Lecture 1: Probabilistic Model

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EE210: Probability and Introductory Random Processes
KAIST EE

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Outline

- Probabilistic Model
- Sample Space, Event, Probability Law
- Probability Axioms

What Do We Want?

Modeling: Approximate reality with a simple (mathematical) model

- Experiment
- Flip two coins

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  - Mathematical description of an uncertain situation
- Which model is good or bad?

Goal: Build up a probabilistic model. Hmm... How?

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Elements of Probabilistic Model

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Question: What are the conditions of Ω and $\mathbb{P}(\cdot)$ under which their induced probability model becomes "legitimate"?

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1. Mutually exclusive
2. Collectively exhaustive
3. At the $\boxed{\text{right granularity}}$ (not too concrete, not too abstract)

1. Toss a coin. What about this?
   $\Omega = \{H, T, HT\}$
2. Toss a coin. What about this? $\Omega = \{H\}$
3. (a) Just figuring out prob. of H or T.
   $\implies \Omega = \{H, T\}$

   (b) The impact of the weather (rain or no rain) on the coin's behavior.

   $$\implies \Omega = \{(H, R), (T, R), (H, NR), (T, NR)t\},$$

   where R(Rain), NR(No Rain).

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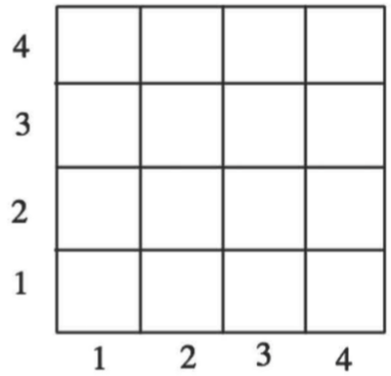
- *Discrete case:* Two rolls of a tetrahedral die

  - $\Omega = \{(1,1),(1,2),\ldots,(4,4)\}$



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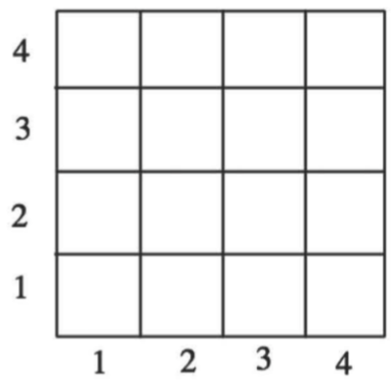
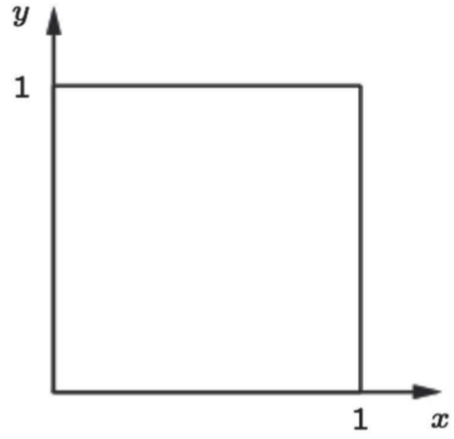


- *Continuous case:* Dropping a needle in a plain

  - $\Omega = \{(x,y) \in \mathbb{R}^2 \mid 0 \leq x, y \leq 1\}$



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- Roll a dice. What is the probability of odd numbers?

  $\mathbb{P}(\{1, 3, 5\})$, where $\{1, 3, 5\} \subset \Omega$ is an event.

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  - many others

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- Note that coming up with the above axioms is far from trivial.

Examples

Prove the following properties using the axioms:

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$$\mathbb{P}(B) \overset{\text{A3}}{=} \mathbb{P}(A) + \mathbb{P}(B \setminus A) \overset{\text{A1}}{\geq} \mathbb{P}(A)$$

Probability Calculation Steps

1. Specify the sample space
2. Specify a probability law
   - from my earlier belief, from data, from expert's opinion
3. Identify an event of interest
4. Calculate

Toss a (biased) coin

1. $\Omega = \{H, T\}$
2. $\mathbb{P}(\{H\}) = 1/4$, $\mathbb{P}(\{T\}) = 3/4$,
3. probability of head or tail
4. $1/4$, $3/4$

Discrete but infinite sample space

- $\Omega = \{1, 2, 3, \ldots\}$, $\mathbb{P}(\{n\}) = \frac{1}{2^n}$, $n = 1, 2, \ldots$
- $\mathbb{P}(\text{even})$?

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- Is the above right? If not, why?

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$$\begin{aligned}\mathbb{P}(\text{even}) &= \mathbb{P}(\{2, 4, 6, \ldots\}) \\ &= \frac{1}{2^2} + \frac{1}{2^4} + \frac{1}{2^6} + \cdots = 1/3\end{aligned}$$

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Probability Axioms Version 2

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A1. Nonnegativity: $\mathbb{P}(A) \geq 0$ for any event $A \subset \Omega$

A2. Normalization: $\mathbb{P}(\Omega) = 1$

A3. $\boxed{\text{Countable additivity:}}$ If $A_1, A_2, A_3, \ldots$ is an infite sequence of disjoint events, then $\mathbb{P}(A_1 \cup A_2 \cup \cdots) = \mathbb{P}(A_1) + \mathbb{P}(A_2) + \cdots$.

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  - axioms $\rightarrow$ theorems

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- Beliefs: $\mathbb{P}(\text{He is reelected}) = 0.7$
  - Subjective, but providing numerical guidance

Questions?

Congratulations! You build up the very basics of a probabilistic model.

What else do we need to build up?

Review Questions

1) Please explain what a probabilistic model is and why we need it.
2) What is the mathematical definition of event?
3) What are the key elements of the probabilistic model?
4) Please list up the probability axioms and explain them.
5) Why do we need countable additivity in the probability axiom?