LINEAR ALGEBRAIC GROUPS AND REPRESENTATION THEORY

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CONTENTS

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1. Categories and Functors

We start with a crash course in category theory. By a *class* we mean a collection of sets, which is not necessarily a set itself, such as the class of all sets. The notion of a category generalizes the idea of a class of structures with structure perserving maps between them.

Definition 1.1. A *category* $\mathcal{C}$ is the data of

(*i*) a class of *objects*, also denoted as $\mathcal{C}$ by abuse of notation
(*ii*) for each pair of objects $X, Y \in \mathcal{C}$, a class $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ of *morphisms* from X to Y, where by a morphism from X to Y, denoted $f : X \to Y$, we mean a triple (X, Y, f) where X is called its *domain*, Y its *codomain*, and f its *mapping*, abusing notation.
(*iii*) for each triple of object $X, Y, Z \in \mathcal{C}$, a *composition* function

$$\begin{aligned}\mathrm{Hom}_{\mathcal{C}}(Y, Z) \times \mathrm{Hom}_{\mathcal{C}}(X, Y) &\to \mathrm{Hom}_{\mathcal{C}}(X, Z)\\ (f, g) &\mapsto f \circ g\end{aligned}$$

such that the following axioms are satisfied

- *identity axiom*: for each $X \in \mathcal{C}$, there exists an *identity morphism* $\mathrm{id}_X \in \mathrm{Hom}_{\mathcal{C}}(X, X)$ such that for all $Y, Z \in \mathcal{C}$ and for all $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ and $g \in \mathrm{Hom}_{\mathcal{C}}(Z, X)$ we have
$$f \circ \mathrm{id}_X = f \quad \text{and} \quad \mathrm{id}_X \circ g = g$$
- *associativity axiom*: for each quadruple of objects $X, Y, Z, W \in \mathcal{C}$ and $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$, $g \in \mathrm{Hom}_{\mathcal{C}}(Y, Z)$, and $h \in \mathrm{Hom}_{\mathcal{C}}(Z, W)$, we have
$$(h \circ g) \circ f = h \circ (g \circ f)$$

When the category $\mathcal{C}$ is clear from context, we write $\mathrm{Hom}(X, Y)$ for $\mathrm{Hom}_{\mathcal{C}}(X, Y)$

Remark 1.2. It is an easy exercise to show the identity morphism for an object is unique.

Example 1.3. In Table 1 we provide a list of common categories.

name of category	*notation*	*objects*	*morphisms*
category of sets	**Set**	sets	functions
category of groups	**Grp**	groups	group homomorphisms
category of abelian groups	**Ab**	abelian groups	group homomorphisms
category of rings	**Ring**	rings	ring homomorphisms
category of algebras over R	$\mathbf{Alg}_R$	R-algebras	R-algebra homomorphisms
category of topological spaces	**Top**	topological spaces	continuous functions
category of vector spaces over K	$\mathbf{Mod}_K$	K-vector spaces	K-linear maps
category of modules over R	$\mathbf{Mod}_R$	R-modules	R-module homomorphisms

TABLE 1. Table of some common categories

Convention 1.4. Unless otherwise specified, by a ring we mean a commutative unital ring, and an R-algebra over a ring R will always mean a commutative, unital, and associative R-algebra.

Definition 1.5. A *subcategory* of a category $\mathcal{C}$ is a category $\mathcal{D}$ such that its objects $\mathcal{D} \subseteq \mathcal{C}$ and its morphisms $\mathrm{Hom}_{\mathcal{D}}(X, Y) \subseteq \mathrm{Hom}_{\mathcal{C}}(X, Y)$ for any $X, Y \in \mathcal{D}$, with the same composition function. We say $\mathcal{D}$ is a *full subcategory* if further that $\mathrm{Hom}_{\mathcal{D}}(X, Y) = \mathrm{Hom}_{\mathcal{C}}(X, Y)$ for any $X, Y \in \mathcal{D}$.

Definition 1.6. Suppose $\mathcal{C}, \mathcal{D}$ are categories, their product category $\mathcal{C} \times \mathcal{D}$ is the category where objects are pairs (X, Y) where $X \in \mathcal{C}$ and $Y \in \mathcal{D}$, and morphisms are

$$\mathrm{Hom}((X_1, Y_1), (X_2, Y_2)) = \mathrm{Hom}(X_1, X_2) \times \mathrm{Hom}(Y_1, Y_2)$$

with element-wise composition.

Definition 1.7. Let $f : X \to Y$ be a morphism in a category then we say f is a

(*i*) *monomorphism* or *mono* if $f \circ g = f \circ h$ implies $g = h$ for any $g, h : Z \to X$,
(*ii*) *epimorphism* or *epi* if $g \circ f = h \circ f$ implies $g = h$ for any $g, h : Y \to Z$
(*iii*) *split monomorphism* or *split mono* if there exists $g : Y \to X$ such that $g \circ f = \mathrm{id}_X$,
(*iv*) *split epimorphism* or *split epi* if there exists $g : Y \to X$ such that $f \circ g = \mathrm{id}_Y$,
(*v*) *bimorphism* if it's both a monomorphism and an epimorphism,
(*vi*) *isomorphism* if it's both a split monomorphism and a split epimorphism,
(*vii*) *endomorphism* if $X = Y$,
(*viii*) *automorphism* if it's both an isomorphism and an endomorphism.

Convention 1.8. For an object X in a category $\mathcal{C}$, we will denote by $\mathrm{End}(X) = \mathrm{Hom}(X, X)$ the endomorphisms of X, and we will denote by $\mathrm{Aut}(X)$ the group of automorphisms of X. If there is an isomorphism $f : X \to Y$ in $\mathcal{C}$, we say X and Y are isomorphic and write $X \cong Y$.

Remark 1.9. It's an easy exercise to show split monos (resp. split epis) are monos (resp. epis). Moreover, suppose $\mathcal{C}$ is a category where one can talk about injective and surjective morphisms, in general, split mono (resp. split epi) is a strictly stronger condition than injective (resp. surjective), and injective (resp. surjective) is a strictly stronger condition than mono (resp. epi).

Definition 1.10. Define the *opposite category* $\mathcal{C}^{\mathrm{op}}$ of a category $\mathcal{C}$ to be the category with the same objects as $\mathcal{C}$ but with its morphisms $\mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(X, Y) = \mathrm{Hom}_{\mathcal{C}}(Y, X)$ for $X, Y \in \mathcal{C}^{\mathrm{op}}$. For each $f : X \to Y$ in $\mathcal{C}$, we denote by $f^{\mathrm{op}} : Y \to X$ the corresponding *opposite morphism* in $\mathcal{C}^{op}$, that is, it has the same mapping as f, but has its domain and codomain swapped.

Definition 1.11. Let $\mathcal{C}, \mathcal{D}$ be categories, a *covariant functor* (or just simply a *functor*) from $\mathcal{C}$ to $\mathcal{D}$, denoted $\mathscr{F} : \mathcal{C} \to \mathcal{D}$, is the collection of the following data

(*i*) for each object $X \in \mathcal{C}$, an object $\mathscr{F}(X) \in \mathcal{D}$
(*ii*) for each morphism $f : X \to Y$ in $\mathcal{C}$, a morphism $\mathscr{F}[f] : \mathscr{F}(X) \to \mathscr{F}(Y)$, such that
  - $\mathscr{F}[\mathrm{id}_X] = \mathrm{id}_{\mathscr{F}(X)}$ for each $X \in \mathcal{C}$,
  - $\mathscr{F}[f \circ g] = \mathscr{F}[f] \circ \mathscr{F}[g]$ for each $f : X \to Y$ and $g : Y \to Z$ in $\mathcal{C}$.

A *contravariant functor* from $\mathcal{C}$ to $\mathcal{D}$ is a covariant functor $\mathscr{F} : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$.

Definition 1.12. Suppose $\mathcal{C}, \mathcal{D}, \mathcal{E}$ are categories and $\mathscr{F} : \mathcal{C} \to \mathcal{D}$ and $\mathscr{G} : \mathcal{D} \to \mathcal{E}$ are functors, we define their *composite functor* $\mathscr{G} \circ \mathscr{F} : \mathcal{C} \to \mathcal{E}$ as the functor that maps $X \mapsto \mathscr{G}(\mathscr{F}(X))$ for objects and maps the morphisms by $(\mathscr{G} \circ \mathscr{F})[f] = \mathscr{G}[\mathscr{F}[f]]$ for each $f : X \to Y$.

Definition 1.13. Let $\mathcal{C}$ be a category, then the *identity functor* $\mathrm{id}_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ for $\mathcal{C}$ is the functor that maps each object $X \in \mathcal{C}$ by $\mathrm{id}_{\mathcal{C}}(X) = X$ and each morphism $f : X \to Y$ by $\mathrm{id}_{\mathcal{C}}[f] = f$.

Example 1.14. Here are some examples of functors in nature

(*i*) the functor $\mathbf{Grp} \to \mathbf{Set}$ that maps a group to its underlying set and sends morphisms to themselves (functor that "forget" data such as this are called *forgetful functors*)
(*ii*) the functor $(-)^\times : \mathbf{Ring} \to \mathbf{Grp}$ which sends a ring R to its multiplicative group of units, and sends a morphism of rings to its restriction on the groups of units,
(*iii*) the functor $\mathrm{GL}_n(-) : \mathbf{Ring} \to \mathbf{Grp}$ which sends a ring R to $\mathrm{GL}_n(R)$ the group of invertible matrices in R, and send a morphism to the obvious entry-wise group homomorphism.
(*iv*) the functor $(- \otimes_R M) : \mathbf{Mod}_R \to \mathbf{Mod}_R$ for an R-module M, which sends a R-module N to $N \otimes_R M$ and a morphism $f : N \to P$ to $f \otimes \mathrm{id}_M$,
(*v*) the contravariant functor $(-)^* : \mathbf{Mod}_R^{\mathrm{op}} \to \mathbf{Mod}_R$ for a ring R which sends an R-module M to its dual module M^*, and a morphism $f^{\mathrm{op}} : M \to N$ to $f^* : N^* \to M^*$ by $g \mapsto g \circ f$,
(*vi*) the functor $\pi_1(-) : \mathbf{PCTop} \to \mathbf{Grp}$ where $\mathbf{PCTop}$ is the full subcategory of $\mathbf{Top}$ of path connected spaces, which sends a space X to its fundamental group $\pi_1(X)$ and a continuous map $f : X \to Y$ to its induced map on fundamental groups $f_* : \pi_1(X) \to \pi_1(Y)$.

Definition 1.15. We say that the category $\mathcal{C}$ is *locally small* if $\mathrm{Hom}(X, Y)$ is a set for all X and Y, and *small* if it is locally small and the class of objects of $\mathcal{C}$ is also a set.

Definition 1.16. Suppose $\mathcal{C}$ is locally small and $X \in \mathcal{C}$. Define the *(covariant) hom-functor* of X

$$\begin{aligned} \mathrm{Hom}(X, -) : \mathcal{C} &\to \mathbf{Set} \\ Y &\mapsto \mathrm{Hom}(X, Y) \\ (f : Y \to Z) &\mapsto (f \circ -) \end{aligned}$$

where $(f \circ -) : \mathrm{Hom}(X, Y) \to \mathrm{Hom}(X, Z)$ maps $g \mapsto f \circ g$. The *contravariant hom-functor* of X is

$$\begin{aligned} \mathrm{Hom}(-, X) : \mathcal{C}^{\mathrm{op}} &\to \mathbf{Set} \\ Y &\mapsto \mathrm{Hom}(Y, X) \\ (f^{\mathrm{op}} : Z \to Y) &\mapsto (- \circ f) \end{aligned}$$

where $(- \circ f) : \mathrm{Hom}(Z, X) \to \mathrm{Hom}(Y, X)$ maps $g \mapsto g \circ f$. The *bivariate hom-functor* of X is

$$\begin{aligned} \mathrm{Hom}(-, -) : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} &\to \mathcal{D} \\ (X, Y) &\mapsto \mathrm{Hom}(X, Y) \\ (f^{\mathrm{op}}, g) : (X_1, Y_1) \to (X_2, Y_2) &\mapsto (g \circ - \circ f) \end{aligned}$$

where $(g \circ - \circ f) : \mathrm{Hom}(X_1, Y_1) \to \mathrm{Hom}(X_2, Y_2)$ maps $h \mapsto g \circ h \circ f$

Remark 1.17. Suppose $\mathcal{C}$ is locally small and $X \in \mathcal{C}$. We also use the notations

$$\mathscr{H}^X := \mathrm{Hom}(X, -) \qquad \mathscr{H}_X := \mathrm{Hom}(-, X) \qquad \mathscr{H} := \mathrm{Hom}(-, -)$$

for the covariant, contravariant, and bivariate hom-functors.

Definition 1.18. Let $\mathcal{I}, \mathcal{C}$ be categories, a *diagram* indexed by $\mathcal{I}$ in $\mathcal{C}$ is simply a functor $\mathscr{F} : \mathcal{I} \to \mathcal{C}$. We say that the diagram $\mathscr{F}$ *commutes* if for each $X, Y \in \mathcal{I}$, if $f, g \in \mathrm{Hom}(X, Y)$ then $\mathscr{F}[f] = \mathscr{F}[g]$.

Example 1.19. We represent a diagram $\mathscr{F} : \mathcal{I} \to \mathcal{C}$ as a directed multigraph in the same shape as the index category $\mathcal{I}$, and label the vertices and edges with their images of $\mathscr{F}$ in $\mathcal{C}$. For example,

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "h"'] & B \arrow[d, "g"] \\ C \arrow[r, "e"'] & D \end{tikzcd}$$

and this square commutes if and only if $g \circ f = e \circ h$.

Remark 1.20. Functors preserve commutative diagrams. More precisely, let $\mathcal{C}, \mathcal{D}$ be categories and $\mathscr{F} : \mathcal{C} \to \mathcal{D}$ a functor. Let $\mathscr{D} : \mathcal{I} \to \mathcal{C}$ be a commutative diagram in $\mathcal{C}$, then $\mathscr{F} \circ \mathscr{D} : \mathcal{I} \to \mathcal{D}$ is a commutative diagram in $\mathcal{D}$. This is straightforward from the definition of commutative diagrams.

Definition 1.21. Let $\mathcal{C}, \mathcal{D}$ be categories, and $\mathscr{F}, \mathscr{G} : \mathcal{C} \to \mathcal{D}$ be functors, a *natural transformation* from $\mathscr{F}$ to $\mathscr{G}$, denoted $\varphi : \mathscr{F} \Rightarrow \mathscr{G}$, is the data of a morphism $\varphi_X : \mathscr{F}(X) \to \mathscr{G}(X)$ for each $X \in \mathcal{C}$ such that the following diagram commutes

$$\begin{tikzcd} \mathscr{F}(X) \arrow[r, "\varphi_X"] \arrow[d, "\mathscr{F}[f]"'] & \mathscr{G}(X) \arrow[d, "\mathscr{G}[f]"] \\ \mathscr{F}(Y) \arrow[r, "\varphi_Y"'] & \mathscr{G}(Y) \end{tikzcd}$$

for all morphisms $f : X \to Y$ in $\mathcal{C}$. In other words, $\mathscr{G}[f] \circ \varphi_X = \varphi_Y \circ \mathscr{F}[f]$ for all $f : X \to Y$ in $\mathcal{C}$. Moreover, if φ_X is an isomorphism for each $X \in \mathcal{C}$, we say φ is a *natural isomorphism*.

Definition 1.22. Let $\mathcal{C}, \mathcal{D}$ be categories, and let $\mathscr{E}, \mathscr{F}, \mathscr{G} : \mathcal{C} \to \mathcal{D}$ be functors. Suppose $\varphi : \mathscr{E} \Rightarrow \mathscr{F}$ and $\psi : \mathscr{F} \Rightarrow \mathscr{G}$ are natural transformations. We define their *(vertical) composition* $\psi \circ \varphi : \mathscr{E} \Rightarrow \mathscr{G}$ as the natural transformation $(\psi \circ \varphi)_X = \psi_X \circ \varphi_X$ for every $X \in \mathcal{C}$.

Definition 1.23. Let $\mathcal{C}, \mathcal{D}, \mathcal{E}$ be categories and $\mathscr{F}_1, G_1 : \mathcal{C} \to \mathcal{D}$ and $\mathscr{F}_2, G_2 : \mathcal{D} \to \mathcal{E}$ functors. Let $\varphi : \mathscr{F}_1 \Rightarrow \mathscr{G}_1$ and $\psi : \mathscr{F}_2 \Rightarrow \mathscr{G}_2$ be natural transformations, then define their *horizontal composition* $\psi * \varphi : \mathscr{F}_2 \circ \mathscr{F}_1 \Rightarrow \mathscr{G}_2 \circ \mathscr{G}_1$ as $(\psi * \varphi)_X = \psi_{G_1(X)} \circ \mathscr{F}_2[\varphi_X]$.

Definition 1.24. Let $\mathcal{C}, \mathcal{D}$ be categories and $\mathscr{F} : \mathcal{C} \to \mathcal{D}$ a functor, then define the *identity natural transformation* $\mathrm{id}_{\mathscr{F}} : \mathscr{F} \Rightarrow \mathscr{F}$ as $(\mathrm{id}_{\mathscr{F}})_X = \mathrm{id}_{\mathscr{F}(X)}$ for each $X \in \mathcal{C}$.

Example 1.25. Here are some examples of natural transformations

(*i*) let $(-)^\times, \mathrm{GL}_n : \mathbf{Ring} \to \mathbf{Grp}$ be functors defined in Example 1.14, then we have the natural transformation $\det : \mathrm{GL}_n(-) \Rightarrow (-)^\times$ where $\det_R : \mathrm{GL}_n(R) \to R^\times$ is the determinant,

(*ii*) let $(-)^{**} = ((-)^*)^* : \mathbf{Mod}_K \to \mathbf{Mod}_K$ be the composition of the dual space functor with itself, then there is natural transformation $\mathrm{eval} : \mathrm{id}_{\mathbf{Mod}_K} \Rightarrow (-)^{**}$ given by $\mathrm{eval}_V : V \to V^{**}$ where $\mathrm{eval}_V(v)(f) = f(v)$ is the evaluation map; it is a natural isomorphism if we replace $\mathbf{Mod}_K$ with $\mathbf{FVect}_K$, its full subcategory of finite dimensional vector spaces,

(*iii*) let $(-)^* \otimes_R M : \mathbf{Mod}_R^{\mathrm{op}} \to \mathbf{Mod}_R$ be the composition of $(-)^*$ with $(-) \otimes_R M$ defined in Example 1.14, for a ring R and an R-module M; let $\mathrm{Hom}(-, M) : \mathbf{Mod}_R^{\mathrm{op}} \to \mathbf{Mod}_R$ be the contravariant hom-functor valued in $\mathbf{Mod}_R$ (with the natural module structure inherited from M), there is a natural isomorphism $\varphi : (-)^* \otimes_R M \Rightarrow \mathrm{Hom}(-, M)$ where for R-module N, the map $\varphi_N : N^* \otimes_R M \to \mathrm{Hom}(N, M)$ is given by $\varphi_N(f \otimes m)(n) = f(n)m$,

Definition 1.26. Let $\mathcal{C}, \mathcal{D}$ be categories, the *functor category* from $\mathcal{C}$ to $\mathcal{D}$, denoted $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$, is the category where objects are functors $\mathscr{F} : \mathcal{C} \to \mathcal{D}$, morphisms are natural transformations, and composition is given by (vertical) composition of natural transformations.

Remark 1.27. Natural isomorphisms are precisely the isomorphisms in the functor category.

Definition 1.28. Let $\mathcal{C}, \mathcal{D}$ be categories, a functor $\mathscr{F} : \mathcal{C} \to \mathcal{D}$ is called an *equivalence* if there is a functor $\mathscr{G} : \mathcal{D} \to \mathcal{C}$ such that there are natural isomorphisms $\eta : \mathrm{id}_{\mathcal{C}} \Rightarrow \mathscr{G} \circ F$ and $\varepsilon : \mathscr{F} \circ G \Rightarrow \mathrm{id}_{\mathcal{D}}$. If there is an equivalence between $\mathcal{C}$ and $\mathcal{D}$, we say they are *equivalent* and write $\mathcal{C} \simeq \mathcal{D}$. Moreover, we call $\mathscr{G}$ the *quasi-inverses* of $\mathscr{F}$ and call the pair of functors an *equivalence of categories*.

Example 1.29. Here are some examples of equivalent categories

(*i*) The category $\mathbf{FVect}_K$ of finite dimensional K-vector spaces is equivalent to the category where objects are K^n for $n \geqslant 0$ and morphisms $\mathrm{Hom}(K^n, K^m)$ are the matrices $\mathrm{M}_{n,m}(K)$.

(*ii*) The category of intemediate extensions of a finite Galois extension $L \mid K$ with L-embeddings as morphisms, is equivalent to the category of finite transitive G-sets where $G = \mathrm{Gal}(L \mid K)$.

(*iii*) The category of complex representations of a finite group G (whose morphisms are equivariant maps) is equivalent to the category of $\mathbb{C}[G]$-modules, and they are also equivalent to $\mathrm{Fun}(\mathrm{B}G, \mathbf{Vect}_{\mathbb{C}})$, where $\mathrm{B}G$ is the category with a single object $\bullet$, with the endomorphisms of $\bullet$ being group elements, and with the group operation as its composition.

It is an easy exercise for the reader to spot what the unstated equivalences in the cases above are.

Definition 1.30. Suppose $\mathcal{C}, \mathcal{D}$ are categories and $\mathscr{F} : \mathcal{C} \to \mathcal{D}$ is a functor. For each $X, Y \in \mathcal{C}$, then

$$\mathrm{Hom}(X, Y) \xrightarrow{f \mapsto \mathscr{F}[f]} \mathrm{Hom}(\mathscr{F}(X), \mathscr{F}(Y))$$

is a map of sets. We say that $\mathscr{F}$ is

(*i*) *faithful* if the above map is injective for all $X, Y \in \mathcal{C}$,

(*ii*) *full* if the above map is surjective for all $X, Y \in \mathcal{C}$,

(*iii*) *fully faithful* if the above map is bijective for all $X, Y \in \mathcal{C}$, Moreover, call $\mathscr{F}$ *essentially surjective* if for all $Y \in \mathcal{D}$ exists $X \in \mathcal{C}$ such that $\mathscr{F}(X) \cong Y$ in $\mathcal{D}$.

Theorem 1.31. *A functor is an equivalence iff it is fully faithful and essentially surjective.*

Proof. Technical. See [KS06, Thm. 1.3.13, p. 22]. $\square$

Definition 1.32. Let $\mathcal{C}$ be a locally small category. Define

$$\widehat{\mathcal{C}} := \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathbf{Set}) \qquad \check{\mathcal{C}} := \mathrm{Fun}(\mathcal{C}, \mathbf{Set})$$

as the *category of presheaves* of $\mathcal{C}$ and *category of copresheaves* of $\mathcal{C}$ respectively. The *Yoneda embedding* is the contravariant functor

$$\begin{aligned} \mathscr{H}^\bullet : \mathcal{C}^{\mathrm{op}} &\to \check{\mathcal{C}} \\ X &\mapsto \mathscr{H}^X = \mathrm{Hom}(X, -) \\ (f^{\mathrm{op}} : Z \to Y) &\mapsto [X \mapsto (- \circ f)] \end{aligned}$$

where $(- \circ f) : \mathrm{Hom}(Z, X) \to \mathrm{Hom}(Y, X)$ is as in Definition 1.16. Dually, the *co-Yoneda embedding* is

$$\begin{aligned} \mathscr{H}_\bullet : \mathcal{C} &\to \widehat{\mathcal{C}} \\ X &\mapsto \mathscr{H}_X = \mathrm{Hom}(-, X) \\ (f : Y \to Z) &\mapsto [X \mapsto (f \circ -)] \end{aligned}$$

where $(f \circ -) : \mathrm{Hom}(X, Y) \to \mathrm{Hom}(X, Z)$ is as in Definition 1.16.

Lemma 1.33 (Yoneda)**.** *Let $\mathcal{C}$ be a locally small category. Suppose $X \in \mathcal{C}$ and $\mathscr{F} : \mathcal{C} \to \mathbf{Set}$ is a functor, then there are bijections Φ and Ψ, shown below,*

$$\begin{tikzcd} \mathrm{Hom}_{\check{\mathcal{C}}}(\mathscr{H}^X, \mathscr{F}) \arrow[r, bend left, "\Phi"] & \mathscr{F}(X) \arrow[l, bend left, "\Psi"] \end{tikzcd}$$

which are inverses of each other.

Proof. For a natural transformation $\varphi : \mathscr{H}^X \Rightarrow \mathscr{F}$ and a morphism $f : X \to Y$ in $\mathcal{C}$, the diagram

$$\begin{tikzcd} \mathrm{Hom}_{\mathcal{C}}(X, X) \arrow[r, "\varphi_X"] \arrow[d, "(f\circ -)"'] & \mathscr{F}(X) \arrow[d, "\mathscr{F}[f]"] \\ \mathrm{Hom}_{\mathcal{C}}(X, Y) \arrow[r, "\varphi_Y"] & \mathscr{F}(Y) \end{tikzcd}$$

commutes. Thus $\varphi_Y \circ (f \circ -) = \mathscr{F}[f] \circ \varphi_X$. Evaluating at id_X on both sides yields the identity

$$\varphi_Y(f) = \mathscr{F}[f](\varphi_X(\mathrm{id}_X))$$

therefore the natural transformation φ is completely determined by $u = \varphi_X(\mathrm{id}_X)$. Therefore, let

$$\Phi : \mathrm{Hom}_{\check{\mathcal{C}}}(\mathscr{H}^X, \mathscr{F}) \to \mathscr{F}(X) \qquad \Phi(\varphi) = \varphi_X(\mathrm{id}_X)$$

and, by the same identity, the inverse of Φ is naturally

$$\Psi : \mathscr{F}(X) \to \mathrm{Hom}_{\check{\mathcal{C}}}(\mathscr{H}^X, \mathscr{F}) \qquad \Psi(u) = \varphi$$

where $\varphi_Y : \mathrm{Hom}(X, Y) \to \mathscr{F}(Y)$ is given by $\varphi_Y(f) = (\mathscr{F}[f])(u)$, for each $Y \in \mathcal{C}$. It is not hard for the reader to verify that indeed Ψ and Φ are inverses of each other. $\square$

Remark 1.34. The preceding Lemma 1.33 is known as *Yoneda lemma* or the *fundamental theorem of category theory*. Notice that the definition of Φ and Ψ does not involve in X and $\mathscr{F}$, therefore one can upgrade the lemma to say that there is a mutually inverse pair of natural isomorphism

$$\begin{tikzcd} \mathrm{Hom}_{\check{\mathcal{C}}}(\mathscr{H}^\bullet, -) \arrow[r, Rightarrow, bend left, "\Phi"] & (-)(\bullet) \arrow[l, Rightarrow, bend left, "\Psi"] \end{tikzcd}$$

where the left hand side functor $\mathrm{Hom}_{\check{\mathcal{C}}}(\mathscr{H}^\bullet, -) : \mathcal{C} \times \check{\mathcal{C}} \to \mathbf{Set}$ is given by the composition

$$\mathcal{C} \times \check{\mathcal{C}} \xrightarrow{\mathscr{H}^\bullet \times \mathrm{id}_{\check{\mathcal{C}}}} \check{\mathcal{C}} \times \check{\mathcal{C}} \xrightarrow{\mathrm{Hom}_{\check{\mathcal{C}}}(-,-)} \mathbf{Set}$$

and the right hand side functor, called the *evaluation functor*, is given by

$$(-)(\bullet) : \mathcal{C} \times \check{\mathcal{C}} \to \mathbf{Set} \qquad (X, \mathscr{F}) \mapsto \mathscr{F}(X)$$

and sends a morphism $(f : X \to Y, \varphi : \mathscr{F} \Rightarrow \mathscr{G})$ to the morphism $\varphi_Y \circ \mathscr{F}[f] = \mathscr{G}[f] \circ \varphi_X$. Same constructions for Φ and Ψ will work, hence we have shown that the Yoneda lemma is *functorial* or *natural* in the variables $X \in \mathcal{C}$ and $\mathscr{F} \in \check{\mathcal{C}}$.

Exercise 1.35. Prove a dual and functorial version of the Yoneda lemma: let $\mathcal{C}$ be a locally small category. Suppose $X \in \mathcal{C}$ and $\mathscr{F} : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$, then construct natural isomorphisms

$$\begin{tikzcd} \mathrm{Hom}_{\hat{\mathcal{C}}}(\mathscr{H}_\bullet, -) \arrow[r, Rightarrow, bend left, "\Pi"] & (-)(\bullet) \arrow[l, Rightarrow, bend left, "\Theta"] \end{tikzcd}$$

which are inverses of each other, where you should define the functor $\mathrm{Hom}_{\hat{\mathcal{C}}}(\mathscr{H}_\bullet, -) : \mathcal{C} \times \hat{\mathcal{C}} \to \mathbf{Set}$ and the *coevaluation functor* $(-)(\bullet) : \mathcal{C} \times \hat{\mathcal{C}} \to \mathbf{Set}$ dually to those in Remark 1.34.

Theorem 1.36. *Let $\mathcal{C}$ be a locally small category, then the Yoneda and co-Yoneda embeddings*

$$\mathscr{H}^\bullet : \mathcal{C}^{\mathrm{op}} \to \check{\mathcal{C}} \quad \text{and} \quad \mathscr{H}_\bullet : \mathcal{C} \to \hat{\mathcal{C}}$$

are fully faithful.

Proof. We will only show $\mathscr{H}^\bullet$ is fully faithful: for the dual $\mathscr{H}_\bullet$, the proof is completely symmetric. Let $X, Y \in \mathcal{C}$. By the Yoneda lemma Lemma 1.33, there is a bijection

$$\Phi : \mathrm{Hom}_{\check{\mathcal{C}}}(\mathscr{H}^X, \mathscr{H}^Y) \to \mathscr{H}^Y(X) = \mathrm{Hom}_{\mathcal{C}}(Y, X)$$

by specifying $\mathscr{F} = \mathscr{H}^Y$. One can verify that this gives the inverse for the map

$$\mathrm{Hom}_{\mathcal{C}}(Y, X) = \mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(X, Y) \xrightarrow{f \mapsto \mathscr{H}^\bullet[f]} \mathrm{Hom}_{\check{\mathcal{C}}}(\mathscr{H}^X, \mathscr{H}^Y)$$

Thus $\mathscr{H}^\bullet$ is fully faithful. $\square$

Lemma 1.37. *Let $\mathcal{C}, \mathcal{D}$ be categories and $\mathscr{F} : \mathcal{C} \to \mathcal{D}$ a fully faithful functor, then the morphism $\mathscr{F}[f] : \mathscr{F}(X) \to \mathscr{F}(Y)$ is an isomorphism in $\mathcal{D}$ if and only if $f : X \to Y$ is an isomorphism in $\mathcal{C}$.*

Proof. Assume $f : X \to Y$ is an isomorphism in $\mathcal{C}$, then there exists $g : Y \to X$ with $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$. Applying functor $\mathscr{F}$, we get $\mathscr{F}[g] \circ \mathscr{F}[f] = \mathrm{id}_{\mathscr{F}(X)}$ and $\mathscr{F}[f] \circ \mathscr{F}[g] = \mathrm{id}_{\mathscr{F}(Y)}$, hence $\mathscr{F}[f]$ has inverse $\mathscr{F}[g]$, which makes it an isomorphism in $\mathcal{D}$. Conversely, by fullness of $\mathscr{F}$, choose $g : Y \to X$ with $\mathscr{F}[g] = \mathscr{F}[f]^{-1}$. Then we have $\mathscr{F}[f \circ g] = \mathscr{F}[f] \circ \mathscr{F}[g] = \mathrm{id}_{\mathscr{F}(X)}$ so by faithfulness of $\mathscr{F}$, we have $f \circ g = \mathrm{id}_X$. By symmetric arguments, we have $g \circ f = \mathrm{id}_Y$, so g is the inverse of f. Therefore f is an isomorphism in $\mathcal{C}$. $\square$

Theorem 1.38. *Let $\mathcal{C}$ be a locally small category, then for a morphism $f : X \to Y$ in $\mathcal{C}$, TFAE*

(i) $f : X \to Y$ *is an isomorphism in* $\mathcal{C}$,

(ii) $\mathscr{H}_\bullet[f] : \mathscr{H}_X \to \mathscr{H}_Y$ *is an isomorphism in* $\widehat{\mathcal{C}}$,

(iii) $\mathscr{H}^\bullet[f^{\mathrm{op}}] : \mathscr{H}^Y \to \mathscr{H}^X$ *is an isomorphism in* $\check{\mathcal{C}}$,

Proof. Straightforward by Theorem 1.36 and Lemma 1.37. $\square$

2. Algebraic Groups

Throughout the text, unless specified otherwise, fix a ring R, which we shall call our *ground ring*.

Convention 2.1. We assume rings are commutative and unital, and algebras are associative, commutative, and unital. In particular, in our convention an algebra A over a ring R is equivalent to the data of a ring A with a ring map $\varphi : R \to A$, viewing the scalar multiplication as $r \cdot a = \varphi(r)a$. We will use this equivalence frequently and often implicitly.

Definition 2.2. A *R-functor* is a functor

$$X : \mathbf{Alg}_R \to \mathbf{Set}$$

and the *category of R-functors* is the functor category $\mathbf{Fun}_R := \mathrm{Fun}(\mathbf{Alg}_R, \mathbf{Set})$.

Definition 2.3. Suppose A is a R-algebra, its *spectrum* is the R-functor

$$\mathrm{Spec}(A) := \mathrm{Hom}_{\mathbf{Alg}_R}(A, -) : \mathbf{Alg}_R \to \mathbf{Set}$$

An *affine R-scheme* is the spectrum of a R-algebra. The *affine n-space* is the affine R-scheme

$$\mathbb{A}^n_R := \mathrm{Spec}(R[x_1, \ldots, x_n])$$

The category $\mathbf{Aff}_R$ of affine R-schemes is the full subcategory of all affine R-schemes of $\mathbf{Fun}_R$.

Remark 2.4. The notion of affine R-schemes generalizes the notion of affine varieties. For readers who have experience in algebraic geometry, you might have been taught that an affine K-variety for some algebraically closed field K is a subset of K^n for some n given by the vanishing set of some polynomials $f_1, \ldots, f_m \in K[x_1, \ldots, x_n]$, that is, the set

$$\{(x_1, \ldots, x_n) \in K^n : f_1(x_1, \ldots, x_n) = \cdots = f_m(x_1, \ldots, x_n) = 0\}$$

For our purposes, however, we take a *functorial* perspective to varieties. To see what this is about, we first make the important observation that the above set can be naturally identified as the set

$$X(K) = \mathrm{Hom}_{\mathbf{Alg}_K}\left(\frac{K[x_1, \ldots, x_n]}{(f_1, \ldots, f_m)}, K\right) \quad \text{where} \quad X = \mathrm{Spec}\left(\frac{K[x_1, \ldots, x_n]}{(f_1, \ldots, f_m)}\right)$$

Namely, each $\phi \in X(K)$ can be identified with $(\phi(x_1), \ldots, \phi(x_n)) \in K^n$. Similarly, for polynomials $f_1, \ldots, f_m \in R[x_1, \ldots, x_n]$, the set $X(S)$ for an R-algebra S can be identified as the set

$$\{(x_1, \ldots, x_n) \in S^n : f_1(x_1, \ldots, x_n) = \cdots = f_m(x_1, \ldots, x_n) = 0\}$$

where $f_1, \ldots, f_m$ are identified as their images along the natural map $R[x_1, \ldots, x_n] \to S[x_1, \ldots, x_n]$ induced by the structure map $R \to S$. Thus, what the spectrum of a R-algebra encodes is the vanishing sets of a set of polynomials over each R-algebra.

Definition 2.5. Suppose that $X = \mathrm{Spec}(A)$ is an affine R-scheme where A is a R-algebra. Let S be another R-algebra then we call the set

$$X(S) = \mathrm{Hom}_{\mathbf{Alg}_R}(A, S)$$

the *S-points* of X. When S is a field, it is also called the *S-rational points* of X.

Remark 2.6. Recall that a R-algebra A is said to be finitely generated if there is a surjective map $R[x_1, \ldots, x_n] \to A$, or equivalently $A = R[x_1, \ldots, x_n]/I$ for some ideal $I \subseteq K[x_1, \ldots, x_n]$. If R is noetherian, that is, each of its ideals is finitely generated, then by Hilbert's basis theorem, so is $R[x_1, \ldots, x_n]$, whence the finitely generated R-algebras are of the form $R[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$.

Remark 2.7. One huge advantage that affine schemes provide is their treatment of *nilpotence*. The vanishing set of a polynomial $f \in K[x_1, \ldots, x_n]$ for K a field is the same as that of f^2. However,

$$\mathrm{Spec}(K[x_1, \ldots, x_n]/(f)) \neq \mathrm{Spec}(K[x_1, \ldots, x_n]/(f^2))$$

with the right hand side being thought of as having an *infinitesimal thickening* or *nilpotent thickening*.

Remark 2.8. Let S be a R-algebra, then $\mathbb{A}^1_R(S) = \mathrm{Hom}_{\mathbf{Alg}_R}(R[x], S)$ is canonically a R-algebra by

$$(\phi + \psi)(f) = \phi(f) + \psi(f)$$
$$(\phi\psi)(f) = \phi(f)\psi(f)$$
$$(r\phi)(f) = r\phi(f)$$

for each $\phi, \psi \in \mathbb{A}^1_R(S)$ and $r \in R$, where $f \in R[x]$. Note that naturally $\mathbb{A}^1_R(S) \cong S$ as R-algebras.

Definition 2.9. For an affine R-scheme $X : \mathbf{Alg}_R \to \mathbf{Set}$, its *coordinate ring* (resp. *coordinate algebra*) is the ring (resp. R-algebra)

$$\mathcal{O}(X) := \mathrm{Hom}_{\mathbf{Aff}_R}(X, \mathbb{A}^1_R)$$

with the algebraic operations given as follows: given $f, g \in \mathcal{O}(X)$ and $r \in R$

$$(f + g)_S(\phi) = f_S(\phi) + g_S(\phi)$$
$$(fg)_S(\phi) = f_S(\phi)f_S(\phi)$$
$$(rf)_S(\phi) = rf_S(\phi)$$

where $\phi \in X(S)$, for each R-algebra S. Moreover, define the contravariant functor

$$\mathcal{O} : \mathbf{Aff}_R^{\mathrm{op}} \to \mathbf{Alg}_R \qquad \mathcal{O} = \mathrm{Hom}_{\mathbf{Aff}_R}(-, \mathbb{A}^1_R)$$

where for a morphism $\varphi : X \to Y$ of affine R-schemes, the induced map

$$\mathcal{O}[\varphi^{\mathrm{op}}] = \varphi^{\#} : \mathcal{O}(Y) \to \mathcal{O}(X) \qquad \psi \mapsto \psi \circ \varphi$$

is called the *induced regular map* of φ.

Definition 2.10. Define the contravariant functor

$$\mathrm{Spec} : \mathbf{Alg}_R^{\mathrm{op}} \to \mathbf{Aff}_R \qquad A \mapsto \mathrm{Spec}(A)$$

where for each morphism $\varphi : A \to B$ of R-algebras, there is the induced natural transformation

$$\varphi^* : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$$

called the *pullback* given by

$$\varphi^*_S : \mathrm{Spec}(B)(S) \to \mathrm{Spec}(A)(S) \qquad \psi \mapsto \psi \circ \varphi$$

for each R-algebras S.

Theorem 2.11. *The pair of functors* Spec *and* $\mathcal{O}$ *is an equivalence of categories*

$$\begin{tikzcd} \mathbf{Alg}_R^{\mathrm{op}} \arrow[r, bend left, "\mathrm{Spec}"] & \mathbf{Aff}_R \arrow[l, bend left, "\mathcal{O}"] \end{tikzcd}$$

Proof. Suppose A is a R-algebra, then applying Lemma 1.33, namely, the Yoneda lemma

$$\mathcal{O}(\mathrm{Spec}(A)) = \mathrm{Hom}_{\mathbf{Aff}_R}(\mathscr{H}^A, \mathbb{A}^1_R) \cong \mathbb{A}^1_R(A) \cong A$$

where we leave it to the reader to check that the relevant bijection is an isomorphism of K-algebras. Conversely, let $X = \mathrm{Spec}(A)$, then there is an induced natural isomorphism of hom-functors

$$\mathrm{Spec}(\mathcal{O}(X)) = \mathrm{Hom}_{\mathbf{Alg}_R}(\mathcal{O}(\mathrm{Spec}(A)), -) \cong \mathrm{Hom}_{\mathbf{Alg}_R}(A, -) = X$$

Therefore Spec and $\mathcal{O}$ are quasi-inverses of each other. □

Definition 2.12. A *R-group functor* is a functor

$$G : \mathbf{Alg}_R \to \mathbf{Grp}$$

The *category of R-group functors* is the functor category $\mathbf{GFun}_R = \mathrm{Fun}(\mathbf{Alg}_R, \mathbf{Grp})$.

Definition 2.13. An *affine R-group scheme* is a R-group functor $G : \mathbf{Alg}_R \to \mathbf{Grp}$ such that its composition with the forgetful functor to $\mathbf{Set}$

$$\widetilde{G} : \mathbf{Alg}_R \xrightarrow{G} \mathbf{Grp} \to \mathbf{Set}$$

is an affine R-scheme. Further, we say an affine R-group scheme G is an *(affine) algebraic group* over R if $\widetilde{G}$ is *of finite type*, i.e. it is the spectrum of a finitely generated R-algebra. Define $\mathbf{GAff}_R$ and $\mathbf{GAlg}_R$, the categories of affine R-group schemes and (affine) algebraic groups over R, as full subcategories of $\mathbf{GFun}_R$ respectively.

Convention 2.14. In this text, "algebraic group" always means affine algebraic group. However, it is important to keep in mind that there are things like abelian varieties which are considered algebraic groups in broader context but are by no means affine.

Example 2.15. Here are some examples of algebraic groups over R,

(*i*) the *additive group*

$$\mathbb{G}_a : \mathbf{Alg}_R \to \mathbf{Grp} \qquad S \mapsto (S, +)$$

where $\widetilde{\mathbb{G}_a} \cong \mathbb{A}^1_R$,

(*ii*) the *multiplicative group*, also known as the 1-*torus*,

$$\mathbb{G}_m : \mathbf{Alg}_R \to \mathbf{Grp} \qquad S \mapsto S^\times$$

where $\widetilde{\mathbb{G}_m} \cong \mathrm{Spec}(R[x, x^{-1}])$,

(*iii*) the *multiplicative group of n-th roots of unity*,

$$\mu_n : \mathbf{Alg}_R \to \mathbf{Grp} \qquad S \mapsto \{a \in S^\times : a^n = 1\}$$

where $\widetilde{\mu_n} \cong \mathrm{Spec}(R[x]/(x^n - 1))$,

Definition 2.16. Let S a R-algebra. If M is an R-module, define the *base change* (or *extension of scalars*) of M from R to S as the S-module $M \otimes_R S$. If M is a S-module, define its *Weil restriction* (or *restriction of scalars*) from S to R as the R-module $\mathrm{Res}_{S/R}M$ with the same additive group as M with scalar multiplication given by $rm = \varphi(r)m$ for $r \in R$ and $m \in M$ where $\varphi : R \to S$ is the structure map. These define functors $(-) \otimes_R S : \mathbf{Mod}_R \to \mathbf{Mod}_S$ and $\mathrm{Res}_{S/R} : \mathbf{Mod}_S \to \mathbf{Mod}_R$. We define the base change and Weil restrictions for algebras in the exact same way, giving rise to functors $(-) \otimes_R S : \mathbf{Alg}_R \to \mathbf{Alg}_S$ and $\mathrm{Res}_{S/R} : \mathbf{Alg}_S \to \mathbf{Alg}_R$.

Convention 2.17. Let S a R-algebra. If M is a R-module, we also use the notation $M_S = M \otimes_R S$. If M is an S-module, we write $M_R = \mathrm{Res}_{S/R}M$.

Convention 2.18. Let $X = (x_{i,j})_{1 \leqslant i,j \leqslant n}$ be a n-by-n matrix of indeterminants. For any ring R, let

$$R[X] := R[x_{1,1}, x_{1,2}, \ldots, x_{n,n}]$$

Abusing notation, identify $X = (x_{i,j})_{1 \leqslant i,j \leqslant n} \in M_n(R[X])$. For any $F = (f_{i,j})_{1 \leqslant i,j \leqslant n} \in M_n(R[X])$, denote $\det F := \sum_{\sigma \in \mathrm{S}_n} \mathrm{sgn}(\sigma) \prod_{i=1}^n f_{i,\sigma(i)} \in R[X]$ and $\mathrm{tr}(A) := \sum_{i=1}^n f_{i,i} \in R[X]$. Moreover, we denote the ideal $(F) := (f_{1,1}, f_{1,2}, \ldots, f_{n,n}) \subseteq R[X]$.

Example 2.19. Here are some more examples of algebraic groups over R,

(*i*) the *general linear group* of a free module V over R of rank n,

$$\mathrm{GL}_V : \mathbf{Alg}_R \to \mathbf{Grp} \qquad S \mapsto \mathrm{Aut}_{\mathbf{Mod}_S}(V_S) = \mathrm{Aut}_{\mathbf{Mod}_S}(V \otimes_R S)$$

where $\widetilde{\mathrm{GL}_V} \cong \mathrm{Spec}\left(\dfrac{R[M][y]}{(y \det(M) - 1)}\right)$ where $M = (m_{i,j})_{1 \leqslant i,j \leqslant n}$ is an indeterminant matrix, (*ii*) the *special linear group* of a free module V over R of rank n,

$$\mathrm{SL}_V : \mathbf{Alg}_R \to \mathbf{Grp} \qquad S \mapsto \mathrm{Ker}\left(\mathrm{GL}_V(S) \xrightarrow{\det_S} S^\times\right)$$

where $\widetilde{\mathrm{SL}_V} \cong \mathrm{Spec}\left(\dfrac{R[M]}{(\det(M)-1)}\right)$ where $M = (m_{i,j})_{1\leqslant i,j\leqslant n}$ is an indeterminant matrix.

Convention 2.20. Let V be a free R-module of rank n. We denote $\mathrm{GL}_n := \mathrm{GL}_V$ and $\mathrm{SL}_n := \mathrm{SL}_V$.

Definition 2.21. Let V be a module over an arbitrary ring R. Suppose $\sigma : R \to R$ is a ring map satisfying $\sigma \circ \sigma = \mathrm{id}$, which we call the *involution* map. A function

$$B : V \times V \to R$$

is called *form* on V, and is said to be

(*i*) *left linear* if $B(rv + u, w) = rB(v, w) + B(u, w)$ for all $r \in R$ and $u, v, w \in V$,
(*ii*) *left σ-linear* if $B(rv + u, w) = \sigma(r)B(v, w) + B(u, w)$ for all $r \in R$ and $u, v, w \in V$,
(*iii*) *right linear* if $B(v, rw + u) = rB(v, w) + B(v, u)$ for all $r \in R$ and $u, v, w \in V$,
(*iv*) *bilinear* if it is left linear and right linear,
(*v*) *σ-sesquilinear* if it is left σ-linear and right linear,
(*vi*) *symmetric* if $B(v, w) = B(w, v)$ for all $v, w \in V$,
(*vii*) *σ-symmetric* if $B(v, w) = \sigma(B(w, v))$ for all $v, w \in V$,
(*viii*) *alternating* if $B(v, v) = 0$ for all $v \in V$,
(*ix*) *nondegenerate* if $B(v, w) = 0$ for all $v \in V$ implies $w = 0$,
(*x*) *orthogonal* if it is a bilinear, symmetric, and nondegenerate,
(*xi*) *symplectic* if it is a bilinear, alternating, and nondegenerate,
(*xii*) *σ-Hermitian* if it is σ-sesquilinear, σ-symmetric, and nondegenerate.

Definition 2.22. Let $B_1 : V \times V \to R$ and $B_2 : V \times V \to R$ be forms on a module V over an arbitrary ring R. We say B_1 is *equivalent* (or *isometric*) to B_2 if there exists an invertible linear map $P : V \to V$, which we call an *isometry*, satisfying $B_1(v, w) = B_2(P(v), P(w))$ for all $v, w \in V$.

Example 2.23. Let $V = R^n$ be the free module of rank n over a ring R with basis $e_1, \dots, e_n$. The standard inner product $\langle \cdot, \cdot \rangle : V \times V \to R$ given by $\langle v, w \rangle = v_1 w_1 + \cdots + v_n w_n$ for each $v, w \in V$, where $v = v_1 e_1 + \cdots + v_n e_n$ and $w = w_1 e_1 + \cdots + w_n e_n$, is an orthogonal form on V.

Example 2.24. Let $V = W \oplus W^\vee$ over a ring R where $W = R^n$ is the free module of rank n, then the *hyperbolic form* $B : V \times V \to R$ given by $((v, \phi), (w, \psi)) \mapsto \phi(w) + \psi(v)$ is an orthogonal form.

Example 2.25. Let $L \mid K$ be a finite separable field extension. The form $\mathrm{Tr}_{L/K} : L \times L \to K$ given by $(a, b) \mapsto \mathrm{tr}_{L/K}(ab)$, where $\mathrm{tr}_{L/K}$ is the field trace, is an orthogonal form called the *trace form*.

Example 2.26. Recall that a quadratic form on a free module $V = R^n$ of rank n over R is a quadratic homogeneous polynomial map $q : V \to R$

$$q(x_1, \dots, x_n) = \sum_{1\leqslant i,j\leqslant n} a_{i,j} x_i x_j$$

Recall the polar form $b_q : V \times V \to R$ of a quadratic form $q : V \to R$ is the symmetric bilinear form

$$b_q(x, y) = q(x + y) - q(x) - q(y)$$

and we say q is nonsingular if b_q is nondegenerate. Suppose 2 is invertible in R, then we can define the polarization $B_q = \frac{1}{2} b_q$ of q, which satisfies $q(x) = B_q(x, x)$. Therefore, when 2 is invertible in R, a quadratic form q corresponds uniquely to a symmetric bilinear form B_q.

Example 2.27. Let $V = R^{2n}$ be the free module of rank $2n$ over a ring R, the form $\omega : V \times V \to R$ given by $\omega(x, y) = x^{\mathrm{T}} J y$, where $J \in M_{2n}(R)$ is the matrix

$$J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$$

with I_n the n-by-n identity matrix, is a symplectic form called the *standard symplectic form*.

Definition 2.28. A *quadratic étale* algebra E over a ring R is a R-algebra such that

$$E \cong R[x]/(x^2 + bx + c)$$

for some $b, c \in R$ where $b^2 - 4c \in R^\times$. Define its *involution* as the unique nontrivial automorphism

$$\sigma : E \to E \qquad x \mapsto -b - x$$

which satisfies $\sigma \circ \sigma = \mathrm{id}_E$ and fixes R.

Example 2.29. Let E be a quadratic étale algebra over R with involution $\sigma : E \to E$.

Example 2.30. scaled Hermitian trace form

Proposition 2.31. *Let $B : V \times V \to R$ be a bilinear form. If B is alternating then it is antisymmetric, i.e. $B(x, y) = -B(y, x)$ for $x, y \in V$. Conversely, if B is antisymmetric and $2 \in R^\times$, then B is alternating.*

Proof. Suppose B is alternating. For $v, w \in V$, we have

$$0 = B(v + w, v + w) = B(v, v) + B(v, w) + B(w, v) + B(w, w) = B(v, w) + B(w, v)$$

Thus $B(v, w) = -B(w, v)$ for all $v, w \in V$. Conversely, suppose $2 \in R^\times$ and B is antisymmetric then for all $v \in V$ we have $B(v, v) = -B(v, v)$ so $2B(v, v) = 0$, thus $B(v, v) = 0$. □

Proposition 2.32. *A finite rank free module admitting a symplectic form is of even rank. All symplectic forms are isometric.*

Proof. Let V be a finite rank free module over a ring R with symplectic form $P : V \times V \to R$. □

Definition 2.33. Let $V = R^n$ be the free module of rank n over a ring R with an ordered basis $e = (e_1, \ldots, e_n)$. Let $B : V \times V \to R$ be a form, then the *Gram matrix* of B with respect to ordered basis e, is the matrix $[B]_e := (B(e_i, e_j))_{1 \leqslant i,j \leqslant n} \in M_n(R)$.

Proposition 2.34. *Let $V = R^n$ be the free module of rank n over a ring R with an ordered basis e. Suppose $B : V \times V \to R$ is a form. Let $\sigma : R \to R$ be an involution. We have the following.*

(i) $[B]_e^* = [B]_e$ *iff B is σ-sesquilinear and σ-symmetric,*
(ii) $[B]_e^{\mathrm{T}} = [B]_e$ *iff B is bilinear and symmetric,*
(iii) $[B]_e^{\mathrm{T}} = -[B]_e$ *iff B is bilinear and alternating,*
(iv) $B(x, y) = [x]_e^*[B]_e[y]_e$ *for all $x, y \in V$, iff B is σ-sesquilinear,*
(v) $B(x, y) = [x]_e^{\mathrm{T}}[B]_e[y]_e$ *for all $x, y \in V$, iff B is bilinear,*
(vi) $[B]_{Pe} = [P]_e^*[B]_e[P]_e$ *for any invertible linear map $P : V \to V$, iff B is σ-sesquilinear,*
(vii) $[B]_{Pe} = [P]_e^{\mathrm{T}}[B]_e[P]_e$ *for any invertible linear map $P : V \to V$, iff B is bilinear,*
(viii) *B is nondegenerate iff $[B]_e$ is invertible,*

where $M^* := (\sigma(m_{j,i})) \in M_{k\times\ell}(R)$ denotes the σ-conjugate transpose of $M = (m_{i,j}) \in M_{\ell\times k}(R)$.

Proof. □

Remark 2.35.

Definition 2.36. Let R be an arbitrary ring, suppose S is a R-algebra, V is a module over R, and $B : V \times V \to R$ a bilinear form on V. The *base change* of B from R to S is the bilinear form

$$B_S : V_S \times V_S \to S \qquad (v \otimes r, w \otimes s) \mapsto B(v, w)rs$$

on $V_S = V \otimes_R S$, extended bilinearly.

Definition 2.37. Base change for

Example 2.38. Let V be a free R-module of rank n with an orthogonal form $B : V \times V \to R$. Here are some more examples of algebraic groups over R.

(*i*) the *orthogonal group* of the pair (V, B)

$$\mathrm{O}_{V,B} : \mathbf{Alg}_R \to \mathbf{Grp}$$

where for each R-algebra S

$$\mathrm{O}_{V,B}(S) = \{g \in \mathrm{Aut}_{\mathbf{Mod}_S}(V_S) : \forall v, w \in V,\ B_S(gv, gw) = B_S(v, w)\}$$

where $\widetilde{\mathrm{O}_{V,B}} = \mathrm{Spec}\left(\dfrac{R[M]}{(M^T BM - B)}\right)$ where $M = (m_{i,j})_{1 \leqslant i,j \leqslant n}$ is an indeterminant matrix

(*ii*) the *special orthogonal group* of the pair (V, B)

(*iii*) the *orthogonal semilitude group*

(*iv*) the *special orthogonal semilitude group*

Example 2.39. Let V be a free R-module of rank $2n$ with an symplectic form $P : V \times V \to R$. Here are some more examples of algebraic groups over R.

(*i*) the *symplectic group* of the pair (V, P)

$$\mathrm{Sp}_{V,P} : \mathbf{Alg}_R \to \mathbf{Grp}$$

where for each R-algebra S

$$\mathrm{Sp}_{V,P}(S) = \{g \in \mathrm{Aut}_{\mathbf{Mod}_S}(V_S) : \forall v, w \in V,\ B_S(gv, gw) = B_S(v, w)\}$$

where $\widetilde{\mathrm{Sp}_{V,P}} = \mathrm{Spec}\left(\dfrac{R[M]}{(M^T BM - B)}\right)$ where $M = (m_{i,j})_{1 \leqslant i,j \leqslant n}$ is an indeterminant matrix

(*ii*) the *symplectic semilitude group*

Example 2.40. Unitary groups

Example 2.41. Pin group, Spin group, GPin, GSpin

Example 2.42. Exceptional groups

3. Basic Properties

sheaf of regular functions dimension connectedness, irreducibility products, semidirect product, identity component, open/closed subgroups, actions, kernel, image, generators

4. Hopf Algebras

5. Jordan Decomposition

6. Lie Algebras

7. Root Systems and Root Datum

8. Isomorphism and Existance Theorems

9. Representations of Split Reductive Groups

10. Tannakian Duality

11. Toric Varieties

12. Flag Varieties

13. Spherical Varieties

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