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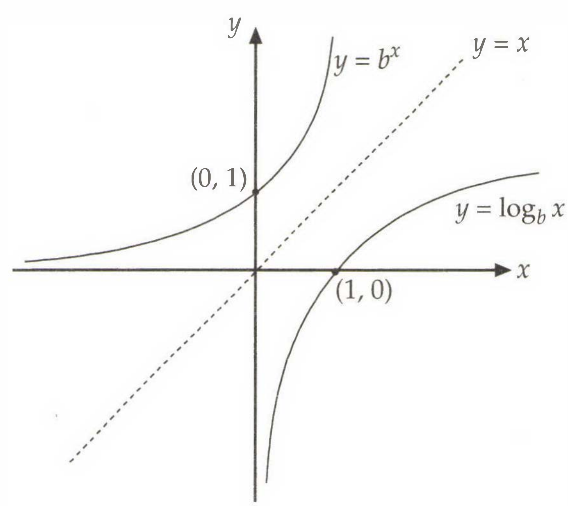
LOGARITHMS

Logarithms are exponents. Given the equation $b^y = x$, the exponent is y, which means that the logarithm is y. More precisely, we'd say that y is the **logarithm** base b of x, and write $y = \log_b x$. The laws of logarithms follow directly from the corresponding laws of exponents. In the equations below, b is a positive number that's not equal to 1.

- $\log_b x = y$ means $b^y = x$
- The function $y = \log_b x$ is the inverse of the exponential function $y = b^x$. The domain of the function $f(x) = \log_b x$ is $x > 0$, and the range is the set of all real numbers. If $b > 1$, the function is increasing (see the diagram below); if $0 < b < 1$, the function is decreasing.



- $\log_b(x_1x_2) = \log_b x_1 + \log_b x_2$
- $\log_b \dfrac{x_1}{x_2} = \log_b x_1 - \log_b x_2$
- $\log_b (x^a) = a \log_b x$
- $b^{\log_b x} = x$
- $(\log_b a)(\log_a x) = \log_b x$ (this is the **change-of-base formula**, with $a \neq 1$)

The two most important bases for logarithms are $b = 10$ [because we use a base-10 (decimal) number system] and $b = e$, where e is an irrational constant, approximately equal to 2.718. The selection of this seemingly unusual number is based on considerations in calculus (which we'll review in the next chapter) and is so important that the function $f(x) = \log_e x$ is called the **natural logarithm function**.

> On the GRE Math Subject Test, the "e" is understood, so $\log x$ means $\log_e x$. It's important to be aware of this, since in many precalculus and calculus texts—and on calculators—$\log x$ denotes $\log_{10} x$ and the abbreviation for $\log_e x$ is $\ln x$.

Example 1.15 Solve for x: $4^x = 2^x + 3$

Solution: Since $4^x = (2^2)^x = 2^{2x} = (2^x)^2$, the equation is equivalent to $(2^x)^2 - 2^x - 3 = 0$, which is quadratic in 2^x. The quadratic formula gives:

$$2^x = \frac{1 \pm \sqrt{13}}{2}$$

Since 2^x cannot be negative, we have to disregard the negative value on the right-hand side and conclude that:

$$2^x = \frac{1 + \sqrt{13}}{2}$$

Therefore, $x = \log_2 \dfrac{1+\sqrt{13}}{2} = \log_2(1+\sqrt{13}) - 1$.

Example 1.16 If $x^2 + y^2 = 14xy$, then $\log[k(x + y)] = \frac{1}{2}(\log x + \log y)$ for some constant k. Find the value of k.

Solution: Adding $2xy$ to both sides of the first equation gives $(x + y)^2 = 16xy$, which is equivalent to $[\frac{1}{4}(x + y)]^2 = xy$. Taking the log of both sides of this equation gives:

$$2 \log[\tfrac{1}{4}(x + y)] = \log x + \log y \;\Rightarrow\; \log\left[\tfrac{1}{4}(x+y)\right] = \tfrac{1}{2}(\log x + \log y)$$

Therefore, $k = \dfrac{1}{4}$.

Example 1.17 Simplify $[\log_{xy}(x^y)][1 + \log_x y]$.

Solution: Since $1 = \log_x x$, the second factor, $1 + \log_x y$, is equal to:

$$\log_x x + \log_x y = \log_x xy$$

Applying the change-of-base formula, $(\log_a b)(\log_b c) = \log_a c$, we find that:

$$[\log_x xy]\,[\log_{xy}(x^y)] = \log_x (x^y) = y$$

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