Lecture 8: Random Processes, Part II

Yi, Yung (이융)

EE210: Probability and Introductory Random Processes
KAIST EE

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Markov Chain

(1) Definition, Transition Probability Matrix, State Transition Diagram

(2) n-step Transition Probability

(3) Classification of States

(4) Steady-state Behaviors and Stationary Distribution

(5) Transient Behaviors

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  - Markov chain
  - One of the most popular random processes in engineering!

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- What will happen at $(n+1)$-th day depends only on what happens at n-th day?

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- Definition. Let $X_1, \ldots, X_n, \ldots$ be a sequence of random variables taking values in some finite space $\mathcal{S} = \{1, 2, \ldots, m\}$, such that for all $i, j \in \mathcal{S}$, $n \geq 0$, the following $\boxed{}$ is satisfied:
for all $n \geq 0$, all $i, j \in \mathcal{S}$, and all possible sequences $i_0, \ldots, i_{n-1}$ of earlier states,

$$\boxed{} = \mathbb{P}(X_{n+1} = j | X_n = i, X_{n-1} = i_{n-1}, \ldots, X_0 = i_0)$$

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- **Alternate definition via conditional independence.** For any fixed n, the future of the process after n is independent of $\{X_1, \ldots, X_{n-1}\}$, given X_n.

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- (Q) Any convenient way of describing a MC for intuitive understanding?

Transition Probability Matrix

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$$p_{11} = \mathbb{P}(X_{n+1} = 1 | X_n = 1) = 1 - b, \qquad p_{12} = \mathbb{P}(X_{n+1} = 2 | X_n = 1) = b$$

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- Property.

$\sum_{j=1}^{m} p_{ij} = 1$ (for each row i, the column sum $= 1$)

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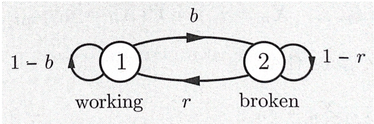

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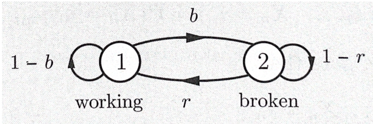

- Transition probability matrix and state transition diagram are the two ways of completely describing a given Markov chain.

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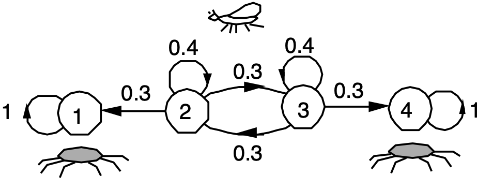

	1	2	3	4
1	1.0	0	0	0
2	0.3	0.4	0.3	0
3	0	0.3	0.4	0.3
4	0	0	0	1.0

p_{ij}

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- $(Y_n : n \geq 0)$ is not a MC.

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  - Infection model. If a person becomes infected during a time slot, then he/she will be in an infectious condition (F) during the following time slot, and from then on will be in an purely infected condition (I).

N $\rightarrow$ F (just one slot) $\rightarrow$ I

COVID-19 Infection Example (1)

- Discrete time slots. N persons. Each person is in one of the three conditions: (F) infectious: infected and infectious, (I) purely infected: infected, but not infectious or (N) noninfected
  - Infection model. If a person becomes infected during a time slot, then he/she will be in an infectious condition (F) during the following time slot, and from then on will be in an purely infected condition (I).

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  - When F meets with N, then N becomes infected, following infection model.

COVID-19 Infection Example (2)

- X_n: number of infectious (F) persons at the beginning of time slot n.
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- Messages
  - Being successful in good modeling depends on the choice of "state" (good modeling sense).
  - Markov chain can be used widely if we choose the state space appropriately.

Markov Chain

(1) Definition, Transition Probability Matrix, State Transition Diagram

(2) n-step Transition Probability

(3) Classification of States

(4) Steady-state Behaviors and Stationary Distribution

(5) Transient Behaviors

Probability of a Sample Path

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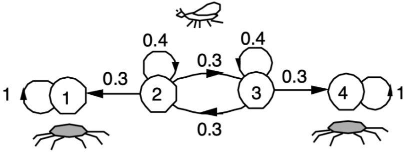

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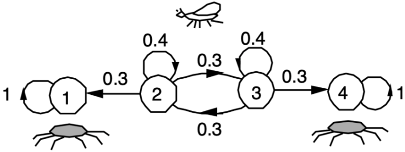

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&= \mathbb{P}(X_0 = 2) p_{22} p_{22} p_{23} p_{34} \\
&= \mathbb{P}(X_0 = 2)(0.4)^2(0.3)^2
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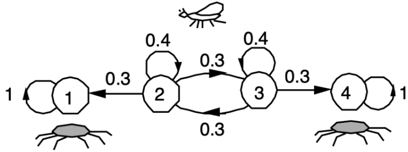

Probability after n Steps

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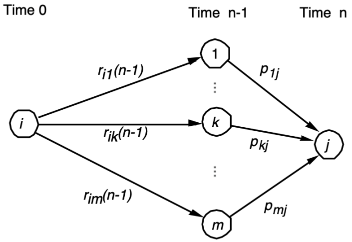

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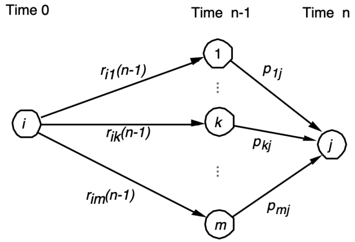

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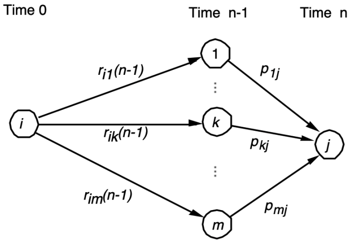

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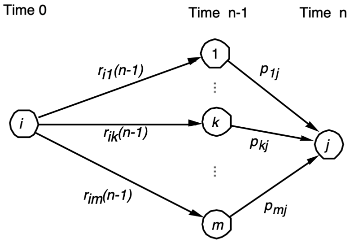

- Possible to compute $r_{ij}(n)$ recursively. This is called Chapman-Kolmogorov equation.

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- (Q) What is the relation between $\boldsymbol{P}^{(n)}$ and $\boldsymbol{P}$? Can we express $\boldsymbol{P}^{(n)}$ with $\boldsymbol{P}$?

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- $r_{ij}(n+l) = \sum_{k=1}^{m} r_{ik}(n) r_{kj}(l)$ and $\boldsymbol{P}^{(n)} \triangleq [r_{ij}(n)]$

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- In other words, n-step transition probability matrix is just a n-time multiplication of the transition probability matrix $\boldsymbol{P}$.

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- Let $A = \{\text{fifth ball is red}\}$.

$$\begin{aligned} \mathbb{P}(A) &= \sum_{i=0}^{2} \mathbb{P}(A|X_4 = i)\mathbb{P}(X_4 = i|X_0 = 2) \\ &= (0)r_{2,0}(4) + (0.5)r_{2,1}(4) + (1)r_{2,2}(4) \end{aligned}$$

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- Solution. Let X_n be the number of red balls after n-th stage (selection and replacement). Then, $\mathcal{S} = \{0, 1, 2\}$.

- $\boldsymbol{P} = \begin{pmatrix} 0.8 & 0.2 & 0 \\ 0.1 & 0.8 & 0.1 \\ 0 & 0.2 & 0.8 \end{pmatrix}$
- Let $A = \{\text{fifth ball is red}\}$.

$$\mathbb{P}(A) = \sum_{i=0}^{2} \mathbb{P}(A|X_4 = i)\mathbb{P}(X_4 = i|X_0 = 2)$$
$$= (0)r_{2,0}(4) + (0.5)r_{2,1}(4) + (1)r_{2,2}(4)$$

- By computing $\boldsymbol{P}^4$, we get $r_{2,1}(4) = 0.4352$ and $r_{2,2}(4) = 0.4872$

Example: Urn with Two Balls

- An urn always contains 2 balls. Ball colors are red and blue.
- At each stage, a ball is randomly chosen, and then replaced by a new ball, which with probability 0.8 is the same color, and with probability 0.2 is the opposite color, as the ball it replaces.
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$$\begin{aligned} \mathbb{P}(A) &= \sum_{i=0}^{2} \mathbb{P}(A | X_4 = i) \mathbb{P}(X_4 = i | X_0 = 2) \\ &= (0) r_{2,0}(4) + (0.5) r_{2,1}(4) + (1) r_{2,2}(4) \end{aligned}$$

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- Thus, $\mathbb{P}(A) = 0.7048$

Markov Chain

(1) Definition, Transition Probability Matrix, State Transition Diagram

(2) n-step Transition Probability

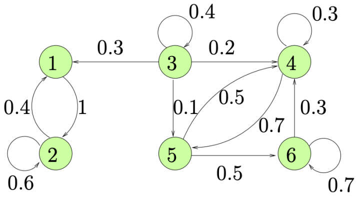
(3) Classification of States

(4) Steady-state Behaviors and Stationary Distribution

(5) Transient Behaviors

Different States and Classes?

- Classes



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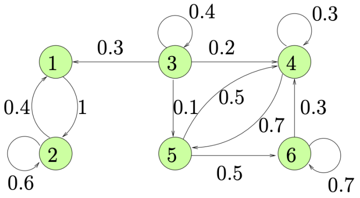

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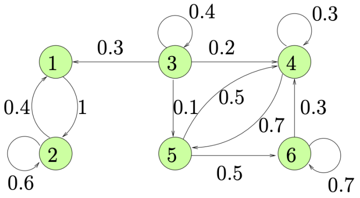

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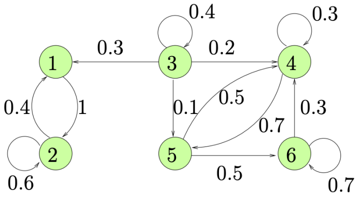

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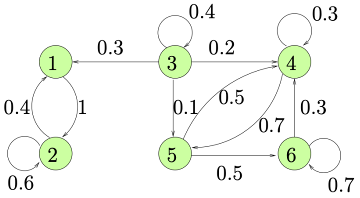

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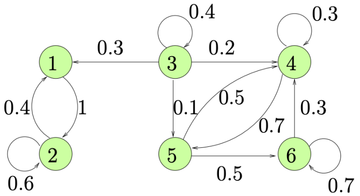

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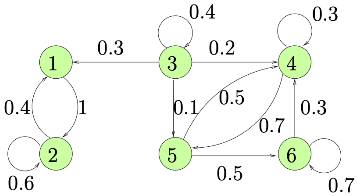

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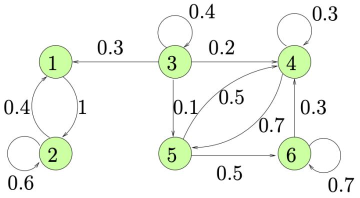

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KAIST EE

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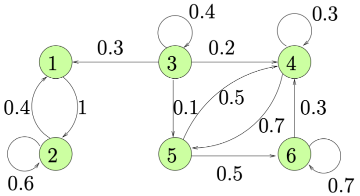

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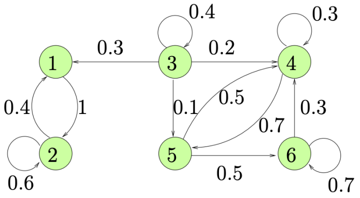

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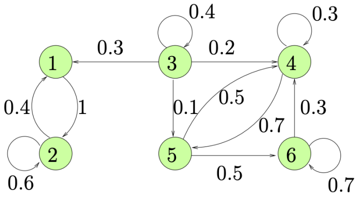

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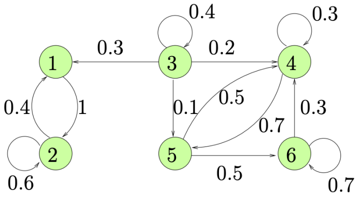

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- Message 3. States in the same class share some properties.

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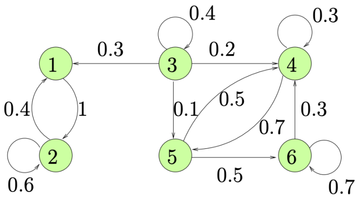

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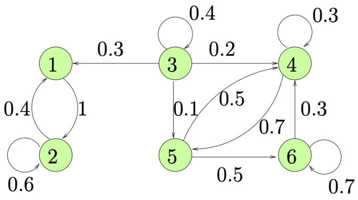

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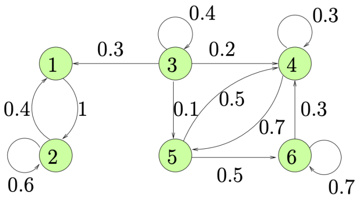

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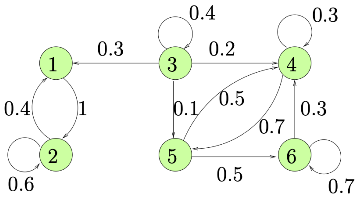

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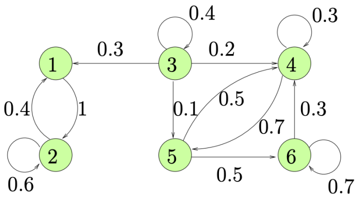

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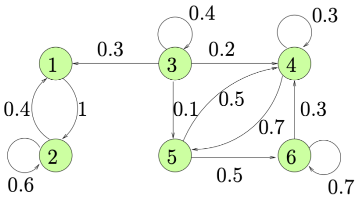

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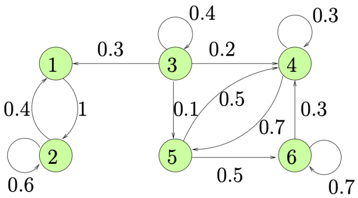

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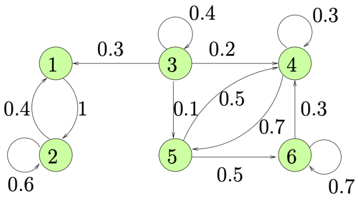

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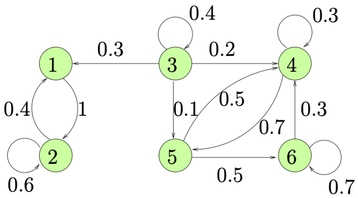

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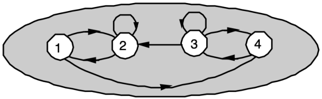
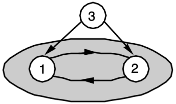
  - If we start from a recurrent state i, then there is always some probability of returning to i. It means that, given enough time, it is certain that it returns to i.

Classification of States (2)

- A set of recurrent states which communicate with each other form a class.



Single class of recurrent states



Single class of recurrent states (1 and 2)
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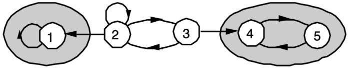

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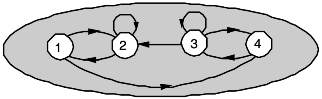

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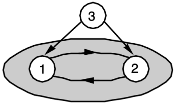

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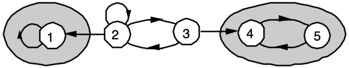

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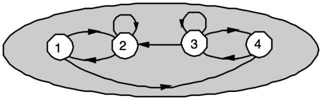

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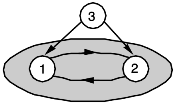

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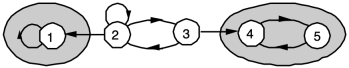

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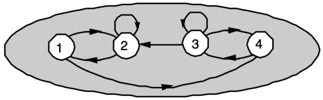

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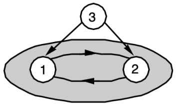

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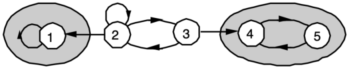

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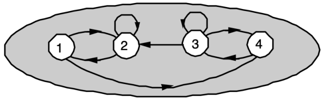

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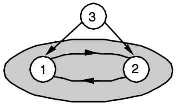

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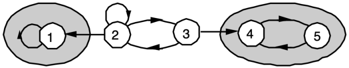

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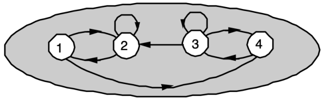
- The MC with only a single recurrent class is said to be irreducible (더이상 분해할 수 없는).



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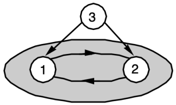

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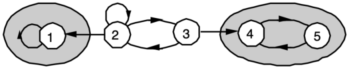

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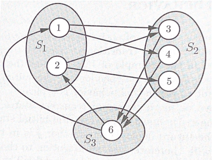
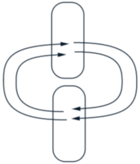
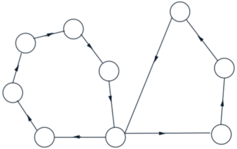
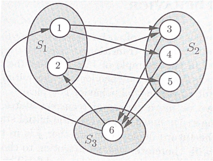
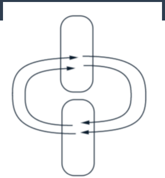
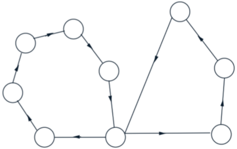

1
2
3
4
5
6
S_1
S_2
S_3











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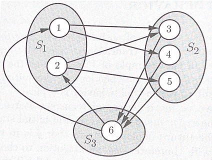

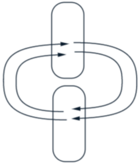

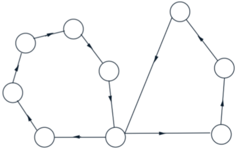

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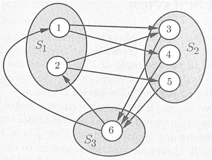

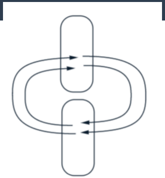

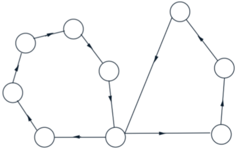

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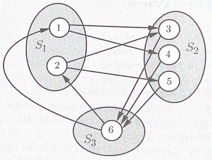

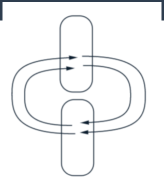

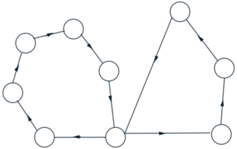

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- For any state i in the d-period recurrent class, $r_{ii}(n) = 0$, whenever n is not divisible by d, where d is the greatest integer with this property.
- Often, it is not easy to see some MC is periodic or not. But, one easy way is to check whether there exists a self-transition or not. An MC with a self-transition must be aperiodic.

Markov Chain

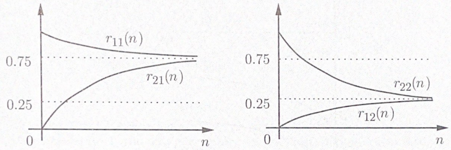
(1) Definition, Transition Probability Matrix, State Transition Diagram

(2) n-step Transition Probability

(3) Classification of States

(4) Steady-state Behaviors and Stationary Distribution

(5) Transient Behaviors



n-step transition probabilities as a function of the number n of transitions

$r_{ij}(1)$

	U	B
U	0.8	0.2
B	0.6	0.4

$r_{ij}(2)$

.76	.24
.72	.28

$r_{ij}(3)$

.752	.248
.744	.256

$r_{ij}(4)$

.7504	.2496
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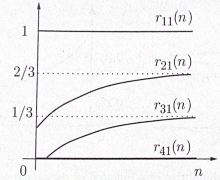
Sequence of n-step transition probability matrices



n-step transition probabilities into state 1

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	1	2	3	4
1	1.0	0	0	0
2	0.3	0.4	0.3	0
3	0	0.3	0.4	0.3
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$r_{ij}(3)$

1.0	0	0	0
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$r_{ij}(4)$

1.0	0	0	0
.55	.12	.12	.21
.21	.12	.12	.55
0	0	0	1.0

....

$r_{ij}(\infty)$

1.0	0	0	0
2/3	0	0	1/3
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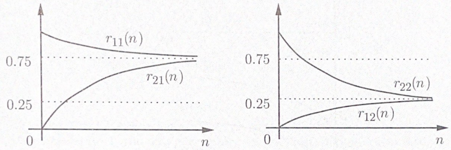
Sequence of transition probability matrices

n-step transition prob.: $r_{ij}(n)$ for large n

- Convergence irrespective of the start state



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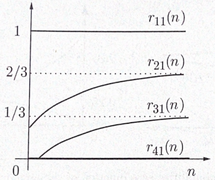

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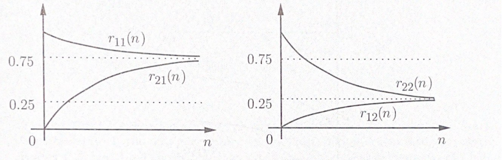

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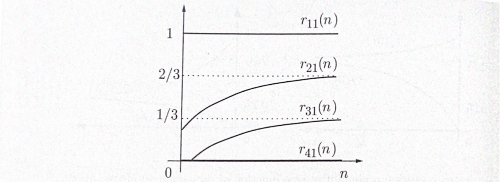

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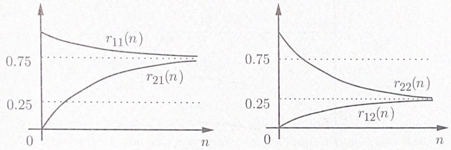

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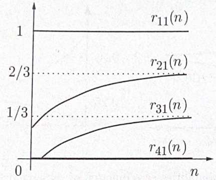

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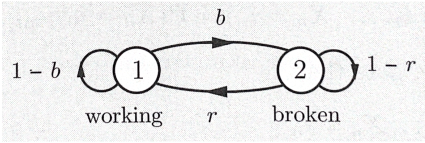
Sequence of transition probability matrices

(Q) Under what conditions, convergence occurs, independent of the start state? If so, how does it depend on the start state and the shape of the MC?

Steady-state behavior: Why Impotant?

- $r_{ij}(n) \xrightarrow{n \to \infty} \pi_j$, for some $\pi_j \leq 1$?



$$\pi_{\text{working}} = \alpha$$

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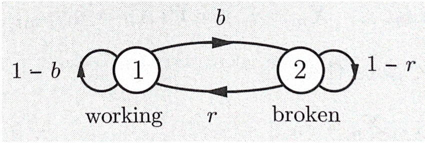
$$\boxed{\pi_j \approx \mathbb{P}(X_n = j) \text{ for large } n}$$



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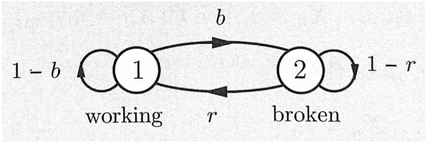

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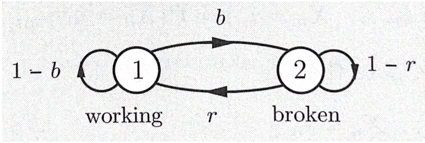
- After running the MC for a long time, we see how long the MC will stay at which state on average.
- Helps in understanding how this MC behaves.



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KAIST EE

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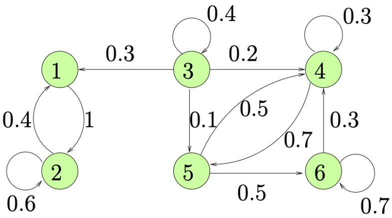
C1. For the case of multiple recurrent classes, one stays at the class including the starting state.



(a) multiple recurrent classes

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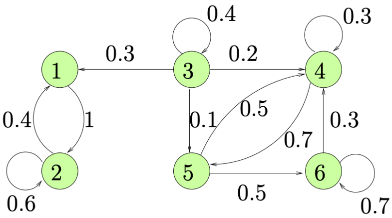

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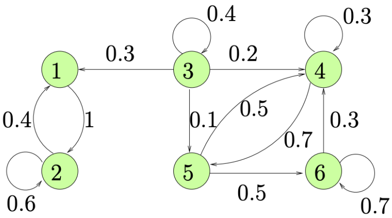
C2. Divergent behavior for periodic recurrent classes.



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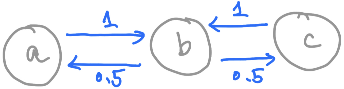

(b) single recurrent, but periodic class

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- If $r_{ij}(n) \xrightarrow{n\to\infty} \pi_j$, for some $\pi_j \leq 1$, from Chapman-Kolmogorov equation,

$$r_{ij}(n) = \sum_{k=1}^{m} r_{ik}(n-1) p_{kj} \Longrightarrow$$

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Steady-state Convergence Theorem

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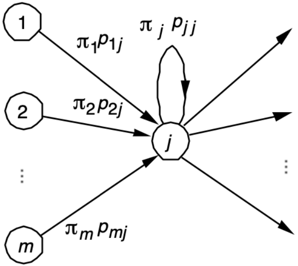
- $\displaystyle\sum_{i=1}^{m} \pi_i = 1$: (Normalization equation)
- Balance eqn. + Normalization eqn. $\Longrightarrow$ Finding the steady-state probabilities $\{\pi_i\}$.
  - Solving linear equations

Long-term Frequency Interpretation

- Probability: often interpreted as the relative frequencies out of many independent trials
- $\pi_j = \lim_{n \to \infty} \frac{v_{ij}(n)}{n}$, where $v_{ij}(n)$ is the expected number of visits to state j up to the first n transitions
- In other words, π_j: long-term expected fraction of time that the MC is at the state j.
- $\pi_j p_{jk}$: the long-term expected fraction of transitions that move the state from j to k.

- Balance equation: $\sum_{k=1}^{m} \pi_k p_{kj} = \pi_j$
  - The expected frequency of visits to j = The sum of the expected frequencies of transitions that lead to j.



Example 1

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- (Balance equation)

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  $\pi_1 = \pi_1 p_{11} + \pi_2 p_{21} = 0.8\pi_1 + 0.6\pi_2,$

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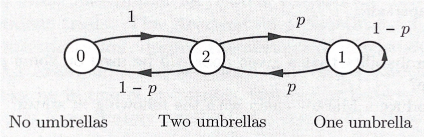

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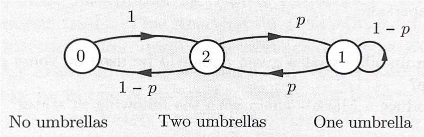

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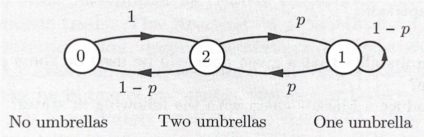

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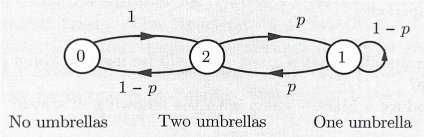

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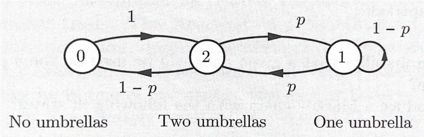

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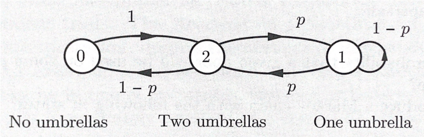

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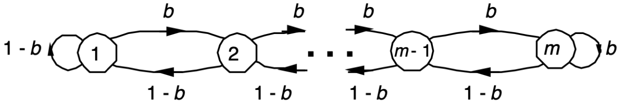

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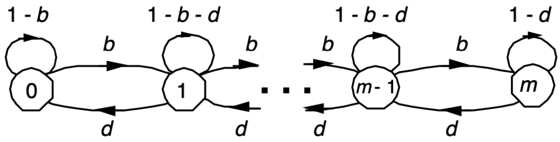

Birth-Death Process (1)

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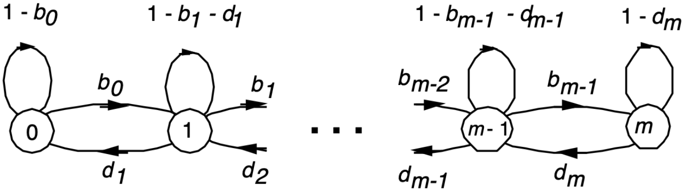

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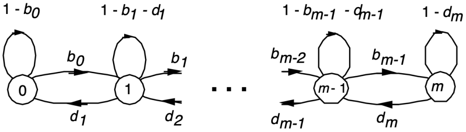
- State transition diagram



Birth-Death Process (2)

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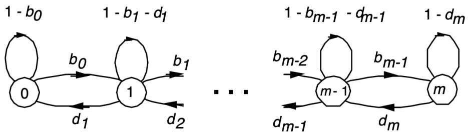

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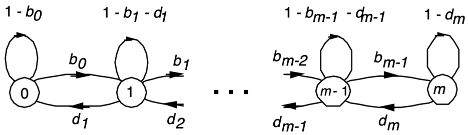

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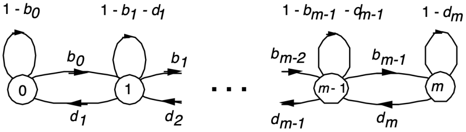

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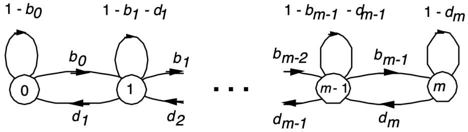

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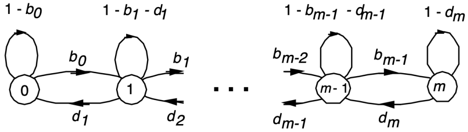

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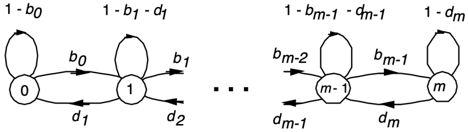

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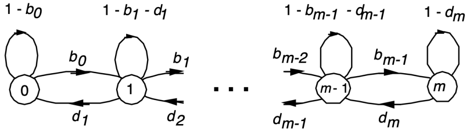

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$$\pi_i b_i=\pi_{i+1} d_{i+1},\ i=0,1,\ldots,m-1$$

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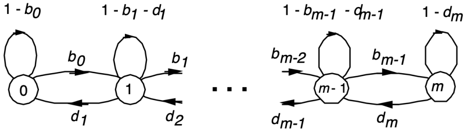

- Balance eqn at state 0

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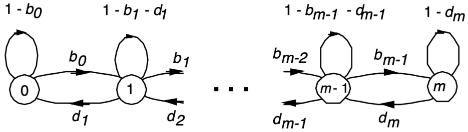

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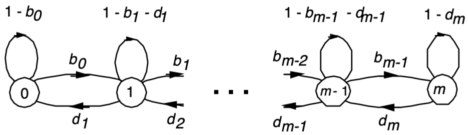

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- Examples 3 and 4 are the special cases of birth-death process. So, please compute the steady-state probabilities for both examples as your homeworks.

Markov Chain

(1) Definition, Transition Probability Matrix, State Transition Diagram

(2) n-step Transition Probability

(3) Classification of States

(4) Steady-state Behaviors and Stationary Distribution

(5) Transient Behaviors

Motivating Questions

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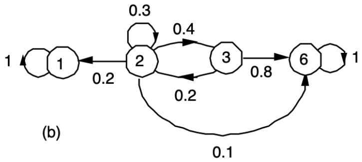

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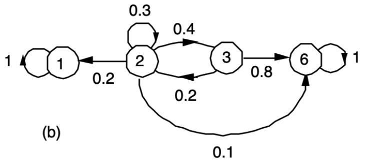

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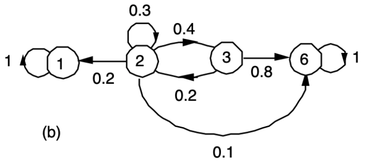

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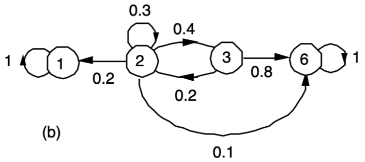

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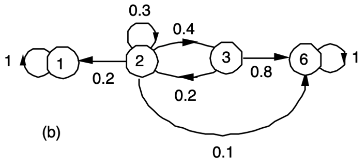

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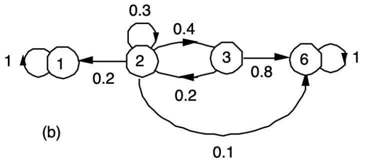

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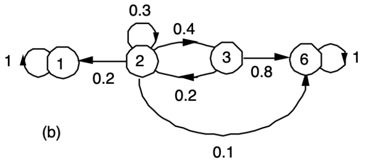

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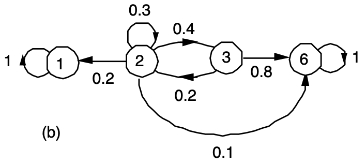

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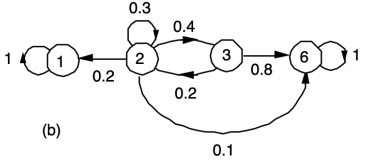

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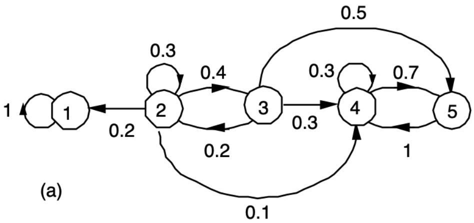
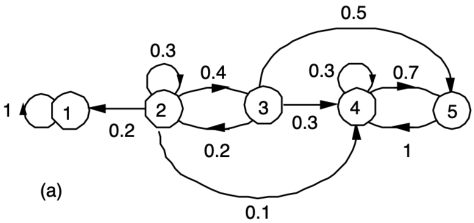
- $a_2 = 21/31$ and $a_3 = 29/31$

- Recurrent classes: $\{1\}$ and $\{4,5\}$



For General MCs

- Recurrent classes: $\{1\}$ and $\{4, 5\}$
- (Q) Probability that the state eventually enters the recurrent class $\{4, 5\}$?



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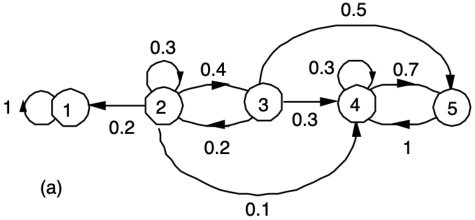

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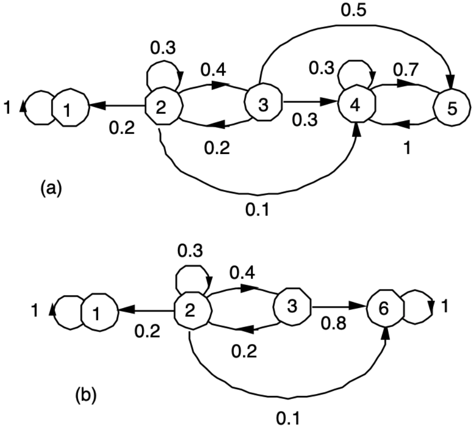
- Thus, convert it into the one only with absorbing recurrent states ((a) $\rightarrow$ (b)).



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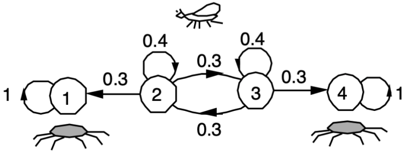

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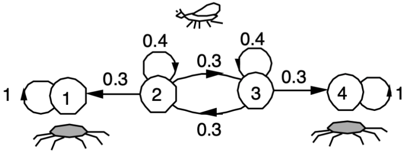

KAIST EE

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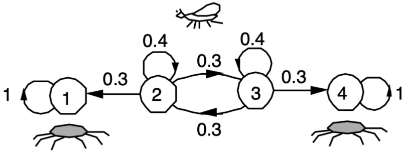

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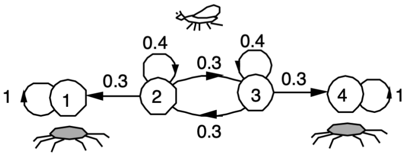

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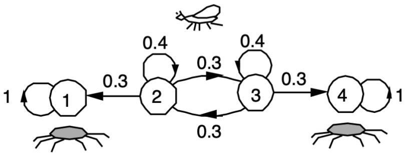

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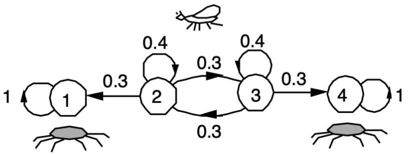

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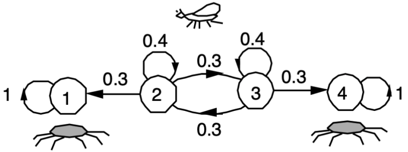
- Again, for general MCs, convert them into the one with only recurrent states that are absorbing



Expected Time to a Particular Recurrent State s

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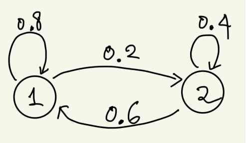

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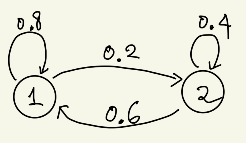

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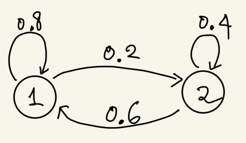

- Mean first passage time from 2 to 1: $t_i = t_i(1)$

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$$t_2 = 1 + p_{21}t_1 + p_{22}t_2 = 1 + 0.4t_2 \Longrightarrow t_2 = 5/3$$

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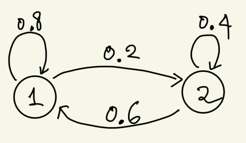

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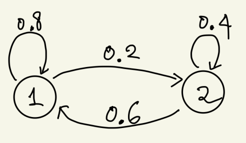

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- Mean first recurrence time from 1 to 1

$$t_1^\star = 1 + p_{11} t_1 + p_{12} t_2 = 1 + 0 + 0.2 \frac{5}{3} = \frac{4}{3}$$

Questions?

1) Why do you think Markov chain (MC) is important?
2) What is the Markov property and its meaning? What's the key difference of MC from Bernoulli processes?
3) What are the limiting distribution and the stationary distribution of MCs?
4) How are you going to compute the stationary distribution, if you are given a transition probability matrix?
5) What are recurrent and transient states in MC?