

## Lecture 8: Random Processes, Part I

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EE210: Probability and Introductory Random Processes  
KAIST EE

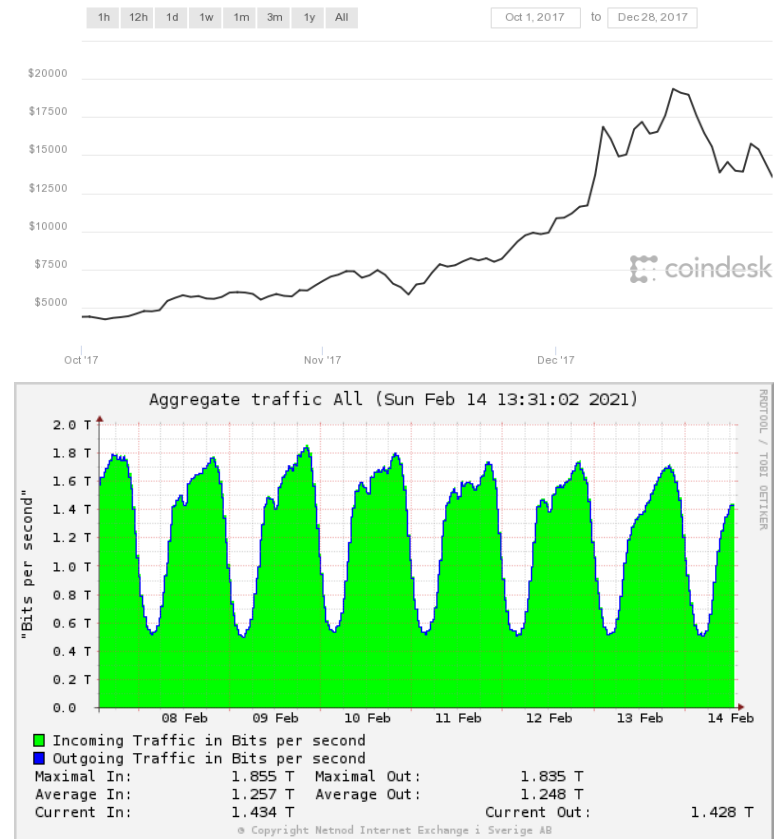
MONTH DAY, 2021

- A lot of applications in engineering systems
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- Bernoulli Process
- Poisson Process
- Use of Bernoulli and Poisson Processes
  - Coding of both processes
  - Merge and Split
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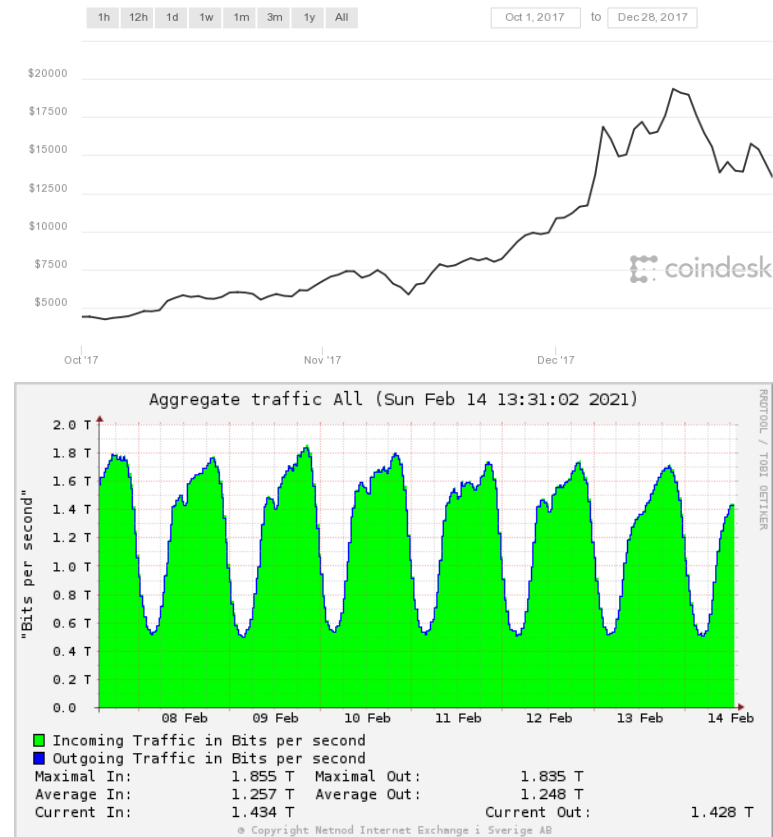
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# Things that evolve in time

- Many probabilistic experiments that **evolve in time**
  - Sequence of daily prices of a stock
  - Sequence of scores in football
  - Sequence of failure times of a machine
  - Sequence of traffic loads in Internet
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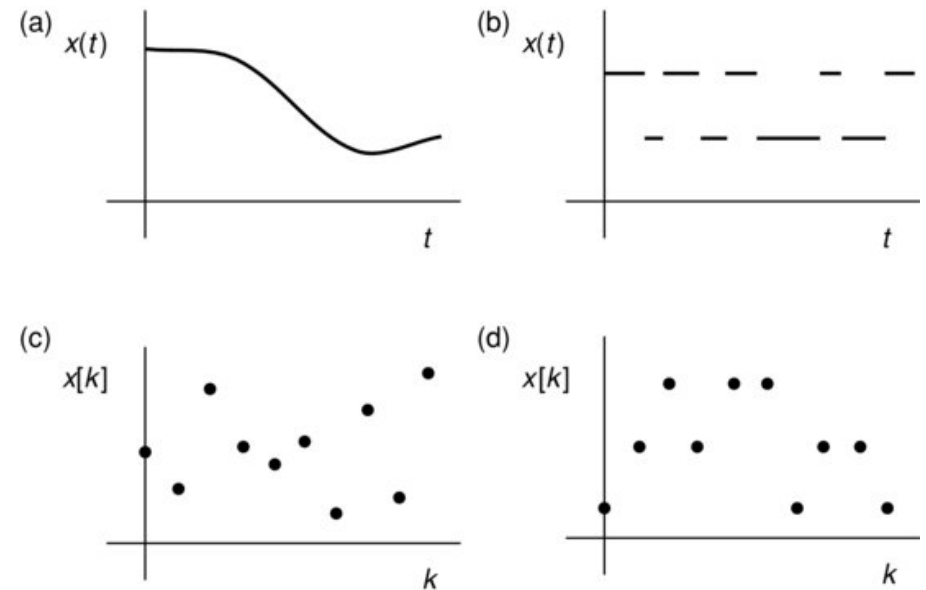
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- The values that  $X_t$  can take: discrete or continuous

## 4 Types of Random Processes

- Types of time and value

- (a) continuous time, continuous value
- (b) continuous time, discrete value
- (c) discrete time, continuous value
- (d) discrete time, discrete value



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- Discrete time, discrete value
- One of the simplest random processes
- A type of “arrival” process

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- Next: Key questions and answers about Bernoulli process

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- But, more than that, as we will see. Independence across time slots leads to many useful properties, allowing the quick solution of many problems.

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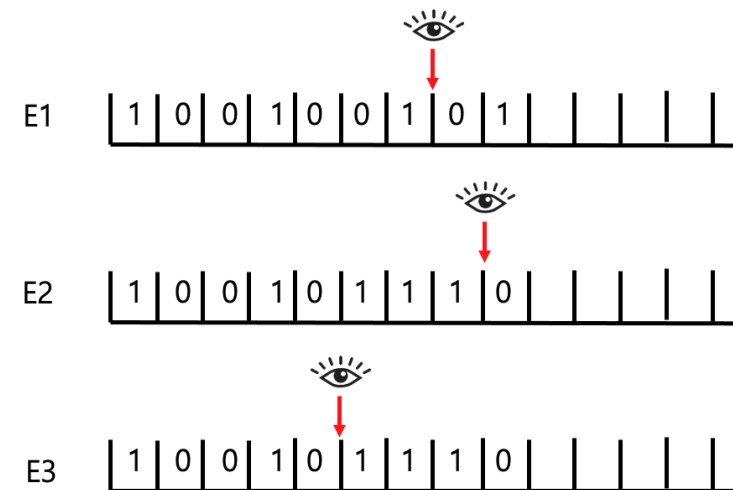
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- $(X_1, \dots, X_5) \perp\!\!\!\perp (X_n)_{n=6}^\infty$
- **Fresh-start** after a deterministic time  $n$
- If you watch the on-going Bernoulli process( $p$ ) from some time  $n$ , you still see the same Bernoulli process( $p$ ).

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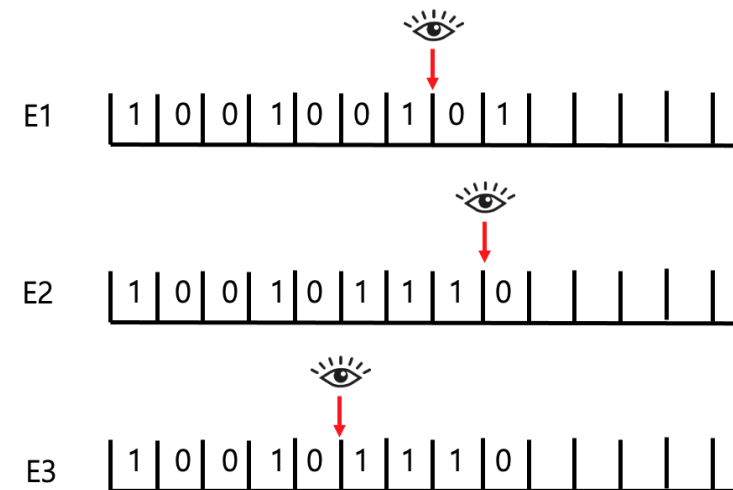
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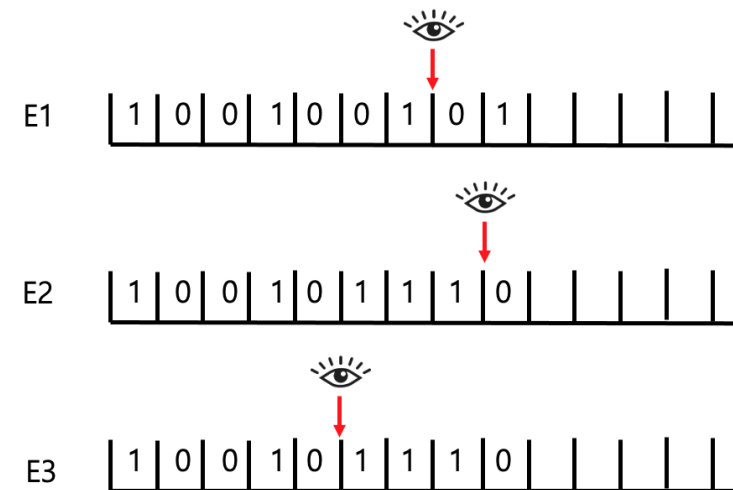


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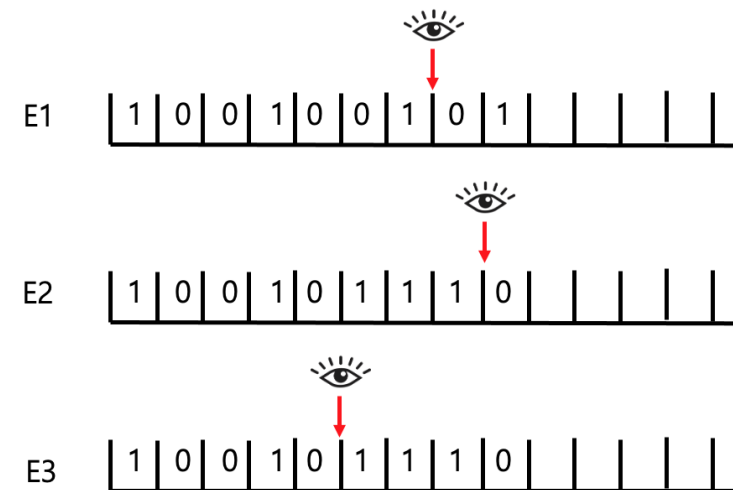
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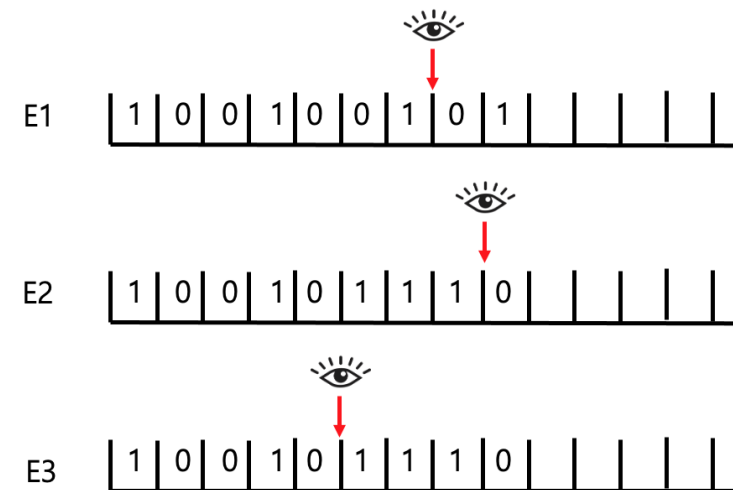
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- Difference of  $N$  from  $n$ 
  - The time when I watch the on-going Bernoulli process is **random**.

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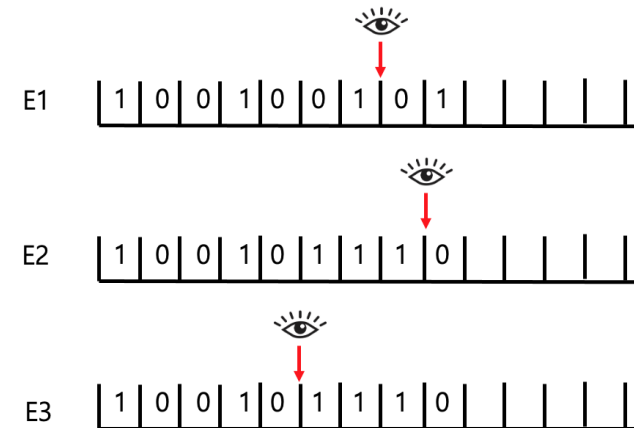
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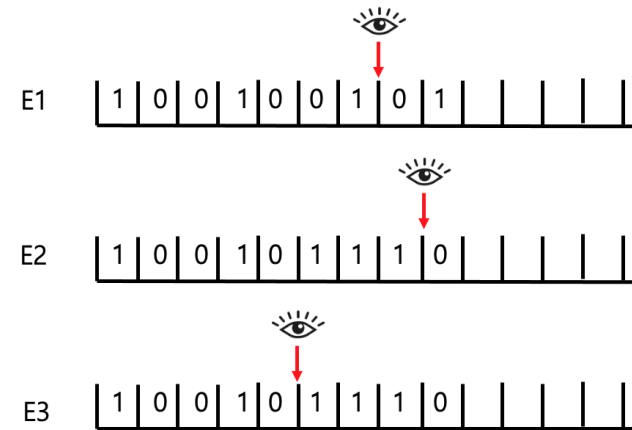
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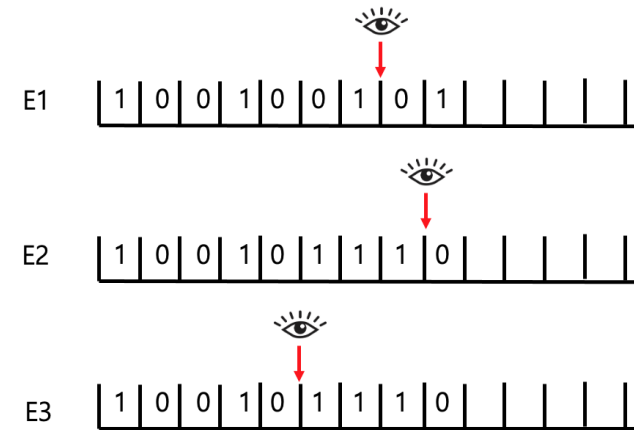
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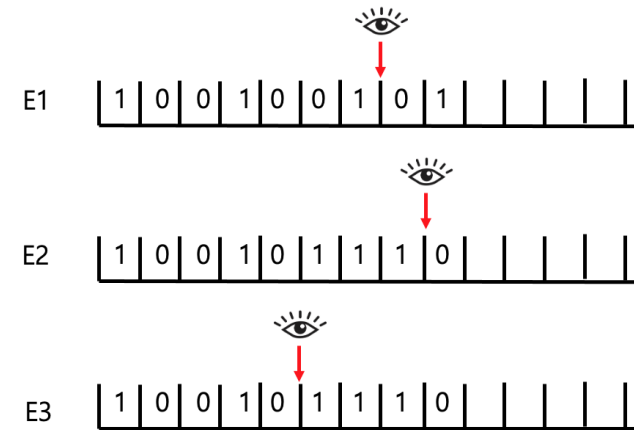
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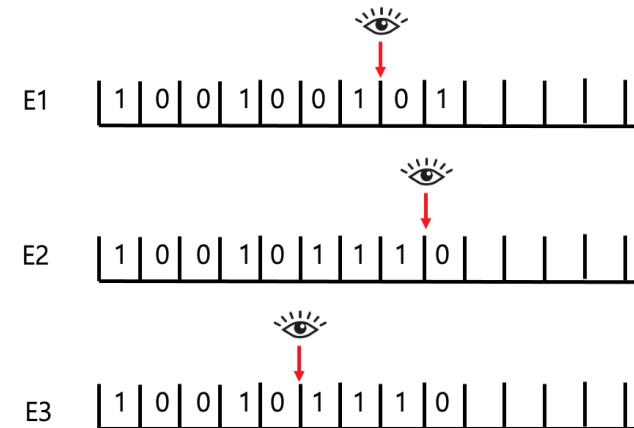
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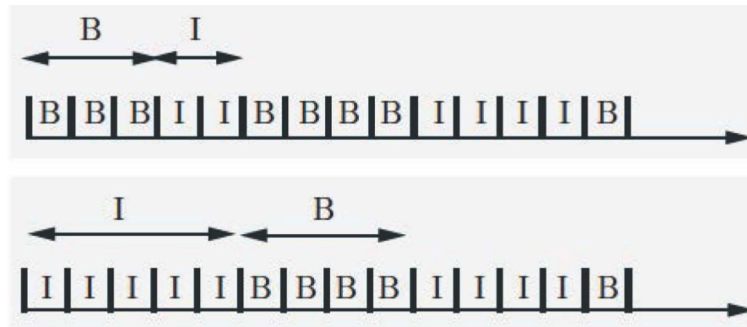
- The question of  $N = n$ ? can be answered just from the knowledge about  $X_1, X_2, \dots, X_n$ ? Then, Yes! (see pp. 301 for more formal description)



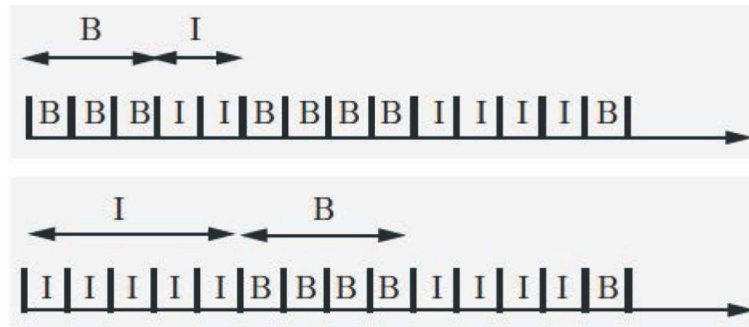
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# Distribution of Busy Periods

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- First busy period  $B_1$ : starts with the first busy slot and ends just before the first subsequent idle slot

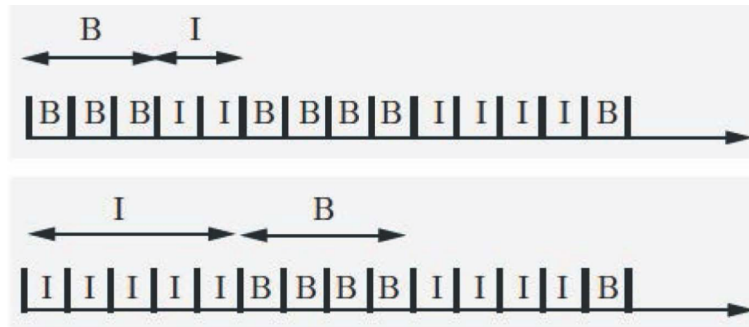


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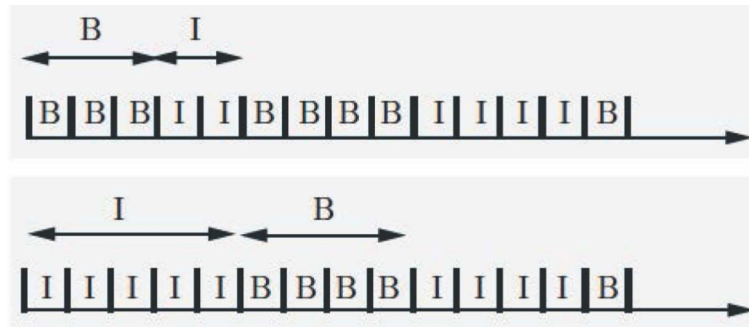
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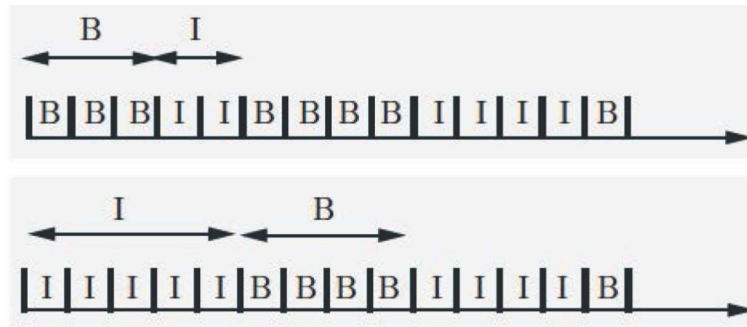
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- (Q6) Distribution of  $B_1$ ?
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- $B_1$  is geometric with parameter  $(1 - p)$
- Question: What about the second busy period  $B_2$ ?  $B_3$ ,  $B_4$ ?

- Time of the first arrival  $Y_1 \sim \text{geom}(p)$

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(Q7) Time of the  $k$ -th arrival  $Y_k$ ?

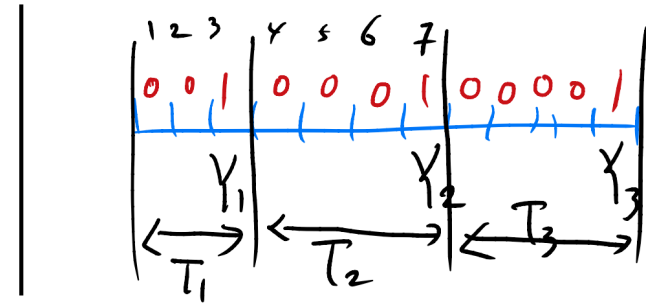


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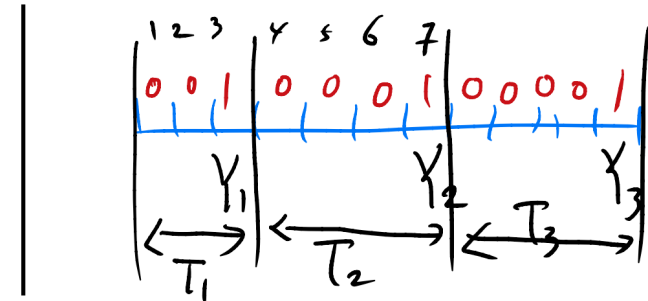


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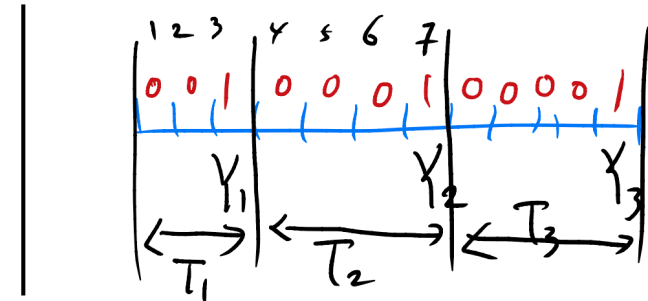


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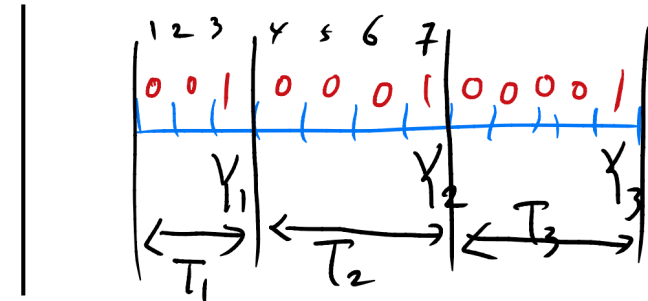
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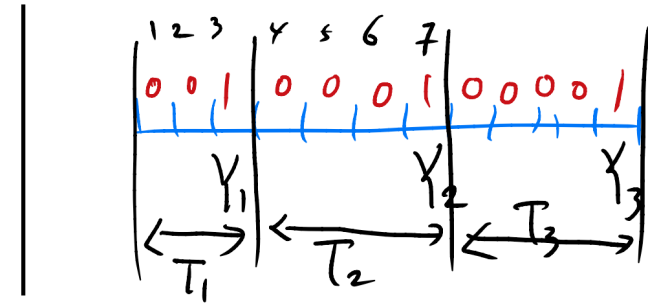
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- A lot of applications in engineering systems
- Basics on Random Process
- Bernoulli Process
- Poisson Process
- Use of Bernoulli and Poisson Processes
  - Coding of both processes
  - Merge and Split
- Markov Chain

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- Need a “modeling sense” to make this possible. It’s a good practice for engineers!

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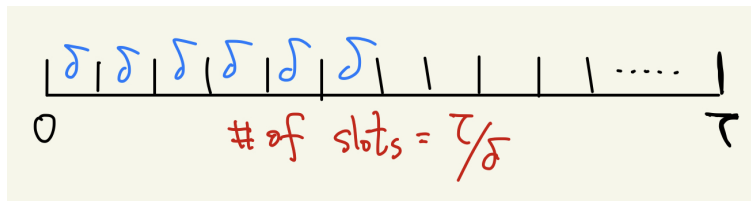
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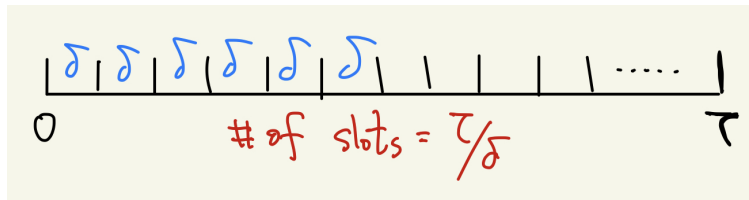
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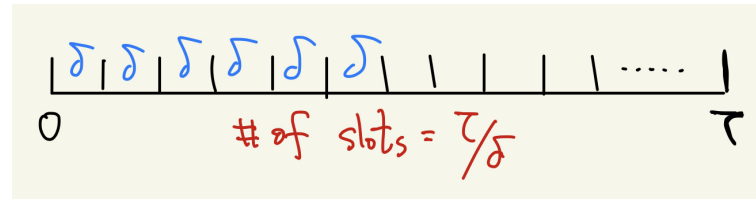
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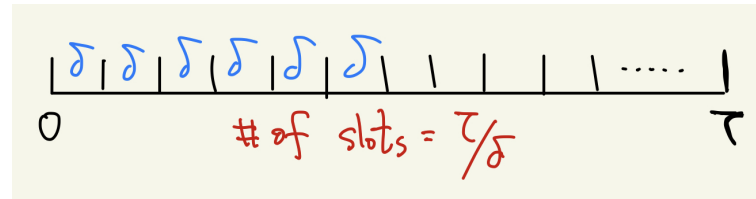
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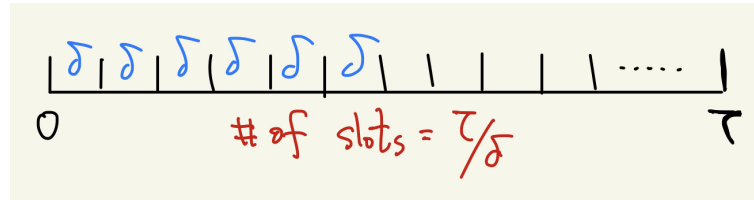
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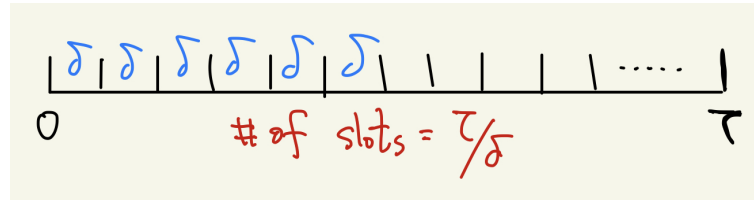


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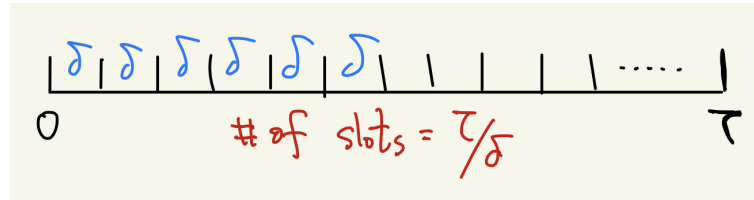
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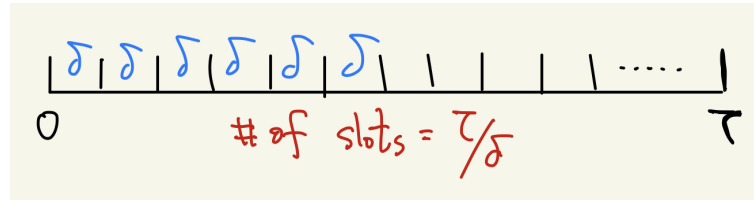
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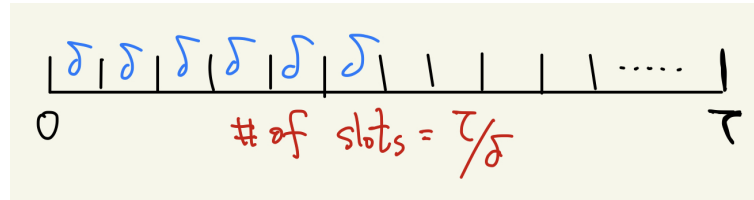
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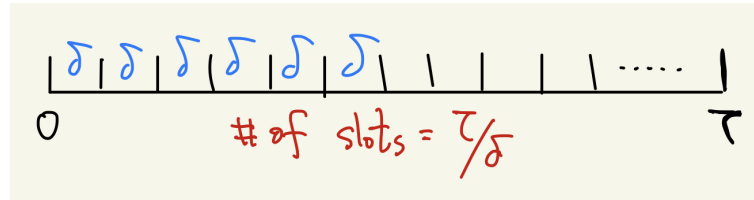


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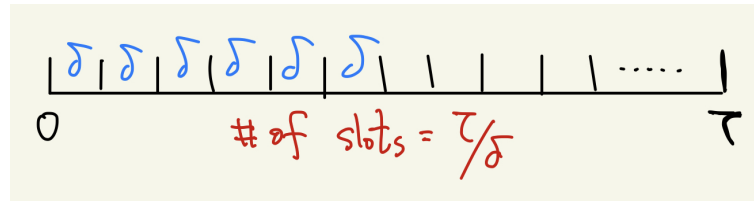
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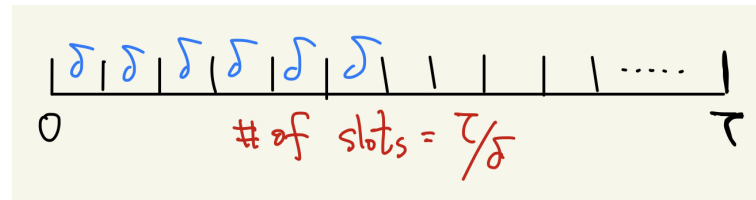
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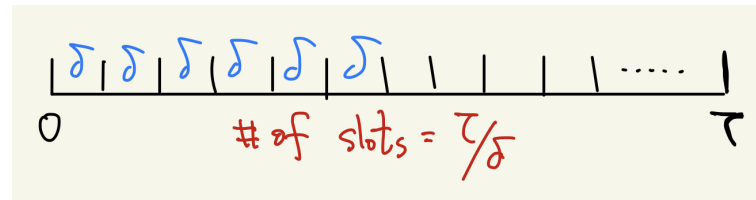
- $o(\delta)$ : some function that goes to zero faster than  $\delta$  goes to zero.
  - Thus, for very small  $\delta$ ,  $o(\delta)$  becomes negligible.
  - Example:  $o(\delta) = \delta^\alpha$ , where any  $\alpha > 1$

## Key Design Idea to Develop a Continuous Twin (3)



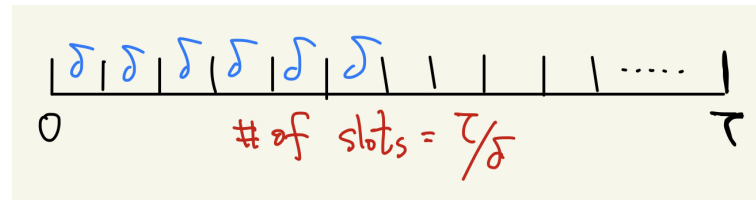
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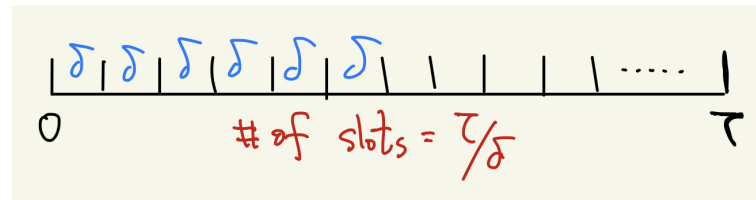
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- # of arrivals over  $[0, \tau]$ ,  $\sim \text{Poisson}(\lambda\tau)$
- This is a continuous twin process of Bernous process, which we call **Poisson process**.





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- **(Small interval probability)** The probabilities  $\mathbb{P}(k, s)$  satisfy:

$$\mathbb{P}(0, s) = 1 - \lambda s + o(s)$$

$$\mathbb{P}(1, s) = \lambda s + o_1(s)$$

$$\mathbb{P}(k, s) = o_k(s) \quad \text{for } k = 2, 3, \dots,$$

where

$$\lim_{s \rightarrow 0} \frac{o(s)}{s} = 0, \quad \lim_{s \rightarrow 0} \frac{o_k(s)}{s} = 0$$



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- $T \sim \text{expo}(\lambda)$ . Thus  $\mathbb{E}[T] = 1/\lambda$  and  $\text{var}[T] = 1/\lambda^2$ 
  - Continuous twin of geometric rv in Bernoulli process
  - Memoryless

- Receive emails according to a Poisson process at rate  $\lambda = 5$  messages per hour
- Mean and variance of mails received during a day
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- Mean and variance of mails received during a day
  - $5 \cdot 24 = 120$
- $\mathbb{P}[\text{one new message in the next hour}]$ 
  - $\mathbb{P}(1, 1) = \frac{(5 \cdot 1)^1 e^{-5 \cdot 1}}{1!} = 5e^{-5}$
- $\mathbb{P}[\text{exactly two msgs during each of the next three hours}]$ 
  - $\left( \frac{5^2 e^{-5}}{2!} \right)^3$



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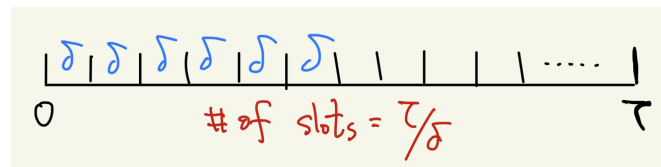
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- Time of  $k$ -th arrivals: Pascal / Erlang



# Poisson Process vs. Bernoulli Process

-  $n = \tau/\delta$ ,  $p = \lambda\delta$ ,  $np = \lambda\tau$



|                       | POISSON              | BERNOULLI      |
|-----------------------|----------------------|----------------|
| Times of Arrival      | Continuous           | Discrete       |
| PMF of # of Arrivals  | Poisson              | Binomial       |
| Interarrival Time CDF | Exponential          | Geometric      |
| Arrival Rate          | $\lambda$ /unit time | $p$ /per trial |

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(Q6)  $\mathbb{E}[\text{number of fish}] = \lambda \cdot 2 + \mathbb{P}(0, 2) \cdot 1$



- A lot of applications in engineering systems
- Basics on Random Process
- Bernoulli Process
- Poisson Process
- Use of Bernoulli and Poisson Processes
  - Coding of both processes
  - Merge and Split
- Markov Chain

- Inter-arrival times facilitates coding of both processes

## Alternative Description of the Bernoulli Process

1. Start with a sequence of independent geometric random variables  $T_1, T_2, \dots$ , with common parameter  $p$ , and let these stand for the interarrival times.
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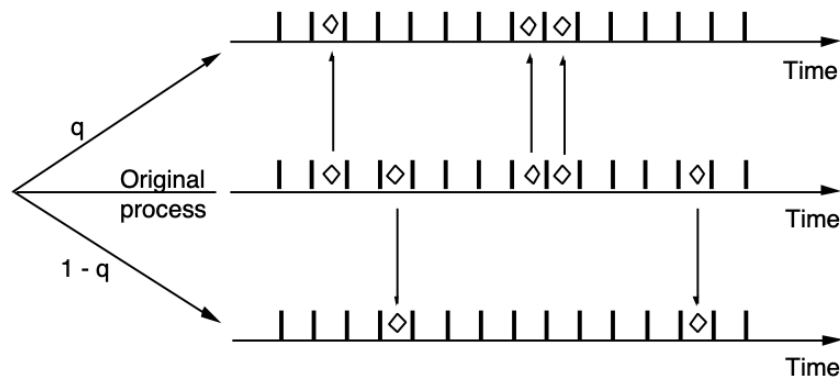
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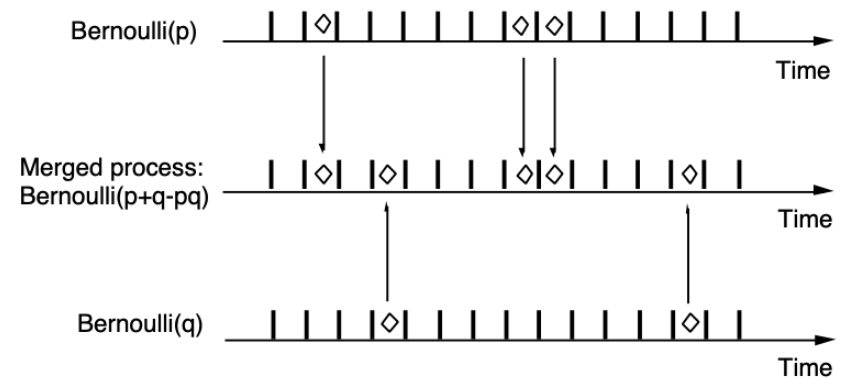
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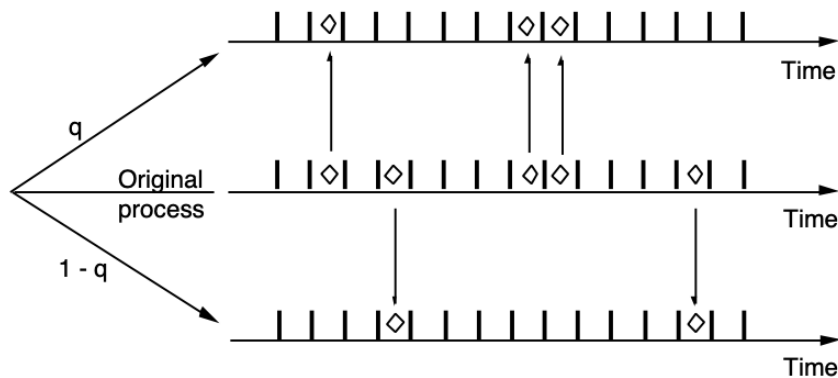


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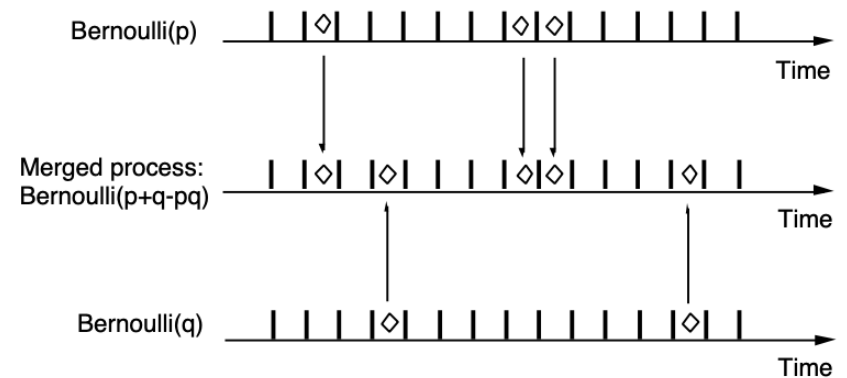


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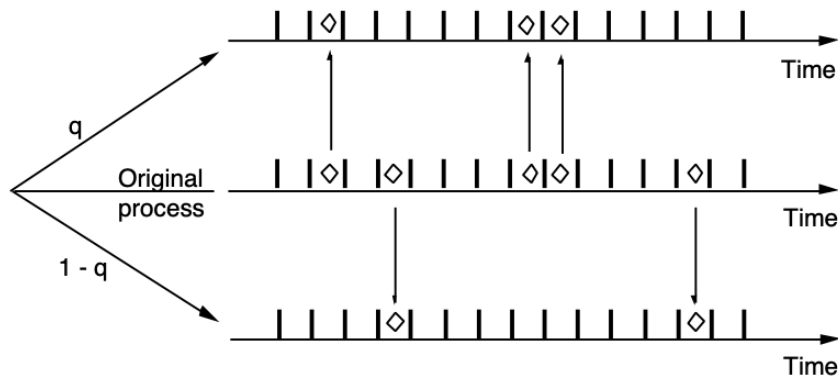


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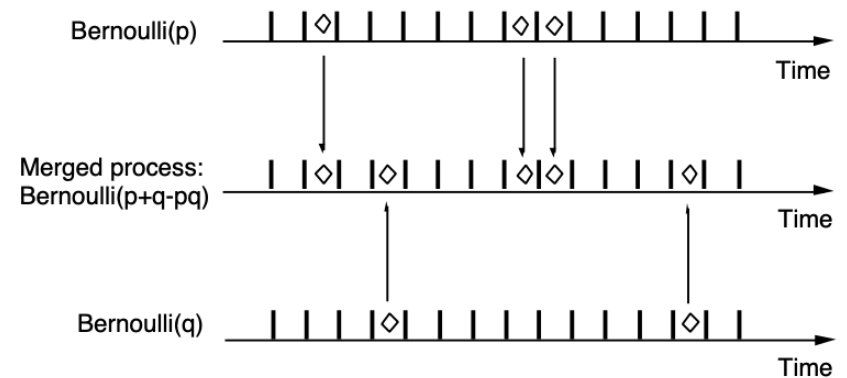


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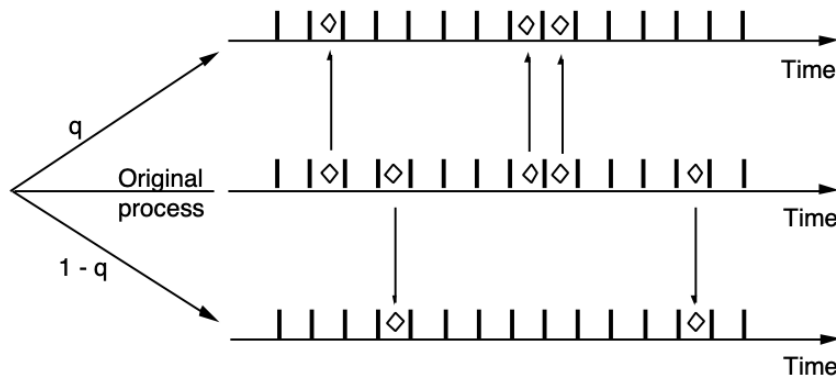


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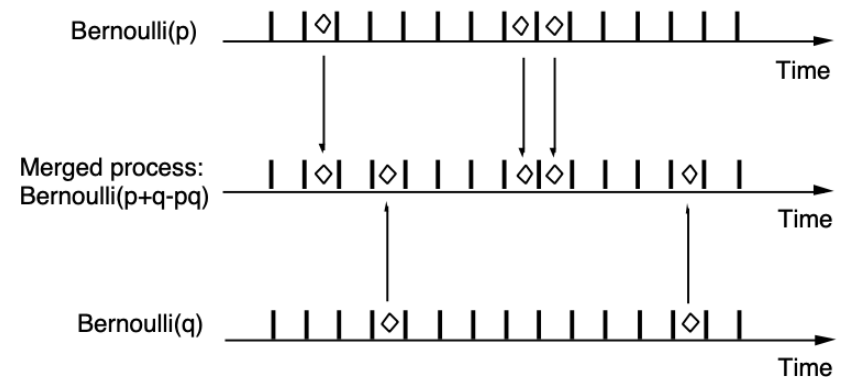


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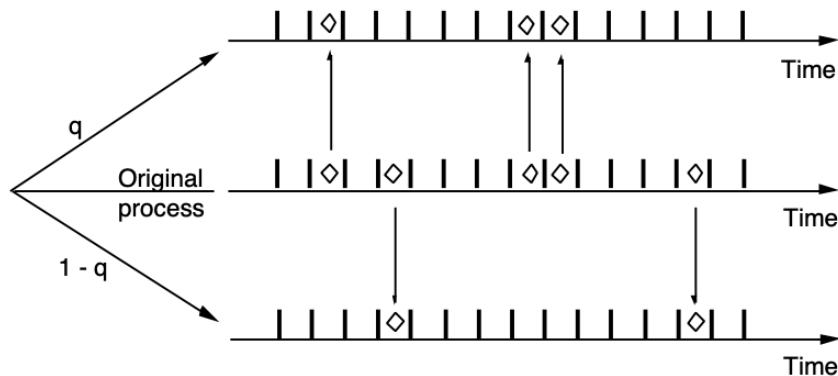


- Merge  $Bernoulli(p)$  and  $Bernoulli(q)$  into one.
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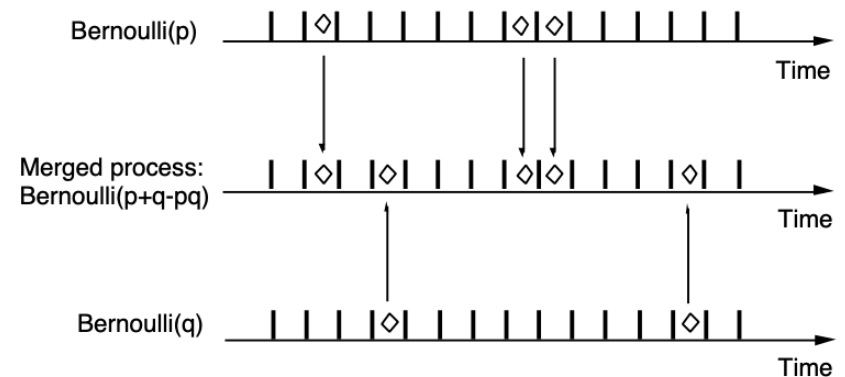


# Split and Merge: Bernoulli Process

- Split  $Bernoulli(p)$  into two processes with biased coin of head probability  $q$
- Split decisions are independent of arrivals
- Split processes: also Bernoulli processes
- $Bernoulli(pq)$  and  $Bernoulli(p(1 - q))$



- Merge  $Bernoulli(p)$  and  $Bernoulli(q)$  into one.
- Collided arrival is regarded just one arrival in the merged process
- Merged process:  
 $Bernoulli(1 - (1 - p)(1 - q))$



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  - Split based on the coin tossing with probability of head  $p$
  - Poisson process  $(\lambda_1 + \lambda_2)$
  - Bernoulli process of small interval  $\delta$

$$\mathbb{P}(0 \text{ arrivals in the merged process}) \approx (1 - \lambda_1\delta)(1 - \lambda_2\delta) = 1 - (\lambda_1 + \lambda_2)\delta + o(\delta)$$

$$\mathbb{P}(1 \text{ arrivals in the merged process}) \approx \lambda_1\delta(1 - \lambda_2\delta) + \lambda_2\delta(1 - \lambda_1\delta) = (\lambda_1 + \lambda_2)\delta + o(\delta)$$

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- $T_1$ : time until the first burn-out,  $T_2$ : time until the second burn-out,  $T_3$ : time until the third burn-out
- $T_1 \sim \exp(3\lambda)$ ,  $T_2 \sim \exp(2\lambda)$ ,  $T_3 \sim \exp(\lambda)$

$$\mathbb{E}[T_1 + T_2 + T_3] = \frac{1}{3\lambda} + \frac{1}{2\lambda} + \frac{1}{\lambda}$$

Questions?



1)