Schubert Calculus and Cohomology of Grassmannians

Yunhai Xiang

University of Waterloo

Canadian Undergraduate Mathematics Conference 2022

Overview

1. Introduction and Motivations
2. Grassmannian
3. Chow ring
4. Chern class
5. Further topics

Introduction and Motivations

Let V be a vector space, its *projectivisation* $\mathbb{P}(V)$ is the space

$$\mathbb{P}(V) = (V \setminus \{0\})/ \sim$$

where $x \sim y$ iff $x = \lambda y$ for some $\lambda \neq 0$.

Define the *complex projective space* $\mathbb{P}^n = \mathbb{P}(\mathbb{C}^{n+1})$, which may be thought of as the affine space $\mathbb{C}^n$ with points, lines, planes, etc., at infinity, i.e. it is a compactification of $\mathbb{C}^n$.

Throughout this talk, we will work in $\mathbb{P}^n$ instead of $\mathbb{R}^n$ or $\mathbb{C}^n$, because the algebraic closedness of $\mathbb{C}$ and the compactness of projective spaces will make our lives easier.

Introduction and Motivations

Some quintessential problems in enumerative geometry and intersection theory:

(i) In $\mathbb{P}^2$, how many points do 2 lines meet?

(ii) In $\mathbb{P}^2$, how many points do a line meet a smooth curve of genus 1?

(iii) In $\mathbb{P}^3$, how many lines will meet 4 general lines?

(iv) In $\mathbb{P}^3$, how many lines are contained on a smooth cubic surface?

Introduction and Motivations

Figure: Parallel lines meet at the point at infinity

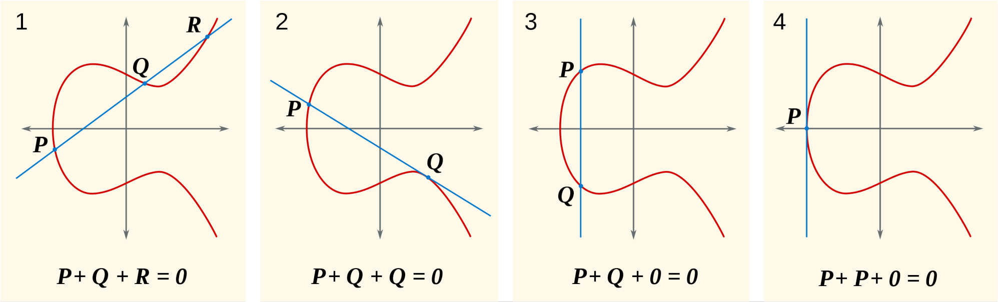


Figure: a line meets an elliptic curve at 3 points

Introduction and Motivations

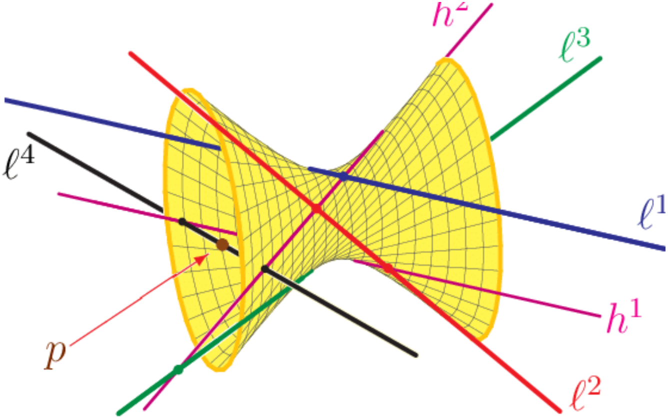


Figure: 2 lines meet 4 general lines (forming a regulus)

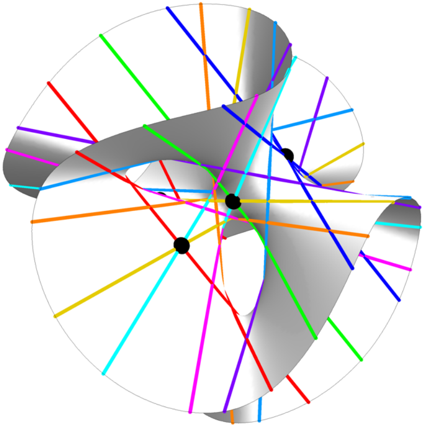


Figure: 27 lines on any smooth cubic surface

Introduction and Motivations

Pioneering work in intersection theory is exemplified by Bézout's theorem.

Theorem 1.1 (Bézout)

Let $X, Y \subseteq \mathbb{P}^2$ be projective curves of degree m and n respectively with no common component, then the number of their intersections counting multiplicity is mn.

Since elliptic curves have degree 3, by Bézout theorem, they meet a line at 3 points.

For more difficult intersection problems, we need methods such as Schubert calculus, which deals with intersection problems involving linear spaces (points, lines, planes, etc).

Introduction and Motivations

Schubert calculus is a branch of enumerative geometry introduced non-rigorously at first by Hermann Schubert. Hilbert's 15th problem is to make Schubert calculus rigorous.

This is eventually done using Grassmannians, which are spaces that "parameterises" linear spaces, i.e. it's a moduli space of linear spaces. Their cohomology ring (Chow ring) encodes the geometric data which we need to compute intersections.

Grassmannian

Definition 2.2 (Grassmannian)

For $0 \leq k \leq n$, the *Grassmannian* $\mathrm{Gr}(k,n)$ is the set of k-dimensional subspaces of $\mathbb{C}^n$.

Grassmannians are usually viewed with some geometric structure such as

(i) a manifold (via the Stiefel manifold or as a homogeneous space),

(ii) a projective variety (via the Plücker embedding),

(iii) others: a scheme, a complex manifold, an affine variety, etc.

We are going to focus on the first two.

Grassmannian

Definition 2.3 (Grassmannian as a manifold)

For $0 \leq k \leq n$, the *Stiefel manifold* $\mathrm{V}_k(\mathbb{C}^n)$ is the set of orthonormal k-frames in $\mathbb{C}^n$ as a submanifold of $(\mathbb{C}^n)^k$, then the map

$$\begin{aligned} \mathrm{V}_k(\mathbb{C}^n) &\longrightarrow \mathrm{Gr}(k,n) \\ (w_1, \ldots, w_k) &\longmapsto \mathrm{Span}\{w_1, \cdots, w_k\} \end{aligned}$$

induces the quotient topology on $\mathrm{Gr}(k,n)$. Alternatively, we can formalise $\mathrm{Gr}(k,n)$ as a homogeneous space. Let $\mathrm{U}(n) \curvearrowright \mathrm{Gr}(k,n)$ in the obvious way, then

$$\mathrm{Gr}(k,n) = \mathrm{U}(n)/\mathrm{Stab}(\Lambda)$$

for any $\Lambda \in \mathrm{Gr}(k,n)$ (note that the action is transitive) as an orbit space, which also endows $\mathrm{Gr}(k,n)$ a smooth structure inherited from the Lie group $\mathrm{U}(n)$.

The smooth manifold $\mathrm{Gr}(k,n)$ is compact, path-connected, and $2k(n-k)$-dimensional.

Grassmannian

Definition 2.4 (Grassmannian as a projective variety)

For $0 \leq k \leq n$, the Grassmannian $\mathrm{Gr}(k,n)$ embeds into $\mathbb{P}\left(\bigwedge^k \mathbb{C}^n\right)$, the projectivisation of the k-th exterior power of $\mathbb{C}^n$, via the map

$$\begin{aligned} \mathrm{Gr}(k,n) &\longrightarrow \mathbb{P}\left(\bigwedge^k \mathbb{C}^n\right) \\ \mathrm{Span}(w_1,\ldots,w_k) &\longmapsto [w_1 \wedge \cdots \wedge w_k] \end{aligned}$$

called the *Plücker embedding*. Identifying $\bigwedge^k \mathbb{C}^n \cong \mathbb{C}^{\binom{n}{k}}$ (note that $e_{i_1} \wedge \cdots \wedge e_{i_k}$ for the indices $1 \leq i_1 < \cdots < i_k \leq n$ forms a basis for $\bigwedge^k \mathbb{C}^n$), $\mathrm{Gr}(k,n)$ embeds in $\mathbb{P}^{\binom{n}{k}-1}$, where its image is closed and irreducible, so it can be viewed as a projective variety.

The projective variety $\mathrm{Gr}(k,n)$ is complete, smooth, and $k(n-k)$-dimensional. With some effort, one can show that $\mathrm{Gr}(k,n)$ is an intersection of quadratics called *Schubert relations*.

Grassmannian

For $0 \leq k \leq n$, we can identify $\mathrm{Gr}(k+1, n+1)$ with

$$\mathbb{G}(k,n) = \{\mathbb{P}(\Lambda) : \Lambda \in \mathrm{Gr}(k+1, n+1)\}$$

via projectivisation.

Example 2.5

(i) $\mathrm{Gr}(1, n+1) = \mathbb{G}(0, n) = \mathbb{P}^n$

(ii) $\mathrm{Gr}(1,3) = \mathrm{Gr}(2,3) = \mathbb{P}^2$

(iii) $\mathrm{Gr}(2,4) = \mathbb{G}(1,3) = \{\text{lines in } \mathbb{P}^3\}$

Grassmannian

Each $\Lambda \in \mathrm{Gr}(k, n)$ can be represented as a k-by-n matrix whose rows are basis vectors of Λ. Two such matrices represent the same space if they are in the same $\mathrm{GL}_k(\mathbb{C}^n)$-orbit. Therefore, each Λ is represented by a RREF matrix. For example,

$$\Lambda = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & * & 0 \\ 0 & 0 & 1 & * & * & 0 & * & 0 \\ 0 & 1 & 0 & * & * & 0 & * & 0 \end{bmatrix}$$

is a point in $\mathrm{Gr}(4, 8)$. It turns out that the Grassmannian is a CW-complex, in fact, the shape of the RREF matrices determines the CW-decomposition of the Grassmannian.

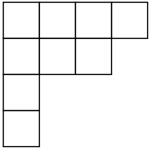
Grassmannian

Definition 2.6 (Partition)

A *partition* is a weakly decreasing sequence $\lambda = (\lambda_1, \ldots, \lambda_k)$ of positive integers with length $\ell(\lambda) = k$, width $w(\lambda) = \lambda_1$, and size $n(\lambda) = \lambda_1 + \cdots + \lambda_k$. For sake of convenience, we set $\lambda_i = 0$ for all $i > k$. Also, assume there is an empty partition ε.

Note that we can encode the shape of the RREF matrix as a partition. For example,

$$\Lambda = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & * & 0 \\ 0 & 0 & 1 & * & * & 0 & * & 0 \\ 0 & 1 & 0 & * & * & 0 & * & 0 \end{bmatrix} \qquad \lambda = (4, 3, 1, 1) =$$



where the blue parts are represented by the Young tableaux of the partition.

Definition 2.7 (Flag)

A *flag* $\mathcal{V}$ of the n-dimensional vector space V is a sequence of subspaces

$$0 \subset V_1 \subset \cdots \subset V_n = V$$

where $\dim V_i = i$. The *standard flag* $\mathcal{E}$ is the flag $0 \subset \mathbb{C}^1 \subset \cdots \subset \mathbb{C}^n$.

Fix a flag $\mathcal{V}$ of $\mathbb{C}^n$, then for $\Lambda \in \mathrm{Gr}(k, n)$,

$$0 \leq \dim(\Lambda \cap V_1) \leq \dim(\Lambda \cap V_2) \leq \cdots \leq \dim(\Lambda \cap V_n) = k$$

where we note that the dimensions of $\Lambda \cap V_i$ and $\Lambda \cap V_{i+1}$ can only differ by at most 1.

Grassmannian

Theorem 2.8 (Grassmannian as a CW-complex)

Let $0 \leq k \leq n$. Fix a flag $\mathcal{V}$ of $\mathbb{C}^n$, then $\mathrm{Gr}(k,n)$ admits a CW-decomposition

$$\mathrm{Gr}(k,n) = \bigcup_{\substack{\lambda \\ \ell(\lambda)\leq k \\ w(\lambda)\leq n-k}} \Sigma_\lambda(\mathcal{V}) = \coprod_{\substack{\lambda \\ \ell(\lambda)\leq k \\ w(\lambda)\leq n-k}} \Omega_\lambda(\mathcal{V})$$

where for each partition λ with $\ell(\lambda) \leq k$ and $w(\lambda) \leq n-k$, the Schubert variety $\Sigma_\lambda(\mathcal{V})$ and Schubert cell $\Omega_\lambda(\mathcal{V})$, both $n(\lambda)$-codimensional, are defined as

$$\Sigma_\lambda(\mathcal{V}) = \{\Lambda \in \mathrm{Gr}(k,n) : \dim(V_{\mu_i} \cap \Lambda) \geq i \ \text{ for } 1 \leq i \leq k\}$$
$$\Omega_\lambda(\mathcal{V}) = \{\Lambda \in \mathrm{Gr}(k,n) : \dim(V_j \cap \Lambda) = i \text{ if } \mu_i \leq j < \mu_{i+1} \text{ for } 1 \leq i \leq k\}$$

for $\mu_0 = 0$ and $\mu_i = n-k+i-\lambda_i$ for $1 \leq i \leq k$. Moreover $\Sigma_\lambda(\mathcal{V}) = \overline{\Omega_\lambda(\mathcal{V})}$.

Grassmannian

Analogous to its CW-decomposition, the Grassmannian as a variety has a stratification.

Definition 2.9

A *stratification* of a variety X is a finite set of locally closed subvarieties U_i such that $\overline{U_i} \cap U_j \neq \emptyset$ implies $U_j \subseteq \overline{U_i}$ and $X = \coprod_i U_i$. The elements U_i of a stratification are called the *strata*, and a stratification is affine if its strata are affine.

Theorem 2.10 (Affine stratification of Grassmannians)

Let $0 \leq k \leq n$. The set of Schubert cells Ω_λ ranging over partitions λ with $\ell(\lambda) \leq k$ and $w(\lambda) \leq n-k$ is an affine stratification of $\mathrm{Gr}(k, n)$, and $\Omega_\lambda \cong \mathbb{A}^{k(n-k)-n(\lambda)}$.

Grassmannian

How do we translate intersection problems into the language of Grassmannians?

For example: how many lines meet four general lines?

Note that the Grassmannian $\mathrm{Gr}(2,4) = \mathbb{G}(1,3)$ parameterises the lines in $\mathbb{P}^3$, and $\Sigma_1(\mathcal{V})$ for a flag $\mathcal{V}$ is the set of Λ s.t. $\dim(\Lambda \cap V_2) \geq 1$. Thus, the problem becomes finding

$$\Sigma_1(\mathcal{V}^1) \cap \Sigma_1(\mathcal{V}^2) \cap \Sigma_1(\mathcal{V}^3) \cap \Sigma_1(\mathcal{V}^4)$$

for 4 general flags $\mathcal{V}^1, \mathcal{V}^2, \mathcal{V}^3, \mathcal{V}^4$. To compute this, we need to introduce the Chow ring.

Chow ring

Recall that given topological space X and coefficient ring R, the cup product

$$H^k(X;R) \times H^\ell(X;R) \longrightarrow H^{k+\ell}(X;R)$$

which, for each singular $(k+\ell)$-chain $\sigma \in C_{k+\ell}(X;R)$, is given by

$$(\phi \smile \psi)(\sigma) = \phi(\sigma|_{[v_0,\dots,v_k]})\psi(\sigma|_{[v_k,\dots,v_{k+\ell}]})$$

gives rise to the grade-commutative (i.e. $\phi \smile \psi = (-1)^{k\ell}(\psi \smile \phi)$) graded ring

$$H^\bullet(X;R) = \bigoplus_{i\in\mathbb{N}} H^i(X;R)$$

called the cohomology ring of X.

Chow ring

Suppose $k \geq \ell$, and recall the cap product

$$H_k(X;R) \times H^\ell(X;R) \longrightarrow H_{k-\ell}(X;R)$$

which, for singular chain $\sigma \in C_k(X;R)$ and singular cochain $\phi \in C^\ell(X;R)$, is given by

$$\sigma \frown \phi = \phi(\sigma|_{[v_0,\ldots,v_\ell]})\sigma|_{[v_\ell,\ldots,v_k]}$$

And recall that if M is a connected orientable compact manifold of dimension n, then a fundamental class of M is a choice of the generator of $H_n(M;\mathbb{Z}) \cong \mathbb{Z}$, denoted $[M]$.

Theorem 3.11 (Poincaré duality)

Let M be a n-dimensional oriented closed manifold, then there is an isomorphism

$$H^k(M;R) \cong H_{n-k}(M;R)$$

given by the map

$$\begin{aligned} H^k(M;R) &\longrightarrow H_{n-k}(M;R) \\ \phi &\longmapsto [M] \frown \phi \end{aligned}$$

for all $0 \leq k \leq n$.

Thus, we may identify these homology and cohomology groups accordingly, and so the notation $[M]$ can also denote the Poincaré dual of the fundamental class of M.

Theorem 3.12 (Cup product is the Poincaré dual of intersection)

Let M be an n-dimensional closed oriented smooth manifold with oriented smooth submanifolds A, B of codimensions i and j resp. and $i + j \leq n$, then

$$[A] \smile [B] = [A \cap B]$$

if A and B intersects transversely.

This allows us to translate questions about intersections into computing cup products.

Chow ring

Definition 3.13 (Schubert cycle)

Let $0 \leq k \leq n$. For a partition λ with $\ell(\lambda) \leq k$ and $w(\lambda) \leq n-k$, define the *Schubert cycle* $\sigma_\lambda = [\Sigma_\lambda(\mathcal{V})] \in H^\bullet(\mathrm{Gr}(k,n);\mathbb{Z})$ where $\mathcal{V}$ is an arbitrary choice of flags of $\mathbb{C}^n$. The cohomology ring $H^\bullet(\mathrm{Gr}(k,n);\mathbb{Z})$ is also known as the *Chow ring* of $\mathrm{Gr}(k,n)$.

Recall the question: how many lines meet 4 general lines?

Choose 4 general flags $\mathcal{V}^1, \mathcal{V}^2, \mathcal{V}^3, \mathcal{V}^4$, then in the Chow ring of $\mathrm{Gr}(2,4)$,

$$[\Sigma_1(\mathcal{V}^1) \cap \Sigma_1(\mathcal{V}^2) \cap \Sigma_1(\mathcal{V}^3) \cap \Sigma_1(\mathcal{V}^4)] = [\Sigma_1]^4 = \sigma_1^4$$

It suffice to find σ_1^4.

Chow ring

The Chow ring can be formalised entirely algebrao-geometrically.

Definition 3.14 (Chow group and Chow ring)

The *Chow group* of a variety X is $Z(X)/\mathrm{Rat}(X)$ where $Z(X)$ is the group of *cycles* (free abelian group generated by subvarieties), and *rational equivalences* $\mathrm{Rat}(X)$ is generated by

$$\langle \Phi \cap (\{t_0\} \times X) \rangle - \langle \Phi \cap (\{t_1\} \times X) \rangle$$

for $t_0, t_1 \in \mathbb{P}^1$ and Φ a subvariety of $\mathbb{P}^1 \times X$ not contained in any $\{t\} \times X$. There is a unique product on the Chow group satisfying $[A][B] = [A \cap B]$ if A, B intersect generically transversely, forming the ring $A(X) = \bigoplus_r A^r(X)$ graded by codimension called *Chow ring*.

For subvarieties A, B, the Chow ring determines the intersection multiplicity $m_C(A, B)$ where C is an irreducible component of $A \cap B$ by $[A][B] = \sum_C m_C(A, B)[C]$.

Chow ring

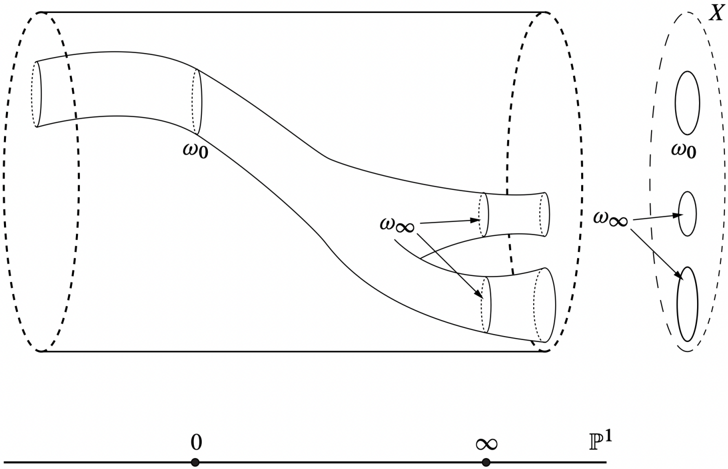


Figure: Rational equivalence between cycles ω_0 and ω_∞

Chow ring

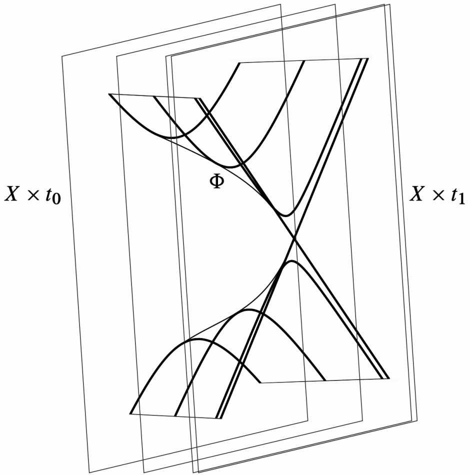


Figure: Rational equivalence between a hyperbola and a union of two lines

Chow ring

How do we ensure the flags are general enough? What if they don't intersect transversely?

Theorem 3.15 (Kleiman)

Suppose an algebraic group G acts on a variety X over $\mathbb{C}$, and $A \subseteq X$ is a subvariety,

(i) *If $B \subseteq X$ is another subvariety, then there is an open dense set of $g \in G$ such that gA is generically transverse to B,*

(ii) *if $\varphi : Y \to X$ is a morphism of varieties, then for general $g \in G$, the preimage $\varphi^{-1}[gA]$ is generically reduced and of same codimension as A,*

(iii) *if G is affine, then $[gA] = [A] \in A(X)$ for any $g \in G$.*

This means we can always perturb them to make them intersect transversely WLOG.

Chow ring

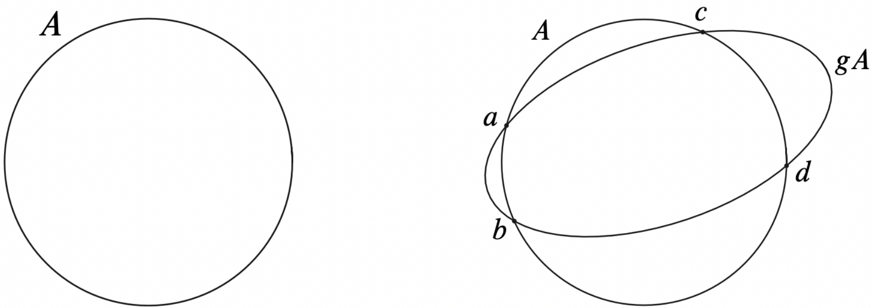


Figure: Perturbing the circle A to make it intersect itself transversely

Chow ring

Theorem 3.16 (Totaro)

If $\{U_i\}$ is a stratification of the variety X, then $A(X)$ is generated by the classes $[\overline{U_i}]$.

In particular, this theorem tells us the Schubert cycles σ_λ generates the Chow ring $A(\mathrm{Gr}(k,n)) = H^\bullet(\mathrm{Gr}(k,n);\mathbb{Z})$. For example, the cohomology of $\mathrm{Gr}(2,4)$ are

$$
\begin{aligned}
A^0(\mathrm{Gr}(2,4)) &= H^0(\mathrm{Gr}(2,4);\mathbb{Z}) = \mathbb{Z}\sigma_\varepsilon \\
A^1(\mathrm{Gr}(2,4)) &= H^2(\mathrm{Gr}(2,4);\mathbb{Z}) = \mathbb{Z}\sigma_1 \\
A^2(\mathrm{Gr}(2,4)) &= H^4(\mathrm{Gr}(2,4);\mathbb{Z}) = \mathbb{Z}\sigma_2 \oplus \mathbb{Z}\sigma_{1,1} \\
A^3(\mathrm{Gr}(2,4)) &= H^6(\mathrm{Gr}(2,4);\mathbb{Z}) = \mathbb{Z}\sigma_{2,1} \\
A^4(\mathrm{Gr}(2,4)) &= H^8(\mathrm{Gr}(2,4);\mathbb{Z}) = \mathbb{Z}\sigma_{2,2}
\end{aligned}
$$

Chow ring

With some effort, one can show that $A(\mathrm{Gr}(2,4))$ has a product structure subjected to

$$\begin{aligned}
\sigma_1^2 &= \sigma_{1,1} + \sigma_2 \\
\sigma_1\sigma_{1,1} &= \sigma_1\sigma_2 = \sigma_{2,1} \\
\sigma_1\sigma_{2,1} &= \sigma_{2,2} \\
\sigma_{1,1}^2 &= \sigma_2^2 = \sigma_{2,2} \\
\sigma_{1,1}\sigma_2 &= 0
\end{aligned}$$

In particular, one obtains $\sigma_1^4 = 2\sigma_{2,2}$. Thus, 2 lines meet 4 general lines in $\mathbb{P}^3$.

There are many ways to compute these products: Pieri's formula, Giambelli's formula, etc.

Chern class

Definition 4.17 (Tautological bundle of Grassmannian)

Let $0 \leq k \leq n$. The *tautological bundle* of a Grassmannian $\mathrm{Gr}(k,n)$ is the vector bundle with total space consisting (Λ, v) where $v \in \Lambda \in \mathrm{Gr}(k,n)$ and projection $\pi(\Lambda, v) = \Lambda$.

If X is a smooth projective variety, there is a way to assign each vector bundle $E \to X$ a *Chern class* $c_i(E) \in A^i(X)$. It turns out that the Chern class $c_4(\mathrm{Sym}^3(E^*))$ where E is the tautological bundle of $\mathrm{Gr}(2,4)$ gives the lines on a smooth cubic surface. Calculating

$$c_4(\mathrm{Sym}^3(E^*)) = 27\sigma_{2,2}$$

we can conclude there are 27 lines on a smooth cubic surface.

Further topics

Some further topics for those interested:

(i) Arakelov geometry: arithmetic version of intersection theory,

(ii) Symmetric function theory: combinatorial theory behind Schubert calculus,

(iii) Gromov–Witten theory: intersection theory on algebraic stacks,

(iv) Connection with string theory: applications of enumerative geometry.

Further topics

Challenge: Prove that there are 3264 plane conics tangent to 5 general conics.

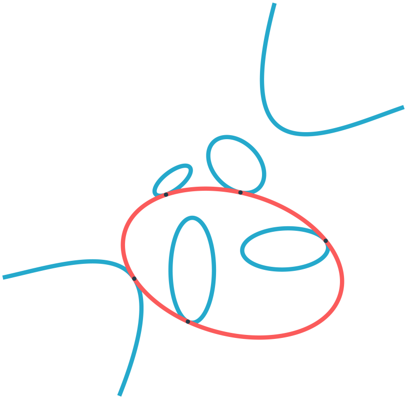


Figure: A plane conic tangent to 5 other conics

Thank you for listening!

References

David Eisenbud, Joe Harris (2016)
3264 and All That: A Second course in Algebraic Geometry
Cambridge University Press

William Fulton (1997)
Intersection Theory
Springer

Letterio Gatto (2005)
Schubert calculus: an Algebraic Introduction
Instituto Nacional de Matemática Pura e Aplicada