AI and Deep Learning

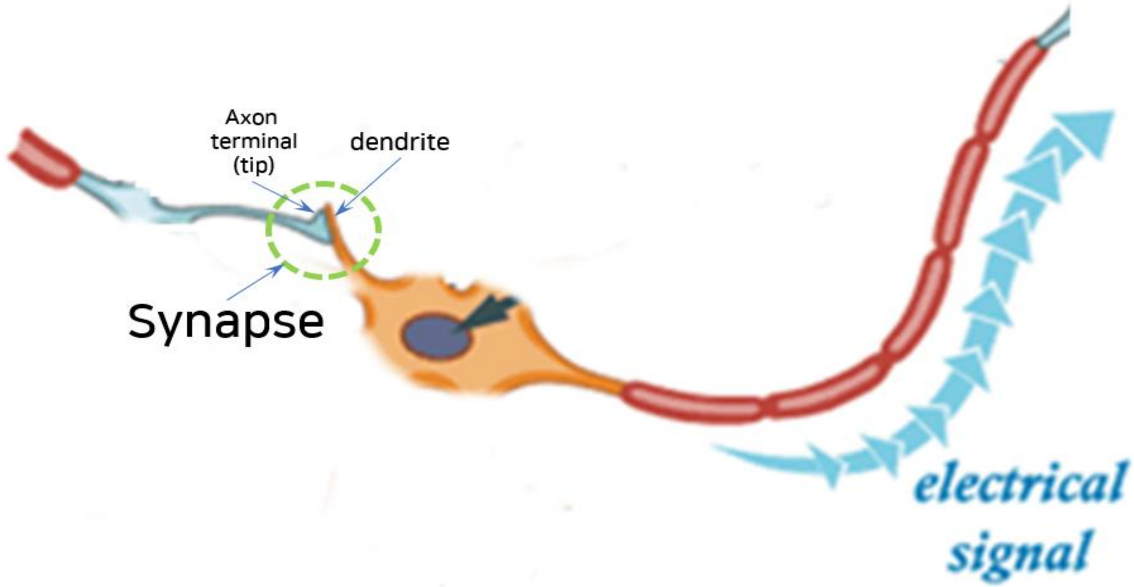
Logistic Regression and Classification

Jeju National University

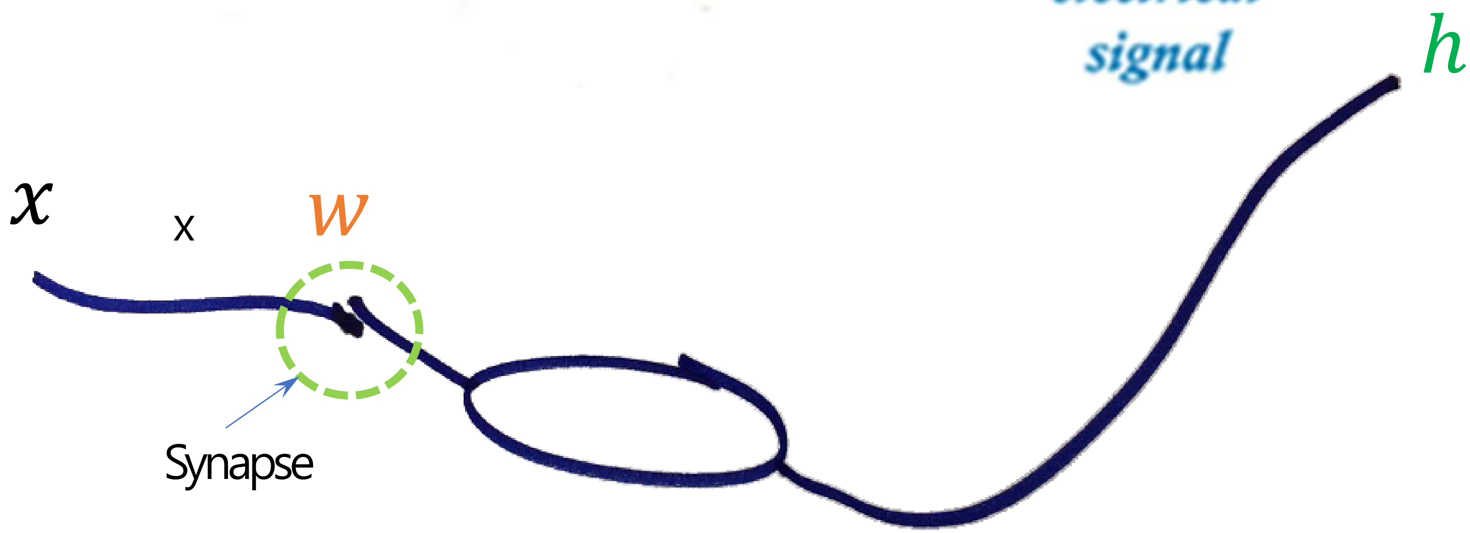
Yung-Cheol Byun

Agenda

- Logistic regression and classification
- **New** loss/cost function
- Decision boundary
- Implementation using TensorFlow
- Multi-class problem

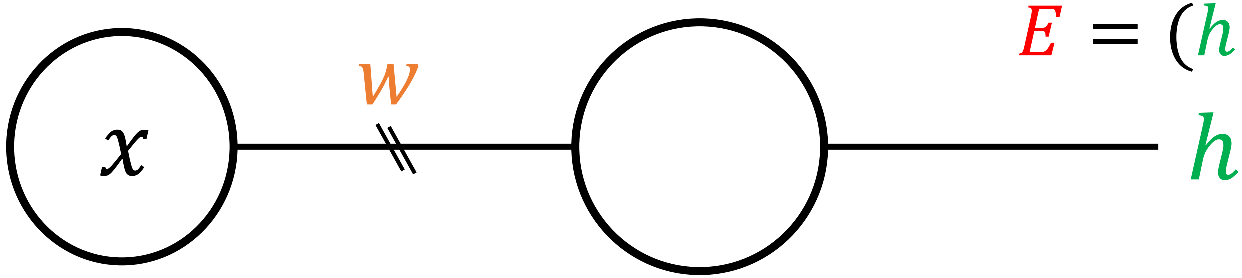


Axon terminal (tip)
dendrite
Synapse
electrical signal



h
x
x
w
Synapse

$$E = (h - y)^2$$



x
w
h

Logistic Regression

Logistic?

What does that mean?

Logistic shape is a common **'s'** curve.

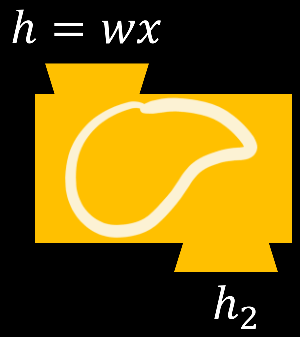
The meaning of logistic or logistic function:

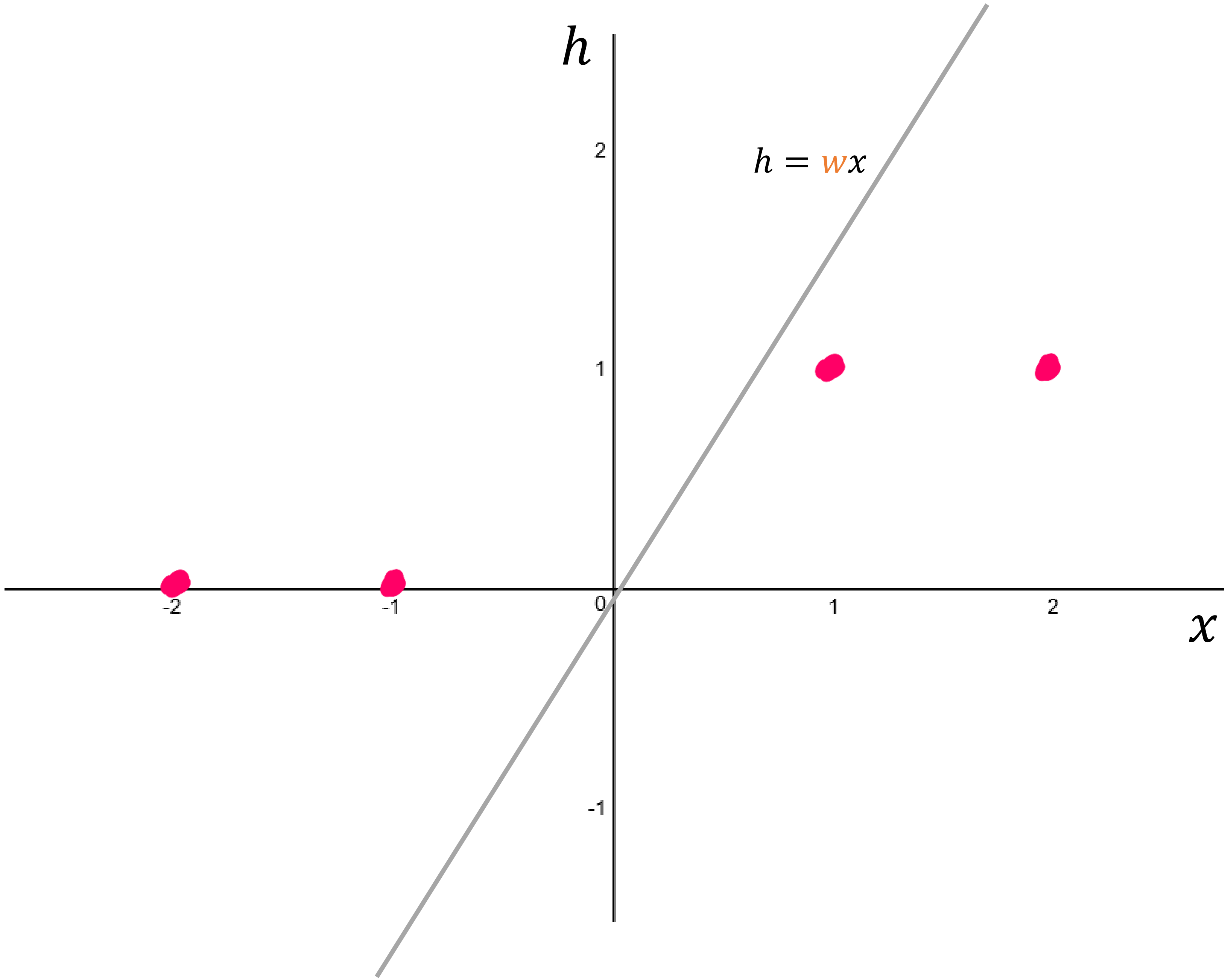
- population growth
- a pattern/shap that "starts slowly, grows rapidly at some point, and then slows down again.

Draw $(-2, 0), (-1, 0), (1, 1), (2, 1)$.

$$h = wx$$

$$h_2 = \frac{1}{1 + e^{-h}}$$



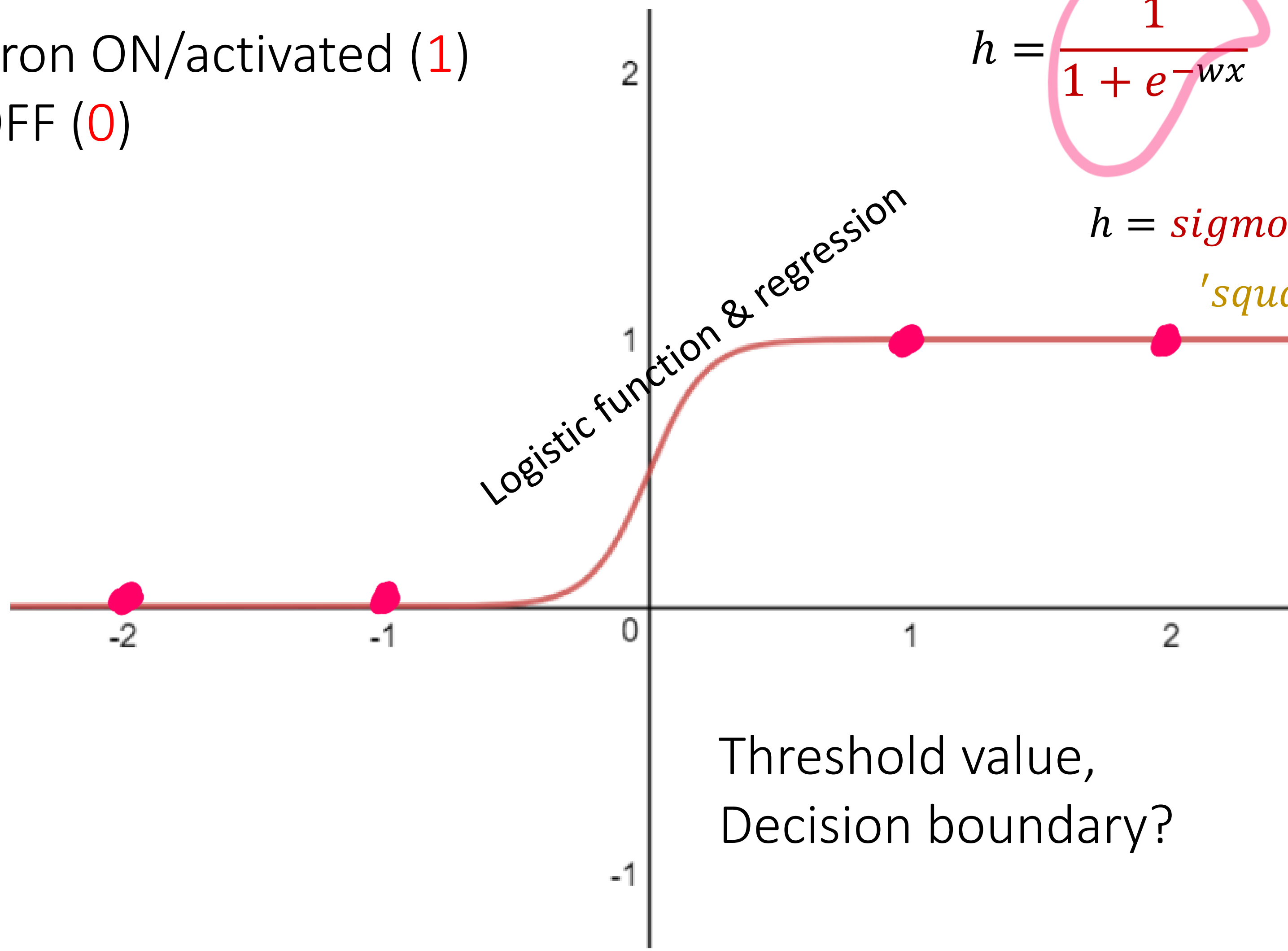

h
2
h = wx
1
-2
-1
0
1
2
x
-1

Neuron ON/activated (1)
or OFF (0)

$$h = \frac{1}{1 + e^{-wx}}$$

$$h = sigmoid(wx)$$

$'squash'$



Threshold value,
Decision boundary?

$$y = \frac{1}{1 + e^{-wx}} \qquad y = \frac{1}{1 + e^{-w(x-0)}}$$

Logistic function

From Wikipedia, the free encyclopedia

For the recurrence relation, see Logistic map.

A **logistic function** or **logistic curve** is a common "S" shape (sigmoid curve), with equation:

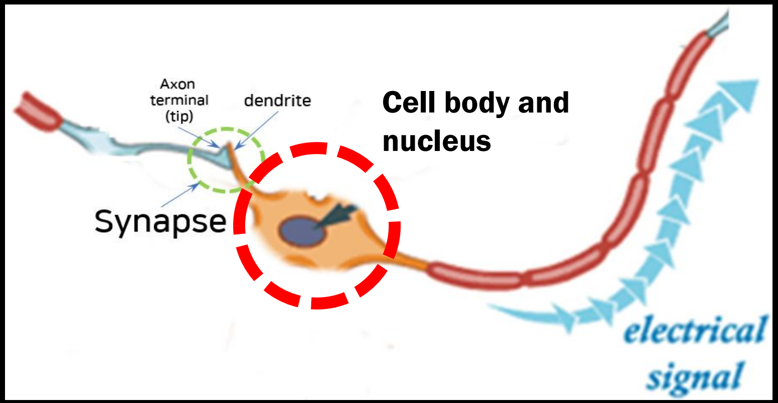
$$f(x) = \frac{L}{1 + e^{-k(x-x_0)}}$$

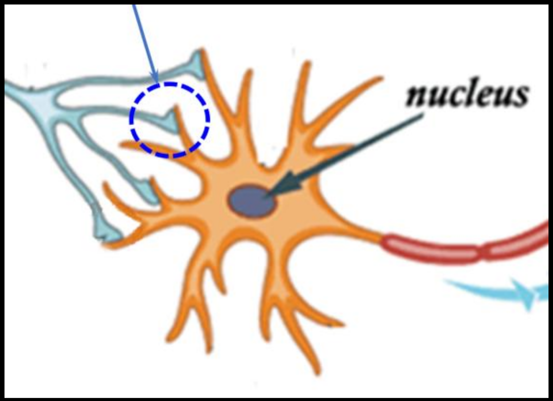
where

- e = the natural logarithm base (also known as Euler's number),
- x_0 = the x-value of the sigmoid's midpoint,
- L = the curve's maximum value, and
- k = the logistic growth rate or steepness of the curve.[1]

Revisited

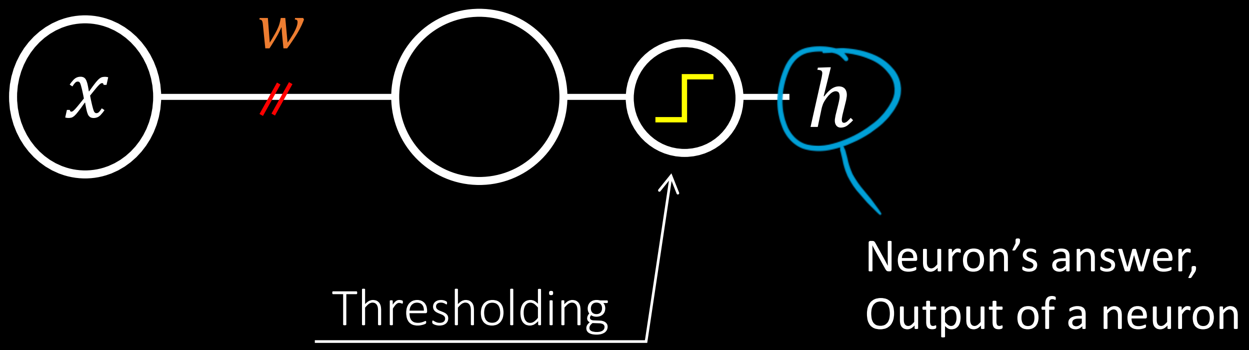
Real operation of a neuron



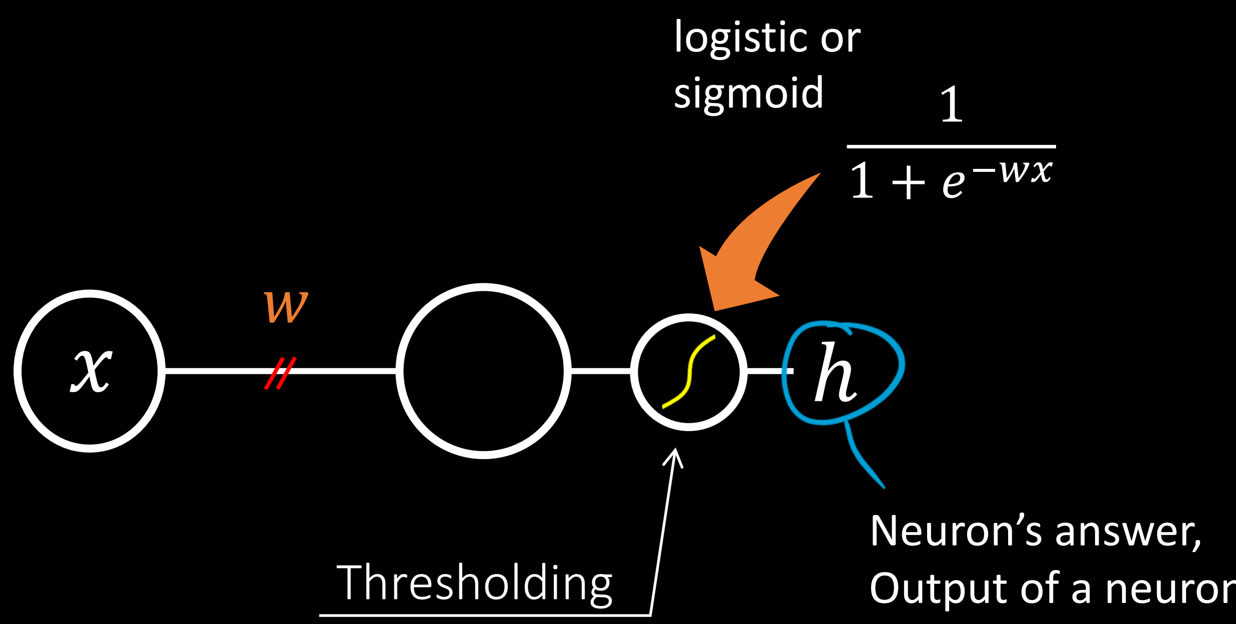


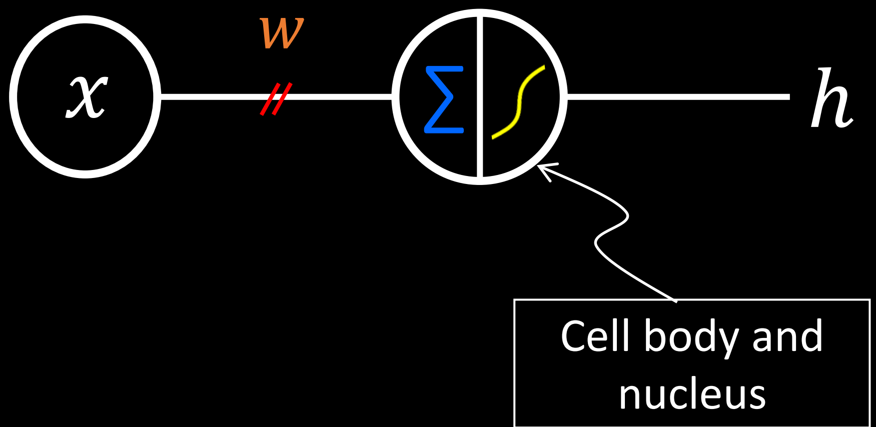
- ON, OFF only
- signal ON if the weighted sum is greater than T
- otherwise signal OFF

Revisited



Revisited


logistic or
sigmoid
$\frac{1}{1+e^{-wx}}$
x
w
h
Thresholding
Neuron's answer,
Output of a neuron

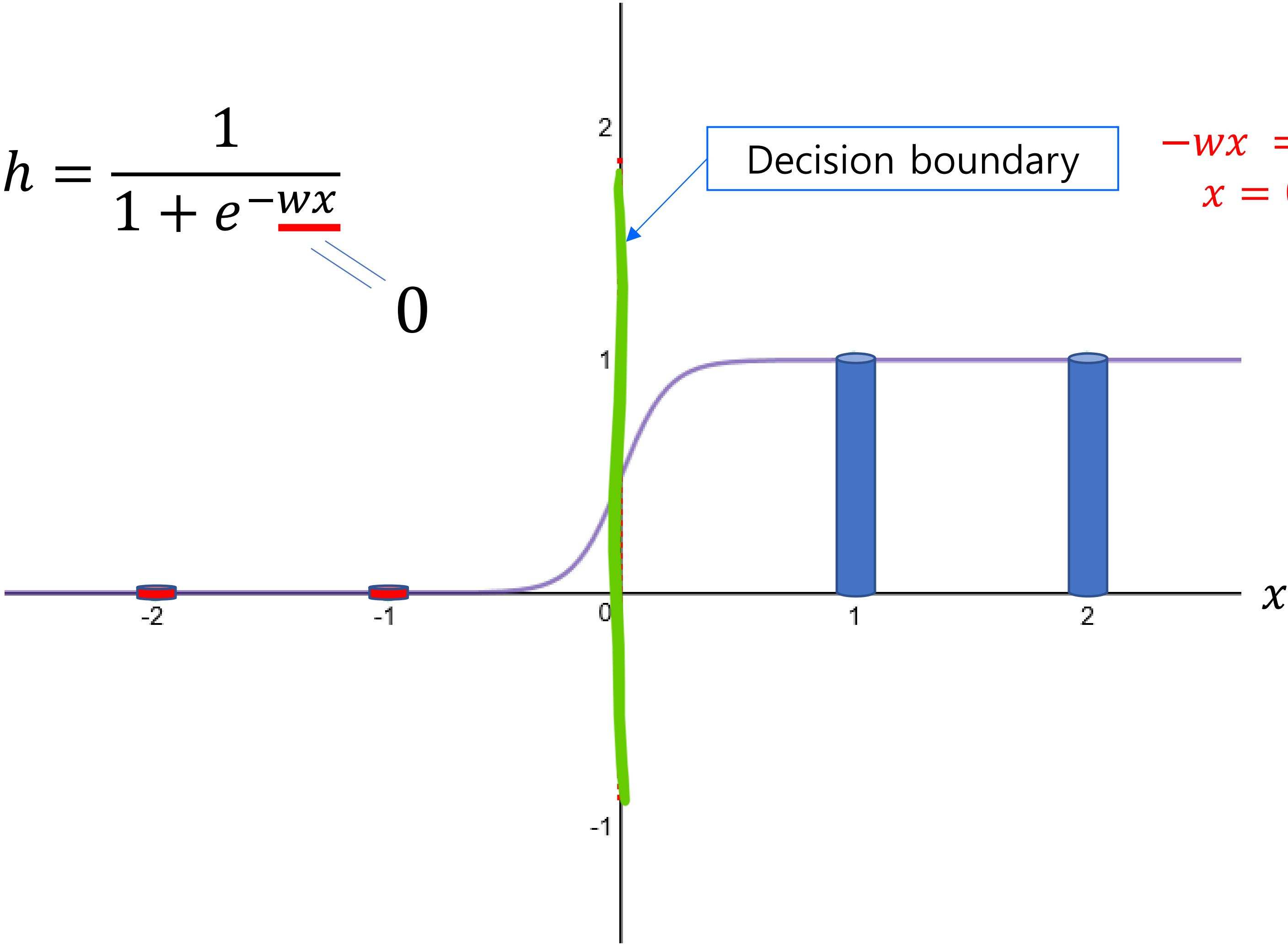


- Activated(1) or not(0) according to the input x
- Find the decision boundary to decide 1 or 0.

Decision boundary

$$h = \frac{1}{1 + e^{-\underline{wx}}}$$

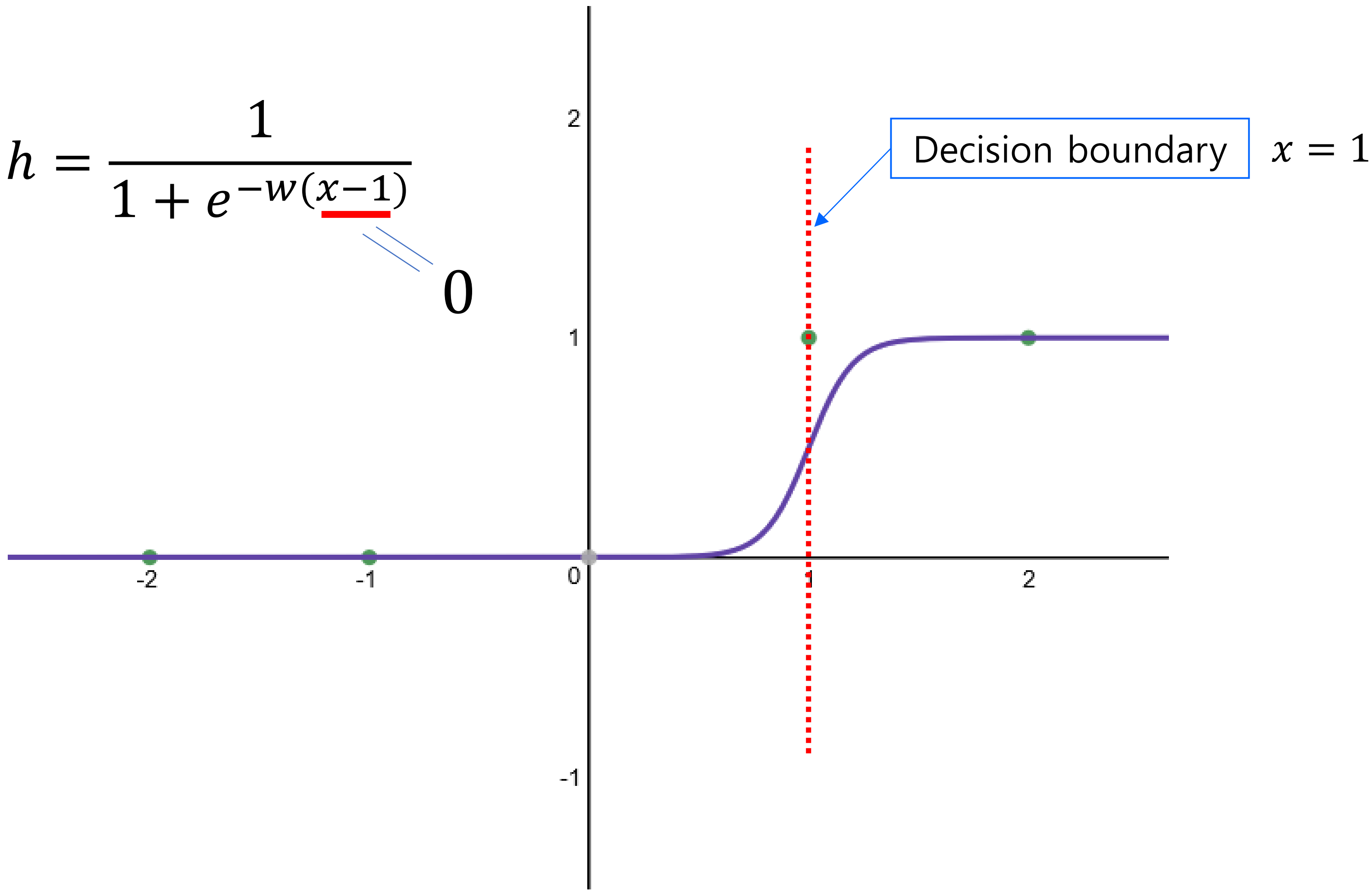
$$0$$



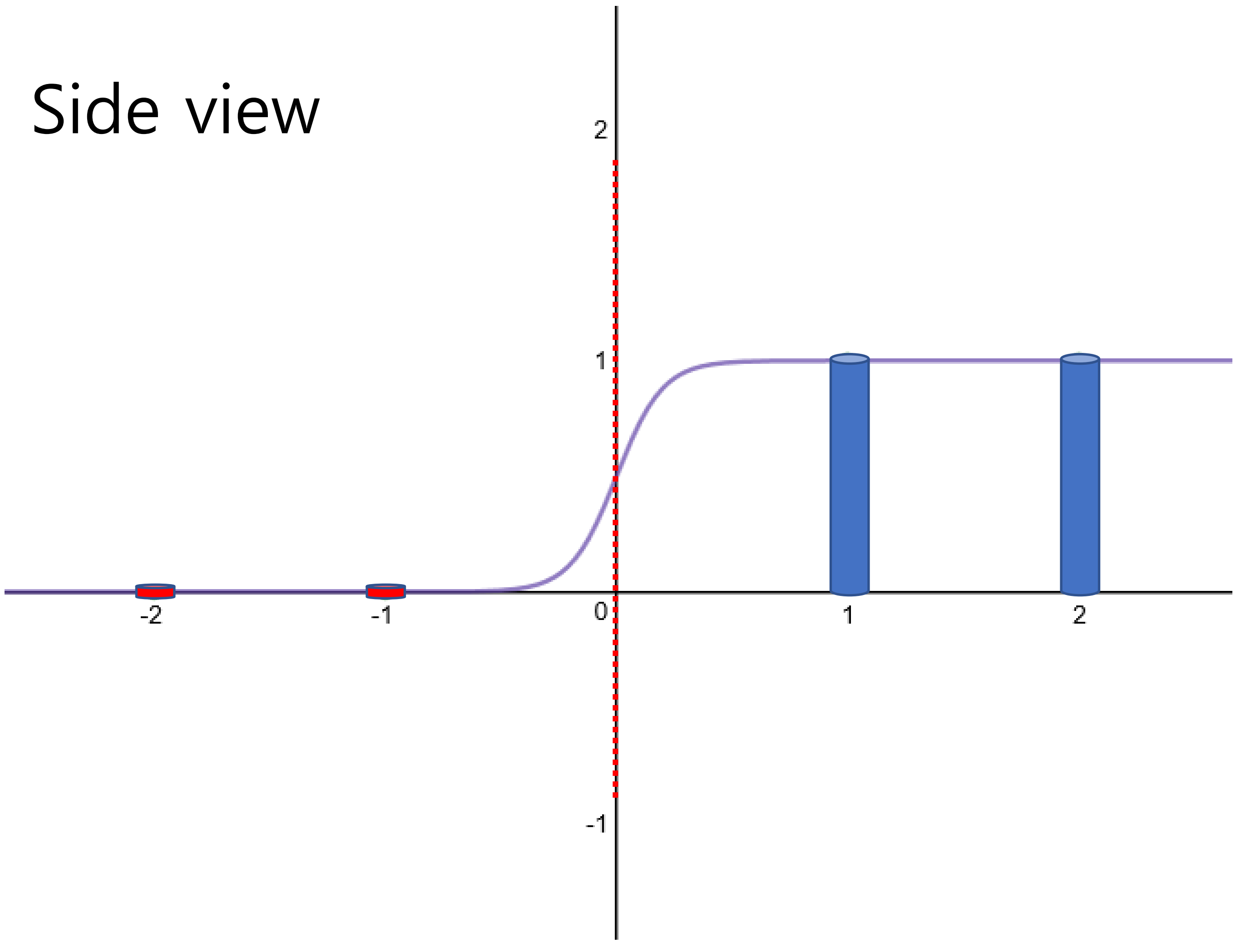
$$-wx = 0$$
$$x = 0$$

Decision boundary

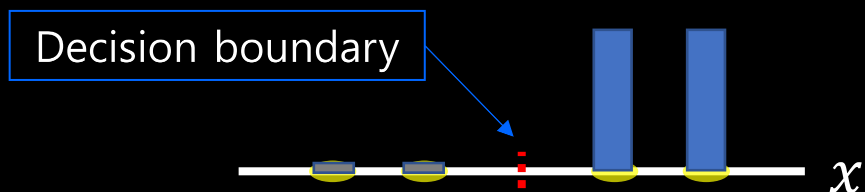
$$h = \frac{1}{1 + e^{-w(\underbrace{x-1}_{=0})}}$$



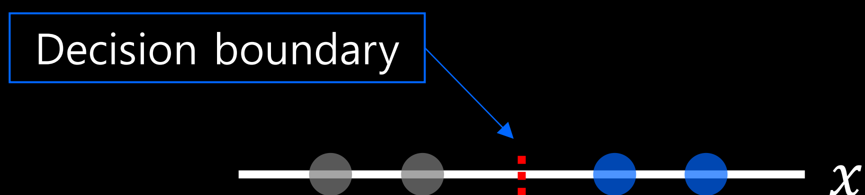
Side view



Decision Boundary



{**Side** view}

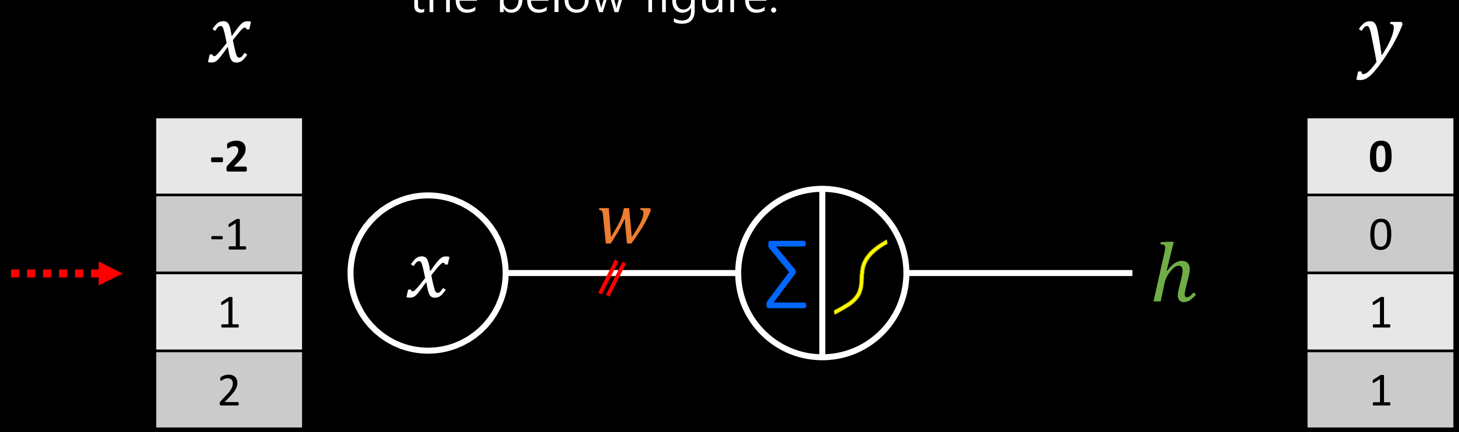


{View from **above**}

Classification using Decision Boundary

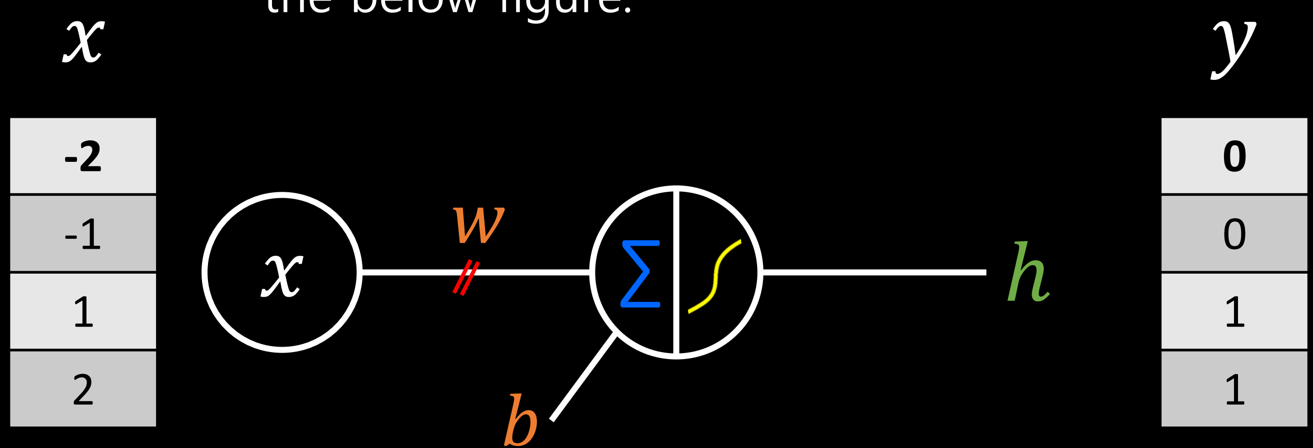
- Pass(1) or Fail(0)
- Spam(1) or Ham(0)
- Scam(fraud, 1) or not(0)
- Safe(1) or Dangerous(0)
- Intrusion/virus(1) or not(0)
- Cancer(1) or not(0)
- Binary classification -> Multiple classification

Guess the decision boundary from the below figure.



$$h = \begin{cases} 1 & if\ wx \geq 0 \\ 0 & otherwise \end{cases}$$

Guess the decision boundary from the below figure.



$$h = \begin{cases} 1 & if\ wx + b \geq 0 \\ 0 & otherwise \end{cases}$$

Hypothesis

- Squash the $wx + b$ line so that the curve passes through all the data points.

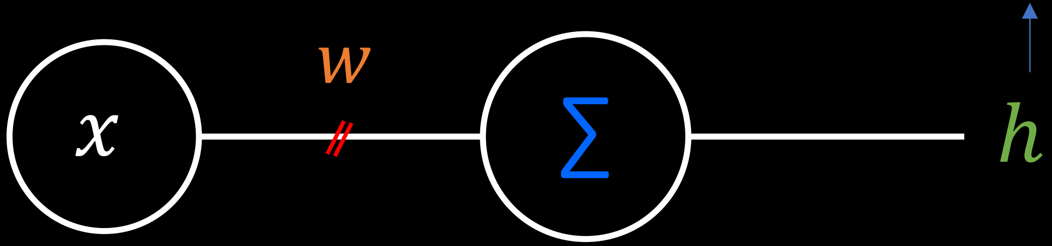
[Q] Find decision boundary from the equation.

$$h = \frac{1}{1 + e^{-wx}}$$

$$h = \frac{1}{1 + e^{-(wx+b)}}$$

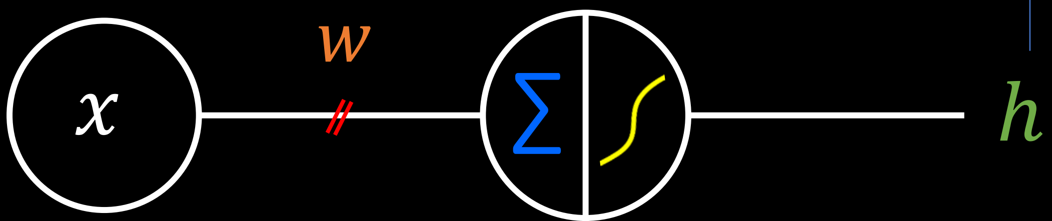
L2 Loss Function

$$E = (h - y)^2$$



L2 Loss Function

$$E = (h - y)^2$$



Does the E work ?

$$h = \frac{1}{1 + e^{-wx}}$$

$$E = (h - y)^2$$

$$E = \left(\frac{1}{1 + e^{-w \cdot x}} - y\right)^2$$

$$E = \left(\frac{1}{1 + e^{-w \cdot 1}} - 1 \right)^2$$

if we have data (1, 1)

Draw $(-2, 0), (-1, 0), (1, 1), (2, 1)$.

$$h = wx$$

$$h = \frac{1}{1 + e^{-wx}}$$

Draw $(1, 1)$ only.

$$E = \left(\frac{1}{1 + e^{-w \cdot 1}} - 1 \right)^2$$

$$(w, E)$$

desmos

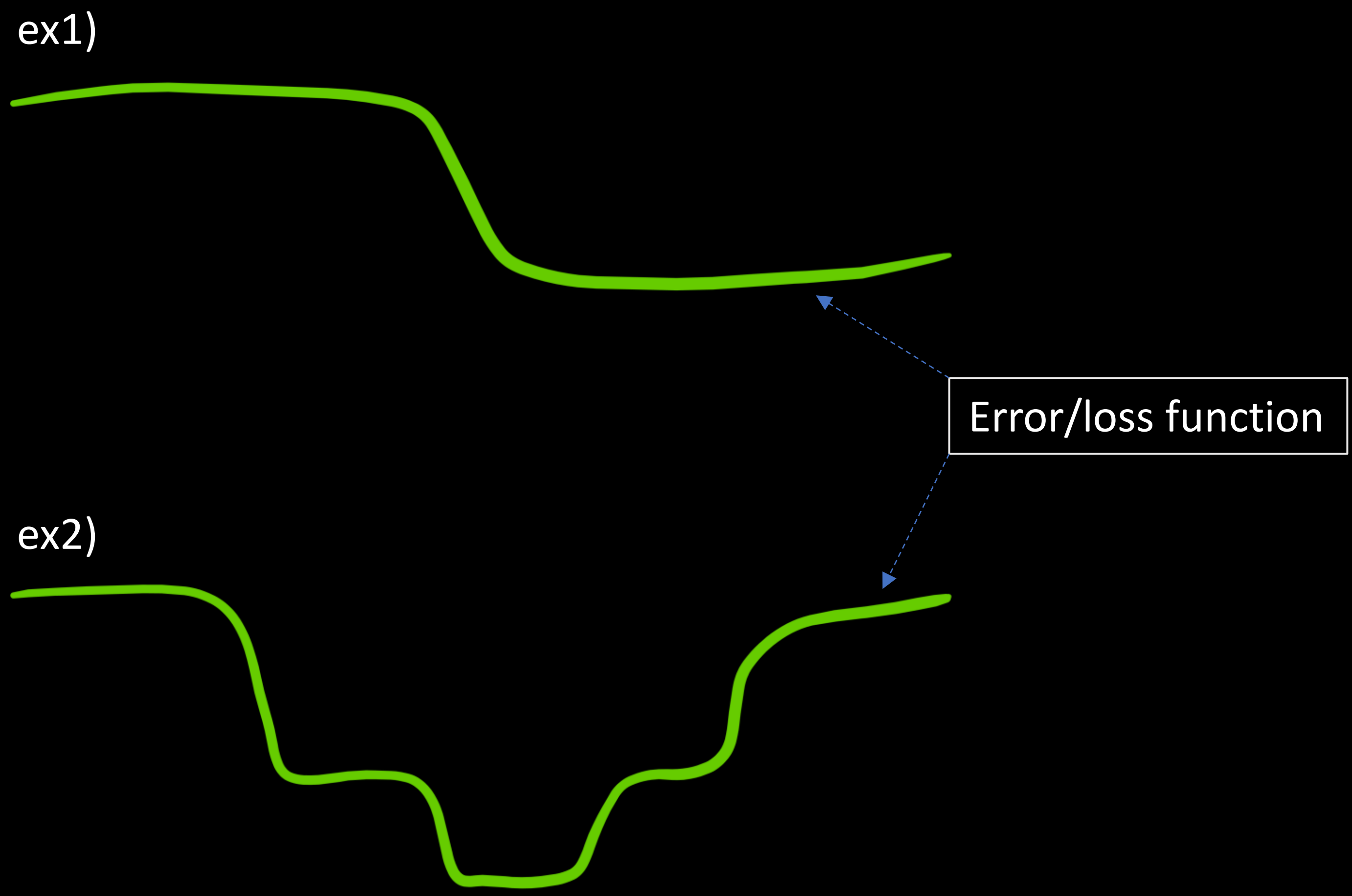
Draw 4 points: $(-1, 0), (1, 1), (-3, 0), (3, 1)$.

$$E = \left(\frac{1}{1 + e^{-w(-1)}} - 0\right)^2 + \left(\frac{1}{1 + e^{-w(1)}} - 1\right)^2 + \left(\frac{1}{1 + e^{-w(-3)}} - 0\right)^2 + \left(\frac{1}{1 + e^{-w(3)}} - 1\right)^2$$

Add bias b.

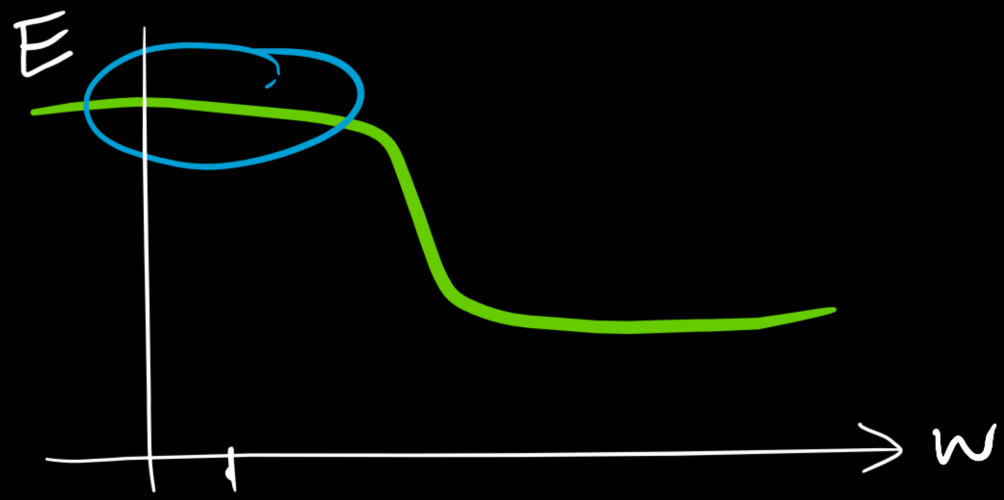
plot the error: (w, E)

L2 Loss for logistic regression



What problem is in the error function?

No gradient decent in some parts!

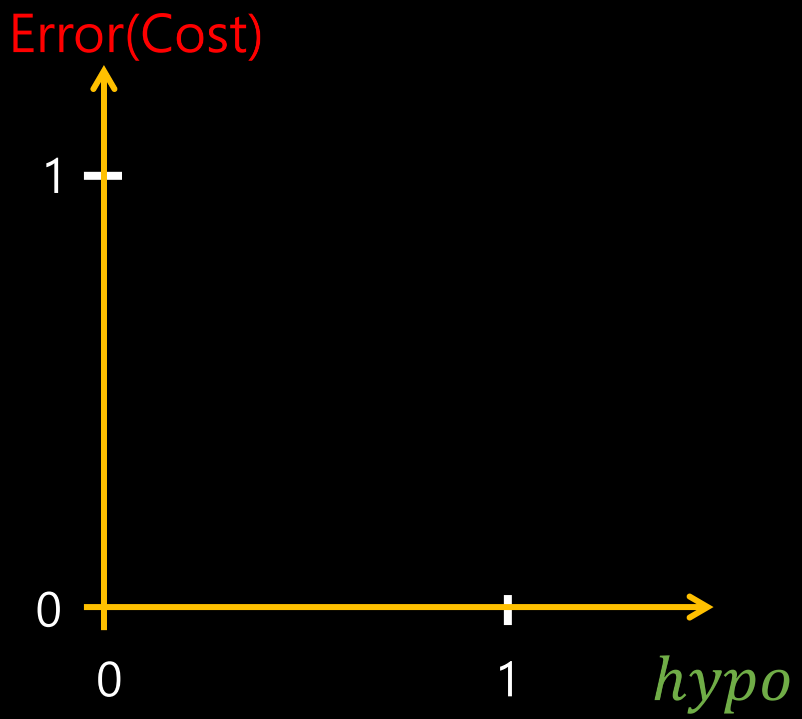


The error function **doesn't work** for the **Logistic** regression.

New Cost/Error Function

When ground truth is 1

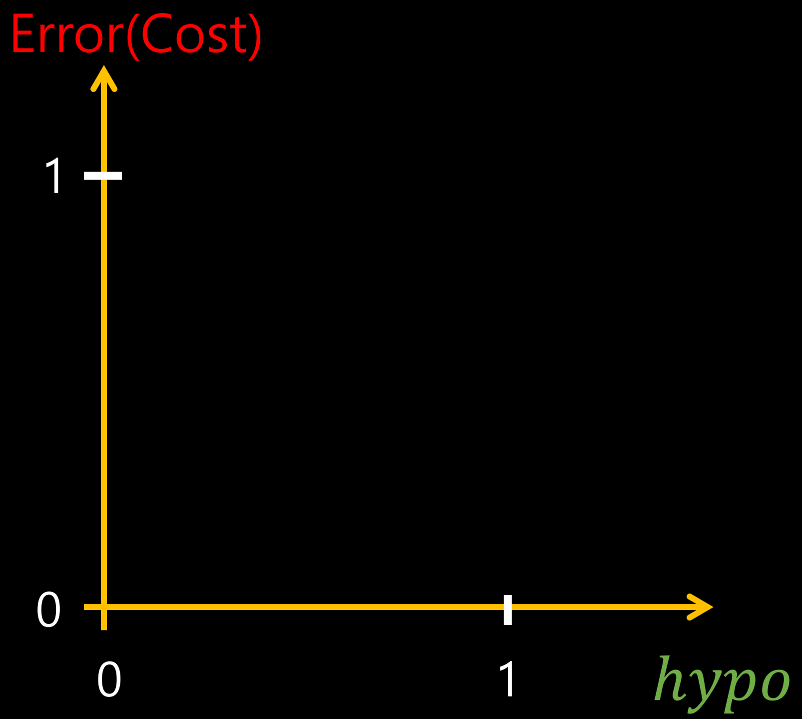
- if hypo is equal to 1, then error = 0
- if hypo is equal to 0 then error = ∞



New Cost/Error Function

When ground truth is 0

- if hypo is equal to 0, then error = 0
- if hypo is equal to 1 then error = ∞



desmos

when ground truth is 1

$$E = -\log(h)$$

when ground truth is 0

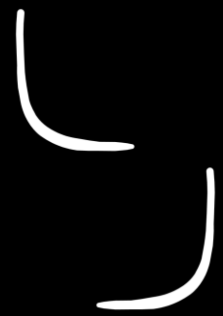
$$E = -\log(1-h)$$

$$E = -\log\left(\frac{1}{1+e^{-wx}}\right)$$

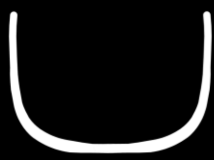
$$E = -\log\left(1-\frac{1}{1+e^{-wx}}\right)$$

New Cost/Error Function

$$E = \begin{cases} -\log(wx) & : y = 1 \\ -\log(1 - wx) & : y = 0 \end{cases}$$



$$E = -y\log(wx) - (1 - \mathrm{y})\log(1 - wx)$$



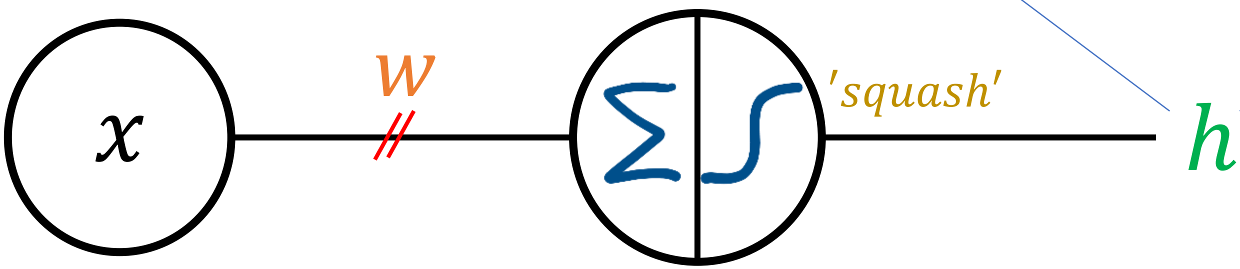
or, $E = -(y\log(wx) + (1 - \mathrm{y})\log(1 - wx))$

$$w = w - \alpha \cdot \frac{\partial E}{\partial w}$$

"Gradient decent"

Binary Cross-Entropy Loss

$$E = -(y\log(h) + (1-y)\log(1-h))$$



If we have multiple data, then

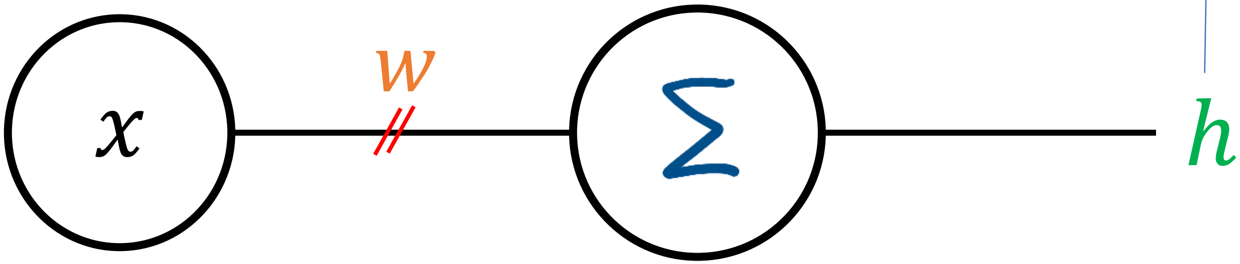
$$E = -\frac{1}{N}\sum(y\log(h) + (1-y)\log(1-h))$$

Example	Probability (확률, 통계)	Entropy (Uncertainty, 불확실성, 물리)
All red balls in a basket	Always, 100%, 1	Stable, 0, fixed, ice
lottery	Almost 0, 0%, 0	Unstable, ∞, steam

- a neuraon
- binary
- logistic regression

L2 Loss

$$E = (h - y)^2$$



- a neuraon
- continuous
- linear regression

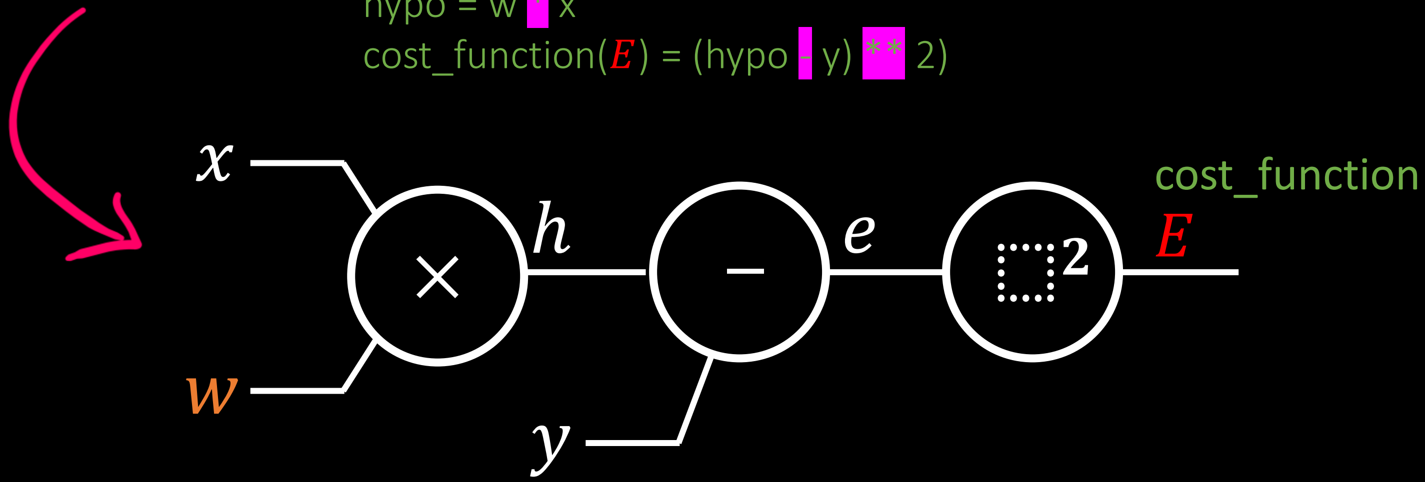
Computational graph

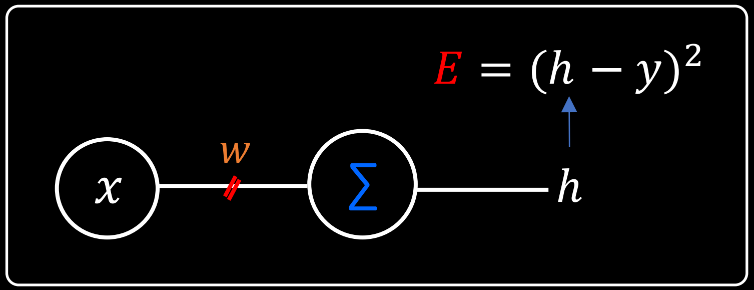
for the new cost function

Computational Graph

$$E = (wx - y)^2$$

```
hypo = w * x
cost_function(E) = (hypo - y) ** 2)
```



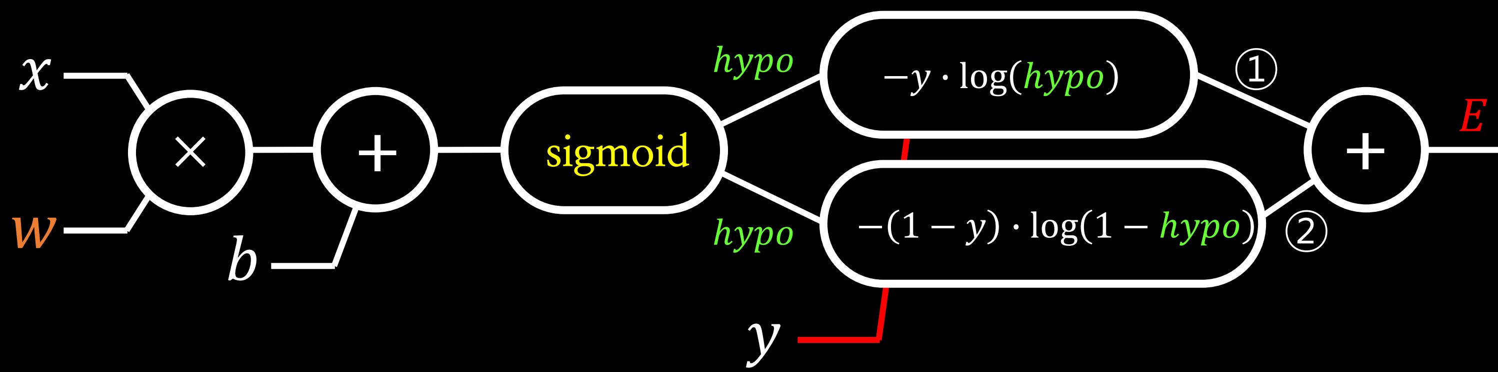


Computational Graph

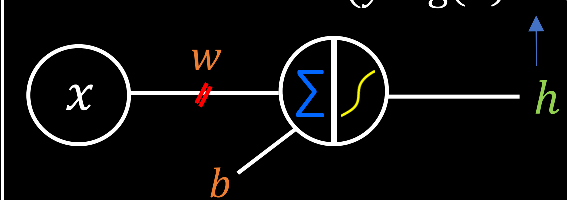
①　　　　　　　　②

$$E = -y \cdot \log(hypo) - (1-y) \cdot \log(1 - hypo)$$

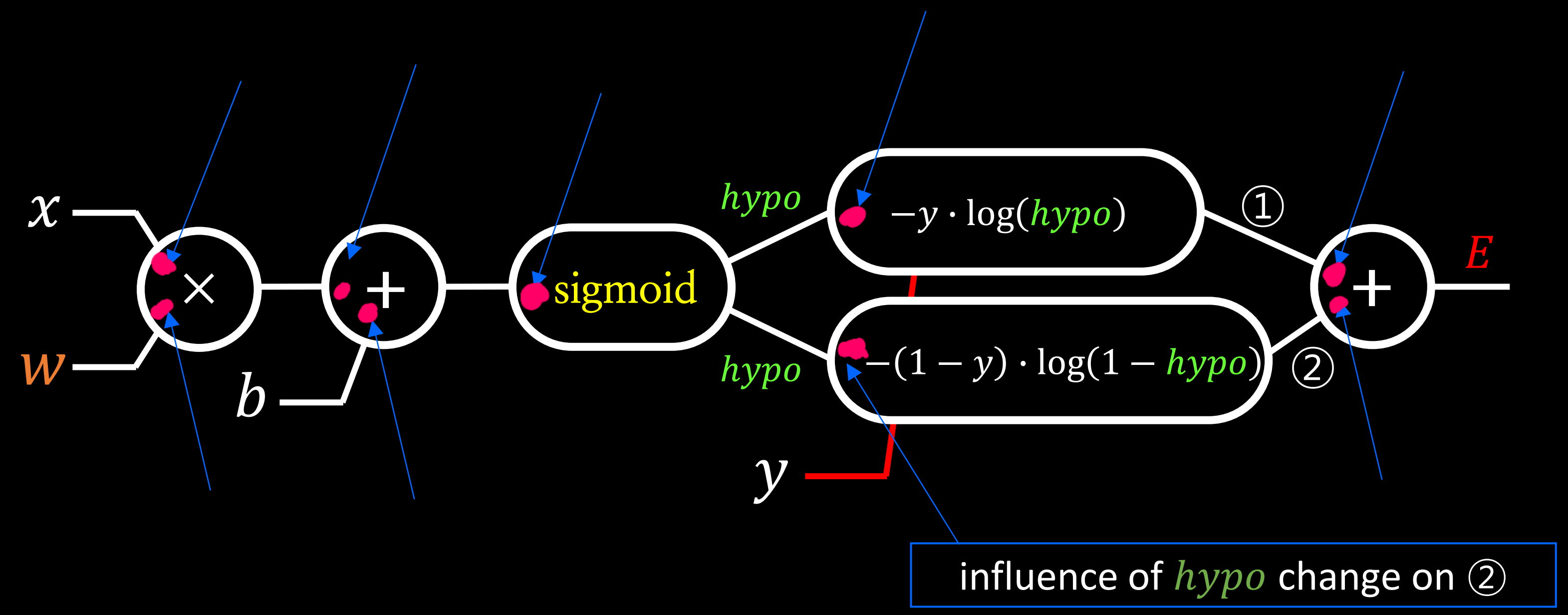
Binary Cross-Entropy Loss



$$E = -(y\log(h) + (1-y)\log(1-h))$$



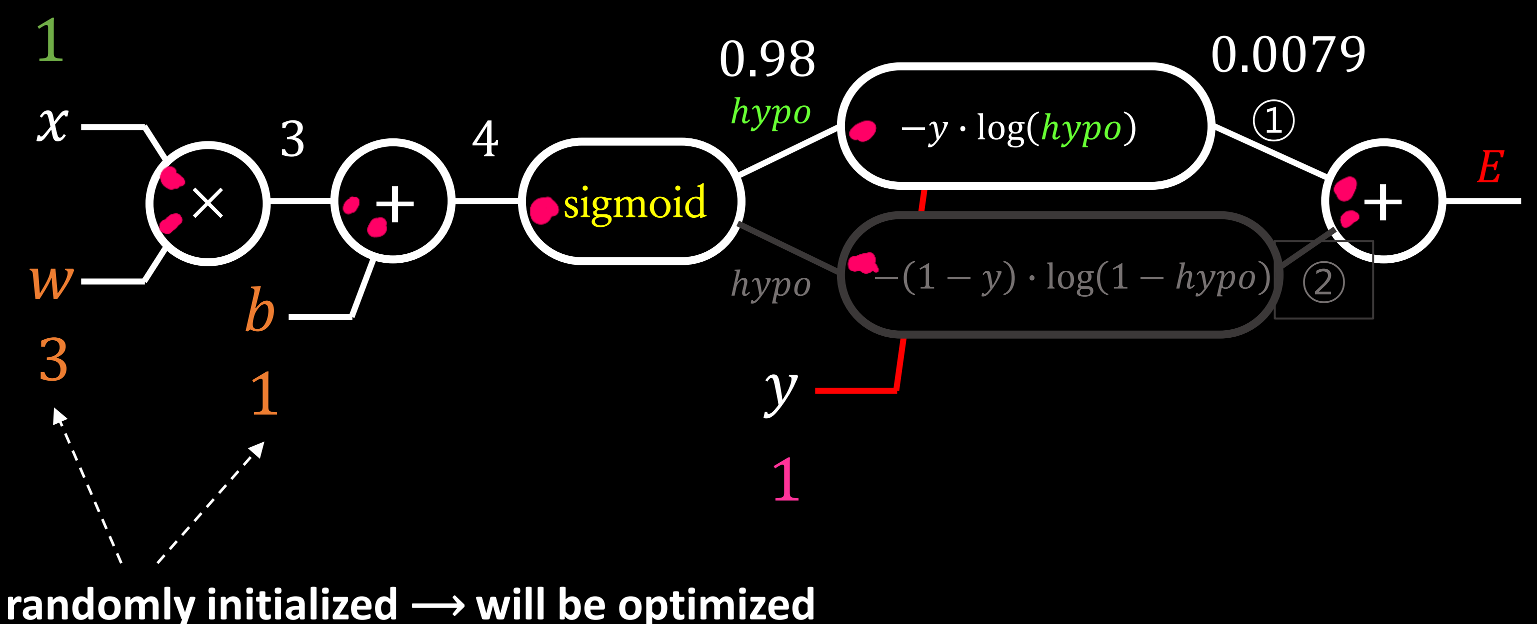
Local Gradients



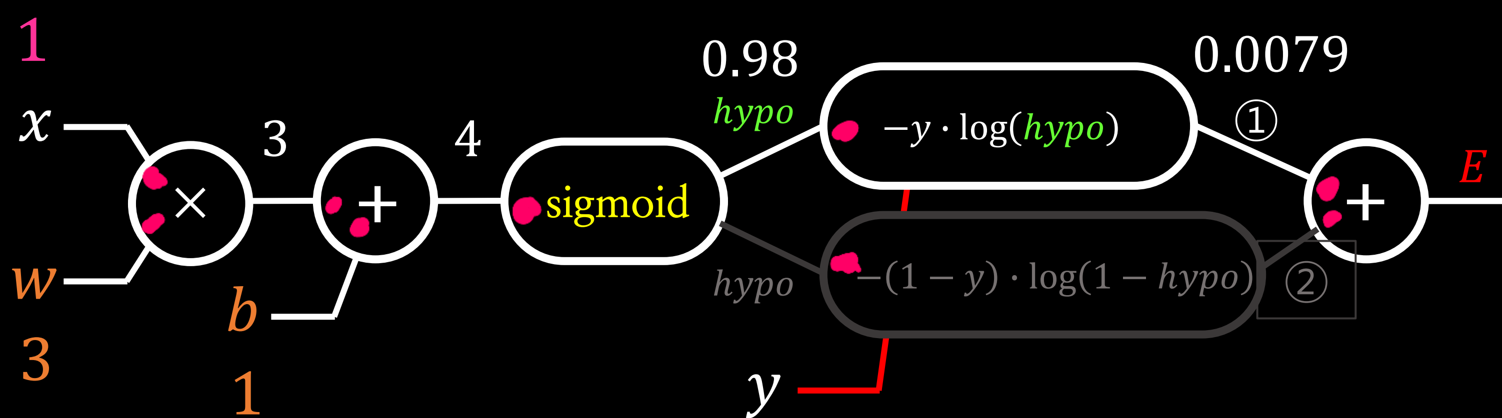
$$\frac{\partial ②}{\partial h}$$

Forward propagation

$$(x, y) \longrightarrow (1, 1)$$



Back-propagation



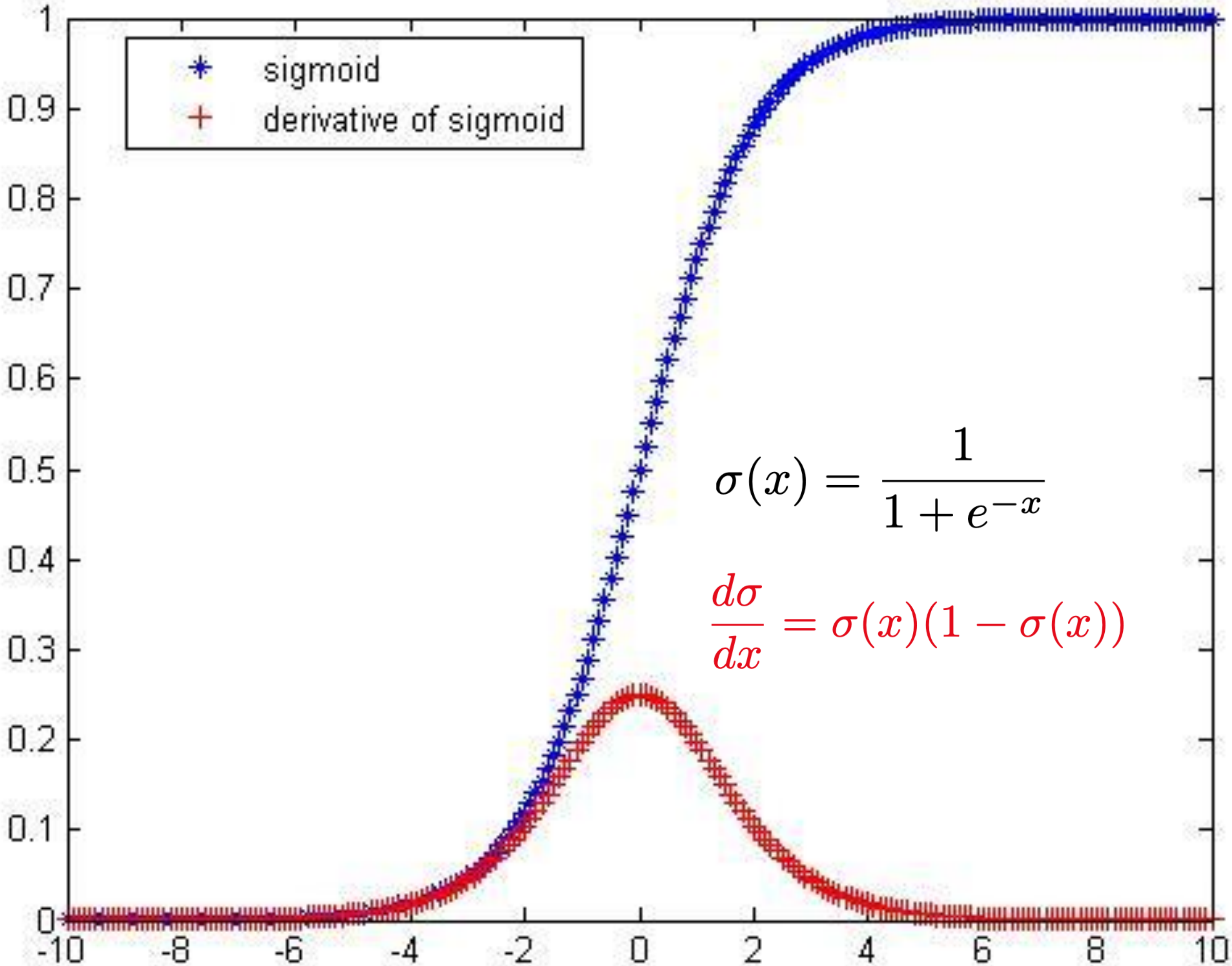
$\frac{\partial E}{\partial w}$

$\frac{\partial E}{\partial b}$

$$w = w - \propto \cdot \frac{\partial E}{\partial w}$$

$$b = b - \propto \cdot \frac{\partial E}{\partial b}$$

Derivative of Sigmoid

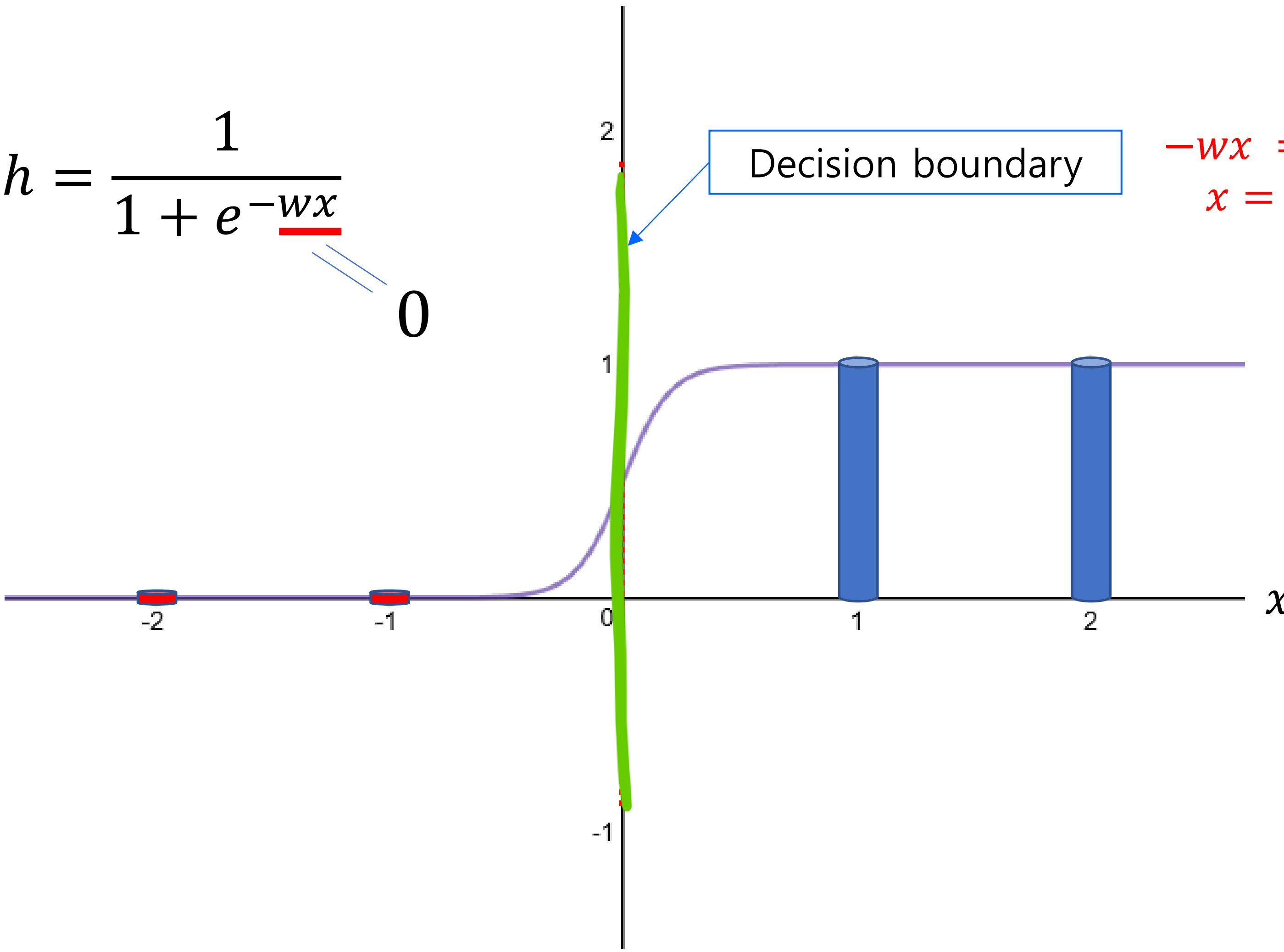

sigmoid
derivative of sigmoid
$\sigma(x) = \frac{1}{1+e^{-x}}$
$\frac{d\sigma}{dx} = \sigma(x)(1-\sigma(x))$
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
-10
-8
-6
-4
-2
0
2
4
6
8
10

Desmos.com

Decision boundary

$$h = \frac{1}{1+e^{-\underline{wx}}}$$

0



$$-wx = 0$$
$$x = 0$$

Parameters(w, b) tuning for what?

decision boundary

$$wx + b = 0$$

for **better** decision boundary

Lab 11.py

Classification of an input as 1 or 0

```
import tensorflow as tf
```

↓

```
import tensorflow.compat.v1 as tf
tf.disable_v2_behavior()
```

$$cost = -(y \log(H(X)) + (1 - y)\log(1 - H(X)))$$

```
x_data = [-2., -1, 1, 2]
y_data = [0., 0, 1, 1]
```

```
#------- a neuron
w = tf.Variable(tf.random_normal([1]))
hypo = tf.sigmoid(x_data * w)
```

```
#----- learning
cost = -tf.reduce_mean(y_data * tf.log(hypo) +
        tf.subtract(1., y_data) * tf.log(tf.subtract(1., hypo)))

train = tf.train.GradientDescentOptimizer(learning_rate=0.01).minimize(cost)

sess = tf.Session()
sess.run(tf.global_variables_initializer())

for step in range(5001):
    sess.run(train)
```

```
#----- testing(classification)
predicted = tf.cast(hypo > 0.5, dtype=tf.float32)
p = sess.run(predicted)
print("Predicted: ", p)
```

Lab 12.py
Adding a bias, *b*

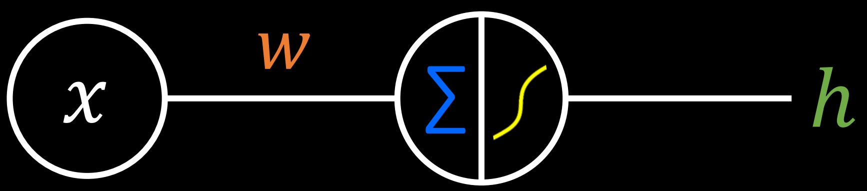
```
import tensorflow as tf
```

↓

```
import tensorflow.compat.v1 as tf
tf.disable_v2_behavior()
```

1-Input Neuron

Guess a decision boundary.

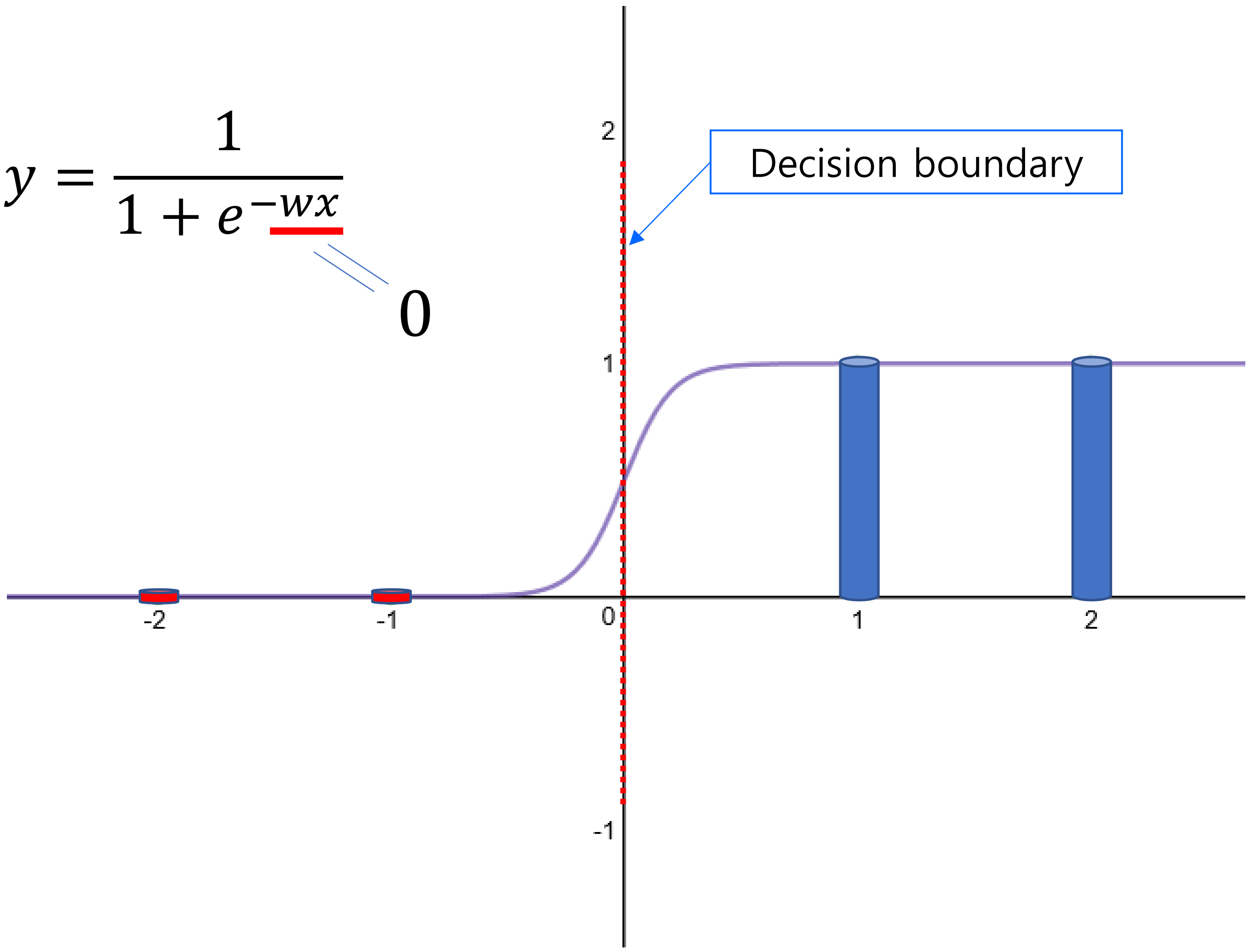


$$h = \frac{1}{1 + e^{-(wx)}}$$

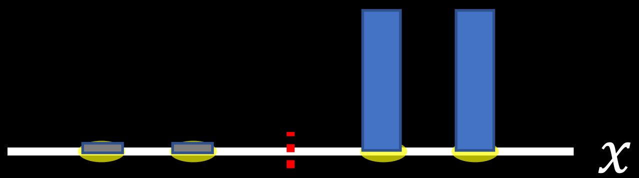
1-Input Neuron

$$y = \frac{1}{1 + e^{-\underline{wx}}}$$

0



1-Input(x) Neuron

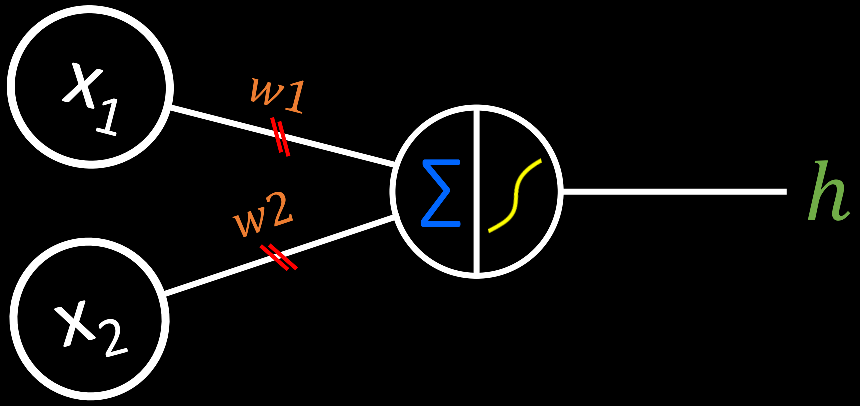


{Side view}

{View from above}

2-Input(x_1, x_2) Neuron

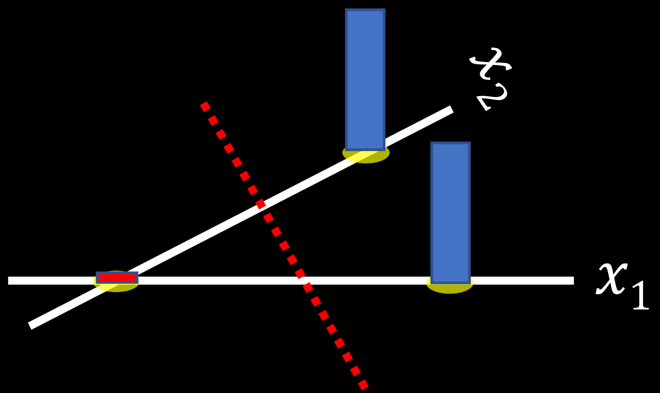
Guess a decision boundary.



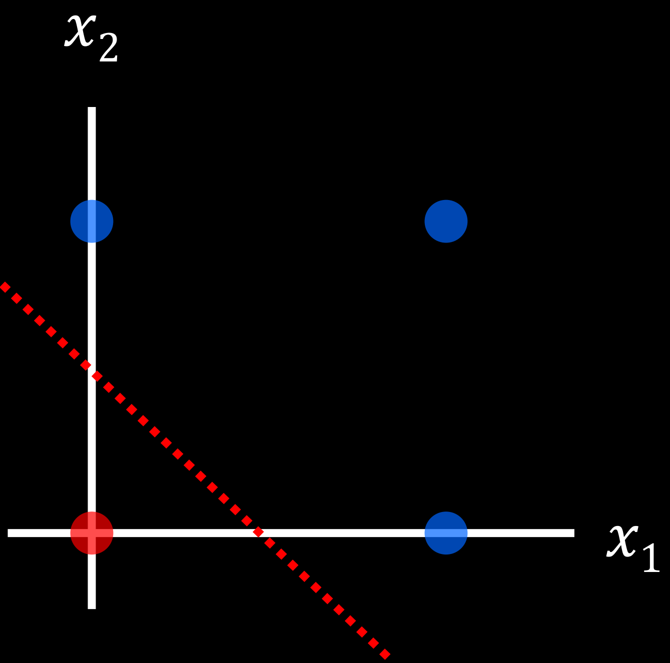
$$h = \frac{1}{1 + e^{-(w_1 x_1 + w_2 x_2)}}$$

2-Input(x_1, x_2) Neuron

x_1	x_2	y
0,	0,	0
0,	1,	1
1,	0,	1
1,	1,	1



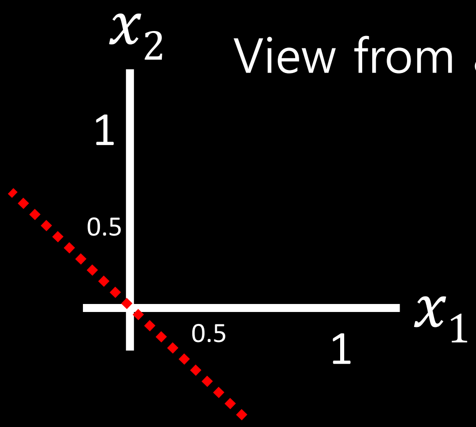
{Side view}



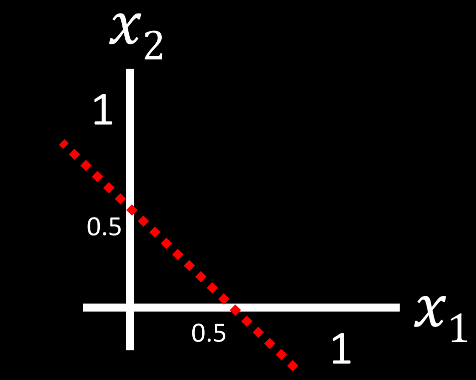
{View from above}

2-Input(x_1, x_2) Neuron

$$x_1 + x_2 = 0$$

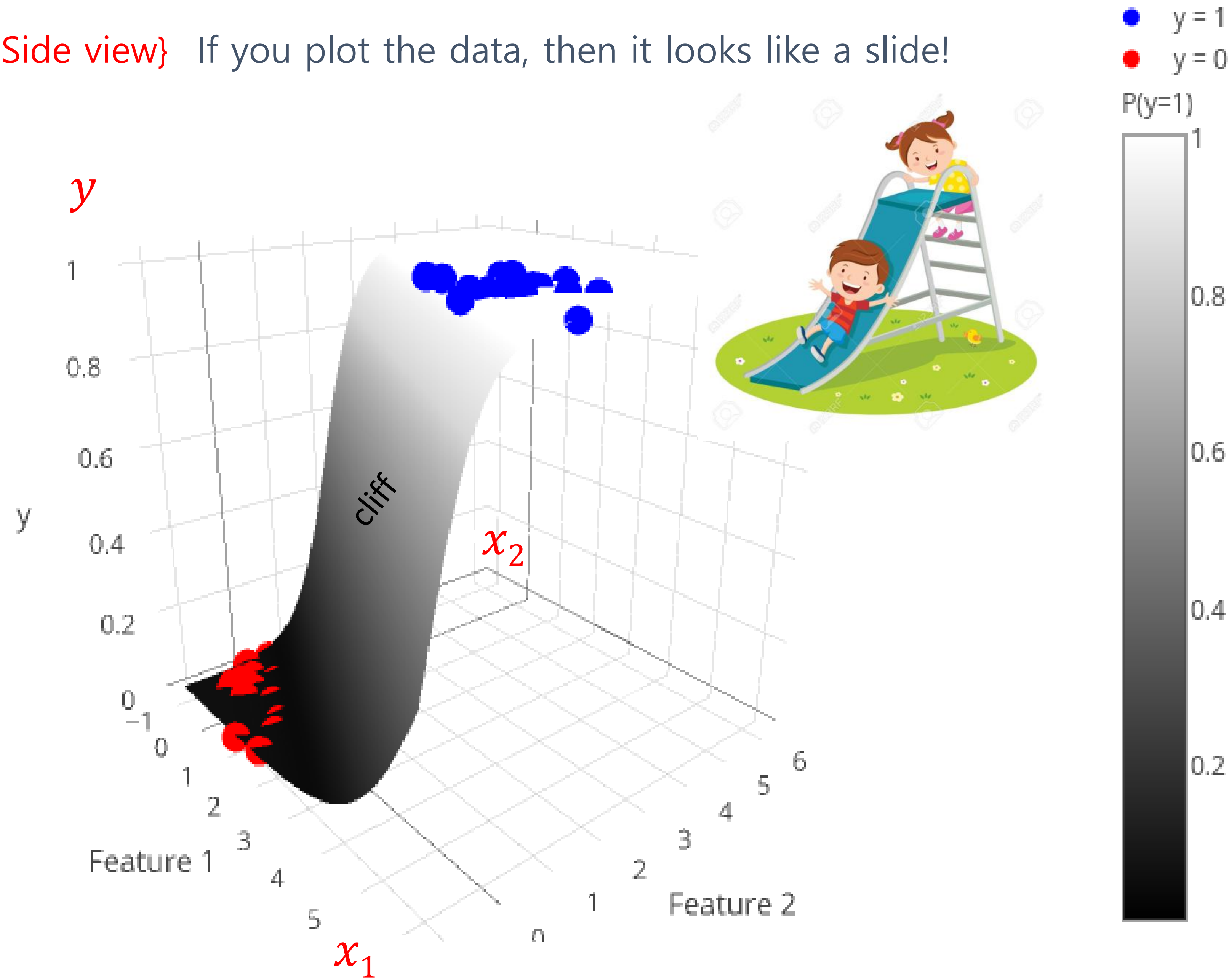


$$x_1 + x_2 - 0.5 = 0$$



Logistic Regression: 2 Features (Inputs)

{Side view} If you plot the data, then it looks like a slide!



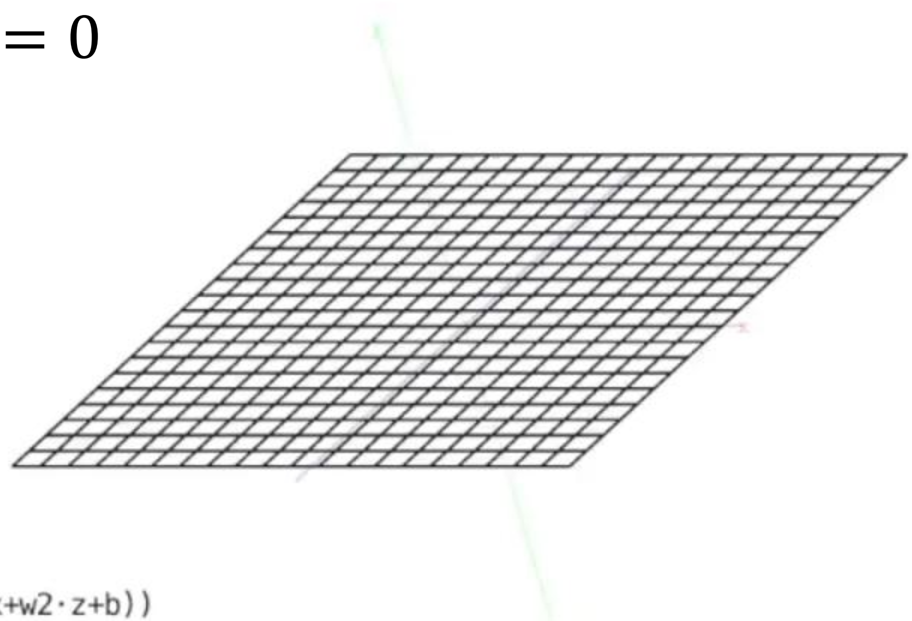
The meaning of parameters

sigmoid(w1·length + w2·width + b)

decision b. slope   decision b. shift

$$w_1 x_1 + w_2 x_2 + b = 0$$

db rotation



surface(f(x,z)=sig(w1·x+w2·z+b))

slope → w1 = 0.00

rotation → w2 = 0.00

shift → b = 0.00

Lab 13.py

Implementation of OR gate with a neuron

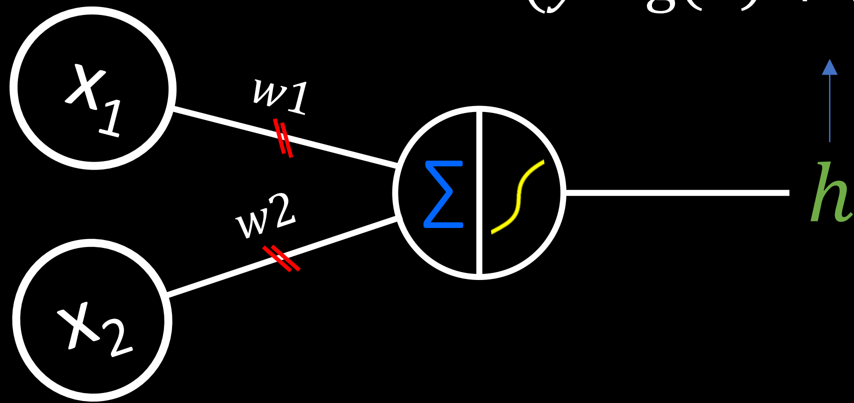
(a decision boundary)

```
import tensorflow as tf
```

↓

```
import tensorflow.compat.v1 as tf
tf.disable_v2_behavior()
```

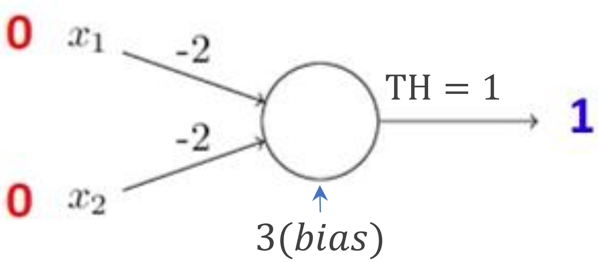
$$E = -(y\log(h) + (1-y)\log(1-h))$$



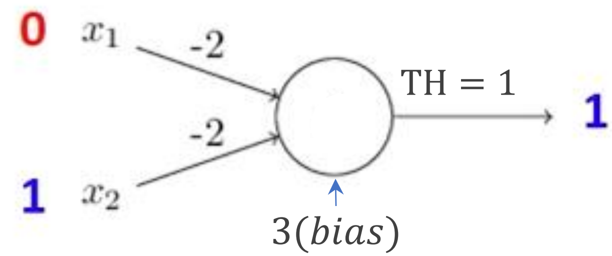
x_1	x_2	*AND(h)*
0	0	0
0	1	0
1	0	0
1	1	1

NAND

- NAND gates are functionally complete.
- We can build any logical functions out of them.



(-2)*0 + (-2)*0+3=3

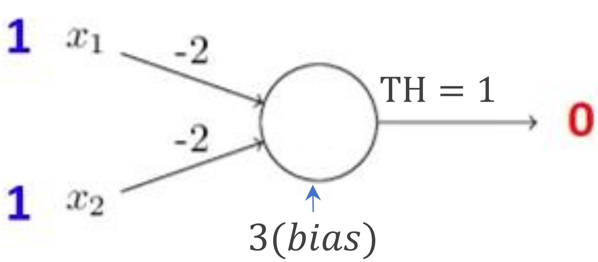


(-2)*0 + (-2)*1+3=1

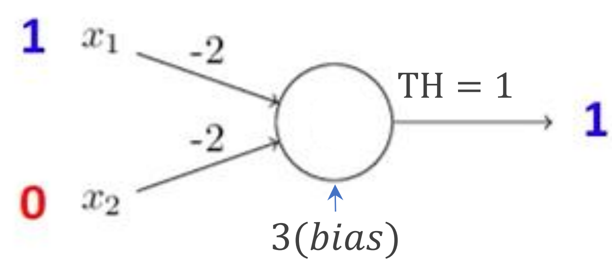
NAND

Truth Table

Input A	Input B		Output Q
0	0		1
0	1		1
1	0		1
1	1		0



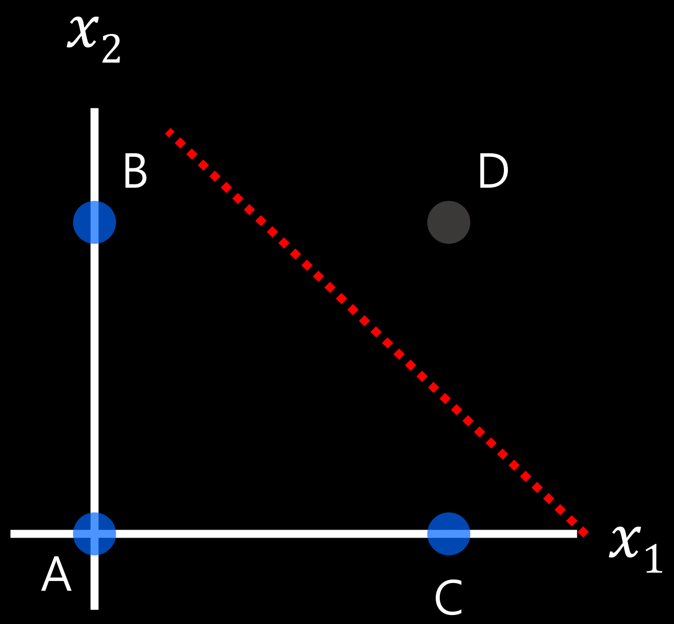
(-2)*1 + (-2)*1+3=-1



(-2)*1 + (-2)*0+3=1

Decision boundary by a neuron

with two inputs



View from above

Decision boundary by a neuron

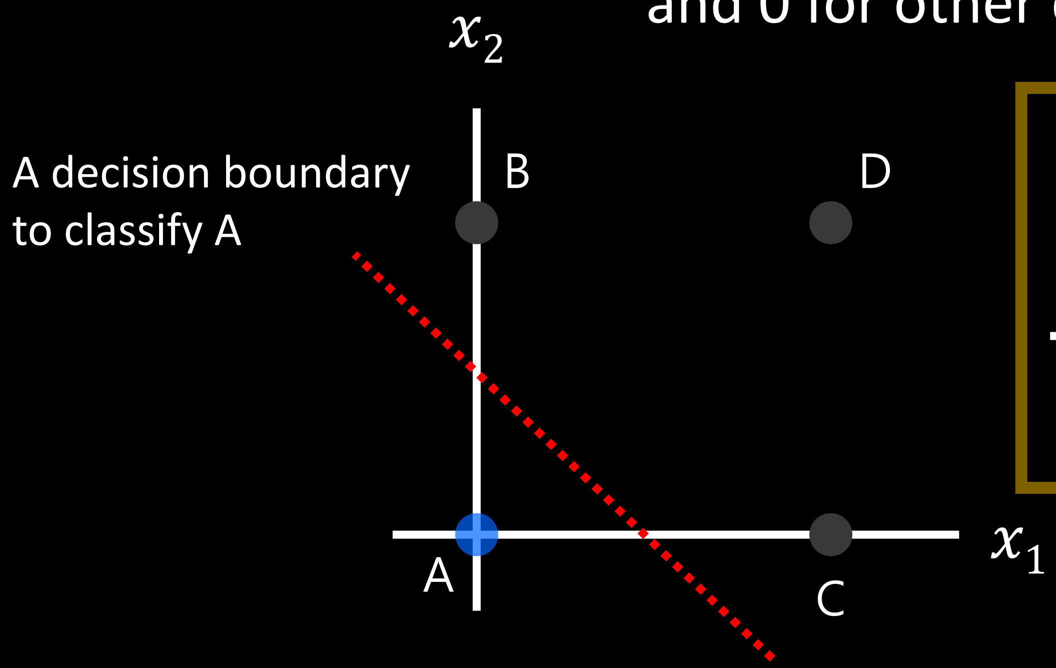
with two inputs

- A neuron, only 1 linear decision boundary
- A decision boundary yielding 2 classes (1 or 0)
- How to solve multiple classes more than 2?

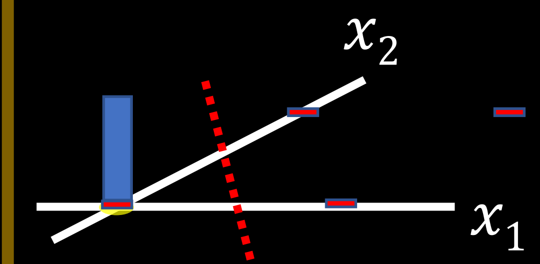
4-Class (A, B, C, D) Classification

Neuron #1

The output of a neuron is 1 for A, and 0 for other cases(B, C, D).



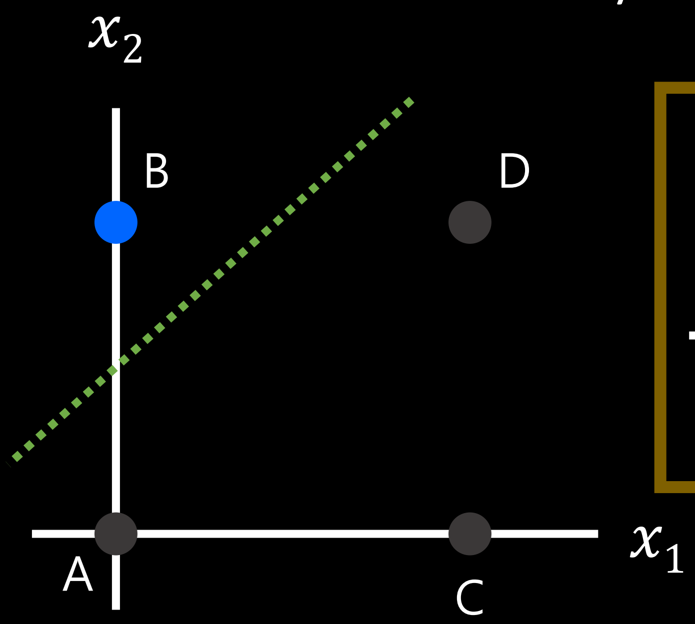
View from above

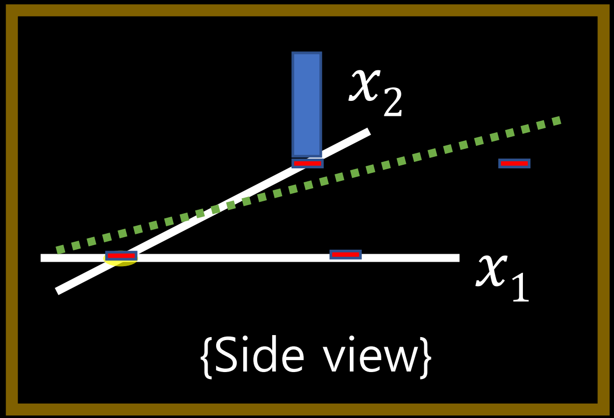


{Side view}

Neuron #2

2nd neuron for 2nd decision boundary to classify B

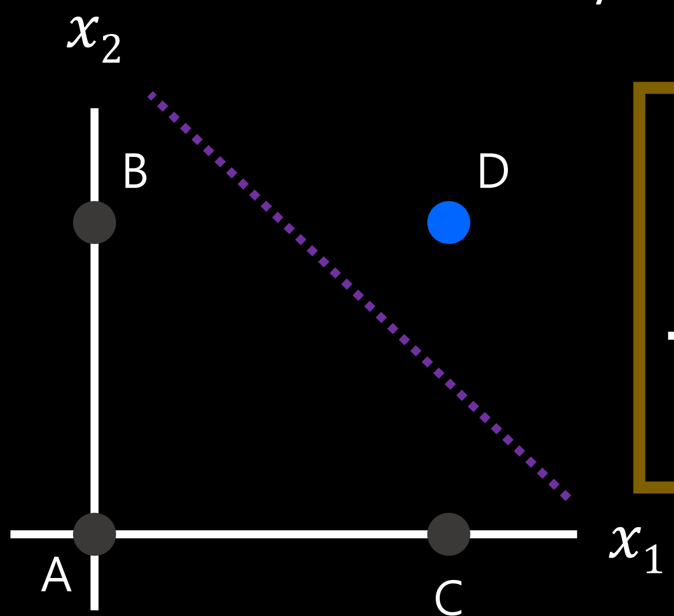




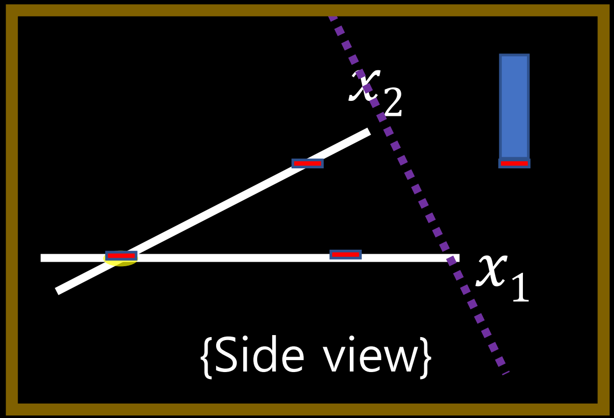
View from above

Neuron #3

3rd neuron for 3rd decision boundary to classify D

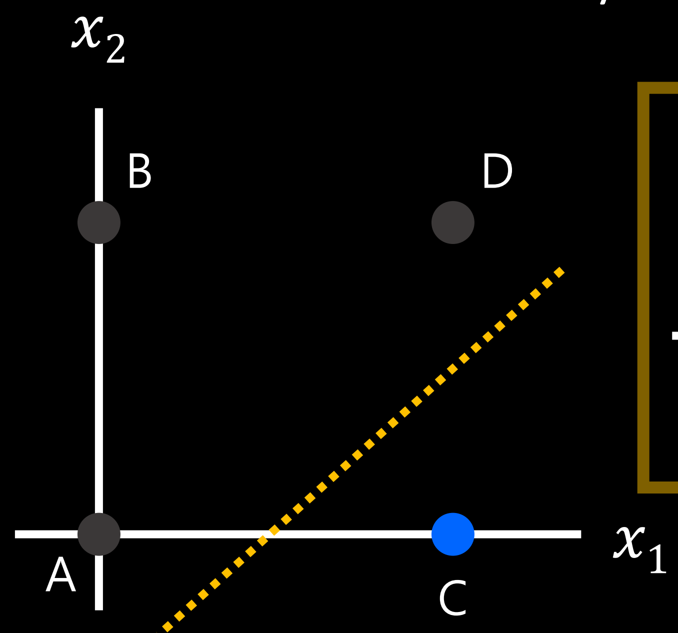


View from above

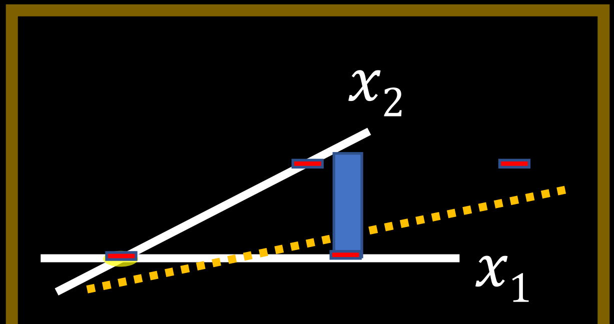


Neuron #4

4th neuron for 4th decision boundary to classify C

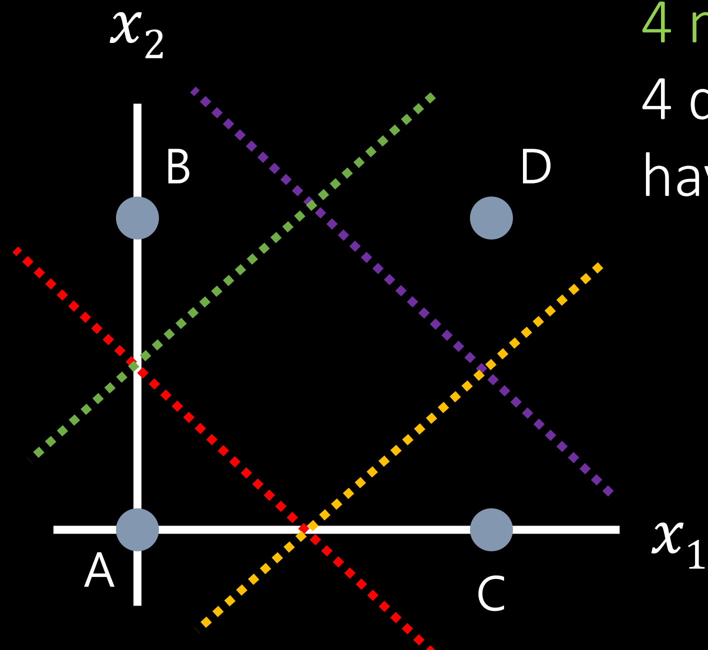


View from above



{Side view}

4 Neurons with 2 inputs



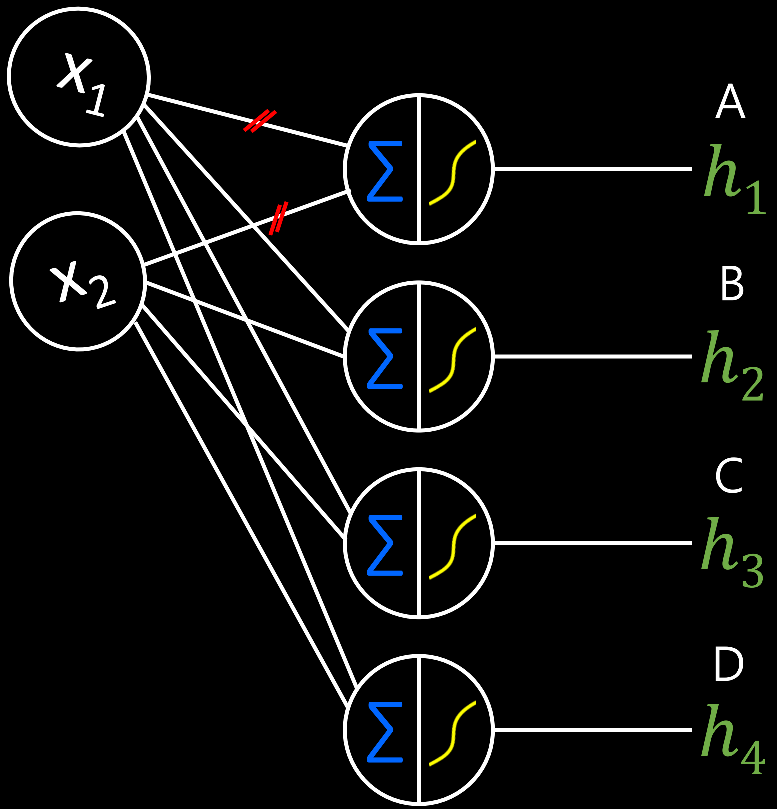
View from above

4 neurons for
4 decision boundaries
having the same inputs

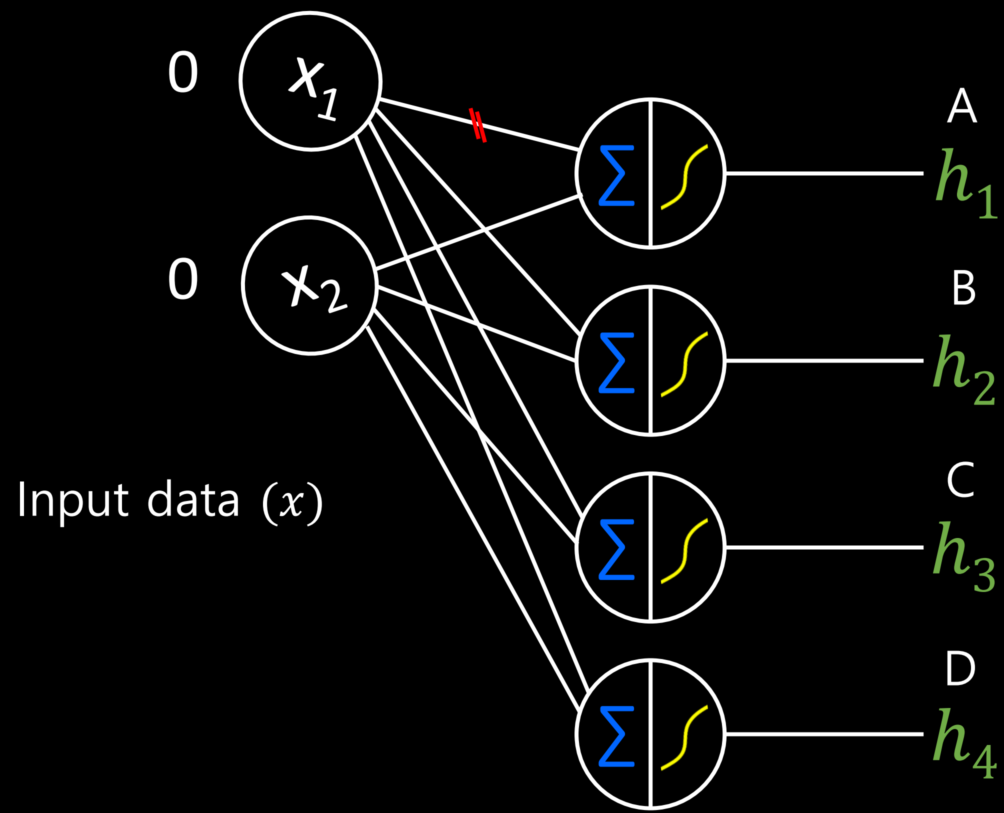
4 Neurons

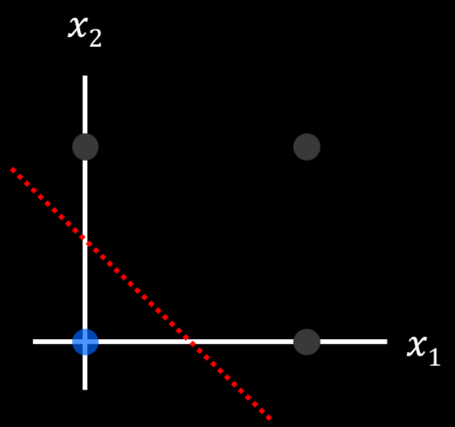
Matrix notation

$$(x_1, x_2)\begin{pmatrix} w_{11}, w_{21}, w_{31}, w_{41} \\ w_{12}, w_{22}, w_{32}, w_{42} \end{pmatrix} \longrightarrow (h_1, h_2, h_3, h_4)$$

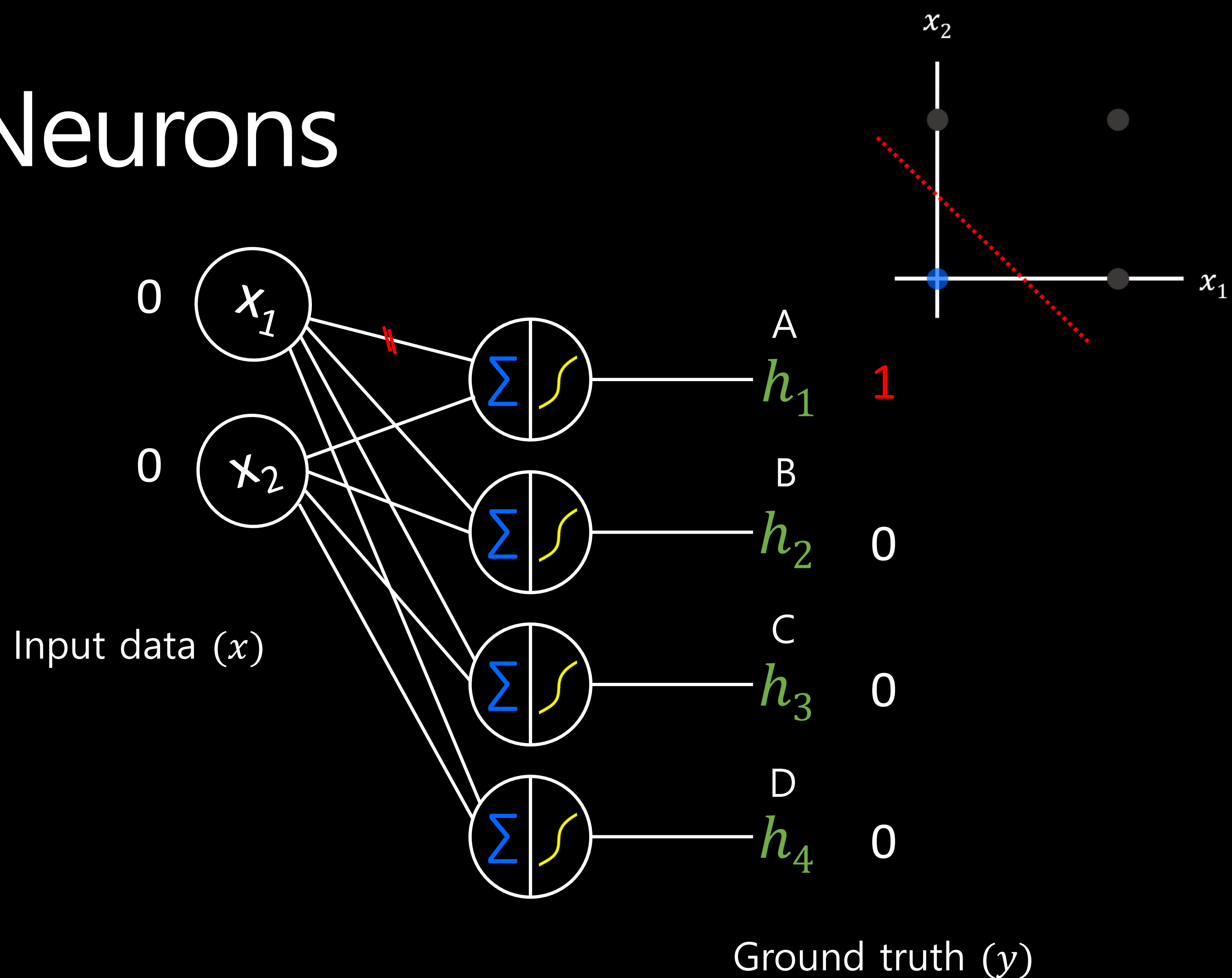


4 Neurons

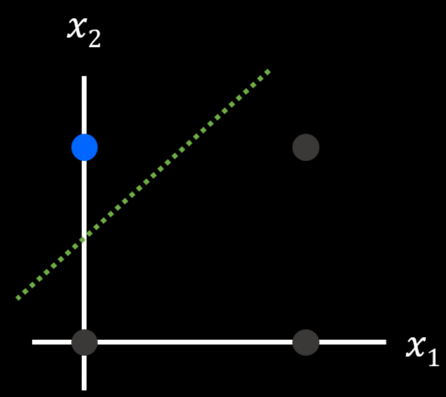


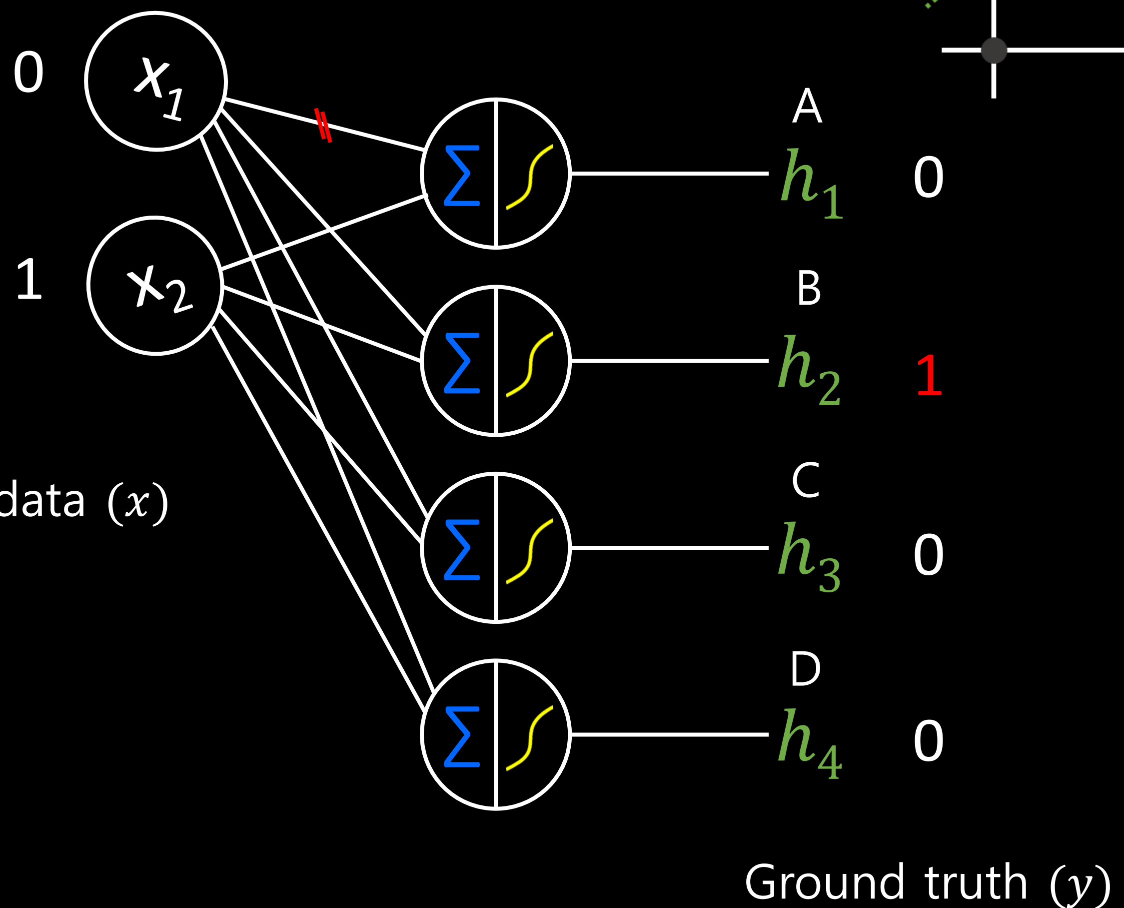


4 Neurons

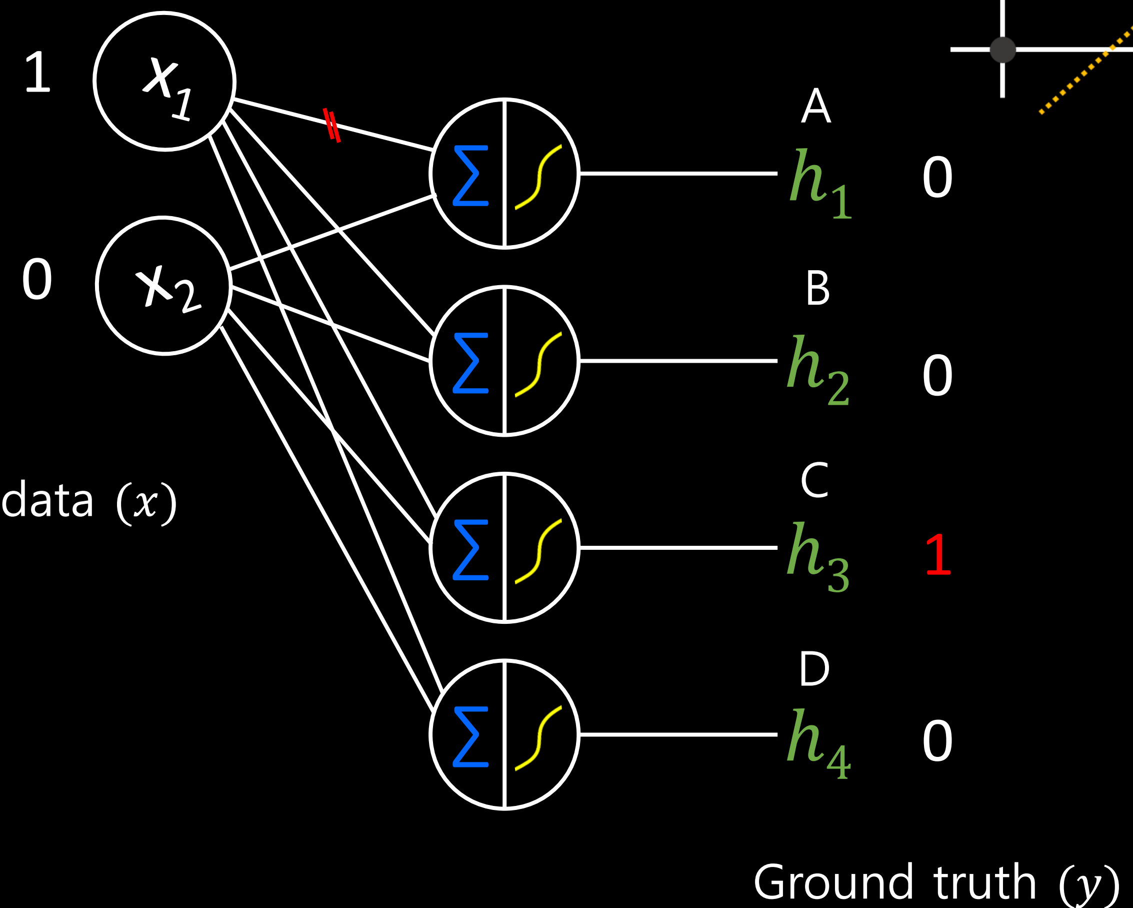


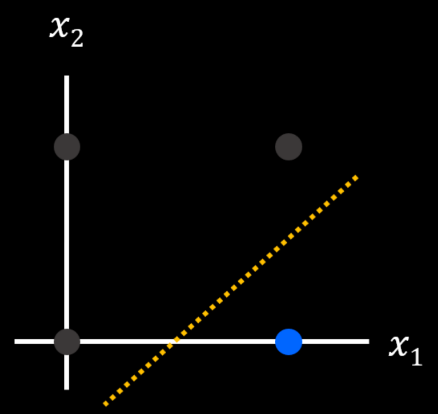
4 Neurons



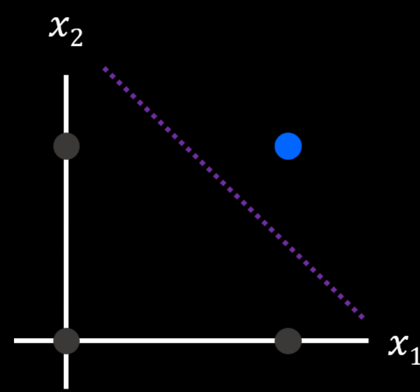


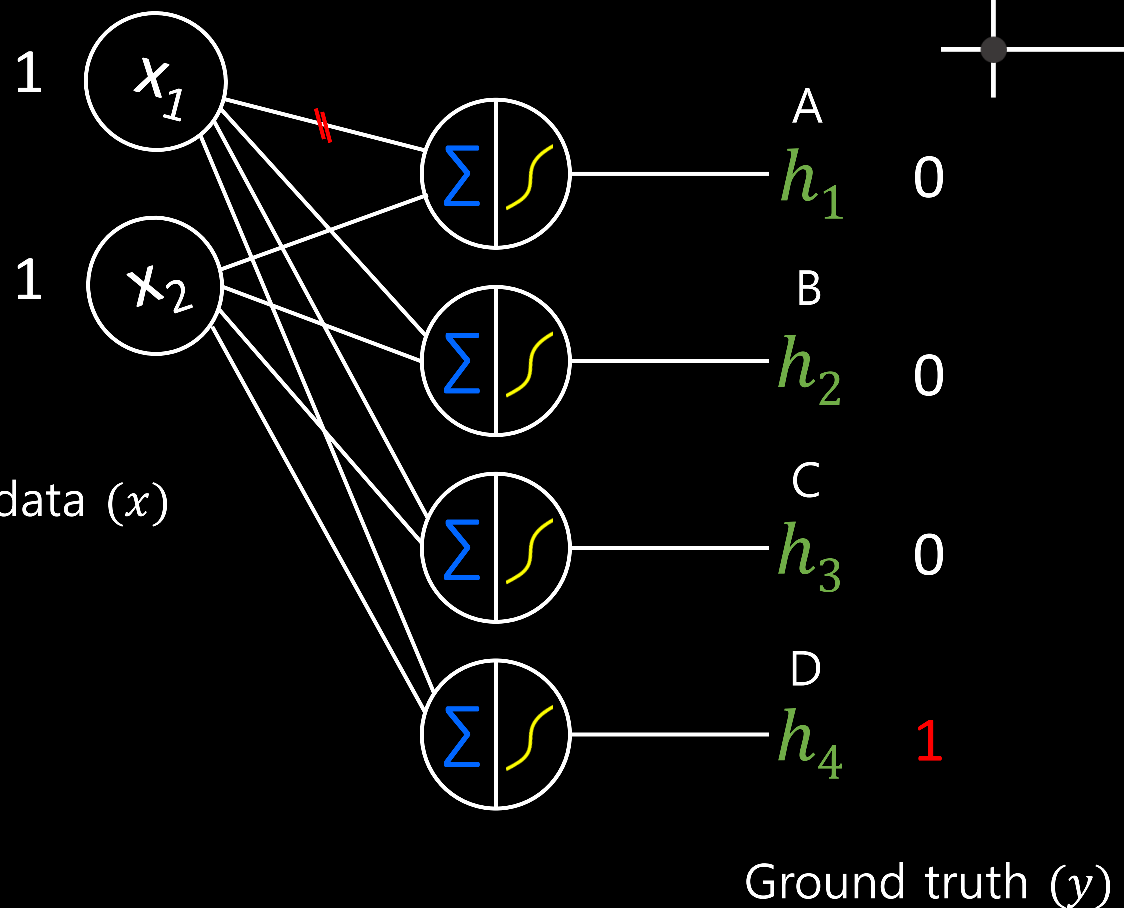
4 Neurons





4 Neurons





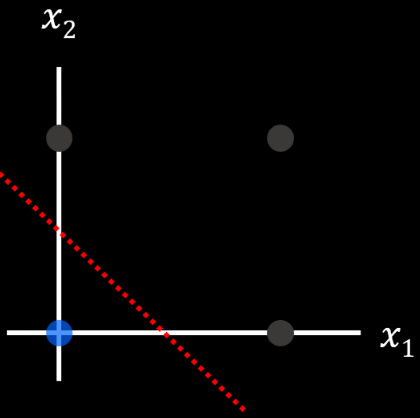
One-hot Encoding

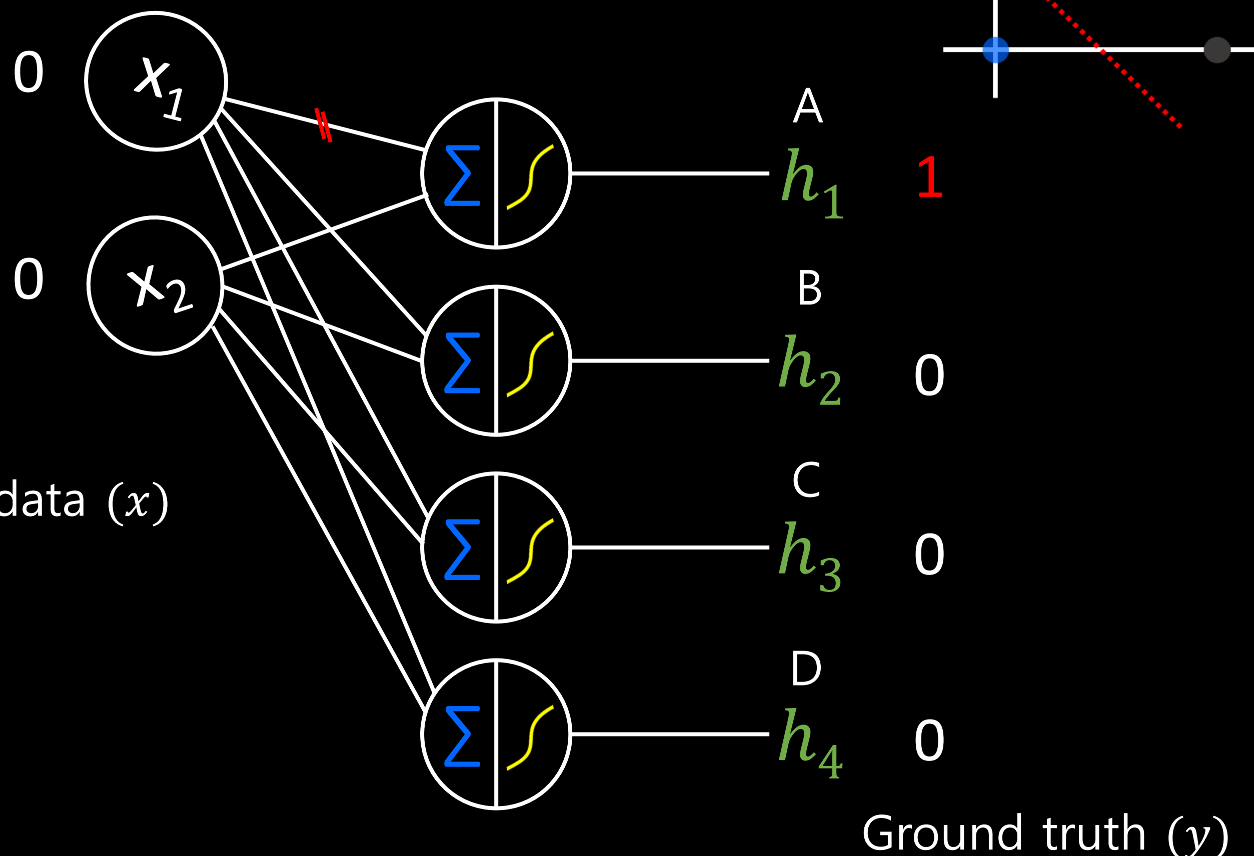
- For the ground truth (y), setting only one output as ON(1) and others as OFF(0) → One-hot encoding

Considerations

- Only one of the neuron's outputs should be 1, and others should be 0.
- However, each neuron produces output independently according to the weights
- No way to control the 4 outputs together to fit it
- A special function introduced → Softmax

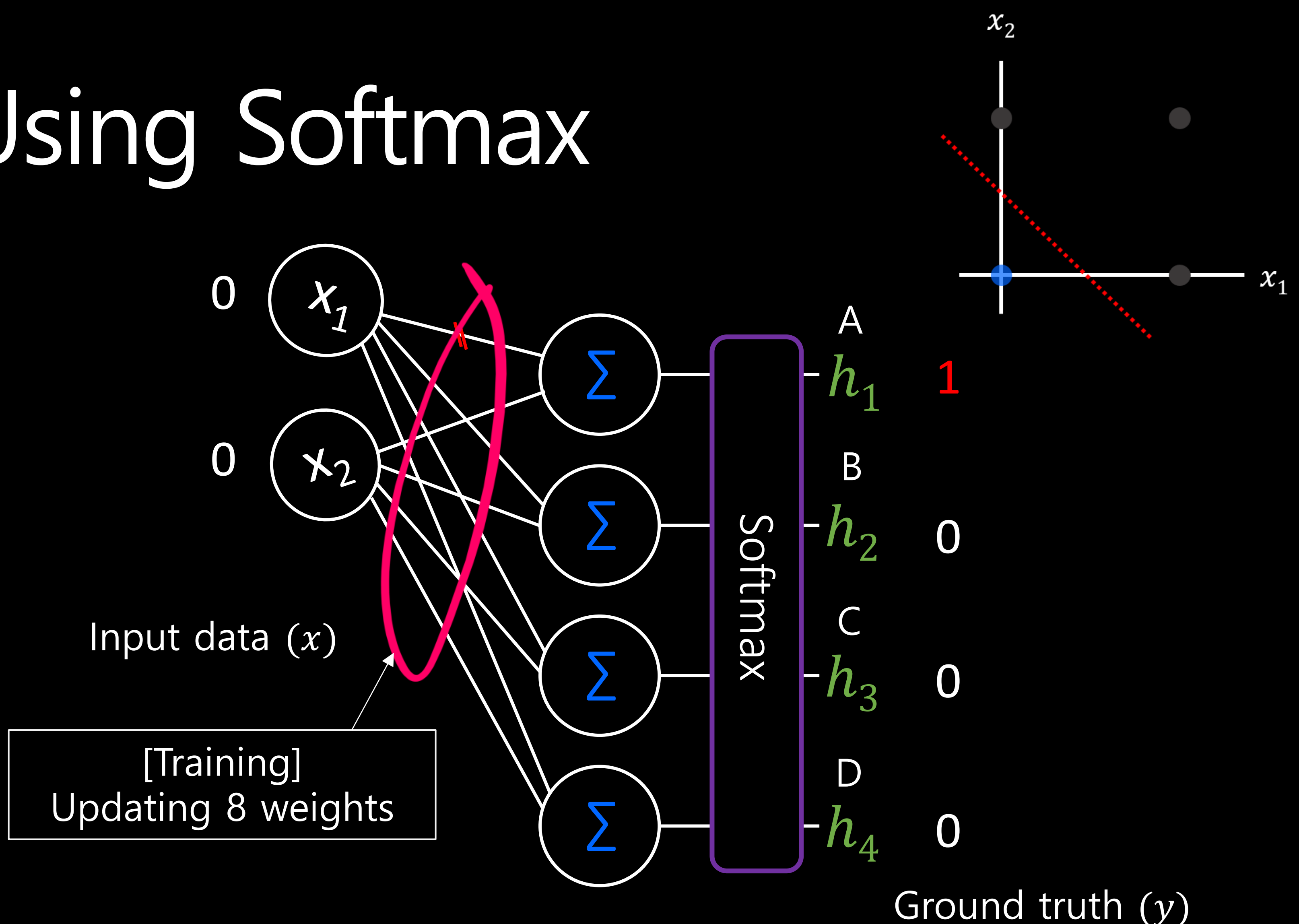
Without Softmax





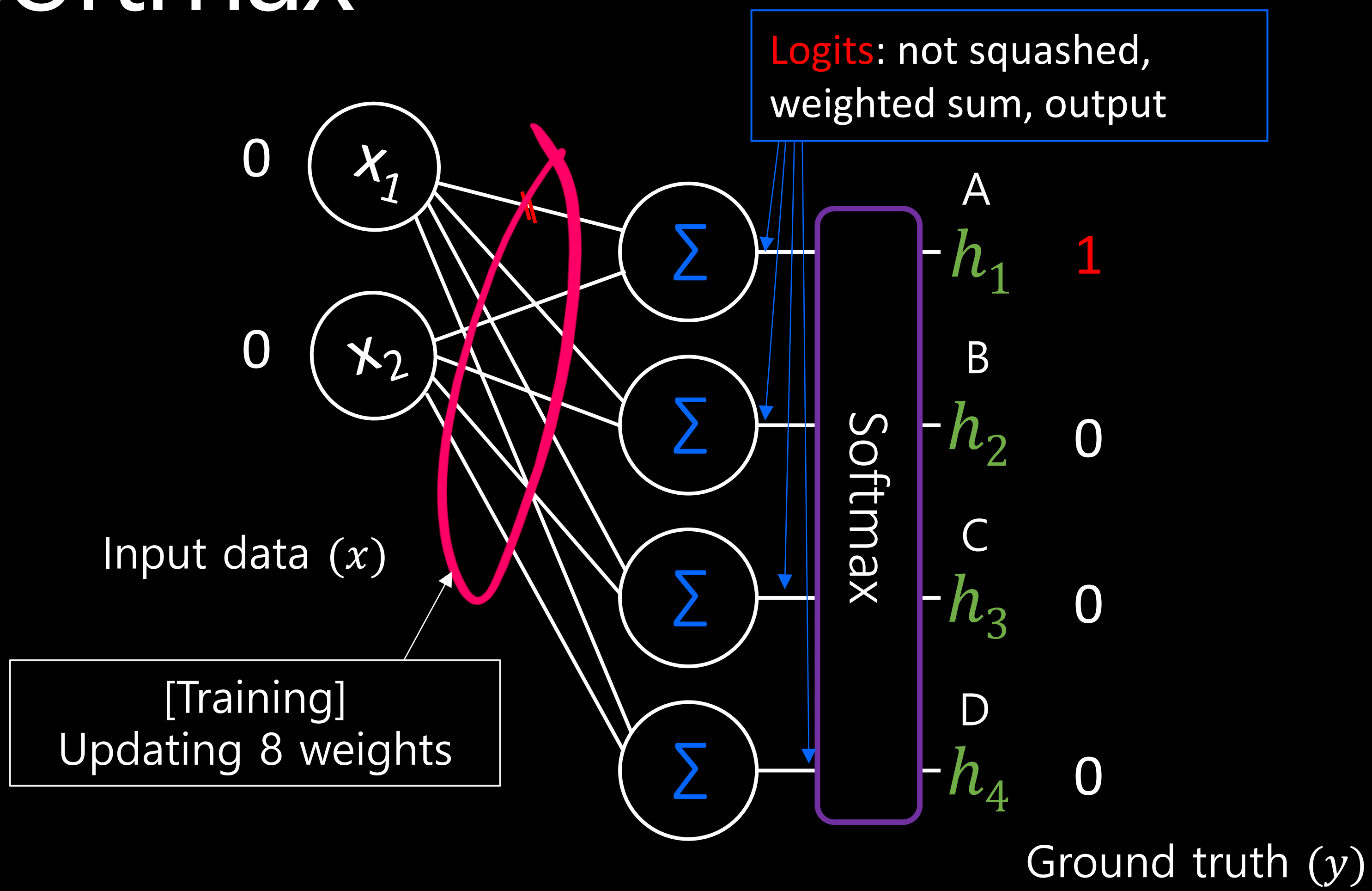
Initial architecture

Using Softmax



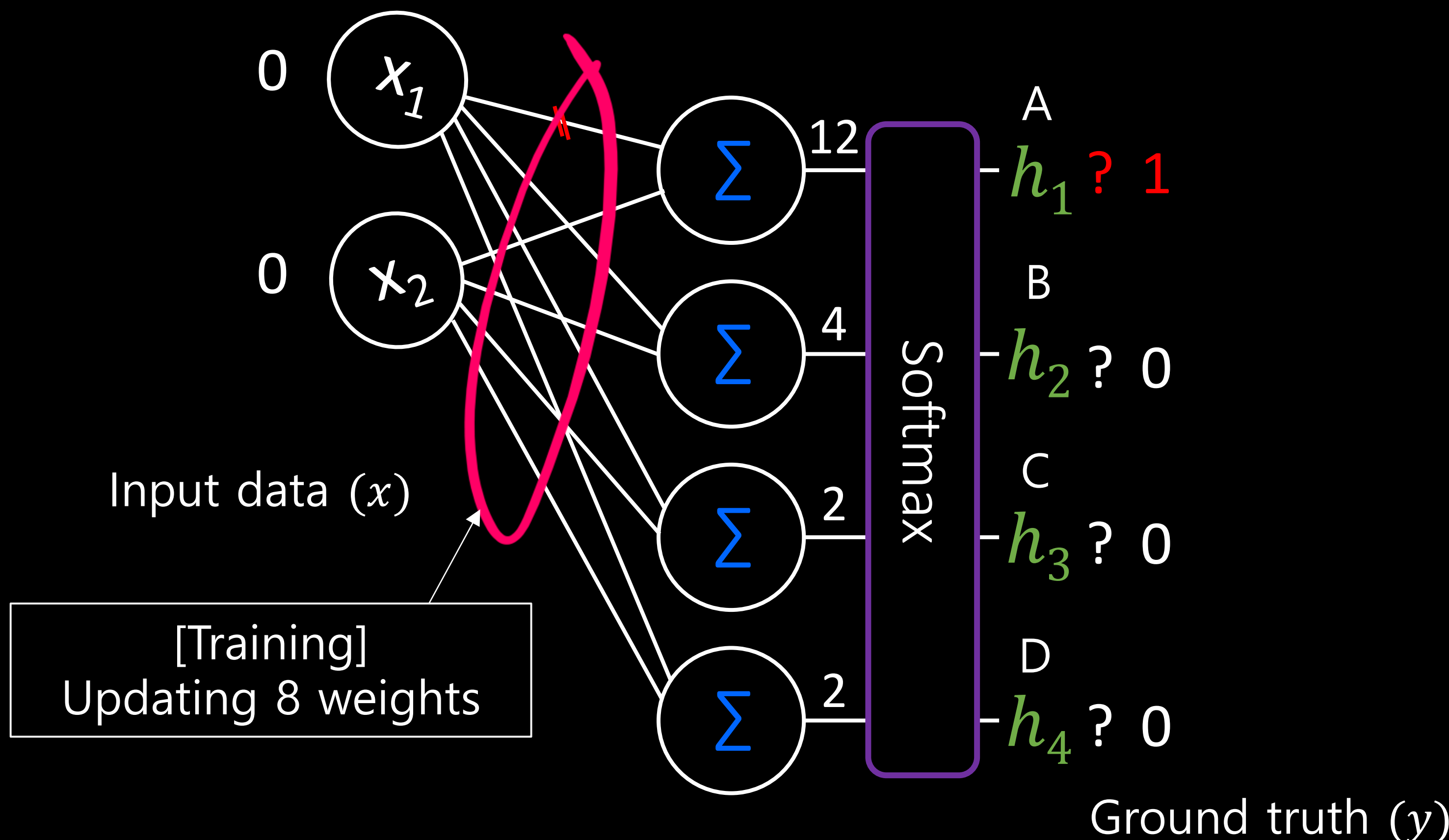
New architecture

Softmax



New architecture

Softmax

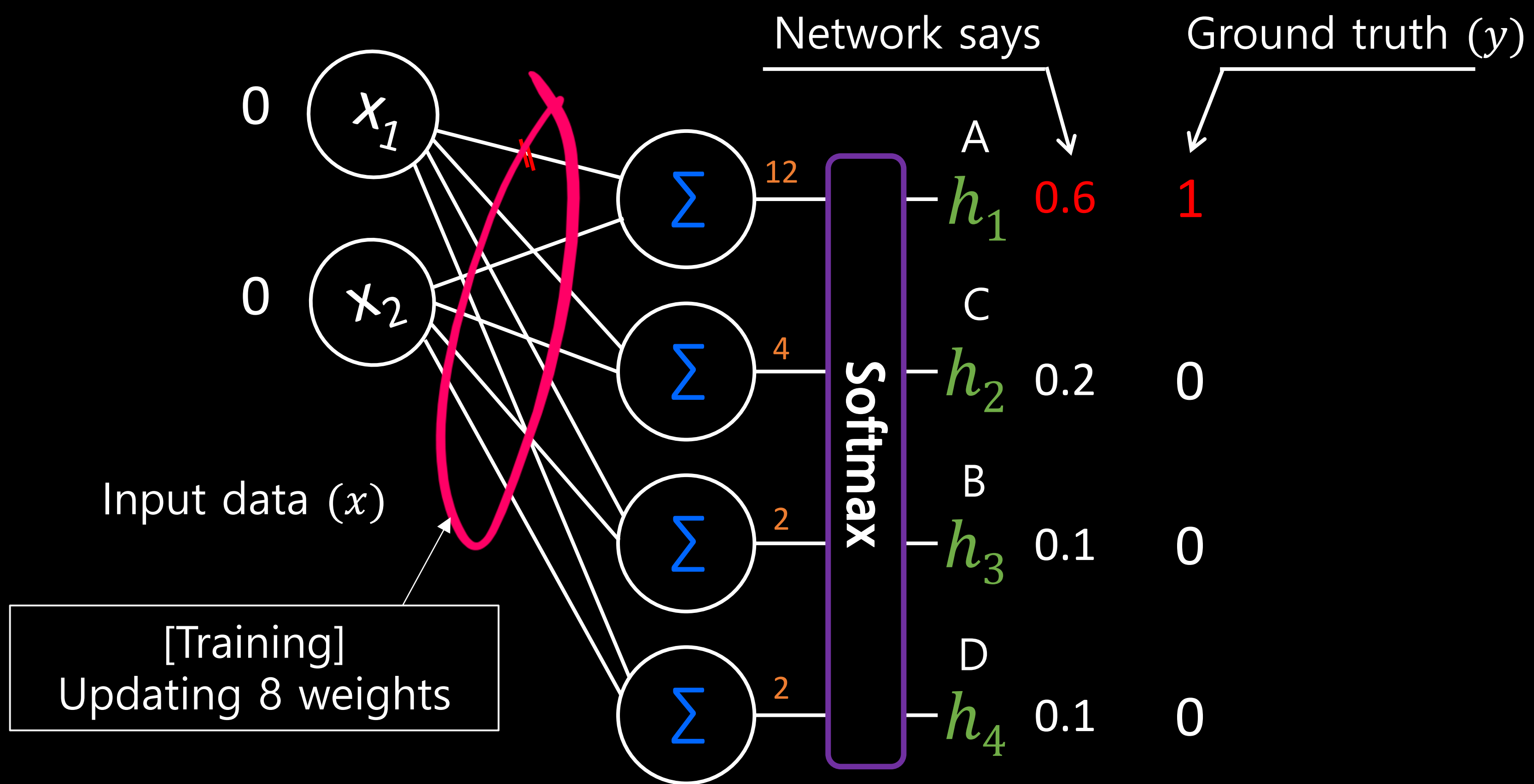


New architecture

Softmax

- for 12, 4, 2, 2, the Softmax function returns $\frac{12}{20}, \frac{4}{20}, \frac{2}{20}, \frac{2}{20}$.
- Normalization of logits values $\longrightarrow$ the probability for each class

Softmax



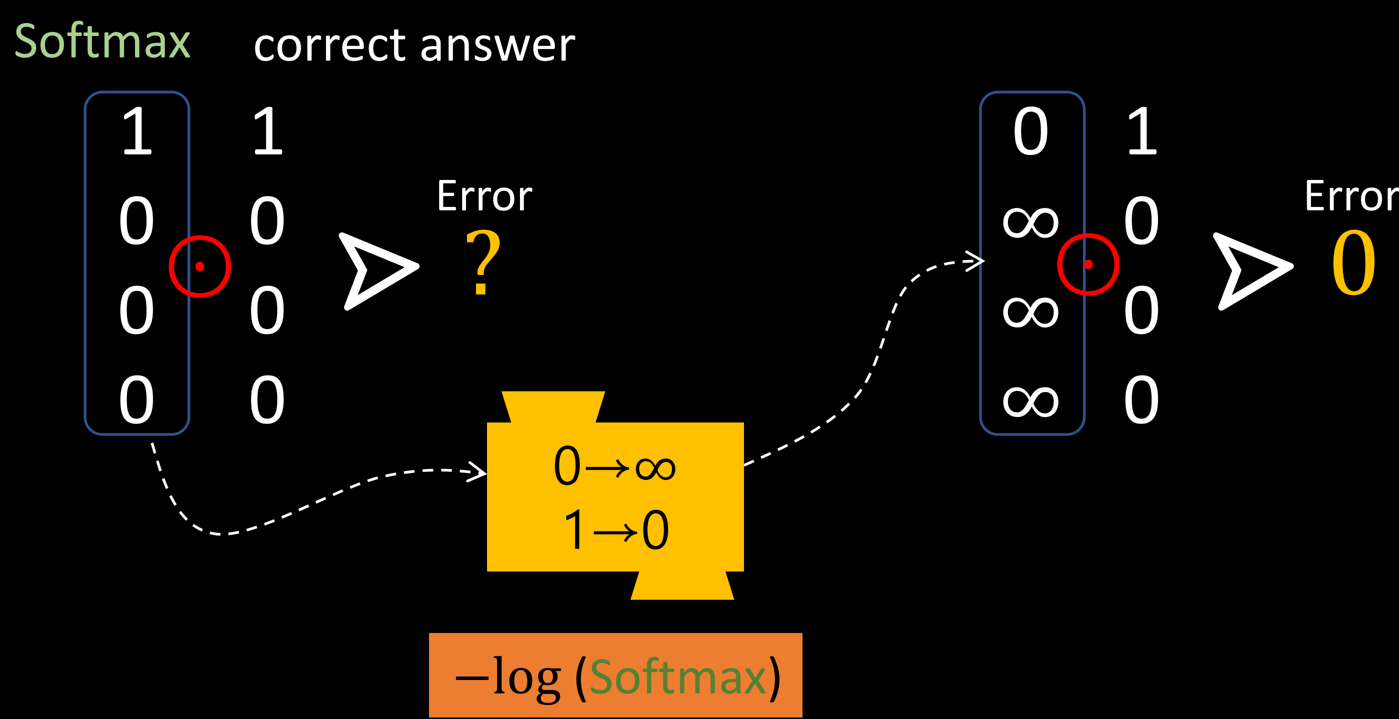
New architecture

New Error Function

- Distance between the output of a network(after Softmax) and the correct answer (ground truth)
- If answer correctly, then the distance is 0,
- If not(incorrect), then the distance would be big or ∞

New Error Function

If the answer is correct, then the error(distance) is 0.

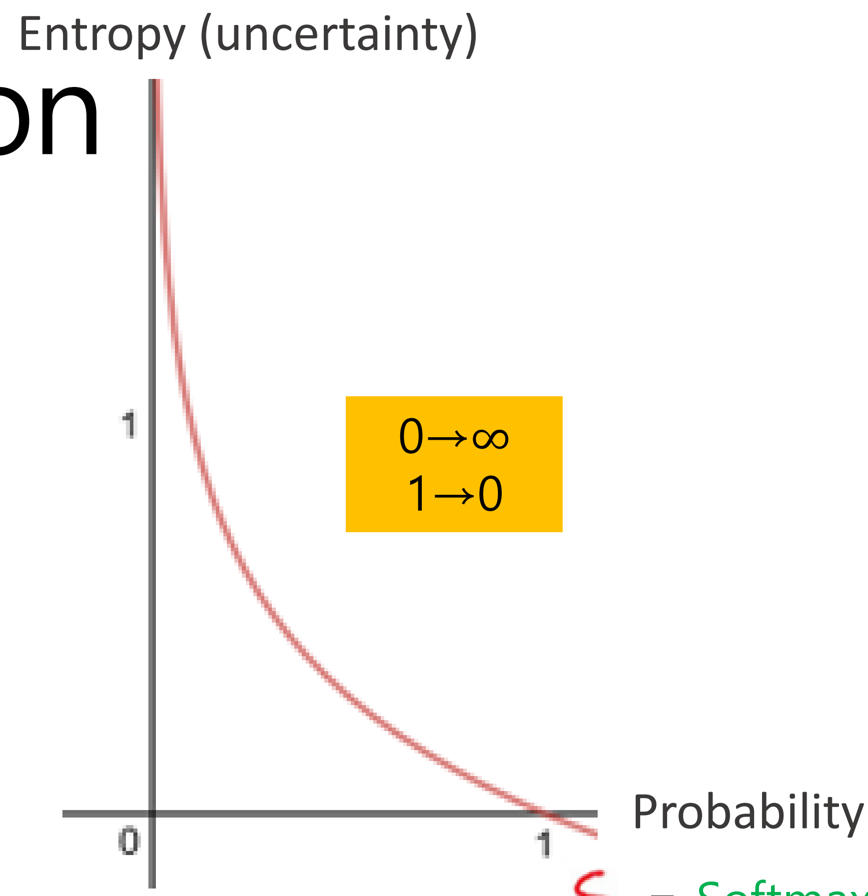


⊙ : element wise product

Probability to Entropy (uncertainty)

$-\log$ function

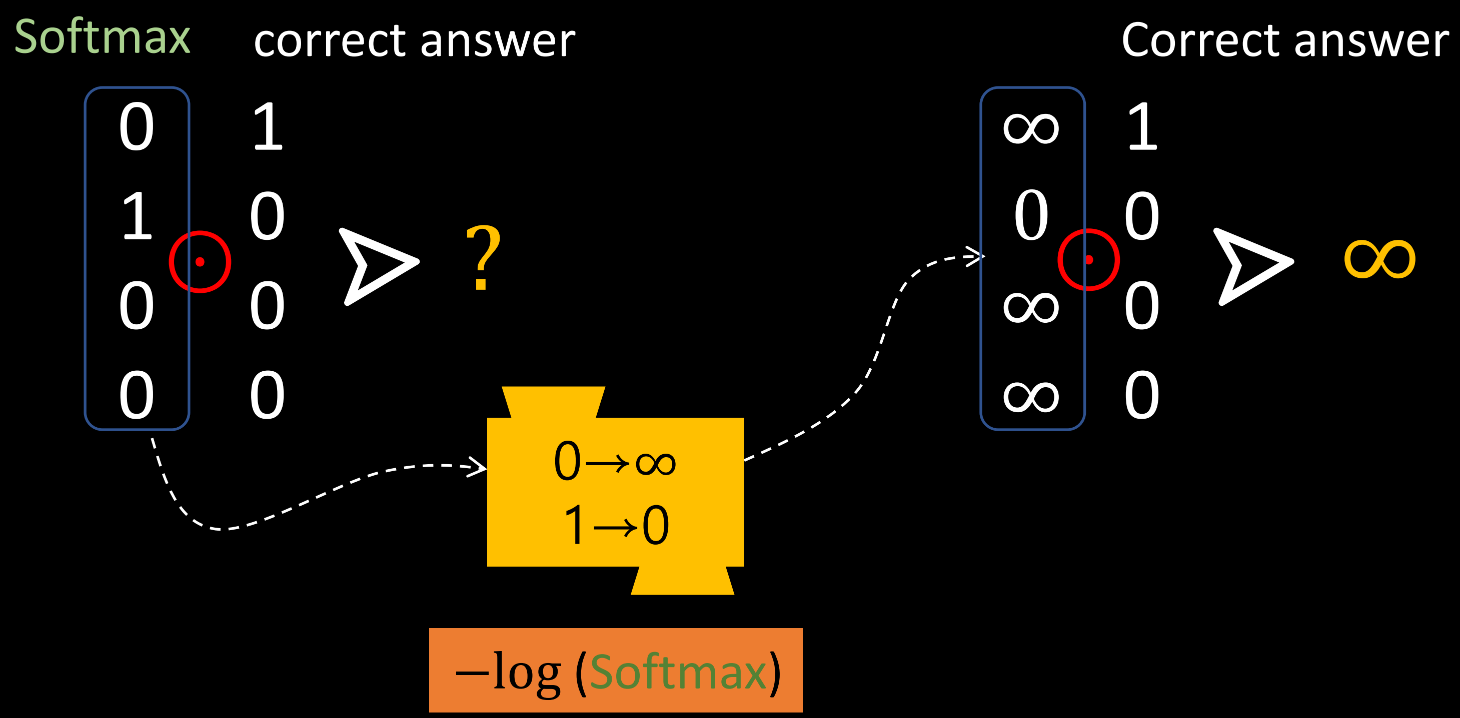
$-\log(S)$



S = Softmax

New Error Function

If incorrect, then the error(distance) is big or ∞.



New Error Function

If incorrect, then the distance(error) is ∞.

Softmax correct answer

$$\boxed{\begin{matrix}0\\1\\0\\0\end{matrix}} \odot \begin{matrix}1\\0\\0\\0\end{matrix} \;\triangleright\; ?$$

$0 \rightarrow \infty$

$1 \rightarrow 0$

$-\log(\text{Softmax})$

Correct answer

$$\boxed{\begin{matrix}\infty\\0\\\infty\\\infty\end{matrix}} \odot \begin{matrix}1\\0\\0\\0\end{matrix} \;\triangleright\; \infty$$

$-L \odot \log(S)$

$\odot$: element wise product

New Error Function

$$-L \log(S)$$

correct answer *L*

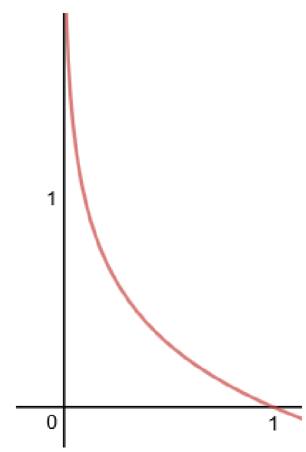
$$-\sum_i L_i \log(S_i)$$

New Error Function

$$D(S, L) = -\sum_i L_i \log(S_i)$$

$$\begin{bmatrix}0.7\\0.2\\0.1\end{bmatrix} \qquad \begin{bmatrix}1.0\\0.0\\0.0\end{bmatrix}$$

$S(Y)$ $\qquad$ L

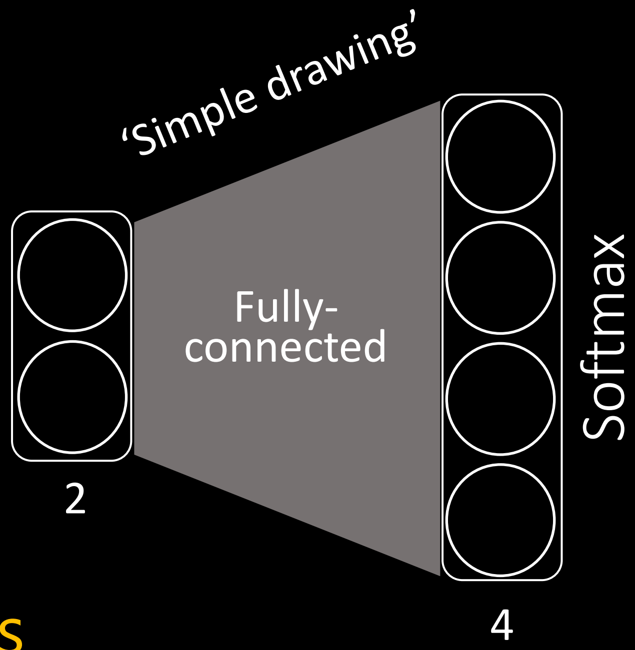


New Error Function

softmax_cross_entropy_with_logits(logits, y_data)

- The function returns 0 if the answer is correct,
- or returns ∞ if the answer is totally incorrect.

Lab 14.py



- Classification into one of four classes
- 4 neurons where each has 2-input
- A bias for each neuron

import tensorflow as tf

↓

import tensorflow.compat.v1 as tf
tf.disable_v2_behavior()