Project Sigma

Algebraic Geometry

Reference & Exercise

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Contents

Chapter 1

Affine Algebraic Sets

Problem 1.0.1. List all points in $V = \mathcal{V}(\{Y - X^2, X - Y^2\})$.

Proof. Since $V = \{(x,y) : y = x^2, x = y^2\}$, we have $x = y^2 = (x^2)^2 = x^4$ if $(x,y) \in V$. By solving $x^4 - x = 0$ we have that $x \in \{0, 1, w, w^2\}$ where $w = e^{2\pi i/3}$. If $x = 0$, then $y = 0$, if $x = 1$ then $y = 1$. We can easily verify that $y = x^2$ and $x = y^2$ in these cases. If $x = w$ then $y = x^2 = w^2$, then $x = w = w^4 = y^2$. If $x = w^2$, then $y = x^2 = w^4 = w$, and $x = w^2 = y^2$. Therefore

$$V = \{(0,0), (1,1), (w, w^2), (w^2, w)\}$$

$\square$

Problem 1.0.2. Show that $W = \{(t, t^2, t^3) : t \in \mathbf{C}\}$ is an algebraic set.

Proof. Consider $V = \mathcal{V}(\{Y - X^2, Z - X^3\})$. For $(x,y,z) \in V$, we have $y = x^2$ and $z = x^3$, therefore $(x,y,z) = (x, x^2, x^3) \in W$. Conversely, let $(x,y,z) = (t, t^2, t^3) \in W$, then $y - x^2 = t^2 - t^2 = 0$ and $z - x^3 = t^3 - t^3 = 0$, hence $(x,y,z) \in V$. Thus $V = W$. $\square$

Problem 1.0.3. Suppose that C is an affine plane curve and L is a line with $L \not\subseteq C$. Suppose that $C = \mathcal{V}(\{F\})$ where $F \in \mathbf{C}[X,Y]$ a polynomial of degree n. Show that $L \cap C$ is a finite set of no more than n points.

Proof. Suppose that $(x,y) \in L \cap C$, since L is a line, we have $y = mx + c$ for some m, c, therefore $F(x, mx + c) = 0$. We note that $\deg F(x, mx + c) \leq n$ since $mx + c$ has degree 1. By the fundamental theorem of algebra, we have $F(x, mx + c) = 0$ has at most n solutions. Hence $L \cap C$ is a finite set of no more than n points. $\square$

Problem 1.0.4. Show that $\mathcal{V}((Y - X^2))$ is irreducible, and that $\mathcal{I}(\mathcal{V}((Y - X^2))) = (Y - X^2)$.

Proof. We will show that $(Y - X^2)$ is prime. Consider $\varphi : \mathbf{C}[X,Y] \to \mathbf{C}[X]$ given by $X \mapsto X$ and $Y \mapsto X^2$ extended to the whole ring, then φ is a homomorphism and $\operatorname{Ker}(\varphi) = (Y - X^2)$. Hence by the first isomorphism theorem, we have $\mathbf{C}[X,Y]/(Y - X^2) \cong \mathbf{C}[X]$ is an integral domain, hence $(Y - X^2)$ is prime. Since prime ideals are radical ideals, we have $\mathcal{I}(\mathcal{V}((Y - X^2))) = (Y - X^2)$ $\square$