Introduction to Rigid Analytic Geometry

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For an algebraic variety $X/\mathbb{C}$, one associates its analytification

$$X^{\mathrm{an}} = X(\mathbb{C})$$

a complex analytic space (a complex manifold when X smooth).

Example

Suppose $X/\mathbb{C}$ a smooth algebraic curve of genus g, then X^{an} is a complex manifold of dimension 1, i.e. a Riemann surface, of genus g. In particular, when X is an elliptic curve, X^{an} is a torus.

Theorem (Uniformization theorem)

The only simply connected Riemann surfaces are the Riemann sphere $\mathbb{CP}^1$*, the complex plane* $\mathbb{C}$*, and the upper-half plane* $\mathbb{H}$*.*

Thus, we have the correspondence

genus	X	X^{an}
0	conic section	$\mathbb{CP}^1$
1	elliptic curve	$\mathbb{C}/\Lambda$
≥ 2	modular curve	$\mathbb{H}/\Gamma$

where Λ a lattice, and Γ a congruence subgroup (congruent to I mod some N) of $\mathrm{SL}_2(\mathbb{Z})$ acting by the Mobius transformation.

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Moreover, Serre showed a collection of results (GAGA) relating coherent sheaves on a variety $X/\mathbb{C}$ to that of X^{an}.

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The nontrivial norms on $\mathbb{Q}$ are precisely the archimedean norm $|\cdot|$ and the nonarchimedean p-adic norms $|\cdot|_p$. Thus the completions of $\mathbb{Q}$ are precisely $\mathbb{R}$ and $\mathbb{Q}_p$ for each prime p.

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However, the algebraic closure $\overline{\mathbb{Q}_p}$ is not complete, so one has to complete it again $\mathbb{C}_p = \widehat{\overline{\mathbb{Q}_p}}$ to get a complete and algebraically closed non-archimedean field (Krasner theorem).

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Is there a uniformization theorem or GAGA for this analogue?

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We will only talk about Tate's approach.

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We must eliminate these bad functions.

Definition

Let K a (complete) non-archimedean field, define the Tate algebra

$$K\langle X_1,\dots,X_r\rangle = \left\{ \sum_{n\in\mathbb{N}^r} a_n \underline{X}^n : a_n \to 0 \text{ as } |n| \to \infty \right\}$$

An affnoid K-algebra is a quotient $K\langle X_1,\dots,X_r\rangle/I$.

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A rigid analytic space defined by gluing affinoid domains, i.e. it is a G-ringed space locally isomorphic to affinoid domains.

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However, there is generally no universal covers in rigid geometry, and almost nothing is simply connected: there is no uniformization!

For an elliptic curve $E/\mathbb{C}$,

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The universal cover factors

$$\mathbb{C} \xrightarrow{e^{2\pi i z}} \mathbb{C}^\times \to \mathbb{C}^\times/q^{\mathbb{Z}} \cong E$$

where $q = e^{2\pi i \tau}$ satisfies $0 < |q| < 1$.

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and many more.