Project Sigma

Algebraic Geometry

Reference & Exercise

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Contents

Chapter 1

Affine Algebraic Sets

Problem 1.0.1. List all points in $V = \mathcal{V}(\{Y - X^2, X - Y^2\})$.

Proof. Since $V = \{(x,y) : y = x^2, x = y^2\}$, we have $x = y^2 = (x^2)^2 = x^4$ if $(x,y) \in V$. By solving $x^4 - x = 0$ we have that $x \in \{0, 1, w, w^2\}$ where $w = e^{2\pi i/3}$. If $x = 0$, then $y = 0$, if $x = 1$ then $y = 1$. We can easily verify that $y = x^2$ and $x = y^2$ in these cases. If $x = w$ then $y = x^2 = w^2$, then we can verify $x = w = w^4 = y^2$. If $x = w^2$, then $y = x^2 = w^4 = w$, and we can verify $x = w^2 = y^2$. Therefore $V = \{(0,0), (1,1), (w, w^2), (w^2, w)\}$. $\square$

Problem 1.0.2. Show that $W = \{(t, t^2, t^3) : t \in \mathbf{C}\}$ is an algebraic set.

Proof. Consider $V = \mathcal{V}(\{Y - X^2, Z - X^3\})$. For $(x,y,z) \in V$, we have $y - x^2 = 0$ and $z - x^3 = 0$, so $y = x^2$ and $z = x^3$, therefore $(x,y,z) = (x, x^2, x^3) \in W$. Conversely, let $(x,y,z) = (t, t^2, t^3) \in W$, then $y - x^2 = t^2 - t^2 = 0$ and $z - x^3 = t^3 - t^3 = 0$, hence $(x,y,z) \in V$. Thus $V = W$. $\square$

Problem 1.0.3. Suppose that C is an affine plane curve and L is a line with $L \not\subseteq C$. Suppose that $C = \mathcal{V}(\{F\})$ where $F \in \mathbf{C}[X,Y]$ a polynomial of degree n. Show that $L \cap C$ is a finite set of no more than n points.

Proof. Suppose that $(x,y) \in L \cap C$, since L is a line, we have $y = mx + c$ for some m, c, therefore $F(x, mx + c) = 0$. We note that $\deg F(x, mx + c) \leq n$ since $mx + c$ has degree 1. By the fundamental theorem of algebra, we have $F(x, mx + c) = 0$ has at most n solutions. Hence $L \cap C$ is a finite set of no more than n points. $\square$

Problem 1.0.4. Show that $\mathcal{V}((Y - X^2))$ is irreducible, and that $\mathcal{I}(\mathcal{V}((Y - X^2))) = (Y - X^2)$.

Proof. We will show that $(Y - X^2)$ is prime. Consider $\varphi : \mathbf{C}[X,Y] \to \mathbf{C}[X]$ given by $X \mapsto X$ and $Y \mapsto X^2$ extended to the whole ring, then φ is a homomorphism and $\mathrm{Ker}(\varphi) = (Y - X^2)$. Hence by the first isomorphism theorem, we have $\mathbf{C}[X,Y]/(Y - X^2) \cong \mathbf{C}[X]$ is an integral domain, hence $(Y - X^2)$ is prime. Since prime ideals are radical ideals, we have $\mathcal{I}(\mathcal{V}((Y - X^2))) = (Y - X^2)$. $\square$

Problem 1.0.5. Let $I = (Y^2 - X^3 - X^2, X)$, then $X \in I$ since it is a generator, hence $X^2, X^3 \in I$ as well. Next, $Y^2 - X^3 - X^2 \in I$ since it is a generator, therefore, $Y^2 - X^3 - X^2 + X^3 + X^2 = Y^2 \in I$. Assume for sake of contradiction that I is a radical ideal, then $Y \in I$ since $Y^2 \in I$. Since $Y \in I$, we have $Y = U(X,Y)(Y^2 - X^3 - X^2) + V(X,Y)X$ for some polynomials $U(X,Y), V(X,Y)$. Let $X = 0$ on both sides, then we have $Y = U(0,Y)Y^2$ as polynomials in the indeterminant Y. This is a contradiction since the dergee of Y on the LHS is 1 and the degree of Y on the RHS is not 1. Since I is not a radical ideal, and $\mathcal{I}(D)$ must be a radical ideal, we have $\mathcal{I}(D) \neq I$.

Problem 1.0.6. Let I be a proper ideal of the commutative ring R, and let $\pi : R \to R/I$ be the natural homomorphism.

(a) For every ideal J' of R/I, show that $J = \pi^{-1}[J']$ is an ideal of R containing I, and for every ideal containing I, the set $J' = \pi(J)$ is an ideal of R/I

(b) Show that J' is a radical/prime/maximal ideal iff J is.

(c) Show that J' is finitely generated if J is. Conclude that R/I is noetherian if R is, and that every ring of the form $\mathbf{C}[X_1, \ldots, X_n]/I$ is noetherian.

Proof.

(a) Let J' be an ideal of R/I and $J = \pi^{-1}[J']$. If $a \in I$, then $\pi(a) = a + I = 0 + I \in J'$ since J' is an ideal, hence $a \in J$. Thus J contains I. Let $a, b \in J$, then $\pi(a), \pi(b) \in J'$, so $\pi(a-b) = \pi(a) - \pi(b) \in J'$, so $a - b \in J$ as well. Similarly, for $r \in R$ and $a \in J$, we have $\pi(a) \in J'$, so $\pi(ra) = \pi(r)\pi(a) \in J'$ since J' is an ideal, thus $ra \in J$. Hence J is an ideal containing I.

Now, let J be an ideal of R containing I, and let $J' = \pi[J]$. Let $a, b \in J'$, then $a = \pi(x)$ and $b = \pi(y)$ for some $x, y \in J$, thus $a - b = \pi(x) - \pi(y) = \pi(x - y)$. Since J is an ideal, we have $x - y \in J$, so $a - b \in J'$. Similarly, for $r' = r + I \in R/I$ and for $a \in J'$, we have $a = \pi(x) = x + I$ for some $x \in J$, so $r'a = (r + I)(x + I) = rx + I = \pi(rx)$. Since J is an ideal, we have $rx \in J$, so $r'a \in J'$. Hence J' is an ideal.

(b)

(c)

$\square$