[redacted]

[redacted]

- [redacted]
- [redacted]

[redacted]

[redacted]

[redacted]

[redacted]

[redacted]

COMBINATORICS

Combinatorics is the branch of mathematics concerned with counting. (Although that may sound fairly easy, the methods of more advanced combinatorics are *far* from trivial.) Here's an example. Let's say we're trying to get from City A to City C via City B. If there are two roads between City A and City B, and three roads from City B to City C, how many different journeys are there from A to B to C? Since there are two choices for the first route, then three possible choices for the second, the total number of journeys is their product: $2 \times 3 = 6$. This illustrates a basic principle:

- If there are a possible choices for the first decision and b possible choices for the second decision, then the total number of possible ways these two independent decisions can be made is the product ab.

- If there's a third independent decision to be made (with, say, c choices), then the total number of ways to make all three decisions is the product abc, and so on.

Example 7.7 Let's say that the personal code for your ATM card consists of six characters, each of which can be any letter or numerical digit.

(a) How many possible personal codes are there?

(b) Suppose you want your personal ATM code to have no repeated characters; now how many possible codes are there?

Solution:

(a) There are twenty-six letters (A through Z) and ten digits (0 through 9), so there are $26 + 10 = 36$ possible choices for each character. Since there are six decisions, any of which can use any of the 36 possible characters, the total number of ways the six decisions can be made is 36^6. (By the way, that's over two *billion* possible codes.)

(b) There are 36 ways to make the first decision, but only 35 for the second (because, to avoid a repeat, you can't use the same choice you made for the first character). Then there are only 34 possible choices for the third character, 33 for the fourth, 32 for the fifth, and 31 for the sixth. Thus, the total number of codes that don't have any repeated characters is $36 \times 35 \times 34 \times 33 \times 32 \times 31$. (This cuts down the total number of codes by roughly one-third.)

PERMUTATIONS AND COMBINATIONS

Let's say we're given a set of n objects; in how many different ways can we choose k of them? The answer depends on whether the order in which we select the objects makes a difference. For example, in a horse race that involves eight horses, how many win, place, and show possibilities are there? Well, we want to see how many ways we can choose three horses out of eight, but here order makes a difference; if horses A, B, and C are the ones that come in first (win), second (place), and third (show), it matters whether the finishing order is A, B, C or, say, B, C, A. On the other hand, if you're playing five-card draw poker, you receive five cards out of a standard deck of 52; the order in which they're dealt to you makes no difference. The number of possible poker hands is found by counting the number of ways we can choose five cards out of 52, without regard to order.

If the order in which the objects are chosen does make a difference, each complete selection of k objects from the original n is called a **permutation** (the variable r is often used in place of k). If the order is irrelevant, then each complete selection is called a **combination**. In general, there are always more permutations than there are combinations. To see why, let's go back to the example of the horse race. Each of the selections

$$A, B, C \quad A, C, B \quad B, A, C \quad B, C, A \quad C, A, B \quad \text{and} \quad C, B, A$$

is a different permutation, but they're all equivalent to a single combination.

Let's figure out the number of ways we can choose k things from n, with regard to order. For our first selection, we can choose any of the given n objects; for the second selection, we can choose any of the remaining $n - 1$ objects; this process continues until, for the k^{th} selection, we can choose any of the remaining $n - (k - 1)$ objects. Since these decisions are independent, the total number of **permutations of n things taken k at a time** is the product:

$$P(n, k) = n(n - 1)(n - 2) \ldots [n - (k - 1)]$$

Now, let's figure out the number of ways we can choose k things from n, without regard to order. Let's say we've chosen k objects from a group of n, with regard to order. That is, assume we have a permutation of n things taken k at a time. In how many ways can we permute these k objects? The answer is $k!$. For example, consider the example of the three horses—A, B, and C—given above. Let's say we selected them in the order A then B then C. Now that we have these three, we can arrange them in 3! = 6 different orders; these orders are displayed above. Each of these six orders is a different permutation, but they collectively represent just one combination. In general, the number of **combinations of n things taken k at a time** is equal to the number of permutations of n things taken k at a time divided by $k!$:

$$C(n,k) = \frac{P(n,k)}{k!} = \frac{n(n-1)(n-2)\cdots[n-(k-1)]}{k!}$$

This expression for $C(n, k)$ can be written in another way:

$$C(n,k) = \frac{n!}{k!(n-k)!}$$

In this form, you should recognize it as the **binomial coefficient**:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

It's called the binomial coefficient because it's precisely the coefficient that appears in the **binomial theorem**:

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

Example 7.8 In a horse race consisting of 8 horses, how many different win-place-show (the top three) finishing orders are there?

Solution: The order in which the horses finish matters here, so we're looking for the number of permutations of 8 things taken 3 at a time; this number is:

$$P(8, 3) = 8 \cdot 7 \cdot 6 = 336$$

Example 7.9 How many 3-element subsets does a set containing 9 elements have?

Solution: The order in which the elements are arranged in a subset is irrelevant. The number of combinations of 9 things taken 3 at a time is:

$$C(9, 3) = \binom{9}{3} = \frac{9!}{3! \cdot 6!} = \frac{9 \cdot 8 \cdot 7}{3!} = \frac{9 \cdot 8 \cdot 7}{2 \cdot 3} = 3 \cdot 4 \cdot 7 = 84$$