AI and Deep Learning

Logistic Regression & Classification

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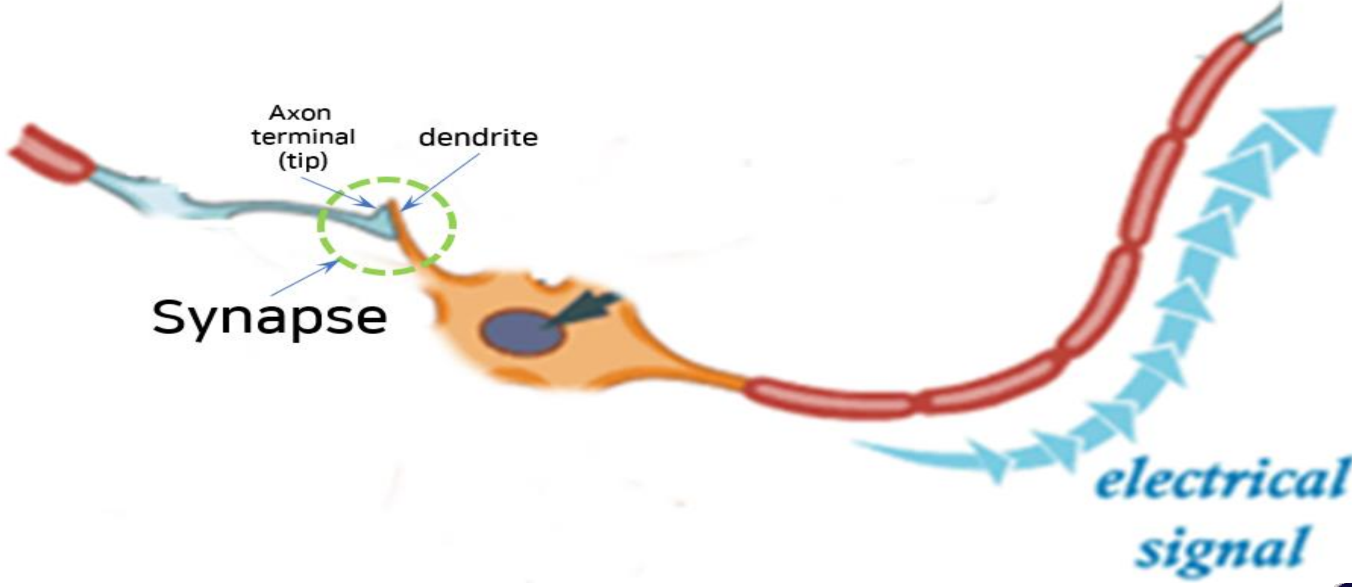
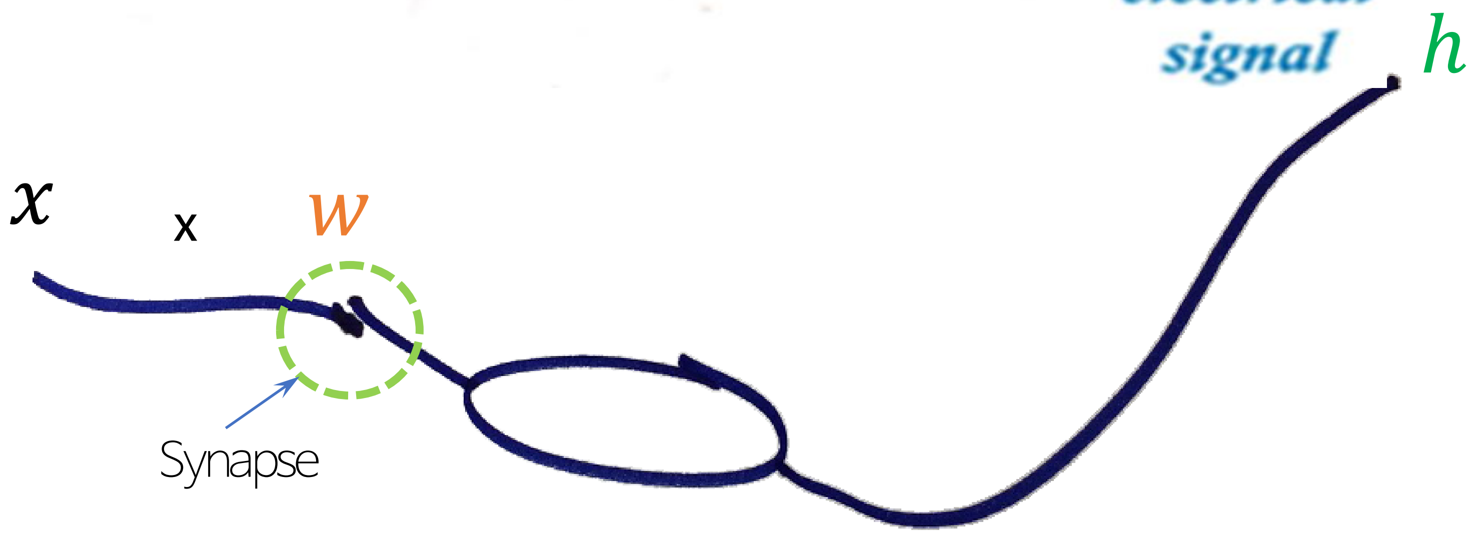
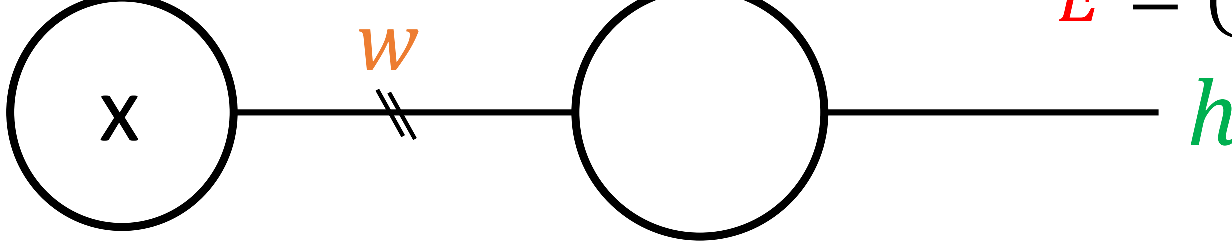
Agenda

- Logistic regression and classification
- New loss/cost function
- Decision boundary
- Implementation using TensorFlow
- Multiple-class problem


Axon terminal (tip)
dendrite
Synapse
electrical signal


h
x
x
w
Synapse

$$E = (h - y)^2$$


w
X
h

Logistic Regression

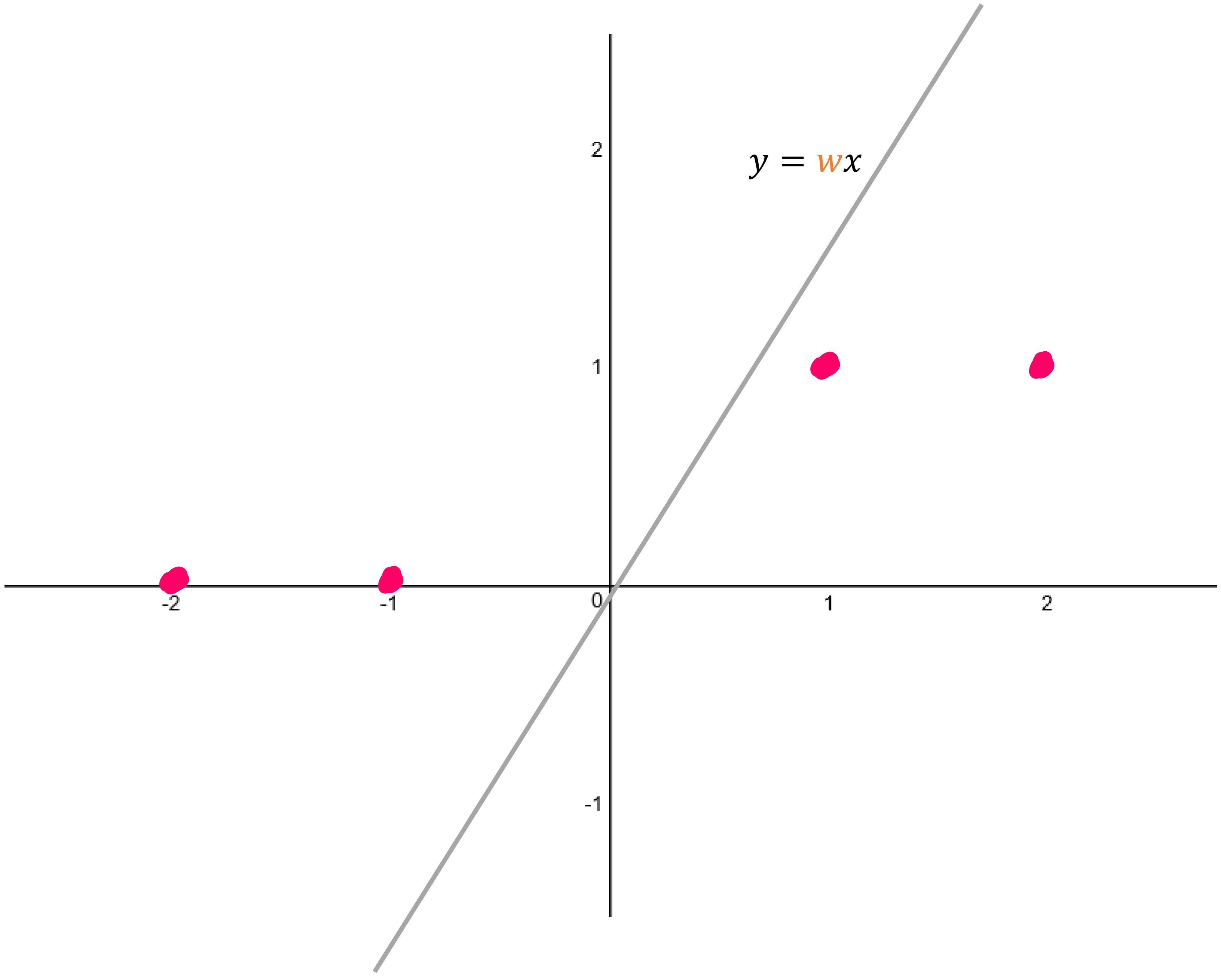
The shape of regression is not linear but logistic.

What does that mean?

Draw $(-2, 0), (-1, 0), (1, 1), (2, 1)$.

$$y = wx$$

$$y = \frac{1}{1 + e^{-wx}}$$



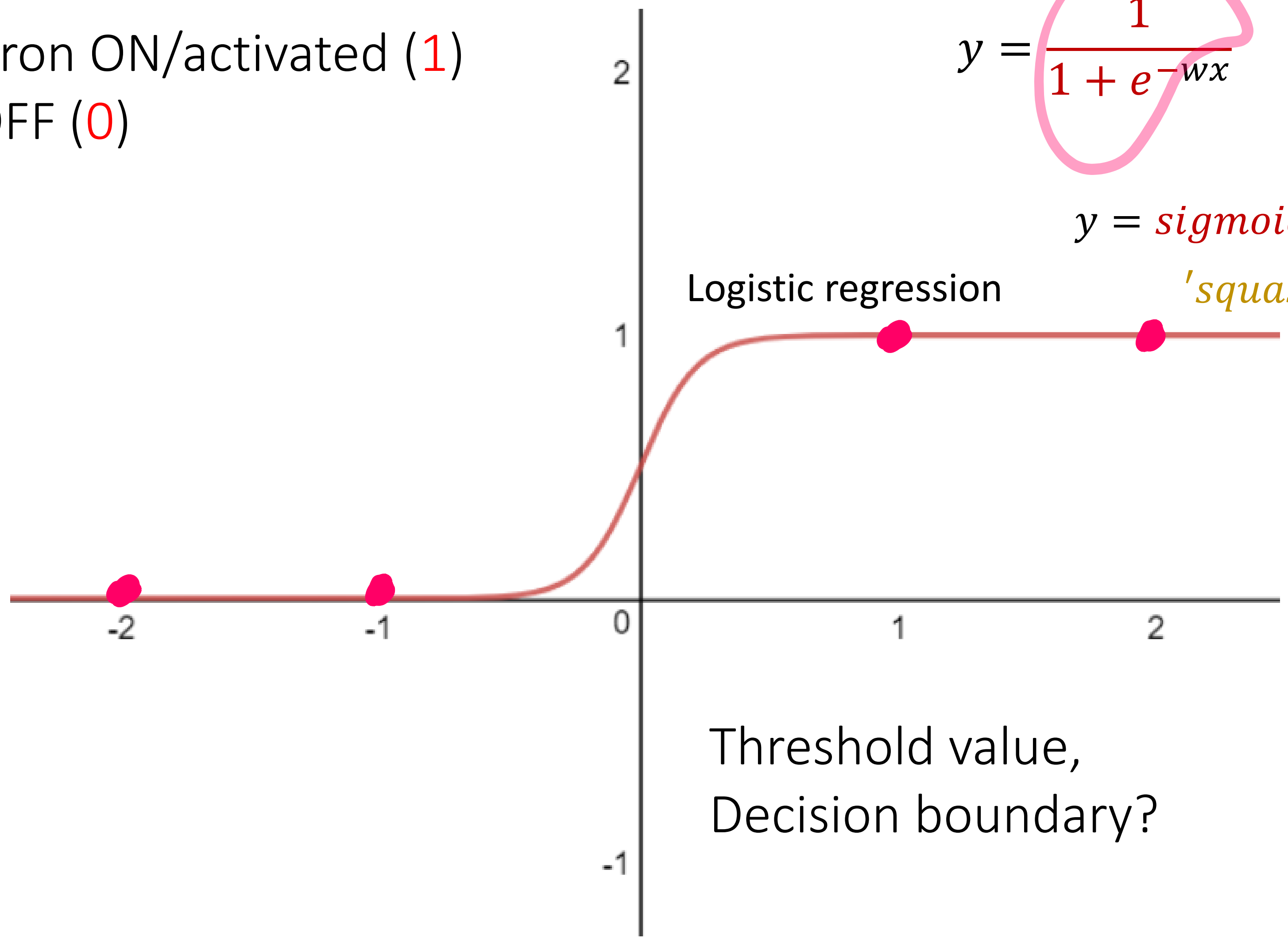
$y = wx$
2
1
0
-1
-2
-1
1
2

Neuron ON/activated (1)
or OFF (0)

$$y = \frac{1}{1 + e^{-wx}}$$

$$y = sigmoid(wx)$$

$'squash'$



Threshold value,
Decision boundary?

$$y = \frac{1}{1 + e^{-wx}} \qquad y = \frac{1}{1 + e^{-w(x-0)}}$$

Logistic function

From Wikipedia, the free encyclopedia

For the recurrence relation, see Logistic map.

A **logistic function** or **logistic curve** is a common "S" shape (sigmoid curve), with equation:

$$f(x) = \frac{L}{1 + e^{-k(x-x_0)}}$$

where

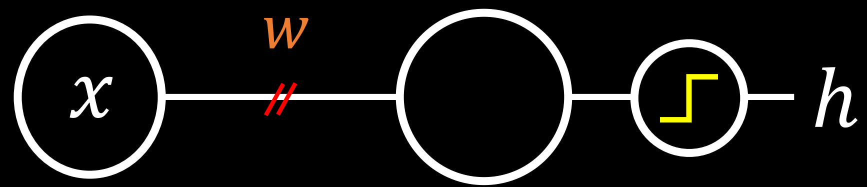
- e = the natural logarithm base (also known as Euler's number),
- x_0 = the x-value of the sigmoid's midpoint,
- L = the curve's maximum value, and
- k = the logistic growth rate or steepness of the curve.[1]

Revisited

Real operation of a neuron

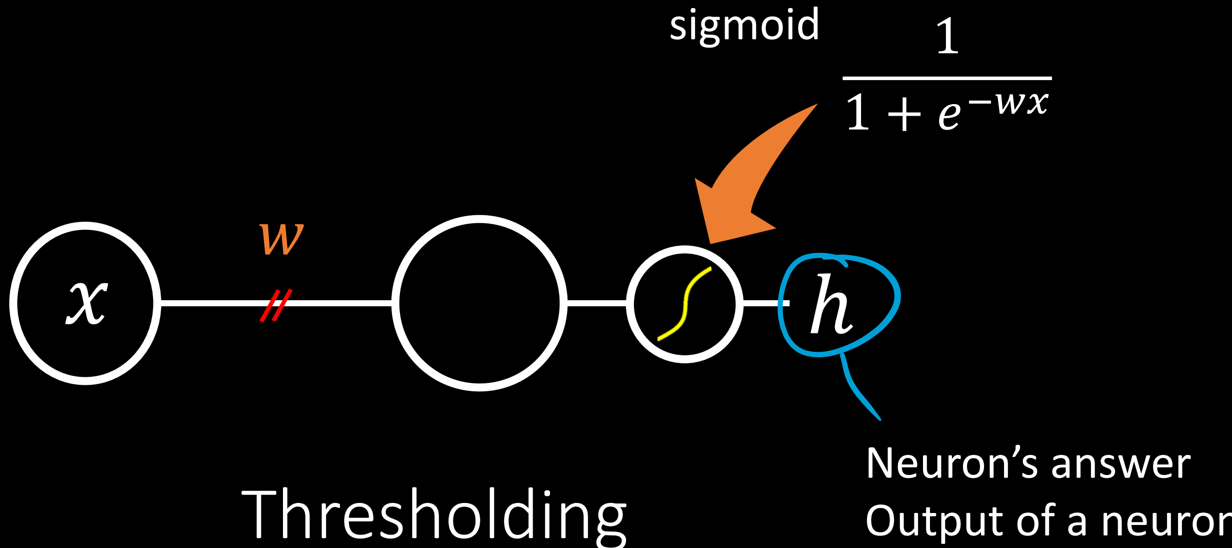
- signal ON if the weighted sum is greater than T
- otherwise signal OFF

Revisited

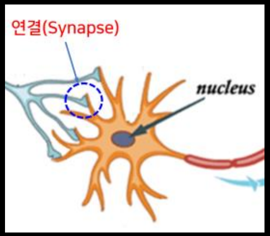


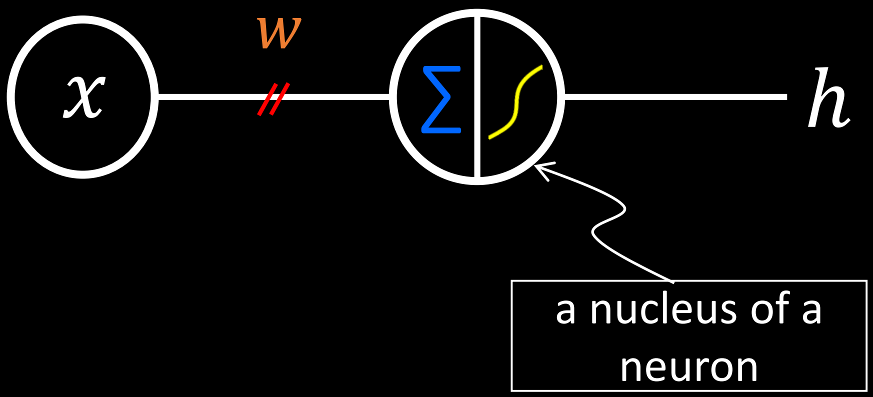
Thresholding

Revisited



Thresholding



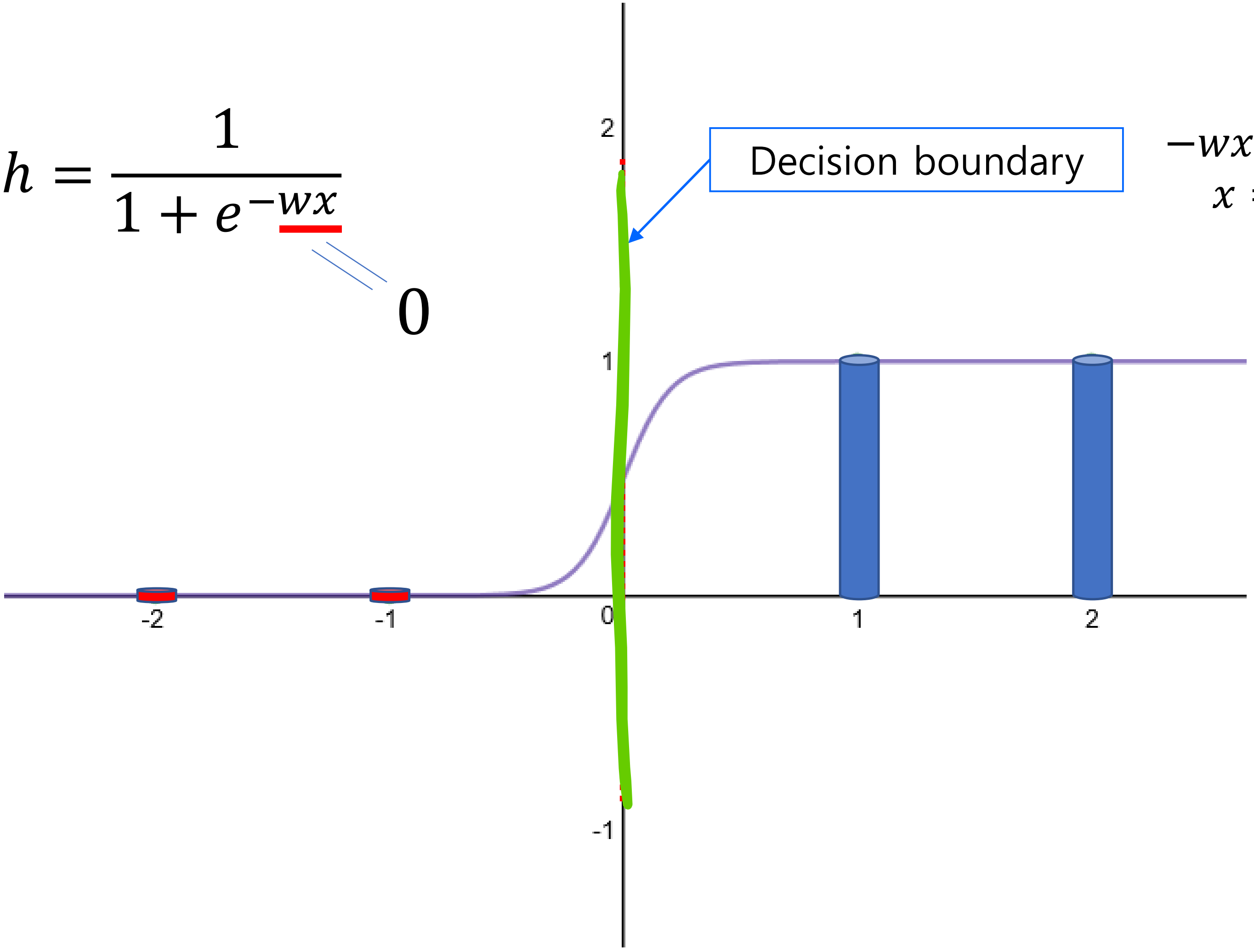


- Activated(1) or not(0) according to the input x
- Let's find the decision boundary to decide 1 or 0.

Decision boundary

$$h = \frac{1}{1 + e^{-\underline{wx}}}$$

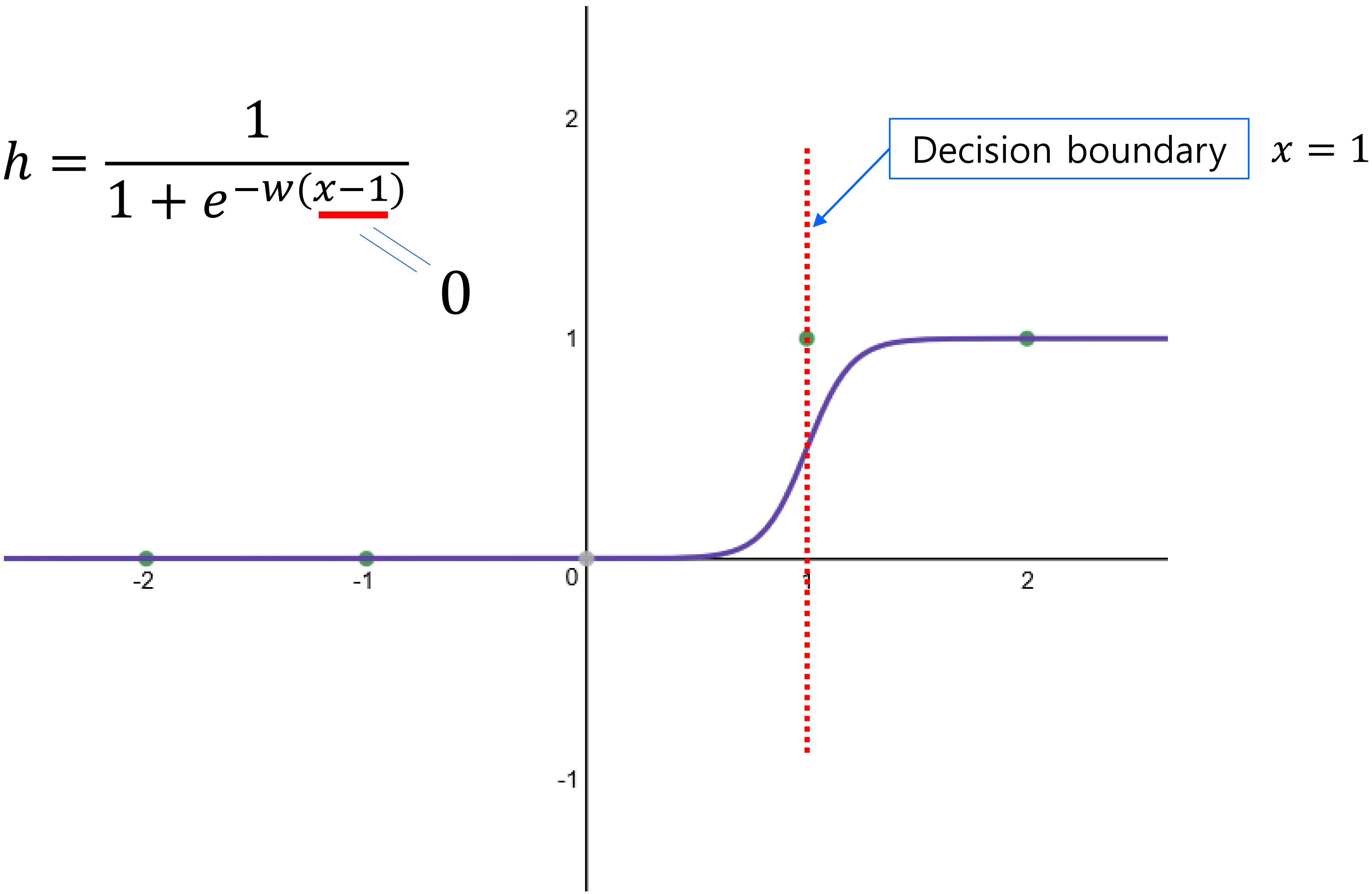
$= 0$



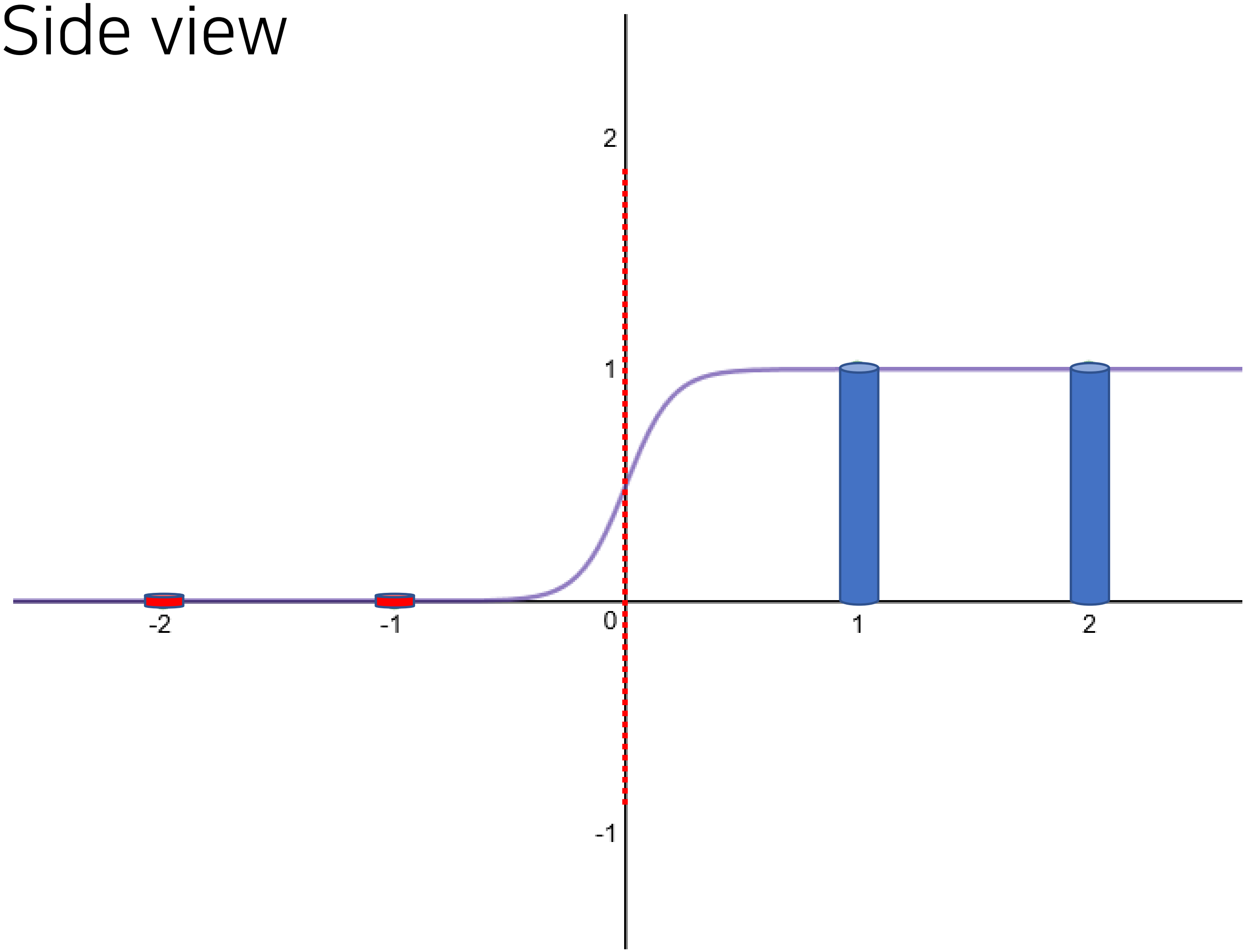
$$-wx = 0$$
$$x = 0$$

Decision boundary

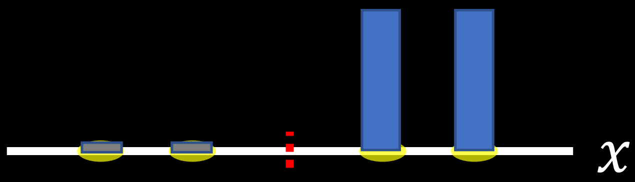
$$h = \frac{1}{1 + e^{-w(\underbrace{x-1}_{0})}}$$



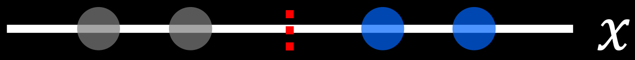
Side view



Decision Boundary



{Side view}



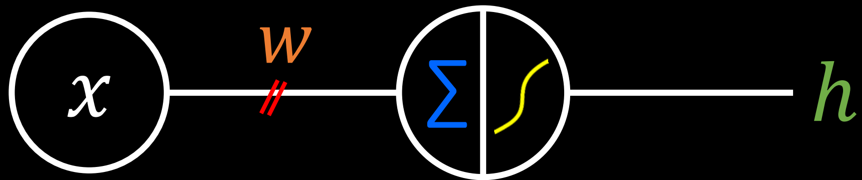
{View from above}

Classification

- Pass(1) or Fail(0)
- Spam(1) or Ham(0)
- Scam(fraud, 1) or not(0)
- Safe(1) or Dangerous(0)
- Intrusion/virus(1) or not(0)
- Cancer(1) or not(0)
- Binary classification -> Multiple classification

Guess the decision boundary from the below figure.

x
-2
-1
1
2

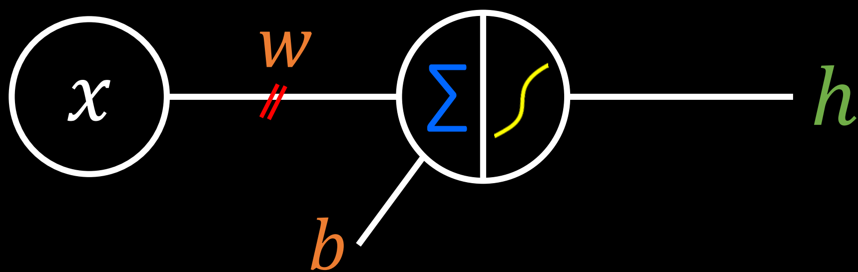


y
0
0
1
1

$$h = \begin{cases} 1 & if\ wx \geq 0 \\ 0 & otherwise \end{cases}$$

Guess the decision boundary from the below figure.

x	y
-2	**0**
-1	0
1	1
2	1



$$h = \begin{cases} 1 & if\ wx + b \geq 0 \\ 0 & otherwise \end{cases}$$

Hypothesis

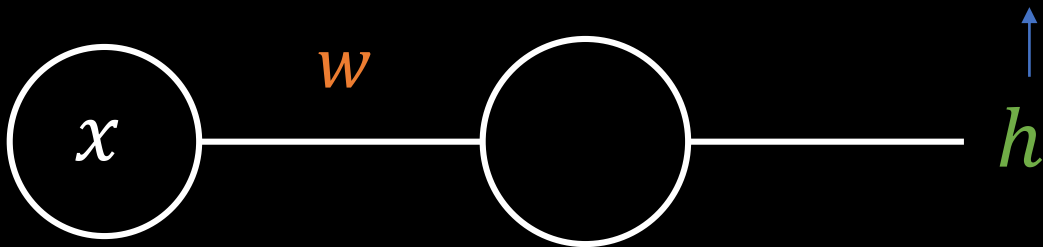
- What is hypothesis? The answer of a neuron
- Find decision boundary from the equation.

$$h = \frac{1}{1 + e^{-wx}}$$

$$h = \frac{1}{1 + e^{-(wx+b)}}$$

Cost/Error Function

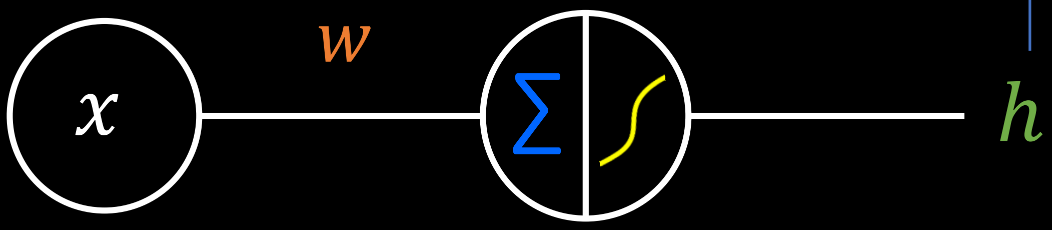
$$E = (h - y)^2$$



Does MSE work?

Cost/Error Function

$$E = (h - y)^2$$



Does MSE work?

Draw $(-2, 0), (-1, 0), (1, 1), (2, 1)$.

$$h = wx$$

$$h = \frac{1}{1 + e^{-wx}}$$

Draw $(1, 1)$ only.

$$E = \left(\frac{1}{1 + e^{-w \cdot 1}} - 1 \right)^2$$

$$(w, E)$$

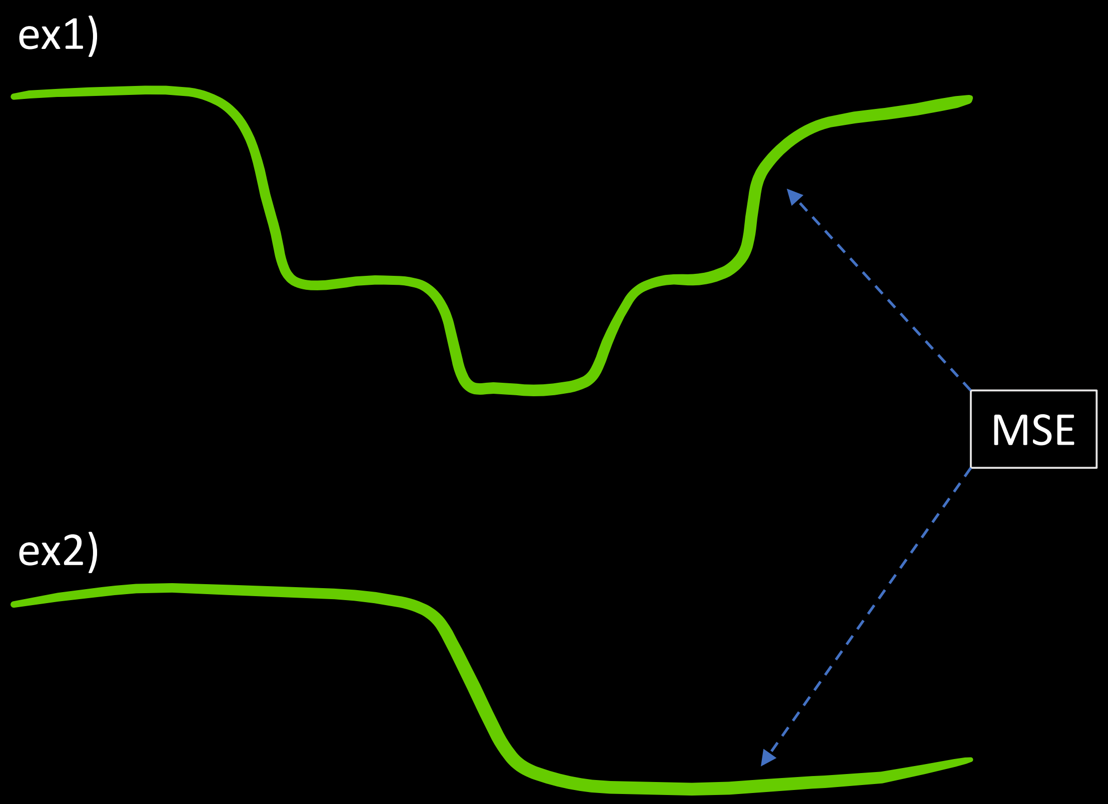
Draw two points: $(-1,0),(1,1),(-3,0),(3,1)$.

$$E=\left(\frac{1}{1+e^{-w(-1)}}-0\right)^2+\left(\frac{1}{1+e^{-w(1)}}-1\right)^2+\left(\frac{1}{1+e^{-w(-3)}}-0\right)^2+\left(\frac{1}{1+e^{-w(3)}}-1\right)^2$$

Add bias b.

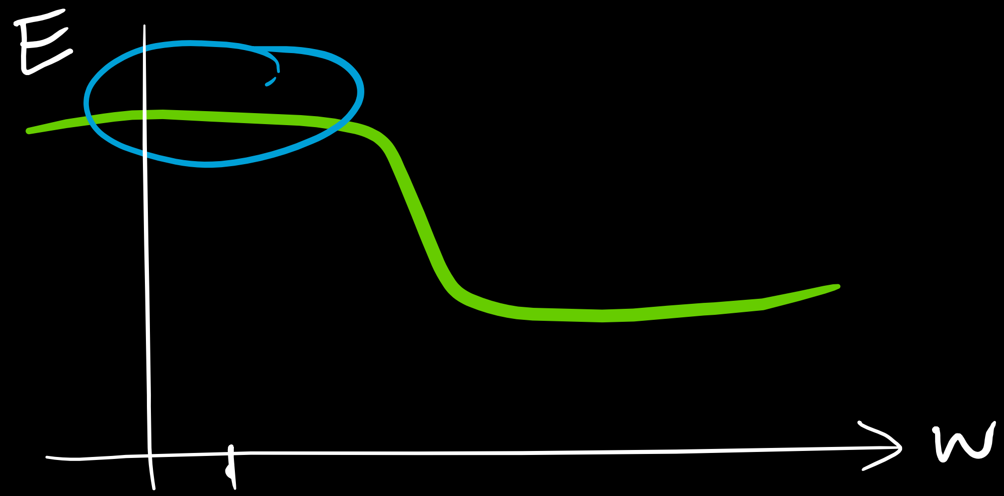
$$\left(w,\frac{E}{2}\right)$$

Cost/Error Function when we use MSE.



What problem in the error function?

No gradient decent
in some parts

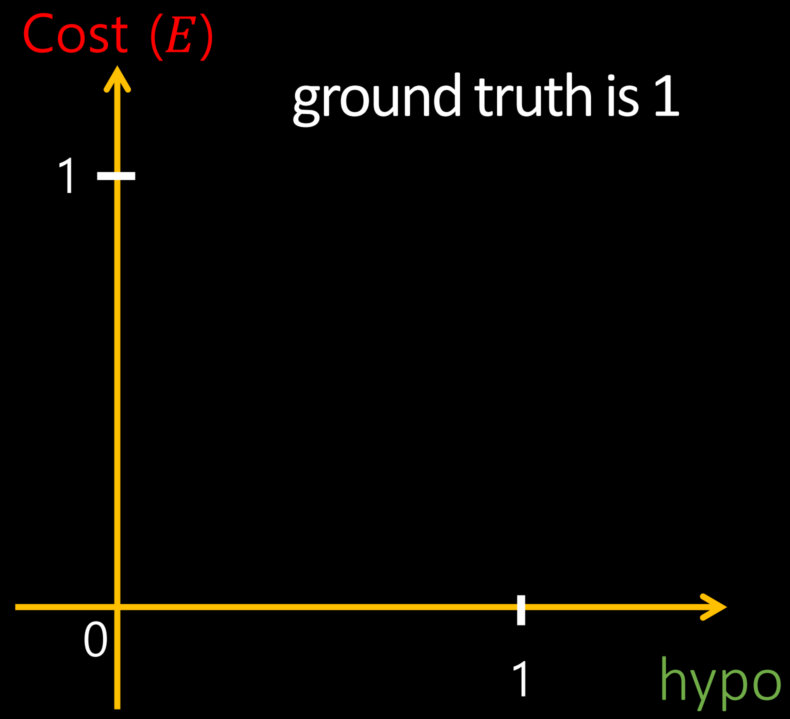


New Cost/Error Function

When ground truth is 1

– if hypo is equal to 1, then error = 0

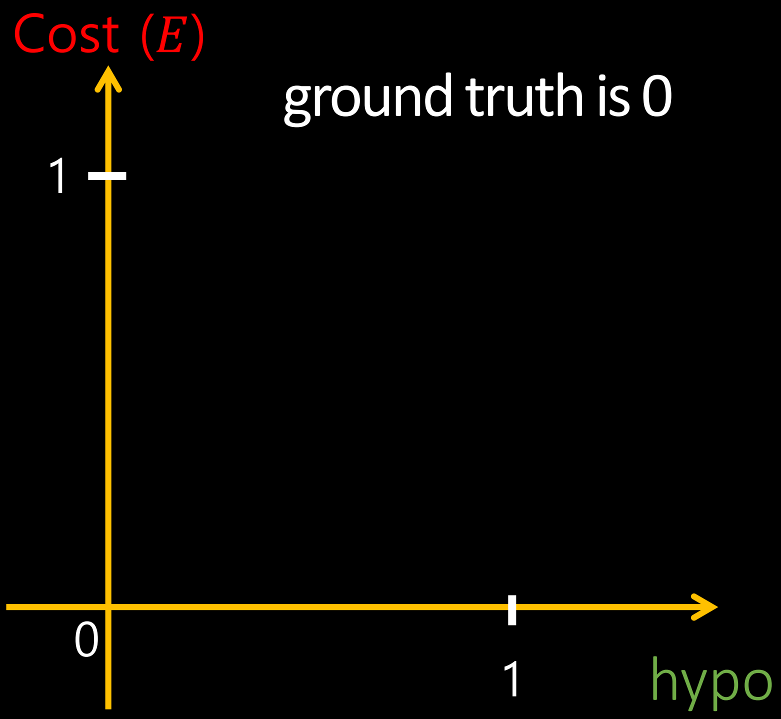
– if hypo is equal to 0 then error = ∞



New Cost/Error Function

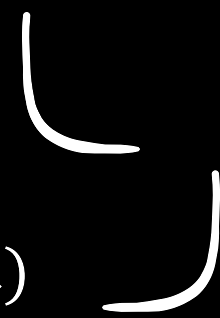
When ground truth is 0

- if hypo is equal to 0, then error = 0
- if hypo is equal to 1 then error = ∞



$$E = -\log(h)$$

$$E = -\log(1-h)$$



$$E = -\log\left(\frac{1}{1+e^{-wx}}\right)$$

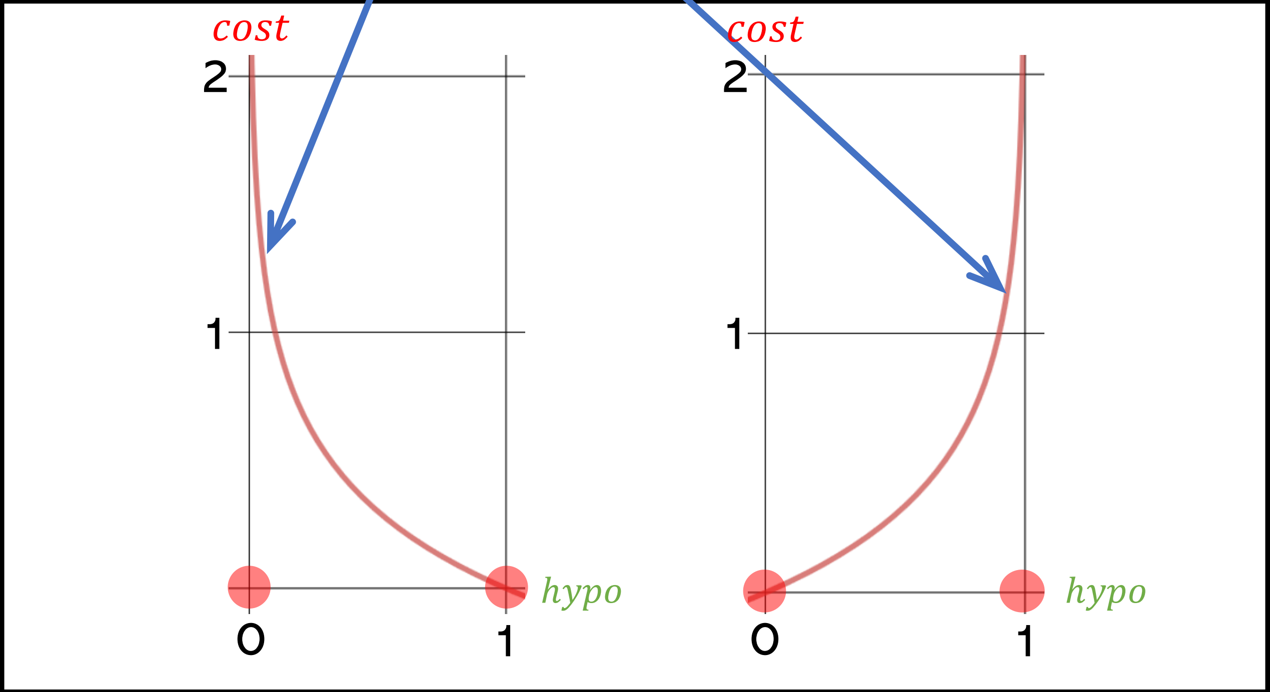
$$E = -\log\left(1 - \frac{1}{1+e^{-wx}}\right)$$

New Cost/Error Function

Prediction by a neuron

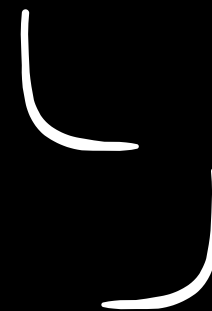
Correct answer

$$E = \begin{cases} -\log(h) & : y = 1 \\ -\log(1-h) & : y = 0 \end{cases}$$

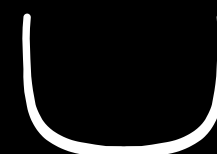


New Cost/Error Function

$$E = \begin{cases} -\log(wx) & : y = 1 \\ -\log(1 - wx) & : y = 0 \end{cases}$$



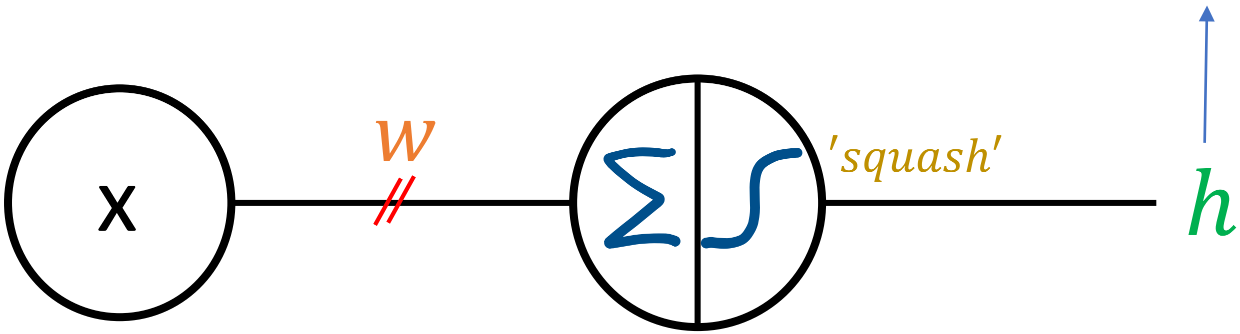
$$E = -y\log(wx) - (1 - \mathrm{y})\log(1 - wx)$$

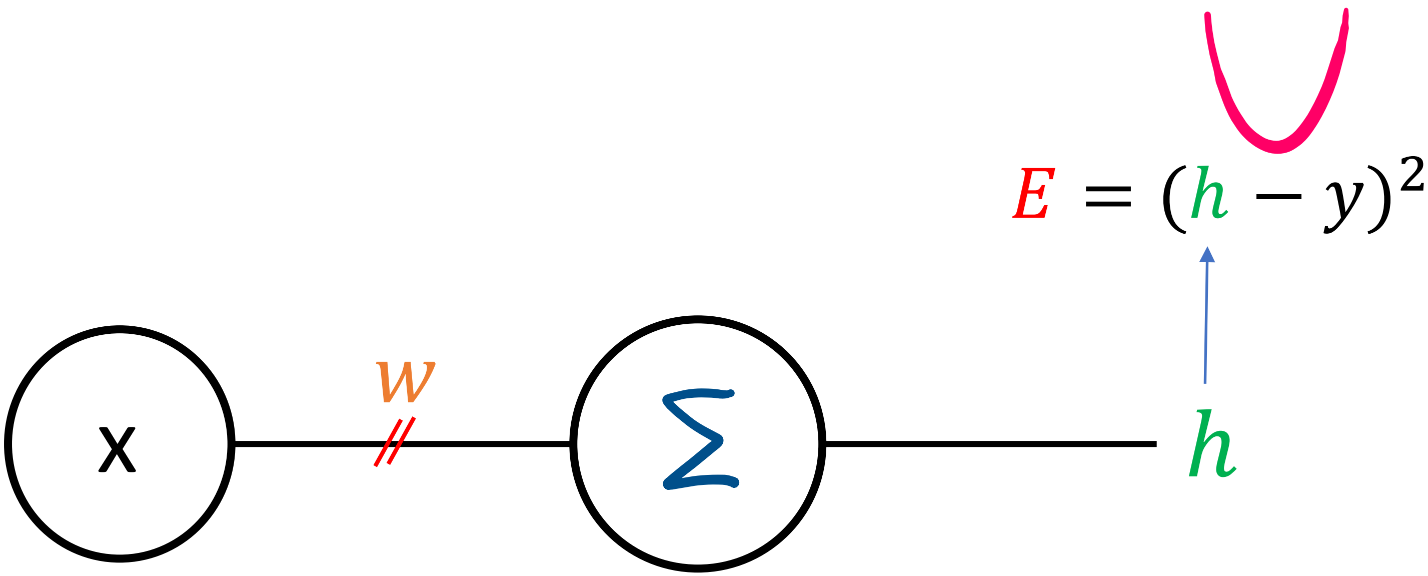


$$E = -(y\log(wx) + (1 - \mathrm{y})\log(1 - wx))$$

$$w = w - \alpha \cdot \frac{\partial E}{\partial w}$$

$$E = -(y\log(h) + (1-y)\log(1-h))$$





$E = (h - y)^2$
x
w
Σ
h

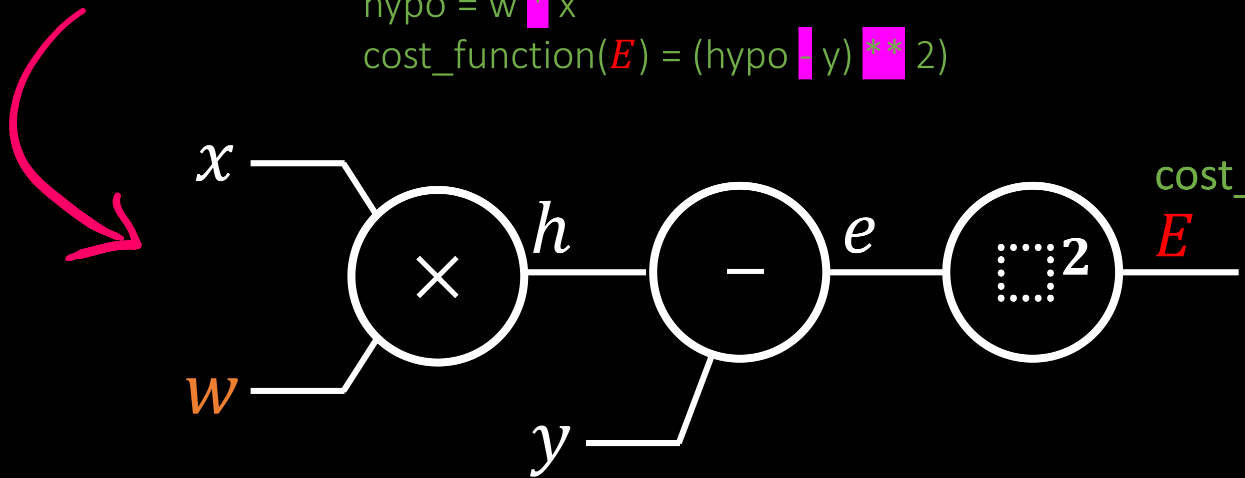
Computational graph

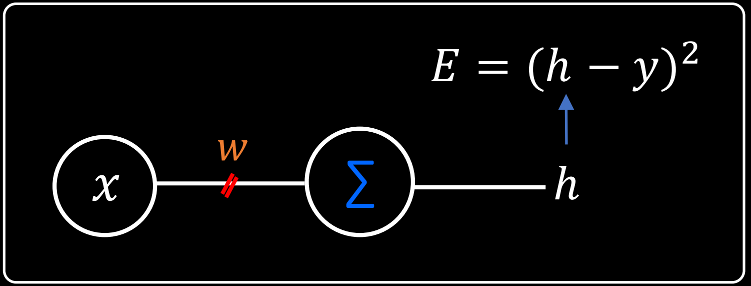
for the new cost function

Computational Graph

$$E = (wx - y)^2$$

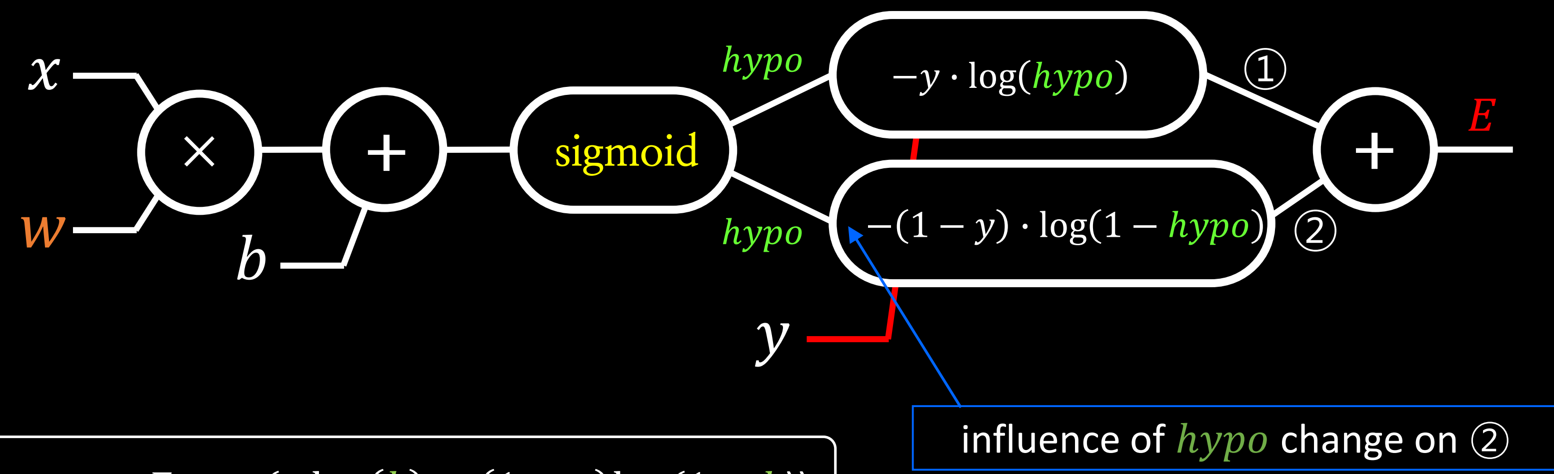
hypo = w * x
cost_function(E) = (hypo - y) ** 2)

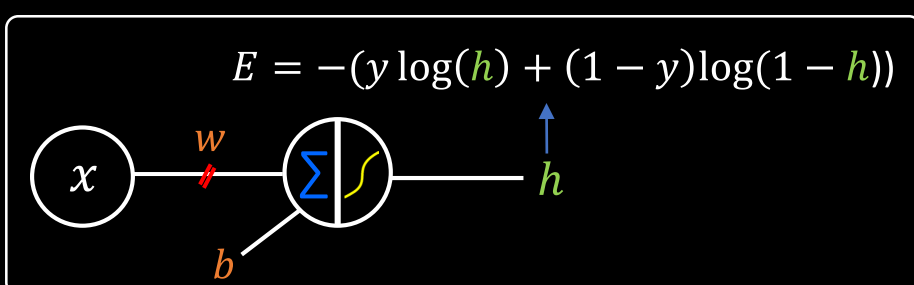




Computational Graph

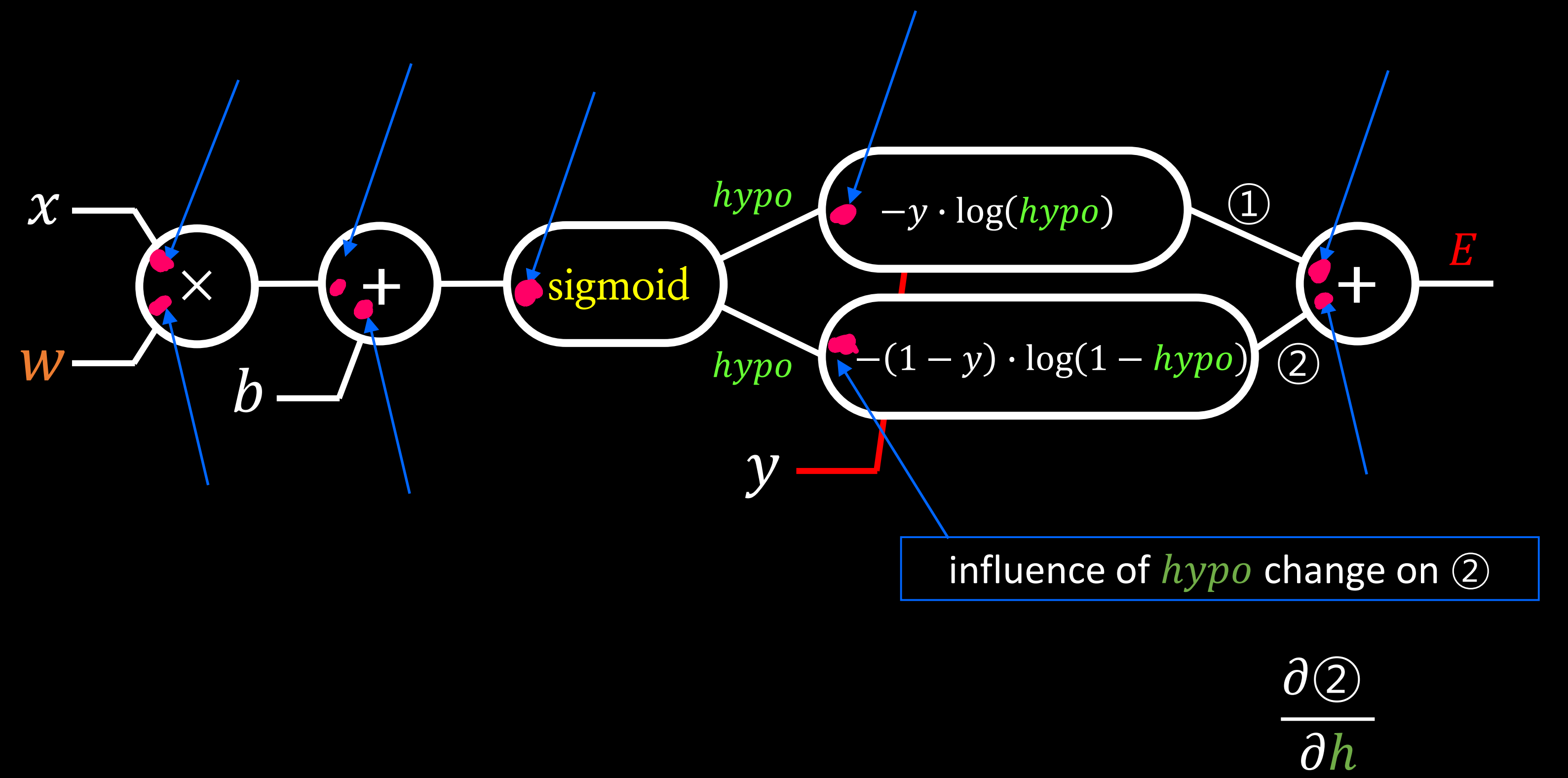
$$E = \underbrace{-y \cdot \log(hypo)}_{①} - \underbrace{(1-y) \cdot \log(1-hypo)}_{②}$$





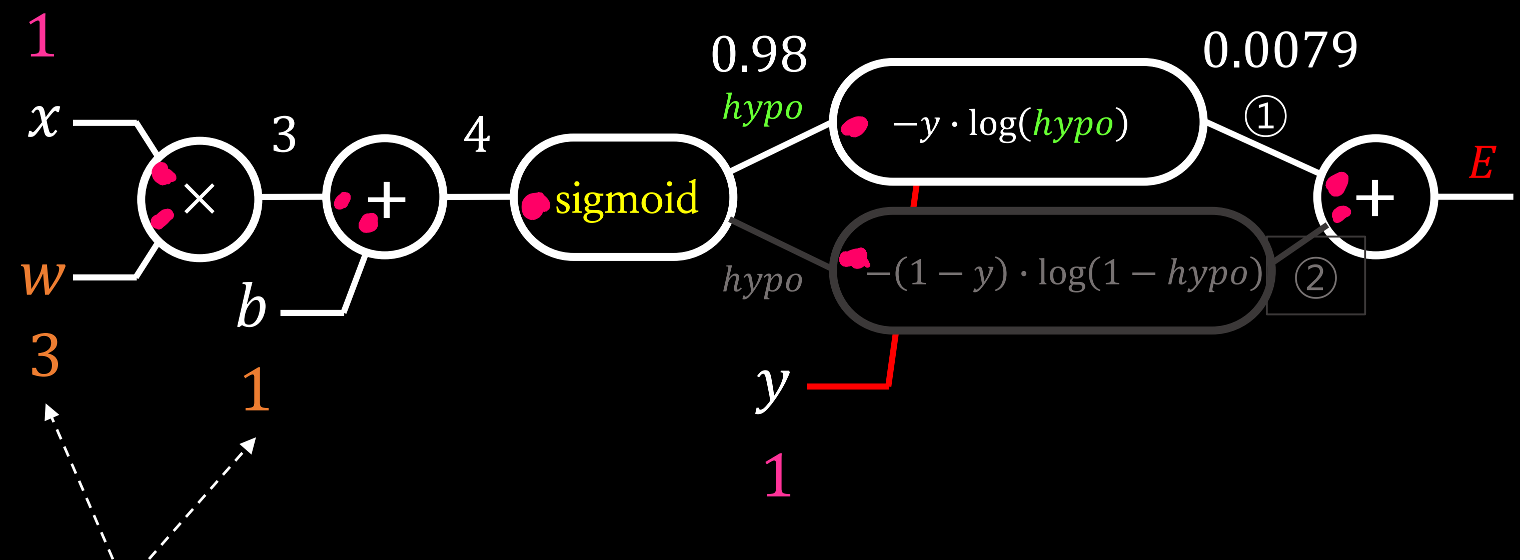
$$\frac{\partial ②}{\partial h}$$

Local Gradients



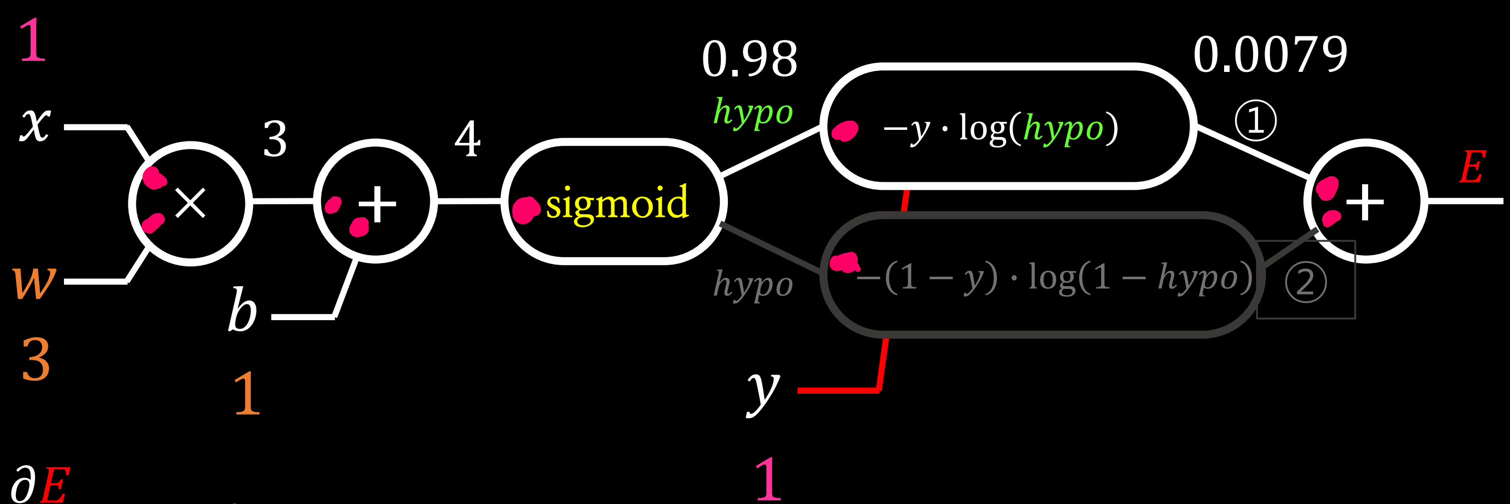
Forward propagation

$$(x, y) \longrightarrow (1, 1)$$



randomly initialized ⟶ should be optimized

Back-propagation



$$\frac{\partial E}{\partial w}$$

$$\frac{\partial E}{\partial b}$$

$$w = w - \propto \cdot \frac{\partial E}{\partial w}$$

$$b = b - \propto \cdot \frac{\partial E}{\partial b}$$

Parameters(w, b) tuning for what?

decision boundary

$$wx + b = 0$$

for better decision boundary

Lab 11.py

Classification of an input as 1 or 0

$$cost = -(y\log(H(X)) + (1-y)\log(1-H(X)))$$

```
x_data = [-2., -1, 1, 2]
y_data = [0., 0, 1, 1]
```

```
#------- a neuron
w = tf.Variable(tf.random_normal([1]))
hypo = tf.sigmoid(x_data * w)
```

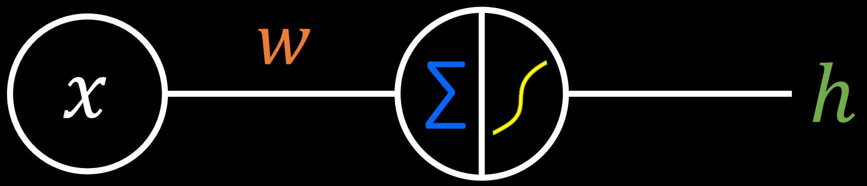
```
#----- learning
cost = -tf.reduce_mean(y_data * tf.log(hypo) +
        tf.subtract(1., y_data) * tf.log(tf.subtract(1., hypo)))

train = tf.train.GradientDescentOptimizer(learning_rate=0.01).minimize(cost)

sess = tf.Session()
sess.run(tf.global_variables_initializer())

for step in range(5001):
    sess.run(train)
```

```
#----- testing(classification)
predicted = tf.cast(hypo > 0.5, dtype=tf.float32)
p = sess.run(predicted)
print("Predicted: ", p)
```

Lab 12.py
With a bias

1-Input Neuron

Guess a decision boundary.

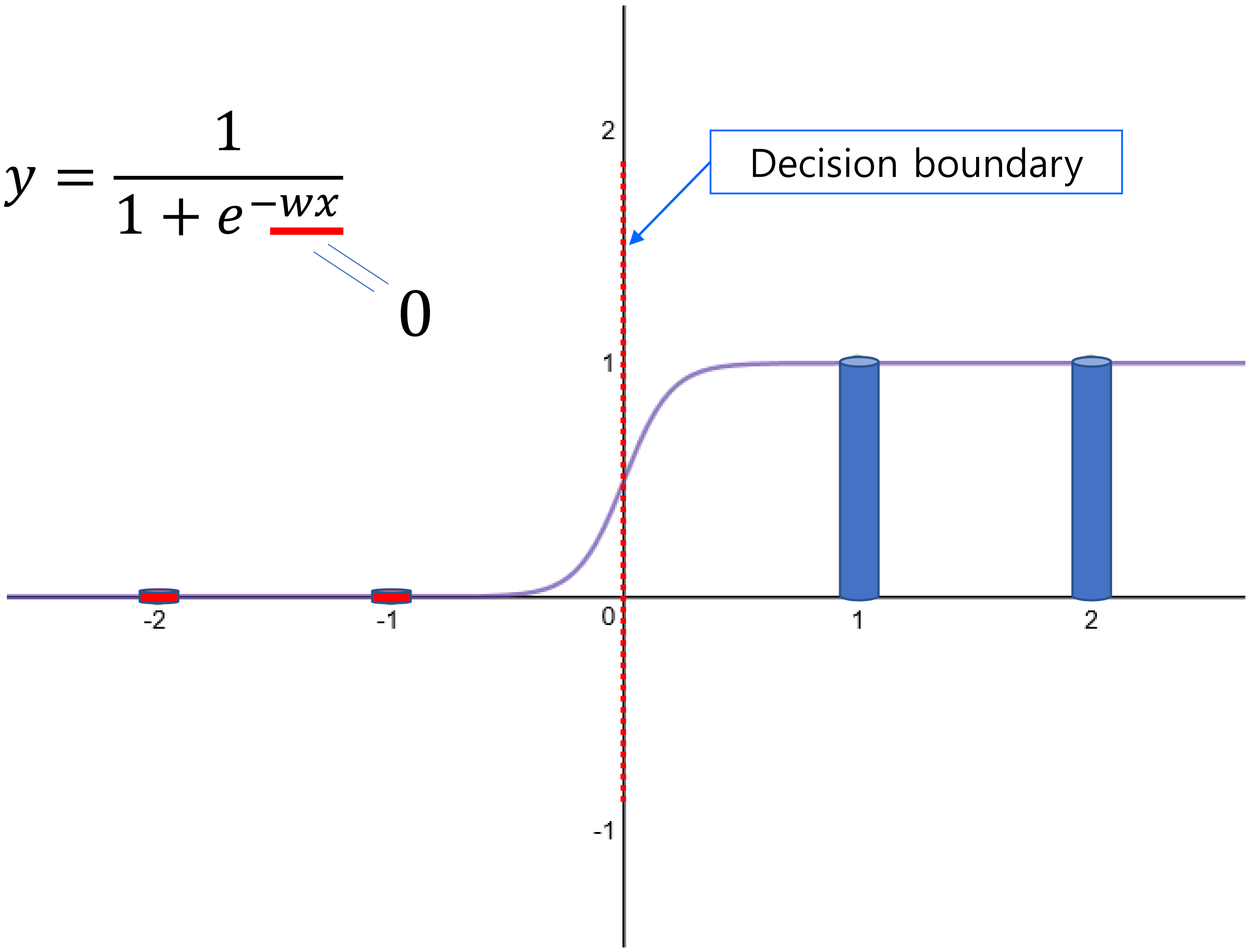


$$h = \frac{1}{1 + e^{-(wx)}}$$

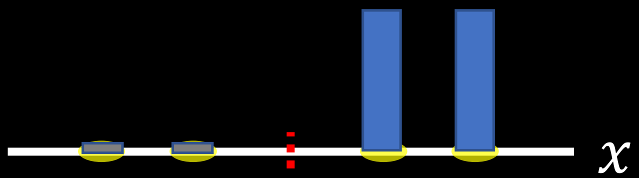
1-Input Neuron

$$y = \frac{1}{1 + e^{-\underline{wx}}}$$

0



1-Input Neuron

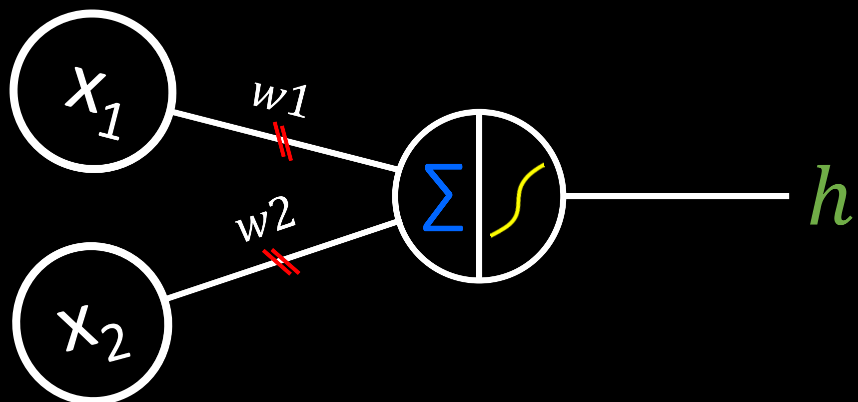


{Side view}

{View from above}

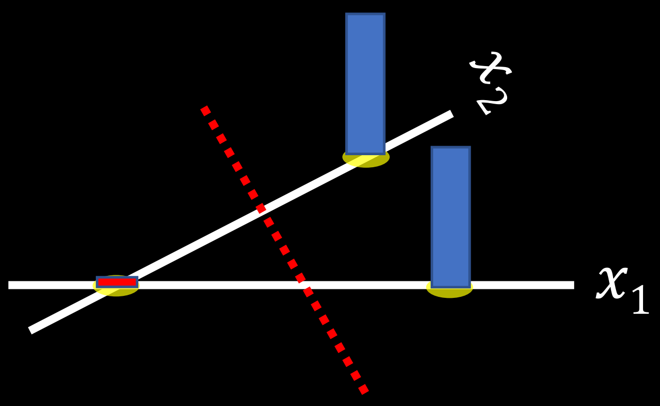
2-Input Neuron

Guess a decision boundary.

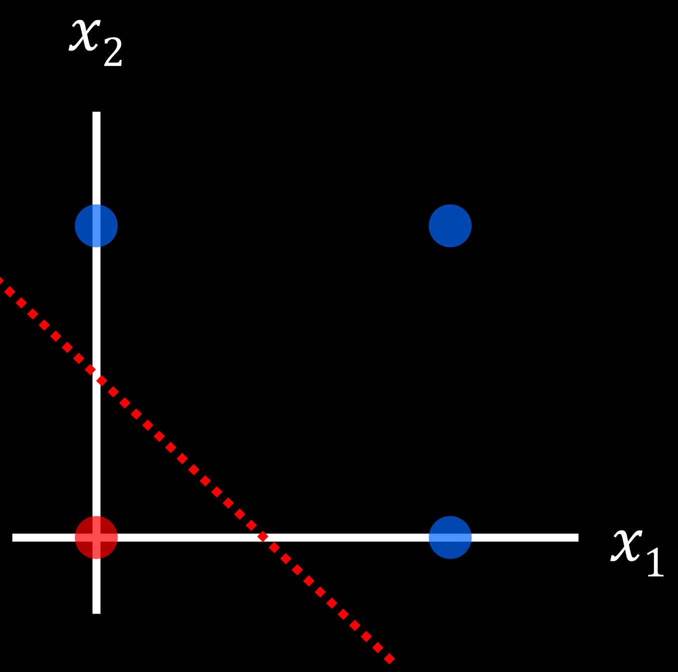


$$h = \frac{1}{1 + e^{-(w_1 x_1 + w_2 x_2)}}$$

2-Input Neuron



{Side view}



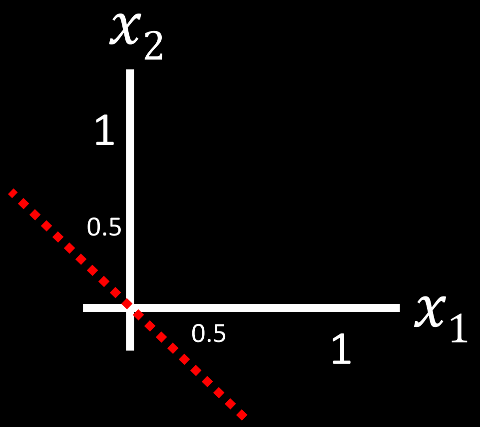
{View from above}

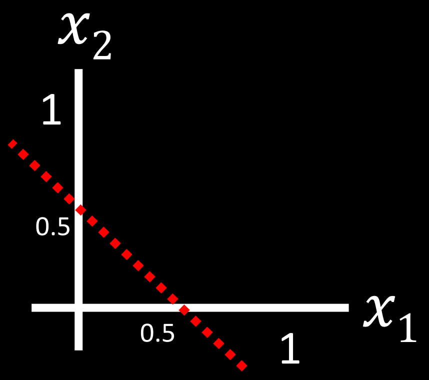
2-Input Neuron

$$x_1 + x_2 = 0$$

$$x_1 + x_2 - 0.5 = 0$$

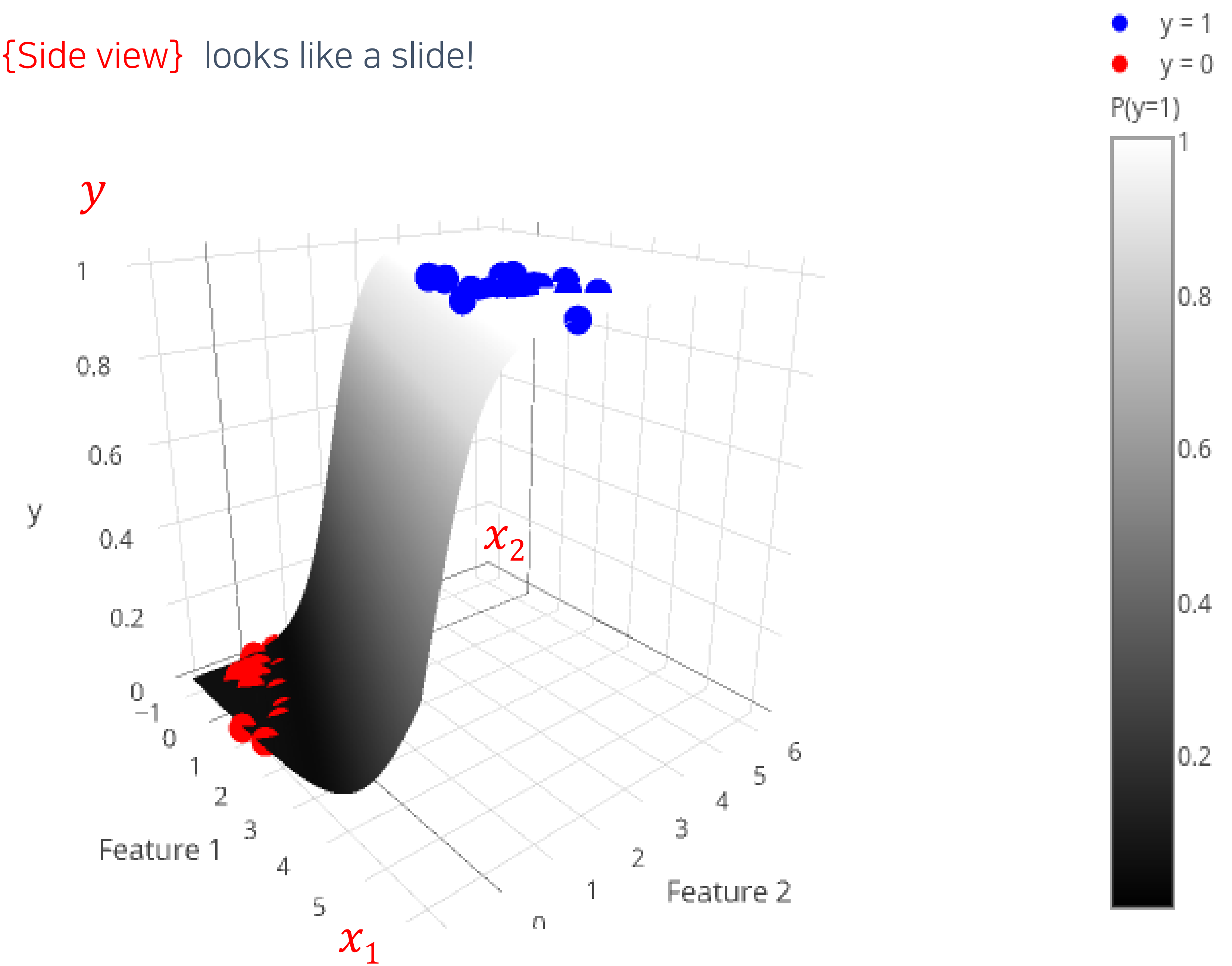
View from above





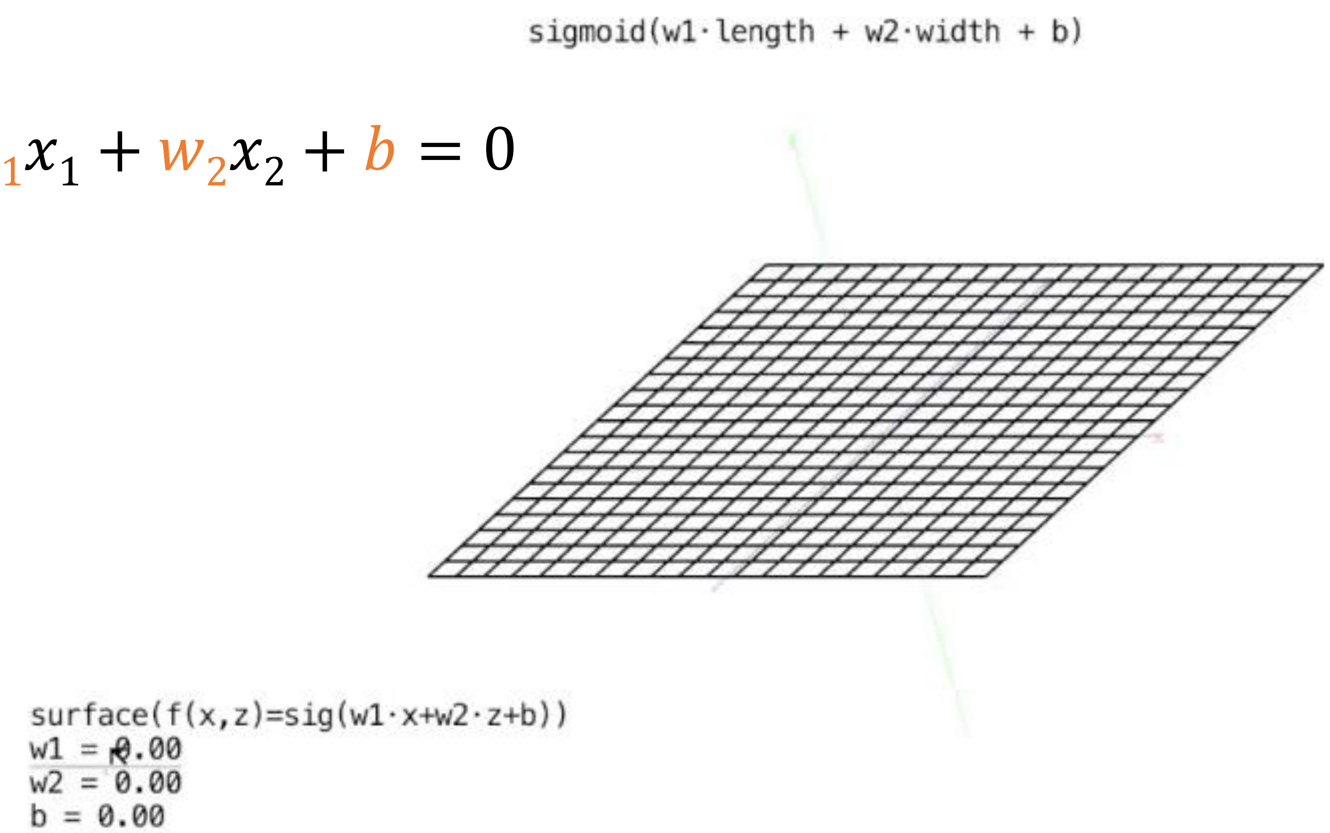
Logistic Regression: 2 Features (Inputs)

{Side view} looks like a slide!



2-Input Neuron

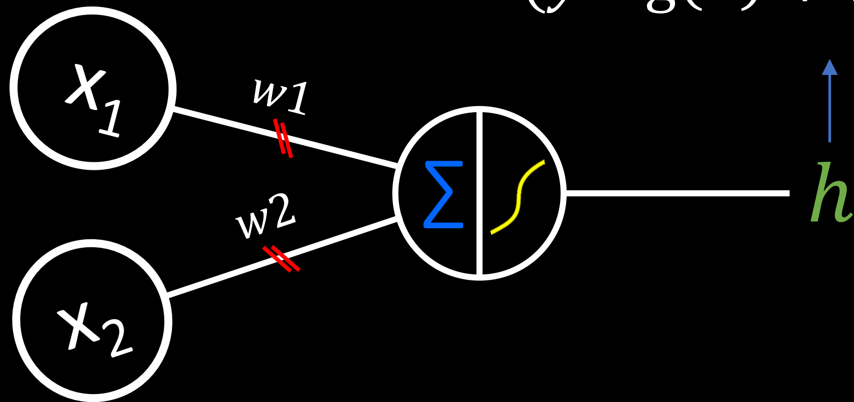
$$w_1 x_1 + w_2 x_2 + b = 0$$



Lab 13.py

Implementation of OR gate with a neuron(a decision boundary)

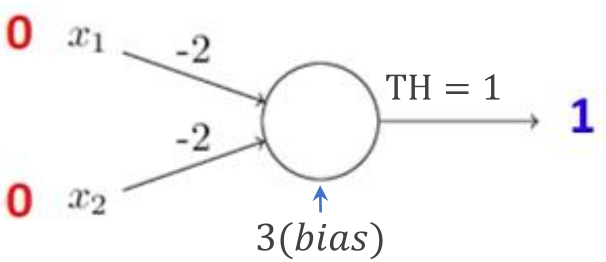
$$E = -(y\log(h) + (1-y)\log(1-h))$$



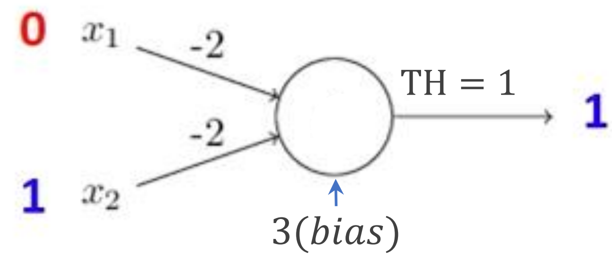
x_1	x_2	$AND(h)$
0	0	0
0	1	0
1	0	0
1	1	1

NAND

- NAND gates are functionally complete.
- We can build any logical functions out of them.



$(-2)*0 + (-2)*0+3=3$

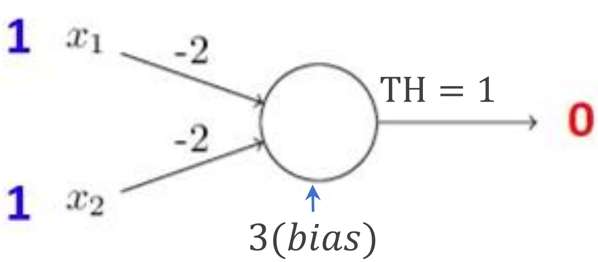


$(-2)*0 + (-2)*1+3=1$

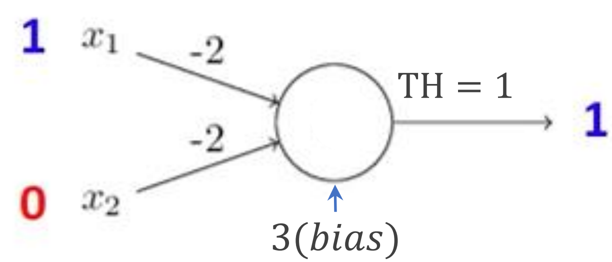
NAND

Truth Table

Input A	Input B	Output Q
0	0	1
0	1	1
1	0	1
1	1	0

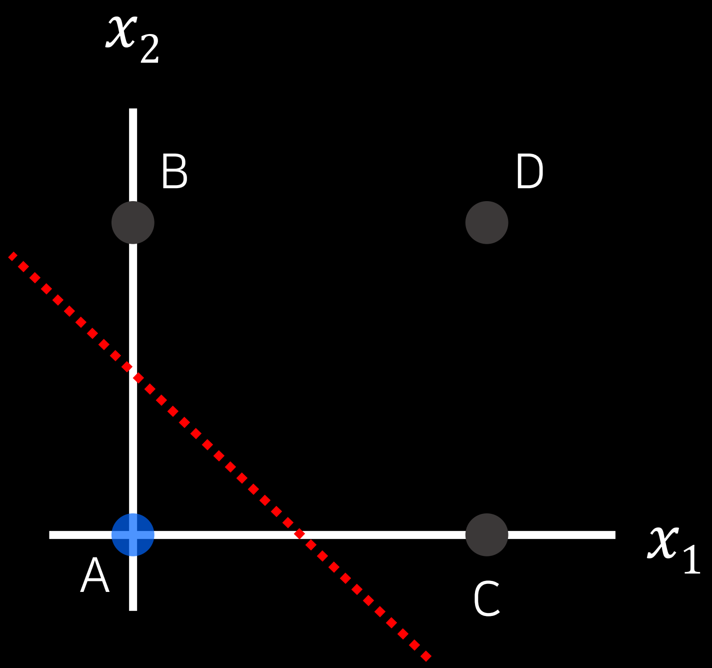


$(-2)*1 + (-2)*1+3=-1$



$(-2)*1 + (-2)*0+3=1$

Decision boundary by a neuron



View from above

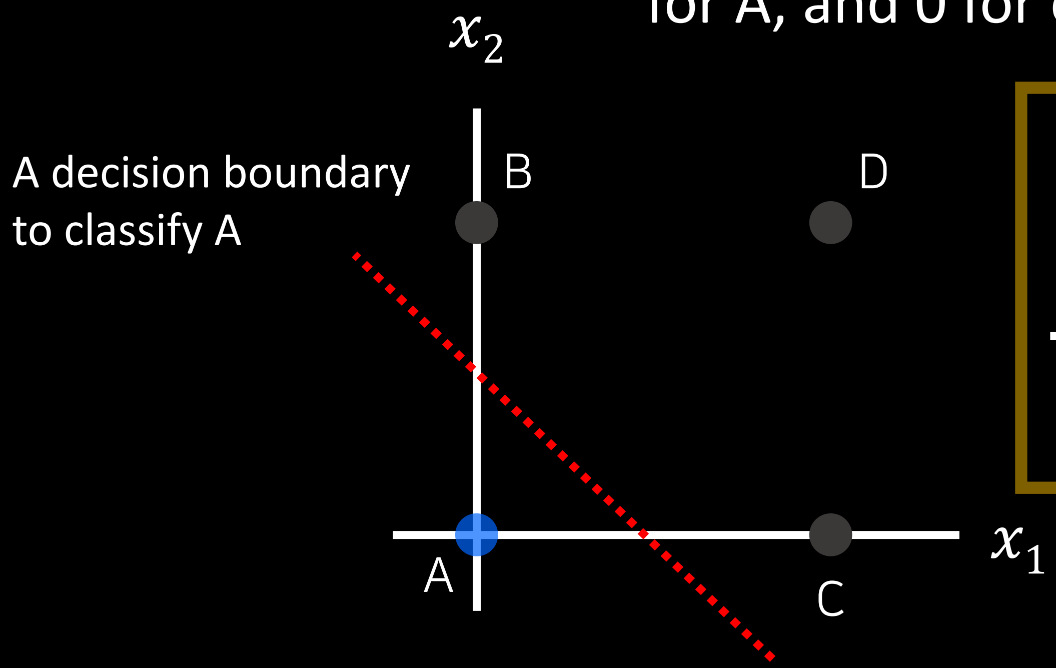
Decision boundary by a neuron

- A neuron, only 1 decision boundary
- A decision boundary yielding 2 classes (1 or 0)
- How to solve multiple classes more than 2

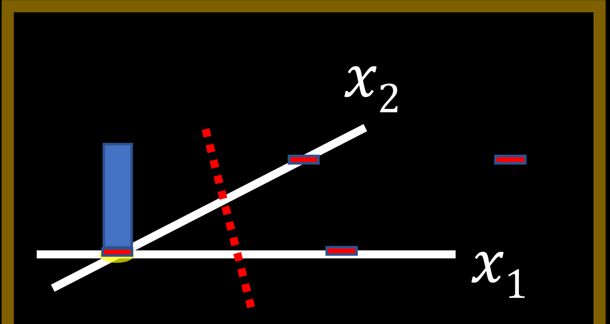
4-Class(A, B, C, D) Classification Problem

Neuron #1

The output of a neuron is 1 for A, and 0 for other cases.



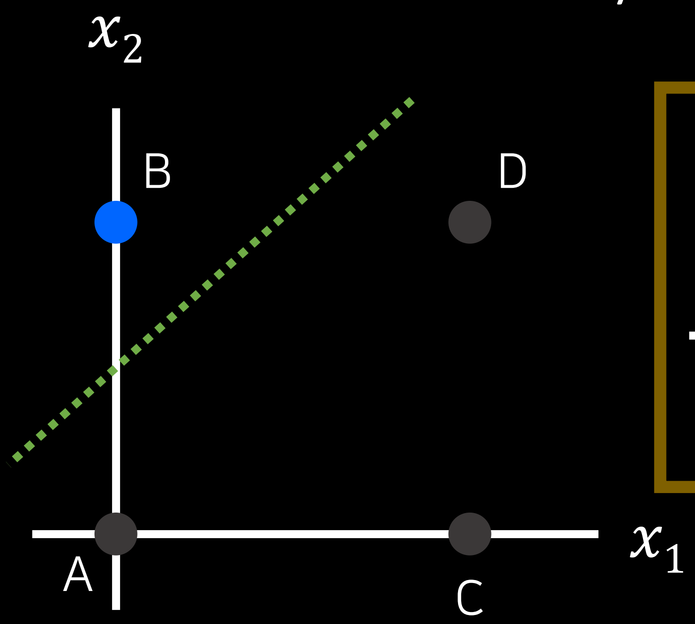
View from above



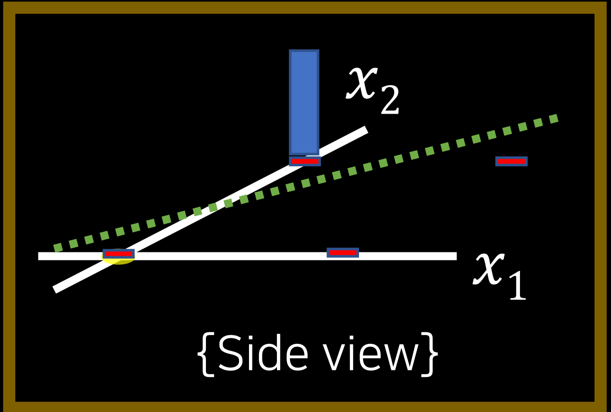
{Side view}

Neuron #2

2nd neuron for 2nd decision boundary to classify B



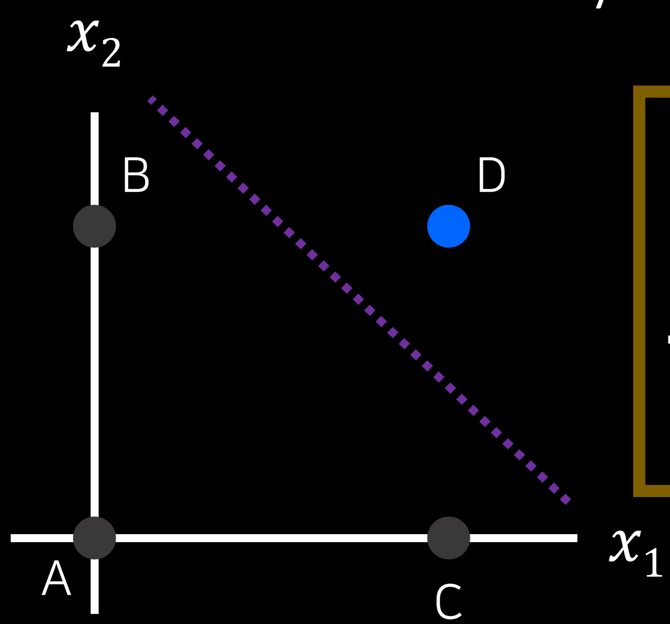
View from above



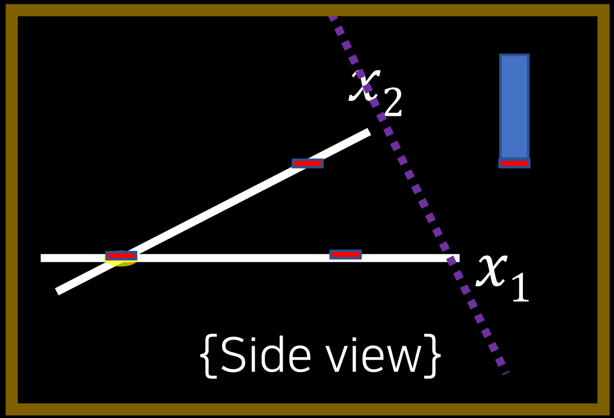
{Side view}

Neuron #3

3rd neuron for 3rd decision boundary to classify D

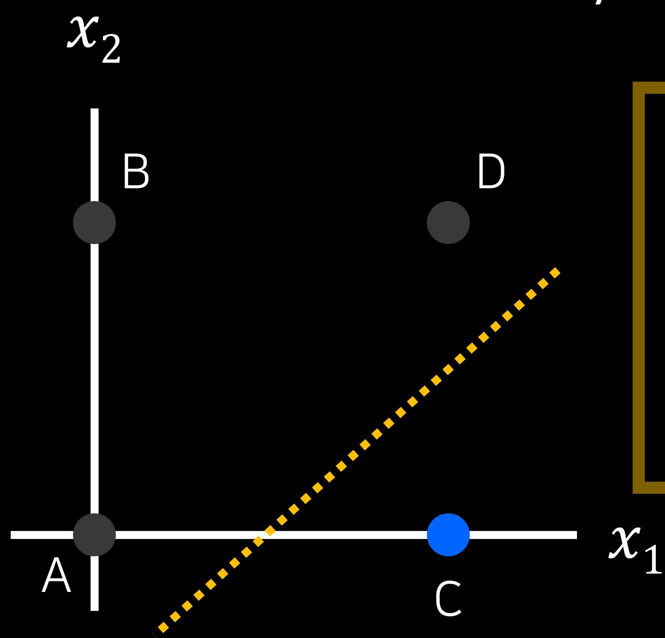


View from above

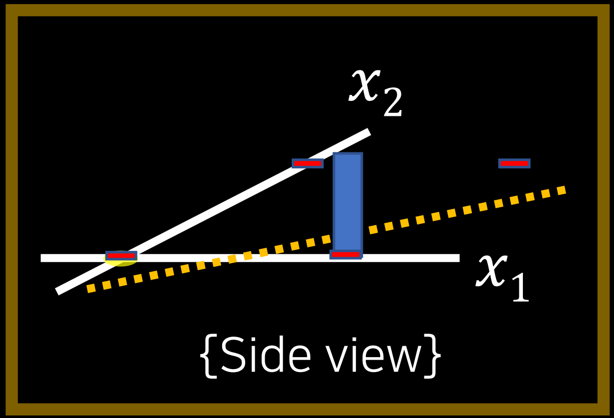


Neuron #4

4^{th} neuron for 4^{th} decision boundary to classify C

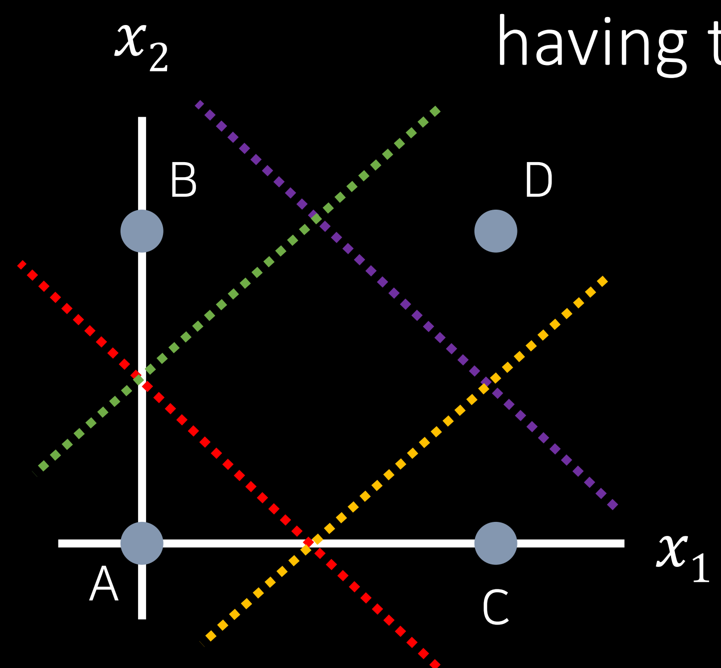


View from above



4 Neurons

4 neurons for
4 decision boundaries
having the same inputs

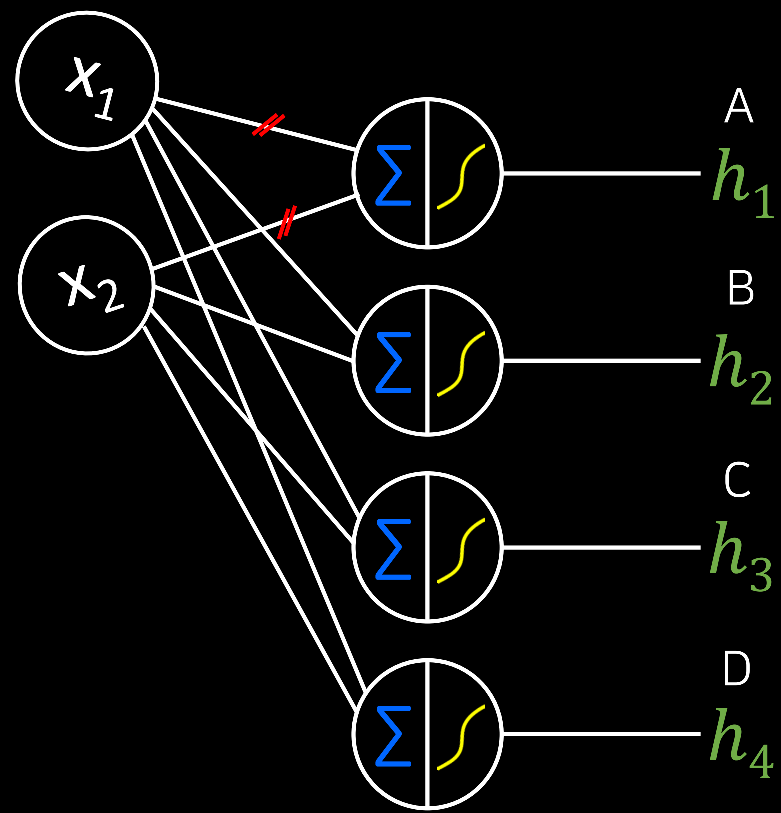


View from above

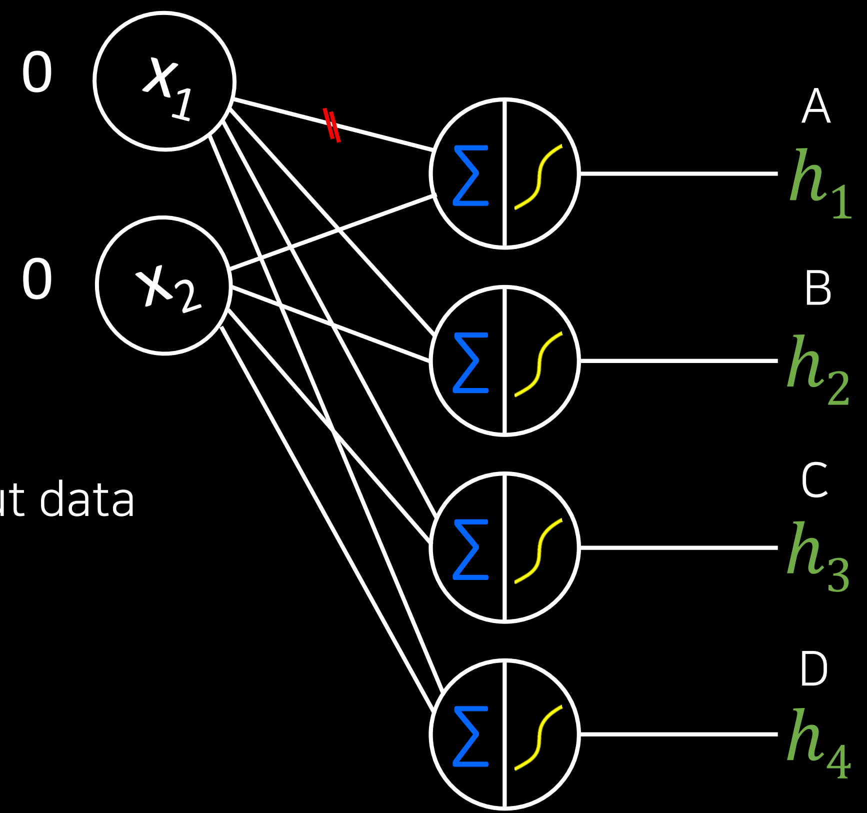
4 Neurons

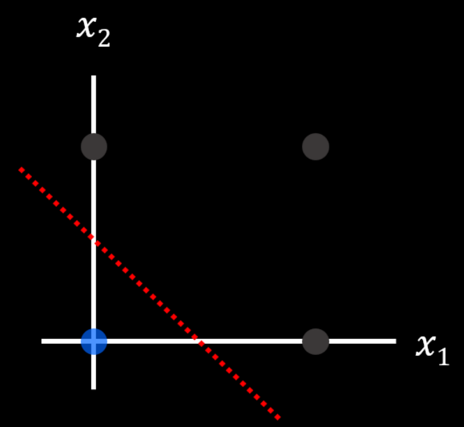
Matrix notation

$$(x_1, x_2)\begin{pmatrix} w_{11}, w_{21}, w_{31}, w_{41} \\ w_{12}, w_{22}, w_{32}, w_{42} \end{pmatrix} \longrightarrow (h_1, h_2, h_3, h_4)$$

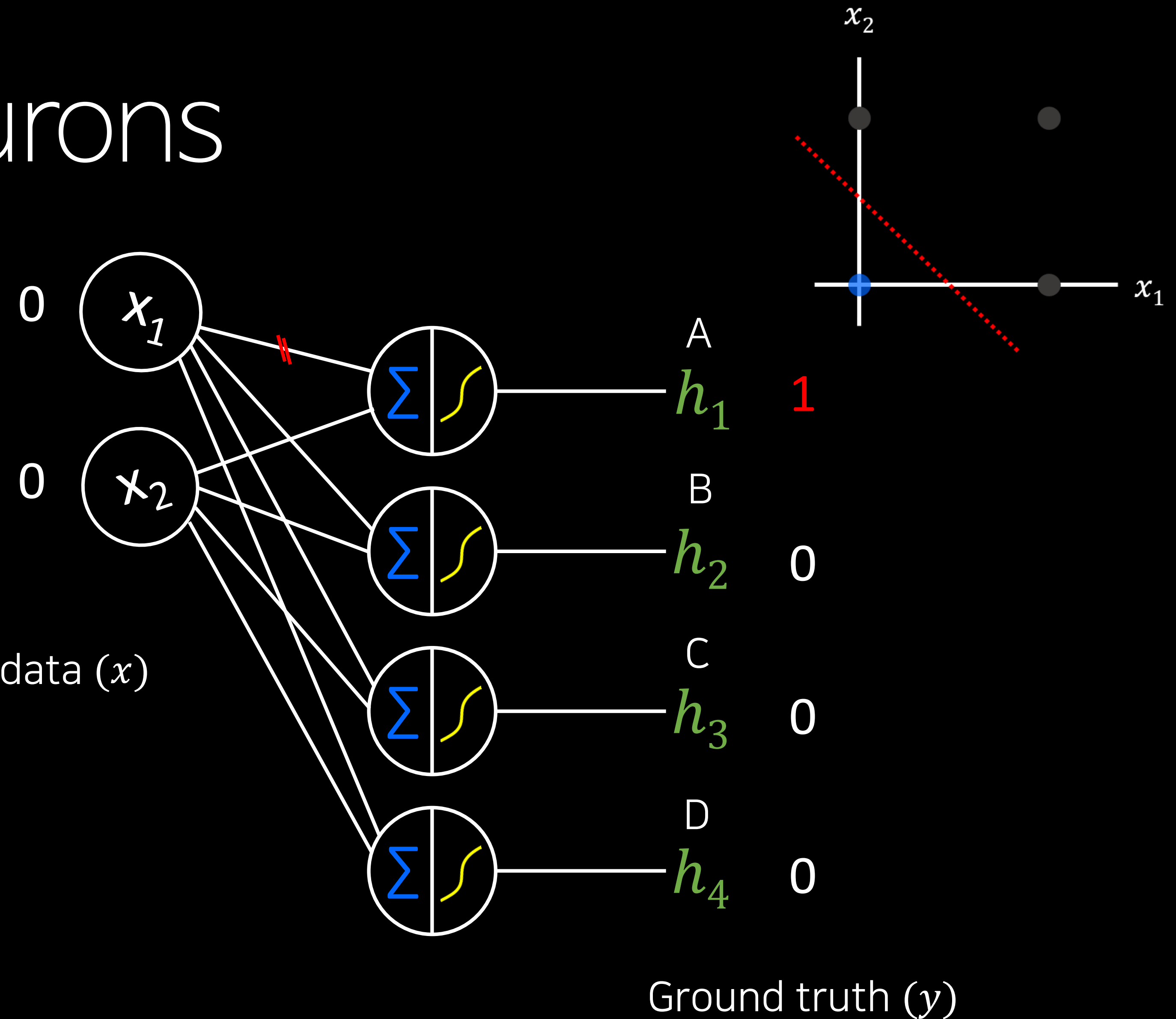


4 Neurons

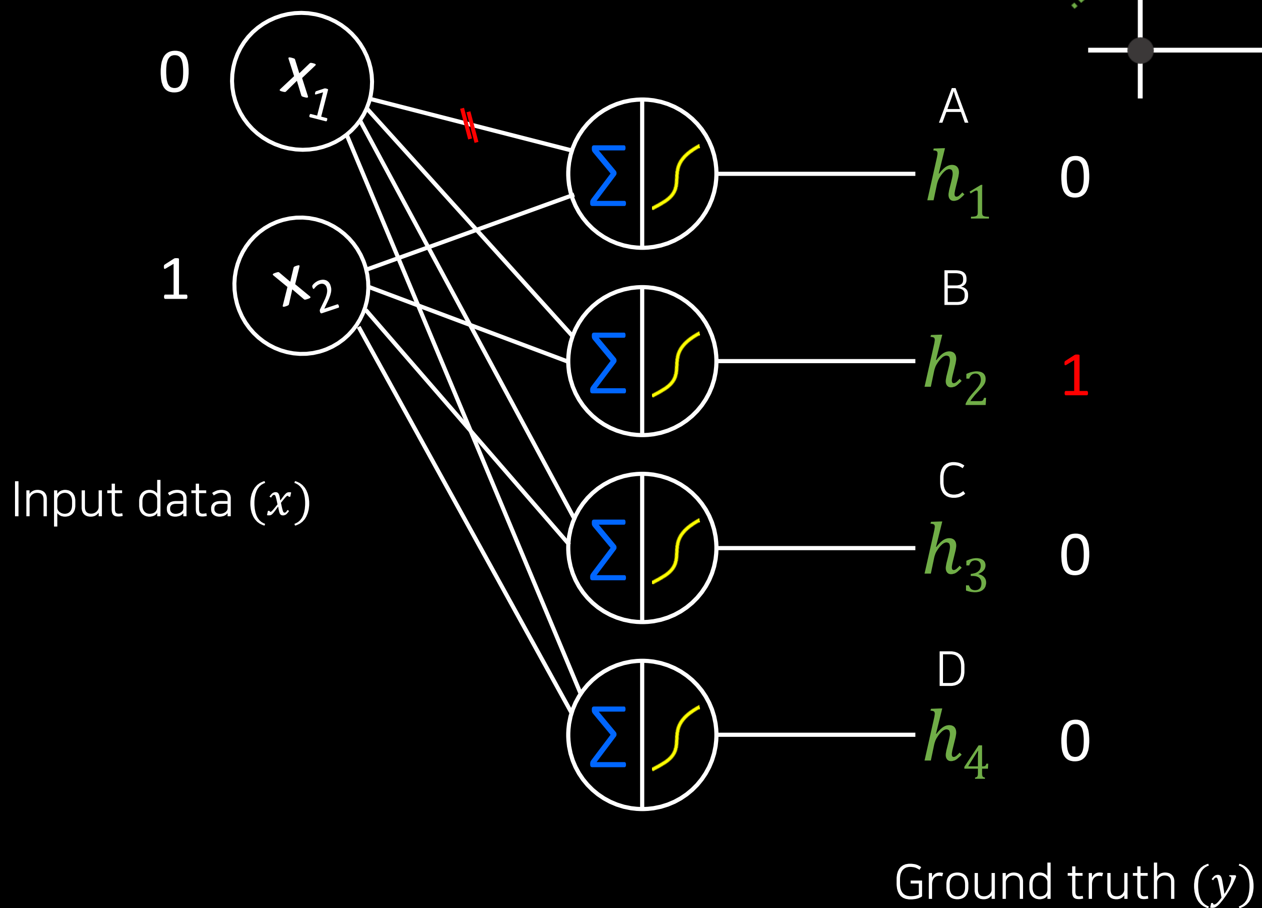


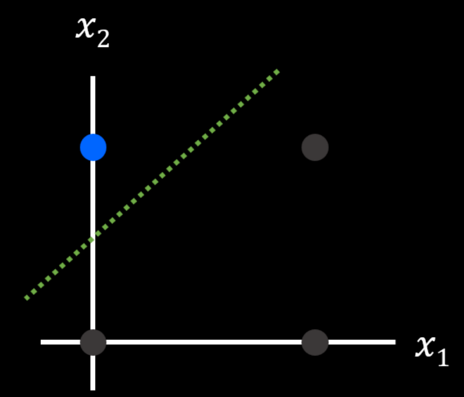


4 Neurons

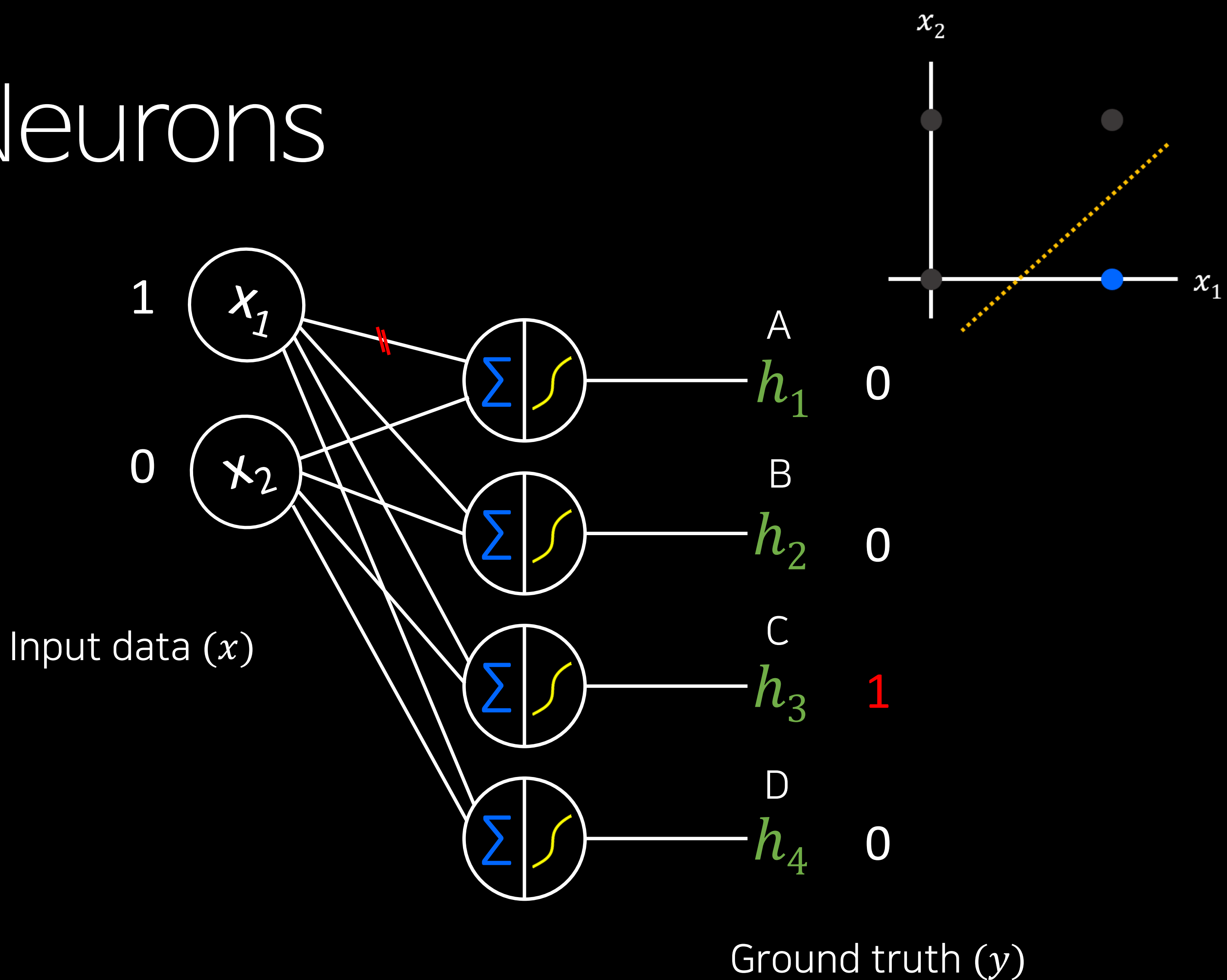


4 Neurons

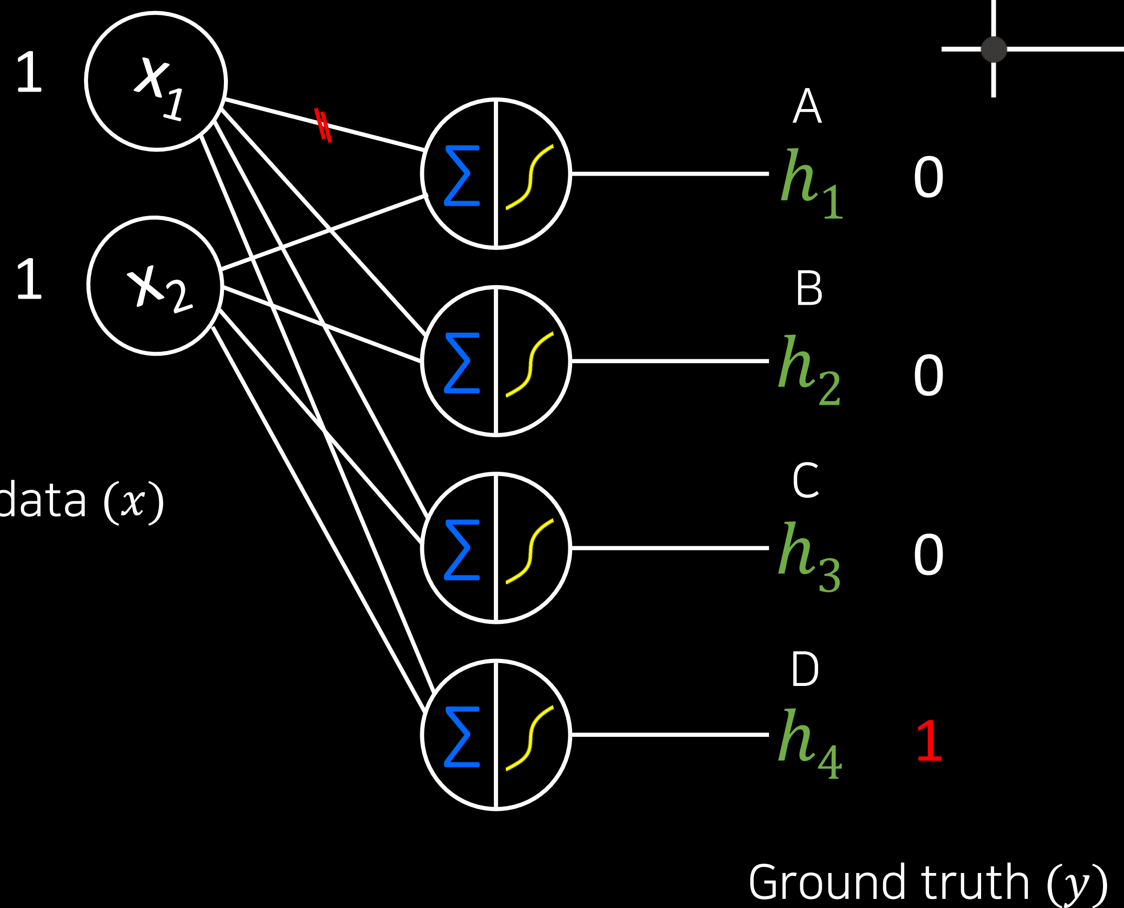


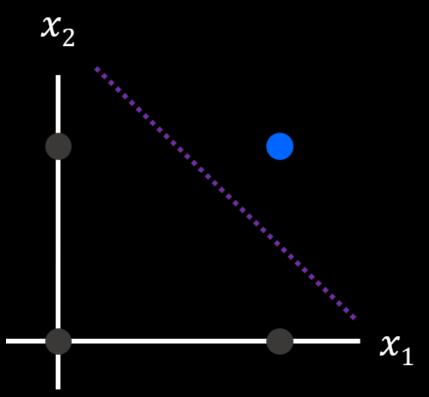


4 Neurons



4 Neurons





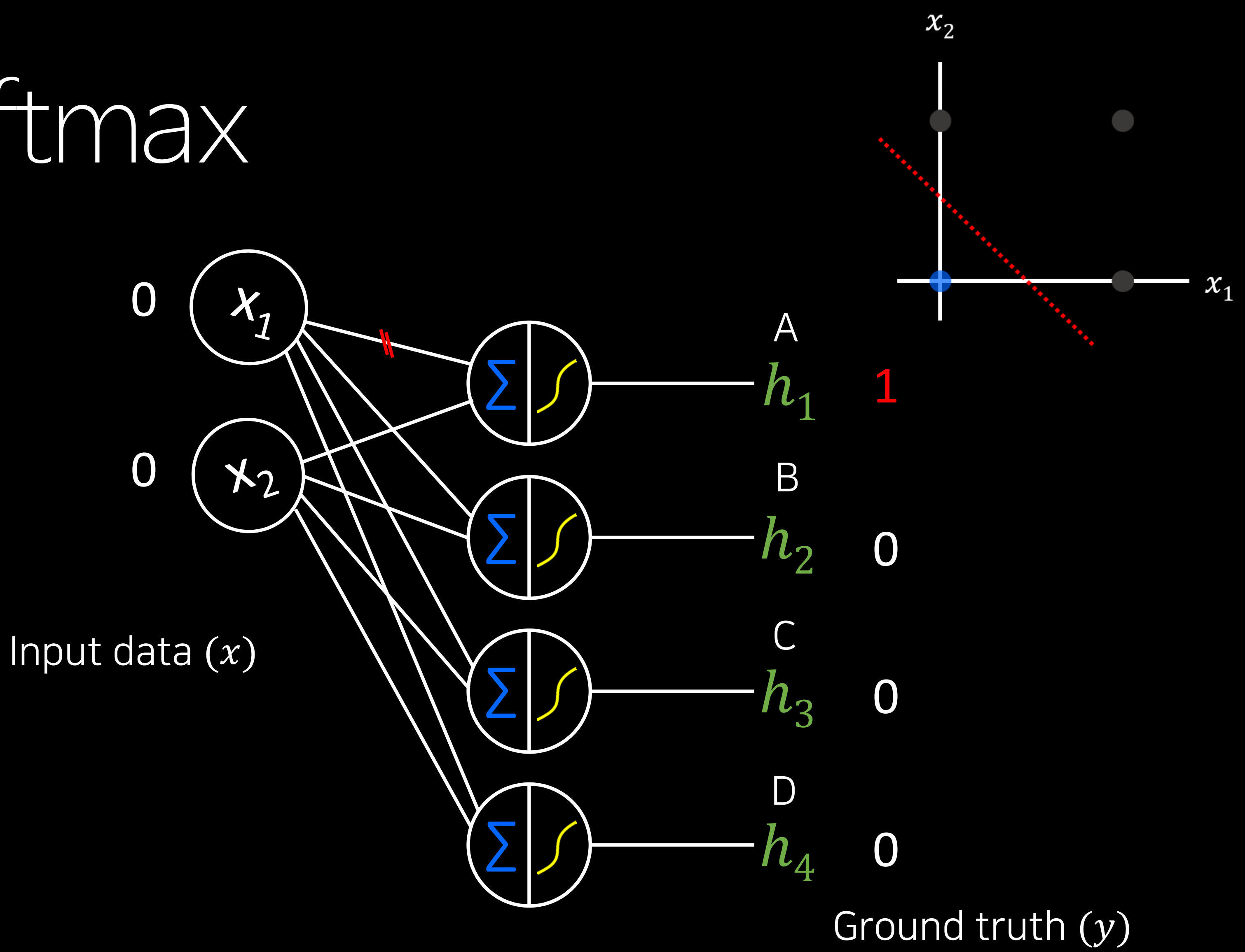
One-hot Encoding

- For the ground truth (y),
- setting only one as ON(1) and others as OFF(0)

Considerations

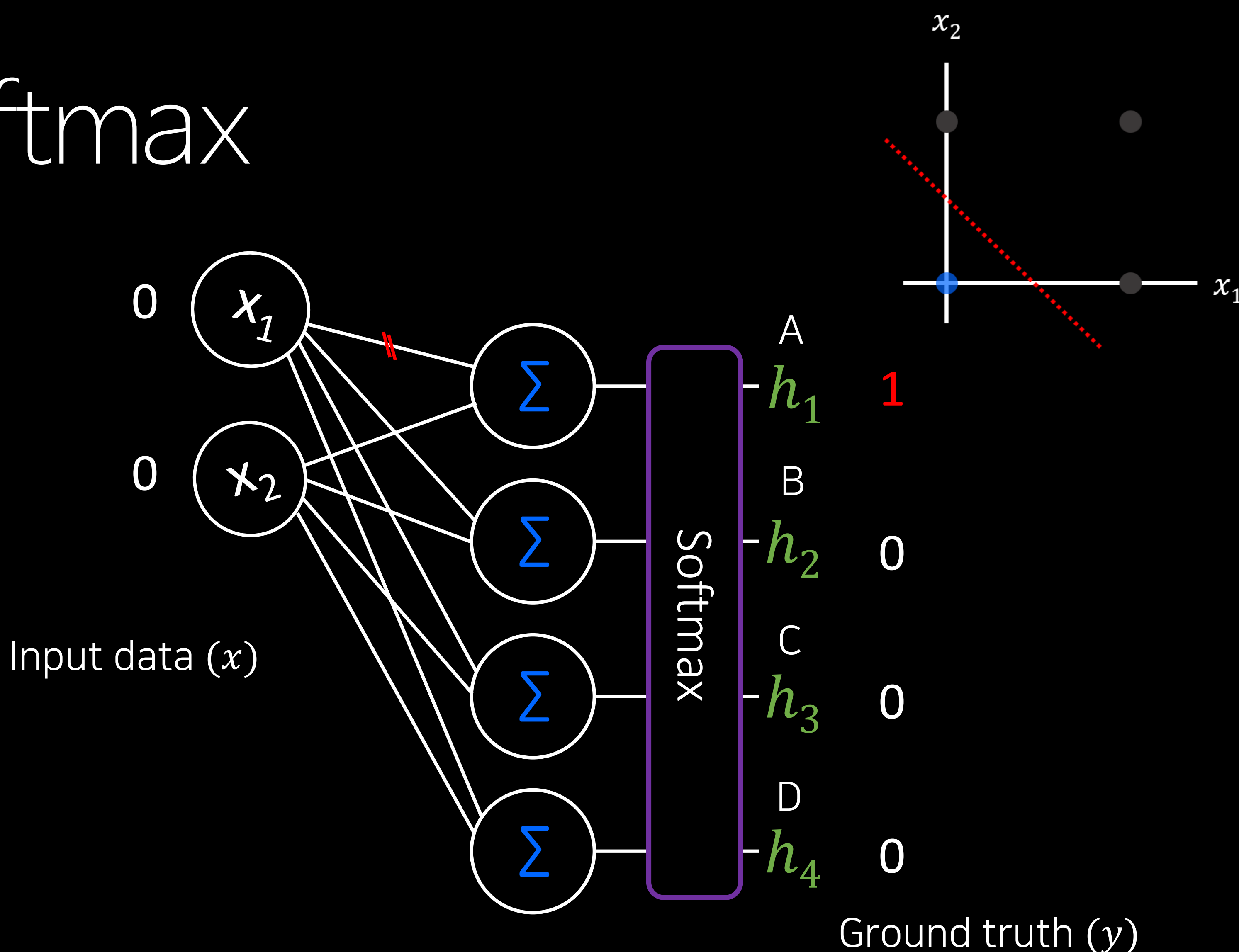
- For the answers of four neurons, if a neuron’s output is 1, and others must be 0.
- However, each neuron produce output independently.
- No way to control the 4 outputs together
- A special function introduced → Softmax

Softmax



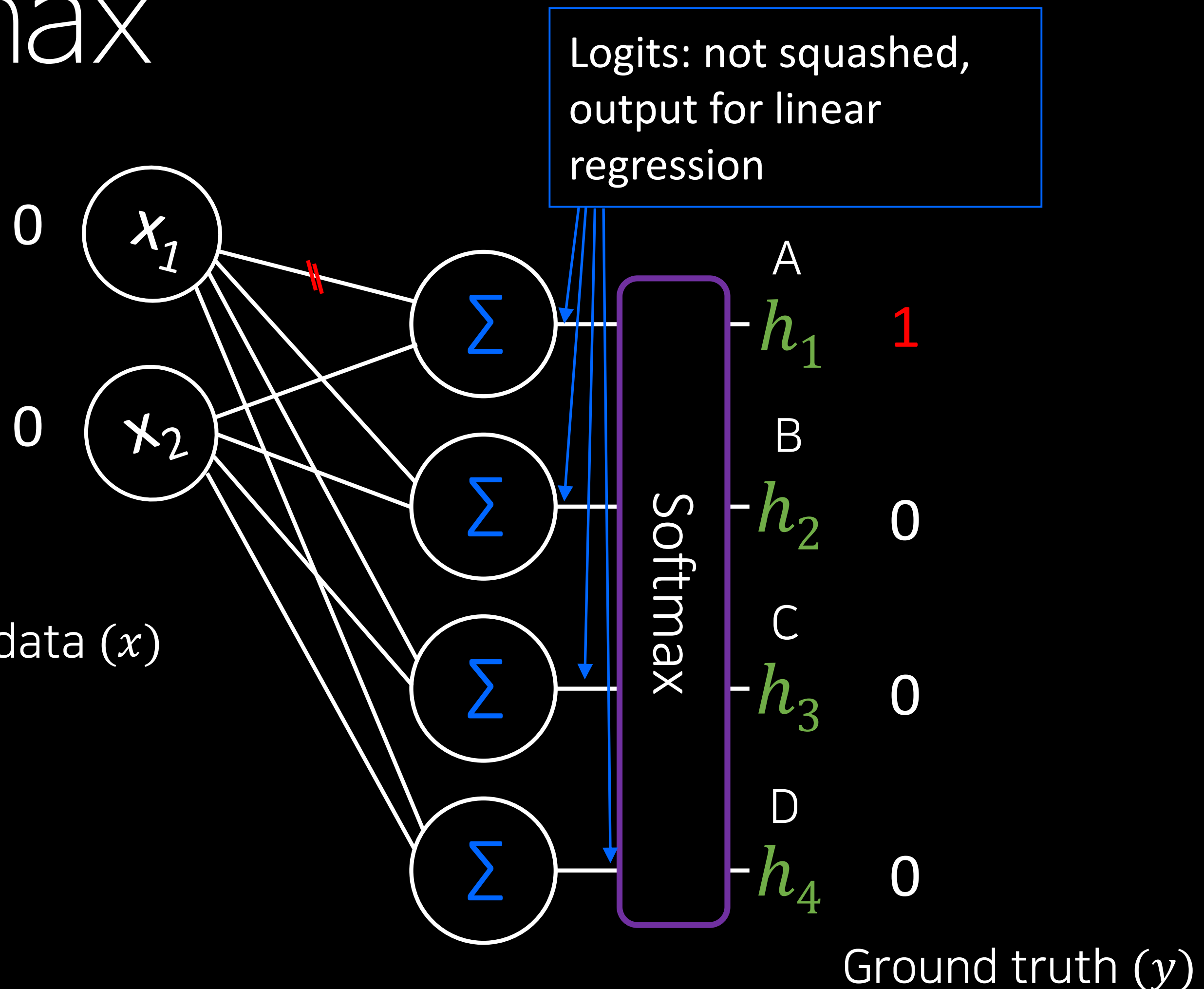
Initial architecture

Softmax



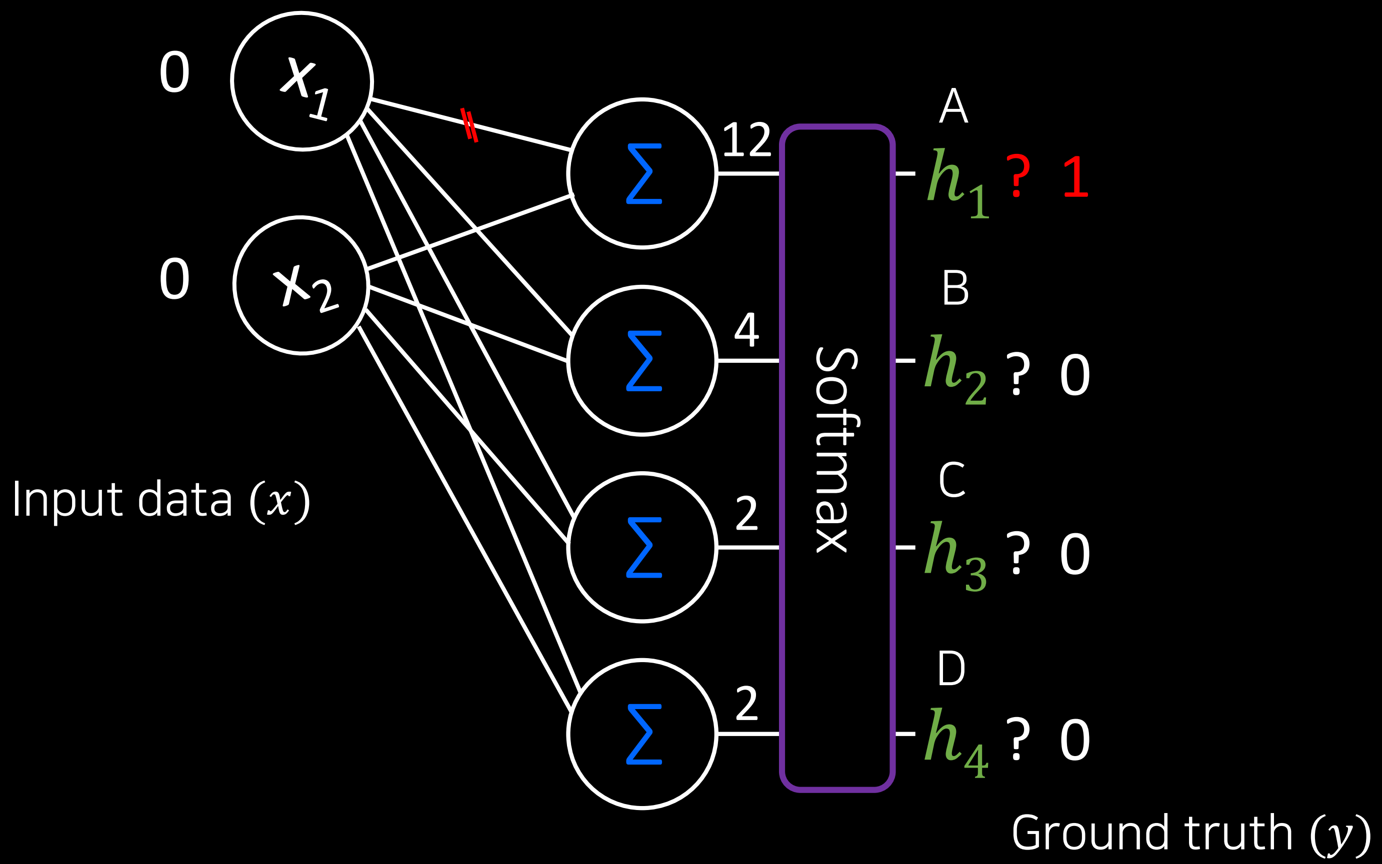
New architecture

Softmax



Initial architecture

Softmax

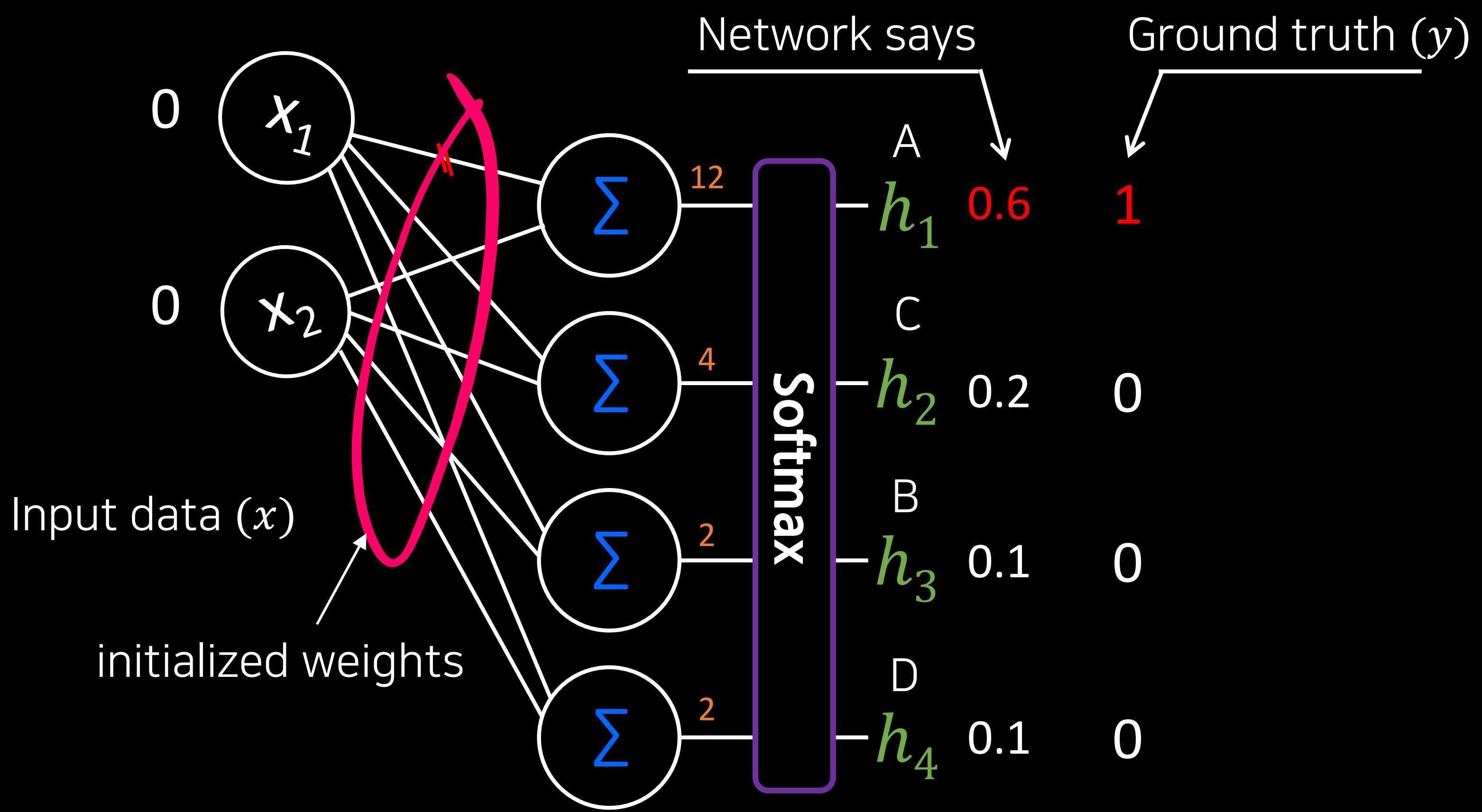


0
x_1
0
x_2
Input data (x)
12
4
2
2
Softmax
A
h_1 ? 1
B
h_2 ? 0
C
h_3 ? 0
D
h_4 ? 0
Ground truth (y)

Softmax

- For example, if the logits are 12, 4, 2, 2, then the Softmax function returns $\frac{12}{20}, \frac{4}{20}, \frac{2}{20}, \frac{2}{20}$ as results.
- Normalization of logits values
- Each value means the probability to be in the class.

Softmax



Network says
Ground truth (y)
0
0
x_1
x_2
Input data (x)
initialized weights
12
4
2
2
Softmax
A
h_1
0.6
1
C
h_2
0.2
0
B
h_3
0.1
0
D
h_4
0.1
0

New Error Function

- Distance between the output of a network(softmax) and correct answer (ground truth)
- If answer correctly, then the distance is 0,
- If not(incorrect), then the distance is ∞

New Error Function

If answer correctly, then the distance(error) is 0.

Softmax correct answer

$$\boxed{\begin{array}{c}1\\0\\0\\0\end{array}} \odot \begin{array}{c}1\\0\\0\\0\end{array} \triangleright \ ?$$

$$\boxed{\begin{array}{l}0\rightarrow\infty\\1\rightarrow0\end{array}}$$

$-\log$ (Softmax) Correct answer

$$\boxed{\begin{array}{c}0\\\infty\\\infty\\\infty\end{array}} \odot \begin{array}{c}1\\0\\0\\0\end{array} \triangleright \ 0$$

New Error Function

If incorrect, then the distance(error) is ∞.

Softmax	correct answer			$-\log$ (Softmax)	Correct answer		
0	1			∞	1		
1	0			0	0		
0	0			∞	0		
0	0			∞	0		

$$\begin{bmatrix}0\\1\\0\\0\end{bmatrix} \odot \begin{matrix}1\\0\\0\\0\end{matrix} \;\triangleright\; ?$$

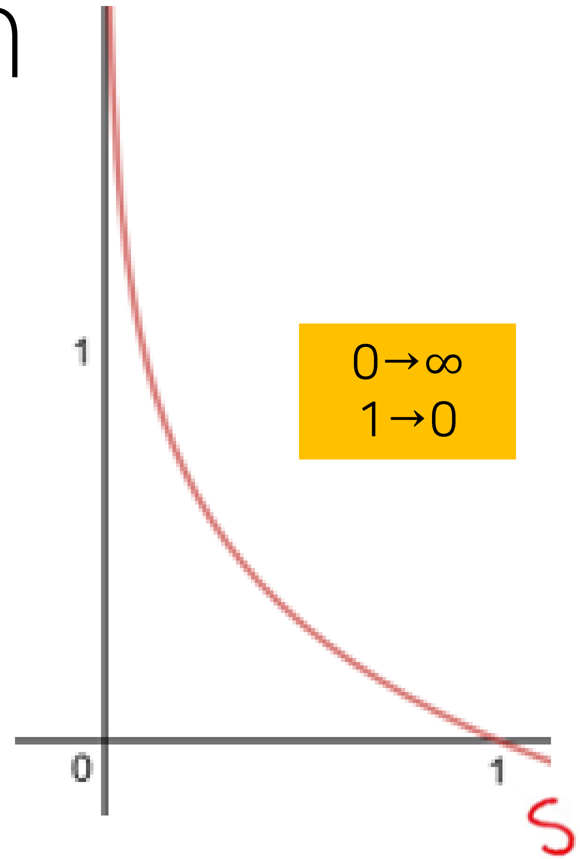
$$\begin{bmatrix}\infty\\0\\\infty\\\infty\end{bmatrix} \odot \begin{matrix}1\\0\\0\\0\end{matrix} \;\triangleright\; \infty$$

$0 \rightarrow \infty$

$1 \rightarrow 0$

$-\log$ function

$$-\log(S)$$



The output of softmax

$$-L\log(S)$$

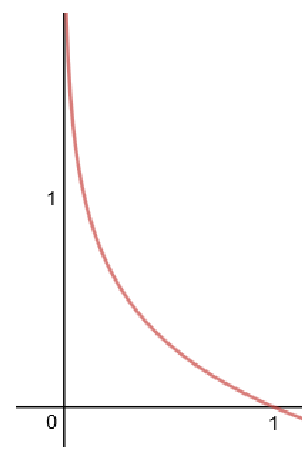
correct answer L

$$-\sum_i L_i \log(S_i)$$

New Error Function

$$D(S, L) = -\sum_i L_i \log(S_i)$$

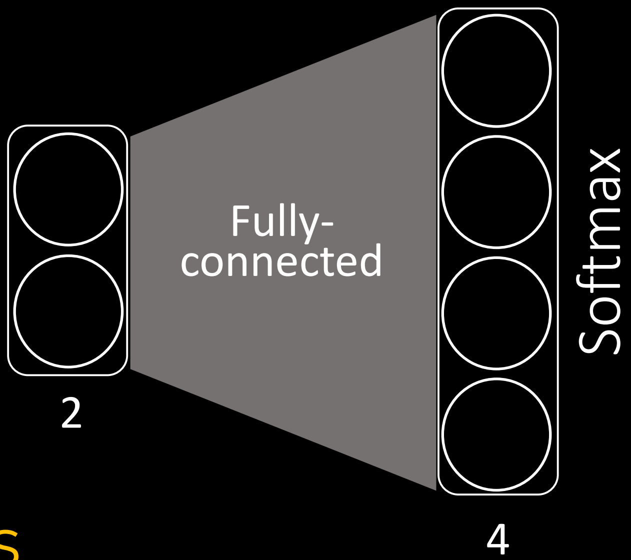
$$\underbrace{\begin{bmatrix} 0.7 \\ 0.2 \\ 0.1 \end{bmatrix}}_{S(Y)} \qquad \underbrace{\begin{bmatrix} 1.0 \\ 0.0 \\ 0.0 \end{bmatrix}}_{L}$$



softmax_cross_entropy_with_logits(logits, y_data)

- The function returns 0 if the answer is correct
- or returns ∞

Lab 14.py



- Classification into one of four classes
- 4 neurons where each has 2-input
- A bias for each neuron