Lecture 2: Conditioning, Bayes' Rule, and Independence

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EE210: Probability and Introductory Random Processes
KAIST EE

MONTH DAY, 2021

Outline

- Conditional Probability
- Bayes' Rule
- Bayesian Inference: Sneak Peek
- Independence, Conditional Independence

Motivating Example

- Pick a person a at random
  - event A: a's age ≤ 20
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    - $\mathbb{P}(\Omega|B) = 1$?
    - $\mathbb{P}(B|B) = 1$ from our common sense. True?

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- Easily check: Nonnegativity and countable additivity.
- All other properties of the law $\mathbb{P}(\cdot)$ is applied to the conditional law $\mathbb{P}(\cdot|B)$.
- For example, finite additivity. For two disjoint events A and C,

$$\mathbb{P}(A \cup C| B) = \mathbb{P}(A| B) + \mathbb{P}(C| B)$$

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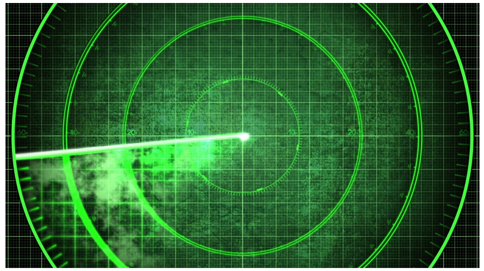

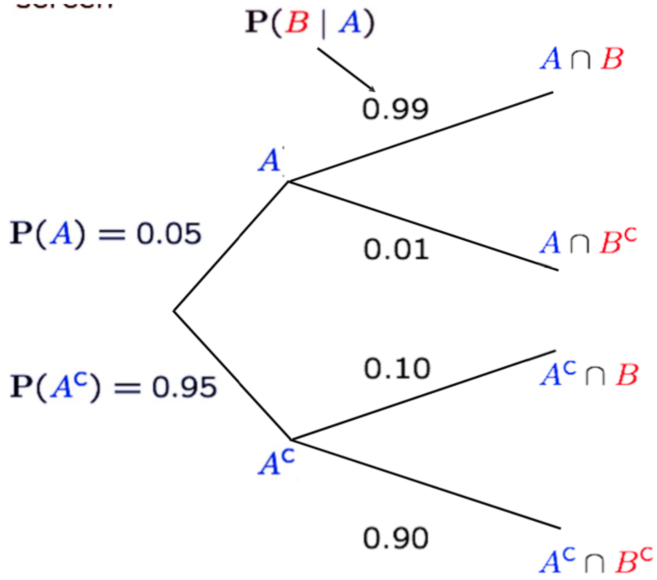

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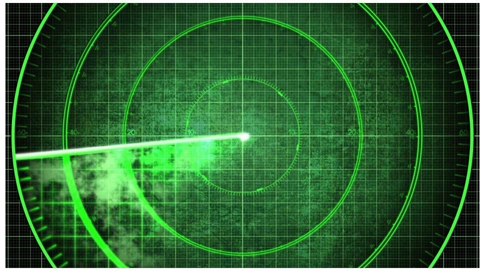

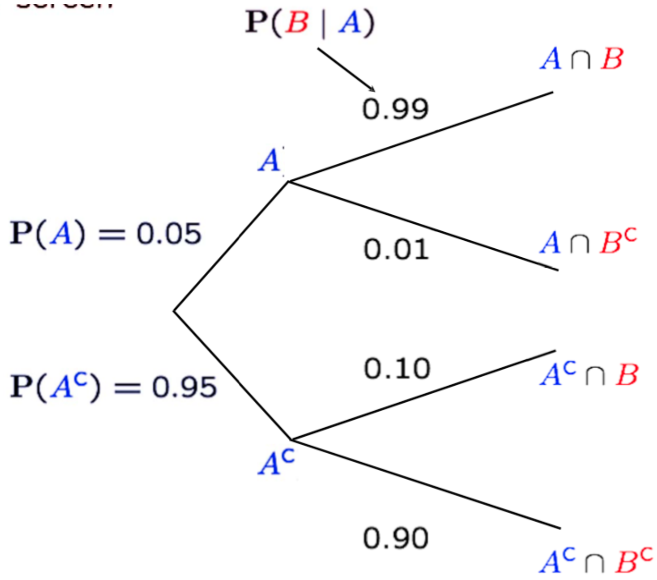

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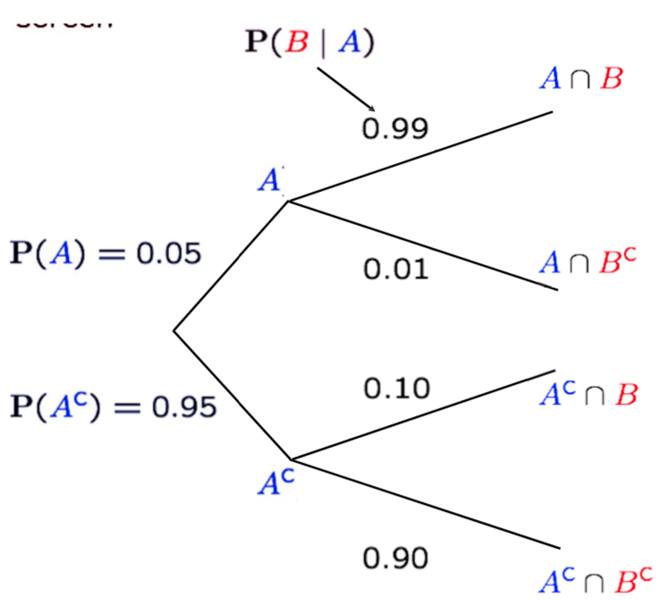

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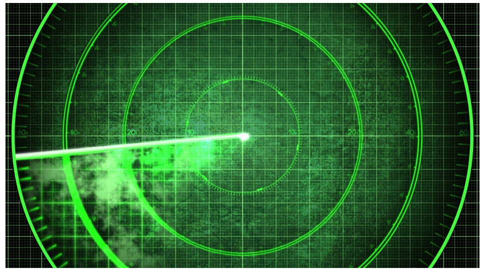

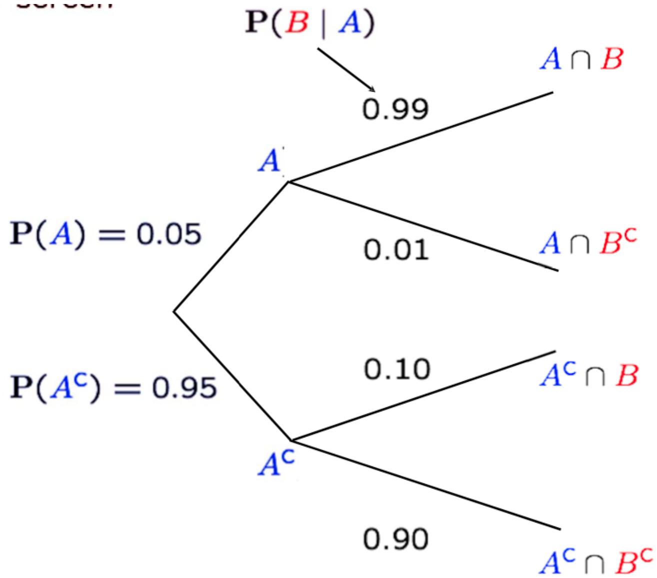

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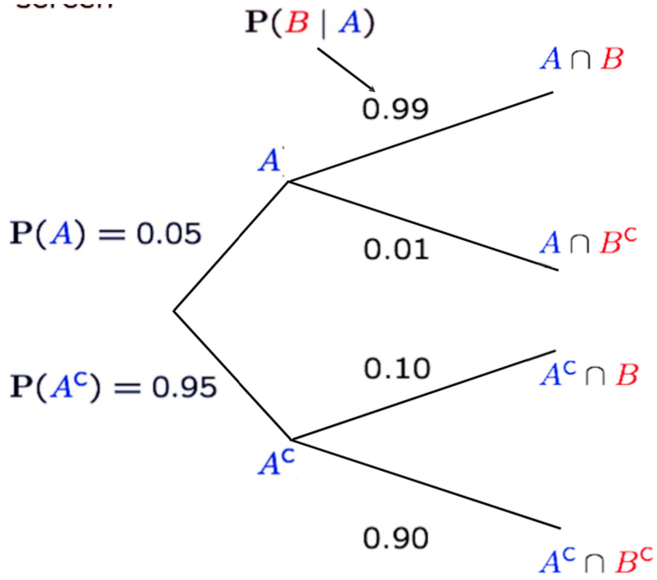

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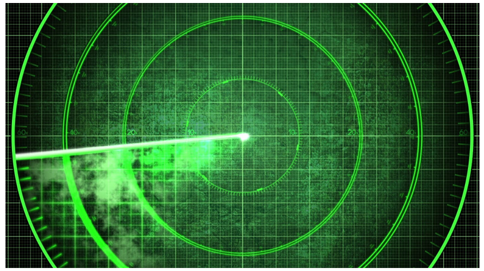

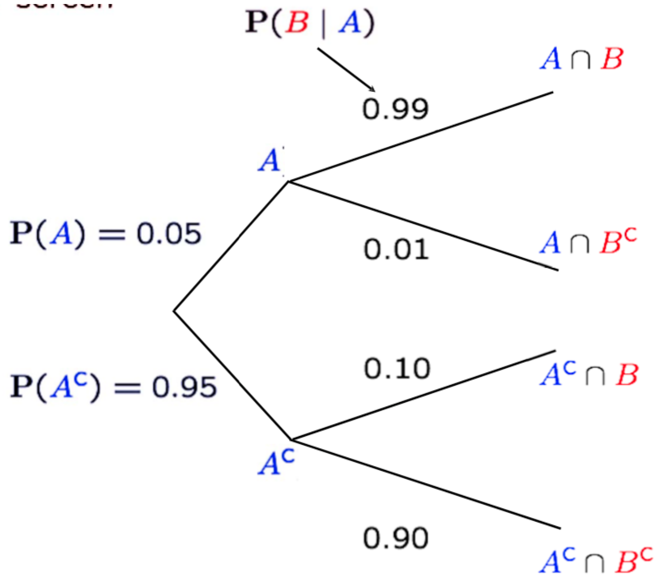

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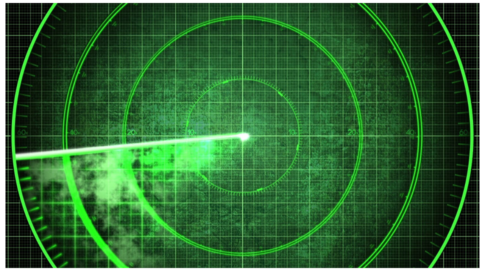

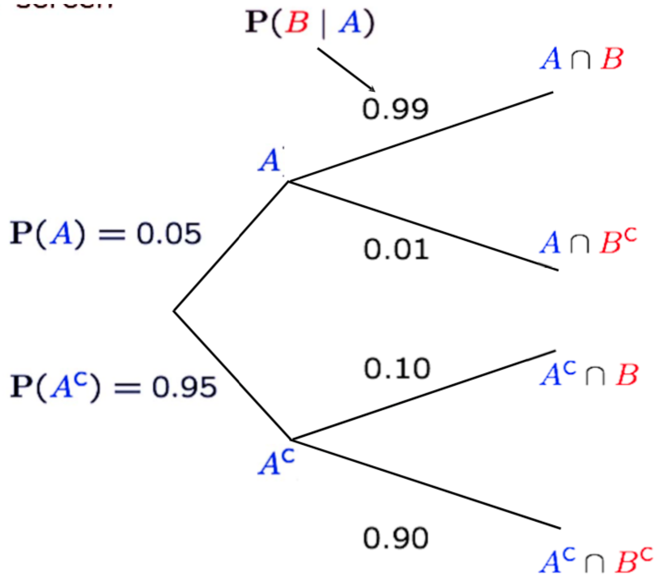

From now on, using the theory of probability and conditional probability constructed so far, we will develop interesting properties and theorems which are very useful to answer some exciting questions.

That is *Bayes' Rule* to make some *inference (추론).*

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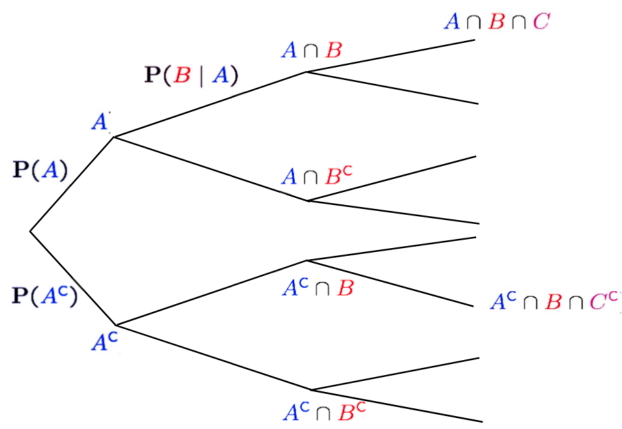
We will study this topic rigorously later in this class (chapter 8).

Multiplication Rule

- $\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$
- $\mathbb{P}(A \cap B) = \boxed{\qquad\qquad} = \boxed{\qquad\qquad}$
- $\mathbb{P}(A^c \cap B \cap C^c) = \boxed{\qquad\qquad\qquad\qquad}$

$= \boxed{\qquad\qquad\qquad\qquad\qquad}$



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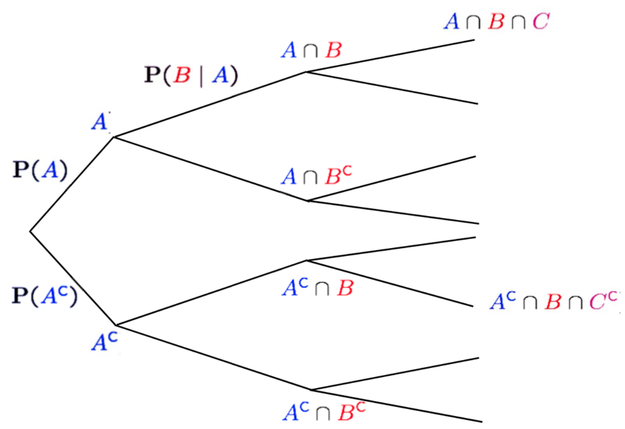

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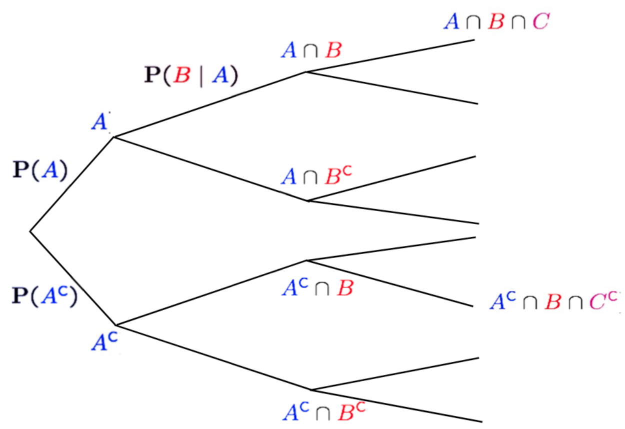

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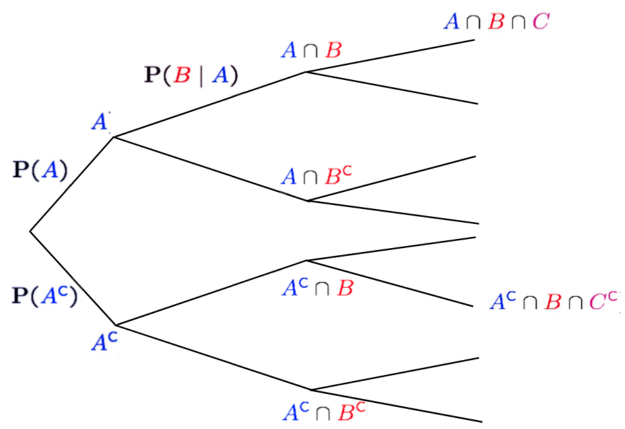

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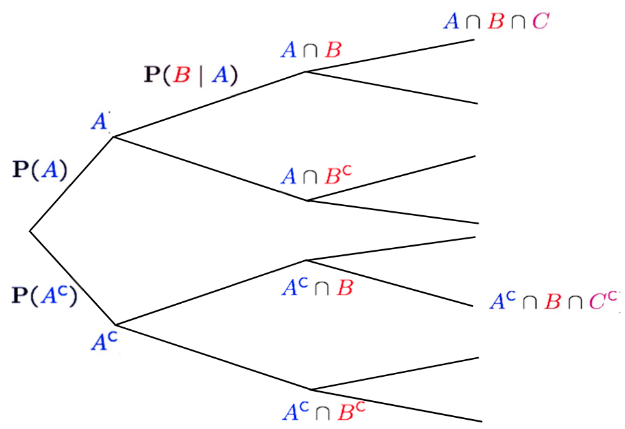

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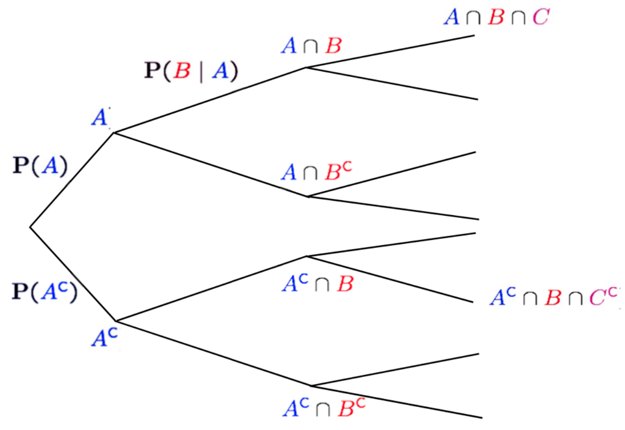

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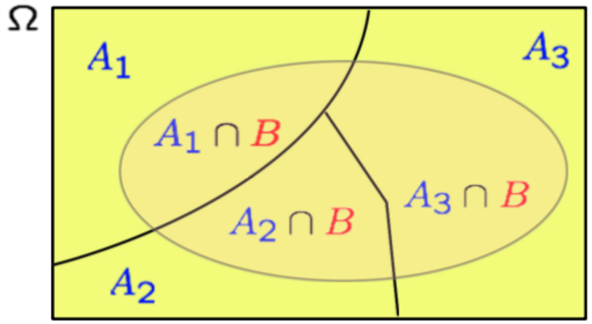
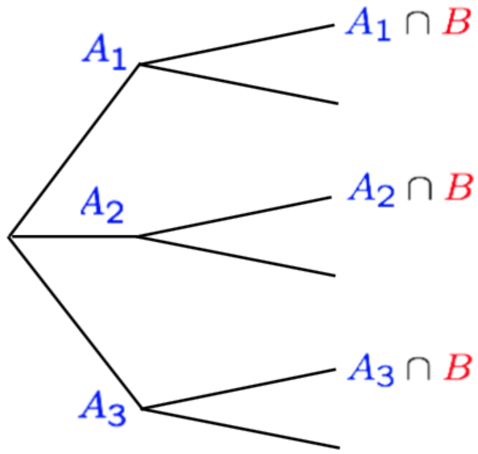
$$\mathbb{P}(A_1 \cap A_2 \cap \cdots A_n) = \mathbb{P}(A_1) \cdot \mathbb{P}(A_2|A_1) \cdot \mathbb{P}(A_3|A_1, A_2) \cdots \mathbb{P}(A_n|A_1, A_2, \ldots, A_{n-1})$$

Total Probability Theorem





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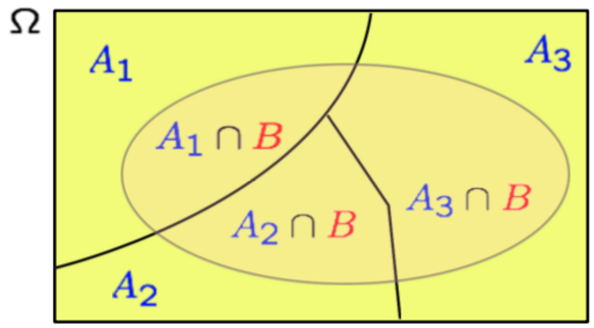

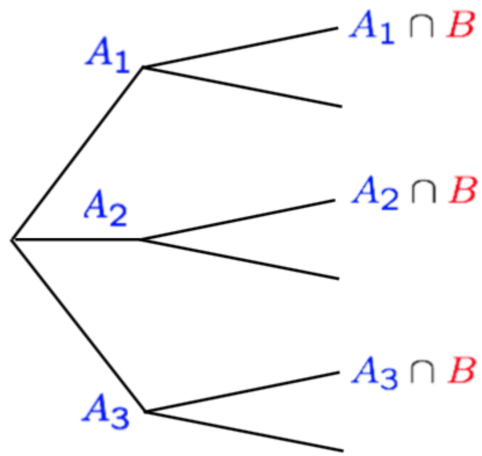

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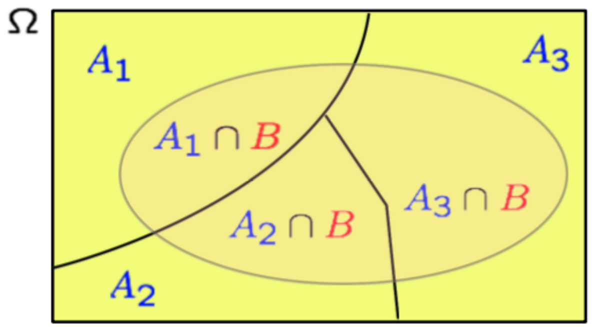

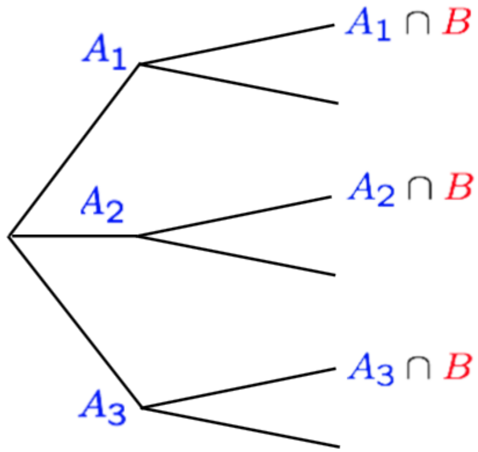

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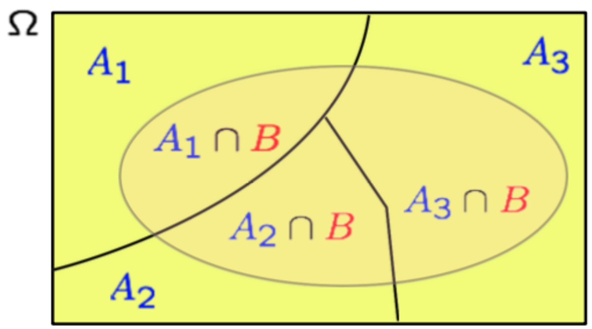

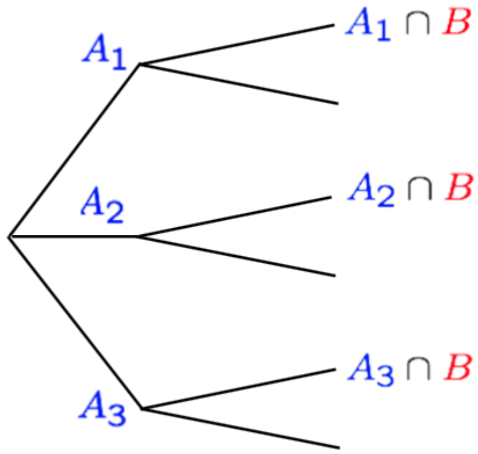

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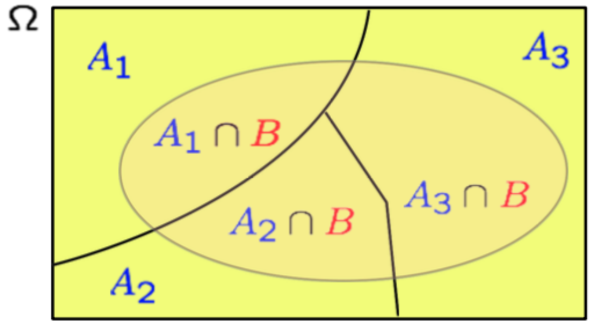

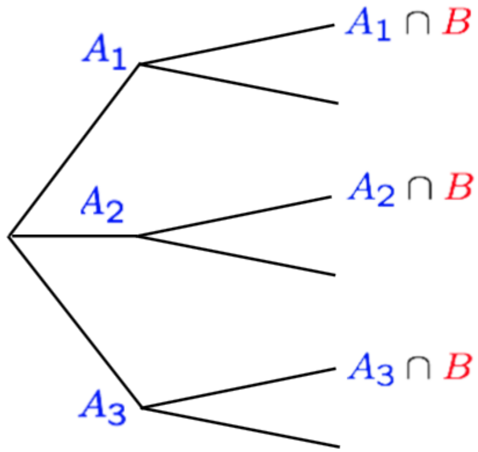

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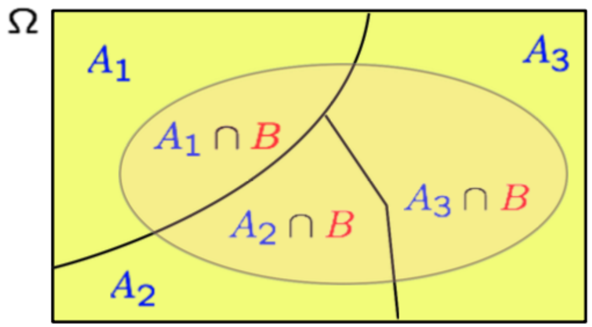

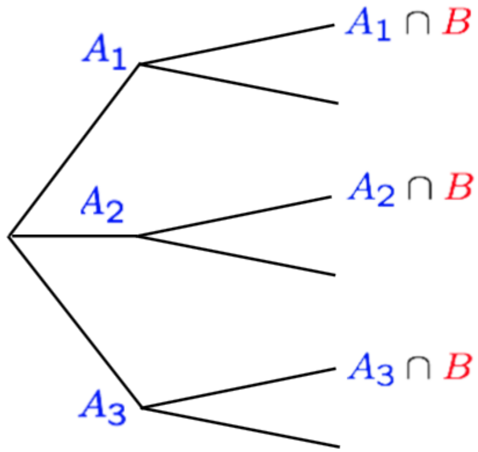

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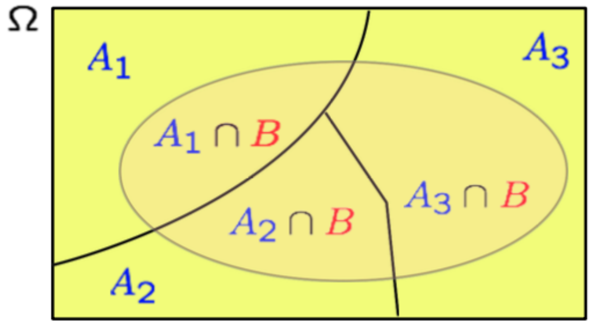

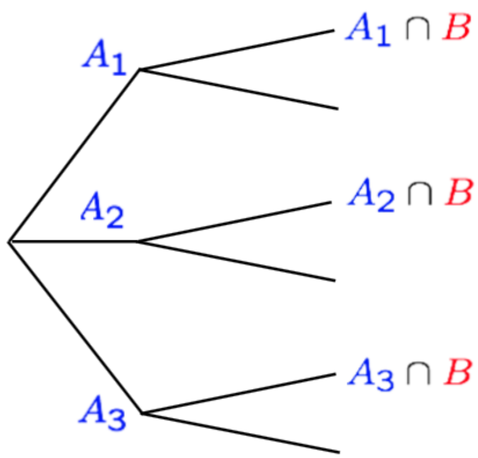

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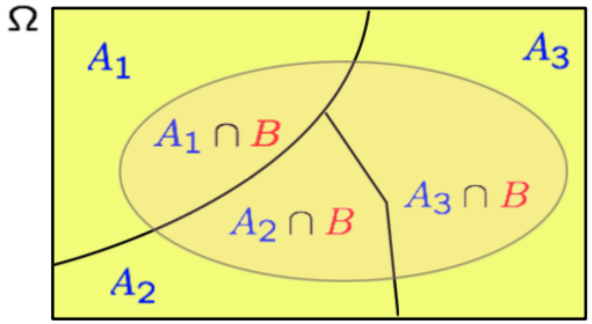

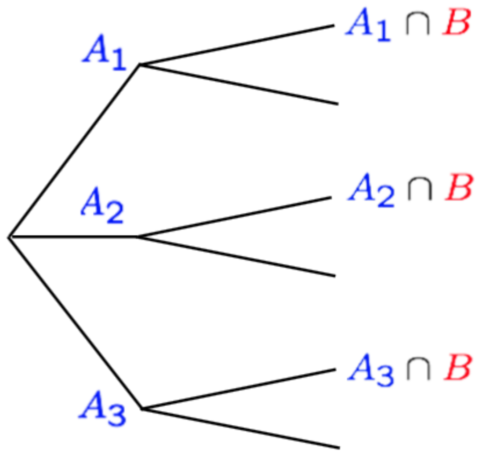

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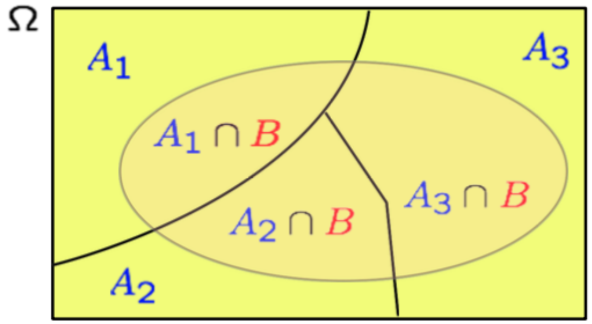

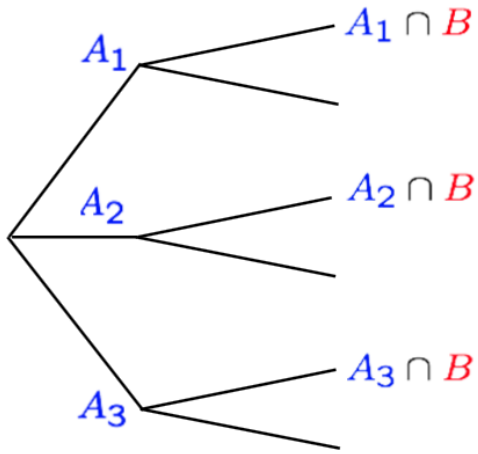

Bayes' Rule: Example

- A_1: you are happy, A_2: you are sad
- B: you shout.
- Assume:

$$\mathbb{P}(A_1) = 0.7,\ \mathbb{P}(A_2) = 0.3,$$

$$\mathbb{P}(B|A_1) = 0.3,\ \mathbb{P}(B|A_2) = 0.5.$$

- Calculate $\mathbb{P}(A_1|B)$ and $\mathbb{P}(A_2|B)$.

$$\mathbb{P}(A_1)\mathbb{P}(B|A_1) = 0.7 \times 0.3 = 0.21$$
$$\mathbb{P}(A_2)\mathbb{P}(B|A_2) = 0.3 \times 0.5 = 0.15$$
$$\mathbb{P}(B) = 0.21 + 0.15 = 0.36$$

$$\mathbb{P}(A_1|B) = \frac{0.21}{0.36} \approx 0.583$$
$$\mathbb{P}(A_2|B) = \frac{0.15}{0.36} \approx 0.417$$

Bayesian inference was really fun.

Now, let's develop a new concept from conditioning.

That is *Independence*.

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- Independence makes our analysis and modeling much simpler, because I can remove independent events in my analysis.

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- Occurrence of A provides no new information about B. Thus, knowledge about A does no change my belief about B.

- Using $\mathbb{P}(B|A) = \mathbb{P}(B \cap A)/\mathbb{P}(A)$,

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$A \perp\!\!\!\perp B \rightarrow A \perp\!\!\!\perp B | C$?

- Two independent coin tosses
  - H_1: 1st toss is a head
  - H_2: 2nd toss is a head
  - D: two tosses have different results.

$A \perp\!\!\!\perp B \to A \perp\!\!\!\perp B | C$?

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- $\mathbb{P}(H_1 \cap H_2|D) = 0$,

- No.

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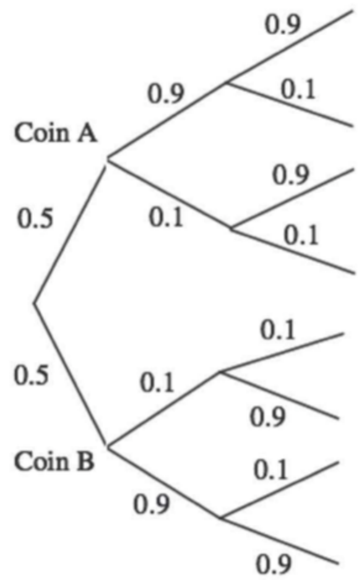

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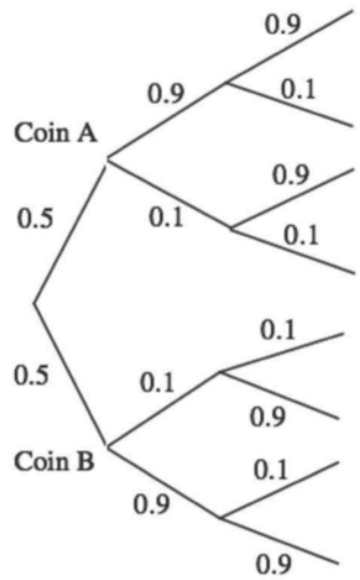

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KAIST EE

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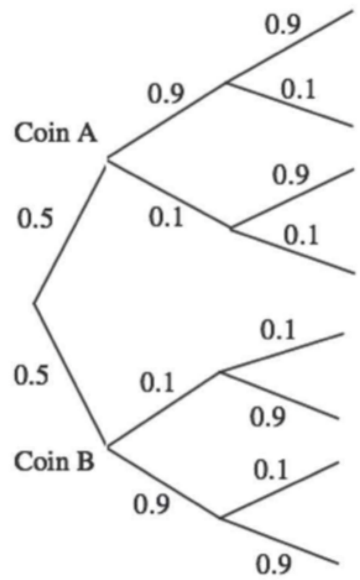

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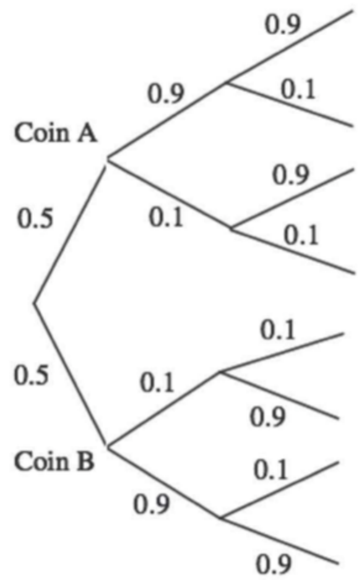

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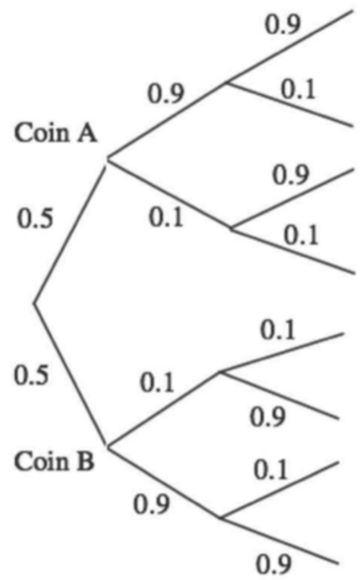
$$\begin{aligned}\mathbb{P}(H_1) &= \mathbb{P}(B)\mathbb{P}(H_1|B) + \mathbb{P}(B^c)\mathbb{P}(H_1|B^c) \\ &= \frac{1}{2}0.9 + \frac{1}{2}0.1 = \frac{1}{2} \\ \mathbb{P}(H_2) &= \mathbb{P}(H_2) \quad \text{(because of symmetry)}\end{aligned}$$



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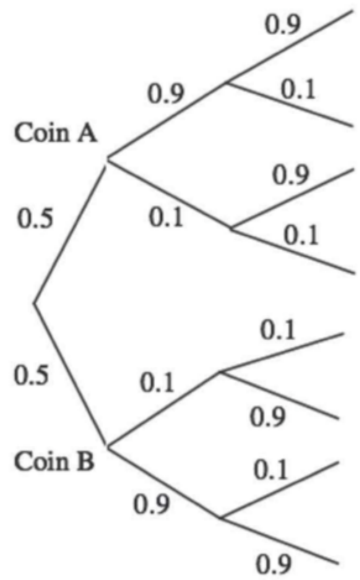

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Independence of Multiple Events

The events $A_1, A_2, \ldots, A_n$ ar said to be independent if

$$\mathbb{P}\left(\bigcap_{i \in S} A_i\right) = \prod_{i \in S} \mathbb{P}(A_i), \quad \text{for every subset } S \text{ of } \{1, 2, \ldots, n\}$$

Questions?

Review Questions

1) What is conditional probability? Why do we need it?
2) Explain the overall framework of Bayesian inference.
3) What is the total probability theorem?
4) What is Bayes' rule? What does it can give us?
5) What's the difference between independence and conditional independence?