Lecture 5: Random Variable, Part III

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EE210: Probability and Introductory Random Processes
KAIST EE

MONTH DAY, 2021

Roadmap

- Famous discrete random variables used in the community
  - Bernoulli, Uniform, Binomial, Geometric, Poisson, etc.
- Summarizing a random variable: Expectation and Variance
- Functions of a single random variable, Functions of multiple random variables
- Conditioning for random variables, Independence for random variables
- Continuous random variables
  - Normal, Uniform, Exponential, etc.
- Bayes' rule for random variables

- (Derived) Distribution of $Y = g(X)$ or $Z = g(X, Y)$
- Quantifying the degree of dependence between two rvs.
- Conditional expectation/variance
- (Random) Sum of random variables

Derived Distribution: $Y = g(X)$

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- Examples: $Y = X$, $Y = X + 1$, $Y = X^2$, etc.
- What are easy or difficult cases?
- Easy cases
  - Discrete
  - Linear: $Y = aX + b$

Discrete Case

- Take all values of x such that $g(x) = y$, i.e.,

$$\begin{aligned} p_Y(y) &= \mathbb{P}(g(X) = y) \\ &= \sum_{x:g(x)=y} p_X(x) \end{aligned}$$

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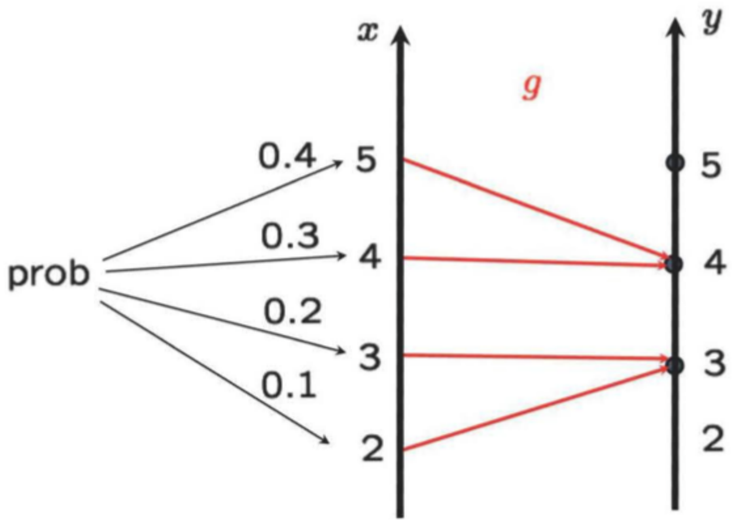

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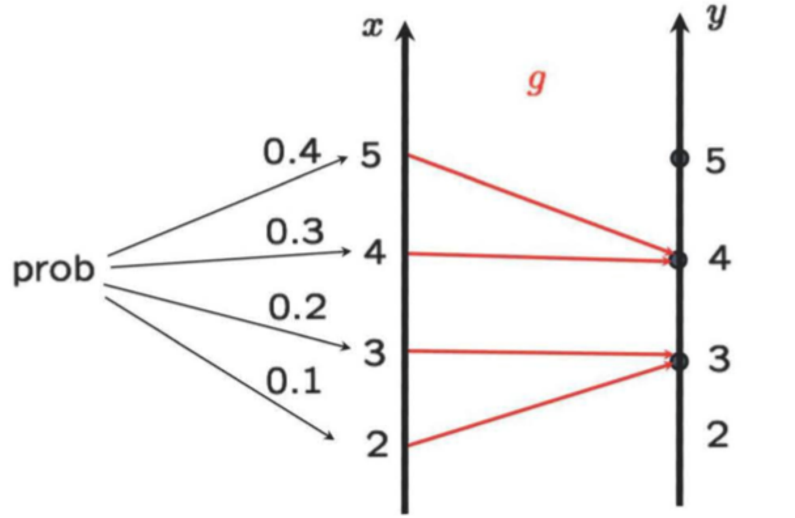
$$p_Y(y) = \mathbb{P}(g(X) = y) = \sum_{x:g(x)=y} p_X(x)$$

$$p_Y(3) = p_X(2) + p_X(3) = 0.1 + 0.2 = 0.3$$
$$p_Y(4) = p_X(4) + p_X(5) = 0.3 + 0.4 = 0.7$$



Linear: $Y = aX + b$, $a \neq 0$

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If $a > 0$, $\quad F_Y(y) = \mathbb{P}(aX + b \leq y) = \mathbb{P}(X \leq \frac{y-b}{a}) = F_X(\frac{y-b}{a})$

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Special case. X is normal. Then, Y is also normal, i.e., $Y \sim N(a\mu + b, a^2\sigma^2)$

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Ex2. $X \sim uniform[0,2]$. $Y = X^3$.

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Ex3. X with $f_X(x)$. $Y = X^2$.

$$F_Y(y) = \mathbb{P}(X^2 \le y) = \mathbb{P}(-\sqrt{y} \le X \le \sqrt{y})$$
$$= F_X(\sqrt{y}) - F_X(-\sqrt{y})$$

$$f_Y(y) = \frac{1}{2\sqrt{y}} f_X(\sqrt{y}) + \frac{1}{2\sqrt{y}} f_X(-\sqrt{y}), \quad y \ge 0$$

Functions of multiple rvs: $Y = g(X, Y)$ (1)

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f_Z(z) = \begin{cases} 2z, & \text{if } 0 \leq z \leq 1 \\ 0, & \text{otherwise} \end{cases}
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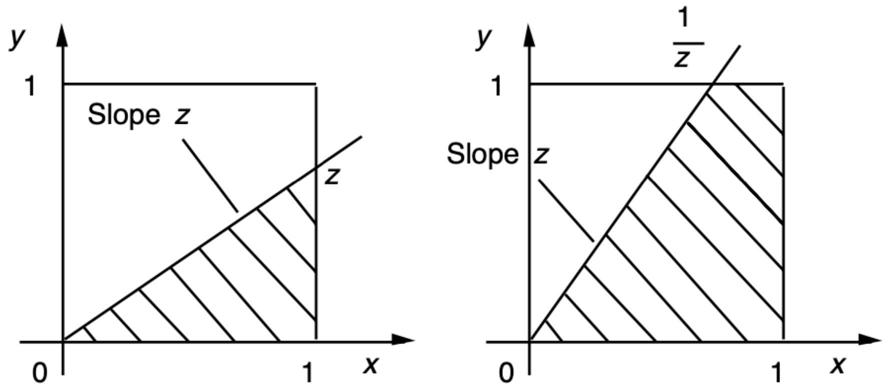

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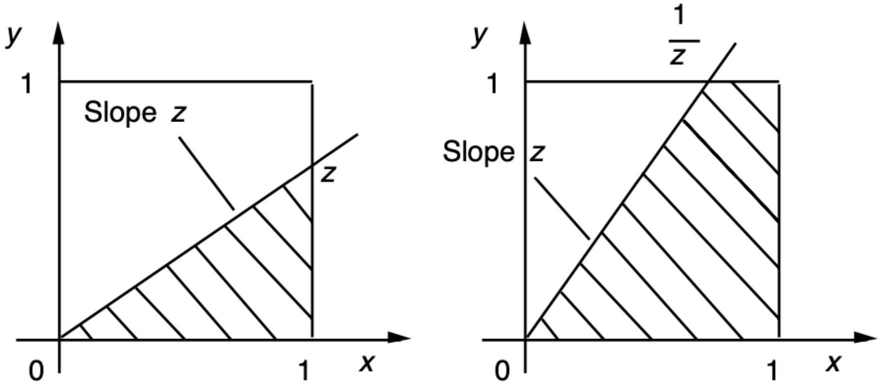

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$$f_Z(z) = \begin{cases} 1/2, & 0 \le z \le 1 \\ 1/(2z^2), & z > 1 \\ 0, & \text{otherwise} \end{cases}$$

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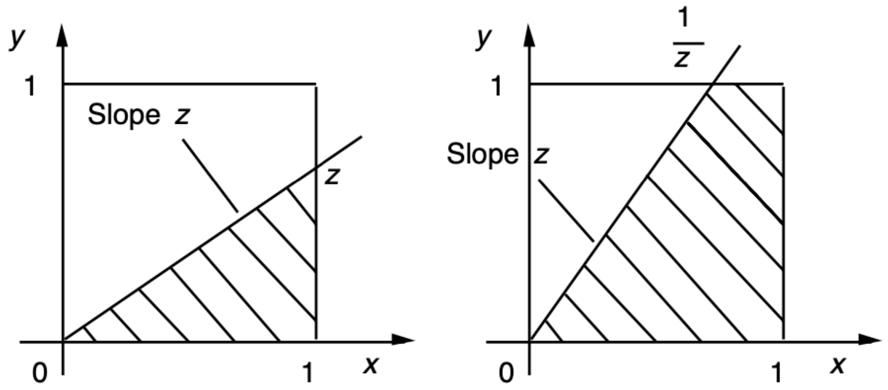

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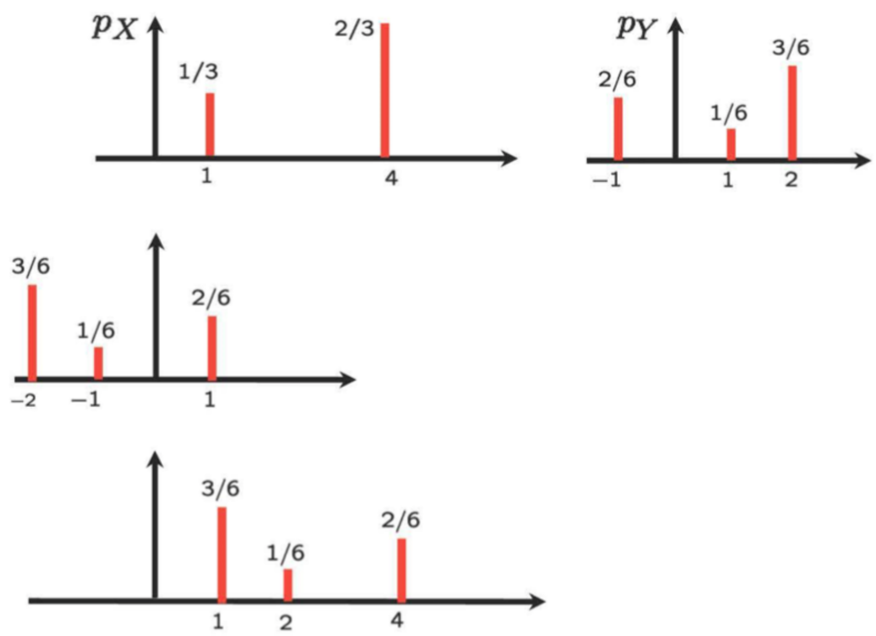
(i) Flip (horizontally) $p_Y(y)$ $(p_Y(-x))$

(ii) Put it underneath $p_X(x)$ $(p_Y(-x+z))$



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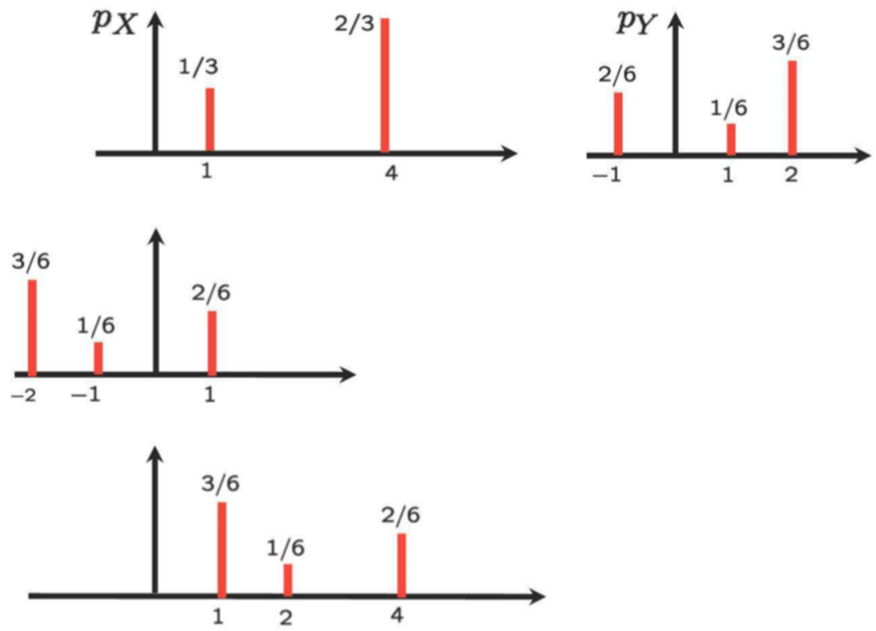

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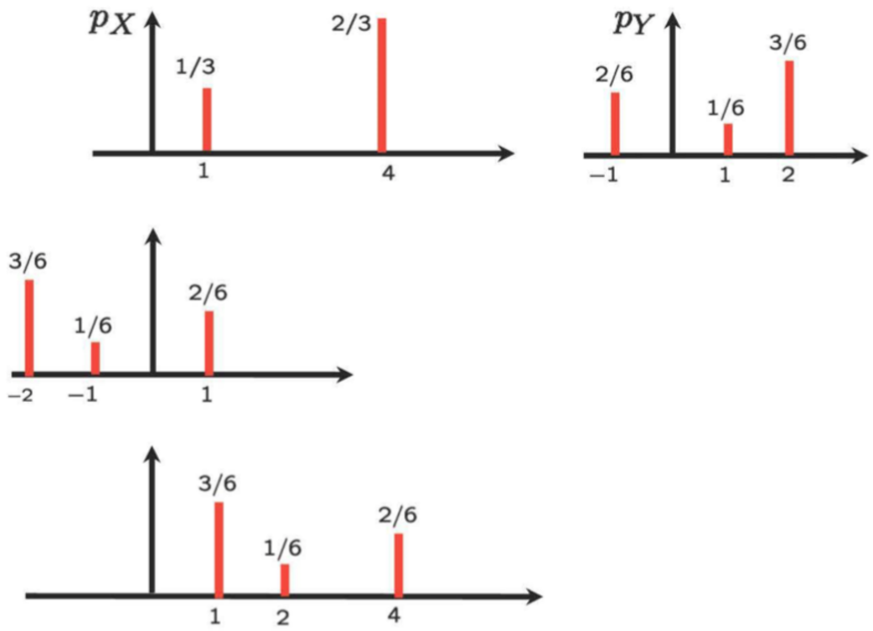

$Y = X + Y$, $X \perp\!\!\!\perp Y$: Continuous

- Same logic as the discrete case

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Sum of two independent normal rvs

$X \sim N(\mu_x, \sigma_x^2)$ and $Y \sim N(\mu_x, \sigma_x^2)$
Then, $X + Y \sim N(\mu_x + \mu_y, \sigma_x^2 + \sigma_y^2)$

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- Why normal rvs are used to model the sum of random noises.
- (Extension) The sum of finitely many independent normals is also normal.

Roadmap

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- Good engineers: Good at making good metrics
  - Metric of how our society is economically polarized
  - A lot of metrics in our professional sports leagues (baseball, basketball, etc)
  - Cybermetrics in MLB (Major League Baseball): `http://m.mlb.com/glossary/advanced-stats`

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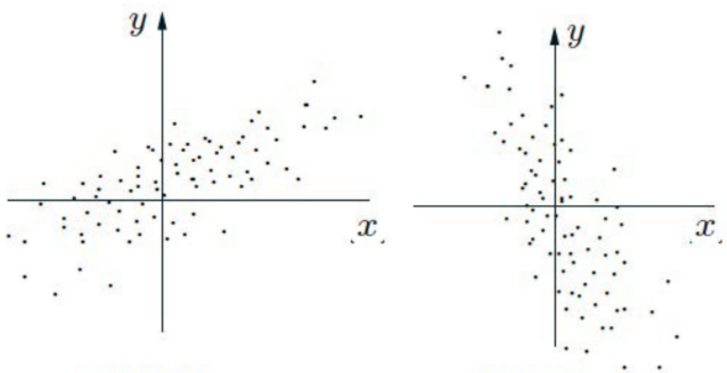
- What about $\mathbb{E}[XY]$? Seems good.
  - $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y] = 0$ when $X \perp\!\!\!\perp Y$
  - More data points (thus increases) when $xy > 0$ (both positive or negative)



$\mathbf{E}[XY]$ $\mathbf{E}[XY]$

OK. Let's Design!

- Simple case: $\mathbb{E}[X] = \mu_X = 0$ and $\mathbb{E}[Y] = \mu_Y = 0$
- Dependent: Positive (If $X \uparrow$, $Y \uparrow$) or Negative (If $X \uparrow$, $Y \downarrow$)
- What about $\mathbb{E}[XY]$? Seems good.
  - $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y] = 0$ when $X \perp\!\!\!\perp Y$
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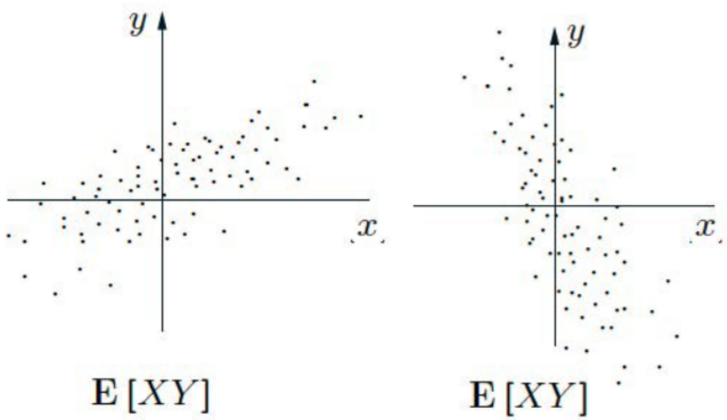

(Q) What about $\mathbb{E}[X+Y]$?

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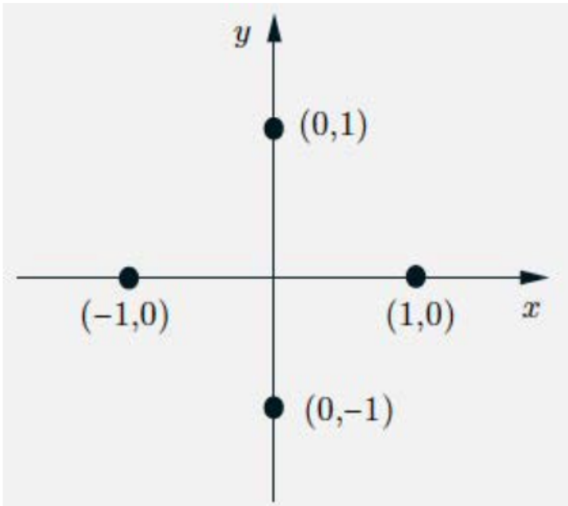
Example: $\text{cov}(X, Y) = 0$, but not independent

- $p_{X,Y}(1,0) = p_{X,Y}(0,1) = p_{X,Y}(-1,0) = p_{X,Y}(0,-1) = 1/4.$



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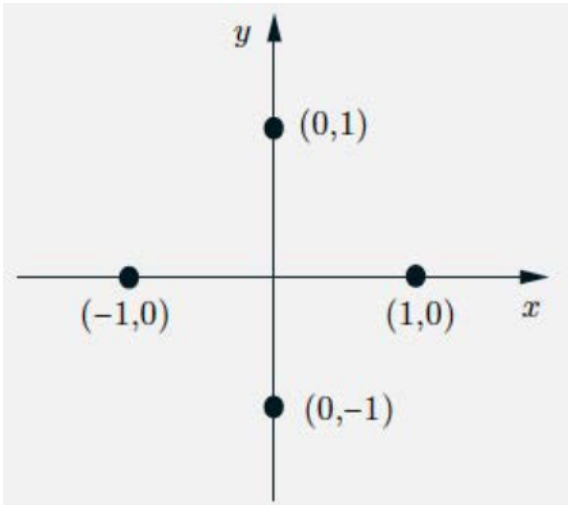

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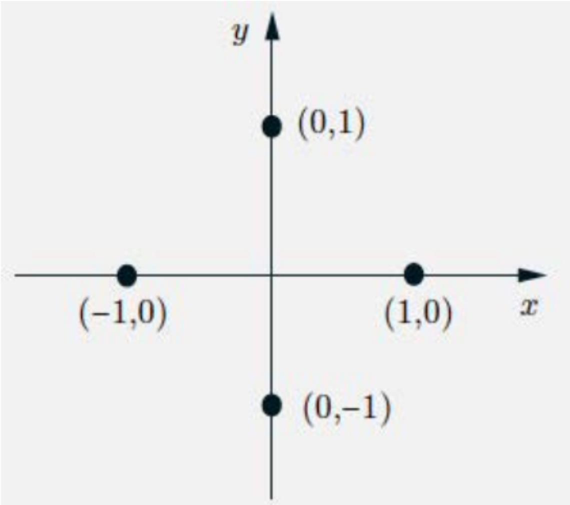
- $\mathbb{E}[X] = \mathbb{E}[Y] = 0$, and $\mathbb{E}[XY] = 0$. So, $\text{cov}(X, Y) = 0$
- Are they independent? No, because if $X = 1$, then we should have $Y = 0$.



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$$\mathrm{var}[X + Y] = \mathbb{E}[(X + Y)^2] - (\mathbb{E}[X + Y])^2 = \mathrm{var}[X] + \mathrm{var}[Y] - 2\mathrm{cov}(X, Y)$$

Example: The hat problem in Lecture 3. Remember?

- n people throw their hats in a box and then pick one at random
- X: number of people with their own hat
- (Q) var[X]
- Key step 1. Define a rv $X_i = 1$ if i selects own hat and 0 otherwise. Then, $X = \sum_{i=1}^{n} X_i$.
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◦ For $i \neq j$,

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&= \mathbb{P}(X_i = 1)\mathbb{P}(X_j = 1 | X_i = 1) - \frac{1}{n^2} \\
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- Reqs. a), b), and c) satisfied.

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- $-1 \leq \rho \leq 1$
- $|\rho| = 1 \implies X - \mu_X = c(Y - \mu_Y)$ (linear relation, VERY related)

Roadmap

- Famous discrete random variables used in the community
  - Bernoulli, Uniform, Binomial, Geometric, Poisson, etc.
- Summarizing a random variable: Expectation and Variance
- Functions of a single random variable, Functions of multiple random variables
- Conditioning for random variables, Independence for random variables
- Continuous random variables
  - Normal, Uniform, Exponential, etc.
- Bayes' rule for random variables

- (Derived) Distribution of $Y = g(X)$ or $Z = g(X, Y)$
- Quantifying the degree of dependence between two rvs.
- Conditional expectation/variance
- (Random) Sum of random variables

A Special Random Variable

- Consider a rv Y, such that

$$Y = \begin{cases} 0, & \text{w.p. } 1/4 \\ 1, & \text{w.p. } 1/4 \\ 2, & \text{w.p. } 1/2 \end{cases}$$

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- What about? $X_{exp}(Y)$, $\mathbb{E}[X_Y]$, $\mathbb{E}_X[Y]$?

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A random variable $g(Y) = \boxed{\phantom{\mathbb{E}[X|Y]}}$, called $\boxed{\phantom{\text{conditional expectation of } X \text{ given } Y}}$, takes the value $g(y) = \mathbb{E}[X|Y = y]$, if Y happens to take the value y.

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- Thus, having a distribution, expectation, variance, all the things that a random variable has
- Often confusing because of the notation

Expectation of $\mathbb{E}[X|Y]$

Expectation of Conditional Expectation

$$\mathbb{E}\Big[\mathbb{E}[X|Y]\Big] = \mathbb{E}[X], \quad \text{Law of iterated expectations}$$

Proof.

$$\mathbb{E}\Big[\mathbb{E}[X|Y]\Big] = \sum_{y} \mathbb{E}[X|Y = y] p_Y(y)$$
$$= \mathbb{E}[X]$$

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- Forecasts on sales: calculating expected value, given any available information
- X : February sales
- Forecast in the beg. of the year: $\mathbb{E}[X]$
- End of Jan. new information $Y = y$ (Jan. sales)
  Revised forecast: $\mathbb{E}[X|Y=y]$
  Revised forecast $\neq \mathbb{E}[X]$

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- Uniformly break at point Y, and break what is left uniformly at point X.
- $\mathbb{E}[X|Y = y] = y/2$
- $\mathbb{E}[X|Y] = Y/2$
- $\mathbb{E}[X] = \mathbb{E}[\mathbb{E}[X|Y]] = \mathbb{E}[Y/2] = \frac{1}{2}\frac{l}{2} = l/4$

- Forecasts on sales: calculating expected value, given any available information
- X : February sales
- Forecast in the beg. of the year: $\mathbb{E}[X]$
- End of Jan. new information $Y = y$ (Jan. sales)
  Revised forecast: $\mathbb{E}[X|Y = y]$
  Revised forecast $\neq \mathbb{E}[X]$
- Law of iterated expectations
  $\mathbb{E}[\text{revised forecast}] = \text{original one}$

Conditional Variance $\mathrm{var}[X|Y]$

$\mathrm{var}[X] = \mathbb{E}[(X - \mathbb{E}[X])^2]$

Conditional Variance var$[X|Y]$

$$\text{var}[X] = \mathbb{E}[(X - \mathbb{E}[X])^2]$$

$$\text{var}[X|Y = y] = \mathbb{E}[(X - \mathbb{E}[X|Y = y])^2|Y = y]$$

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A random variable $g(Y) =$ $\boxed{}$ and called $\boxed{}$, takes the value $g(y) = \mathrm{var}[X|Y = y]$, if Y happens to take the value y.

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- A function of Y
- A random variable
- Thus, having a distribution, expectation, variance, all the things that a random variable has

Expectation and Variance of $\mathbb{E}[X|Y]$ and $\text{var}[X|Y]$

	$\mathbb{E}[X\|Y]$	$\text{var}[X\|Y]$
Expectation	$\mathbb{E}\left[\mathbb{E}(X\|Y)\right]$	$\mathbb{E}\left[\text{var}(X\|Y)\right]$
Variance	$\text{var}\left[\mathbb{E}(X\|Y)\right]$	$\text{var}\left[\text{var}(X\|Y)\right]$

Law of Total Variance

Law of total variance

$$\mathrm{var}[X] =$$

Proof.

(1)

(2)

Law of Total Variance

Law of total variance

$$\mathrm{var}[X] = \mathbb{E}\Big[\mathrm{var}(X|Y)\Big] + \mathrm{var}[\mathbb{E}(X|Y)]$$

Proof.

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Law of Total Variance

Law of total variance

$$\mathrm{var}[X] = \mathbb{E}\Big[\mathrm{var}(X|Y)\Big] + \mathrm{var}[\mathbb{E}(X|Y)]$$

Proof.

$$\mathrm{var}(X|Y) = \mathbb{E}[X^2|Y] - (\mathbb{E}[X|Y])^2$$

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$$(1) + (2) = \mathbb{E}[X^2] + (\mathbb{E}[X])^2 = \mathrm{var}[X]$$

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$$\begin{aligned}
\text{var}(\mathbb{E}[Y|N]) &= \text{var}(N\mu) = \mu^2\text{var}[N] \\
\text{var}[Y|N] &= N\text{var}[X_i] \\
\mathbb{E}[\text{var}(Y|N)] &= \mathbb{E}[N\text{var}[X_i]] = \mathbb{E}[N]\text{var}[X_i]
\end{aligned}$$

Sum of a random number of rvs

- N : number of stores visited (random)
- X_i: money spent in store i, independent of other X_j and N, X_is are identically distributed with $\mathbb{E}[X_i] = \mu$
- $Y = X_1 + X_2 + \dots X_N$. What are $\mathbb{E}[Y]$ and var$[Y]$?
- $\mathbb{E}[Y] = \mathbb{E}[\mathbb{E}[Y|N]] = \mathbb{E}[N\mathbb{E}[X_i]] = \mathbb{E}[N]\mathbb{E}[X_i] = \mu\mathbb{E}[N]$
- $\text{var}[Y] = \mathbb{E}\Big[\text{var}(Y|N)\Big] + \text{var}[\mathbb{E}(Y|N)] = \mathbb{E}[N]\text{var}[X_i] - \mu^2\text{var}[N]$

$$\text{var}(\mathbb{E}[Y|N]) = \text{var}(N\mu) = \mu^2\text{var}[N]$$

$$\text{var}[Y|N] = N\text{var}[X_i]$$

$$\mathbb{E}[\text{var}(Y|N)] = \mathbb{E}[N\text{var}[X_i]] = \mathbb{E}[N]\text{var}[X_i]$$

Questions?

Review Questions

1) What are the key steps to get the derived distributions of $Y = g(X)$ or $Z = g(X, Y)$?

2) How can we compute the distribution of $Z + X + Y$ when X and Y are independent?

3) What are covariance and correlation coefficient? Why do we need them?

4) Please explain the concepts of conditional expectation and conditional variance.