Introduction to Physics– Modern Physics

taught by

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Textbook: Physics for Computer Science Students, by N. Garcia, A. Damask, and S. Schwarz, 2^{nd}, 1998.

Chapter Thirteen: Electrostatics

- The interaction of charges at rest is called *electrostatics*.
- *Superposition principle*: The behavior of multiple charges on one another is a simple sum of the one-to-one interactions (pairwise).

Charges

- There are two different types of charges called *positive* and *negative*.
- Metals have been known as electrical *conductors* and nonmetals as *insulators*.

Coulomb's Law

- Attraction and repulsion of charges: see Fig. 13-2.
- Coulomb's experiments concerning the forces between charges : see Fig. 13-3. The force diagram for q_2 is shown in Fig. 13-3b and the system is in the equilibrium state. Thus,

$$T_1 \cos\theta + F_{\text{electrostatic}} = Mg$$

- Coulomb's law:

$$F \propto \frac{q_1 q_2}{r^2}$$

where the sign of q_1 and q_2 may be wither plus or minus, and r is the distance between q_1 and q_2.

- In the SI system the constant in Coulomb's law is taken as

$$\frac{1}{4\pi\epsilon_0} = 9 \times 10^9 \text{ N-m}^2/\text{C}^2$$

The symbol C stands for Coulomb and is the unit of charge.

- The charge of the electron in coulombs is $e = -1.6 \times 10^{-19}$ C.
- Coulomb's law:

$$F = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r^2}$$

- The direction of the force that q_1 exerts on q_2 is along the line joining the two charges, pointing away from q_1 if the force is repulsive or toward q_1 if the force is attractive.

Example 13-1

Two pith balls of mass 0.1 g each are suspended on 50-cm threads. They are given equal charges and assume a position in which each makes an angle of 20° with vertical, as in Fig. 13-4a. What is the charge on each?

Sol: The vector diagram of the forces on the right-hand ball is shown in Fig. 13-4b, where F is the coulombic force of repulsion between the two charged pith balls.

$$\begin{aligned}
\sum F_x &= 0 \\
F - T\cos 70^\circ &= 0 \\
F &= 0.34T \\
T\sin 70^\circ - mg &= 0 \\
T &= \frac{mg}{\sin 70^\circ}
\end{aligned}$$

$$\begin{aligned} &= \frac{0.1 \times 10^{-3}\ \text{kg} \times 9.8\ \text{m/sec}}{0.94} \\ &= 1.04 \times 10^{-3}\ \text{N} \end{aligned}$$

Substituting this value of T in the equation for F, we have

$$F = 0.34T = 0.34 \times 1.04 \times 10^{-3}\ \text{N} = 3.5 \times 10^{-4}\ \text{N}$$

$$\begin{aligned} r &= 2\ell \sin 20^\circ \\ &= 2 \times 0.5\ \text{m} \times \sin 20^\circ \\ &= 0.34\ \text{m} \end{aligned}$$

Using Coulomb's law,

$$F = 9 \times 10^9 \frac{\text{N-m}^2}{\text{C}^2} \frac{q^2}{r^2}$$

because

$$q_1 = q_2$$

and substituting for F and r we have

$$3.5 \times 10^{-4}\ \mathrm{N} = \frac{9 \times 10^{9}\ \frac{\mathrm{N\text{-}m^2}}{\mathrm{C^2}} q^2}{(0.34\ \mathrm{m})^2}$$

or

$$q = 6.7 \times 10^{-8}\ \mathrm{C}$$

Charge of an Electron

- In the years 1909 through 1913 R. Millikan measured the charge on an electron by the system shown in Fig. 13-5.
- With a spray he introduced fine oil drops between two parallel metal plates and observed the motion of a single drop through a telescope.
- He found the drops usually acquired a negative charge.
- He also found that the smallest charge that was ever acquired by the drop had a magnitude of 1.6×10^{-19} C and that larger charges were always integral multiples of this quantity.

Superposition Principle

- If one selects a given charge in a group and asks for the total force on it, this force would be the resultant of the individual vector forces on it from each of the charges. This is called the *superposition principle* of charges.

Example 13-2

Three charges are arranged in a triangle as shown in Fig. 13-6a. What is the direction and the magnitude of the resultant force on the 1×10^{-8} C charge?

Sol:

$$F_1 = 9 \times 10^9 \frac{\text{N-m}^2}{\text{C}^2} \frac{1 \times 10^{-8}\ \text{C} \times 4 \times 10^{-8}\ \text{C}}{10^{-2}\ \text{m}^2} = 3.6 \times 10^{-4}\ \text{N}$$

at 30° above the positive x axis.

$$F_2 = 9 \times 10^9 \frac{\text{N-m}^2}{\text{C}^2} \frac{1 \times 10^{-8}\ \text{C} \times 2 \times 10^{-8}\ \text{C}}{10^{-2}\ \text{m}^2} = 1.8 \times 10^{-4}\ \text{N}$$

at 30° below the positive x axis.

We now use the vector diagram of these two forces and find the resultant by the component method of Chapter 2. We have

$$\begin{aligned}
\boldsymbol{F}_1 &= 3.6 \times 10^{-4}\ \mathrm{N}\cos 30^\circ \boldsymbol{i} + 3.6 \times 10^{-4}\ \mathrm{N}\sin 30^\circ \boldsymbol{j} \\
\boldsymbol{F}_2 &= 1.8 \times 10^{-4}\ \mathrm{N}\cos 30^\circ \boldsymbol{i} - 1.8 \times 10^{-4}\ \mathrm{N}\sin 30^\circ \boldsymbol{j} \\
\boldsymbol{R} &= 4.7 \times 10^{-4}\ \mathrm{N}\boldsymbol{i} + 0.9 \times 10^{-4}\ \mathrm{N}\boldsymbol{j} \\
|R| &= \sqrt{(4.7 \times 10^{-4}\ \mathrm{N})^2 + (0.9 \times 10^{-4}\ \mathrm{N})^2} = 4.8 \times 10^{-4}\ \mathrm{N} \\
\theta &= \arctan \frac{0.9}{4.7} = 10.8^\circ \text{ above the positive } x \text{ axis}
\end{aligned}$$

Homework: 13.4, 13.9, 13.10, 13.11, 13.12, 13.13, 13.14.

Chapter Fourteen: The Electric Field and the Electric Potential

- The idea of an *electric field* is introduced to describe the effect in all space around a charge so that if another charge is present we can predict the effect on it.
- The concept of separating the calculation into the formation of an electric field and the response to the electric field by a given charge placed in it greatly simplifies the calculations.

The Electric Field

- The *electric field*, with symbol $\mathcal{E}$, at a point in space as the vector resultant force experienced by a *positive* test charge of magnitude 1 C placed at that point.
- If an arbitrary test charge q' is placed at that point, the charge experience a force

$$\boldsymbol{F} = q'\mathcal{E}$$

- The magnitude of the force between two charges, q and q', is

$$F = \frac{1}{4\pi\epsilon_0}\frac{qq'}{r^2}$$

- The magnitude of the electric field produced by q at P is given by

$$\mathcal{E} \quad = \quad \frac{1}{4\pi\epsilon_0}\frac{qq'}{q'r^2}$$

$$= \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}$$

- The electric field produced by a positive charge q at a point P is along the line joining the charge q and the point P and directed away from q. See Fig. 14-1.
- On the other hand, for a negative point charge q, the electric field that it produces is directed radially toward it.
- The electric field obeys the superposition principle.
- Not only is there no electric field when there no charges, but there is no electric field at a point when the force from an assembly of charges on a test charge is zero at that point.

Example 14-1

A charge $q_1 = 3 \times 10^{-6}$ C is located at the origin of the x axis. A second charge $q_2 = -5 \times 10^{-6}$ C is also on the x axis 4 m from the origin in the positive x direction (see Fig. 14-2).

(a) Calculate the electric field at the midpoint P of the line joining the two charges.

(b) At what point P' on that line is the resultant field zero?

Sol:

(a) Since q_1 is positive and q_2 is negative, at any point between them, both electric fields produced by them are the same direction which is toward to q_2. Thus,

$$|\mathcal{E}_1| \quad = \quad 9 \times 10^9 \frac{\text{N-m}^2}{\text{C}^2} \frac{3 \times 10^{-6}\ \text{C}}{(2\ \text{m})^2} = 6.75 \times 10^3\ \text{N/C}$$

$$|\mathcal{E}_2| = 9\times 10^9 \frac{\text{N-m}^2}{\text{C}^2}\frac{5\times 10^{-6}\ \text{C}}{(2\ \text{m})^2} = 11.25\times 10^3\ \text{N/C}$$

The resultant electric field $\mathcal{E}$ at P is

$$\mathcal{E} = \mathcal{E}_1 + \mathcal{E}_2 = 18\times 10^3\ \text{N/C}$$

(b) It is clear that the resultant $\mathcal{E}$ can not be zero at any point between q_1 and q_2 because both $\mathcal{E}_1$ and $\mathcal{E}_2$ are in the same direction. Similarly $\mathcal{E}$ can not be zero to the right of q_2 because the magnitude of q_2 is greater then q_1 and the distance r is smaller for q_2 than q_1. Thus, $\mathcal{E}$ can only be zero to the left of q_1 at some point P' to be found. Let the distance from P' to q_1 be x.

$$\begin{aligned}\frac{1}{4\pi\epsilon_0}\frac{q_1}{x^2} &= \frac{1}{4\pi\epsilon_0}\frac{q_2}{(x+4)^2}\\ 3(x+4)^2 &= 5x^2\end{aligned}$$

$$\begin{aligned} 2x^2 - 24x - 48 &= 0 \\ x &= 13.75 \text{ m},\ x = -1.75 \text{ m} \end{aligned}$$

Apparently, we need to take x which is positive.

Electrical Potential Energy

- The magnitude of the electric field at a point P resulting from a point charge is independent of the angular position of the point P.
- The direction of the electric field is radially away from the charge producing the field if the charge is positive or radially toward it if the charge is negative. See Fig. 14-3.
- By definition work involves the dot product of the force vector $\boldsymbol{F}$ and displacement vector $\Delta\boldsymbol{s}$, that is, $W = \boldsymbol{F} \cdot \Delta\boldsymbol{s}$.
- In Fig. 14-3 the same amount of work is done in moving a charge from point A to point B either by path 1 or by path 2 or by any other path.
- The work done in moving the test charge q' from point A to point B is

$$W_{A\to B} = \int_A^B \boldsymbol{F} \cdot d\boldsymbol{s}$$

- The force $\boldsymbol{F}$ needed to move q' at constant velocity must be equal and opposite to the force exerted by the electric field of q.

$$W_{A\to B} = -q' \int_A^B \mathcal{E} \cdot d\boldsymbol{s}$$

- See Fig. 14-4. We may evaluate $W_{A\to B}$ by moving tangentially from point A to C and then radially from point C to point B.

$$W_{A\to B} = W_{C\to B} = q' \int_C^B \mathcal{E} ds$$

since $\mathcal{E} \cdot d\boldsymbol{s} = \mathcal{E} ds \cos 180^\circ = -\mathcal{E} ds$.

- As we move a distance ds toward B from point C, the radius r

decreases, that is, $ds = -dr$.

$$\begin{aligned} W_{A\to B} &= -q' \int_C^B \mathcal{E}\; dr \\ &= -\frac{qq'}{4\pi\epsilon_0} \int_{r_A}^{r_B} \frac{dr}{r^2} \\ &= \frac{qq'}{4\pi\epsilon_0} \left(\frac{1}{r_B} - \frac{1}{r_A} \right) \end{aligned}$$

- By definition, the work done in moving an object between two points in a force is equal to the difference in the potential energy E_p between the two points, that is,

$$E_p(B) - E_p(A) = \frac{qq'}{4\pi\epsilon_0} \left(\frac{1}{r_B} - \frac{1}{r_A} \right)$$

- The reference point at which the potential energy is chosen to be zero is $r = \infty$.

- The potential energy of our two charges system q and q' when they are separated by a distance r is simply the work done in bringing one of them from infinity to r. That is,

$$E_p(r) = \frac{1}{4\pi\epsilon_0}\frac{qq'}{r}$$

  This potential energy is called *electric potential energy*.

- If both q and q' are positive, then E_p is also positive. To move q' from infinity to r we have to do positive work, we have to overcome the repulsive force between the two charges. The same is true if both charges are negative.

- If the charges are of unlike sign, they will attract each other and, consequently, to move q' at constant velocity, we will have to hold it back. We will then do negative work and the potential energy will be negative.

- By the superposition principle, the total energy of the three-charge system shown in Fig. 14-5 is obtained as

$$E_p = \frac{1}{4\pi\epsilon_0}\left(\frac{q_1 q_2}{r_{12}} + \frac{q_1 q_3}{r_{13}} + \frac{q_2 q_3}{r_{23}}\right)$$

  Thus, for a system of charges, the procedure to follow is to calculate the potential energy separately for the pairs and then to add these algebraically.

Example 14-2

Three charges-$q_1 = 3 \times 10^{-6}$ C, $q_2 = -5 \times 10^{-6}$ C, and $q_3 = -8 \times 10^{-6}$ C- are positioned on a straight line as shown in Fig. 14-6. Find the potential energy of the charges.

Sol:

$$\begin{aligned} E_p &= 9 \times 10^9 \frac{\text{N-m}^2}{\text{C}^2} \left[\frac{(3 \times 10^{-6}\ \text{C})(-5 \times 10^{-6}\ \text{C})}{4\ \text{m}} \right. \\ &\quad + \frac{(3 \times 10^{-6}\ \text{C})(-8 \times 10^{-6}\ \text{C})}{9\ \text{m}} \\ &\quad \left. + \frac{(-5 \times 10^{-6}\ \text{C})(-8 \times 10^{-6}\ \text{C})}{5\ \text{m}} \right] \\ &= 1.43 \times 10^{-2}\ \text{J} \end{aligned}$$

Electric Potential

- The *electric potential* at a point P is defined as the work done in bringing a unit positive charge from infinity to the point. That is,

$$W_{\infty \to P} = V(P) = -\int_{\infty}^{P} \mathcal{E} \cdot d\boldsymbol{s}$$

- The work done in bringing a charge q' of arbitrary magnitude or sign to P is $W = q'V(P)$ and we have

$$E_p = q'V$$

- The SI unit of potential is *volt* and *one volt* can be defined as *one joule per coulomb*.
- The potential resulting from a point charge q at a distance r

away from it is

$$V(r) = \frac{1}{4\pi\epsilon_0}\frac{q}{r}$$

- The potential resulting from several point charges is simply equal to the *algebraic* sum.

$$V = \frac{1}{4\pi\epsilon_0}\frac{q_1}{r_1} + \frac{1}{4\pi\epsilon_0}\frac{q_2}{r_2} + \cdots$$

where $r_1, r_2, \ldots$ are the distances from q_1 and q_2, respectively, to the point where the potential is being evaluated.

- A potential difference between two points is commonly referred to as a *voltage difference* or simply *voltage*.
- The potential difference can be calculated directly from the electric field.

$$W_{A\to B} = -q' \int_A^B \mathcal{E} \cdot d\boldsymbol{s}$$

$$\frac{W_{A\to B}}{q'} = \Delta V = V(B) - V(A) = -\int_A^B \mathcal{E}\cdot d\boldsymbol{s}$$

$$\Delta V = V(B) - V(A) = \int_B^A \mathcal{E}\cdot d\boldsymbol{s}$$

- Consider two plates, B which is positively charged and A which is negatively charged (see Fig. 14-7). The electric field is directed away from the positive charges and toward the negative charges.
- A unit of positive charge placed at B will be accelerated toward A. Objects are accelerated when they move from a point to another of lower potential energy. Thus,

$$V(B) > V(A)$$

- The electric field is directed from high potential points to low potential points, and that positive charges, if free to move, do so from high potential points to low potential points.

- By using the conservation of total mechanical energy, we have

$$E_k(B) + qV(B) = E_k(A) + qV(A)$$

Example 14-3

A potential difference of 100 V is established between the two plates of Fig. 14-7, B being the high potential plate. A proton of charge $q = 1.6 \times 10^{-19}$ C is released from plate B. What will be the velocity of the proton when it reaches plate A? The mass of the proton is 1.67×10^{-27} kg.

Sol: Because the proton is released with no initial velocity, $E_k(B)$ is zero. Thus,

$$E_k(A) = q[V(B) - V(A)] = q\Delta V$$

or

$$\frac{1}{2}mv_A^2 = q\Delta V$$

Solving for v_A,

$$v_A = \sqrt{\frac{2q\Delta V}{m}}$$

$$
\begin{aligned}
&= \sqrt{\frac{2 \times 1.6 \times 10^{-19}\ \mathrm{C} \times 100\ \mathrm{V}}{1.67 \times 10^{-27}\ \mathrm{kg}}} \\
&= 1.38 \times 10^{5}\ \mathrm{m/sec}
\end{aligned}
$$

The Electron Volt and Capacitance

- The charge of the electron is $q = e = -1.6 \times 10^{-19}$ C.
- If an electron is moved through a potential difference of 1 V (1 J/C) the energy change is

$$|e|\Delta V = 1.6 \times 10^{-19} \text{ C} \times 1 \text{ J/C} = 1.6 \times 10^{-19} \text{ J}$$

- We define 1 electron volt (eV) as 1.6×10^{-19} J.
- The battery maintaining a constant potential difference between plates connected to it is called an *electromotive force*, or simply *emf*.
- Suppose we connect the terminals of a battery to two parallel metal plates, as Fig. 14-8. The plate on the left will quickly attain a negative charge of $-q$ and the one on the right a positive charge of $+q$.

- Experiments show that the charge is proportional to the potential difference, ΔV,
$$q \propto V$$
where V actually means ΔV or voltage difference between the two terminals of the battery.
- $q = CV$.
- The arrangement of such a set of plates as in Fig. 14-8 is called a *capacitor*, and the constant is called the *capacitance*. The constant has unit *farad* where
$$1 \text{ farad} = 1 \text{ coulomb/volt}$$
- The material placed between the plates is called a *dielectric* and
$$q = \kappa CV$$
where the factor κ is called the *dielectric constant* and it is

dependent on the dielectric. For example, for air or vacuum the κ is 1 and for paper it is 3.5.

Homework: 14.2, 14.3, 14.4, 14.5, 14.6, 14.7, 14.9, 14.10, 14.20, 14.22, 14.23.

Chapter Fifteen: Electric Current

- We consider the motion of electrons in a conductor (a metal) when there is a voltage difference applied between the ends of the conductor.
- We will limit our discussion mostly to *direct currents*, that is, currents whose magnitude and direction do not change with time.

Motion of Charges in an Electric Field

- By Newton's second law $\boldsymbol{F} = m\boldsymbol{a}$ we have

$$\begin{aligned}\mathcal{E} &= \frac{\boldsymbol{F}}{q} \\ &= \frac{m\boldsymbol{a}}{q} \\ \boldsymbol{a} &= \frac{q\mathcal{E}}{m}\end{aligned}$$

  where q represents an arbitrary charge.

- In the case of an electron, $q = -1.6 \times 10^{-19}$ C and the mass of an electron is $m = 9.1 \times 10^{-31}$ kg. Thus

$$\boldsymbol{a} = 1.76 \times 10^{11}\mathcal{E}$$

- We may calculate the velocity of the electrons after they travel a distance s assuming that no scattering (or collisions) occurs over that distance.

$$v^2 - v_0^2 = 2ax$$

When

$$v_0 = 0,\ x = s$$

and

$$a = \frac{e\mathcal{E}}{m}$$

the velocity is

$$v = \sqrt{\frac{2e\mathcal{E}s}{m}}$$

Electric Current

- The motion of an electron in an electric field is a series of short accelerations interrupted by collisions that scatter the electron. It has a random path, although there is a slow net velocity opposite to the field direction (see Fig. 15-1). It is the net velocity of the electrons, called the *drift velocity*, that gives rise to the current, not the brief accelerations.
- The charge Δq that flows by in time Δt through a plane perpendicular to a wire is defined as *electric current* i, where

$$i = \frac{\Delta q}{\Delta t} \ \mathrm{C/sec}$$

- When i is not constant we define electric current as

$$i = \frac{dq}{dt}$$

- In the SI units, current is measured in amperes, or amps. One ampere (1 A) is equal to one coulomb per second and is a relatively large quantity.
- We use the milliampere (1 mA $= 10^{-3}$ A) or the microampere (1μ A $= 10^{-6}$ A).
- See Fig. 15-2. Assume that there are both positive and negative charges, both of which are mobile in the presence of an electric field $\mathcal{E}$ with a vector direction from left to right. Assume that there are N_p (N_n) positive (negative) charges per unit volume with drift velocity of v_p (v_n).
- In time Δt the positive charges will move from left to right a distance of $v_p \Delta t$. If each charge has a charge q_p, the charge flowing across the right end of the cylinder is

$$\Delta q_p = q_p N_p A v_p \Delta t$$

Thus,

$$\begin{aligned} i_p &= \frac{\Delta q_p}{\Delta t} \\ &= \frac{q_p N_p A v_p \Delta t}{\Delta t} \\ &= q_p N_p A v_p \end{aligned}$$

- In the same way, the negative particles, each with charge q_n, flow from right to left given rise to a current

$$i_n = q_n N_n A v_n$$

  Both the sign of the charge q_n and the sign of the drift velocity v_n are negative and therefore their product is positive.

- A flow of negative charges to the left is equivalent to a flow of positive charges to the right. Thus,

$$\begin{aligned} i &= i_p + i_n \\ &= A(q_p N_p v_p + q_n N_n v_n) \end{aligned}$$

- The direct current i in a conductor has the same direction as that of the electric field $\mathcal{E}$.
- There is no pileup of electric charges in the wire at any point. If we connect a wire between the terminals of a battery, it is therefore reasonable to conclude that charge flows at a steady rate throughout the wire.
- The *current density* is defined as the current per unit cross-sectional area, that is

$$\mathcal{J} = \frac{i}{A} \ \mathrm{A/m^2 (amp/m^2)}$$

Example 15-1

Suppose a copper wire carries 10 A (amps) of current and has a cross-section of 10^{-6} m^2. As will be seen later, each atom of copper contributes one electron that is free to move, so the electron carrier density N_n is about the same as the density of atoms, which is about 7×10^{28} atoms per m^3. The charge on an electron is -1.6×10^{-19} C. (a) What is the drift velocity v_n of the electrons? (b) How long would it take an electron to move from one terminal of a battery to the other if this wire were 1 m long?

Sol:

(a)

$$
\begin{aligned}
i &= Aq_nN_nv_n \\
v_n &= \frac{i}{Aq_nN_n}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{10 \text{ A}}{10^{-6} \text{ m}^2 \times 1.6 \times 10^{-19} \text{ C} \times 7 \times 10^{28} \text{ m}^{-3}} \\
&= 9 \times 10^{-4} \text{ m/sec}
\end{aligned}
$$

(b) $t = \frac{x}{v_n} = \frac{1 \text{ m}}{9\times 10^{-4} \text{ m/sec}} = 1.1 \times 10^3 \text{ sec} = 18 \text{ min}$

So the actual drift velocity of a given electron is very small. The speed of propagation of the electric field along the wire is that of the speed of light in the wire.

Resistance and Resistivity

- Experiment shows that in many cases the electric current i, hence the current density $\mathcal{J}$, are proportional to $\mathcal{E}$.
- Define *electrical resistivity* ρ as

$$\mathcal{E} = \rho\mathcal{J}$$

  The resistivity is a property of a given material and is indenpendent of its shape.
- The resistivity was found to be a constant for a given metal at a given temperature by G. Ohm. Thus, the above equation is called Ohm's law.
- A material obeying Ohm's law is called an *ohmic conductor*.

- The units of ρ (called ohm meter, Ω-m) is

$$\rho = \frac{\mathcal{E}(\text{ N/C})}{\mathcal{J}(\text{C/sec-m}^2)} = \frac{\mathcal{E}}{\mathcal{J}}\left(\frac{\text{N-sec-m}^2}{\text{C}^2}\right)$$

  See Table 15-1.

- The *conductivity* σ is defined as

$$\sigma = \frac{1}{\rho}$$

- Suppose we have a given metal wire with cross section A, length l, and resistivity ρ with an applied electric field $\mathcal{E}$ (see Fig. 15-3). The potential difference between the two ends of the conductor, point 1 and 2 is

$$\Delta V = V_1 - V_2 = \int_{s_1}^{s_2} \mathcal{E} \cdot d\boldsymbol{s}$$

If the electric field inside the conductor is uniform,

$$\Delta V = \mathcal{E} l \text{ or } \mathcal{E} = \frac{\Delta V}{l}$$

where $l = s_2 - s_1$. Thus,

$$\Delta V = i\frac{\rho l}{A}$$

which can be written as

$$V = iR \text{ (Ohm's law)}$$

where V means ΔV and $R = \frac{\rho l}{A}$.

- R is called *resistance* of the wire and has units of Ω (ohms).
- The current in a resistance (resistor) is from its high potential side to its low potential side.

Resistances in Series and Parallel

- See Fig. 15-4. The voltage difference across a resistance (resistor) is called *voltage drop*.
- See Fig. 15-5. The electric potential at point A is the same as that at the left side of the battery (emf), and that at point D is the same as the right side of the battery. The same current must pass through each of these resistances at that which passes between points A and D. This combination is called *series* resistances.
- $V_{AB} = iR_1$, $V_{BC} = iR_2$, $V_{CD} = iR_3$. Thus,

$$\begin{aligned} V &= V_{AB} + V_{BC} + V_{CD} \\ &= iR_1 + iR_2 + iR_3 = i(R_1 + R_2 + R_3) \\ &= iR_{\text{eq}} \end{aligned}$$

where R_{eq} is the equivalent resistance of the three.

- It is obvious that $R_{\text{eq}} = \sum_i R_i$ will be true regardless of the number of resistances in series.
- See Fig. 15-6. The resistances is arranged in *parallel*.
- The left side of each resistance is at the same potential and the right side is at the same potential; hence, the same voltage drop V must occur across each.
- $i = i_1 + i_2 + i_3$. By Ohm's law,

$$V = i_1 R_1,\ V = i_2 R_2,\ V = i_3 R_3$$

and

$$i_1 = \frac{V}{R_1},\ i_2 = \frac{V}{R_2},\ i_3 = \frac{V}{R_3}$$

Thus,

$$
\begin{aligned}
i &= \frac{V}{R_1} + \frac{V}{R_2} + \frac{V}{R_3} \\
&= V\left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right) \\
V &= \frac{i}{\left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right)} \\
&= \frac{i}{\frac{1}{R_{\mathrm{eq}}}} = iR_{\mathrm{eq}}
\end{aligned}
$$

where

$$\frac{1}{R_{\mathrm{eq}}} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$$

- See Fig. 15-9. The current through R_1 is the same as that through R_2, and

$$i = \frac{V_1}{R_1},\ i = \frac{V_2}{R_2}$$

where V_1 and V_2 are the voltage drops across R_1 and R_2, respectively. Equating the i's gives

$$\begin{aligned} \frac{V_1}{R_1} &= \frac{V_2}{R_2} \\ \frac{V_1}{V_2} &= \frac{R_1}{R_2} \text{ (series)} \end{aligned}$$

- In a series circuit the ratio of the voltage drops is equal to the ratio of the resistances.
- See Fig. 15-10. The voltage across each resistance is the same and

$$V_1 = i_1 R_1,\ V_2 = i_2 R_2$$

Equating V_1 and V_2 gives

$$\begin{aligned} i_1 R_1 &= i_2 R_2 \\ \frac{i_1}{i_2} &= \frac{R_2}{R_1} \text{ (parallel)} \end{aligned}$$

- In a parallel circuit the ration of the currents through each resistor is inversely proportional to the resistances.

Example 15-2

Suppose in Fig. 15-5 the voltage $V = 1.5$ V and the resistances are $R_1 = 5\ \Omega$, $R_2 = 10\ \Omega$, and $R_3 = 15\ \Omega$. What are the voltages V_{AB}, V_{BC}, and V_{CD}?

Sol:

$$\begin{aligned} V &= iR_{\text{eq}} = i(R_1 + R_2 + R_3) \\ i &= \frac{1.5\ \text{V}}{(5+10+15)\ \Omega} = 0.05\ \text{A} = 50\ \text{mA} \end{aligned}$$

Then applying Ohm's law to each resistance

$$\begin{aligned} V_{AB} &= iR_1 = 0.05\ \text{A} \times 5\ \Omega = 0.25\ \text{V} \\ V_{BC} &= iR_2 = 0.05\ \text{A} \times 10\ \Omega = 0.5\ \text{V} \\ V_{CD} &= iR_3 = 0.05\ \text{A} \times 15\ \Omega = 0.75\ \text{V} \end{aligned}$$

Example 15-3

Suppose two resistors, $R_1 = 5\ \Omega$ and $R_2 = 10\ \Omega$, are connected in parallel to a 1.5 V battery as in Fig. 15-7. (a) What is the current through each? (b) What is the total current in the circuit?

Sol: (a) Using Ohm's law

$$\begin{aligned} V &= i_1 R_1,\ V = i_2 R_2 \\ i_1 &= \frac{V}{R_1} = \frac{1.5\ \text{V}}{5\ \Omega} = 0.3\ \text{A} = 300\ \text{mA} \\ i_2 &= \frac{V}{R_2} = \frac{1.5\ \text{V}}{10\ \Omega} = 0.15\ \text{A} = 150\ \text{mA} \end{aligned}$$

(b) $i = i_1 + i_2 = 300\ \text{mA} + 150\ \text{mA} = 450\ \text{mA}$. We may check this answer by solving the equivalent circuit.

$$\frac{1}{R_{\text{eq}}} = \frac{1}{R_1} + \frac{1}{R_2} = \frac{1}{5\ \Omega} + \frac{1}{10\ \Omega} = 0.2\ \Omega^{-1} + 0.1\ \Omega^{-1} = 0.3\ \Omega^{-1}$$

$$
\begin{aligned}
R_{\mathrm{eq}} &= \frac{1}{0.3\ \Omega^{-1}} = 3.33\ \Omega \\
i &= \frac{V}{R_{\mathrm{eq}}} = \frac{1.5\ \mathrm{V}}{3.33\ \Omega} = 0.45\ \mathrm{A} = 450\ \mathrm{mA}
\end{aligned}
$$

Example 15-4

Three resistors are connected in a combination of series and parallel as in Fig. 15-8. What is the current through each?

Sol: First we find $R_{\text{eq(p)}}$ for the parallel combination

$$\begin{aligned} \frac{1}{R_{\text{eq(p)}}} &= \frac{1}{R_1} + \frac{1}{R_2} = \frac{1}{2\ \Omega} + \frac{1}{4\ \Omega} = 0.5\ \Omega^{-1} + 0.25\ \Omega^{-1} = 0.75\ \Omega^{-1} \\ R_{\text{eq(p)}} &= 1.33\ \Omega \end{aligned}$$

We then have the equivalent circuit, Fig. 15-8b.

$$R_{\text{eq(s)}} = R_{\text{eq(p)}} + R_3 = 1.33\ \Omega + 3\ \Omega = 4.33\ \Omega$$

Now we have the simpler equivalent circuit of Fig. 15-8c.

$$i = \frac{V}{R_{\text{eq(s)}}} = \frac{1.5\ \text{V}}{4.33\ \Omega} = 0.346\ \text{A} = 346\ \text{mA}$$

By the relation given previously we have

$$\frac{i_1}{i_2} = \frac{R_2}{R_1} = 2$$

Furthermore, $i = i_1 + i_2 = 346$ mA. Thus,

$$\begin{aligned} 2i_2 + i_2 &= 346 \text{ mA} \\ i_2 &= \frac{346 \text{ mA}}{3} = 115 \text{ mA} \\ i_1 &= 346 \text{ mA} - 115 \text{ mA} = 231 \text{ mA} \end{aligned}$$

Kirchhoff's Rules

- Two fundamental rules established by G. R. Kirchhoff that aid in the solution of electrical networks are
  1. The algebraic sum of currents toward any branch point is zero.
  2. The algebraic sum of all potential changes in a closed loop is zero.
- Charge can not accumulate in a dc circuit: If it did, there would be a larger electric field at that region which would exert a larger force and thereby redistribute the charge evenly.
- Rule 2 is a statement of the conservation of energy.
- In applying rule 2, it is useful to follow certain guidelines that will prevent errors in the signs of the potential changes.

  (a) As indicated in connection with rule 1, we first assume a direction for the current through each branch of the circuit.

(b) We then choose any closed loop in the circuit and designate the direction in which we wish to mentally traverse it.

(c) We now go around the loop in the chosen direction adding algebraically all the potential changes and setting the sum equal to zero.

- When we meet an emf source, its voltage V is taken as positive if we cross the source from the negative (low potential) side to the positive (high potential) side.
- If in our mental trip around the circuit loop we cross a resistor in the same direction as the current, we must take the iR drop as negative because we are going from high to low potential-a decrease.
- Consider the circuit of Fig. 15-12. We apply rule 2 and write

$$-iR_1 - iR_2 + V = 0$$

$$V = iR_1 + iR_2$$

- Consider the circuit of Fig. 15-13a. We apply rule 2 and write

$$\begin{aligned} -iR_1 - iR_2 + V_2 + V_1 &= 0 \\ V_2 + V_1 &= iR_1 + iR_2 \end{aligned}$$

- Consider the circuit of Fig. 15-13b. We apply rule 2 and write

$$\begin{aligned} -iR_1 - iR_2 + V_2 - V_1 &= 0 \\ V_2 - V_1 &= iR_1 + iR_2 \end{aligned}$$

Example 15-5

In the circuit of Fig. 15-14, (a) Find the currents i_C, i_E, and i_B and the voltage drop across resistors R_1 and R_2. (b) Find the voltage difference between points C and D and between D and F.

Sol:

(a) From the first rule at branch point B

$$i_C + i_B = i_E$$

For the right-hand loop, if we traverse it in the counterclockwise direction starting at point D, we have

$$\begin{aligned} V_2 - i_C R_2 - i_C R_C + i_B R_B &= 0 \\ 9\ \text{V} - i_C 500\ \Omega - i_C 1000\ \Omega + i_B 60\ \Omega &= 0 \\ 9\ \text{V} - i_C 1500\ \Omega + i_B 60\ \Omega &= 0 \end{aligned}$$

For the left-hand loop, traversing it counterclockwise, we write

$$\begin{aligned} V_1 - i_B R_B - i_E R_E - i_E R_1 &= 0 \\ 1.5\ \mathrm{V} - i_B 60\ \Omega - i_E 100\ \Omega - i_E 50\ \Omega &= 0 \\ 1.5\ \mathrm{V} - i_B 60\ \Omega + i_E 150\ \Omega &= 0 \end{aligned}$$

We now have three equations to be solved simultaneously for i_C, i_E, and i_B.

$$\begin{aligned} i_C + i_B &= i_E \\ 9\ \mathrm{V} - i_C 1500\ \Omega + i_B 60\ \Omega &= 0 \\ 1.5\ \mathrm{V} - i_B 60\ \Omega - i_E 150\ \Omega &= 0 \end{aligned}$$

We can use the first equation to eliminate i_B from the last two.

$$\begin{aligned} 9\ \mathrm{V} - i_C 1500\ \Omega + (i_E - i_C) 60\ \Omega &= 0 \\ 9\ \mathrm{V} - i_C 1560\ \Omega + i_E 60\ \Omega &= 0 \end{aligned}$$

$$
\begin{aligned}
1.5\ \mathrm{V} - (i_E - i_C)60\ \Omega - i_E 150\ \Omega &= 0 \\
1.5\ \mathrm{V} - i_E 210\ \Omega + i_C 60\ \Omega &= 0 \\
7 \times (9\ \mathrm{V} - i_C 1560\ \Omega + i_E 60\ \Omega) &= 0 \\
2 \times (1.5\ \mathrm{V} - i_E 210\ \Omega + i_C 60\ \Omega) &= 0 \\
63\ \mathrm{V} - i_C 10,920\ \Omega + 3\ \mathrm{V} + i_C 120\ \Omega &= 0 \\
i_C = \frac{66\ \mathrm{V}}{10,800\ \Omega} = 6.1 \times 10^{-3}\ \mathrm{A} &= 6.1\ \mathrm{mA}
\end{aligned}
$$

We can now solve for i_E.

$$
\begin{aligned}
1.5\ \mathrm{V} - i_E 210\ \Omega + i_C 60\ \Omega &= 0 \\
i_E &= \frac{1.5\ \mathrm{V} + i_C 60\ \Omega}{210\ \Omega} \\
&= \frac{1.5\ \mathrm{V} + (6.1 \times 10^{-3}\ \mathrm{A})(60\ \Omega)}{210\ \Omega} \\
&= 8.9 \times 10^{-3}\ \mathrm{A} = 8.9\ \mathrm{mA}
\end{aligned}
$$

Finally we can obtain i_B

$$i_B = i_E - i_C = 8.9 \text{ mA} - 6.1 \text{ mA} = 2.8 \text{ mA}$$

The voltage drop across R_1 is

$$V = i_E R_1 = 8.9 \times 10^{-3} \text{ A} \times 50\ \Omega = 0.45 \text{ V}$$

and the voltage drop across R_2 is

$$V = i_C R_2 = 6.1 \times 10^{-3} \text{ A} \times 500\ \Omega = 3.1 \text{ V}$$

(b)

$$\begin{aligned} V_D + V_2 - i_C R_2 &= V_C \\ V_C - V_D &= V_2 - i_C R_2 \\ &= 9 \text{ V} - 6.1 \times 10^{-3} \text{ A} \times 500\ \Omega \\ &= 6 \text{ V} \end{aligned}$$

$$
\begin{aligned}
V_D - i_B R_B - i_E R_E &= V_E \\
V_E - V_D &= -i_B R_B - i_E R_E \\
&= -2.8 \times 10^{-3} \text{ A} \times 60 \ \Omega \\
&\quad -8.9 \times 10^{-3} \text{ A} \times 100 \ \Omega \\
&= -1.1 \text{ V}
\end{aligned}
$$

Galvanometers and Voltmeters

- See Fig. 15-15. Electric current passing through a wire produces a magnetic field. If a loop of wire is used then, on the passage of current, one end of the loop becomes the north pole of a magnet and the other end becomes the south pole.
- The larger the number of loops, the stronger the magnet for a given current. Similarly, the larger current, the stronger the magnet for a given number of loops.
- A full-scale deflection of a instrument needle can be established for a given amount of current through the coil. This instrument is called a *galvanometer*. The current for full-scale deflection is called the *current rating* of a meter.
- The common current rating is 0.1 mA.
- To extend the range of the meter, a lower resistance, called a

shunt, is placed in parallel with the meter (see Fig. 15-16).

- The resistance of the coil R_c is commonly 1000 Ω. From Ohm's law the voltage drop across the galvanometer in Fig. 15-16 must be

$$V = iR_c = 10^{-4}\ \text{A} \times 10^3\ \Omega = 0.1\ \text{V}$$

- In Fig. 15-16b

$$R_s = \frac{V}{i_s} = \frac{0.1\ \text{V}}{9.9 \times 10^{-3}\ \text{A}} = 10.1\ \Omega$$

- In Fig. 15-16c

$$R_s = \frac{V}{i_s} = \frac{0.1\ \text{V}}{99.9 \times 10^{-3}\ \text{A}} = 1.001\ \Omega$$

- An instrument to measure the voltage difference between two points in a circuit is called a *voltmeter* (see 15-17). The idea instrument would be one that had infinite resistance since we do not want such a voltmeter to disturb the current flow through the resistor.

•

$$\begin{aligned} V_m + 10^{-4}\ \mathrm{A} \times R_2 &= 10\ \mathrm{V} \\ R_2 &= \frac{9.9\ \mathrm{V}}{10^{-4}\ \mathrm{A}} = 9.9 \times 10^4\ \Omega \end{aligned}$$

Power Dissipation by Resistors

- In an elastic collision between an electron and an atom, very little energy is transferred to the atom-most of the kinetic energy is retained by the electron in its recoil. Because many collisions are taken place, each small energy loss adds to a considerable amount.
- Since temperature is a measure of the average kinetic energy of the atoms of a system, we expect any conductor to heat up when an electric current is passed through it.
- Let V_A and V_B represent the potentials of points A and B, respectively, and V_{AB} the potential difference. The change in potential energy of a charge Δq entering at A and leaving at B is

$$\Delta E_p = \Delta q(V_B - V_A)$$

This represents an energy loss because V_A is greater than V_B.

$$\begin{aligned} \Delta q &= i\Delta t \\ \Delta E_p &= V_{AB} i \Delta t \\ P_{AB} &= \frac{\Delta E_p}{\Delta t} \\ &= V_{AB} i \end{aligned}$$

- In general

$$\begin{aligned} P &= Vi \\ &= i^2 R \\ &= \frac{V^2}{R} \end{aligned}$$

Charging a Capacitor-RC Circuits

- See Fig. 15-19.

$$\begin{aligned} V_{AD} &= V_{AB} + V_{BD} \\ V &= iR + \frac{q}{C} \end{aligned}$$

- Since $i = \frac{dq}{dt}$, we have

$$\begin{aligned} V &= R\frac{dq}{dt} + \frac{q}{C} \\ \frac{V}{R} - \frac{q}{RC} &= \frac{dq}{dt} \\ dt &= \frac{dq}{\frac{V}{R} - \frac{q}{RC}} \\ -\frac{1}{RC}\int_0^t dt &= \int_0^q \frac{-\frac{1}{RC}dq}{\frac{V}{R} - \frac{q}{RC}} \end{aligned}$$

$$\begin{aligned}
-\frac{t}{RC} &= \ln\left(\frac{V}{R}-\frac{q}{RC}\right)\Bigg|_0^q \\
-\frac{t}{RC} &= \ln\left(\frac{V}{R}-\frac{q}{RC}\right)-\ln\left(\frac{V}{R}\right) \\
-\frac{t}{RC} &= \ln\left(1-\frac{q}{VC}\right) \\
e^{-t/RC} &= 1-\frac{q}{CV} \\
q &= CV(1-e^{-t/RC})
\end{aligned}$$

- See Fig. 15-20. At $t=0$, $q=CV(1-e^{-0})=CV(1-1)=0$. This agrees with the fact that at $t=0$ the capacitor was unchanged. As t increases, the exponential term in the parenthesis decreases and consequently q increases. As $t\to\infty, e^{-t/RC}\to 0$ and $q\to CV$, the ultimate charge on the capacitor.

- The time of charging rate is determined by the product RC, which is called the *time constant* of the circuit.
- The current i passing the capacitor is

$$i = \frac{dq}{dt} = \frac{V}{R}e^{-t/RC}$$

- At $t = 0$, $i = \frac{V}{R}$ and the capacitor acts as if it were a wire with no resistance. As $t \to \infty, e^{-t/RC} \to 0$ and $i \to 0$, the ultimate current on the capacitor.

Homework: 15.4, 15.6, 15.8, 15.9, 15.11, 15.12, 15.13, 15.14, 15.15, 15.18, 15.20, 15.21, 15.23, 15.24.

Chapter Sixteen: Magnetic Fields and Electromagnetic Waves

- If the bar magnet is suspended by a thread or supported by a pivot, one of the ends will point in a northerly direction. This end of the magnet is called the *north pole* of the magnet, with symbol N. The opposite end of the magnetic is called the *south pole*, with symbol S.
- Elementary experiments show that like poles repel and unlike poles attract.
- There is something that we call a *magnetic field* by which poles can exert forces on each other.
- If we break a bar magnet in half, we can not make single poles, but instead we will have two bar magnets.
- There is an intimate relation between the motion of electric

charges and magnetic field.

Magnet Fields

- The magnet fields are indicated by continuous lines from the north to the south pole, and the number of lines is arbitrary (see Fig. 16-1) .
- The direction of the magnet field for a long, straight wire is schematically represented in Fig. 16-2. The field becomes weaker as we move away from the wire.
- See Fig. 16-3. The arrows represent the magnetic field direction inside the loop. These field lines return outside the loop so that the loop itself becomes a magnet.

Force on Current-Carrying Wire

- Experiment shows that when a wire carrying a current is placed in a magnetic field, it will experience a force.
- Fig. 16-4 is a schematic drawing of an experiment which shows that when a wire carrying a current is placed in a magnetic field $\boldsymbol{B}$ the force $\boldsymbol{F}$ is in a direction that is perpendicular to the plane defined by the field and the direction of the current.
- The force $\boldsymbol{F}$ is proportional to the sine of the angle θ between the field and the wire.
- $$B = \frac{F}{i\Delta l \sin\theta}$$
- The SI unit for B is newtons/ampere-meter (N/A-m) and named Tesla (T). An old unit for the magnetic field is the Gauss (G), where 1 tesla = 10^4 gauss.

- The earth's magnetic field is about 10^{-4} T.
- The direction of the force induced by magnetic field and a current-carrying wire is the perpendicular both to the magnetic field and to the direction of the current. Thus,

$$\boldsymbol{F} = i\Delta \boldsymbol{l} \times \boldsymbol{B}$$

- It is conventional to let the current i be a scalar quantity and to let Δl be a vector pointing in the direction of the current.

Torque on a Current Loop

- Consider a single rectangular loop of wire connected to a pivot rod, as shown in Fig. 16-5. For sides 1 and 3 the magnitudes of the forces are the same because the angle between $\boldsymbol{\Delta l}$ and $\boldsymbol{B}$ is 90° and both wires have the same length a, that is,

$$F_1 = F_3 = i\Delta l B \sin 90^\circ = iaB$$

- From the definition of the cross product, $\boldsymbol{F}_1$ is out of the page toward the reader whereas $\boldsymbol{F}_3$ is into the page.
- Similarly, the magnitudes of the forces on sides 2 and 4 are equal, that is,

$$F_2 = F_4 = i\Delta l B \sin(90^\circ - \theta) = ibB \sin(90^\circ - \theta)$$

- The direction of $\boldsymbol{F}_2$ is upward, while that of $\boldsymbol{F}_4$ is downward.

- See Fig.16-5b. If we look at the top view along the pivot rod we see that there exists a torque that tends to rotate the loop about the pivot rod.
- Since $\boldsymbol{F}_1$ and $\boldsymbol{F}_3$ exert a torque on the loop, we have

$$\begin{aligned}\tau &= 2\boldsymbol{r} \times \boldsymbol{F} \\ &= 2rF\sin\theta \\ &= 2riaB\sin\theta \\ &= iabB\sin\theta \text{ (since } r = b/2) \\ &= iAB\sin\theta\end{aligned}$$

where $A = ab$.

- The result is the same for any other geometric configuration.
- The torque is a maximum when $\theta = 90^\circ$, that is, when the plane of the loop lies in the direction of $\boldsymbol{B}$.

- The torque is zero when $\theta = 0^\circ$, that is, when $\boldsymbol{B}$ is perpendicular to the plane of the loop.
- The magnitude of the angle of rotation is a function of the current passing through the coil.
- A way of increasing the torque on the coil is by using a coil made of several loops.

Magnetic Dipole Moment

- The *magnetic dipole moment* or simply the *magnetic moment* μ of the coil is defined as

$$\mu = iA$$

- The expression for the torque can now be written as

$$\tau = \mu B \sin\theta$$

- We define the direction of μ is the perpendicular to the plane of the loop according to the right-hand rule. Thus,

$$\tau = \mu \times \boldsymbol{B}$$

- Because a magnetic dipole experiences a torque when placed in an external magnetic field, work must be done by an external agent to change its orientation. This work becomes the potential energy E_p of the dipole.

- It is customary to set $E_p = 0$ when $\theta = 90^\circ$, that is, the dipole vector is perpendicular to the magnetic field.
-

$$
\begin{aligned}
W_\theta &= \int_{\theta_0}^{\theta_f} \tau d\theta \\
E_p &= \int_{90^\circ}^{\theta} \mu B \sin\theta d\theta \\
E_p &= -\mu B \cos\theta \Big|_{90^\circ}^{\theta} \\
&= -\mu B \cos\theta \\
&= -\mu \cdot \boldsymbol{B}
\end{aligned}
$$

- Since the $\cos\theta$ varies between 1 and -1, the maximum energy and minimum energy are μB and $-\mu B$, respectively.

Example 16-1

Assume that the electron in a hydrogen atom is essentially in a circular orbit of radius 0.5×10^{-10} m, and rotates about the nucleus at the rate of 10^{14} times per second. What is the magnetic moment of the hydrogen atom due to the orbital motion of the electron?

Sol:

$$\begin{aligned} \mu &= \text{area} \times \text{current} \\ &= \pi r^2 i \end{aligned}$$

where i is the current due to a single electron. Because current is defined as the amount of charge passing per unit time, we have

$$i = e\nu$$

where ν is the frequency of rotation and e is the magnitude of the charge of the electron.

$$\begin{aligned}\mu &= \pi r^2 e\nu \\ &= \pi(0.5\times 10^{-10}\ \text{m})^2(1.6\times 10^{-19}\ \text{C})(10^{14}\ \text{Hz}) \\ &= 1.26\times 10^{-25}\ \text{A-m}^2\end{aligned}$$

Force on a Moving Charge

- We know that

$$i = qNAv$$

where q is the magnitude of a charge, N the number of charge carries per unit volume, A the cross-sectional area, and v the average drift velocity of the charge carriers.

-

$$\begin{aligned} \boldsymbol{F} &= i\Delta\boldsymbol{l} \times \boldsymbol{B} \\ &= qNAv\Delta\boldsymbol{l} \times \boldsymbol{B} \end{aligned}$$

- The force per charge carrier $\boldsymbol{F}_q$ is

$$\boldsymbol{F}_q = \frac{\boldsymbol{F}}{NA\Delta\boldsymbol{l}} = q\boldsymbol{v} \times \boldsymbol{B}$$

We assume that $\boldsymbol{v}$ is in the same direction as $\Delta\boldsymbol{l}$ and therefore the current.

- Because $\boldsymbol{F}$ is perpendicular to $\boldsymbol{v}$ the magnetic field does no work on the charge. The magnetic field does not change the magnitude of the velocity of the charged particle.

The Hall Effect

- Consider the system shown in Fig. 16-6. Let the direction of the magnetic field be into the paper, indicated by the symbol $\otimes$, which suggests the tail of an arrow.
- See Fig. 16-7.

$$\boldsymbol{F}_B = q\boldsymbol{v} \times \boldsymbol{B} = qvB$$

  This force causes the positive charges to move to the upper part of the conducting strip while they are moving to the right.
- Because the sample as a whole must remain neutral, the lower part of the strip will become negatively charged.
- The accumulation of positive charges along the upper part and of negative charges along the lower part creates an electric field $\mathcal{E}_y$ that opposes the further upward drift of positive charges.
- There will be a potential difference V_H between D and C

associated with this electric field.

$$V_H = V_D - V_C = \mathcal{E}_y d$$

where it is assumed that in equilibrium $\mathcal{E}_y$ is constant and d is the width of the strip.

- This voltage difference is called the *Hall voltage* and the phenomenon is called the *Hall effect*.
- The equilibrium Hall voltage V_H will be established when the downward force $\mathcal{E}_y$ equal the upward force resulting from the magnetic field.
- At equilibrium

$$\begin{aligned} F_{\mathcal{E}} &= F_B \\ q\mathcal{E}_y &= qvB \\ \mathcal{E}_y &= vB \end{aligned}$$

$$V_H = vBd$$

- Because the Hall voltage can be readily measured by connecting a voltmeter between D and C, the Hall effect permits the experimental determination of the drift velocity of the charge carriers.
- An alternative form of V_H:

$$\begin{aligned} i &= qNAv \\ v &= \frac{i}{qNA} \\ V_H &= \frac{iBd}{qNA} \\ &= \frac{1}{qN}\frac{iB}{t} \text{ (since } A = t \times d) \\ V_H &= R_H \frac{iB}{t} \end{aligned}$$

where $R_H = 1/qN$ is called the *Hall coefficient*.

- Because i, B, and t are measurable, the magnitude of the Hall voltage will yield the value of N, the density of charge carriers.
- Let us consider the negative charge carriers in the Hall effect (see Fig. 16-8). Although the velocity vector v is reversed, the direction of the force given is still upward because the charge q is negative.
- The upper part of the strip will have an accumulation of negative charges and the lower part an accumulation of positive charges.
- The polarity of the Hall voltage will tell which type of carriers is responsible for conduction.

Example 16-2

A current of 50 A is established in a slab of copper 0.5 cm thick and 2 cm wide. The slab is placed in a magnetic field B of 1.5 T. The magnetic field is perpendicular to the plane of the slab and to the current. The free electron concentration in copper is 8.4×10^{28} electrons/m^3. What will be the magnitude of the Hall voltage across the width of the slab?

Sol:

$$\begin{aligned} V_H &= \frac{1}{Nq}\frac{iBd}{A} \\ &= \frac{50 \text{ A} \times 1.5 \text{ T} \times 2 \times 10^{-2} \text{ m}}{8.4 \times 10^{28} \text{ m}^{-3} \times 1.6 \times 10^{-19} \text{ C} \times 10^{-4} \text{ m}^2} \\ &= 1.12 \times 10^{-6} \text{ V} \end{aligned}$$

Electromagnetic Waves: The Nature of Light

- Starting with the fundamental laws of electromagnetism, J. Maxwell showed that accelerated charges would produce electromagnetic waves whose velocity of propagation c through free space should be

$$c = 3 \times 10^8 \text{ m/sec}$$

- An electromagnetic wave consists of an electric field $\mathcal{E}$ and a magnetic field B perpendicular to each other with both $\mathcal{E}$ and B perpendicular to the direction of their propagation.
- If we measure the value of $\mathcal{E}$ and B at difference points along the x axis, at some fixed times t we will observe that both $\mathcal{E}$ and B vary sinusoidally with x.
- If we sit at a fixed point in space and measure $\mathcal{E}$ and B at that

point as a function of time, we observe that both vary sinusoidally with time.

-
$$\begin{aligned}\mathcal{E} &= \mathcal{E}_0 \sin(kx - \omega t)\\ B &= B_0 \sin(kx - \omega t)\end{aligned}$$

- Any charge distribution that oscillates sinusoidally with time should produce electric and magnetic fields that behave as described by the above equations.
- The frequency ω of the electromagnetic wave should be the same as the frequency of oscillation of the charges producing it.
- No motion of material particles is involved in the electromagnetic wave, hence, there is no need for a medium of propagation.
- All electromagnetic waves should travel with velocity

$c = 3 \times 10^8$ m/sec.

- See Table on page 245 of the textbook.
- The laws of electromagnetic waves apply to waves of the entire electromagnetic spectrum.

Homework: 16.3, 16.4, 16.5, 16.6, 16.8, 16.10, 16.11, 16.14, 16.15, 16.17, 16.18, 16.19, 16.20.

The Doppler Effect

- If you are driving toward the police car that is parked by the side of the highway and sounding its siren, you will hear a higher frequency. If you are driving away from the police car you will hear a lower frequency.
- These motion-related frequency changes are examples of the *Doppler effect*.
- The Doppler effect holds not only for sound waves but also for electromagnetic waves, including microwaves, radio waves, and visible light.
- Police use the Doppler effect with microwaves to determine the speed of a car: a radar unit beams microwaves of a certain frequency ν toward the oncoming car.
- We will measure the speeds of sources S of sound waves and a

detector D of those waves relative to that body of air.

- We shall assume that S and D move either directly toward or directly away from each other, at speeds less than the speed of sound.

Detector Moving; Source stationary

- See Fig. 16-s1. A detector D is moving at speed v_D toward a stationary source S that emits spherical wavefronts, of wavelength λ and frequency ν, moving at the speed v of sound in air.
- Let us for the moment consider the situation in which D is stationary (Fig. 16-s2). The number of wavelengths in the distance vt is the number of wavelengths intercepted by D in time t, and that number is vt/λ. The rate at which D intercepts wavelengths, which is the frequency ν detected by D, is

$$\nu = \frac{vt/\lambda}{t} = \frac{v}{\lambda}$$

- See Fig. 16-s3. Let us again consider the situation in which D moves opposite the wavefronts. In time t, the distance moved by the wavefronts relative to D is $vt + v_D t$. The number of wavelengths in this relative distance $vt + v_D t$ is the number of wavelengths intercepted by D in time t, and is $(vt + v_D t)/\lambda$. The rate at which D intercepts wavelengths in this situation is the frequency ν', given by

$$\begin{aligned} \nu' &= \frac{(vt + v_D t)/\lambda}{t} = \frac{v + v_D}{\lambda} \\ &= \frac{v + v_D}{v/\nu} = \nu \frac{v + v_D}{v} \end{aligned}$$

- The frequency detected by D if D moves away from the source is

$$\nu' = \nu \frac{v - v_D}{v}$$

Source Moving; Detector Stationary

- Let detector D be stationary with respect to the body of air, and let source S move toward D at speed v_S (Fig. 16-s4).
- The motion of S changes the wavelength of the sound waves it emits, and the frequency detected by D.
- Let T $(= 1/\nu)$ be the time between the emission of any pair of successive wavefronts W_1 and W_2. During T, wavefront W_1 moves a distance vT and the source moves a distance $v_S T$. At the end of T, wavefront W_2 is emitted. In the direction in which S moves, the distance between W_1 and W_2, which is the wavelength λ' of the waves moving in that direction, is $vT - v_S T$. If D detects those waves, it detects frequency ν' given by

$$\nu' \quad = \quad \frac{v}{\lambda'} = \frac{v}{vT - v_S T}$$

$$= \quad \frac{v}{v/\nu - v_S/\nu} = \nu \frac{v}{v - v_S}$$

- In the direction opposite that taken by S, the wavelength λ' of the waves is $vT + v_S T$. If D detects those waves, it detects frequency ν' given by

$$\nu' = \nu \frac{v}{v + v_S}$$

- When both the source and the detector can be moving with respect to the air mass we have

$$\nu' = \nu \frac{v \pm v_D}{v \mp v_S}$$

which is the general Doppler effect.

The Doppler Effect at Low Speeds

- The Doppler effects for a moving detector and for a moving source are different, even through the detector and the source may be moving at the same speed.
- If the speeds are low enough (that is, if $v_D \ll v$ and $v_S \ll v$), the frequency changes produced by these two motions are essentially the same.
- Let $b \ll a$ and we want to approximate the value $(a+b)^n$. We have

$$(a+b)^n = a^n(1+b/a)^n = (a^n)(1+x)^n$$

where $x = b/a$. By binomial theorem,

$$(1+x)^n = 1 + \frac{n}{1!}x + \frac{n(n-1)}{2!}x^2 + \cdots$$

and

$$(1+x)^n \approx 1 + nx$$

when x is small.

- By using the binomial theorem, we have

$$\nu' \approx \nu \left(1 \pm \frac{u}{v}\right)$$

in which $u(= |v_S \pm v_D|)$ is the relative speed of the source with respect to the detector. If the source and the detector are moving toward each other, we anticipate a greater frequency; this requires that we choose the plus sign. On the other hand, if the source and the detector are moving away from each other, we anticipate a frequency decrease and choose the minus sign.

Example 16-s1

A toy rocket at a speed of 242 m/sec directly toward a stationary pole (through stationary air) while emitting sound waves at frequency $\nu = 1250$ Hz. (a) What frequency ν' is sensed by a detector that is attached to the pole? (b) Some of the sound reaching the pole reflects back to the rocket, which has an onboard detector. What frequency ν'' does it detect? (Assume that the sound speed is 343 m/sec.)

Sol: (a) The detected frequency is

$$\begin{aligned} \nu' &= \nu \frac{v}{v - v_S} = (1250\ \text{Hz}) \frac{343\ \text{m/sec}}{343\ \text{m/sec} - 242\ \text{m/sec}} \\ &= 4245\ \text{Hz} \end{aligned}$$

(b) The pole now acts as the source of sound in that it reflects sound waves, producing an echo. The frequency of the waves from the pole is the same as the frequency of the waves sensed by the pole, namely

$\nu' = 4245$ Hz. Thus,

$$\begin{aligned}\nu'' &= \nu' \frac{v + v_D}{v} = (4245 \text{ Hz}) \frac{343 \text{ m/sec} + 242 \text{ m/sec}}{343 \text{ m/sec}} \\ &= 7240 \text{ Hz}\end{aligned}$$

Example 16-s2

Bats navigate and search out prey by emitting, and then detecting reflections of, ultrasonic waves, which are sound waves with frequencies greater than what can be heard by a human. Suppose a horseshoe bat flies toward a moth at speed $v_b = 9.0$ m/sec, while the moth flies toward the bat with speed $v_m = 8.0$ m/sec. From its nostrils, the bat emits ultrasonic waves of frequency ν_{be} that reflect from the moth back to the bat with frequency ν_{bd}. The bat adjusts the emitted frequency ν_{be} until the returned frequency ν_{bd} is 83 kHz, at which the bat's hearing is best. (a) What is the frequency ν_m of the waves heard and reflected by the moth when ν_{bd} is 83 kHz? (b) What is the frequency ν_{be} emitted by the bat when ν_{bd} is 83 kHz?

Sol: (a) To find ν_m, we treat the moth as the source (of reflected waves with frequency ν_m) and the bat as the detector (of the echo with frequency $\nu_{bd} = 83$ kHz). Then we have

$$\begin{aligned}
\nu_{bd} &= \nu_m \frac{v + v_b}{v - v_m} \\
(83000 \text{ Hz}) &= \nu_m \frac{343 \text{ m/sec} + 9 \text{ m/sec}}{343 \text{ m/sec} - 8 \text{ m/sec}} \\
\nu_m &= 79000 \text{ Hz}
\end{aligned}$$

(b) To find ν_{be}, now we treat the bat as the source (of frequency ν_{be}) and the moth as the detector (of frequency ν_m kHz). Then we have

$$\begin{aligned}
\nu_m &= \nu_{be} \frac{v + v_m}{v - v_b} \\
(79000 \text{ Hz}) &= \nu_{be} \frac{343 \text{ m/sec} + 8 \text{ m/sec}}{343 \text{ m/sec} - 9 \text{ m/sec}} \\
\nu_{be} &= 75000 \text{ Hz}
\end{aligned}$$

The Doppler Effect for Light

- Sound waves require a medium through which to travel, but light waves do not.
- The speed of light always has the same value c, in all directions and in all inertial frames.
- Einstein's theory of special relativity shows that the Doppler effect for light depends only on the relative motion between the source of the light and a detector.
- At low enough speeds, doppler equations for light and for sound reduce to the same approximate result. When the relative speed of the source and the detector $u \ll c$, we have

$$\nu' = \nu(1 \pm u/c)$$

Chapter Seventeen: The Beginning of the Quantum Story

- The principle of relativity seemed to fail when applied to electromagnetism.
- The principle states that the laws of physics should be the same in all inertial frames of reference.
- This mathematical invariance was shown to be preserved with the laws of mechanics, but it broke down with the laws of electricity and magnetism.
- This problem eventually led to development of Einstein's *special theory of relativity*.
- Another problem that baffled physicists at the beginning of the twentieth century was the nature of the spectrum emitted by a class of objects called *blackbodies*.

- The predictions of classical ideas did not fit the experimental results. This problem led to the development of what we now call *quantum mechanics*.
- We will address only the quantum mechanical part of modern physics in the remainder of the class.

Blackbody Radiation

- All substances at finite temperatures radiate electromagnetic waves.
- Isolated atoms emit discrete frequencies, molecules emit bands of frequencies, and solids radiate a continuous spectrum of frequencies.
- The details of the spectrum emitted by a solid depend on its temperature and to some extent on its composition.
- Most of the radiation emitted lies in the infrared part of electromagnetic spectrum.
- As the temperature of the solid increases, more and more of the emitted radiation is in the visible.
- Objects that emit a spectrum of universal character, one that

does not depend on the composition of the object, are called *blackbodies*.

- A type of blackbody is a metallic cavity with a small hole (Fig. 17-1).
- After multiple partial reflections by the inner walls of the cavity, the radiation is eventually absorbed by the atoms in the walls of the cavity.
- These atoms, in turn, will reradiate electromagnetic waves into the cavity and some of it will leak out through the hole.
- The main features of the spectrum emitted by a blackbody are:
  1. The spectrum is continuous with a broad maximum. See Fig. 17-2.
  2. The integral of $I(\nu)$ over all ν, which we call I_T, represents the energy emitted per unit time per unit area, regardless of

the frequency, and it is found to increase with the fourth power of the temperature. That is (Stefan-Boltzmann law)

$$I_T = \int_0^\infty I(\nu)d\nu = \sigma T^4$$

where the constant $\sigma = 5.67 \times 10^{-8}\ \mathrm{W/m^2 - K^4}$.

3. The spectrum shifts toward higher frequencies as the temperature increases. That is,

$$\nu_{max} \propto T$$

- Attempts by physicists to explain the blackbody spectrum using the laws of classical electromagnetism and thermodynamics proved unsuccessful.
- Planck presented a paper entitled, "On the Theory of the Energy Distribution Law of the Normal Spectrum," which is considered

the birthday of quantum mechanics.

- The revolutionary physical hypothesis Plank made: A system undergoing simple harmonic motion with frequency ν can only have and therefore can only emit energies given by $E = nh\nu$, where $n = 1, 2, 3, \ldots$ and h is a constant known as Planck's constant.
- $h = 6.63 \times 10^{-34}$ Joule second (J-sec).
- Plank postulated that atomic oscillators can have only *discrete* energy values. See Fig. 17-3.
- An oscillator can take only certain values for the energy, when they lose that energy; they lose it in multiples of $h\nu$. these small quantities of energy are called *quanta* (singular, *quantum*).
- Plank derived a rather complicated expression for $I(\nu)$ that

matched the experimental data:

$$I(\nu) = \frac{2\pi h\nu^3}{c^2}\frac{1}{\exp(h\nu/k_BT) - 1}$$

where c is the velocity of light, k_B is the Boltzmann constant, ν is the frequency of the electromagnetic wave, and T is the absolute temperature of the black body.

Example 17-s1

Let us consider a mass $m = 10$ kg, attached to a spring of force constant $k = 10^3$ N/m. Let the initial amplitude of the motion be $A = 0.1$ m. What is the separation between adjacent energy levels?

Sol: We know that

$$E = \frac{1}{2}kA^2 = \frac{1}{2} \times 10^3 \text{ N/m} \times (0.1 \text{ m})^2 = 5 \text{ J}$$

and

$$\nu = \frac{1}{2\pi}\sqrt{\frac{k}{m}} = 1.59 \text{ Hz}$$

By Plank's hypothesis

$$\Delta E = h\nu = 6.63 \times 10^{-34} \text{ J-sec} \times 1.59 \text{ sec}^{-1} \approx 10^{-33} \text{ J}$$

The Photoelectric Effect

- The quantum idea was further expanded by Albert Einstein in connection with the photoelectric effect.
- Under certain conditions, light incident on a metal will cause electrons to be ejected from the surface of the metal. This is known as the *photoelectric effect*.
- An experimental arrangement that can be used to study some of the properties of the photoelectric effect is shown in Fig. 17-4.
- Because the anode is at a negative potential V with respect to the cathode, the electrons, on being emitted by the incident light striking the cathode, face a *retarding* voltage V.
- To reach the anode the photoelectron must be ejected with a kinetic energy E_k that is greater than the difference in potential energy, $|e|V$, between the anode and the cathode.

- A summary of the experimental results is shown in Fig. 17-5.
- When $V = V_0$ (V_0 is called the *stopping potential*), all the electrons, including the most energetic ones, are turned back and the current drops to zero.
- V_0 is a measure of the maximum energy with which the electrons are ejected from the cathode,

$$|e|V_0 = E_{k\ max}$$

- Figure 17-5b shows that V_0 is independent of the intensity of the light.
- $V_0 = a\nu$, where $a = 4.1 \times 10^{-15}$ J-sec/C.
- Fig 17-5c also shows that for frequencies $\nu \leq \nu_c$ V_0 is zero.
- When the conditions for photoemission are favorable (high enough ν, low enough V), the emission is almost instantaneous

(within 10^{-9} sec). This instantaneous emission has been observed to take place with extremely low intensities of light, as low as 10^{-10} W/m^2.

- According to classical physics, light is an electromagnetic wave. Two facts about waves are:
  1. The energy of a wave is continuously distributed over the entire space traversed by the wave.
  2. The intensity of a wave, which represents the energy carries by the wave per unit area perpendicular to the direction of propagation of the wave per unit time is proportional to the direction of the amplitude of the wave. In the case of electromagnetic waves,

$$I = \frac{1}{2}\epsilon_0 c \mathcal{E}_0^2$$

where ϵ_0 is the permittivity of free space, c is the velocity of light, and $\mathcal{E}_0$ is the amplitude of the electric field of the wave.

- The results of Fig. 17-5a can be explained in terms of classical concepts.
- The fact that $E_{k\ max}$ is independent of the intensity is difficult to explain by classical theory.
- The fact that $E_{k\ max}$ increases with ν can not be accounted for by classical physics. Since the energy of the electromagnic wave depends on its intensity, not on its frequency, V_0 should not depend on ν.
- The fact that the emission is almost instantaneous plays a key role in teh rejection of the classical ideas about the nature of electromagnetic radiation. See the next example.

Example 17-s2

Let us consider a sheet of some metal with an area of 1 m^2, as shown in Fig. 17-6. Assume that light of intensity $I = 10^{-10}$ W/m^2 shines on it. By the classical physics, how long that an electron will be emitted? Assume that the interatomic separation d in a metal is about $2\mathring{A}$ ($1\mathring{A} = 10^{-10}$ m).

Sol: Let us be optimistic and assume that all the energy falling on a certain atomic site of the metal sheet is absorbed by only one of the electrons of the atom, the most loosely bound. The number of atoms in a 1-m-long row is

$$\frac{1 \text{ m}}{d} = \frac{1 \text{ m}}{2 \times 10^{-10} \text{ m}} = 5 \times 10^9 \text{ atoms/row}$$

For simplicity, let the metal has cubic structure. Consequently, the number of atoms in the first layer of the metal sheet is

$(5 \times 10^9)^2 = 2.5 \times 10^{19}$ atoms/layer. That is,

$$\text{Energy/second-electron} = \frac{10^{-10}\ \text{J/sec}}{2.5 \times 10^{19} e} = 4 \times 10^{-30}\ \text{J/sec-e}$$

Assume that the minimum binding energy of an electron in a metal is 1 eV. The time required for an electron to collect 1 eV from the electromagnetic wave is

$$\begin{aligned} t &= \frac{1\ \text{eV} \times 1.6 \times 10^{-19}\ \text{J/eV}}{4 \times 10^{-30}\ \text{J/sec}} \\ &= 4 \times 10^{10}\ \text{sec} \\ &\sim 10^5\ \text{days} \end{aligned}$$

We see that classical physics also fails to explain the short time release of electrons in the photoelectric effect.

Einstein's Theory

- According to Einstein, the energy of an electromagnetic wave of frequency ν is not continuously distributed over the entire wave front, but instead it is localized in small bundles called *photons*.
- The energy of each photon is $E_{photon} = h\nu$.
- Basically a beam of electromagnetic radiation carries energy like a beam of particles, not like a wave.
- Einstein visualized the photoelectric effect as a particle-particle collision in which a photon of energy $h\nu$ collides with an electron in the metal and imparts all its energy to the electron.
- $$h\nu = E_k + E_b$$

  where E_b is the energy with which the particular electron is bound to the metal and E_k is the kinetic energy with which that electron is ejected.

- $$E_{k\ max} = h\nu - \phi$$

  where ϕ is the minimum binding energy and is called the *work function* of the metal.

- $E_{k\ max}$ is measured by determining the stopping potential V_0 and $E_{k\ max} = eV_0$. Thus,

  $$V_0 = \frac{h}{e}\nu - \frac{\phi}{e}$$

- $h/e = 4.1 \times 10^{-15}$ J-sec/C is equal to the observed value.
- If $h\nu < \phi$ or $\nu < \frac{\phi}{h} = \nu_c$, then no photoemission will take place.
- The emission is instantaneous because the process is not one in which the electrons progressively gather energy until they have enough to come out. It is a particle-particle collision.

Example 17-1

The eye is capable of detecting 10 eV of light energy. If we take as the average wavelength of light $6000\mathring{A}$, how many photons is the eye capable of detecting.

Sol: $\lambda\nu = c$. We obtain

$$\begin{aligned}\text{Energy/photon} &= h\nu = \frac{hc}{\lambda} \\ &= \frac{6.63 \times 10^{-34}\ \text{J-sec} \times 3 \times 10^{8}\ \text{m/sec}}{6000 \times 10^{-10}\ \text{m}} \\ &= 3.32 \times 10^{-19}\ \text{J} = 2.07\ \text{eV} \\ \text{Number of photons} &= \frac{10\ \text{eV}}{2.07\ \text{eV/photon}} = 5\ \text{photons}\end{aligned}$$

Example 17-2

The cut-off frequency for photoemission in copper is 1.0×10^{15} Hz. What is the maximum kinetic energy of the photoelectrons emitted when light of wavelength $1000\mathring{A}$ is shone on a copper surface?

Sol: The work function is

$$\phi = h\nu_c = 6.63 \times 10^{-34} \text{ J-sec} \times 1.0 \times 10^{15} \text{ Hz} = 6.63 \times 10^{-19} \text{ J}$$

$$\begin{aligned} E_{k\ max} &= h\nu - \phi = \frac{hc}{\lambda} - \phi \\ &= \frac{6.63 \times 10^{-34} \text{ J-sec} \times 3 \times 10^{8} \text{ m/sec}}{1000 \times 10^{-10} \text{ m}} - 6.63 \times 10^{-19} \text{ J} \\ &= 13.26 \times 10^{-19} \text{ J} = 8.29 \text{ eV} \end{aligned}$$

Mass-Energy Equivalence

- Einstein proposed that the mass of a particle can be treated as total energy of it dividing by c^2, where c is the speed of the light.
- Let E be the total energy of a particle and m be mass of this particle when the particle is of speed $\boldsymbol{v}$. Thus,

$$\frac{dE}{dt} = \boldsymbol{F} \cdot \boldsymbol{v}. \tag{1}$$

Since $\boldsymbol{F} = d(m_v \boldsymbol{v})/dt$, Equation 1 becomes

$$\frac{d(m_v c^2)}{dt} = \boldsymbol{v} \cdot \frac{d(m_v \boldsymbol{v})}{dt}. \tag{2}$$

Multiplying $2m_v$ on both sizes of the above equation, we have

$$c^2 (2m_v) \frac{dm_v}{dt} \quad = \quad 2m_v \boldsymbol{v} \cdot \frac{d(m_v \boldsymbol{v})}{dt}.$$

$$= \quad 2m_v v \frac{d(m_v v)}{dt}. \tag{3}$$

Equation 3 can be rewritten as

$$c^2 \frac{d(m_v^2)}{dt} = \frac{d(m_v^2 v^2)}{dt}. \tag{4}$$

Integrating both sizes of Equation 4, we have

$$m_v^2 c^2 = m_v^2 v^2 + C, \tag{5}$$

where C is a constant.

- When $\boldsymbol{v} = 0$, we have

$$m_0^2 c^2 = 0 + C,$$

where m_0 is the mass when the particle is at rest.

- Substituting C into Equation 5, we have

$$m_v^2 c^2 = m_v^2 v^2 + m_0^2 c^2.$$

That is

$$m_v = \frac{m_0}{\sqrt{1 - v^2/c^2}}.$$

Momentum of the Photon

- Momentum and energy for all possible speed from Einstein's theory of special relativity:
  1. $p = \gamma m_0 v$ (momentum);
  2. $E_k = m_0 c^2(\gamma - 1)$ (kinetic energy);
  3. $E = \gamma m_0 c^2 = m_0 c^2 + E_k$ (total energy, single particle);
  4. $(pc)^2 = E_k^2 + 2E_k m_0 c^2$ (the relation between momentum and kinetic energy);

     where
     $$\gamma = \frac{1}{\sqrt{1 - (v/c)^2}},$$
     v is the speed of the particle, m_0 is the rest mass of the particle, and c is the velocity of light.
- E_k is the kinetic energy of the particle, $E_0 = m_0 c^2$ is the rest

energy (m_0 is the rest mass of the particle, that is, the mass of the particle when its velocity is zero).

- For the photon, the rest energy and therefore the rest mass is zero.
- By Einstein's postulate,

$$E_{photon} = E_k = h\nu$$

Thus

$$\begin{aligned}(p_{photon}c)^2 &= E_k^2 + 2E_k m_0 c^2 = E_k^2 \\ &= (h\nu)^2\end{aligned}$$

Therefore, the momentum p of the photon is

$$p_{photon} = \frac{h\nu}{c} = \frac{h}{\lambda}$$

Compton Effect

- In 1916 Einstein extended his photon concept by asserting that when light interacts with matter, ont only energy but also linear momentum is transferred via photon.
- Like energy, momentum is transferred in discrete amounts and at pointlike locations instead of broad regions.
- A. Compton carried out an experiment that gave solid support to the view that both momentum and energy are transferred via photon. See Fig. 17-10.
- Fig. 17-11 shows his results.
- Although there is only a single wavelength in the incident x-ray beam, the scattered x-ray contain a range of wavelengths with two prominent intensity peaks.

- One peak is centered about the incident wavelength λ, the other about a wavelength λ' that is larger than λ by an amount $\Delta\lambda$, which is called the *Compton shift*.
- The value of the Compton shift varies with the angle at which the scattered x-ray are detected.
- Classical electromagnetic theory can not explain the presence of a longer wavelength in the scattered beam.
- Compton interpreted the scattering of x-rays from carbon in terms of energy and momentum transfers, via photons, between the incident x-ray beam and loosely bound electrons in the carbon target. See Fig. 17-12.
- As a result of collision, an x-ray of wavelength λ' moves off at an angle θ and the electron moves off at an angle ϕ.

- The conservation of energy gives

$$h\nu = h\nu' + E_k$$

where $h\nu$ is the energy of the incident x-ray photon, $h\nu'$ is the energy of the scattered x-ray photon, and E_k is the kinetic energy of the recoiling electron.

- Because the electron may recoil with a speed comparable to that of light, we must use the relativistic expression of its kinetic energy. That is,

$$E_k = mc^2(\gamma - 1)$$

where m is the rest mass of an electron.

$$\begin{aligned} h\nu &= h\nu' + mc^2(\gamma - 1) \\ \frac{h}{\lambda} &= \frac{h}{\lambda'} + mc(\gamma - 1) \end{aligned}$$

- The momentum of the incident and scattered photons is $p_{photon} = h/\lambda$ and that of the scattered electron is $p_e = \gamma m v$.
- We need to apply the law of conservation of momentum to the x-ray-electron collision.

$$\begin{aligned} \frac{h}{\lambda} &= \frac{h}{\lambda'}\cos\theta + \gamma m v \;\; \cos\phi \text{ (x direction)} \\ 0 &= \frac{h}{\lambda'}\sin\theta - \gamma m v \;\; \sin\phi \text{ (y direction)} \end{aligned}$$

- If we eliminate v and ϕ which deal only with the recoiling electron, then

$$\Delta\lambda = \lambda' - \lambda = \frac{h}{mc}(1 - \cos\theta)$$

- The photon can also collide with the atoms in the graphite. Since

, for graphite, $m_{\text{atom}} \approx 24000\ m_{\text{electron}}$, we have

$$\Delta\lambda_{\text{max}} \approx 2 \times 10^{-6} \mathring{A}$$

which is an unobservable amount when we compare it with $\lambda = 0.709\mathring{A}$.

Example 17-3

X-rays of wavelength $\lambda = 0.700\mathring{A}$ are Compton-scattered by the electrons in a graphite target. (a) What is the wavelength of the X-rays scattered at an angle $\theta = 120°$? (b) What is the kinetic energy of the scattering electrons if they were originally at rest? (c) What is the scattering angle of the electrons?

sol:

(a)

$$\begin{aligned} \lambda' &= \lambda + \frac{h}{mc}(1 - \cos 120°) \\ &= 0.7 \times 10^{-10} \text{ m} + \frac{6.63 \times 10^{-34} \text{ J-sec}}{9.1 \times 10^{-31} \text{ kg} \times 3 \times 10^{8} \text{ m/sec}}(1 + 0.5) \\ &= 0.736 \times 10^{-10} \text{ m} = 0.736\mathring{A} \end{aligned}$$

(b) From conservation of energy principles, the kinetic energy of the electron is equal to the energy lost by the photon, that is,

$$\begin{aligned} E_k &= h\nu - h\nu' \\ &= \frac{hc}{\lambda} - \frac{hc}{\lambda'} = hc\left(\frac{1}{\lambda} - \frac{1}{\lambda'}\right) \\ &= 6.63 \times 10^{-34}\text{J-sec} \times 3 \times 10^{8}\ \text{m/sec} \times \\ &\quad \left[\frac{1}{0.7 \times 10^{-10}\ \text{m}} - \frac{1}{0.736 \times 10^{-10}\ \text{m}}\right] \\ &= 1.39 \times 10^{-16}\ \text{J} = 869\text{eV} \end{aligned}$$

(c) From conservation of linear momentum principles on x direction we have

$$\frac{h\nu}{c} = p_e \cos\phi - \frac{h\nu'}{c}\cos 60^\circ$$

Therefore,

$$p_e \cos\phi = \frac{h\nu}{c} + \frac{h\nu'}{c}\cos 60^\circ$$

$$\begin{aligned} &= \frac{h}{\lambda} + \frac{h}{\lambda'}\cos 60^\circ \\ &= 6.63 \times 10^{-34}\text{J-sec}\left[\frac{1}{0.7 \times 10^{-10}\ \text{m}} + \frac{0.5}{0.736 \times 10^{-10}\ \text{m}}\right] \\ &= 13.98 \times 10^{-24}\ \text{kg m/sec} \end{aligned}$$

From conservation of linear momentum principles on y direction we have

$$0 = -p_e \sin\phi + \frac{h\nu'}{c}\sin 60^\circ$$

Therefore,

$$\begin{aligned} p_e \sin\phi &= \frac{h\nu'}{c}\sin 60^\circ = \frac{h}{\lambda'}\sin 60^\circ \\ &= \frac{6.63 \times 10^{-34}\text{J-sec}}{0.736 \times 10^{-10}\ \text{m}} \times 0.866 \\ &= 7.8 \times 10^{-24}\ \text{kg m/sec} \end{aligned}$$

Thus,

$$\tan\phi = \frac{7.8 \times 10^{-24} \text{ kg m/sec}}{13.98 \times 10^{-24} \text{ kg m/sec}} = 0.56$$

$$\phi = 29.2^\circ$$

Homework: 17.1, 17.3, 17.5, 17.6, 17.8, 17.9, 17.10, 17.12, 17.14, 17.15, 17.17, 17.18, 17.19.

Chapter Eighteen: Atomic Models

- The atom contains electrons. On the other hand, the atoms were known to be electrically neutral.
- The atom carries an amount of positive charges equal in magnitude to the charge of its electrons.
- The mass of the atom is thousands of times greater than that of the electron.
- most of the mass of the atom resides either with the positive charged matter or with neutral matter.

The Rutherford Model

- In 1911, Rutherford and two co-workers did a series of experiments in which positively charged α particles, emitted by certain radioactive materials were sent through a collimator and then shot through thin metallic foils. See Fig. 18-1.
- They studied the angular dependence of the scattered α particles, that is, what fraction of the α particles were scattered at a given angle θ.
- Rutherford suggested that the experimentally observed large-angle scattering of α particles could be explained on the basis of a "nuclear planetary model".
- The problems caused by the model, such as stability of the model which can not be explained by classical physics, required a drastic modification of physical concepts and led to the first

quantum mechanical model, the Bohr model.

- Let us consider the simplest of atoms, the hydrogen atom. An electron of charge $q = -e$ rotates in a circular orbit of radius r under the electrostatic attraction of the nucleus $q = +e$. See Fig. 18-2.

-

$$\begin{aligned} E_{\text{TOTAL}} &= \text{kinetic energy} + \text{potential energy} \\ &= \frac{1}{2}mv^2 - \frac{1}{4\pi\epsilon_0}\frac{e^2}{r} \end{aligned}$$

- Since the electrostatic attraction of the nucleus supplies the centripetal force that causes the circular motion of the electron, we have

$$F_{\text{radial}} = ma_{\text{radial}}$$

$$\begin{aligned}
\frac{1}{4\pi\epsilon_0}\frac{e^2}{r^2} &= m\frac{v^2}{r} \\
\frac{1}{2}mv^2 &= \frac{1}{2}\frac{1}{4\pi\epsilon_0}\frac{e^2}{r} \\
E_{\text{TOTAL}} &= -\frac{1}{2}\frac{1}{4\pi\epsilon_0}\frac{e^2}{r}
\end{aligned}$$

- The frequency of rotation ν can be expressed as

$$\nu = \frac{\text{velocity}}{\text{distance}} = \frac{v}{2\pi r}$$

$$\nu = \left\{\frac{e^2}{16\pi^3\epsilon_0 m}\right\}^{1/2}\frac{1}{r^{3/2}}$$

- A particle moving in a circle with constant frequency $\omega = 2\pi\nu$ is equivalent to a particle simultaneously undergoing simple harmonic motion in two mutually perpendicular directions.

- By classical electromagnetism, the electron moving in an orbit of radius r should radiate electromagnetic radiation.
- Because this electromagnetic radiation carries away energy, the energy of the atom must decrease as energy is radiated from it.
- According to principles of classical physics, the Rutherford model is unstable.
- By the classical model the spectrum emitted by such an atom should be a continuous one.
- The experimental facts are: The hydrogen atom is not unstable and the spectrum of hydrogen is not continuous.

The Spectrum of Hydrogen

- A typical experimental arrangement used to study the electromagnetic spectrum emitted by an element is shown in Fig. 18-3.
- The interference pattern of a grating is similar to that of a double slit. It consists of a central maximum that contains all the wavelengths present in the incident radiation. Secondary maxima will be observed on the screen where

  $$d \sin\theta = n\lambda \; n = 1, 2, 3 \ldots$$

  and d is the separation between slits in the double slit condition.
- In the case of hydrogen, the spectrum consists of families of lines ; the lines of a given family could be fitted to a simple empirical relation known as the Rydberg-Ritz formula, which yields a

series of lines

$$\frac{1}{\lambda} = R\left\{\frac{1}{n_k^2} - \frac{1}{n_j^2}\right\}$$

where $R = 1.0967757 \times 10^7\ \mathrm{m}^{-1}$ and n_k and n_j are integers.

- For a given n_k, $n_j = n_k + 1, n_k + 2, n_k + 3, \ldots$. For example,

  If $n_k = 1$, then $n_j = 2, 3, 4, \ldots$ This family is known as the Lyman series;

  If $n_k = 2$, then $n_j = 3, 4, 5, \ldots$ This family is known as the Balmer series;

  If $n_k = 3$, then $n_j = 4, 5, 6, \ldots$ This family is known as the Paschen series;

  If $n_k = 4$, then $n_j = 5, 6, 7, \ldots$ This family is known as the Brackett series;

- The Lyman series corresponds to frequencies in the ultraviolet part of the spectrum, the Balmer series to frequencies in the visible, and the other series to those in the infrared.

Example 18-1

What are the shortest and longest wavelengths of the Lyman series?

Sol: The longest wavelength corresponds to the smallest value of n_j, which for the Lyman series is 2.

$$\begin{aligned}
\frac{1}{\lambda_{\text{longest}}} &= R\left(\frac{1}{1^2} - \frac{1}{2^2}\right) \\
&= 1.0968 \times 10^7 \text{ m}^{-1}[1 - 0.25] \\
&= 0.8226 \times 10^7 \text{ m}^{-1} \\
\lambda_{\text{longest}} &= \frac{1}{0.8226 \times 10^7 \text{ m}^{-1}} \\
&= 1.215 \times 10^{-7} \text{ m} = 1215\mathring{A}
\end{aligned}$$

The shortest wavelength corresponds to the largest value of n_j that is

∞.

$$\begin{aligned}\frac{1}{\lambda_{\text{shortest}}} &= R\left(\frac{1}{1^2}-\frac{1}{\infty}\right)\\ &= 1.0968\times 10^{7}\ \text{m}^{-1}\\ \lambda_{\text{shortest}} &= \frac{1}{1.0968\times 10^{7}\ \text{m}^{-1}}\\ &= 0.9117\times 10^{-7}\ \text{m}\approx 912\mathring{A}\end{aligned}$$

The Bohr Atom

- To avoid the two problems encountered by the Rutherford model and to explain the spectral data that we have discussed before, Bohr proposed a model of the hydrogen atom that can be summarized in three postulates.

  Postulate 1. The electron can taken only certain orbits for which the angular momentum L takes values given by

  $$L = mvr = n\hbar \; n = 1, 2, 3, \ldots$$

  where $\hbar$ is Plank's constant divided by 2π.

  Postulate 2. An electron in one of the allowed orbits does not radiate electromagnetic radiation.

  Postulate 3. If an electron is initially in an allowed orbit of energy E_i and goes into another orbit of lower energy E_f, electromagnetic radiation will be emitted with a precise frequency given by

$$\nu = \frac{E_i - E_f}{h}$$

- Bohr quantizes the orbital angular momentum, not the energy; however, the energy will also be quantized.
- The second postulate is needed to prevent the instability predicted by electromagnetic theory.
- The third postulate is basically a reaffirmation of the photon concept.
-

$$\begin{aligned} v &= n\frac{\hbar}{mr} \\ \frac{1}{4\pi\epsilon_0}\frac{e^2}{r^2} &= \frac{m}{r}n^2\frac{\hbar^2}{m^2r^2} \\ r_n &= n^2 r_0 \end{aligned}$$

where

$$r_0 = \frac{4\pi\hbar^2\epsilon_0}{e^2 m}$$

- We see that only orbits with radius $r_0, 4r_0, 9r_0, \ldots$, are allowed.
- Any system, if left alone, will tend to go to the lowest energy state available to it.
- The lowest state corresponds to the orbit of smallest radius.
- According to Bohr model, the normal state of the hydrogen atom is a circular orbit of radius r_0, or (recall that $1/4\pi\epsilon_0 = 9 \times 10^9$ N-m^2/C^2)

$$\begin{aligned} r_0 &= \frac{4\pi\hbar^2\epsilon_0}{e^2 m} \\ &= \frac{(1.11 \times 10^{-10}\ \mathrm{C}^2/\ \mathrm{N\text{-}m}^2)(1.05 \times 10^{-34}\ \mathrm{J\text{-}sec})^2}{(1.6 \times 10^{-19}\ \mathrm{C})^2(9.1 \times 10^{-31}\ \mathrm{kg})} \end{aligned}$$

$$= \quad 0.53 \times 10^{-10} \text{ m} = 0.53\mathring{A}$$

This result is in agreement with the experimentally known size of the atom.

- The restriction on the allowed radius for the orbit leads immediately to the quantization of the energy spectrum.
-

$$\begin{aligned} E &= -\frac{1}{2}\frac{1}{4\pi\epsilon_0}\frac{e^2}{r} \\ &= -\frac{e^2}{8\pi\epsilon_0 n^2 r_0} \\ E_n &= -\frac{e^4 m}{8\epsilon_0^2 h^2}\frac{1}{n^2} \\ &= -\frac{E_0}{n^2} \quad n = 1, 2, 3, \ldots \end{aligned}$$

where

$$E_0 = \frac{e^4 m}{8\epsilon_0^2 h^2}$$

- $$\begin{aligned} E_0 &= \frac{(1.6 \times 10^{-19}\ \text{C})^4 (9.1 \times 10^{-31}\ \text{kg})}{(8)(8.84 \times 10^{-12}\ \text{C}^2/\ \text{N-m}^2)(6.63 \times 10^{-34}\ \text{J-sec})^2} \\ &= 2.17 \times 10^{-18}\ \text{J} = 13.56\ \text{eV} \end{aligned}$$

- $E_1 = -13.56$ eV.
- A few of the energy levels of the hydrogen spectrum are shown in Fig. 18-4.
- The energy of $n = 1$ level is called *ground state.* The other levels, corresponding to higher values of n, are called the *excited states.*
- To ionize the atom (to separate the electron from the nucleus), 13.56 eV of energy must be given to it according to the model.

- Once the atom has been ionized, a free electron near the nucleus can be captured by it. This capture can take place in one step-namely, the electron in one jump falls down to the ground state-or in a series of steps in which the electron fails into one or more excited states before it winds up in the ground state.
- In each transition toward a lower energy state the electron loses a precise quantity of energy, called a *quantum*.
- The energy will depend on the energy difference of the levels between which the transition takes place.
- Because the energy difference between levels is not continuous, the frequency spectrum of the emitted photons will not be continuous either.
- Suppose that an electron is initially in a state of energy $E_i = -\frac{E_0}{n_i^2}$ and then makes a transition to a state of energy

$E_f = -\frac{E_0}{n_f^2}$. Thus, a photon of frequency

$$\nu = \frac{E_i - E_f}{h}$$

will be emitted. See Fig. 18-5.

$$\nu = \frac{\left\{-\frac{E_0}{n_i^2}\right\} - \left\{-\frac{E_0}{n_f^2}\right\}}{h} = \frac{E_0}{h}\left(\frac{1}{n_f^2} - \frac{1}{n_i^2}\right)$$

Since $\lambda\nu = c$ we have

$$\begin{aligned} \frac{1}{\lambda} &= \frac{E_0}{hc}\left(\frac{1}{n_f^2} - \frac{1}{n_i^2}\right) \\ &= R_{\text{Bohr}}\left(\frac{1}{n_f^2} - \frac{1}{n_i^2}\right) \end{aligned}$$

where R_{Bohr} is a constant and

$$\begin{aligned} & R_{\text{Bohr}} \\ = \quad & \frac{E_0}{hc} = \frac{e^4 m}{8\epsilon_0^2 h^3 c} \\ = \quad & \frac{(1.6 \times 10^{-19}\ \text{C})^4 \times (9.1 \times 10^{-31}\ \text{kg})}{8 \times (8.84 \times 10^{-12}\ \text{C}^2/\ \text{N-m}^2)^2 \times (6.63 \times 10^{-34}\ \text{J-sec})^3 \times (3 \times 10^8\ \text{m/sec})} \\ = \quad & 1.0974 \times 10^7\ \text{m}^{-1} \end{aligned}$$

- We see that above equation is similar to the Rydberg-Ritz formula and the constant R_{Bohr} is almost identical to the Rydberg constant R. See Fig. 18-6.

Example 18-2

After begin excited, the electron of a hydrogen atom eventually falls back to the ground state. This can take place in one jump or in a series of jumps, the electron falling into lower excited states before it ends up in the ground state. Consider a hydrogen atom that has been raised to the second excited state, that is, $n = 3$. Calculate the different photon energies that may be emitted as the atom returns to the ground state.

Sol: The possible transitions are shown in Fig. 18-7.

$$\begin{aligned} h\nu_{31} &= E_3 - E_1 = E_0\left(\frac{1}{1^2} - \frac{1}{3^2}\right) \\ &= 13.56\ \text{eV}\left(1 - \frac{1}{9}\right) = 12.05\ \text{eV} \end{aligned}$$

$$\begin{aligned}
h\nu_{32} &= E_3 - E_2 = E_0\left(\frac{1}{2^2} - \frac{1}{3^2}\right) \\
&= 13.56 \text{ eV}\left(\frac{1}{4} - \frac{1}{9}\right) = 1.88 \text{ eV} \\
h\nu_{21} &= E_2 - E_1 = E_0\left(\frac{1}{1^2} - \frac{1}{2^2}\right) \\
&= 13.56 \text{ eV}\left(\frac{1}{1} - \frac{1}{4}\right) = 10.17 \text{ eV}
\end{aligned}$$

The Franck-Hertz Experiment

- The existence of discrete energy levels in atoms was demonstrated directly by Franck and Hertz in 1914.
- A schematic of the experimental set-up is shown in Fig. 18-8.
- See Fig. 18-9 and 18-10.
- The fact that there is no drop in the current until $V_0 = 4.9$ V indicates that the electrons do not lose energy through collisions until they have 4.9 eV of kinetic energy.

Homework: 18.2, 18.4, 18.5, 18.8, 18.9, 18.10, 18.11, 18.13, 18.14, 18.15, 18.16, 18.17, 18.18, 18.19.

Chapter Nineteen: Fundamental Principles of Quantum Mechanics

- In this chapter we will present two principles that form the cornerstones of quantum mechanics: *de Broglie's hypothesis* and the *uncertainty principle*.
- We must consider electromagnetic radiation as having dual properties: a wave and a particle.
- These two models are complementary.

De Broglie's Hypothesis and its Experimental Verification

- In 1925, de Broglie assumed the existence of a natural symmetry in nature and proposed that the dual character exhibited by photons should equally apply to all material particles.
- The motion of a particle is governed by the wave propagation properties of a *pilot wave* (also called *matter wave*).
- The wavelength λ and the frequency ν of the pilot wave associated with a particle of momentum p and energy E are

$$\lambda = \frac{h}{p} \text{ and } \nu = \frac{E}{h}$$

- To prove de Broglie's hypothesis we have to show experimentally that a beam of particles exhibits wave-like properties, such as interference and diffraction.

- Consider a bullet of mass $m = 0.1$ kg moving with a velocity $v = 10^3$ m/sec. According to de Broglie,

$$\lambda = \frac{h}{p} = \frac{6.63 \times 10^{-34} \text{ J-sec}}{0.1 \text{ kg} \times 10^3 \text{ m/sec}} = 6.63 \times 10^{-36} \text{ m}$$

which is 26 orders of magnitude smaller than the diameter of an atom.

- In order to observe interference effects, the size of the diffracting slit d must be comparable to the wavelength of the wave.
- The smallest slit or diffraction grating available in nature is a crystalline solid where d is a few angstroms.
- In 1927, Davisson and Germer performed an experiment that was successful to observe the wave nature of material particles.
- The experimental setup used is shown in Fig. 19-1 and the electron path must be through a vacuum.

- If the propagation of the beam is particle-like, we may expect that the intensity I of the scattered beam will have a smooth monotonic dependence of both ϕ and V because only elastic collisions with the atoms of the crystal are involved.
- If the incoming electrons are not particles but are actually waves, we would expect a diffraction effect like the one observed with X rays when the Bragg scattering condition is satisfied, that is, when

$$n\lambda = 2d\sin\theta$$

- The d for nickel is 0.91 $\mathring{A}$ and the wavelength of X rays is 1.65 $\mathring{A}$.
- The angle of incidence to the plane of atoms for first-order diffraction is

$$\sin\theta \quad = \quad \frac{1.65\ \mathring{A}}{2\times 0.91\ \mathring{A}} = 0.907$$

$$\theta = 65^\circ$$

- Since $2\theta + \phi = 180^\circ$, $\phi = 50^\circ$.
- The experimental results of Davisson and Germer are shown in Fig. 19-2b.
- The angle ϕ between the incoming and the scattered beams of electrons was set at 50°. A maximum intensity was observed when the voltage was $V = 54$ V.
- Davisson and Germer calculated the wavelength that the electrons would have from the de Broglie hypothesis as follows:

$$\begin{aligned} E_k &= \frac{1}{2}mv^2 \\ &= \frac{p^2}{2m} \\ p &= \sqrt{2mE_k} \end{aligned}$$

Since $E_k = eV$, we have

$$p = \sqrt{2meV}$$

$$\begin{aligned}\lambda &= \frac{h}{p} = \frac{h}{\sqrt{2meV}} \\ &= \frac{6.63 \times 10^{-34} \text{ J-sec}}{(2 \times 9.1 \times 10^{-31} \text{ kg} \times 1.6 \times 10^{-19} \text{ C} \times 54 \text{ V})^{1/2}} \\ &= 1.67 \times 10^{-10} \text{ m} = 1.67 \text{ \AA}\end{aligned}$$

The wavelength for the electrons is similar to that of X rays. Thus, the experiment confirms de Broglie hypothesis that particles have wave properties.

- The existence of matter waves rests on solid experimental foundations, and electron and neutron diffraction measurements are standard techniques for the study of crystalline materials.

- The design and construction of the electron microscope are based on the principle that electrons propagate as waves.

Example 19-1

A beam of monochromatic neutrons is incident on a KCl crystal with lattice spacing of 3.14 $\mathring{A}$. The first-order diffraction maximum is observed when the angle θ between the incident beam and the atomic planes is 37°. What is the kinetic energy of the neutrons?

Sol: Using the Bragg condition we can find the wavelength of the neutron beam.

$$\begin{aligned}\lambda &= 2d\sin\theta \\ &= 2\times 3.14\ \mathring{A}\ \sin 37^\circ = 3.78\ \mathring{A}\end{aligned}$$

From de Broglie's hypothesis, the momentum of the neutrons is

$$p = \frac{h}{\lambda} = \frac{6.63\times 10^{-34}\ \text{J-sec}}{3.78\times 10^{-10}\ \text{m}} = 1.75\times 10^{-24}\ \text{kg}-\text{m/sec}$$

The kinetic energy of the neutrons will be

$$\begin{aligned} E_k &= \frac{1}{2}mv^2 = \frac{p^2}{2m} \\ &= \frac{(1.75 \times 10^{-24} \text{ kg} - \text{m/sec})^2}{2 \times 1.67 \times 10^{-27} \text{ kg}} = 9.21 \times 10^{-22} \text{ J} \\ &= 5.75 \times 10^{-3} \text{ eV} \end{aligned}$$

Nature of the Wave

- Classically, electromagnetic (EM) radiation is an electric and a magnetic field whose behavior in time and space is comparable to that of a wave in a string. However, this model of EM radiation cannot explain the photoelectric effect or the Compton effect.
- The link between the classical picture and the photon picture is the intensity I. The intensity has meaning in both models; it is the energy per time, per unit area of the beam.
- The intensity of a wave is proportional to the square of the amplitude of the wave.
- In the photon model, the intensity is

$$I = (c)(N)(h\nu)$$

where N is the number of photons per unit volume of the beam.

- After comparing the intensity of two models, we conclude that the number of photons in the beam must be proportional to the square of the amplitude of the wave.
- If one were to shine light on a screen with an intensity corresponding to 100 photons per second per meter, one would make the following observations:
  1. The 100 photons per second is an average value.
  2. One can not predict where the next photon striking the screen will land.
- The reason for this randomness in the behavior of photons is the process of photon emission itself.
- As a result of this randomness we can not state with certainty whether or not we will find a photon within a small volume of the beam: We can nevertheless talk about the probability of

finding a photon within such a volume.

- The probability of finding a photon within a given volume of the beam is proportional to the square of the amplitude of the wave associated with the beam.
- Within the photon model the wave is not an electromagnetic field; it is represented by a mathematical function that measures the photon probability density.
- In 1926, Born extended this probabilistic interpretation to the matter waves proposed by de Broglie for material particles.
- The guiding wave is represented by a mathematical function, $\psi(r,t)$, called a *wavefunction*.
- At some instant t, a measurement is made to locate the particle associated with the wavefunction ψ. The probability $P(r,t)dV$ that the particle will be found within a small volume dV

centered around a point with position vector r (with respect to a prechosen set of coordinates) is equal to $| \psi |^2 \, dV$, that is,

$$P(r,t)dV = | \psi |^2 \, dV$$

- The probability of finding the particle somewhere in space must be unity. Therefore,

$$\int_{-\infty}^{\infty} | \psi |^2 \, dV = 1$$

- The methods of quantum mechanics consist in first finding the wave-function associated with a particle or a system of particle.

The Double-Slit Experiment Revisited

- If the detector detects electrons at different positions for a sufficiently long period of time, one finds an interference pattern representing the number of electrons arriving at any position along the detector line. See Fig. 19-s1.
- Although the electrons are detected as particles at a localized spot at some instant of time, the probability of arrival at that spot is determined by the intensity of two interfering matter waves.
- If one slit is covered during the experiment, the result is a symmetric curve that peaks around the center of the open slit, much like the pattern formed by bullets shot through a hole in armor plate.
- The two overlapping blue curves in the center of Fig. 19-s2 are

plots of electrons detected per minute with only one slit open.

- This curves are expressed as $\mid \psi_1 \mid^2$ and $\mid \psi_2 \mid^2$, where ψ_1 and ψ_2 represent the matter waves of the electron passing through slit 1 and slit 2, respectively.
- In the single-slit curve, a maximum probability of arrival no longer occurs at $\theta = 0$. In fact, the interference patterns has been lost, and the accumulated result is simply the sum of the individual results.
- To find the probability of detecting the electron at a particular points at the detector when both slits are open, we may say the electron is in a superposition state, given by

$$\psi = \psi_1 + \psi_2.$$

- The probability of detecting the electron at the detector is

$|\psi_1 + \psi_2|^2$, and not $|\psi_1|^2 + |\psi_2|^2$.

- Because in general matter waves that start out in phase at the slits travel different distances to the detector, ψ_1 and ψ_2 have a relative phase difference ϕ at the detector.
- Using a phasor diagram (Fig 19-s3) to find $|\psi_1 + \psi_2|^2$ yields

$$|\psi|^2 = |\psi_1 + \psi_2|^2 = |\psi_1|^2 + |\psi_2|^2 + 2|\psi_1||\psi_2|\cos\phi.$$

- We conclude that an electron's wave probability interacts with both slits simultaneously.

The Uncertainty Principle

- The use of probability functions to describe the behavior of a system is not foreign to classical physics.
- In statistical mechanics, where one often deals with a very large number of particles, we do not deal with the individual velocities, energy, or such, of the particles.
- In quantum mechanics, the use of probability functions is one of fundamental necessity.
- The famous *uncertainty principle* by Heisenberg: An experiment cannot simultaneously determine a component of the momentum of a particle, for example p_x and the exact value of the corresponding coordinate x.

- The best one can do is

$$\Delta p_x \Delta x \geq \hbar$$

  where Δ represent the uncertainty of measurement.

- There are three points that we should note about the uncertainty principle.
  1. The limitations imposed by the uncertainty principle have nothing to do with the quality of the experimental instrument.
  2. The uncertainty principle does not say that one can not determine the position or the momentum exactly.
  3. The uncertainty principle is a direct consequence of de Broglie's hypothesis, which is confirmed by experiment.
- Let us try to predict the trajectory of the moon around the earth. Suppose that we are able to determine the position of the

moon with an uncertainty $\Delta x = 10^{-6}$ m, a very small uncertainty in this case. What is the limit of our ability to determine its velocity simultaneously?

$$\Delta p_x \geq \frac{\hbar}{\Delta x} = \frac{10^{-34} \text{ J-sec}}{10^{-6} \text{ m}} = 10^{-28} \text{ kg m/sec}$$

Since $p_x = mv_x$, we have

$$\Delta v_x = \frac{\Delta p_x}{m} \geq \frac{10^{-28} \text{ kg m/sec}}{6 \times 10^{22} \text{ kg}} = 10^{-50} \text{ m/sec}$$

which is an insignificant error when we compare it with the measured value of $v = 10^3$ m/sec.

- Let us consider an electron in the hydrogen atom. Let $\Delta x \approx 10^{-10}$ m. With what accuracy can the x component of p

be determined?

$$\Delta p_x \geq \frac{\hbar}{\Delta x} = \frac{10^{-34} \text{ J-sec}}{10^{-10} \text{ m}} = 10^{-24} \text{ kg m/sec}$$

$$E_k = 13.6 \text{ eV} = 13.6 \text{ eV} \times 1.6 \times 10^{-19} \text{ J/eV} = 2.18 \times 10^{-18} \text{ J}$$

The corresponding momentum is

$$\begin{aligned} p &= (2mE_k)^{1/2} = (2 \times 9.1 \times 10^{-31} \text{ kg} \times 2.18 \times 10^{-18} \text{ J})^{1/2} \\ &= 2 \times 10^{-24} \text{ kg m/sec} \end{aligned}$$

Then the ratio of the uncertainty of the momentum Δp_x to the momentum itself is

$$\frac{\Delta p_x}{p} = \frac{10^{-24} \text{ kg m/sec}}{2 \times 10^{-24} \text{ kg m/sec}} = 0.5$$

or 50% uncertainty.

- With such a limitation on simultaneous knowledge of position and momentum, we must develop and use the concept of the probability of location and momentum.

Physical Origin of the Uncertainty Principle

- Suppose that we want to determine the position of an electron and that we can look at the electron with a hypothetical microscope. See Fig. 19-3.
- The smallest intensity we can use, according to the quantum mechanical model of light, is one photon. This photon may enter the objective lens anywhere within the angular range $+\phi$ to $-\phi$.
- We conclude that the uncertainty of the x-component of the momentum of the photon is

$$\Delta p_x(\text{photon}) = 2p_{\text{photon}} \sin\phi$$

- Conservation of linear momentum necessitates that if the photon acquires a certain momentum in the x direction, the electron must acquire the same amount in the opposite direction.

$$\begin{aligned}\Delta p_x(\text{electron}) &= 2p_{\text{photon}}\sin\phi \\ &= 2\frac{h}{\lambda}\sin\phi\end{aligned}$$

Matter Waves and the Uncertainty Principle

- The spacial propagation of a particle is governed by a wave $\psi(x,t)$.
- The sinusoidal travelling wave discussed before, $A\sin(kx-\omega t)$, can not be used as $\psi(x,t)$. The reason is as follows: Because the amplitude of the wave is the same for all values of x, the particle can be found with equal probability at any point in space, that is the particle is completely unlocalized, $\Delta x = \infty$.
- The velocity of the wave given above is not the same as the velocity of the particle that it guides. Recall that $v = \lambda\nu$. By de Broglie's relations,
$$\lambda = \frac{h}{p} \text{ and } \nu = \frac{E}{h}$$
where p is the momentum of the particle and is equal to mv_{particle},

and E is the relativistic energy of the particle mc^2. Thus,

$$v = \frac{h}{p} \times \frac{E}{h} = \frac{mc^2}{mv_{\text{particle}}} = \frac{c^2}{v_{\text{particle}}}$$

The phase velocity of the wave is greater than the velocity of the particle.

- To describe a particle that is partially localized, we need a wave with an amplitude that is different from zero only over a small region of space where there is a chance of finding the particle.
- We need a wave that looks like the *wave packet* shown in Fig. 19-4.
- The more localized the particle is, the narrower the wave packet must be. Mathematically, such a wave package is obtained by mixing together an infinitely large number of sinusoidal travelling waves.

- Alternatively, we can express the wave packet as an integral with respect to k and ω, where $\lambda = 2\pi/k$ and $\nu = \omega/2\pi$. That is,

$$\psi(x,t) = \int_0^\infty \int_0^\infty A(k,\omega)\sin(kx - \omega t)dk\, d\omega$$

  See Fig. 19-5.

- For a given k and ω, the amplitude A has a definite value. The coefficient A basically determines how much of a particular wavelength and frequency we mix with the others. These coefficients determine the shape of the wave packet.

Velocity of the Wave Packet: Group Velocity

- Because the wave package accompanies the particle and tells us approximately where the particle may be found, it must travel with the same velocity as the particle.
- Let us consider a simpler case that is mathematically easier to handle.
- Let

$$\psi(x,t) = \psi_1 + \psi_2 = A\sin[kx - \omega t] + A\sin[(k+\Delta k)x - (\omega + \Delta\omega)t]$$

  where $\Delta k \ll k$ and $\Delta\omega \ll \omega$.

- If we use the trigonometric relation

$$\sin A + \sin B = 2\cos\frac{A-B}{2}\sin\frac{A+B}{2}$$

we get

$$\psi(x,t) = 2A\cos\left(\frac{\Delta kx - \Delta\omega t}{2}\right)\sin(kx - \omega t)$$

where we have used the approximations $2k + \Delta k \approx 2k$, $2\omega + \Delta\omega \approx 2\omega$. See Fig. 19-6.

- Such a ψ could be used to describe a beam of particles, with one particle in each wave packet.
- The velocity of the wave inside the envelopes is the same as the velocity of the individual waves, $v = \omega/k$.
- The envelopes (that is , the wave packets) travel with a velocity, called the group velocity, v_{group}, given by

$$v_{\text{group}} \frac{\frac{\Delta\omega}{2}}{\frac{\Delta k}{2}} = \frac{\Delta\omega}{\Delta k} \approx \frac{d\omega}{dk}$$

- We can show that v_{group} is the same as the velocity of the particle, by using de Broglie's relations,

$$\lambda = \frac{h}{p} \text{ or } k = \frac{p}{\hbar} \text{ hence } dk = \frac{dp}{\hbar}$$

and

$$\nu = \frac{E}{h} \text{ or } \omega = \frac{E}{\hbar} \text{ hence } d\omega = \frac{dE}{\hbar}$$

- Thus,

$$v_{\text{group}} = \frac{dE}{dp}$$

But we know that $E = \frac{1}{2}mv_{\text{particle}}^2 = p^2/2m$; it follows that

$$v_{\text{group}} = \frac{dE}{dp} = \frac{p}{m} = \frac{mv_{\text{particle}}}{m} = v_{\text{particle}}$$

- The wave packet moves with the particle. Although we have derived this result for a particularly simple case, it holds for the more general case.

The Principle of Complementarity

- When we summarize our knowledge of nature, we see that energy appears in two forms; it is either *wave-like*, such as in the case of the ripples in a pond, or *particle-like*, such as in the transfer of energy from a gun to the target by the bullets.
- We need a more complex model, one that encompasses aspects of both the particle and the wave models. This dual character of matter is summarized in Bohr's principle of complementarity: The particle and the wave model are complementary.
- Certain measurements reveal the wave aspects of electromagnetic radiation and of material particles; measurements that involve their spacial distribution.
- Other measurements reveal their particle aspects; measurements dealing with the interaction with one another or with other

entities.

Homework: 19.3, 19.4, 19.5, 19.7, 19.8, 19.9, 19.11, 19.12, 19.13, 19.14, 19.15, 19.17, 19.20, 19.21, 19.22.

Chapter Twenty: An Introduction to the Methods of Quantum Mechanics

- We have seen how the introduction of the quantization postulates explained the experimental facts concerning blackbody radiation, the photoelectric effect, and the hydrogen spectrum. These theories constitute what we call today the *old quantum theory* (OQT).
- Despite its successes, the OQT has some series deficiencies:
  1. The theory can be applied only to periodic systems (harmonic oscillators, circular motion, and such).
  2. The Bohr theory does not explain why certain wavelengths are more intense than others.
  3. The Bohr theory fails to explain the spectrum of even the simplest of the multielectron atoms, He.

4. It is not a unified or a general theory.

The Schrödinger Theory of Quantum Mechanics

- The basis of the modern theory of quantum mechanics was developed in 1925 by E. Schrödinger; an equivalent, but mathematically different, theory was presented about the same time by W. Heisenberg.
- The most important fact that we have presented so far is that the behavior of a microscopic particle is governed by the wave associated with it.
- If we want to describe a free particle, which is partially localized, we could use a wave packet.
- De Broglie's hypothesis does not tell us what type of wave one can associate with a particle that is not free and that is acted on by a a force.
- The Schrödinger theory tells us how to obtain the wavefunction

$\psi(x, t)$ associated with a particle, when we specify the forces acting on the particle, by given the potential energy associated with the forces.

- The Schrödinger theory also tells us how to extract information about the particle from the associated wavefunction.
- The *Schrödinger equation* tells us how the wavefunction changes as a result of the forces acting on the particle.
- Because the wavefunction ψ is a function of space and time, the equation contains derivatives with respective to x, y, and z and with respective to t.
- We will primarily consider motion in only the x direction.
- The partial derivative of ψ with respective to x is written as $\partial\psi/\partial x$.
- i is the imaginary number $\sqrt{-1}$.

- The total energy of a particle is equal to the kinetic energy plus the potential energy,

$$\begin{aligned} E &= \frac{1}{2}mv^2 + E_p \\ &= \frac{p^2}{2m} + E_p \end{aligned}$$

- Multiplying both sides of this equation by the wavefunction ψ we obtain

$$E\psi = \frac{p^2}{2m}\psi + E_p\psi$$

- We will see later that there is a relation between the energy E and the operator $i\hbar\ \partial/\partial t$ and between the momentum p and the operator$-i\hbar\ \partial/\partial x$.
- Operating on the function with $i\hbar\ \partial/\partial t$ and with $-i\hbar\ \partial/\partial x$ is the same as multiplying the function by E and p, respectively.

Thus, we have

$$\begin{aligned} i\hbar\frac{\partial\psi}{\partial t} &= \frac{1}{2m}\left(-i\hbar\frac{\partial}{\partial x}\right)\left(-i\hbar\frac{\partial}{\partial x}\right)\psi + E_p\psi \\ i\hbar\frac{\partial\psi}{\partial t} &= -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + E_p\psi \end{aligned}$$

which is conventionally written as

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + E_p\psi = i\hbar\frac{\partial\psi}{\partial t}$$

- The above equation is known as the *one-dimensional time-dependent Schrödinger equation.*
- The Schrödinger equation is to quantum mechanics what Newton's second law is to classical physics.
- When the particle is not free, the relationship (between the energy E and the operator $i\hbar\,\partial/\partial t$ and between the momentum

p and the operator$-i\hbar\ \partial/\partial x$) can not be proved, but we postulate that it still holds and experiment bears this out.

- The Schrödinger equation can not be derived from first principles; it is a first principle, which can not be mathematically derived, just as Newton's laws of motion are not derivable.
- The justification lies in the fact that its predictions agree with the experiment.

The Schrödinger Equation for a Free Particle

- Let us consider a free particle moving along the x axis with definite momentum $p = mv$ and definite energy $E = 1/2mv^2$.
- If no force acts on the particle, the potential energy is $E_p =$ constant, which we can choose to be 0.
- The Schrödinger equation in this case is

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} = i\hbar\frac{\partial\psi}{\partial t}$$

- Let

$$\psi = Ae^{i(kx-\omega t)} = A[\cos(kx-\omega t) + i\sin(kx-\omega t)]$$

$$\frac{\partial\psi}{\partial x} = \frac{\partial}{\partial x}\left[Ae^{i(kx-\omega t)}\right] = ikAe^{i(kx-\omega t)}$$

$$\begin{aligned}\frac{\partial^2\psi}{\partial x^2} &= \frac{\partial}{\partial x}\left[ikAe^{i(kx-\omega t)}\right] = (ik)^2Ae^{i(kx-\omega t)} = -k^2Ae^{i(kx-\omega t)}\\ \frac{\partial\psi}{\partial t} &= \frac{\partial}{\partial t}\left[Ae^{i(kx-\omega t)}\right] = -i\omega Ae^{i(kx-\omega t)}\end{aligned}$$

Thus,

$$\frac{\hbar^2k^2}{2m}Ae^{i(kx-\omega t)} = \hbar\omega Ae^{i(kx-\omega t)}$$

$$\frac{\hbar^2k^2}{2m} = \hbar\omega$$

- We now show that the function given above is consistent with de Broglie's hypothesis. From de Broglie's hypothesis,

$$\begin{aligned}E &= h\nu = h\frac{\omega}{2\pi} = \hbar\omega\\ p &= \frac{h}{\lambda} = h\frac{2\pi}{k} = \hbar k\end{aligned}$$

Substituting these relations into the expression for the energy,

$E = \frac{p^2}{2m}$, we get

$$\hbar\omega = \frac{\hbar^2 k^2}{2m}$$

which is the same as the previous result.

- For a free particle of definite momentum and energy, the wavefunction ψ that satisfies the Schrödinger equation and de Broglie's postulate is $Ae^{i(kx-\omega t)}$.
- The wave-function itself has no physical meaning. It is $|\psi|^2$ that has physical significance.
- When ψ is a complex function we have

$$\begin{aligned} |\psi|^2 &= \psi^*\psi = A^*e^{-i(kx-\omega t)}Ae^{i(kx-\omega t)} \\ &= A^*A \end{aligned}$$

- In this equation the probability of finding the particle at any point in space is given as $|\psi|^2 dV$, and therefore $|\psi|^2$, when properly normalized, $|\psi|^2$, may be considered a probability density that must be both real and positive.

- Average values in quantum mechanics are called *expectation value*. If we measure the value of a particular dynamical quantity, the quantity is different for different particles even though all are described by the same $\psi(x, t)$.
- Expectation value do have physical significance as a statistical average even in cases where the exact value of the individual dynamical quantity is not well defined.
- The average particle position is

$$\overline{x} = \int_{-\infty}^{\infty} xP(x)dx$$

where $p(x)$ is the probability that the particle occurs at position

x.

- If the particle's position varies with time, then the average value of x is
$$\overline{x} = \frac{\int_{-\infty}^{\infty} xP(x,t)dx}{\int_{-\infty}^{\infty} P(x,t)dx}$$
- As mentioned before, the probability $P(x,t)dV$ that a particle is found within a small volume dV centered around a point with position vector $\boldsymbol{r}$ was $|\psi|^2 dV$.
- When the particle is restricted to the x axis, the probability $P(x,t)dx$ of finding the particle at time t between x and $x+dx$ will be $|\psi|^2 dx$.
- Thus,
$$\overline{x} = \frac{\int_{-\infty}^{\infty} x\psi^*\psi dx}{\int_{-\infty}^{\infty} \psi^*\psi dx}$$

- If the wavefunction is normalized, we have

$$\int_{-\infty}^{\infty} \psi^* \psi dx = 1$$

and

$$\overline{x} = \int_{-\infty}^{\infty} x\psi^* \psi dx$$

- The average value of the potential energy $E_p(x)$ is

$$\overline{E}_p = \int_{-\infty}^{\infty} \psi^* E_p(x) \psi dx$$

- According to the uncertainty principle, we can not express the momentum as a function of x.
- Let us differentiate the wavefunction given before with respect to

x and then multiply both sides by $-i\hbar$.

$$\begin{aligned} \frac{\partial\psi}{\partial x} &= ikAe^{i(kx-\omega t)} \\ -i\hbar\frac{\partial\psi}{\partial x} &= (-i\hbar)ikAe^{i(kx-\omega t)} = \hbar kAe^{i(kx-\omega t)} \\ -i\hbar\frac{\partial\psi}{\partial x} &= p\psi \end{aligned}$$

- This result tells us that there is an association between the dynamical quantity p and the operator $-i\hbar\frac{\partial}{\partial x}$.
- The effect of multiplying the wavefunction ψ by the momentum p is the same as operating on ψ with the operator $-i\hbar\frac{\partial}{\partial x}$.
- We obtain the expectation value of p for a particle moving along the x axis as follows

$$\overline{p} = \int_{-\infty}^{\infty} \psi^* \left(-i\hbar\frac{\partial}{\partial x}\right)\psi dx$$

$$
\begin{aligned}
&= \int_{-\infty}^{\infty} \psi^* \left(-i\hbar \frac{\partial}{\partial x}\right) Ae^{i(kx-\omega t)} dx \\
&= \int_{-\infty}^{\infty} \psi^* \left(-i\hbar ik Ae^{i(kx-\omega t)}\right) dx \\
&= \hbar k \int_{-\infty}^{\infty} \psi^* \psi dx \\
&= \hbar k
\end{aligned}
$$

- A similar method can be used to find the average value of the energy.

$$
\begin{aligned}
\psi &= Ae^{i(kx-\omega t)} \\
\frac{\partial \psi}{\partial t} &= -i\omega Ae^{i(kx-\omega t)} \\
i\hbar \frac{\partial \psi}{\partial t} &= (i\hbar)(-i\omega) Ae^{i(kx-\omega t)} = \hbar\omega Ae^{i(kx-\omega t)}
\end{aligned}
$$

$$i\hbar\frac{\partial}{\partial t}\psi = E\psi$$

- There is an association between the energy of the particle and the operator $i\hbar\frac{\partial}{\partial t}$.
- The effect of multiplying the wavefunction ψ by the energy E is the same as operating on ψ with the operator $i\hbar\frac{\partial}{\partial t}$.
- The average value of E is

$$\begin{aligned}
\overline{E} &= \int_{-\infty}^{\infty} \psi^* E\psi dx \\
&= \int_{-\infty}^{\infty} \psi^* \left(i\hbar\frac{\partial}{\partial t}\right)\psi dx \\
&= \int_{-\infty}^{\infty} \psi^* \left(i\hbar\frac{\partial}{\partial t}\right) Ae^{i(kx-\omega t)} dx
\end{aligned}$$

$$
\begin{aligned}
&= \int_{-\infty}^{\infty} \psi^*(i\hbar)(-i\omega)Ae^{i(kx-\omega t)}dx \\
&= \hbar\omega \int_{-\infty}^{\infty} \psi^*\psi dx \\
&= \hbar\omega
\end{aligned}
$$

Time-Independent Schrödinger Equation

- The Schrödinger equation in one dimension is a partial differential equation that involves t and x as the independent variables.
- If the potential energy E_p is a function of x alone, the wavefunction ψ will be the form

$$\psi(x,t) = \chi(x)\Gamma(t)$$

- We will show that the above wavefunction satisfies the Schrödinger equation.

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + E_p(x)\psi = i\hbar\frac{\partial\psi}{\partial t}$$

$$-\frac{\hbar^2}{2m}\Gamma(t)\frac{d^2\chi(x)}{dx^2} + E_p(x)\chi(x)\Gamma(t) = i\hbar\chi(x)\frac{d\Gamma(t)}{dt}$$

$$-\frac{\hbar^2}{2m}\frac{1}{\chi(x)}\frac{d^2\chi(x)}{dx^2} + E_p(x) = i\hbar\frac{1}{\Gamma(t)}\frac{d\Gamma(t)}{dt}$$

- Because x and t are independent variables, the equation will be correct if both sides are equal to the same constant G which is called the *separation constant.*

$$\begin{aligned} -\frac{\hbar^2}{2m}\frac{1}{\chi(x)}\frac{d^2\chi(x)}{dx^2} + E_p(x) &= G \\ i\hbar\frac{1}{\Gamma(t)}\frac{d\Gamma(t)}{dt} &= G \end{aligned}$$

- The solution for $\Gamma(t)$ is the same in all physical situations where E_p does not depend on time.

$$\frac{d\Gamma(t)}{\Gamma(t)} = \frac{G}{i\hbar}dt = -\frac{iG}{\hbar}dt$$

$$\begin{aligned} \ln \Gamma(t) &= -i\frac{G}{\hbar}t + \text{ constant} \\ \Gamma(t) &= Ke^{-iGt/\hbar} \end{aligned}$$

where K is the constant. Because it is an arbitrary constant, we will set it equal to unity (1).

- We may evaluate the constant G by the following method.

$$\begin{aligned} i\hbar\frac{\partial}{\partial t}\psi(x,t) &= i\hbar\frac{\partial}{\partial t}\chi(x)\Gamma(t) \\ &= i\hbar\frac{\partial}{\partial t}\chi(x)e^{-iGt/\hbar} \\ &= i\hbar\chi(x)\left(-i\frac{G}{\hbar}\right)e^{-iGt/\hbar} \\ &= G\chi(x)\Gamma(t) \\ i\hbar\frac{\partial}{\partial t}\psi(x,t) &= G\psi(x,t) \end{aligned}$$

We see that the separation constant G is the total energy E of the system. Thus,

$$\Gamma(t) = e^{-iEt/\hbar}$$

- If we wish to find what the space-dependent part of the wavefunction is, we have to solve the differential equation

$$-\frac{\hbar^2}{2m}\frac{d^2\chi(x)}{dx^2} + E_p(x)\chi(x) = E\chi(x)$$

  The above equation is called the time-independent Schrödinger equation.
- In the Schrödinger theory, energy quantization is not introduced as an arbitrary postulate; it is a consequence of the fact that the wavefunction associated with the particle must be well-behaved, a term defined later.
- The solutions χ of the above equation are called the

eigenfunctions or *eigenstates*; the corresponding values of E are called the *eigenvalues*.

Required Properties of the Eigenfunction and Its Derivative

- The eigenfunction $\chi(x)$ and its derivative $d\chi(x)/dx$ must have the following properties:
  1. They must be finite everywhere. See Fig. 20-1a.
  2. They must be single-valued everywhere. See Fig. 20-1b.
  3. They must be continuous everywhere. See Fig. 20-1c.
- When these conditions are satisfied, the eigenfunction and the associated wavefunction are said to be *well-behaved.*
- If χ or $d\chi/dx$ are not finite the average value of the momentum will be infinite.
- Recall that

$$\overline{x} = \int_{-\infty}^{\infty} \psi^* x \psi dx$$

and

$$\overline{p} = \int_{-\infty}^{\infty} \psi^* \left(-i\hbar \frac{\partial}{\partial x}\right) \psi dx$$

- If χ or $d\chi/dx$ were not single-valued, the average position and the average momentum would not be defined. The exact value of x and p may be uncertain, but the averages have to be defined.
- If χ is not continuous at a given point in space, then $d\chi/dx$ would be infinite at that point. This is not allowed. If $d\chi/dx$ is discontinuous at some point, then $d^2\chi/dx^2$ would be infinite at that point. From the time-independent Schrödinger equation, this would imply that either E or E_p is infinite.
- In the case of bound systems (one in which the particle is restricted to move over a finite region of space), these requirements on χ lead to the quantization of the energy and of other physical quantities of the system.

Particle Inside an Infinite Potential Well

- Let us consider a particle such as an electron confined inside a one-dimensional well of length a with infinitely high walls (see Fig. 20-2).

$$E_p(x) = \begin{cases} \infty & \text{for } 0 > x > a \\ 0 & \text{for } 0 \leq x \leq a \end{cases}$$

- The eigenfunction outside the well is zero since for the particle to get out, an infinite amount of energy must be provided. Thus,

$$\chi(x) = 0 \text{ for } 0 \geq x \geq a$$

- To evaluate χ inside, we must solve the following equation

$$-\frac{\hbar^2}{2m}\frac{d^2\chi(x)}{dx^2} = E\chi(x)$$

That is

$$\frac{d^2\chi(x)}{dx^2} + \frac{2mE}{\hbar^2}\chi(x) = 0$$

or

$$\frac{d^2\chi(x)}{dx^2} + k^2\chi(x) = 0$$

where

$$k^2 = \frac{2mE}{\hbar^2}$$

- Since in this case the total energy E is simply the kinetic energy,

$$E = \frac{1}{2}mv^2 = \frac{p^2}{2m}$$

From de Broglie's hypothesis,

$$\begin{aligned} p &= \frac{h}{\lambda} = \frac{h}{\frac{2\pi}{k}} = \hbar k \\ E &= \frac{p^2}{2m} = \frac{\hbar^2 k^2}{2m} \end{aligned}$$

$$k^2 = \frac{2mE}{\hbar^2}$$

which is the same relation as the above one.

- The solution for χ will be

$$\begin{aligned}\chi(x) &= ae^{ikx} + be^{-ikx} \\ &= (a+b)\cos kx + i(a-b)\sin kx\end{aligned}$$

- Because a and b are arbitrary constants, we may rewrite the equation as

$$\chi(x) = A\cos kx + B\sin kx$$

- The condition that χ be continuous requires that $\chi(0) = 0$ and $\chi(a) = 0$ because χ is zero outside the well and from this we evaluate the constant A and the values of k. Thus,

$$0 = A(1) + B(0)$$

and therefore

$$A = 0$$

$$\chi(x) = B \sin kx$$

Furthermore, because

$$\chi(a) = 0$$

then

$$0 = B \sin ka$$

- Apparently, $B \neq 0$. Thus,

$$\sin ka = 0$$

that is

$$ka = 0, \pi, 2\pi, 3\pi, \ldots$$

or

$$k = n\frac{\pi}{a} \quad n = 1, 2, 3, \ldots$$

Notice that $n = 0$ is not an acceptable choice.

- We get a set of eigenfunctions and the corresponding set of eigenvalues that we can associated with the particle in the well.

$$\chi_n(x) = B \sin n \frac{\pi}{a} x$$

and

$$E = \frac{k^2 \hbar^2}{2m} = \frac{n^2 \pi^2}{a^2} \frac{\hbar^2}{2m}$$

and the possible values of E are

$$E = n^2 E_0$$

where $E_0 = \frac{\pi^2 \hbar^2}{2ma^2}$.

- When the $\chi(x)$ is multiplied by the time part of the wavefunction, $\Gamma(t)$, the resulting wavefunction $\psi(x, t)$ represents a standing wave of the type discussed in Chapter 12. See Fig. 20-4.

Example 20-1

The arbitrary constant B is determined by the normalization condition; that is, the probability of finding the particle somewhere in space must be 1. In mathematical terms this fact is stated as follows:

$$\int_{-\infty}^{\infty} \psi^* \psi dx = 1$$

Evaluate the constant B for the ground state wavefunction of a particle in a one-dimensional well, that is, for the wavefunction

$$\psi_1(x,t) = \chi_1(x)\Gamma(t) = B \sin\left(\frac{\pi x}{a}\right) \exp\left(-\frac{iE_0}{\hbar}t\right)$$

.

Sol: For the particle in the well the limits of integration must be from 0 to a. Therefore,

$$\int_0^a B^2 \sin^2\left(\frac{\pi x}{a}\right) \exp\left(\frac{iE_0}{\hbar}t\right) \exp\left(-\frac{iE_0}{\hbar}t\right) dx = 1$$

$$\int_0^a B^2 \sin^2\left(\frac{\pi x}{a}\right) dx = 1$$

$$\frac{a}{\pi}\int_0^a B^2 \sin^2\left(\frac{\pi x}{a}\right)\left(\frac{\pi}{a}dx\right) = 1$$

The integral can be found in standard integration tables. From them, we get

$$\begin{aligned} \frac{a}{\pi}B^2\left\{\frac{\pi x}{2a} - \frac{1}{4}\sin\left(\frac{2\pi x}{a}\right)\right\}_0^a &= 1 \\ \frac{a}{\pi}B^2\left(\frac{\pi}{2}\right) &= 1 \\ B &= \left(\frac{2}{a}\right)^{1/2} \end{aligned}$$

Example 20-2

We indicated previously that all the information about the particle can be found from the wavefunction. Consider a particle in the ground state, that is, one represented by the wavefunction of Example 20-1. What is (a) the average position, (b) the average momentum, and (c) the average energy of such a particle?

Sol:

(a) The average position is given by

$$\overline{x} = \int_0^a \psi_1^* x \psi_1 dx$$

which in the present case becomes

$$\overline{x} = \frac{2}{a}\int_0^a \sin^2\left(\frac{\pi}{a}x\right) x dx$$

$$= \frac{2a^2}{a\pi^2} \int_0^a \left(\sin^2\left(\frac{\pi x}{a}\right)\right)\left(\frac{\pi x}{a}\right)\left(\frac{\pi}{a} dx\right)$$

From the integration tables, we get

$$\begin{aligned}
\overline{x} &= \frac{2a}{\pi^2}\left[\frac{1}{4}\left(\frac{\pi x}{a}\right)^2 - \frac{\pi x \sin\left(\frac{2\pi x}{a}\right)}{4a} - \frac{\cos\left(\frac{2\pi x}{a}\right)}{8}\right]_0^a \\
&= \frac{2a}{\pi^2}\left\{\left(\frac{\pi^2}{4} - 0 - \frac{1}{8}\right) - \left(0 - 0 - \frac{1}{8}\right)\right\} \\
\overline{x} &= \frac{a}{2}
\end{aligned}$$

(b) The average value of the momentum can be found as follows:

$$\begin{aligned}
\overline{p} &= \int_0^a \psi_1^* \left(-i\hbar \frac{\partial}{\partial x}\right) \psi_1 dx \\
&= \frac{2}{a}\int_0^a \left(\sin\frac{\pi x}{a}\right)\left(-i\hbar\frac{\partial}{\partial x}\right)\left(\sin\frac{\pi x}{a}\right) dx
\end{aligned}$$

$$
\begin{aligned}
&= \frac{2}{a}\int_0^a \left(\sin\frac{\pi x}{a}\right)\left(-i\hbar\frac{\pi}{a}\cos\frac{\pi x}{a}\right)dx \\
&= \frac{2}{a}(-i\hbar)\int_0^a \left(\sin\frac{\pi x}{a}\right)\left(\cos\frac{\pi x}{a}\right)\left(\frac{\pi}{a}dx\right) \\
&= \frac{2}{a}(-i\hbar)\left.\frac{\sin^2\frac{\pi x}{a}}{2}\right|_0^a \\
&= 0
\end{aligned}
$$

(c) The average value of the energy E can be calculated by

$$
\begin{aligned}
\overline{E} &= \int_0^a \psi_1^*\left(i\hbar\frac{\partial}{\partial t}\right)\psi_1 dx \\
&= \int_0^a \psi_1^*\left(i\hbar\frac{\partial}{\partial t}\right)\chi_1(x)\exp\left(-i\frac{E_0}{\hbar}t\right)dx \\
&= \int_0^a \psi_1^*(i\hbar)\left(-i\frac{E_0}{\hbar}\right)\chi_1(x)\exp\left(-i\frac{E_0}{\hbar}t\right)dx
\end{aligned}
$$

$$
\begin{aligned}
&= \int_0^a E_0 \psi_1^* \psi_1 dx \\
&= E_0
\end{aligned}
$$

<u>Homework:</u> 20.1, 20.2, 20.3, 20.5, 20.6, 20.7, 20.9, 20.12, 20.14, 20.15, 20.17, 20.18, 20.19, 20.21.

Chapter Twenty One: Quantum Mechanics of Atoms

- The great success of the Schrödinger theory, which is a postulate, but a fundamental one, is that one is able to derive from mathematical principles the postulates that had originally been presented to explain the classically unexplainable experimental results.
- For a classical harmonic oscillator with potential energy $E_p = \frac{1}{2}kx^2$, the only physically acceptable solutions are those for which the energy E has values given by

$$E_n = \left(n + \frac{1}{2}\right) h\nu \text{ where } n = 1, 2, 3, \ldots$$

- Note that the solution of the spectrum given here has a factor of $\frac{1}{2}$ in it. This means that in the lowest energy state $n = 0$ a system still has vibrational energy.

- When one solves the Schrödinger equation for an electron in the potential of a proton, one again finds that the only physically acceptable solutions for the wave functions are those where the energy E takes certain discrete values.
- The Schrödinger theory provides much more information than only the quantization of the energy spectrum. It explains why certain spectral lines are brighter than others; that is, it can be used to calculate transition rates.
- The mathematical complexities do not permit us to solve in detail the Schrödinger equation for the coulomb potential of the hydrogen atom. We will outline the solution of the problem and present the results that are obtained by solving the Schrödinger equation.

Outline of the Solution of the Schrödinger Equation for the H Atom

- The *atomic number*, Z, represents the number of protons in the nucleus and also the number of electron in neutral atoms.
- The potential energy of the electron in the electric field of a nucleus is

$$E_p = -\frac{1}{4\pi\epsilon_0}\frac{Ze^2}{r}$$

  where $r = (x^2 + y^2 + z^2)^{1/2}$ is the distance between the electron and the protons.
- We extend the time-independent Schrödinger equation to a situation where a particle is free to move in three dimensions.

$$-\frac{\hbar^2}{2m}\left[\frac{\partial^2\chi}{\partial x^2} + \frac{\partial^2\chi}{\partial y^2} + \frac{\partial^2\chi}{\partial z^2}\right] + E_p(x, y, z)\chi = E\chi$$

- We may written the above equation in spherical coordinates (see Fig. 21-1)

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \chi}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\chi}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2\chi}{\partial\phi^2}$$
$$+\frac{2m}{\hbar^2}[E-E_p(r)]\chi=0$$

- We try a solution of the form

$$\chi(r,\theta,\phi)=R(r)\Theta(\theta)\Phi(\phi)$$

where R, Θ, and Φ are functions of only one coordinate.

- We have

$$-\frac{\sin^2\theta}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)-\frac{2m}{\hbar^2}r^2\sin^2\theta[E-E_p(r)]$$

$$-\frac{\sin\theta}{\Theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)=\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2}$$

- The left side of the above equation is a function of r and θ alone, whereas the right side is a function of ϕ alone. The only way the equation can be valid is when both sides of the equation are equal to the same constant.
- Thus,

$$\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2}=-m_l^2 \tag{6}$$

The constant is chosen as $-m_l^2$ in order that m_l will be an integer.

-

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)+\frac{2m}{\hbar^2}r^2\left[E-E_p(r)\right]$$

$$= \quad \frac{m_l^2}{\sin^2\theta} - \frac{1}{\Theta}\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)$$

- Again, one side of this equation is a function of the variable r only, whereas the other side is a function of the variable θ only. The only way the equality can be valid is when both sides are equal to the same constant. If we call this constant $l(l+1)$, l will be an integer.
- we have

$$\frac{m_l^2}{\sin^2\theta} - \frac{1}{\Theta}\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) = l(l+1) \tag{7}$$

and

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{2m}{\hbar^2}r^2[E - E_p(r)] = l(l+1) \tag{8}$$

- We will simply mention what happens when one solves the three differential equations (6, 7, and 8).

1. When we solve Eq. 6, we find that the only solutions for Φ that are single-valued are those for which

$$m_l = 0, \pm 1, \pm 2, \ldots$$

2. When we solve Eq. 7 for Θ, we find that the only solutions that are finite everywhere are those for which

$$l = 0, 1, 2, \ldots$$

and

$$l \geq |m_l|$$

3. When we solve Eq. 8, we find that the only solutions for R that remain finite everywhere are those for which

$$E_n = -\frac{Z^2 e^4 m}{8\epsilon_0^2 h^2} \frac{1}{n^2} \; n = 1, 2, 3, \ldots$$

and

$$l < n$$

- The restrictions on the values that m_l and l can take now be restated as follows:

$$m_l = 0, \pm 1, \pm 2, \ldots, \pm l$$

and

$$l = 0, 1, 2, \ldots, \ldots, n - 1$$

Physical Significance of the Results

- The most important result of the solution of the Schrödinger equation for the H atom is the fact that the energy of the atom is quantized.
- Because for a given n the other two numbers can take several values, this means that it is possible for the electron to have quite different characteristics while maintaining the same energy.
- States χ having the same energy but different values for the quantum numbers l and m_l are called *degenerate*.
- This degeneracy is often represented in a two-dimensional diagram (see Fig. 21-2).
- States for which $l = 0$ are called s states, those with $l = 1$ are called p states, those with $l = 2$ are called d states, those with $l = 3$ are called f states.

- The quantum number l is called the *orbital quantum number*, because l determines the magnitude of the angular momentum L of the atom.
- The quantum number m_l is called the *magnetic quantum number*, because m_l determines the orientation of the angular momentum L in a magnetic field.
- It can be shown that if an atom is placed in a magnetic field directed along the z direction, the z component of the angular momentum L of the atom is given by

$$L_z = m_l \hbar$$

- In an atom, L can not have any arbitrary orientation with respect to the z axis, but rather it can have only certain discrete orientations. This is known as *space quantization*. See Fig. 21-3.
- If you perform an experiment in which a particular direction

becomes preferred, such as by the application of a magnetic field, then direction becomes meaningful.

- If the field tends to align the angular momentum of the atom along a given direction, then that direction becomes the z axis.

The Zeeman Effect

- If a current i flows through a loop of area A, a magnetic dipole moment $\mu = iA$ is associated with it.
- If we place this dipole in an external magnetic field B (see Fig. 21-4), it will experience a torque $\tau = \mu \times \boldsymbol{B} = \mu B \sin\theta$, where θ is the angle between the two vectors.
- We can associate with such a dipole μ in a magnetic field $\boldsymbol{B}$ a potential energy that will depend on the orientation of μ with respect to $\boldsymbol{B}$. The potential energy E_p is,

$$E_p = -\mu \cdot \boldsymbol{B} = -\mu B \cos\theta$$

- Let us consider an electron in one of the Bohr orbits (see Fig.

21-5); the circulating electron represents a current i given by

$$i = \frac{|e|}{T} = \frac{|e|}{\frac{2\pi r}{v}} = \frac{|e|v}{2\pi r}$$

where T is the period of rotation, v is the tangential speed of the electron, and r is the radius of the orbit.

- Then

$$\mu = Ai = \pi r^2 i = \pi r^2 \frac{|e|v}{2\pi r} = \frac{|e|rv}{2}$$

- In quantum mechanics we can not talk about a particle moving in a circular loop of radius with a velocity v. Thus,

$$\begin{aligned} \mu &= \frac{|e|rv}{2} = \frac{|e|mvr}{2m} = \frac{|e|}{2m}L \\ \mu_l &= -\frac{|e|}{2m}\boldsymbol{L} \end{aligned}$$

where L is the angular momentum of a particle of mass m

rotating with a tangential velocity v about a point a distance r away.

- We have included a minus sign because the direction of i is opposite to the direction of motion of the electron, hence of L (see Fig. 21-6).
- $E_p = -\mu_l \cdot \boldsymbol{B} = \frac{|e|}{2m}\boldsymbol{L} \cdot \boldsymbol{B} = \frac{|e|}{2m}BL\cos\theta$.
- Since $L\cos\theta = L_z$, thus

$$E_p = \frac{|e|}{2m}BL_z$$

- The total energy of the atom, E_{Total}, will be the sum of the energy resulting from the interaction of the electron with the nucleus, that is , E_n, and the energy resulting from the interaction of the dipole moment with the magnetic field B, that is , E_p.

$$E_{\text{Total}} = E_n + E_p$$

- Quantum mechanically, L_z can take only certain values, that is, $L_z = m_l \hbar$. Thus,

$$E_{\text{Total}} = E_n + \frac{|e|}{2m} B \hbar m_l$$

(see Fig. 21-8)

- If we now reexamine the spectrum of hydrogen, the $n = 2$ to $n = 1$ transition should give rise to three different lines (three different λ's).
- There is a selection rule that initially was found experimentally and was later derived from the Schrödinger theory which states that the only allowed transitions are those for which $\Delta m_l = 0$ or ± 1.
- Thus, regardless of the number of sublevels, transitions from a given n to another should give rise only three lines. This effect is

known as the *normal Zeeman effect*.

- It had been observed experimentally in the spectrum of such as helium, calcium, zinc, and a few others long before quantum mechanics was formulated. Often more than three lines are observed. This effect is known as the *anomalous Zeeman effect*.
- The experiment shows that the idea of space quantization is correct. However, it also shows that the Schrödinger theory as originally formulated was incomplete.

Example 21-1

Calculate the normal Zeeman splitting of the calcium 4226 $\mathring{A}$ line when the atoms are placed in a magnetic field of 1.2 T (tesla).

Sol: The energy between adjacent Zeeman levels is

$$\begin{aligned}\Delta E &= \frac{|e|}{2m}B\hbar \\ &= \frac{1.6\times 10^{-19}\ \mathrm{C}\times 1.2\ \mathrm{T}\times 1.05\times 10^{-34}\ \mathrm{J\text{-}sec}}{2\times 9.1\times 10^{-31}\ \mathrm{kg}} \\ &= 1.11\times 10^{-23}\ \mathrm{J} = 6.92\times 10^{-5}\ \mathrm{eV}\end{aligned}$$

By Einstein's relation: $E_{\mathrm{photon}} = h\nu = hc/\lambda$. Differentiating this expression with respect to the wavelength, we have

$$dE = -\frac{hc}{\lambda^2}d\lambda$$

or

$$|dE| = \frac{hc}{\lambda^2}|d\lambda|$$

Because the shift in energy, ΔE, is small compared with the energy of the level, we can make the approximation $dE \approx \Delta E$. Thus

$$\begin{aligned} |\Delta\lambda| &= \frac{\lambda^2}{hc}|\Delta E| \\ &= \frac{(4.226 \times 10^{-7}\ \text{m})^2 \times 1.11 \times 10^{-23}\ \text{J}}{6.63 \times 10^{-34}\ \text{J-sec} \times 3 \times 10^{8}\ \text{m/sec}} \\ &= 9.96 \times 10^{-12}\ \text{m} = 0.0996\ \mathring{A} \end{aligned}$$

Stern-Gerlach Experiment

- If the magnetic field $\boldsymbol{B}$ is not uniform, the dipole will also experiences a net force.
- It can be shown that if there is a magnetic gradient along the z axis the force will be directed along the z axis and the force will depend on the magnitude of the gradient and the orientation of μ_l with respect to the gradient.
- The actual value of the force is

$$\begin{aligned} F_z &= \mu_l \cdot \frac{d\boldsymbol{B}}{dz} \\ &= \mu_l \frac{dB}{dz} \cos\theta \end{aligned}$$

where θ is the angle between μ_l and the gradient $d\boldsymbol{B}/dz$.

- Since $\mu_l \cos\theta = \mu_{lz}$, we have

$$\begin{aligned} F_z &= \mu_{lz}\frac{dB}{dz} \\ &= -\frac{|e|}{2m}L_z\frac{dB}{dz} \end{aligned}$$

- In the experiment, Stern and Gerlach sent a beam of neutral silver atoms from a heated oven through a collimator and then through a region with a nonuniform magnetic field $B(z)$ with a gradient along the z axis (see Fig. 21-11).
- The stronger the force, the greater the deflection, and therefore the farther away from the center they would land on the screen.
- If the Schrödinger theory is correct, and L_z is quantized, then the force in the z direction, F_z, can have only a discrete number of values, one for each m_l.

- The experimental results for silver(Ag) atoms were as shown in Fig. 21-13.
- The experiment confirmed the concept of space quantization.
- Although the experimental results showed splitting, which demonstrates space quantization, they were in quantitative disagreement with the Schrödinger theory.
- Because the Phipps-Taylor experiment on hydrogen showed two well-defined z spacing and no value at $z = 0$, it is evident that there is a magnetic dipole other than the orbital dipole that has been overlooked.
- In the absence of an external magnetic field, the transition in hydrogen from $n = 2$ to $n = 1$ should give rise to just one line, one λ.
- Actually, the experimental observation of this spectral line shows

two very closely spaced wavelengths that correspond to two very close frequencies, that is, energies. This phenomenon is known as the *fine structure* of a spectral line.

The Spin

- While studying the fine structure of the spectrum Uhlenbeck and Goudsmit proposed the idea of the electron spin: the electron has an intrinsic angular momentum called the spin S.
- The electron has a spin dipole moment $\mu_s = -e/mS$.
- The magnitude of S and its z component are quantized as follows

$$S = [s(s+1)]^{1/2}\hbar \text{ where } s = \frac{1}{2}$$

$$S_z = m_s\hbar, \text{ where } m_s = \pm\frac{1}{2}$$

- In the relativistic quantum mechanical theory developed later by Dirac, the existence of the spin and the rules governing its behavior are a natural consequence of its formulation.
- Using the spin postulate, the experimental results concerning the

anomalous Zeeman effect, the Stern-Gerlach experiment and the Phipps-Taylor experiment, as well as the fine structure, can be explained.

- The state of an electron is now specified by four quantum numbers: n, l, m_l, m_s.

Example 21-2

A beam of hydrogen atoms is used in a Stern-Gerlach type experiment. The atoms emerge from the oven with a velocity $v = 10^4$ m/sec. They enter a region 20 cm long where there is a magnetic field gradient $dB/dz = 3 \times 10^4$ T/m. The field gradient is perpendicular to the incident velocity of the atoms. The mass of the hydrogen atom is 1.67×10^{-27} kg. What is the separation of the two components of the beam as they emerge from the magnet?

Sol: In the ground state, hydrogen atoms have no net orbital magnetic dipole moment. The only dipole moment is the one associated with the spin of the electron in the 1s state, that is, $\mu_s = -|e|/mS$, where m is the mass of the electron.

$$F_z = \mu_s \cdot \frac{dB}{dz} = -\frac{|e|}{m} S_z \frac{dB}{dz} = \pm \frac{1}{2} \frac{|e|}{m} \hbar \frac{dB}{dz}$$

We can use Newton's second law to find the acceleration a_z of the hydrogen atoms as they traverse the magnet.

$$\begin{aligned} a_z &= \frac{F_z}{m_{atom}} = \frac{|e|\hbar \frac{dB}{dz}}{2mm_{atom}} \\ &= \frac{1.60 \times 10^{-19} \text{ C} \times 1.05 \times 10^{-34} \text{ J-sec} \times 3 \times 10^4 \text{ T/m}}{2 \times 9.1 \times 10^{-31} \text{ kg} \times 1.67 \times 10^{-27} \text{ kg}} \\ &= 1.65 \times 10^8 \text{ m/sec}^2 \end{aligned}$$

The deflection of each component in the direction of the force will be

$$\Delta z = \frac{1}{2} a_z t^2$$

where t is the time that the atoms spend in the magnet.

$$t = \frac{0.20 \text{ m}}{10^4 \text{ m/sec}} = 2 \times 10^{-5} \text{ sec}$$

Therefore

$$\begin{aligned}\Delta z &= \frac{1}{2} \times 1.65 \times 10^{8}\ \text{m/sec}^{2} \times 4 \times 10^{-10}\ \text{sec}^{2} \\ &= 3.3 \times 10^{-2}\ \text{m} = 3.3\ \text{cm}\end{aligned}$$

Because of the two possible values for m_s, some will be deflected upward and some downward. Therefore, the separation between the two components of the beam will be $2\Delta z$ or 6.6 cm.

The Pauli's Exclusion Principle

- The lowest energy state of an electron is called the ground state.
- Pauli's exclusion principle: No two electrons in a system (be it an atom or a solid) can be in the same quantum state.
- No two electrons can have identical values for the set of quantum number specifying the state, which in the case of an atom are n, l, m_l, m_s. At least one of the quantum numbers must be different.

Homework: 21.1, 21.2, 21.3, 21.5, 21.6, 21.7, 21.8, 21.11, 21.12, 21.13.