

Lecture 3: Random Variable, Part I

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EE210: Probability and Introductory Random Processes
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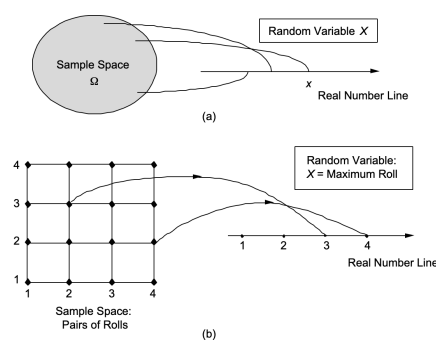
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- Random Variable: Discrete
- PMF (Probability Mass Function)
- Representative Discrete Random Variables
- Expectation and Variance
- Functions of Random Variables
- Conditioning and Independence for Random Variables

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Random Variable: Idea

- In reality, many outcomes are **numerical**, e.g., stock price.
- Even if not, very convenient if we map numerical values to random outcomes, e.g., '0' for male and '1' for female.



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Random Variable: More Formally

- Mathematically, a random variable X is a **function** which maps from Ω to \mathbb{R} .
- **Notation.** Random variable X , numerical value x .
- Different random variables X , Y , etc can be defined on the same sample space.
- For a fixed value x , we can associate an **event** that a random variable X has the value x , i.e., $\{\omega \in \Omega \mid X(\omega) = x\}$
- Assume that values x are discrete¹ such as $1, 2, 3, \dots$.
For notational convenience,

$$p_X(x) \triangleq \mathbb{P}(X = x) \triangleq \mathbb{P}(\{\omega \in \Omega \mid X(\omega) = x\})$$

- For a discrete random variable X , we call $p_X(x)$ **probability mass function** (PMF).

¹Finite or countably infinite.

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- Famous discrete random variables used in the community
 - Bernoulli, Uniform, Binomial, Geometric, Poisson, etc.
- Summarizing a random variable: Expectation and Variance
- Functions of a single random variable, Functions of multiple random variables
- Conditioning for random variables, Independence for random variables
- Continuous random variables
 - Normal, Uniform, Exponential, etc.
- Bayes' rule for random variables

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Bernoulli X with parameter $p \in [0, 1]$

- Only **binary** values

$$X = \begin{cases} 0, & \text{w.p.}^2 \ 1 - p, \\ 1, & \text{w.p.} \ p \end{cases}$$

In other words, $p_X(0) = 1 - p$ and $p_X(1) = p$ from our PMF notation.

- Models a trial that results in binary results, e.g., success/failure, head/tail
- Very useful for an **indicator rv** of an event A . Define a rv 1_A as:

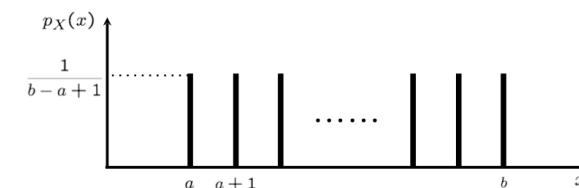
$$1_A = \begin{cases} 1, & \text{if } A \text{ occurs,} \\ 0, & \text{otherwise} \end{cases}$$

²with probability

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Uniform X with parameter a, b

- integers a, b , where $a \leq b$
- Choose a number of $\Omega = \{a, a + 1, \dots, b\}$ uniformly at random.
- $p_X(i) = \frac{1}{b - a + 1}, i \in \Omega$.



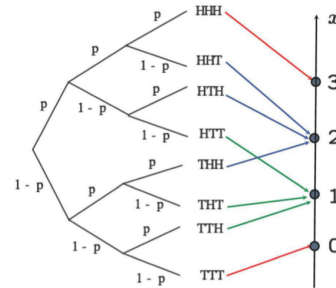
- Models complete ignorance (I don't know anything about X)

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Binomial X with parameter n, p

- Models the number of successes in a given number of independent trials
- n independent trials, where one trial has the success probability p .

$$p_X(k) = \binom{n}{k} p^k (1-p)^{n-k}$$



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Poisson X with parameter λ

- $\text{Binomial}(n, p)$: Models the number of successes in a given number of independent trials with success probability p .
- Very large n and very small p , such that $np = \lambda$

$$p_X(k) = e^{-\lambda} \frac{\lambda^k}{k!}, \quad k = 0, 1, \dots$$

- Is this a legitimate PMF?

$$\sum_{k=0}^{\infty} e^{-\lambda} \frac{\lambda^k}{k!} = e^{-\lambda} \left(1 + \lambda + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} \dots \right) = e^{-\lambda} e^{\lambda} = 1$$

- Prove this:

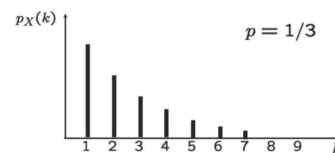
$$\lim_{n \rightarrow \infty} p_X(k) = \binom{n}{k} (1/n)^k (1 - 1/n)^{n-k} = e^{-\lambda} \frac{\lambda^k}{k!}$$

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Geometric X with parameter p

- Experiment: infinitely many independent Bernoulli trials, where each trial has success probability p
- Random variable: number of trials until the **first success**.
- Models waiting times until something happens.

$$p_X(k) = (1-p)^{k-1} p$$



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Roadmap

- Famous discrete random variables used in the community
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- Average.

Definition

$$\mathbb{E}[X] = \sum_x x p_X(x)$$

- $p_X(x)$: relative frequency of value x (trials with x /total trials)
- Example 1: Bernoulli r.v. with p

$$\mathbb{E}[X] = 1 \times p + 0 \times (1 - p) = p_X(1)$$

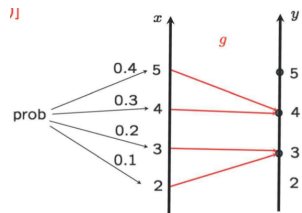
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Not very surprising. Easy to prove using the definition.

- If $X \geq 0$, $\mathbb{E}[X] \geq 0$.
- If $a \leq X \leq b$, $a \leq \mathbb{E}[X] \leq b$.
- For a constant c , $\mathbb{E}[c] = c$.

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- For a rv X , $Y = g(X)$ is also a r.v.
- $\mathbb{E}[Y] = \mathbb{E}[g(X)] = \sum_x g(x) p_X(x)$
- Compute $\mathbb{E}[Y]$ for the following:



$$4 \times (0.4 + 0.3) + 3 \times (0.1 + 0.2) = 2.8 + 0.9 = 3.7$$

Linearity of Expectation

$$\mathbb{E}[aX + b] = a\mathbb{E}[X] + b$$

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- Measures how much the **spread** of a PMF is.
- What about $\mathbb{E}[X - \mu]$, where $\mu = \mathbb{E}[X]$? Then, what about $\mathbb{E}[(X - \mu)^2]$?

Variance, Standard Deviation

$$\text{var}[X] = \mathbb{E}[(X - \mu)^2]$$

$$\sigma_X = \sqrt{\text{var}[X]}$$

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- $\text{var}[X] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$
 $\text{var}[X] = \mathbb{E}[X^2 - 2\mu X + \mu^2]$
 $= \mathbb{E}[X^2] - 2\mu\mathbb{E}[X] + \mu^2 = \mathbb{E}[X^2] - \mu^2$
- $Y = X + b, \text{var}[Y] = \text{var}[X]$
 $\text{var}[Y] = \mathbb{E}[(X + b)^2] - (\mathbb{E}[X + b])^2$
- $Y = aX, \text{var}[Y] = a^2\text{var}[X]$
 $\text{var}[Y] = \mathbb{E}[a^2 X^2] - (a\mathbb{E}[X])^2$

Example: Variance of a Bernoulli rv (p)

$$\begin{aligned}\mathbb{E}[X] &= 1 \times p + 0 \times (1 - p) = p \\ \mathbb{E}[X^2] &= 1 \times p + 0 \times (1 - p) = p \\ \text{var}[X] &= \mathbb{E}[X^2] - \mu^2 = p - p^2 \\ &= p(1 - p)\end{aligned}$$

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- Joint PMF.** For two random variables X, Y , consider two events $\{X = x\}$ and $\{Y = y\}$, and

$$p_{X,Y}(x,y) \triangleq \mathbb{P}(\{X=x\} \cap \{Y=y\})$$

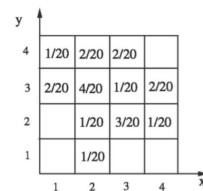
- $\sum_x \sum_y p_{X,Y}(x,y) = 1$

- Marginal PMF.

$$p_X(x) = \sum_y p_{X,Y}(x,y),$$

$$p_Y(y) = \sum_x p_{X,Y}(x,y)$$

Example.



$$p_{X,Y}(1,3) = 2/20$$

$$p_X(4) = 2/20 + 1/20 = 3/20$$

$$\mathbb{P}(X = Y) = 1/20 + 4/20 + 3/20 = 8/20$$

- Consider a rv $Z = g(X, Y)$. (Ex) $X + Y, X^2 + Y^2$. Then, PMF of Z is:

$$p_Z(z) = \mathbb{P}(g(X, Y) = z) = \sum_{(x,y): g(x,y)=z} p_{X,Y}(x, y)$$

- Similarly,

$$\mathbb{E}[Z] = \mathbb{E}[g(X, Y)] = \sum_x \sum_y g(x, y) p_{X, Y}(x, y)$$

Linearity of Expectation for multiple rvs

- Remember: $\mathbb{E}[aX + b] = a\mathbb{E}[X] + b$
- Similarly,

$$\mathbb{E}[X + Y] = \mathbb{E}[X] + \mathbb{E}[Y]$$
 (easy to prove, using the definition.)
- $\mathbb{E}[X_1 + \dots + X_n] = \mathbb{E}[X_1] + \dots + \mathbb{E}[X_n]$
- $\mathbb{E}[2X + 3Y - Z] = 2\mathbb{E}[X] + 3\mathbb{E}[Y] - \mathbb{E}[Z]$

- Example.** Mean of a binomial rv Y with (n, p)
- Y : number of successes in n Bernoulli trials with p
- $Y = X_1 + \dots + X_n$, where X_i is a Bernoulli rv.
- $\mathbb{E}[Y] = n\mathbb{E}[X_i] = n\mathbb{P}(X_i = 1) = np$

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Conditional PMF: Conditioning on an event

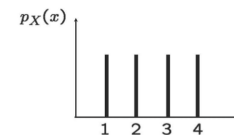
Remember two probability laws: $\mathbb{P}(\cdot)$ and $\mathbb{P}(\cdot|A)$, for an event A .

- $p_X(x) = \mathbb{P}(X = x)$
- $\mathbb{E}[X] = \sum_x x p_X(x)$
- $\mathbb{E}[g(X)] = \sum_x g(x) p_X(x)$
- $\text{var}[X] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$
- $p_{X|A}(x) \triangleq \mathbb{P}(X = x|A)$
- $\mathbb{E}[X|A] \triangleq \sum_x x p_{X|A}(x)$
- $\mathbb{E}[g(X)|A] \triangleq \sum_x g(x) p_{X|A}(x)$
- $\text{var}[X|A] \triangleq \mathbb{E}[X^2|A] - (\mathbb{E}[X|A])^2$
- Note.** $p_{X|A}(x)$, $\mathbb{E}[X|A]$, $\mathbb{E}[g(X)|A]$, and $\text{var}[X|A]$ are all just notations!

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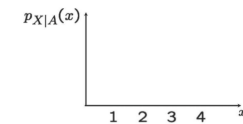
Example: Conditional PMF

$$A = \{X \geq 2\}$$



$$\mathbb{E}[X] = \frac{1}{4}(1 + 2 + 3 + 4) = 2.5$$

$$\begin{aligned} \text{var}[X] &= \mathbb{E}[X^2] - (\mathbb{E}[X])^2 \\ &= \frac{1}{4}(1^2 + 2^2 + 3^2 + 4^2) - 2.5^2 \end{aligned}$$



$$\mathbb{E}[X|A] = \frac{1}{3}(2 + 3 + 4) = 3$$

$$\begin{aligned} \text{var}[X|A] &= \mathbb{E}[X^2|A] - (\mathbb{E}[X|A])^2 \\ &= \frac{1}{3}(2^2 + 3^2 + 4^2) - 3^2 = 2/3 \end{aligned}$$

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Conditional PMF: Conditioning on a RV

What do we mean by "conditioning on a rv"? Consider $A = \{Y = y\}$ for a rv Y .

- $p_{X|A}(x) \triangleq \mathbb{P}(X = x|A)$
 - $\mathbb{E}[X|A] \triangleq \sum_x x p_{X|A}(x)$
 - $\mathbb{E}[g(X)|A] \triangleq \sum_x g(x) p_{X|A}(x)$
 - $\text{var}[X|A] \triangleq \mathbb{E}[X^2|A] - (\mathbb{E}[X|A])^2$
- $p_{X|Y}(x|y) \triangleq \mathbb{P}(X = x|Y = y)$
 - $\mathbb{E}[X|Y = y] \triangleq \sum_x x p_{X|Y}(x|y)$
 - $\mathbb{E}[g(X)|Y = y] \triangleq \sum_x g(x) p_{X|Y}(x|y)$
 - $\text{var}[X|Y = y] \triangleq \mathbb{E}[X^2|Y = y] - (\mathbb{E}[X|Y = y])^2$

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Conditional PMF

Conditional PMF

$$p_{X|Y}(x|y) \triangleq \mathbb{P}(X = x|Y = y) = \frac{p_{X,Y}(x,y)}{p_Y(y)}$$

for y such that $p_Y(y) > 0$.

$$\sum_x p_{X|Y}(x|y) = 1$$

Multiplication rule.

$$p_{X,Y}(x,y) = p_Y(y) p_{X|Y}(x|y) \\ = p_X(x) p_{Y|X}(y|x)$$

$$p_{X,Y,Z}(x,y,z) = \\ p_X(x) p_{Y|X}(y|x) p_{Z|X,Y}(z|x,y)$$

y				
4	1/20	2/20	2/20	
3	2/20	4/20	1/20	2/20
2		1/20	3/20	1/20
1		1/20		
	1	2	3	4
	x			

$$p_{X|Y}(2|2) = \frac{1}{1+3+1}$$

$$p_{X|Y}(3|2) = \frac{3}{1+3+1}$$

$$\mathbb{E}[X|Y = 3] = 1(2/9) + 2(4/9) + 3(1/9) + 4(2/9)$$

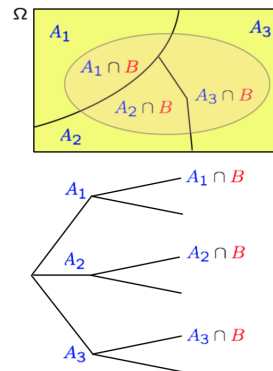
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Remind: Total Probability Theorem (from Lecture 2)

- Partition of Ω into A_1, A_2, A_3
- Known: $\mathbb{P}(A_i)$ and $\mathbb{P}(B|A_i)$
- What is $\mathbb{P}(B)$? (probability of result)

Total Probability Theorem

$$\mathbb{P}(B) = \sum_i \mathbb{P}(A_i) \mathbb{P}(B|A_i)$$



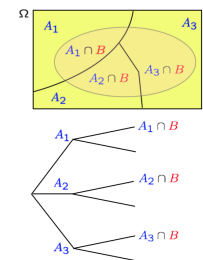
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Total Probability Theorem: $B = \{X = x\}$

- Partition of Ω into A_1, A_2, A_3

Total Probability Theorem

$$p_X(x) = \sum_i \mathbb{P}(A_i) \mathbb{P}(X = x|A_i) = \sum_i \mathbb{P}(A_i) p_{X|A_i}(x)$$



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Total Expectation Theorem for $\{A_i\}$

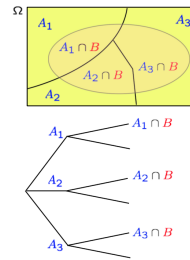
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Total Probability Theorem

$$p_X(x) = \sum_i \mathbb{P}(A_i) \mathbb{P}(X = x | A_i) = \sum_i \mathbb{P}(A_i) p_{X|A_i}(x)$$

Total Expectation Theorem

$$\mathbb{E}[X] = \sum_i \mathbb{P}(A_i) \mathbb{E}[X | A_i]$$



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Total Expectation Theorem for $\{Y = y\}$

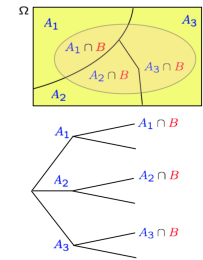
- Partition of Ω into A_1, A_2, A_3

Total Expectation Theorem

$$\mathbb{E}[X] = \sum_i \mathbb{P}(A_i) \mathbb{E}[X | A_i]$$

Total Expectation Theorem

$$\mathbb{E}[X] = \sum_y \mathbb{P}(Y = y) \mathbb{E}[X | Y = y] = \sum_y p_Y(y) \mathbb{E}[X | Y = y]$$



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Example 1: Total Expectation Theorem

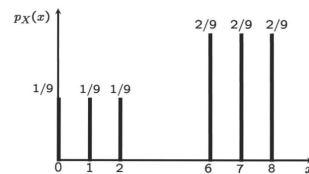
- $A_1 = \{X \in \{0, 1, 2\}\}$, $A_2 = \{X \in \{6, 7, 8\}\}$

- Using TET,

$$\begin{aligned} \mathbb{E}[X] &= \sum_{i=1,2} \mathbb{P}(A_i) \mathbb{E}[X | A_i] \\ &= 1/3 \cdot 1 + 2/3 \cdot 7 = 7 \end{aligned}$$

- Without using TET,

$$\mathbb{E}[X] = \frac{1}{9}(0 + 1 + 2) + \frac{2}{9}(6 + 7 + 8)$$



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Background: Memoryless Property of Geometric rv (1)

- Some random variable often does not have **memory**.

- Definition.** A random variable X is called **memoryless** if, for any $n, m \geq 0$,

$$\mathbb{P}(X > n + m | X > m) = \mathbb{P}(X > n)$$

- Meaning.** Conditioned on $X > m$, $X - m$'s distribution is the same as the original X .

- Remind.** Geometric rv X with parameter p

$$\mathbb{P}(X = k) = (1 - p)^{k-1} p$$

$$\mathbb{P}(X > k) = 1 - \sum_{k'=1}^k (1 - p)^{k'-1} p = (1 - p)^k$$

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- **Theorem.** Any geometric random variable is **memoryless**.

$$\begin{aligned}\mathbb{P}(X > n + m | X > m) &= \frac{\mathbb{P}(X > n + m \text{ and } X > m)}{\mathbb{P}(X > m)} \\ &= \frac{\mathbb{P}(X > n + m)}{\mathbb{P}(X > m)} \\ &= \frac{(1 - p)^{n+m}}{(1 - p)^m} = (1 - p)^n = \mathbb{P}(X > n)\end{aligned}$$

- **Meaning.** Conditioned on $X > m$, $X - m$ is geometric with the same parameter.

- Write softwares over and over, and each time w.p. p of working correctly (independent from prev. programs).
- X : number of tries until the program works correctly.
- **Q) mean and variance of X**
- X is geometric
- Direct computation is boring.

$$\mathbb{E}[X] = \sum_{k=1}^{\infty} k(1-p)^{k-1}p$$

- Total expectation theorem and memorylessness helps a lot.

- $A_1 = \{X = 1\}$ (first try is success),
 $A_2 = \{X > 1\}$ (first try is failure).

$$\begin{aligned}\mathbb{E}[X] &= 1 + \mathbb{E}[X - 1] \\ &= 1 + \mathbb{P}(A_1)\mathbb{E}[X - 1 | X = 1] \\ &\quad + \mathbb{P}(A_2)\mathbb{E}[X - 1 | X > 1] \\ &= 1 + (1 - p)\mathbb{E}[X]\end{aligned}$$

$$\mathbb{E}[X] = 1 + (1 - p)\frac{1}{p} = 1/p.$$

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- Two events

$$\mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B)$$

$$\mathbb{P}(A \cap B | C) = \mathbb{P}(A | C) \cdot \mathbb{P}(B | C)$$

- A rv and an event

$$\mathbb{P}(\{X = x\} \cap B) = \mathbb{P}(X = x) \cdot \mathbb{P}(B), \quad \text{for all } x$$

$$\mathbb{P}(\{X = x\} \cap B | C) = \mathbb{P}(X = x | C) \cdot \mathbb{P}(B | C), \quad \text{for all } x$$

- Two rvs

$$\mathbb{P}(\{X = x\} \cap \{Y = y\}) = \mathbb{P}(X = x) \cdot \mathbb{P}(Y = y), \quad \text{for all } x, y$$

$$p_{X,Y}(x, y) = p_X(x) \cdot p_Y(y)$$

$$\mathbb{P}(\{X = x\} \cap \{Y = y\} | C) = \mathbb{P}(X = x | C) \cdot \mathbb{P}(Y = y | C), \quad \text{for all } x, y$$

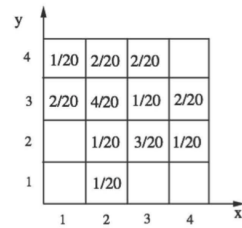
$$p_{X,Y|C}(x, y) = p_{X|C}(x) \cdot p_{Y|C}(y)$$

Example

- $X \perp\!\!\!\perp Y$?

$$p_{X,Y}(1,1) = 0, \quad p_X(1) = 3/20 \\ p_Y(1) = 1/20.$$

- $X \perp\!\!\!\perp Y | \{X \leq 2 \text{ and } Y \geq 3\}$?
- Yes.



$Y = 4 \ (1/3)$	$1/9$	$2/9$
$Y = 3 \ (2/3)$	$2/9$	$4/9$
	$X = 1 \ (1/3)$	$X = 2 \ (2/3)$

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Expectation and Variance

- Always true.
 $\mathbb{E}[aX + b], \mathbb{E}[X + Y] = \mathbb{E}[X] + \mathbb{E}[Y]$
- Generally, $\mathbb{E}[g(X, Y)] \neq g(\mathbb{E}[X], \mathbb{E}[Y])$
- However, if $X \perp\!\!\!\perp Y$,
 $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$
 $\mathbb{E}[g(X)h(Y)] = \mathbb{E}[g(X)]\mathbb{E}[h(Y)]$

- Proof.

$$\begin{aligned} \mathbb{E}[g(X)h(Y)] &= \sum_x \sum_y g(x)h(y)p_{X,Y}(x,y) \\ &= \sum_x xp_X(x) \sum_y yp_Y(y) \end{aligned}$$

- Always true.
 $\text{var}[aX] = a^2\text{var}[X], \text{var}[X + a] = \text{var}[X]$
- Generally, $\text{var}[X + Y] \neq \text{var}[X] + \text{var}[Y]$
- However, if $X \perp\!\!\!\perp Y$,
 $\text{var}[X + Y] = \text{var}[X] + \text{var}[Y]$
- Practice.
 - $X = Y \implies \text{var}[X + Y] = 4\text{var}[X]$
 - $X = -Y \implies \text{var}[X + Y] = 0$
 - $X \perp\!\!\!\perp Y \implies \text{var}[X - 3Y] = \text{var}[X] + 9\text{var}[Y]$

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$$\text{var}[X + Y] \neq \text{var}[X] + \text{var}[Y]$$

- Why not generally true?

$$\begin{aligned} \text{var}[X + Y] &= \mathbb{E}[(X + Y)^2] - (\mathbb{E}[X + Y])^2 \\ &= \mathbb{E}[X^2 + Y^2 + 2XY] - ((\mathbb{E}[X])^2 + (\mathbb{E}[Y])^2 + 2\mathbb{E}[X]\mathbb{E}[Y]) \\ &= \text{var}[X] + \text{var}[Y] + 2(\mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]) \end{aligned}$$

- $X \perp\!\!\!\perp Y$ is a sufficient condition for $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$
- Also, a necessary condition? we will see later, when we study **covariance**.

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Example: The hat problem (1)

- n people throw their hats in a box and then pick one at random
- X : number of people with their own hat
- $\mathbb{E}[X]$? $\text{var}[X]$?
- All permutations are equally likely as $1/n!$. Thus, this equals to picking one hat at a time.
- Key step 1. Define a rv $X_i = 1$ if i selects own hat and 0 otherwise.

$$X = \sum_{i=1}^n X_i.$$

- $\{X_i\}, i = 1, 2, \dots, n$: identically distributed (symmetry)

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Example: The hat problem (2)

- $\mathbb{E}[X] = n\mathbb{E}[X_1] = n\mathbb{P}(X_1 = 1) = n \times \frac{1}{n} = 1$.
- **Key step 2.** Are X_i s are independent? If yes, easy to get $\text{var}(X)$.
- Assume $n = 2$. Then, $X_1 = 1 \rightarrow X_2 = 1$, and $X_1 = 0 \rightarrow X_2 = 0$. Thus, **dependent**.

$$\text{var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$$

$$= \mathbb{E}\left[\sum_i X_i^2 + \sum_{i,j:i \neq j} X_i X_j\right] - (\mathbb{E}[X])^2$$

$$\mathbb{E}[X_i^2] = 1 \times \frac{1}{n} + 0 \times \frac{n-1}{n} = \frac{1}{n}$$

$$\mathbb{E}[X_i X_j] = \mathbb{E}[X_1 X_2] = 1 \times \mathbb{P}(X_1 X_2 = 1) = \mathbb{P}(X_1 = 1)\mathbb{P}(X_2 = 1|X_1 = 1), \quad (i \neq j)$$

- $\mathbb{E}[X^2] = n\mathbb{E}[X_1^2] + n(n-1)\mathbb{E}[X_1 X_2] = n\frac{1}{n} + n(n-1)\frac{1}{n(n-1)} = 2$
- $\text{var}(X) = 2 - 1 = 1$

Questions?

Review Questions

- 1) What is Random Variable? Why is it useful?
- 2) What is PMF (Probability Mass Function)?
- 3) Explain Bernoulli, Binomial, Poisson, Geometric rvs, when they are used and what their PMFs are.
- 4) What are joint and marginal PMFS?
- 5) Describe and explain the total probability/expectation theorem for random variables?
- 6) When is it useful to use total probability/expectation theorem?
- 7) What is conditional independence?