

Lecture 8: Random Processes, Part II

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EE210: Probability and Introductory Random Processes
KAIST EE

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Markov Chain

- (1) Definition, Transition Probability Matrix, State Transition Diagram
- (2) n -step Transition Probability
- (3) Classification of States
- (4) Steady-state Behaviors and Stationary Distribution
- (5) Transient Behaviors

Markov Chain

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- Assume discrete times $n = 1, 2, \dots$
 - Random process: A sequence of X_1, X_2, X_3, \dots
 - “Simplest” random process
 - Process without memory

$$\mathbb{P}(X_n = i_n \mid X_{n-1} = i_{n-1}, X_{n-2} = i_{n-2}, X_{n-3} = i_{n-3}, \dots, X_1 = i_1) = \mathbb{P}(X_n = i_n)$$

- Bernoulli process
- A random process that is just a little more general than the above?
 - Process that depends only on “yesterday”, not the entire history

$$\mathbb{P}(X_n = i_n \mid X_{n-1} = i_{n-1}, X_{n-2} = i_{n-2}, X_{n-3} = i_{n-3}, \dots, X_1 = i_1) = \mathbb{P}(X_n = i_n \mid X_{n-1} = i_{n-1})$$

- Markov chain
 - One of the most popular random processes in engineering!

- A machine: working or broken down on a given day.
 - If working, break down in the next day w.p. b , and continue working w.p. $1 - b$.
 - If broken down, it will be repaired and be working in the next day w.p. r , and continue to be broken down w.p. $1 - r$.
- $X_n \in \{1, 2\}$: status of the machine, 1: working and 2: broken down
- $(X_n)_{n=1}^{\infty}$: A random process satisfying: for any $n \geq 1$,
 $\mathbb{P}(X_{n+1} = 1 | X_n = 1) = 1 - b, \quad \mathbb{P}(X_{n+1} = 2 | X_n = 1) = b$
 $\mathbb{P}(X_{n+1} = 1 | X_n = 2) = r, \quad \mathbb{P}(X_{n+1} = 2 | X_n = 2) = 1 - r$
- What will happen at $(n + 1)$ -th day depends only on what happens at n -th day?

- **Definition.** Let X_1, \dots, X_n, \dots be a sequence of random variables taking values in some finite space $\mathcal{S} = \{1, 2, \dots, m\}$, such that for all $i, j \in \mathcal{S}$, $n \geq 0$, the following **Markov property** is satisfied:

for all $n \geq 0$, all $i, j \in \mathcal{S}$, and all possible sequences i_0, \dots, i_{n-1} of earlier states,

$$\mathbb{P}(X_{n+1} = j | X_n = i) = \mathbb{P}(X_{n+1} = j | X_n = i, X_{n-1} = i_{n-1}, \dots, X_0 = i_0)$$

- **Alternate definition via conditional independence.** For any fixed n , the future of the process after n is **independent** of $\{X_1, \dots, X_{n-1}\}$, **given** X_n .

⁰A Markov chain can be also defined for the infinite $\mathcal{S} = \{1, 2, \dots\}$, but we just focus on the finite state space in this lecture.

- The value that X_n can take is called *state* (e.g., working or broken down in the previous example). Thus, the space $\mathcal{S} = \{1, \dots, m\}$ is called *state space*.
- We will focus on the MC of *time homogeneity*. The probability $\mathbb{P}(X_{n+1} = j | X_n = i)$ does NOT depends on n .
 - In the machine failure example, $\mathbb{P}(X_{100} = 1 | X_{99} = 1) = \mathbb{P}(X_{200} = 1 | X_{199} = 1) = 1 - b$.
- Thus, for any $n \geq 0$, we introduce a simple notation p_{ij}
$$p_{ij} \triangleq \mathbb{P}(X_{n+1} = j | X_n = i)$$
- (Q) Any convenient way of describing a MC for intuitive understanding?

- Machine example: $\mathcal{S} = \{1, 2\}$

$$p_{11} = \mathbb{P}(X_{n+1} = 1 | X_n = 1) = 1 - b, \quad p_{12} = \mathbb{P}(X_{n+1} = 2 | X_n = 1) = b$$

$$p_{21} = \mathbb{P}(X_{n+1} = 1 | X_n = 2) = r, \quad p_{22} = \mathbb{P}(X_{n+1} = 2 | X_n = 2) = 1 - r$$

$$\boldsymbol{P} = \begin{pmatrix} 1 - b & b \\ r & 1 - r \end{pmatrix}$$

- Transition Probability Matrix

The $m \times m$ matrix $\boldsymbol{P} = [p_{ij}]$, where $p_{ij} \triangleq \mathbb{P}(X_{n+1} = j | X_n = i)$

- Property.

$$\sum_{j=1}^m p_{ij} = 1 \text{ (for each row } i, \text{ the column sum} = 1)$$

- Machine example.

$$p_{11} = \mathbb{P}(X_{n+1} = 1 | X_n = 1) = 1 - b,$$

$$p_{12} = \mathbb{P}(X_{n+1} = 2 | X_n = 1) = b$$

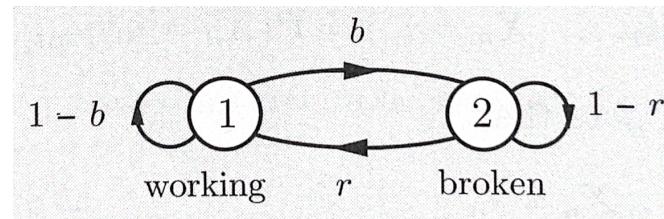
$$p_{21} = \mathbb{P}(X_{n+1} = 1 | X_n = 2) = r,$$

$$p_{22} = \mathbb{P}(X_{n+1} = 2 | X_n = 2) = 1 - r$$

- Transition probability matrix

$$\mathbf{P} = \begin{pmatrix} 1 - b & b \\ r & 1 - r \end{pmatrix}$$

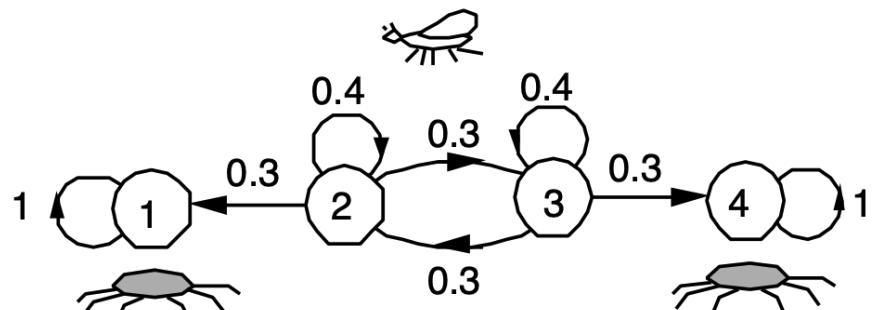
- Any other way? **State Transition Diagram**



- Transition probability matrix and state transition diagram are the two ways of completely describing a given Markov chain.

Spider-Fly Example (Example 7.2)

- A fly moves along a line in unit increments.
- At each time, it moves one unit (i) left w.p. 0.3, (ii) right w.p. 0.3 and (iii) stays in place w.p. 0.4, independent of the past history of movements.
- Two spiders lurk at positions 1 and 4: if the fly lands there, it is captured by the spider, and the process terminates. Assume that the fly starts in a position between 1 and 4.
- X_n : position of the fly. Please draw the state transition diagram and find the transition probability matrix.



	1	2	3	4
1	1.0	0	0	0
2	0.3	0.4	0.3	0
3	0	0.3	0.4	0.3
4	0	0	0	1.0

p_{ij}

- Assume that the process starts at any of the four positions with equal probability 1/4.
- Let $Y_n = 1$ whenever the MC is at position 1 or 2, and $Y_n = 2$ whenever the MC is at position 3 or 4.
- Is $(Y_n : n \geq 0)$ a Markov chain? VIDEO PAUSE 1
- The key is the Markov property. In other words, given $Y_1, Y_2 \perp\!\!\!\perp Y_0$ or not?
- For example, compare $\mathbb{P}(Y_2 = 2 | Y_1 = 2, Y_0 = 1)$ and $\mathbb{P}(Y_2 = 2 | Y_1 = 2, Y_0 = 2)$.
- Given $Y_1 = 2$ (i.e., at time 1, I am at position 3 or 4), the event that I am still at position 3 or 4 at time 2 depends on where I was at time 0 or not? VIDEO PAUSE 2
- $\mathbb{P}(Y_2 = 2 | Y_1 = 2, Y_0 = 1) = \mathbb{P}(X_2 \in \{3, 4\} | X_1 = 3) = 0.7$ (because $Y_0 = 1$ implies that X_1 has to be 3).
- $\mathbb{P}(Y_2 = 2 | Y_1 = 2, Y_0 = 2) > 0.7$, because $X_1 = 3$ or $X_1 = 4$.
- $(Y_n : n \geq 0)$ is not a MC.

- Discrete time slots. N persons. Each person is in one of the three conditions: (F) **infectious**: infected and infectious, (I) **purely infected**: infected, but not infectious or (N) **noninfected**
 - **Infection model.** If a person becomes infected during a time slot, then he/she will be in an infectious condition (F) during the following time slot, and from then on will be in an purely infected condition (I).

$$N \rightarrow F \text{ (just one slot)} \rightarrow I$$

- **Contact model.** During every time slot, each of the $\binom{N}{2}$ pairs of persons are independently in contact w.p. p
 - When F meets with N , then N becomes infected, following **infection model**.

- X_n : number of infectious (F) persons at the beginning of time slot n .
- Y_n : number of noninfected (N) persons at the beginning of time slot n .

Q1. Is $(X_n : n \geq 0)$ a MC?

- X_n only depends on X_{n-1} ?
- X_n also depends on the number of noninfected persons at $n - 1$ time slot. Thus, No.

Q2. Is $(Y_n : n \geq 0)$ a MC?

- Y_n only depends on Y_{n-1} ?
- Y_n also depends on the number of infections persons at $n - 1$ time slot. Thus, No.

Q3. Is $((X_n, Y_n) : n \geq 0)$ a MC?

- (X_n, Y_n) only depends on (X_{n-1}, Y_{n-1}) ?
- Yes.

- **Messages**

- Being successful in good modeling depends on the choice of “state” (good modeling sense).
- Markov chain can be used widely if we choose the state space appropriately.

Markov Chain

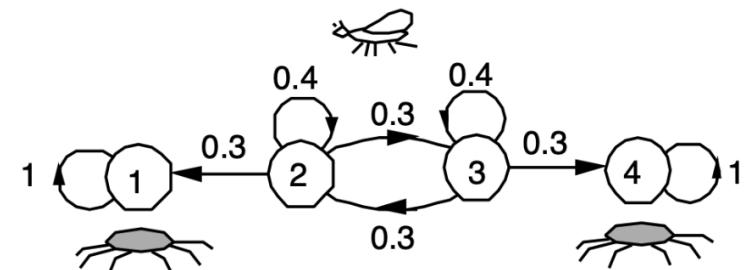
- (1) Definition, Transition Probability Matrix, State Transition Diagram
- (2) *n*-step Transition Probability
- (3) Classification of States
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- (5) Transient Behaviors

(Q) What is the probability of a sample path in a Markov chain with the transition probability matrix $P = [p_{ij}]$?

$$\begin{aligned} & \mathbb{P}(X_0 = i_0, X_1 = i_1, X_2 = i_2, \dots, X_n = i_n) \\ &= \mathbb{P}(X_n = i_n | X_0 = i_0, X_1 = i_1, \dots, X_{n-1} = i_{n-1}) \cdot \mathbb{P}(X_0 = i_0, X_1 = i_1, \dots, X_{n-1} = i_{n-1}) \\ &= p_{i_{n-1}i_n} \cdot \mathbb{P}(X_0 = i_0, X_1 = i_1, \dots, X_{n-1} = i_{n-1}) = \mathbb{P}(X_0 = i_0) \cdot p_{i_0i_1} \cdot p_{i_1i_2} \cdots p_{i_{n-1}i_n} \end{aligned}$$

- Spider-Fly example

$$\begin{aligned} & \mathbb{P}(X_0 = 2, X_1 = 2, X_2 = 2, X_3 = 3, X_4 = 4) \\ &= \mathbb{P}(X_0 = 2)p_{22}p_{22}p_{23}p_{34} \\ &= \mathbb{P}(X_0 = 2)(0.4)^2(0.3)^2 \end{aligned}$$



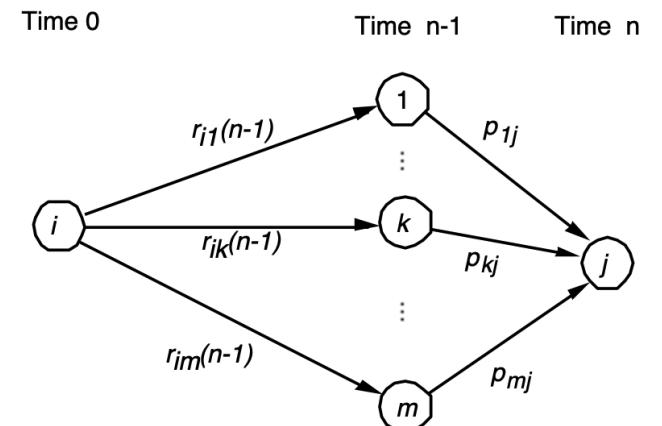
(Q) What is the probability that my state is j , after n steps, starting from i ?

- n -step transition probability: $r_{ij}(n) \triangleq \mathbb{P}(X_n = j | X_0 = i)$
- Recursive formula, starting with $r_{ij}(1) = p_{ij}$,

$$r_{ij}(n) = \mathbb{P}(X_n = j | X_0 = i) = \sum_{k=1}^m \mathbb{P}(X_n = j, X_{n-1} = k | X_0 = i)$$

$$\sum_{k=1}^m \mathbb{P}(X_{n-1} = k | X_0 = i) \mathbb{P}(X_n = j | X_{n-1} = k, X_0 = i)$$

$$= \sum_{k=1}^m r_{ik}(n-1) p_{kj}$$



- Possible to compute $r_{ij}(n)$ recursively. This is called Chapman-Kolmogorov equation.

$${}^0\mathbb{P}(A, C | B) = \mathbb{P}(C | B) \mathbb{P}(A | C, B)$$

L8(2)

$$\bullet \ r_{ij}(n+1) = \sum_{k=1}^m r_{ik}(n)r_{kj}(1)$$

$$r_{ij}(n+1) = \mathbb{P}(X_{n+1} = j \mid X_0 = i) = \sum_{k=1}^m \mathbb{P}(X_{n+1} = j, X_n = k \mid X_0 = i)$$

$$= \sum_{k=0}^m \mathbb{P}(X_{n+1} = j \mid X_n = k, X_0 = i) \mathbb{P}(X_n = k \mid X_0 = i) = \sum_{k=1}^m r_{ik}(n)r_{kj}(1)$$

- Let $\mathbf{P}^{(n)}$ be the matrix of n -step transition probability, i.e., $\mathbf{P}^{(n)} \triangleq [r_{ij}(n)]$
- (Q) What is the relation between $\mathbf{P}^{(n)}$ and \mathbf{P} ? Can we express $\mathbf{P}^{(n)}$ with \mathbf{P} ?

- $r_{ij}(n+l) = \sum_{k=1}^m r_{ik}(n)r_{kj}(l)$ and $\mathbf{P}^{(n)} \triangleq [r_{ij}(n)]$
- Then, by letting $n = 1, l = 1$,
$$\mathbf{P}^{(2)} = \left[\sum_{k=1}^m r_{ik}(1)r_{kj}(1) \right] = \left[\sum_{k=1}^m p_{ik}p_{kj} \right] = \mathbf{P} \times \mathbf{P} = \mathbf{P}^2.$$
- By letting $n = 2, l = 1$,
$$\mathbf{P}^{(3)} = \left[\sum_{k=1}^m r_{ik}(2)r_{kj}(1) \right] = \left[\sum_{k=1}^m r_{ik}(2)p_{kj} \right] = \mathbf{P}^{(2)} \times \mathbf{P} = \mathbf{P}^3$$
- Then, by induction, $\mathbf{P}^{(n)} = \mathbf{P}^n$
- In other words, n -step transition probability matrix is just a **n -time multiplication** of the transition probability matrix \mathbf{P} .

Example: Urn with Two Balls

- An urn always contains 2 balls. Ball colors are **red** and **blue**.
- At each stage, a ball is randomly chosen, and then replaced by a new ball, which with probability 0.8 is the same color, and with probability 0.2 is the opposite color, as the ball it replaces.
- If initially both balls are red, find the probability that the **fifth ball** selected is red.
- **Solution.** Let X_n be the number of red balls after n -th stage (selection and replacement). Then, $\mathcal{S} = \{0, 1, 2\}$.

- $\mathbf{P} = \begin{pmatrix} 0.8 & 0.2 & 0 \\ 0.1 & 0.8 & 0.1 \\ 0 & 0.2 & 0.8 \end{pmatrix}$
- Let $A = \{\text{fifth ball is red}\}$.

$$\begin{aligned}\mathbb{P}(A) &= \sum_{i=0}^2 \mathbb{P}(A|X_4 = i)\mathbb{P}(X_4 = i|X_0 = 2) \\ &= (0)r_{2,0}(4) + (0.5)r_{2,1}(4) + (1)r_{2,2}(4)\end{aligned}$$

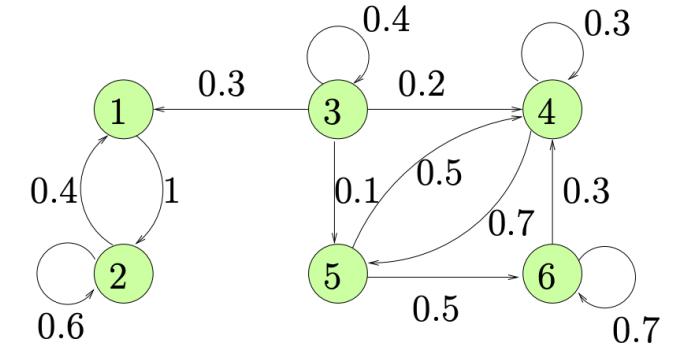
- By computing \mathbf{P}^4 , we get $r_{2,1}(4) = 0.4352$ and $r_{2,2}(4) = 0.4872$
- Thus, $\mathbb{P}(A) = 0.7048$

Markov Chain

- (1) Definition, Transition Probability Matrix, State Transition Diagram
- (2) n -step Transition Probability
- (3) **Classification of States**
- (4) Steady-state Behaviors and Stationary Distribution
- (5) Transient Behaviors

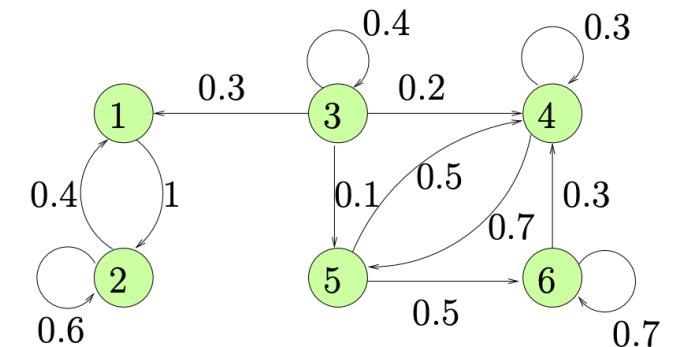
Different States and Classes?

- Classes
 - 3 can only be reached from 3
 - 1 and 2 can reach each other but no other state
 - 4, 5, and 6 all reach each other.
 - Divide into three classes: $\{3\}$, $\{1, 2\}$, $\{4, 5, 6\}$
 - **Message 1.** Multiple classes may exist.
- Difference between 1 and 3
 - 1: If I start from 1, visit 1 infinite times.
 - 3: If I start from 3, visit 3 only finite times (move to other classes and don't return).
 - **Message 2.** Some states are visited infinite times, but some states are not.
- State 2 will share the above properties with 1 (similarly, $\{4, 5, 6\}$)
- **Message 3.** States in the same class share some properties.



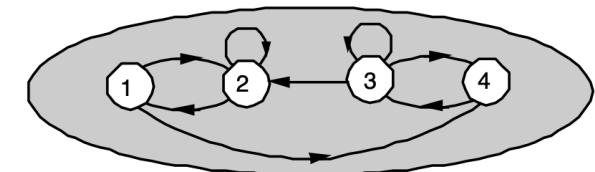
Classification of States (1)

- **Definition.** State j is **accessible** from state i , if for some n $r_{ij}(n) > 0$. denoted by $i \dashrightarrow j$
 - 6 is accessible from 3, but not the other way around.
- **Definition.** If i is accessible from j and j is accessible from i , we say that i **communicates** with j , denoted by $i \leftrightarrow j$.
 - $1 \leftrightarrow 2$, but 3 does not communicate with 5.
- **Definition.** Let $A(i) = \{\text{states accessible from } i\}$. State i is **recurrent**, if $\forall j \in A(i)$, i is also accessible from j . In other words, “I communicate with all of my (direct/indirect) friends!”
 - A state that is not recurrent is **transient**.
 - 2 is recurrent? Yes. 3 is recurrent? No.
 - If we start from a recurrent state i , then there is always some probability of returning to i . It means that, given enough time, it is certain that it returns to i .

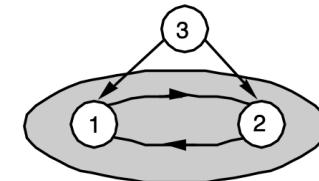


Classification of States (2)

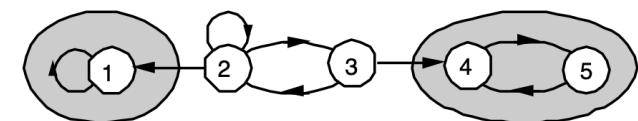
- A set of recurrent states which communicate with each other form a **class**.
- Markov chain decomposition
 - A MC can be decomposed into **one or more recurrent classes**, plus possibly **some transient states**.
 - A recurrent state is accessible from all states in its class, but it not accessible from recurrent states in other classes.
 - A transient state is not accessible from any recurrent state.
 - At least one, possibly more, recurrent states are accessible from a given transient state.
- The MC with only a single recurrent class is said to be **irreducible** (더이상 분해할 수 없는).



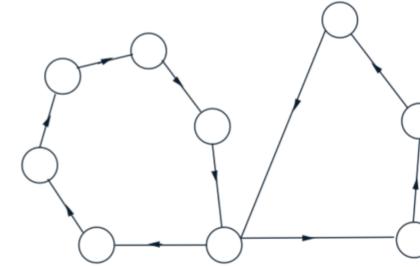
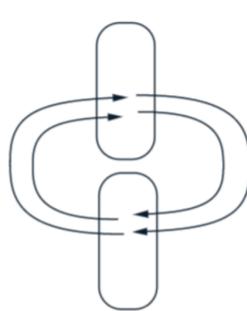
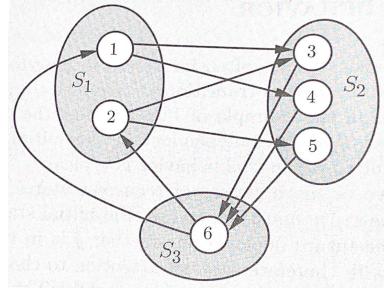
Single class of recurrent states



Single class of recurrent states (1 and 2)
and one transient state (3)



Two classes of recurrent states
(class of state 1 and class of states 4 and 5)
and two transient states (2 and 3)



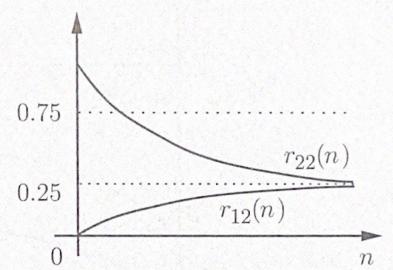
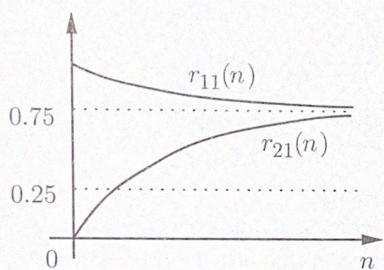
- **Definition.** A recurrent class is said to be **periodic**, if its states can be grouped in $d > 1$ disjoint subsets S_1, \dots, S_d so that all transitions from S_k lead to S_{k+1} (or to S_1 if $k = d$). We call d the **period** of the recurrent class.
- A recurrent class that is not periodic (i.e., period $d = 1$) is said to be **aperiodic**.
- For any state i in the d -period recurrent class, $r_{ii}(n) = 0$, whenever n is not divisible by d , where d is the greatest integer with this property.
- Often, it is not easy to see some MC is periodic or not. But, one easy way is to check whether there exists a self-transition or not. **An MC with a self-transition must be aperiodic.**

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n -step transition prob.: $r_{ij}(n)$ for large n

- Convergence **irrespective of** the start state

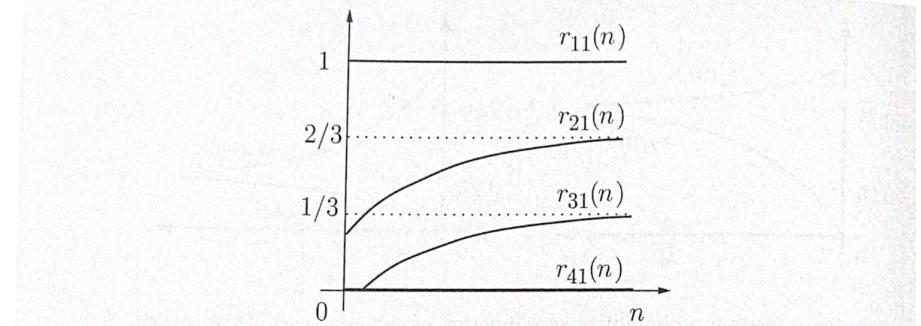


n -step transition probabilities as a function of the number n of transitions

	U	B				
U	0.8	0.2	.76	.24	.752	.248
B	0.6	0.4	.72	.28	.744	.256
	$r_{ij}(1)$		$r_{ij}(2)$		$r_{ij}(3)$	
						$r_{ij}(4)$
						$r_{ij}(5)$

Sequence of n -step transition probability matrices

- Convergence **depending on** the start state



n -step transition probabilities into state 1

	1	2	3	4						
1	1.0	0	0	0						
2	0.3	0.4	0.3	0	.42	.25	.24	.09		
3	0	0.3	0.4	0.3	.09	.24	.25	.42		
4	0	0	0	1.0	0	0	0	1.0		
	$r_{ij}(1)$				$r_{ij}(2)$				$r_{ij}(3)$	
									$r_{ij}(4)$	
										$r_{ij}(\infty)$

Sequence of transition probability matrices

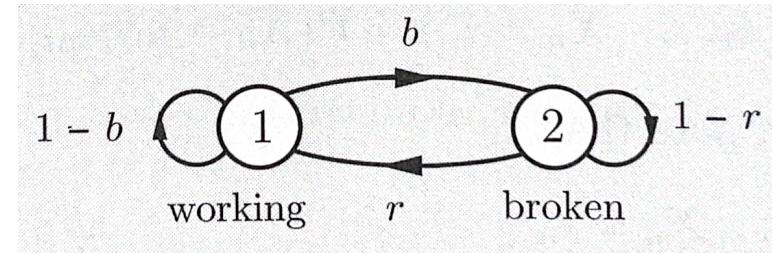
(Q) Under what conditions, convergence occurs, **independent of the start state**? If so, how does it depend on the start state and the shape of the MC?

Steady-state behavior: Why Important?

- $r_{ij}(n) \xrightarrow{n \rightarrow \infty} \pi_j$, for some $\pi_j \leq 1$?
- Interpretation.

$$\pi_j \approx \mathbb{P}(X_n = j) \text{ for large } n$$

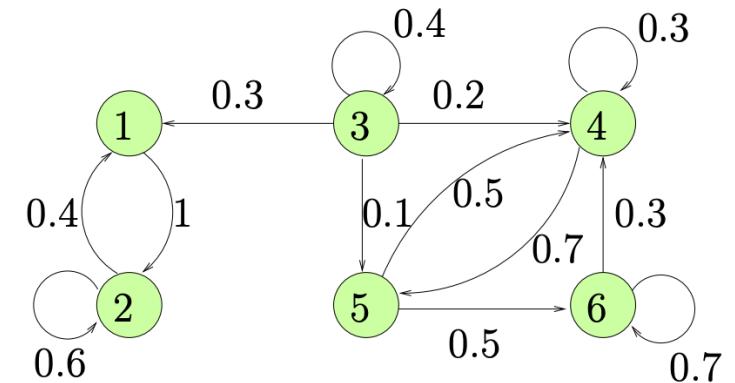
- After running the MC for a long time, we see how long the MC will stay at which state on average.
- Helps in understanding how this MC behaves.



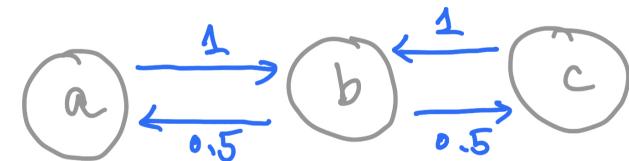
$$\pi_{\text{working}} = \alpha$$

$$\pi_{\text{broken}} = \beta$$

- $r_{ij}(n) \xrightarrow{n \rightarrow \infty} \pi_j$, for some $\pi_j \leq 1$?
- Convergence occurs, **independent of** the starting state, if:
 - C1.** Only a **single recurrent class**
 - C2.** such recurrent class is **aperiodic**
- C1.** For the case of multiple recurrent classes, one stays at the class including the starting state.
- C2.** Divergent behavior for periodic recurrent classes.



(a) multiple recurrent classes



(b) single recurrent, but periodic class

(Q) How to easily compute $(\pi_1, \pi_2, \dots, \pi_m)$ rather than taking the limit?

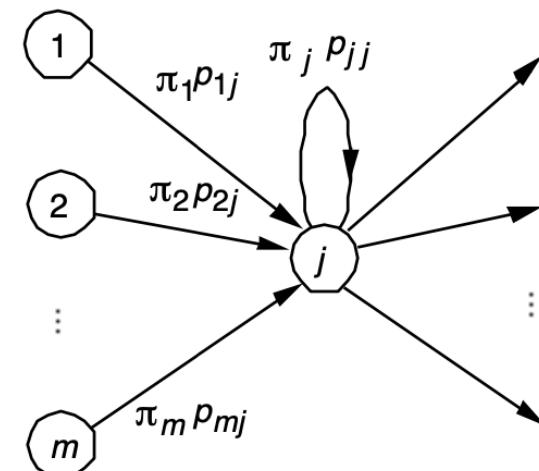
- If $r_{ij}(n) \xrightarrow{n \rightarrow \infty} \pi_j$, for some $\pi_j \leq 1$, from Chapman-Kolmogorov equation,

$$r_{ij}(n) = \sum_{k=1}^m r_{ik}(n-1)p_{kj} \implies \pi_j = \sum_{k=1}^m \pi_k p_{kj} \quad (\text{Balance equation})$$

- $\sum_{i=1}^m \pi_i = 1$: (Normalization equation)
- Balance eqn. + Normalization eqn. \implies Finding the steady-state probabilities $\{\pi_i\}$.
 - Solving linear equations

- Probability: often interpreted as the **relative frequencies** out of many independent trials
- $\pi_j = \lim_{n \rightarrow \infty} \frac{v_{ij}(n)}{n}$, where $v_{ij}(n)$ is the expected number of visits to state j up to the first n transitions
- In other words, π_j : long-term **expected fraction of time** that the MC is at the state j .
- $\pi_j p_{jk}$: the long-term expected **fraction of transitions** that move the state from j to k .

- Balance equation: $\sum_{k=1}^m \pi_k p_{kj} = \pi_j$
 - The expected frequency of visits to j = The sum of the expected frequencies of transitions that lead to j .



- A two-state MC with: $\begin{pmatrix} 0.8 & 0.2 \\ 0.6 & 0.4 \end{pmatrix}$
- (Balance equation)

$$\pi_1 = \pi_1 p_{11} + \pi_2 p_{21} = 0.8\pi_1 + 0.6\pi_2,$$

$$\pi_2 = \pi_2 p_{22} + \pi_1 p_{12} = 0.4\pi_2 + 0.2\pi_1$$

- (Normalization equation) $\pi_1 + \pi_2 = 1$
- Steady-state probabilities: $\pi_1 = 0.25$, $\pi_2 = 0.75$.

- $\{\pi_j\}$ is also called a **stationary distribution**. Why?
- **Distribution**, because $\sum_{j=1}^m \pi_j = 1$.
- **Stationary**, because, if you choose the initial state according to $\{\pi_j\}$, then for any $j \in \{1, \dots, m\}$

$$\mathbb{P}(X_0 = j) = \pi_j \xrightarrow{\text{total prob. theorem}} \mathbb{P}(X_1 = j) = \sum_{k=1}^m \mathbb{P}(X_0 = k) p_{kj} = \sum_{k=1}^m \pi_k p_{kj} = \pi_j$$

- Similarly, we have $\mathbb{P}(X_n = j) = \pi_j$, for all n and j .
- If the initial state is chosen according to $\{\pi_j\}$, the state at any future time will have the same distribution (i.e., the distribution does not change over time).
- We say that "the limiting distribution (steady-state distribution) is equal to the stationary distribution"

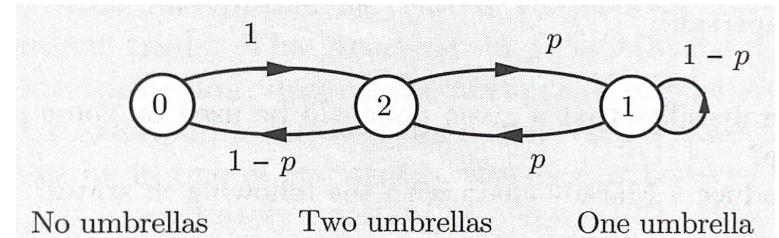
⁰stationary: not moving or not intended to be moved.

Example 2

- An absent-minded professor: two umbrellas from home to office and back.
- If it rains and an umbrella is available, she takes it. If it is not raining, she always forgets to take an umbrella.
- Suppose that it rains w.p. $0 < p < 1$ each time when she commutes, independent of other times.
- (Q) What is the steady-state probability that she gets wet during a commute?
- (Hint) If you think that this can be modeled by a MC, think about what should be chosen as states. What is changing over time?

VIDEO PAUSE

- state $i \in \{0, 1, 2\}$: i umbrellas available in her location.
- Transition diagram



- Single recurrent class and aperiodic
- Balance and normalization equation

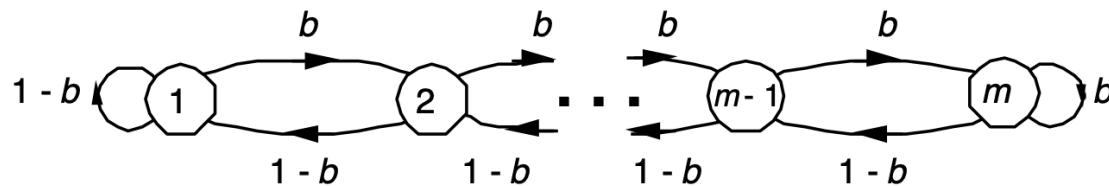
$$\pi_0 = (1 - p)\pi_2, \quad \pi_1 = (1 - p)\pi_1 + p\pi_2$$

$$\pi_2 = \pi_0 + p\pi_1, \quad \pi_0 + \pi_1 + \pi_2 = 1$$

- $\pi_0 = \frac{1-p}{3-p}, \pi_1 = \frac{1}{3-p}, \pi_2 = \frac{1}{3-p}.$
- The answer is $p \times \pi_0$.

Example 3: Random Walk with Reflecting Barriers

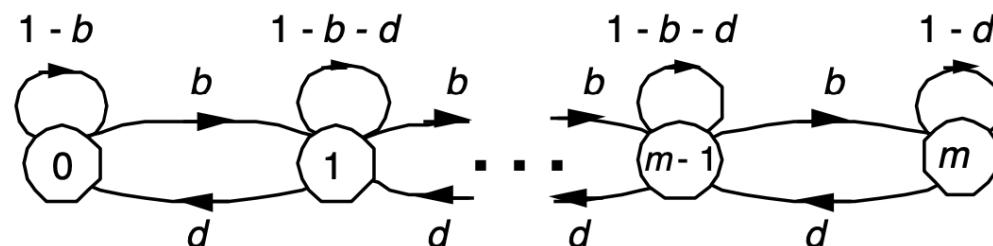
- A person walks along a straight line, and at each time, moves right w.p. b and moves left w.p. $1 - b$.
- Starts in one of the positions $1, 2, \dots, m$.
- If he reaches position 0 (or position $m + 1$), his step is instantly reflected back to position 1 (or position m , respectively).



Example 4: Queueing

- Customers arrive at the supermarket counter. If there are some customers at the counter, then new customers should wait in a line whose capacity is m .
- If there are m customers, then new customer cannot wait in the line, and is discarded.
- We assume discrete time slots. We assume that at each time slot, exactly one of the followings (a), (b), and (c) occurs

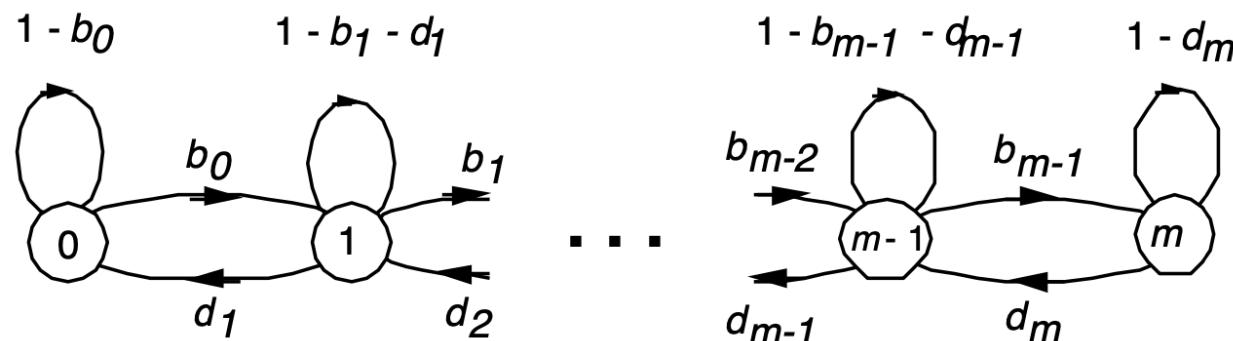
- (a) A new customer arrives w.p. $b > 0$
- (b) One existing customer at the counter leaves w.p. $d > 0$. If there are no customers, nothing happens.
- (c) No new customer and no existing customer leaves w.p. $1 - b - d$, if there is at least one customer at the counter and w.p. $1 - b$ otherwise.



- A special type of Markov chain where the states are **linearly arranged** and transitions can occur only to a **neighboring** state.
- Birth and Death

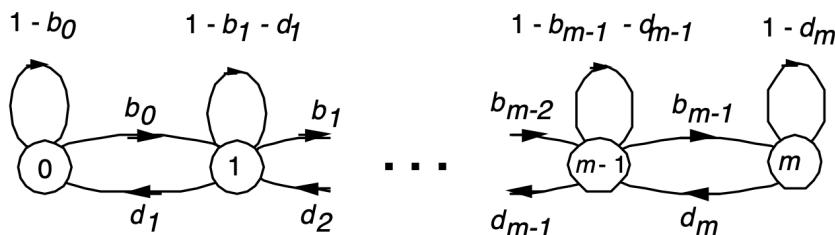
$b_i = \mathbb{P}(X_{n+1} = i + 1 | X_n = i)$, **birth** probability at state i

$d_i = \mathbb{P}(X_{n+1} = i - 1 | X_n = i)$, **death** probability at state i



Birth-Death Process (2)

- State transition diagram



- Balance eqn at state 0

$$\pi_0(1 - b_0) + \pi_1 d_1 = \pi_0 \leftrightarrow \pi_0 b_0 = \pi_1 d_1$$

- Balance eqn at state 1

$$\begin{aligned} \pi_0 b_0 + \pi_1 (1 - b_1 - d_1) + \pi_2 d_2 &= \pi_1 \\ \leftrightarrow \pi_1 d_1 + \pi_1 (1 - b_1 - d_1) + \pi_2 d_2 &= \pi_1 \\ \leftrightarrow \pi_1 b_1 &= \pi_2 d_2 \end{aligned}$$

- By induction, we have the following: called **local balance equation**:

$$\pi_i b_i = \pi_{i+1} d_{i+1}, i = 0, 1, \dots, m-1$$

- Using the above local balance eqn,

$$\pi_i = \pi_0 \frac{b_0 b_1 \cdots b_{i-1}}{d_1 d_2 \cdots d_i}, \quad i = 1, \dots, m$$

- Using the above and $\sum \pi_i = 1$, we can easily compute the $[\pi_i]$.
- Examples 3 and 4 are the special cases of birth-death process. So, please compute the steady-state probabilities for both examples as your homeworks.

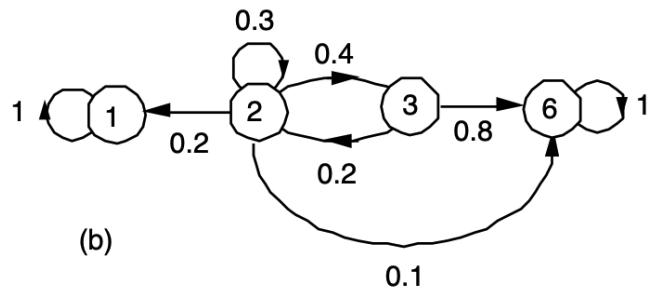
Markov Chain

- (1) Definition, Transition Probability Matrix, State Transition Diagram
- (2) n -step Transition Probability
- (3) Classification of States
- (4) Steady-state Behaviors and Stationary Distribution
- (5) Transient Behaviors

- In the previous lecture,
 - A MC with a single recurrent, aperiodic class $\mathcal{R} = \{1, 2, \dots, m\}$
 - Every state will be visited an infinite number of times
- (Q) Steady-state behavior: what are the long-term average frequencies of states?
- In this lecture,
 - A MC with multiple recurrent classes, say, $\mathcal{R}_1, \dots, \mathcal{R}_k$ and a set of transient states \mathcal{T}
 - Assume that we start from a state $i \in \mathcal{T}$.
 - Transient states will be visited a finite number of times. Then, the MC will enter a recurrent class whose states are visited infinite number of times, but the states in other recurrent classes will not be visited.
- (Q) Transient behavior: what is the first recurrent state to be entered as well as the time until this happens?

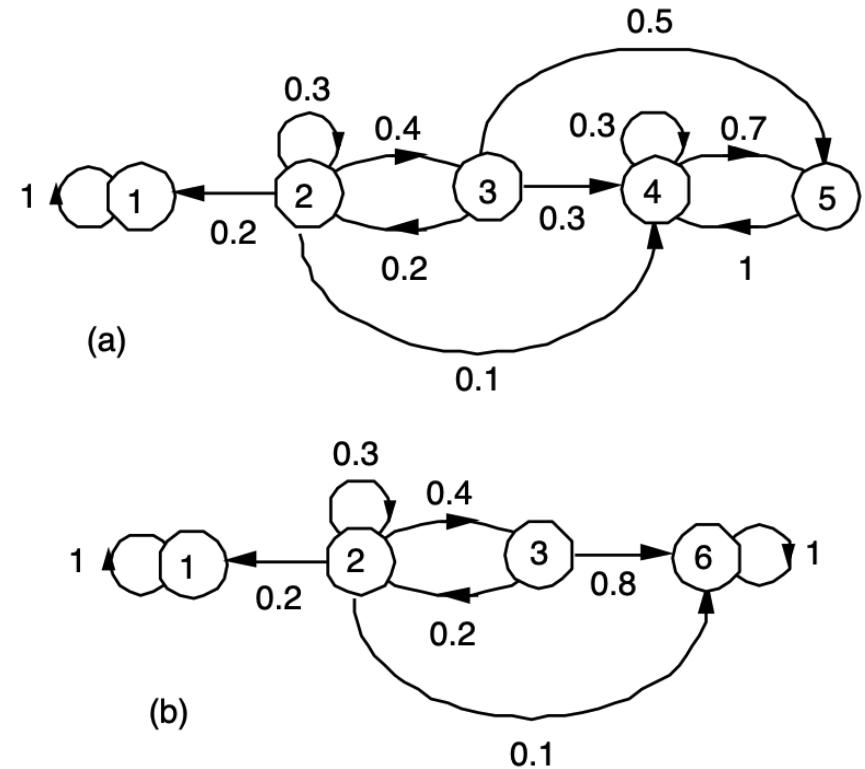
- Rather than dealing with a general MC, let's focus on the Markov chain that **every recurrent state is absorbing**.
- **Definition.** A state k is **absorbing**, if $p_{kk} = 1$, and $p_{kj} = 0$ for all $j \neq k$.
 - states 1 and 6 are absorbing
- For a given absorbing state s , the probability $a_i = a_i(s)$ of reaching s , starting from a state i ?
- Fix $s = 6$:

$$a_1 = 0, \quad a_6 = 1, \quad a_2 = 0.2a_1 + 0.3a_2 + 0.4a_3 + 0.1a_6, \quad a_3 = 0.2a_2 + 0.8a_6$$



- Our interest: a_2 and a_3
- $a_2 = 21/31$ and $a_3 = 29/31$

- Recurrent classes: $\{1\}$ and $\{4, 5\}$
- (Q) Probability that the state eventually enters the recurrent class $\{4, 5\}$?
- Possible transitions within the class $\{4, 5\}$ are NOT important. Why?
- Thus, convert it into the one only with absorbing recurrent states ((a) \rightarrow (b)).



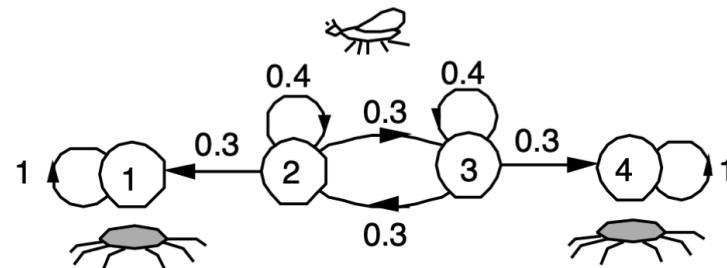
(Q) Starting from a transient state i , what is the expected number of steps until a recurrent state is entered (which we call **absorption**)?

- Special case when all recurrent states are absorbing
- μ_i : expected number of transitions until absorption, starting from i
- Spider-fly example

$$\mu_1 = \mu_4 = 0 \quad (\text{for recurrent states})$$

$$\mu_2 = 1 + 0.4\mu_2 + 0.3\mu_3, \quad \mu_3 = 1 + 0.3\mu_2 + 0.4\mu_3 \quad (\text{for transient states})$$

- Again, for general MCs, convert them into the one with only recurrent states that are absorbing



- Assume a single recurrent class for simplicity

(Q) Mean first passage time. Starting from i , expected number of transitions t_i to reach s for the first time?

(Q) Mean first recurrence time. Starting from s , expected number of transitions t_s^* to reach s for the first time?

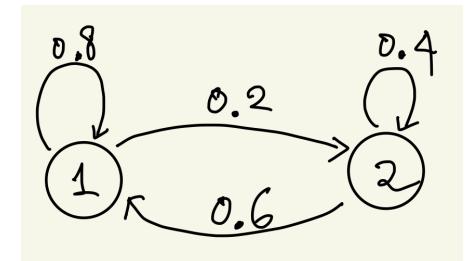
- Mean first passage time from 2 to 1: $t_2 = t_1(1)$

$$t_1 = 0$$

$$t_2 = 1 + p_{21}t_1 + p_{22}t_2 = 1 + 0.4t_2 \implies t_2 = 5/3$$

- Mean first recurrence time from 1 to 1

$$t_1^* = 1 + p_{11}t_1 + p_{12}t_2 = 1 + 0 + 0.2 \frac{5}{3} = \frac{4}{3}$$



Questions?

- 1) Why do you think Markov chain (MC) is important?
- 2) What is the Markov property and its meaning? What's the key difference of MC from Bernoulli processes?
- 3) What are the limiting distribution and the stationary distribution of MCs?
- 4) How are you going to compute the stationary distribution, if you are given a transition probability matrix?
- 5) What are recurrent and transient states in MC?