Project Sigma

Algebraic Geometry

Reference & Exercise

Yunhai Xiang

June 6, 2021

Contents

Chapter 1

Affine Algebraic Sets

Problem 1.0.1. List all points in $V = \mathcal{V}(\{Y - X^2, X - Y^2\})$.

Proof. Since $V = \{(x,y) : y = x^2, x = y^2\}$, we have $x = y^2 = (x^2)^2 = x^4$ if $(x,y) \in V$. By solving $x^4 - x = 0$ we have that $x \in \{0, 1, w, w^2\}$ where $w = e^{2\pi i/3}$. If $x = 0$, then $y = 0$, if $x = 1$ then $y = 1$. We can easily verify that $y = x^2$ and $x = y^2$ in these cases. If $x = w$ then $y = x^2 = w^2$, then we can verify $x = w = w^4 = y^2$. If $x = w^2$, then $y = x^2 = w^4 = w$, and we can verify $x = w^2 = y^2$. Therefore $V = \{(0,0), (1,1), (w, w^2), (w^2, w)\}$. $\square$

Problem 1.0.2. Show that $W = \{(t, t^2, t^3) : t \in \mathbf{C}\}$ is an algebraic set.

Proof. Consider $V = \mathcal{V}(\{Y - X^2, Z - X^3\})$. For $(x,y,z) \in V$, we have $y - x^2 = 0$ and $z - x^3 = 0$, so $y = x^2$ and $z = x^3$, therefore $(x,y,z) = (x, x^2, x^3) \in W$. Conversely, let $(x,y,z) = (t, t^2, t^3) \in W$, then $y - x^2 = t^2 - t^2 = 0$ and $z - x^3 = t^3 - t^3 = 0$, hence $(x,y,z) \in V$. Thus $V = W$. $\square$

Problem 1.0.3. Suppose that C is an affine plane curve and L is a line with $L \not\subseteq C$. Suppose that $C = \mathcal{V}(\{F\})$ where $F \in \mathbf{C}[X,Y]$ a polynomial of degree n. Show that $L \cap C$ is a finite set of no more than n points.

Proof. Suppose that $(x,y) \in L \cap C$, since L is a line, we have $y = mx + c$ for some m, c, therefore $F(x, mx + c) = 0$. We note that $\deg F(x, mx + c) \leq n$ since $mx + c$ has degree 1. By the fundamental theorem of algebra, we have $F(x, mx + c) = 0$ has at most n solutions. Hence $L \cap C$ is a finite set of no more than n points. $\square$

Problem 1.0.4. Show that $\mathcal{V}((Y - X^2))$ is irreducible, and that $\mathcal{I}(\mathcal{V}((Y - X^2))) = (Y - X^2)$.

Proof. We will show that $(Y - X^2)$ is prime. Consider $\varphi : \mathbf{C}[X,Y] \to \mathbf{C}[X]$ given by $X \mapsto X$ and $Y \mapsto X^2$ extended to the whole ring, then φ is a homomorphism and $\mathrm{Ker}(\varphi) = (Y - X^2)$. Hence by the first isomorphism theorem, we have $\mathbf{C}[X,Y]/(Y - X^2) \cong \mathbf{C}[X]$ is an integral domain, hence $(Y - X^2)$ is prime. Since prime ideals are radical ideals, we have $\mathcal{I}(\mathcal{V}((Y - X^2))) = (Y - X^2)$. $\square$

Problem 1.0.5. Let $I = (Y^2 - X^3 - X^2, X)$, then $X \in I$ since it is a generator, hence $X^2, X^3 \in I$ as well. Next, $Y^2 - X^3 - X^2 \in I$ since it is a generator, therefore, $Y^2 - X^3 - X^2 + X^3 + X^2 = Y^2 \in I$. Assume for sake of contradiction that I is a radical ideal, then $Y \in I$ since $Y^2 \in I$. Since $Y \in I$, we have $Y = U(X,Y)(Y^2 - X^3 - X^2) + V(X,Y)X$ for some polynomials $U(X,Y), V(X,Y)$. Let $X = 0$ on both sides, then we have $Y = U(0,Y)Y^2$ as polynomials in the indeterminant Y. This is a contradiction since the dergee of Y on the LHS is 1 and the degree of Y on the RHS is not 1. Since I is not a radical ideal, and $\mathcal{I}(D)$ must be a radical ideal, we have $\mathcal{I}(D) \neq I$.

Problem 1.0.6. Show that $\mathcal{V}(F) \cong \mathcal{V}(G)$ where $F(X,Y) = X^2 + Y^2 - 1$ and $G(X,Y) = X^2 - Y^2 - 1$.

Proof. We let $\varphi : \mathcal{V}(F) \to \mathcal{V}(G)$ be $(x,y) \mapsto (x, iy)$ which is obviously a polynomial map with an inverse $\varphi^{-1} : \mathcal{V}(G) \to \mathcal{V}(F)$ given by $(x,y) \mapsto (x, -iy)$ which is also a polynomial map. We easily verify that $\varphi(\varphi^{-1}(x,y)) = (x,y)$ and $\varphi^{-1}(\varphi(x,y)) = (x,y)$. We note that if $(x,y) \in \mathcal{V}(F)$ then $x^2 + y^2 - 1 = 0$, and we have $G(\varphi(x,y)) = x^2 - (iy)^2 - 1 = x^2 + y^2 - 1 = 0$. And if $(x,y) \in \mathcal{V}(G)$ then $x^2 - y^2 - 1 = 0$ then $F(\varphi^{-1}(x,y)) = x^2 + (-iy)^2 - 1 = x^2 - y^2 - 1 = 0$. Therefore φ, φ^{-1} are well-defined. Therefore $\mathcal{V}(F) \cong \mathcal{V}(G)$. $\square$

Problem 1.0.7. Let $V = \mathcal{V}(Y^2 - X^3)$ and let $\phi : \mathbf{A}^1 \to V$ be $\phi(t) = (t^2, t^3)$, show that ϕ is a bijective polynomial map which is not an isomorphism.

Proof. Assume $s \neq t$ and $(t^2, t^3) = (s^2, s^3)$ then we have $s^2 = t^2$ and $s^3 = t^3$. Since $s \neq t$ we have $s - t \neq 0$. Since $s^2 = t^2$ we have $s^2 - t^2 = (s+t)(s-t) = 0$. Since $s - t \neq 0$, we have $s + t = 0$, thus $s = -t$, hence $s^3 = (-t)^3 = -t^3$. Since $s^3 = t^3$ and $s^3 = -t^3$, we have $t^3 = -t^3$, so $t = 0$. Since $t = 0$ we have $s = -t = 0 = t$ which contradicts the hypothesis that $s \neq t$. This shows that ϕ is injective. Next, for each $(x,y) \in \mathcal{V}(\{Y^2 - X^3\})$, we have $y^2 - x^3 = 0$ and thus $y^2 = x^3$. We know that x has square roots α and $-\alpha$ for some α. We show that one of them is also a cube root of y. We have $\alpha^6 = (\alpha^2)^3 = x^3 = y^2$, therefore $y = \alpha^3$ or $y = -\alpha^3$. Since $y = \alpha^3$ or $y = (-\alpha)^3$, we have one of $\pm\alpha$ is a cube root of y. Let $t = \alpha$ if α is a cube root of y and $t = -\alpha$ otherwise. We then have $\phi(t) = (t^2, t^3) = (x,y)$. Thus ϕ is surjective, hence bijective. Suppose for contradiction that there is a polynomial map inverse $\phi^{-1} : V \to \mathbf{A}^1$ which can be represented by a polynomial $f \in \mathbf{C}[X,Y]$. Then have $\phi^{-1}(\phi(t)) = t$, so $f(t^2, t^3) = t$. We note that $[t^1]f(t^2, t^3) = 0$, since for each term aX^nY^m, substituding $X = t^2$ and $Y = t^3$ gives at^{2n+3m}, and there is no n, m with $2n + 3m = 1$. This is a contradiction since $[t^1]t = 1$. $\square$

Problem 1.0.8. Let $\phi : \mathbf{A}^1 \to V$ be $\phi(t) = (t^2 - 1, t(t^2 - 1))$ where $V = \mathcal{V}(\{Y^2 - X^2(X+1)\})$. Show that ϕ is one-to-one and onto except at $\phi(\pm 1) = (0,0)$.

Proof. Suppose that $s \neq t$ and $(s^2 - 1, s(s^2 - 1)) = (t^2 - 1, t(t^2 - 1))$, we then have $s^2 - 1 = t^2 - 1$ thus $s^2 - t^2 = (s-t)(s+t) = 0$. Since $s \neq t$, we have $s = -t$. Next, since $s(s^2 - 1) = t(t^2 - 1)$ we have $-t(t^2 - 1) = t(t^2 - 1)$. Thus $t = 0$ or $t^2 = 1$. If $t = 0$ then $s = -t = 0 = t$ which contradicts $s \neq t$, so $t^2 = 1$. Thus $t = \pm 1$ and $s = \mp 1$. Thus ϕ is injective except at $t = \pm 1$. Next, let $(x,y) \in V$ then $y^2 - x^2(x+1) = 0$ so $y^2 = x^2(x+1)$. Let α and $-\alpha$ be the square roots of $x + 1$. By $y^2 = x^2(x+1)$, we have $y = \alpha x$ or $y = -\alpha x$. Let $t = \alpha$ if $y = \alpha x$ and $t = -\alpha$ otherwise. We thus have $y = tx$. Since t is a square root of $x + 1$, we have $t^2 = x + 1$, so $x = t^2 - 1$. Thus $x = t^2 - 1$ and $y = t(t^2 - 1)$. Hence $\phi(t) = (x,y)$. Thus ϕ is surjective. $\square$

Problem 1.0.9. Let $V = \mathcal{V}(\{X^2 - Y^3, Y^2 - Z^3\})$, and let $\overline{\alpha} : \Gamma(V) \to \mathbf{C}[T]$ be given by $\overline{\alpha}(X) = T^9$, $\overline{\alpha}(Y) = T^6$ and $\overline{\alpha}(Z) = T^6$. Then

(a) What is the polynomial map $f : \mathbf{A}^1 \to V$ with $f^* = \overline{\alpha}$

(b) Show that f is bijective but not an isomorphism

Proof.

(a) Define the polynomial map $f : \mathbf{A}^1 \to V$ by $f(t) = (t^9, t^6, t^4)$ as in the proof of Theorem 1.6. We can verify that this is well-defined since $X^2 - Y^3 = t^{18} - t^{18} = 0$ and $Y^2 - Z^3 = t^{12} - t^{12} = 0$. We verify that the pullback $f^*(X) = [(x,y,z) \mapsto x] \circ f = T^9$, $f^*(Y) = [(x,y,z) \mapsto y] \circ f = T^6$, and $f^*(Z) = [(x,y,z) \mapsto z] \circ f = T^4$. Thus $f^* = \overline{\alpha}$.

(b) We note that $f(t) = (0,0,0)$ iff $t = 0$, so we can assume $t \neq s$ are nonzero and $(t^9, t^6, t^4) = (s^9, s^6, s^4)$. Since $t^4 = s^4$, we have $t \in \{s\zeta_4, s\zeta_4^2, s\zeta_4^3\}$. Since $t^6 = s^6$, we have $t \in \{s\zeta_6, \dots, s\zeta_6^5\}$. Since $t^9 = s^9$, we have $t \in \{s\zeta_9, \dots, s\zeta_9^8\}$. Since $\gcd(9,6,4) = 1$, this is a contradiction. To explain in simpler language, $t^4 = s^4$ implies that the angle between t, s is $90^\circ, 180^\circ$ or 270°; $t^6 = s^6$ implies that the angle between t, s is $60^\circ, 120^\circ, 180^\circ, 240^\circ$ or 300°; $t^9 = s^9$ implies that the angle between t, s is $40^\circ, 80^\circ, 120^\circ, 160^\circ, 200^\circ, 240^\circ, 280^\circ$ or 320°. There is no angle between t, s that satisfies our requirement. Thus f is injective. Next, let $(x,y,z) \in V$, we then have $x^2 - y^3 = 0$ and $y^2 - z^3 = 0$, thus $x^2 = y^3$ and $y^2 = z^3$. The 6-th roots of y are $\{\alpha, \alpha\omega, \dots, \alpha\omega^5\}$ for some α where $\omega = e^{\frac{2\pi i}{6}}$. Let s be a 6-th roots of y. Thus $s^{18} = (s^6)^3 = y^3 = x^2$, so $x = \pm s^9$, so $x \in \{s^9, s^9\omega^3\}$. Similarly, $s^{12} = (s^6)^2 = y^2 = z^3$, therefore $\{z, z\omega^2, z\omega^4\} = \{s^4, s^4\omega^2, s^4\omega^4\}$, hence $z \in \{s^4, s^4\omega^2, s^4\omega^4\}$. Suppose that $x = s^9\omega^{3n}$ for $n \in \{0,1\}$ and $z = s^4\omega^{2m}$ for $m \in \{0,1,2\}$. Let $t = s\omega^k$ then t is also a 6-th root of unity, so $y = t^6$. Also, $x = t^9\omega^{3n-9k}$ and $z = t^4\omega^{2m-4k}$. I claim that we can always choose k such that $3n \equiv 9k \pmod 6$ and $2m \equiv 4k \pmod 6$. Note that $3n \equiv 9k \pmod 6$ iff $k \equiv n \pmod 2$, and note that $2m \equiv 4k \pmod 6$ iff $k \equiv 2m \pmod 3$. By the Chinese remainder theorem, such k can always be chosen. Hence we have $x = t^9$, $y = t^6$ and $z = t^4$. Thus $f(t) = (x,y,z)$. Thus f is surjective, so f is bijective.

We see that f is not an isomorphism, since if so there is a polynomial map $g : V \to \mathbf{A}^1$ which can be viewed as a polynomial $g \in \mathbf{C}[X,Y,Z]$ which is the inverse of f, then by $g \circ f = \mathrm{id}$, we have $g(t^9, t^6, t^4) = t$. We note that $[t^1]g(t^9, t^6, t^4) = 0$ since if $aX^pY^qZ^r$ is a term in $g(X,Y,Z)$, then substituding $X = t^9, Y = t^6, Z = t^4$ gives $at^{9p+6q+4r}$, and there is no p, q, r such that $9p + 6q + 4r = 1$. This contradicts the fact that $[t^1]t = 1$.

□

Problem 1.0.10. If $\phi : V \subseteq \mathbf{A}^n \to W \subseteq \mathbf{A}^m$ is an onto polynomial map, show that if X is an algebraic subset of W then $\phi^{-1}[X]$ is an algebraic subset of V, and that X is irreducible if $\phi^{-1}[X]$ is irreducible.

Proof. Suppose that $X = \mathcal{V}(I)$ for some $I \subseteq \mathbf{C}[X_1, \dots, X_m]$, then for $x \in V$, we have

$$x \in \phi^{-1}[X] \iff \phi(x) \in X \iff f(\phi(x)) = 0, \forall f \in I \iff x \in \mathcal{V}(\{f \circ \phi : f \in I\})$$

Therefore $\phi^{-1}[X] = \mathcal{V}(\{f \circ \phi : f \in I\})$ is algebraic. If $X = U \cup V$ where algebraic sets $U, V \subset X$ properly, then $\phi^{-1}[X] = \phi^{-1}[U] \cup \phi^{-1}[V]$. Choose $p \in X \setminus U$, and let x be such that $\phi(x) = p$, then $x \in \phi^{-1}[X] \setminus \phi^{-1}[U]$, so $\phi^{-1}[U] \subset \phi^{-1}[X]$ properly, and similarly $\phi^{-1}[V] \subset \phi^{-1}[X]$ properly. Since $\phi^{-1}[U], \phi^{-1}[V]$ are algebraic as U, V are algebraic, we have $\phi^{-1}[X]$ is reducible. Thus $\phi^{-1}[X]$ is irreducible implies X is irreducible. □

Problem 1.0.11. Let $V \subseteq \mathbf{A}^n$ be a variety, show that TFAE

(i) V is a point

(ii) $\Gamma(V) = \mathbf{C}$

(iii) $\dim_{\mathbf{C}} \Gamma(V)$ is finite

Proof. Assume (i), then let $V = \{(x_1, \ldots, x_n)\}$. We claim that $\mathcal{I}(V) = (X_1 - x_1, \ldots, X_n - x_n)$. Note that $\mathcal{V}((X_1 - x_1, \ldots, X_n - x_n)) = V$ which is straightforward. Next, since $x_1, \ldots, x_n \in \mathbf{C}$, we have

$$\mathbf{C}[X_1, \ldots, X_n]/(X_1 - x_1, \ldots, X_n - x_n) \cong \mathbf{C}[x_1, \ldots, x_n] \cong \mathbf{C}$$

which is an integral domain, so $(X_1 - x_1, \ldots, X_n - x_n)$ is prime, so it's also a radical ideal. Therefore we have $\mathcal{I}(V) = \mathcal{I}(\mathcal{V}((X_1 - x_1, \ldots, X_n - x_n))) = (X_1 - x_1, \ldots, X_n - x_n)$ by Nullstellensatz. Thus, we indeed have $\Gamma(V) = \mathbf{C}[X_1, \ldots, X_n]/\mathcal{I}(V) = \mathbf{C}$. Next, assume (ii), then $\dim_{\mathbf{C}} \Gamma(V) = \dim_{\mathbf{C}} \mathbf{C} = 1 < \infty$ straightforwardly. Assume (iii), then $\Gamma(V) = \mathbf{C}[X_1, \ldots, X_n]/\mathcal{I}(V)$ has finite dimension over $\mathbf{C}$. Let $i \in \{1, \ldots, n\}$. We note that if $\{1, X_i, X_i^2, X_i^3, \ldots\}$ is linearly independent then we cannot have $\dim_{\mathbf{C}} \Gamma(V) < \infty$, thus they are linearly dependent. This means that there exists some polynomial $f_i \in \mathbf{C}[X_i] \subseteq \mathbf{C}[X_1, \ldots, X_n]$ with coefficients not all zero for which $f_i(X_i) \equiv 0 \pmod{\mathcal{I}(V)}$. Hence $f_i \in \mathcal{I}(V)$ for each i. By Hilbert's Nullstellensatz, we have $\mathcal{V}(\mathcal{I}(V)) = V$ as V is an algebraic set. Thus for each $p \in V$, we have $p \in \mathcal{V}(\mathcal{I}(V))$, so $f_i(p) = 0$ for each i. The fact that each f_i is a single-variable polynomial over $\mathbf{C}$ means that it has finitely many roots. Therefore we only have finitely many choices for each coordinate of p. Thus V is a finite set. Since V is a variety, it is irreducible, therefore it must be a single point. $\square$

Problem 1.0.12. Decompose $\mathcal{V}(Y^4 - X^2, Y^4 - X^2Y^2 + XY^2 - X^3)$ into irreducible components.

Proof. We note that $Y^4 - X^2 = (Y^2 - X)(Y^2 + X)$ and $Y^4 - X^2Y^2 + XY^2 - X^3 = (X + Y)(Y - X)(X + Y^2)$. We note that $X + Y^2$ and $Y^2 - X$ are irreducible. Let $V = \mathcal{V}(Y^4 - X^2, Y^4 - X^2Y^2 + XY^2 - X^3)$ then

$$\begin{aligned} V &= \mathcal{V}(Y^4 - X^2) \cap \mathcal{V}(Y^4 - X^2Y^2 + XY^2 - X^3) \\ &= (\mathcal{V}(Y^2 - X) \cup \mathcal{V}(Y^2 + X)) \cap (\mathcal{V}(X + Y) \cup \mathcal{V}(Y - X) \cup \mathcal{V}(X + Y^2)) \\ &= (\mathcal{V}(Y^2 - X) \cap (\mathcal{V}(X + Y) \cup \mathcal{V}(Y - X) \cup \mathcal{V}(X + Y^2))) \cup \mathcal{V}(Y^2 + X) \end{aligned}$$

We note that if $(x, y) \in \mathcal{V}(Y^2 - X) \cap \mathcal{V}(X + Y)$ then $x + y = 0$ and $y^2 - x = 0$ so $y^2 + y = 0$, which implies that $(x, y) \in \{(0, 0), (1, -1)\}$. If $(x, y) \in \mathcal{V}(Y^2 - X) \cap \mathcal{V}(Y - X)$ then $y - x = 0$ and $y^2 - x = 0$, which implies $x^2 - x = 0$ so $(x, y) \in \{(1, 1), (0, 0)\}$. If $(x, y) \in \mathcal{V}(Y^2 - X) \cap \mathcal{V}(X + Y^2)$ then $y^2 - x = 0$ and $x + y^2 = 0$ so $2y^2 = 0$ so $(x, y) = (0, 0)$. Therefore

$$\begin{aligned} V &= \{(0, 0), (1, -1)\} \cup \{(1, 1), (0, 0)\} \cup \{(0, 0)\} \cup \mathcal{V}(Y^2 + X) \\ &= \mathcal{V}(X, Y) \cup \mathcal{V}(X - 1, Y + 1) \cup \mathcal{V}(X - 1, Y - 1) \cup \mathcal{V}(Y^2 + X) \end{aligned}$$

The first three components are irreducible since they are single points. The last component $\mathcal{V}(Y^2 + X)$ is irreducible since $(Y^2 + X)$ is prime, and hence also radical, so $\mathcal{I}(\mathcal{V}(Y^2 + X)) = (Y^2 + X)$ which is prime. We note that $(Y^2 + X)$ is prime since $Y^2 + X$ is prime, and $Y^2 + X$ is prime since $Y^2 + X$ is irreducible and $\mathbf{C}[X, Y]$ is a ufd. $\square$

Problem 1.0.13. Find all irreducible components of $\mathcal{V}(2X^3 - X^2Y - 2XY + Y^2)$.

Proof. We note that if $(x, y) \in \mathcal{V}(2X^3 - X^2Y - 2XY + Y^2)$, then $2x^3 - (x^2 + 2x)y + y^2 = 0$, so $y = \frac{x^2+2x\pm\sqrt{(x^2+2x)^2-8x^3}}{2} = \frac{x(x+2)\pm x(x-2)}{2}$. Therefore $y = x^2$ or $y = 2x$. Conversely, if $y = 2x$, then $2x^3 - x^2y - 2xy + y^2 = 2x^3 - 2x^3 - 4x^2 + 4x^2 = 0$. If $y = x^2$ then $2x^3 - x^2y - 2xy + y^2 = 2x^3 - x^4 - 2x^3 + x^4 = 0$. Therefore we have $\mathcal{V}(2X^3 - X^2Y - 2XY + Y^2) = \mathcal{V}(Y - 2X) \cup \mathcal{V}(Y - X^2)$. We note that $\mathcal{V}(Y - X^2)$ is irreducible by a previous assignment. Also $\mathcal{V}(Y - 2X)$ is irreducible since $(Y - 2X)$ is prime and hence also radical so by Nullstellensatz we have $\mathcal{I}(\mathcal{V}(Y - 2X)) = (Y - 2X)$ which is prime. We note that $(Y - 2X)$ is prime since $Y - 2X$ is prime, and $Y - 2X$ is prime since $Y - 2X$ is irreducible and $\mathbf{C}[X, Y]$ is a ufd. □

Problem 1.0.14. Let $V = \mathcal{V}(Y^2 - X^2(X + 1))$ and $z = Y/X \in K(V)$, find the pole sets of z and z^2.

Proof. First, we note that $z = Y/X = X(X + 1)/Y$ since $Y^2 \equiv X^2(X + 1) \pmod{\mathcal{I}(V)}$. Thus if $x \neq 0$ then the expression $z = Y/X$ is defined, and if $y \neq 0$ then the expression $z = X(X + 1)/Y$ is defined. Thus z is defined for all $(x, y) \neq (0, 0)$. For $(x, y) = (0, 0)$, suppose that z is defined then exists expression $r(X, Y)/s(X, Y) = X/Y$ with $s(0, 0) \neq 0$. Since $s(0, 0) \neq 0$, we know that $s(X, Y)$ has a nonzero constant term. Next, $Yr(X, Y) \equiv Xs(X, Y) \pmod{\mathcal{I}(V)}$ so $Yr(X, Y) - Xs(X, Y) = h(X, Y)(Y^2 - X^2(X + 1))$. Consider the coefficient of $X = X^1Y^0$ on both sides. For the LHS, since $s(X, Y)$ has a nonzero constant term, the coefficient of $X = X^1Y^0$ in $-Xs(X, Y)$ is -1. Since $Yr(X, Y)$ has Y as a factor, the coefficient of $X = X^1Y^0$ in $Yr(X, Y)$ is 0. Next, for the RHS. Since $Y^2h(X, Y)$ has Y as a factor, coefficient of $X = X^1Y^0$ in $Y^2h(X, Y)$ is 0. Since $-X^2(X + 1)h(X, Y) = -X^3h(X, Y) - X^2h(X, Y)$, we have the coefficient of $X = X^1Y^0$ in $-X^2(X + 1)h(X, Y)$ is 0. Thus the coefficient of $X = X^1Y^0$ in the LHS is -1 but it is 0 in the RHS, contradiction. Next, we note that $z^2 = Y^2/X^2 = (X + 1)$ since $Y^2 \equiv X^2(X + 1) \pmod{\mathcal{I}(V)}$. The denominator of $X + 1$ is 1, so z^2 is defined on all points in V. Hence z has pole at $(0, 0)$ only and z^2 has no pole. □

Problem 1.0.15. Let $F(X, Y) = Y^2 - X^3 + X$, $W = \mathcal{V}(F)$ and $P = (0, 0)$.

(a) Show that $aX + bY$ is an element of the maximal ideal $\mathcal{M}_P(W)$ of the local ring $\mathcal{O}_P(W)$,

(b) Show that $aX + bY$ is an element of $\mathcal{M}_P(W)^2$ iff $aX + bY$ is tangent to W at P.

Proof.

(a) We recall that
$$\mathcal{M}_P(W) = \left\{ f \in K(W) \mid f = \frac{r}{s} \text{ for } r(P) = 0, s(P) \neq 0 \right\}$$
and in this case where $P = (0, 0)$, the denominator of $aX + bY$ is always 1 which is nonzero and when $(x, y) = (0, 0)$ the numerator is $a0 + b0 = 0$. Thus $aX + bY \in \mathcal{M}_P(W)$.

(b) We note that the line $aX + bY$ is tangent to W at P iff their intersection $\mathcal{V}(aX + bY) \cap W = \{P\}$. We also note that $\mathcal{M}_P(W) = \mathcal{I}(P)\mathcal{O}_P(W)$. Since $\mathcal{I}(P) = (X, Y)$, we have $\mathcal{I}(P)^2 = (X^2, XY, YX, Y^2) = (X^2, XY, Y^2)$. We note that $\mathcal{I}(P) = \mathcal{I}(\mathcal{V}(aX + bY)) \cup \mathcal{I}(W)$.

□

Problem 1.0.16. Let $W = \mathcal{V}(Y^2 - X^2(X + 1))$, and $P = (0, 0)$, show that for each $a, b \in \mathbf{C}$, $aX + bY$ not an element of $\mathcal{M}_P(W)^2$ unless $a = b = 0$.

Problem 1.0.17. Let $C = \mathcal{V}(Y^2 - X^3)$, show that the function field $K(C)$ of C is isomorphic to $\mathbf{C}(T)$ but $\Gamma(C)$ is not isomorphic to $\mathbf{C}[T]$.

Proof. First, we claim that $(Y^2 - X^3)$ is a prime ideal in $\mathbf{C}[X,Y]$. We claim that $Y^2 - X^3$ is irreducible. We note that if $Y^2 - X^3 = f(X,Y)g(X,Y)$ then the degree of Y in $f(X,Y), g(X,Y)$ are 1. If $f(X,Y) = f_0(X) + f_1(X)Y$ and $g(X,Y) = g_0(X) + g_1(X)Y$ then $f(X,Y)g(X,Y) = f_0(X)g_0(X) + (f_0(X)g_1(X) + g_0(X)f_1(X))Y + f_1(X)g_1(X)Y^2$. Thus $f_0(X)g_0(X) = -X^3$ and we have $f_0(X)g_1(X) = g_0(X)f_1(X)$ and $f_1(X)g_1(X) = 1$. By $f_1(X)g_1(X) = 1$, we have f_1, g_1 are nonzero constants, thus by $f_0(X)g_1(X) = g_0(X)f_1(X)$, we have $f_0(X) = cg_0(X)$ for a nonzero constant c. Thus $f_0(X)g_0(X) = cg_0(X)^2$ which has a even degree, but $f_0(X)g_0(X) = -X^3$ which has odd degree, contradiction. Since $Y^2 - X^3$ is irreducible, it is a prime as $\mathbf{C}[X,Y]$ is a ufd. Thus $(Y^2 - X^3)$ is a prime ideal. Hence $\Gamma(C) = \mathbf{C}[X,Y]/(Y^2 - X^3)$ by Nullstellensatz. We claim that $\Gamma(C)$ is not isomorphic to $\mathbf{C}[T]$. We note that $\mathbf{C}[T]$ is integrally closed since it is a ufd. However $\Gamma(C) \cong \mathbf{C}[\alpha^2, \alpha^3]$ a trancedental extension of $\mathbf{C}$ where α is not a root of any polynomial. Therefore $\Gamma(C)$ is not integrally closed, so it is not isomorphic to $\mathbf{C}[T]$. On the other hand, the field of fraction of $\mathbf{C}[\alpha^2, \alpha^3]$, which is $K(C)$, is isomorphic to $\mathbf{C}(T)$ as we can set $T = \alpha = \alpha^3/\alpha^2$ □

Problem 1.0.18. Let $V = \mathcal{V}(Y - X^2)$ and $P = (1,1)$, which of the three rational functions are equal in $\mathcal{O}_P(V)$?

1. $F_1(X,Y) = \frac{1}{X+1}$,
2. $F_2(X,Y) = \frac{X}{X+Y}$,
3. $F_3(X,Y) = \frac{X^2}{X+Y^2}$.

Proof. First, we note that $F_1 = F_2$ since $\frac{X}{X+Y} = \frac{X}{X+X^2} = \frac{1}{X+1}$. On the other hand, if $F_1 = F_3$ then $\frac{1}{X+1} = \frac{X^2}{X+Y^2}$ so $X^2(X+1) \equiv X + Y^2 \pmod{\mathcal{I}(V)}$. Therefore $X^3 + X^2 - X - Y^2 \in \mathcal{I}(V)$. Therefore, we have $X^3 + X^2 - X - Y^2 = h(X,Y)(Y - X^2)$ for some h. Considering both sides as polynomials of Y, since Y has degree 2 in the LHS and Y has degree 1 in $Y - X^2$, we have Y has degree 1 in $h(X,Y)$. Since the coefficient of Y^2 is -1 in the LHS, and the coefficient of Y in $Y - X^2$ is 1, we have $h(X,Y) = -Y + f(X)$ for some f. Next, we multiply out and get $(-Y + f(X))(Y - X^2) = -Y^2 + (X^2 + f(X))Y - X^2 f(X)$. Since this is equal to $X^3 + X^2 - X - Y^2$, we have $X^2 + f(X) = 0$ so $f(X) = -X^2$. Thus $-X^2 f(X) = X^4$. However, the coefficient of Y^0 in $X^3 + X^2 - X - Y^2$ is $X^3 + X^2 - X$, contradiction. Thus $F_1 = F_2$ but F_3 is not equal to F_1 or F_2. □

Problem 1.0.19. Let $V \subseteq \mathbf{A}^n$ be a variety, and $\phi : \mathcal{O}_P(V) \to \mathbf{C}$ a $\mathbf{C}$-algebra homomorphism, then show that $\phi(f) = f(P)$ for all f.

Proof. First, for some $f(X_1, \ldots, X_n) = \frac{p(X_1,\ldots,X_n)}{q(X_1,\ldots,X_n)}$, we note that $\phi(f) = \frac{\phi(p)}{\phi(q)}$. Next, for any $a \in \mathbf{C}$ and $k_1, \ldots, k_n \in \mathbf{N}$ we have $\phi(aX_1^{k_1} \cdots X_n^{k_n}) = a\phi(X_1)^{k_1} \cdots \phi(X_n)^{k_n}$. Since also $\phi(f+g) = \phi(f) + \phi(g)$, we have that ϕ is completely determined by $\phi(X_1), \ldots, \phi(X_n)$. Assume that $\phi(X_i) \neq P_i$ where $P = (P_1, \ldots, P_n)$, then $\frac{1}{X_i - P_i}$ is defined at P, so it is in $\mathcal{O}_P(V)$.

□

Problem 1.0.20. Let $F \in \mathbf{C}[X,Y]$ be nonzero and $U = \mathbf{A}^2 \setminus \mathcal{V}(F)$. Show that $\Gamma(U) = \mathbf{C}[X,Y,1/F]$.

Proof. First, $\mathbf{A}^2$ is a variety since it is equal to $\mathcal{V}(0)$ and (0) is prime, and its coordinate ring is just $\mathbf{C}[X,Y]$. By theorem 1.5 with $V = \mathbf{A}^2$, we have $\Gamma(U) = \mathbf{C}[X,Y][1/F] = \mathbf{C}[X,Y,1/F]$. □

Problem 1.0.21. Let $V = \mathcal{V}(Y^2 - X^3 - X)$ and $P = (0,0)$. Let $\mathcal{M} = \mathcal{M}_P(V) \subseteq \mathcal{O}_P(V)$. Show that $\dim_{\mathcal{C}} \mathcal{M}/\mathcal{M}^2 = 1$.

Proof. First, we note that $Y^2 - X^3 \in \mathcal{M}^2$. Note that $Y^2 - X^3 = YY + (-X^2)X$. Since the rational functions $Y, X, -X^2$ all have denominator 1, and they all vanish at $P = (0,0)$, we have $Y, X, -X^2 \in \mathcal{M}$ and $Y^2, -X^3 \in \mathcal{M}^2$, thus $Y^2 - X^3 \in \mathcal{M}^2$. Therefore $X \in \mathcal{M}^2$ as we have $X \equiv Y^2 - X^3 \pmod{\mathcal{I}(V)}$. Thus $X = 0$ in $\mathcal{M}/\mathcal{M}^2$. Next, since $Y^2 \in M^2$, we have $Y^2 = 0$ in $\mathcal{M}/\mathcal{M}^2$. Thus for some rational function f in $\mathcal{M}/\mathcal{M}^2$, all terms with X or Y^2 as a factor can be erased, so it leaves us with $f(X,Y) = \frac{aY+b}{cY+d}$. Since the denominator is nonzero at P and the numerator is 0 at P, we have $b = 0$ and $d \neq 0$. Thus $f(X,Y) = 0$ or $f(X,Y) = \frac{Y}{cY+d}$ for some c and some $d \neq 0$ (where we assume $a = 1$ without loss of generality). Since $\frac{1}{d}Y(cY+d) = \frac{c}{d}Y^2 + Y \equiv Y$, we have $f(X,Y) = \frac{Y}{cY+d} \equiv \frac{1}{d}Y$. Thus $\mathcal{M}/\mathcal{M}^2$ is spanned by Y, so $\dim_{\mathbf{C}} \mathcal{M}/\mathcal{M}^2 \leq 1$. Next, we claim that $\dim_{\mathbf{C}} \mathcal{M}/\mathcal{M}^2 \neq 0$. Assume that $\dim_{\mathbf{C}} \mathcal{M}/\mathcal{M}^2 = 0$, then $Y \in \mathcal{M}^2$, so □

Problem 1.0.22. Let $V = \mathcal{V}(X^2 - Y^3, Y^2 - Z^3)$ and $P = (0,0,0)$. Write $\mathcal{M} = \mathcal{M}_P(V)$, show that $\dim_{\mathbf{C}} \mathcal{M}/\mathcal{M}^2 = 3$.

Problem 1.0.23. If $\phi : \mathbf{A}^2 \to \mathbf{A}^2$ is a polynomial map given by $\phi(X,Y) = (f(X,Y), g(X,Y))$, then if ϕ is an isomorphism, then $\det D\phi$ is a nonzero constant, where $D\phi$ is the Jacobian matrix.

Proof. Since ϕ is an isomorphism, there exists a polynomial map inverse μ. Since $\mu \circ \phi = \mathbf{1}$,

$$1 = \det D(\mathbf{1}) = \det D(\mu(\phi(X,Y))) = \det((D\mu)(\phi(X,Y))(D\phi)) = \det((D\mu)(\phi(X,Y))) \det D\phi$$

by chain rule. Thus $\det D\phi \in \mathbf{C}[X,Y]$ is a unit which is just a nonzero constant.

□