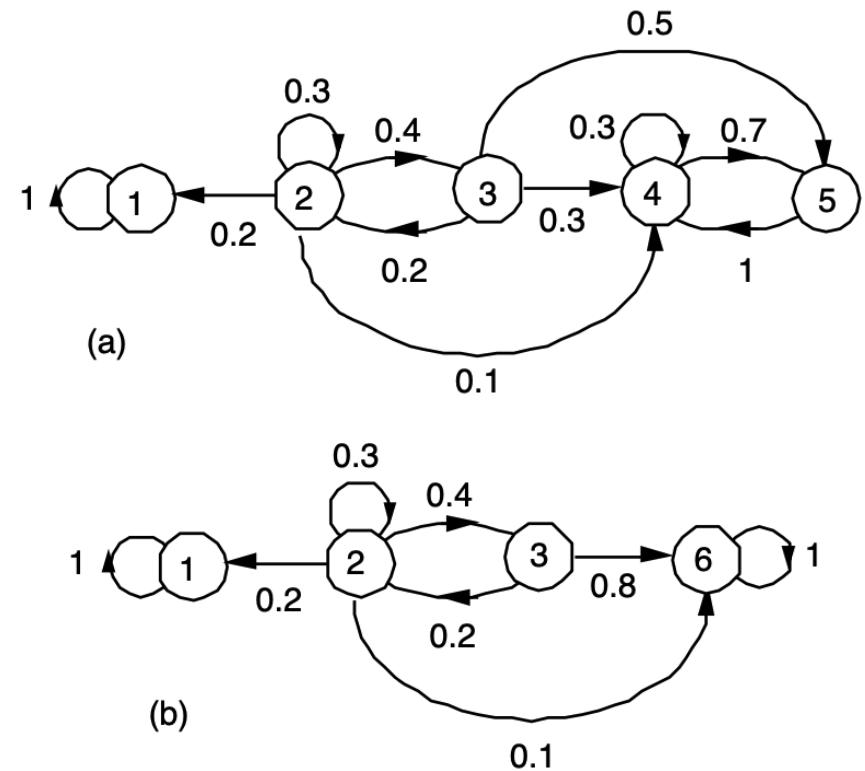
- Rather than dealing with a general MC, let's focus on the Markov chain that **every recurrent state is absorbing**.
- **Definition.** A state  $k$  is **absorbing**, if  $p_{kk} = 1$ , and  $p_{kj} = 0$  for all  $j \neq k$ .
  - states 1 and 6 are absorbing
- For a given absorbing state  $s$ , the probability  $a_i = a_i(s)$  of reaching  $s$ , starting from a state  $i$ ?
- Fix  $s = 6$ :

$$a_1 = 0, \quad a_6 = 1, \quad a_2 = 0.2a_1 + 0.3a_2 + 0.4a_3 + 0.1a_6, \quad a_3 = 0.2a_2 + 0.8a_6$$



- Our interest:  $a_2$  and  $a_3$
- $a_2 = 21/31$  and  $a_3 = 29/31$

- Recurrent classes:  $\{1\}$  and  $\{4, 5\}$
- (Q) Probability that the state eventually enters the recurrent class  $\{4, 5\}$ ?
- Possible transitions within the class  $\{4, 5\}$  are NOT important. Why?
- Thus, convert it into the one only with absorbing recurrent states ((a)  $\rightarrow$  (b)).



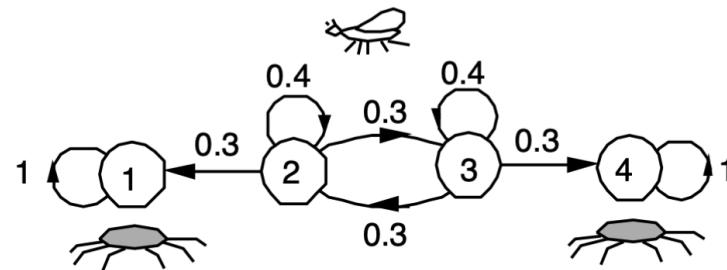
(Q) Starting from a transient state  $i$ , what is the expected number of steps until a recurrent state is entered (which we call **absorption**)?

- Special case when all recurrent states are absorbing
- $\mu_i$ : expected number of transitions until absorption, starting from  $i$
- Spider-fly example

$$\mu_1 = \mu_4 = 0 \quad (\text{for recurrent states})$$

$$\mu_2 = 1 + 0.4\mu_2 + 0.3\mu_3, \quad \mu_3 = 1 + 0.3\mu_2 + 0.4\mu_3 \quad (\text{for transient states})$$

- Again, for general MCs, convert them into the one with only recurrent states that are absorbing



- Assume a single recurrent class for simplicity

(Q) Mean first passage time. Starting from  $i$ , expected number of transitions  $t_i$  to reach  $s$  for the first time?

(Q) Mean first recurrence time. Starting from  $s$ , expected number of transitions  $t_s^*$  to reach  $s$  for the first time?

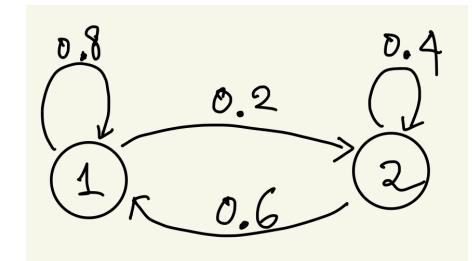
- Mean first passage time from 2 to 1:  $t_2 = t_1(1)$

$$t_1 = 0$$

$$t_2 = 1 + p_{21}t_1 + p_{22}t_2 = 1 + 0.4t_2 \implies t_2 = 5/3$$

- Mean first recurrence time from 1 to 1

$$t_1^* = 1 + p_{11}t_1 + p_{12}t_2 = 1 + 0 + 0.2 \frac{5}{3} = \frac{4}{3}$$



# Questions?

- 1) Why do you think Markov chain (MC) is important?
- 2) What is the Markov property and its meaning? What's the key difference of MC from Bernoulli processes?
- 3) What are the limiting distribution and the stationary distribution of MCs?
- 4) How are you going to compute the stationary distribution, if you are given a transition probability matrix?
- 5) What are recurrent and transient states in MC?