Project Sigma

Algebraic Geometry

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Chapter 1

Affine Algebraic Sets

Let k be a field and $n \in \mathbf{N}$, then the *affine space* $\mathbf{A}^n(k)$ of dimension n, or simply $\mathbf{A}^n$ if it does not cause confusion, is the same structure as the n-dimensional vector space k^n over k, except with affine maps as morphisms, where an affine map is a linear map shifted by a constant. In this course note, unless otherwise specified, I will take $k = \mathbf{C}$ the set of complex numbers.

Definition 1.0.1. Suppose that $S \subseteq k[X_1, \ldots, X_n]$ for some $n \in \mathbf{N}$, we define

$$\mathcal{V}(S) = \{x \in \mathbf{A}^n(k) : \forall f \in S, \, f(x) = 0\}$$

as the *zero-locus* of S. Subsets of $\mathbf{A}^n(k)$ that are the *zero-loci* of some S are called (affine) *algebraic*.

For example, in $\mathbf{A}^2(\mathbf{R})$, the sets $\mathcal{V}(\{Y\})$ and $\mathcal{V}(\{X\})$ are the x-axis and the y-axis, the set $\mathcal{V}(\{X, Y\})$ is the origin. The set $\mathcal{V}(\{f\})$, where $f \in k[X, Y]$ is polynomial of degree 2, is known as a *conic section*, for example, the circle $\mathcal{V}(\{X^2 + Y^2 - 1\})$, the parabola $\mathcal{V}(\{Y - X^2\})$, and the hyperbola $\mathcal{V}(\{XY - 1\})$. These are all examples of algebraic sets. We should also mention examples of non-algebraic sets. The set $\mathbf{Z}$ considered as a subset of $\mathbf{A}^1(\mathbf{R})$ is obviously not algebraic. Next, we claim that to find all algebraic sets, we need not consider all subsets of $k[X_1, \ldots, X_n]$. Let R be a commutative ring, we recall the following theorems.

Theorem 1.0.2. R is noetherian iff all ideals $I \subseteq R$ are finitely generated.

Theorem 1.0.3 (Hilbert's Basis theorem). If R is noetherian, then so is $R[X_1, \ldots, X_n]$.

Therefore $k[X_1, \ldots, X_n]$ is noetherian, and hence every for each $S \subseteq k[X_1, \ldots, X_n]$, the ideal $\langle S \rangle$ is generated by some $f_1, \ldots, f_m \in k[X_1, \ldots, X_n]$. We claim that $\mathcal{V}(S) = \mathcal{V}(\langle S \rangle) = \mathcal{V}(\{f_1, \ldots, f_m\})$. First, we know that $\mathcal{V}(\langle S \rangle) \subseteq \mathcal{V}(S)$ since $S \subseteq \langle S \rangle$. Conversely, suppose that $f \in \langle S \rangle$, then there exists $g_1, \ldots, g_\ell \in S$ and $\lambda_1, \ldots, \lambda_\ell \in k[X_1, \ldots, X_n]$ with $f = \lambda_1 g_1 + \cdots + \lambda_\ell g_\ell$. Suppose that $p \in \mathcal{V}(S)$, then $f(p) = \lambda_1 g_1(p) + \cdots + \lambda_\ell g_\ell(p) = 0$. Since all $p \in \mathcal{V}(S)$ is a zero of all $f \in \langle S \rangle$, we must have $\mathcal{V}(S) \subseteq \mathcal{V}(\langle S \rangle)$, and hence $\mathcal{V}(S) = \mathcal{V}(\langle S \rangle)$. The equality $\mathcal{V}(\langle S \rangle) = \mathcal{V}(\{f_1, \ldots, f_m\})$ is derived similarly. Hence all algebraic sets are the zero loci of ideals, and also all algebraic sets are the zero loci of finite sets. Conversely, we can define an ideal $\mathcal{I}(X)$ for each $X \subseteq \mathbf{A}^n(k)$.

Definition 1.0.4. Let $X \subseteq \mathbf{A}^n(k)$, then define the ideal $\mathcal{I}(X)$ of $k[X_1, \ldots, X_n]$ as

$$\mathcal{I}(X) = \{f \in k[X_1, \ldots, X_n] : \forall p \in X, \, f(p) = 0\}$$

which is a well-defined ideal as we can verify easily.

In fact, not only is $\mathcal{I}(X)$ an ideal, it is also a radical ideal. We recall that an radical ideal of a commutative ring R is an ideal $I \subseteq R$ with $I = \sqrt{I}$ where $\sqrt{I} = \{r \in R : \exists m > 0, r^m \in I\}$. In other words, a radical ideal is an ideal $I \subseteq R$ where for all $r \in R$, if $r^m \in I$ for some $m > 0$, then $r \in I$. To see that $\mathcal{I}(X)$ is a radical ideal, note that if $f^m \in \mathcal{I}(X)$ for some $m > 0$, then $f^m(p) = 0$ for all $p \in X$, then $f(p) = 0$ for all $p \in X$ as k is a field, hence $f \in \mathcal{I}(X)$.

Similar to how $\mathcal{V}(S) \subseteq \mathcal{V}(T)$ when $T \subseteq S \subseteq k[X_1, \ldots, X_n]$, we easily have $\mathcal{I}(X) \subseteq \mathcal{I}(Y)$ when $Y \subseteq X \subseteq \mathbf{A}^n(k)$. Let $f \in S \subseteq k[X_1, \ldots, X_n]$, then by definition f vanishes on all of $\mathcal{V}(S)$, therefore $f \in \mathcal{I}(\mathcal{V}(S))$. Let $p \in X \subseteq \mathbf{A}^n(k)$, then by definition p is a zero of all polynomials of $\mathcal{I}(X)$, therefore $p \in \mathcal{V}(\mathcal{I}(X))$. Hence we have $S \subseteq \mathcal{I}(\mathcal{V}(S))$ and $X \subseteq \mathcal{V}(\mathcal{I}(X))$. From these facts, we derive that $\mathcal{I}(X) = \mathcal{I}(\mathcal{V}(\mathcal{I}(X)))$ and $\mathcal{V}(S) = \mathcal{V}(\mathcal{I}(\mathcal{V}(S)))$. In fact, we have the following.

Theorem 1.0.5 (Hilbert's Nullstellensatz). There is a bijective correspondance

$$\{\text{radical ideals of } k[X_1, \ldots, X_n]\} \longleftrightarrow \{\text{algebraic sets of } \mathbf{A}^n(k)\}$$

given by $I \mapsto \mathcal{V}(I)$ and $X \mapsto \mathcal{I}(X)$.

whose proof we will provide later. It is really this observation that is central to algebraic geometry. Next, we observe that $\mathcal{V}()$