Project Sigma

Algebraic Geometry

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May 9, 2021

Contents

Chapter 1

Affine Algebraic Sets

Let k be a field and $n \in \mathbf{N}$, then the *affine space* $\mathbf{A}^n(k)$ of dimension n, or simply $\mathbf{A}^n$ if it does not cause confusion, is the same structure as the n-dimensional vector space k^n over k, except with affine maps as morphisms, where an affine map is a linear map shifted by a constant. In this course note, unless otherwise specified, I will take $k = \mathbf{C}$ the set of complex numbers.

Definition 1.0.1. Suppose that $S \subseteq k[X_1, \dots, X_n]$ for some $n \in \mathbf{N}$, we define

$$\mathcal{V}(S) = \{x \in \mathbf{A}^n(k) : \forall f \in S,\, f(x) = 0\}$$

as the *set of zeros* of S. Subsets of $\mathbf{A}^n(k)$ that are sets of zeros of some S are called (affine) *algebraic*.

For example, in $\mathbf{A}^2(\mathbf{R})$, the sets $\mathcal{V}(Y)$ and $\mathcal{V}(X)$ are the x-axis and the y-axis, the set $\mathcal{V}(X, Y)$ is the origin. The set $\mathcal{V}(f)$, where $f \in k[X, Y]$ is polynomial of degree 2, is known as a *conic section*, for example, the circle $\mathcal{V}(X^2 + Y^2 - 1)$, the parabola $\mathcal{V}(Y - X^2)$, and the hyperbola $\mathcal{V}(XY - 1)$. These are all examples of algebraic sets. We should also mention examples of non-algebraic sets. The set $\mathbf{Z}$ considered as a subset of $\mathbf{A}^1(\mathbf{R})$ is obviously not algebraic. Next, we claim that to find all algebraic sets, we need not consider all subsets of $k[X_1, \dots, X_n]$. Let R be a commutative ring.

Theorem 1.0.2. R is noetherian iff all ideals $I \subseteq R$ are finitely generated.

Theorem 1.0.3 (Hilbert's Basis theorem)**.** If R is noetherian, then so is $R[X_1, \dots, X_n]$.

Corollary 1.0.4. The ring $k[X_1, \dots, X_n]$ is noetherian.

Thus for $S \subseteq k[X_1, \dots, X_n]$, the ideal $\langle S \rangle = (f_1, \dots, f_m)$ for some $f_1, \dots, f_m \in k[X_1, \dots, X_n]$. We claim that $\mathcal{V}(S) = \mathcal{V}(\langle S \rangle) = \mathcal{V}(f_1, \dots, f_m)$. Obviously

Definition 1.0.5. Let $X \subseteq \mathbf{A}^n(k)$, then define the ideal $\mathcal{I}(X)$ of $k[X_1, \dots, X_n]$ as

$$\mathcal{I}(X) = \{f \in k[X_1, \dots, X_n] : \forall p \in X,\, f(p) = 0\}$$

which is a well-defined ideal as we can verify easily.

Lemma 1.0.6. Suppose that I is an ideal of $k[X_1, \dots, X_n]$, then

$$\mathcal{I}(\mathcal{V}(I)) = \sqrt{I}$$

where the radical $\sqrt{I} = \{r \in R : \exists n > 0,\, r^n \in I\}$.

Theorem 1.0.7 (Hilbert's Nullstellensatz)**.** There is a bijective correspondance

$$\{\text{radical ideals of } k[X_1, \ldots, X_n]\} \longleftrightarrow \{\text{algebraic sets of } \mathbf{A}^n(k)\}$$

given by $I \mapsto \mathcal{V}(I)$ and $X \mapsto \mathcal{I}(X)$.

Proof. Note that each $\mathcal{V}(I)$ is an algebraic set, and each $\mathcal{I}(X)$ is a radical ideal, so these maps are well defined. Next, for an ideal I, $\square$