

Lecture 8: Random Processes, Part II

Yi, Yung (이웅)

EE210: Probability and Introductory Random Processes
KAIST EE

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Roadmap

- Basics on Random Process
- Bernoulli Process
- Poisson Process
- Use of Bernoulli and Poisson Processes
- **Markov Chain**
 - Definition, Transition Probability Matrix, State Transition Diagram
 - Classification of States
 - Steady-state Behaviors and Stationary Distribution
 - Transient Behaviors

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- Assume discrete times $n = 1, 2, \dots$

- Random process: A sequence of X_1, X_2, X_3, \dots

- “Simplest” random process

- Process without memory

$$\mathbb{P}(X_n = i_n \mid X_{n-1} = i_{n-1}, X_{n-2} = i_{n-2}, X_{n-3} = i_{n-3}, \dots, X_1 = i_1) = \mathbb{P}(X_n = i_n)$$

- Bernoulli process

- A random process that is a little more complex than the above?

- Process that depends only on “yesterday”, not the entire history

$$\mathbb{P}(X_n = i_n \mid X_{n-1} = i_{n-1}, X_{n-2} = i_{n-2}, X_{n-3} = i_{n-3}, \dots, X_1 = i_1) = \mathbb{P}(X_n = i_n \mid X_{n-1} = i_{n-1})$$

- Markov chain

- One of the most popular random processes in engineering

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Example: Machine Failure, Repair, and Replacement

- A machine: working or broken down on a given day.
 - If working, break down in the next day w.p. b , and continue working w.p. $1 - b$.
 - If broken down, it will be repaired and be working in the next day w.p. r , and continue to be broken down w.p. $1 - r$.
- $X_n \in \{1, 2\}$: status of the machine, 1: working and 2: broken down
- $(X_n)_{n=1}^{\infty}$: A random process satisfying: for any $n \geq 1$,

$$\mathbb{P}(X_{n+1} = 1 \mid X_n = 1) = 1 - b, \quad \mathbb{P}(X_{n+1} = 2 \mid X_n = 1) = b$$

$$\mathbb{P}(X_{n+1} = 1 \mid X_n = 2) = r, \quad \mathbb{P}(X_{n+1} = 2 \mid X_n = 2) = 1 - r$$
- What will happen at $(n + 1)$ -th day depends only on what happens at n -th day?

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- **Definition.** Let X_1, \dots, X_n, \dots be a sequence of random variables taking values in some finite space $\mathcal{S} = \{1, 2, \dots, m\}$, such that for all $i, j \in \mathcal{S}$, $n \geq 0$, the following **Markov property** is satisfied:

$$\mathbb{P}(X_{n+1} = j | X_n = i) = \mathbb{P}(X_{n+1} = j | X_n = i, X_{n-1} = i_{n-1}, \dots, X_0 = i_0),$$

- For any fixed n , the future of the process after n is **independent** of $\{X_1, \dots, X_n\}$, **given** X_n (i.e., depends only on X_n)
- The value that X_n can take is called '**state**'. Thus, the space \mathcal{S} is called **state space**.
- **Time homogeneity.** The probability $\mathbb{P}(X_{n+1} = j | X_n = i)$ does NOT depends on n .

Thus, for any $n \geq 0$, we introduce a simple notation p_{ij}

$$p_{ij} \triangleq \mathbb{P}(X_{n+1} = j | X_n = i)$$

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Transition Prob. Matrix and State Transition Diagram

- **Transition Probability Matrix.** Consider a $m \times m$ matrix $\mathbf{P} = [p_{ij}]$, where $p_{ij} \triangleq \mathbb{P}(X_{n+1} = j | X_n = i)$

- Machine example.

$$p_{11} = \mathbb{P}(X_{n+1} = 1 | X_n = 1) = 1 - b,$$

$$p_{12} = \mathbb{P}(X_{n+1} = 2 | X_n = 1) = b$$

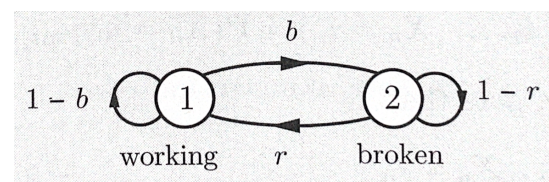
$$p_{21} = \mathbb{P}(X_{n+1} = 1 | X_n = 2) = r,$$

$$p_{22} = \mathbb{P}(X_{n+1} = 2 | X_n = 2) = 1 - r$$

- Transition probability matrix

$$\begin{bmatrix} 1-b & b \\ r & 1-r \end{bmatrix}$$

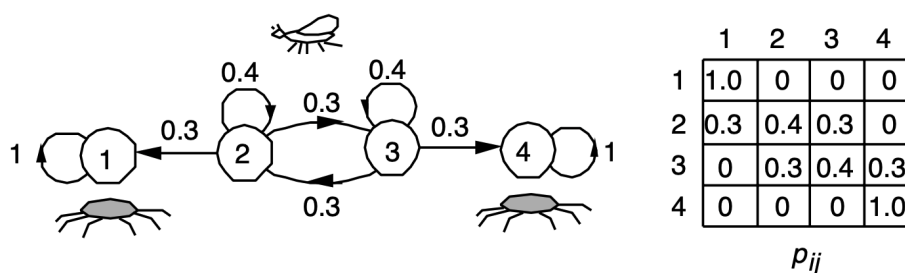
- State transition diagram



- Both are the complete description of Markov chain.
- $\sum_{j=1}^m p_{ij} = 1$ (for each row i , the column sum = 1)

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- A fly moves along a line in unit increments.
- At each time, it moves one unit (i) left w.p. 0.3, (ii) right w.p. 0.3 and (iii) stays in place w.p. 0.4, independent of the past history of movements.
- Two spiders lurk at positions 1 and 4: if the fly lands there, it is captured by the spider, and the process terminates. Assume that the fly starts in a position between 1 and 4.
- X_n : position of the fly. Please draw the state transition diagram and find the transition probability matrix.



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(Q) What is the probability of a sample path in a Markov chain?

$$\begin{aligned}
 & \mathbb{P}(X_0 = i_0, X_1 = i_1, X_2 = i_2, \dots, X_n = i_n) \\
 &= \mathbb{P}(X_n = i_n | X_0 = i_0, X_1 = i_1, \dots, X_{n-1} = i_{n-1}) \cdot \mathbb{P}(X_0 = i_0, X_1 = i_1, \dots, X_{n-1} = i_{n-1}) \\
 &= p_{i_{n-1}i_n} \cdot \mathbb{P}(X_0 = i_0, X_1 = i_1, \dots, X_{n-1} = i_{n-1}) = \mathbb{P}(X_0 = i_0) \cdot p_{i_0i_1} \cdot p_{i_1i_2} \cdots p_{i_{n-1}i_n}
 \end{aligned}$$

- Spider-Fly example

$$\mathbb{P}(X_0 = 2, X_1 = 2, X_2 = 2, X_3 = 3, X_4 = 4) = \mathbb{P}(X_0 = 2) p_{22} p_{22} p_{23} p_{34} = \mathbb{P}(X_0 = 2) (0.4)^2 (0.3)^2$$

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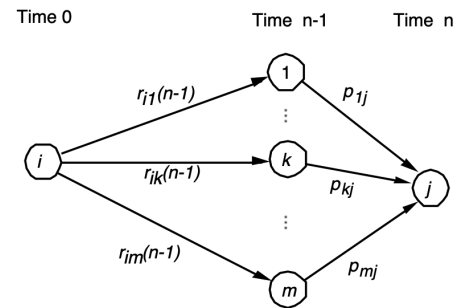
(Q) What is the probability that my state is j , starting from i after n steps?

- n -step transition probability

$$r_{ij}(n) \triangleq \mathbb{P}(X_n = j \mid X_0 = i)$$

- Recursive formula, starting with $r_{ij}(1) = p_{ij}$

$$\begin{aligned} r_{ij}(n) &= \mathbb{P}(X_n = j \mid X_0 = i) = \\ &= \sum_{k=1}^m \mathbb{P}(X_{n-1} = k \mid X_0 = i) \mathbb{P}(X_n = j \mid X_{n-1} = k, X_0 = i) \\ &= \sum_{k=1}^m r_{ik}(n-1) p_{kj} \end{aligned}$$

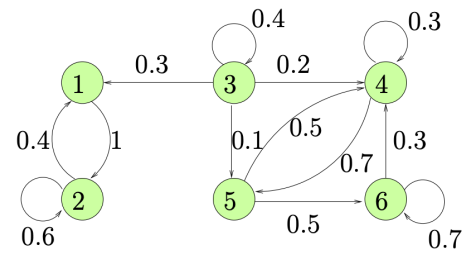


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 - Definition, Transition Probability Matrix, State Transition Diagram
 - Classification of States**
 - Steady-state Behaviors and Stationary Distribution
 - Transient Behaviors

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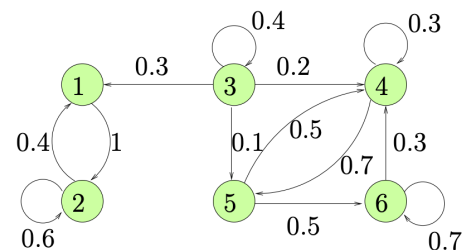
- **Classes**
 - 3 can only be reached from 3
 - 1 and 2 can reach each other but no other state
 - 4, 5, and 6 all reach each other.
 - Divide into three classes: $\{3\}$, $\{1, 2\}$, $\{4, 5, 6\}$
 - **Insight 1.** Multiple classes may exist.
- **Difference between 1 and 3**
 - 1: If I start from 1, visit 1 infinite times.
 - 3: If I start from 3, visit 3 only finite times (move to other classes and don't return).
 - **Insight 2.** Some states are visited infinite times, but some states are not.
- State 2 will share the above properties with 1 (similarly, 4, 5, and 6)
- **Insight 3.** States in the same class share some properties.



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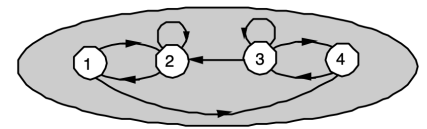
Classification of States (1)

- **Definition.** State j is **accessible** from state i , if for some n $r_{ij}(n) > 0$.
 - 6 is accessible from 3, but not the other way around.
- **Definition.** If i is accessible from j and j is accessible from i , we say that i communicates with j .
 - $1 \leftrightarrow 2$, but 3 does not communicate with 5.
- **Definition.** Let $A(i) = \{\text{states accessible from } i\}$. State i is **recurrent**, if $\forall j \in A(i)$, i is also accessible from j . In other words, "I communicate with all of my neighbors!"
 - A state that is not recurrent is **transient**.
 - 2 is recurrent? Yes. 3 is recurrent? No.
 - If we start from a recurrent state i , then there is always some probability of returning to i . It means that, given enough time, it is certain that it returns to i .

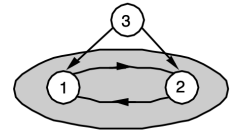


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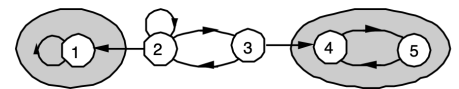
- A set of recurrent states which communicate with each other form a **class**.
- Markov chain decomposition
 - A MC can be decomposed into one or more recurrent classes, plus possibly some transient states.
 - A recurrent state is accessible from all states in its class, but it not accessible from recurrent states in other classes.
 - A transient state is not accessible from any recurrent state.
 - At least one, possibly more, recurrent states are accessible from a given transient state.
- The MC with only a single recurrent class is said to be **irreducible** (더이상 분해할 수 없는).



Single class of recurrent states



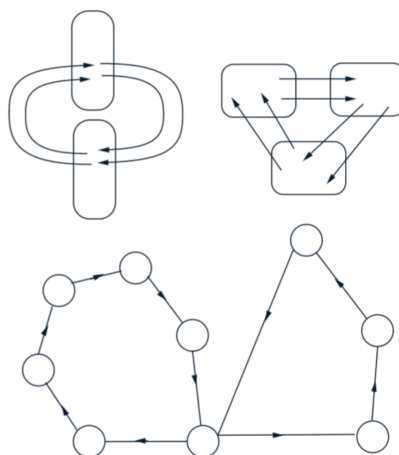
Single class of recurrent states (1 and 2) and one transient state (3)



Two classes of recurrent states (class of state 1 and class of states 4 and 5) and two transient states (2 and 3)

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- The states in a recurrent class are periodic if they can be grouped into $d > 1$ groups so that all transitions from one group lead to the next group.
- A recurrent class that is not periodic is said to be aperiodic.



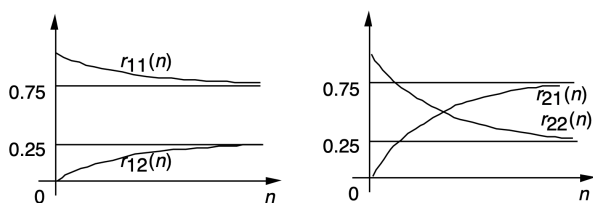
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n -step transition prob.: $r_{ij}(n)$ for large n

- Convergence irrespective of the starting state

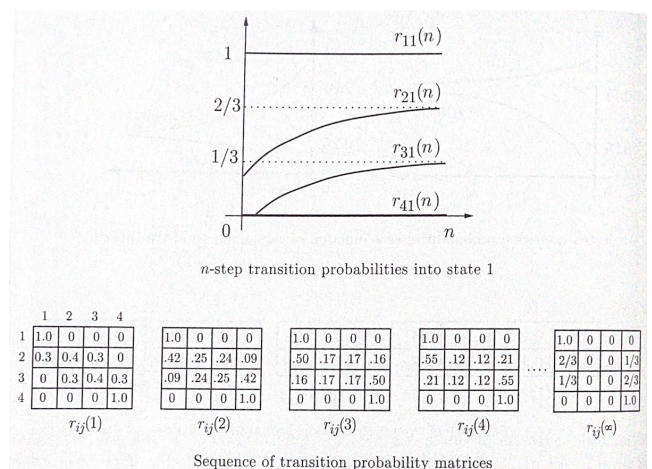


n -step transition probabilities as a function of the number n of transitions

	UpD	B						
UpD	0.8	0.2	.76	.24	.752	.248	.7504	.2496
B	0.6	0.4	.72	.28	.744	.256	.7488	.2512
	$r_{ij}(1)$	$r_{ij}(2)$	$r_{ij}(3)$	$r_{ij}(4)$	$r_{ij}(5)$			

Sequence of n -step transition probability matrices

- Convergence depending on the starting state



n -step transition probabilities into state 1

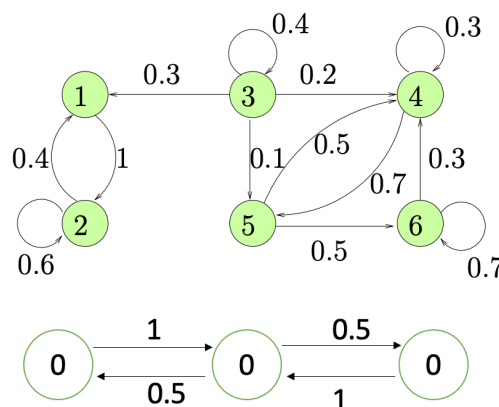
	1	2	3	4										
1	1.0	0	0	0	1.0	0	0	0	1.0	0	0	0	1.0	0
2	0.3	0.4	0.3	0	.42	.25	.24	.09	.50	.17	.17	.16	.55	.12
3	0	0.3	0.4	0.3	.09	.24	.25	.42	.16	.17	.17	.50	.21	.12
4	0	0	0	1.0	0	0	0	1.0	0	0	0	1.0	0	0
	$r_{ij}(1)$	$r_{ij}(2)$	$r_{ij}(3)$	$r_{ij}(4)$					$r_{ij}(4)$					

Sequence of transition probability matrices

(Q) Under what conditions, convergence occurs? If so, how does it depend on the starting state?

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- $r_{ij}(n) \xrightarrow{n \rightarrow \infty} \pi_j$, for some $\pi_j \leq 1$?
- Convergence occurs, independent of the starting state, if:
 - C1.** Only a single recurrent class
 - C2.** such recurrent class is aperiodic
- C1.** For the case of multiple recurrent classes, one stays at the class including the starting state.
- C2.** Divergent behavior for periodic recurrent classes.



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- If $r_{ij}(n) \xrightarrow{n \rightarrow \infty} \pi_j$, for some $\pi_j \leq 1$,

$$r_{ij}(n) = \sum_{k=1}^m r_{ik}(n-1)p_{kj} \implies \pi_j = \sum_{k=1}^m \pi_k p_{kj} \text{ (Balance equation)}$$

- Normalization equation

$$\sum_{i=1}^m \pi_i = 1$$

- Balance equation + Normalization equation \implies Finding the steady-state probabilities $\{\pi_i\}$.

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- A two-state MC with:

$$p_{11} = 0.8, \quad p_{12} = 0.2,$$

$$p_{21} = 0.6, \quad p_{22} = 0.4.$$

- Balance equation:

$$\pi_1 = \pi_1 p_{11} + \pi_2 p_{21}$$

$$\pi_2 = \pi_2 p_{22} + \pi_1 p_{12}$$

- Normalization equation: $\pi_1 + \pi_2 = 1$
- The stationary distribution is: $\pi_1 = 0.25, \pi_2 = 0.75$.

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Stationary Distribution

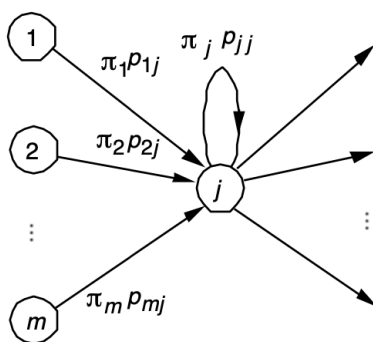
- $\{\pi_j\}$ is also called a **stationary distribution**. Why?
- **Distribution**, because $\sum_{j=1}^m \pi_j = 1$.
- **Stationary**, because, if you choose the starting state according to $\{\pi_j\}$, then

$$\mathbb{P}(X_0 = j) = \pi_j, \quad j = 1, \dots, m \implies \mathbb{P}(X_1 = j) = \sum_{k=1}^m \mathbb{P}(X_0 = k) p_{kj} = \sum_{k=1}^m \pi_k p_{kj} = \pi_j$$

- Then, $\mathbb{P}(X_n = j) = \pi_j$, for all n and j .
- If the initial state is chosen according to $\{\pi_j\}$, the state at any future time will have the same distribution (i.e., the distribution does not change over time).
- We say that "the limiting distribution is equal to the stationary distribution"

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- π_j : the long-term **expected fraction of time** that the state is equal to j .
- Balance equation: $\sum_{k=1}^m \pi_k p_{kj} = \pi_j$ means:
 - The expected frequency π_j of visits to j is equal to the sum of the expected frequencies $\pi_k p_{kj}$ of transitions that lead to j .



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- **Definition.** A state k is **absorbing**, if $p_{kk} = 1$, and $p_{kj} = 0$ for all $j \neq k$.
- states 1 and 6 are absorbing

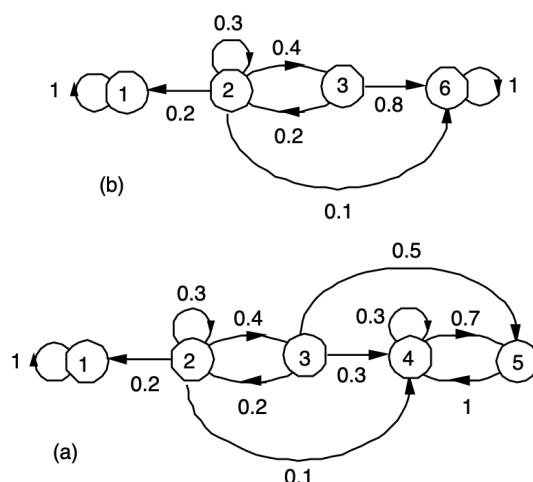
(Q) For a fixed absorbing state s , the probability a_i of reaching s , starting from a transient state i ?

- Fix $s = 6$.

$$a_1 = 0, \quad a_6 = 1$$

$$a_2 = 0.2a_1 + 0.3a_2 + 0.4a_3 + 0.1a_6$$

$$a_3 = 0.2a_2 + 0.8a_6$$

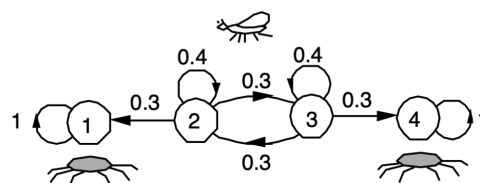


(Q) What if there are some non-absorbing recurrent state?

- Convert it into the one only with absorbing recurrent states (from (a) to (b)).

⁰The notation a_i should have dependence on s , but we omit it for simplicity.

(Q) Starting from a transient state i , expected number of transitions μ_i until absorption to any absorbing state?



- Spider-fly example

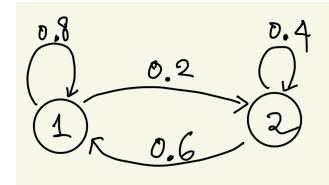
$$\mu_1 = \mu_4 = 0 \quad (\text{for recurrent states})$$

$$\mu_2 = 1 + 0.4\mu_2 + 0.3\mu_3, \quad \mu_3 = 1 + 0.3\mu_2 + 0.4\mu_3 \quad (\text{for transient states})$$

- For generalized description, please see the textbook (pp. 367).

- Assume a single recurrent class

- (Q) **First passage time.** Starting from a i , expected number of transitions t_i to reach s for the first time?
- (Q) **First recurrence time.** Starting from a s , expected number of transitions t_s^* to reach s for the first time?



- Mean first passage time from 2 to 1

$$t_1 = 0$$

$$t_2 = 1 + p_{21}t_1 + p_{22}t_2 = 1 + 0.4t_2 \implies t_2 = 5/3$$

- Mean first recurrence time from 1 to 1

$$t_1^* = 1 + p_{11}t_1 + p_{12}t_2 = 1 + 0 + 0.2 \frac{5}{3} = \frac{4}{3}$$

- For generalized description, please see the textbook (pp. 368)

⁰The notation t_i should have the dependence on s , but we omit it for simplicity.

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Questions?

- 1) Why do you think Markov chain (MC) is important?
- 2) What is the Markov property and its meaning? What's the key difference of MC from Bernoulli processes?
- 3) What are the limiting distribution and the stationary distribution of MCs?
- 4) How are you going to compute the stationary distribution, if you are given a transition probability matrix?
- 5) What are recurrent and transient states in MC?