Project Sigma

Algebraic Geometry

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Chapter 1

Affine Algebraic Sets

Let k be a field and $n \in \mathbf{N}$, then the *affine space* $\mathbf{A}^n(k)$ of dimension n, or simply $\mathbf{A}^n$ if it does not cause confusion, is the same structure as the n-dimensional vector space k^n over k, except with affine maps as morphisms, where an affine map is a linear map shifted by a constant. In this course note, unless otherwise specified, I will take $k = \mathbf{C}$ the set of complex numbers.

Definition 1.0.1. Suppose that $S \subseteq k[X_1, \ldots, X_n]$ for some $n \in \mathbf{N}$, we define

$$\mathcal{V}(S) = \{x \in \mathbf{A}^n(k) : \forall f \in S,\ f(x) = 0\}$$

as the *set of zeros* of S. Subsets of $\mathbf{A}^n(k)$ that are sets of zeros of some S are called *algebraic*.

For example, in $\mathbf{A}^2(k)$, the sets $\mathcal{V}(Y)$ and $\mathcal{V}(X)$ are the x-axis and the y-axis, the set $\mathcal{V}(X, Y)$ is the origin. The set $\mathcal{V}(f)$, where $f \in k[X, Y]$ is polynomial of degree 2, is known as a *conic section*, for example, the circle $\mathcal{V}(X^2 + Y^2 - 1)$, the parabola $\mathcal{V}(Y - X^2)$, and the hyperbola $\mathcal{V}(XY - 1)$.

Definition 1.0.2. Let $X \subseteq \mathbf{A}^n(k)$, then define the ideal corresponding to X

$$\mathcal{I}(X) = \{f \in k[X_1, \ldots, X_n] : \forall p \in X,\ f(p) = 0\}$$

which is the set of polynomials vanishing on all of X.

We can easily verify that $\mathcal{I}(X)$ is indeed an ideal of $k[X_1, \ldots, X_n]$, and it is a radical ideal. We recall that a radical ideal $I \subseteq R$ is an ideal that satisfies $I = \sqrt{I} := \{r \in R : \exists n > 0,\ r^n \in I\}$. Note that we always have $I \subseteq \sqrt{I}$ and $\sqrt{I}$ is an ideal if I is. Thus, to see that $\mathcal{I}(X)$ is a radical ideal, note that $f^n(p) = 0$ for all $p \in X$ implies $f(p) = 0$ for all $p \in X$, as k is a field.

Theorem 1.0.3 (Hilbert's Nullstellensatz)**.** There is a bijective correspondance

$$\{\text{radical ideals of } k[X_1, \ldots, X_n]\} \longleftrightarrow \{\text{algebraic sets of } \mathbf{A}^n(k)\}$$

given by $I \mapsto \mathcal{V}(I)$ and $X \mapsto \mathcal{I}(X)$.

Proof. Note that each $\mathcal{V}(I)$ is an algebraic set, and each $\mathcal{I}(X)$ is a radical ideal, so these maps are well defined. $\square$