

CATEGORY THEORY DEMYSTIFIED

A Friendly Introduction to Abstract Nonsense

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Date: 2023/02/09



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Rule 1. Don't be intimidated by categories (or the fancy diagrams or buzzwords).

Introduction

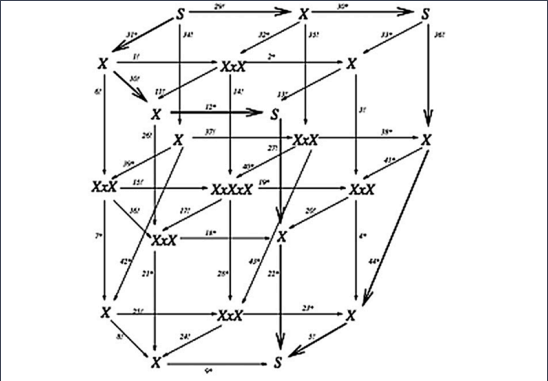


Figure: Voevodsky's 2-theory

Category theory originated from Eilenberg and Mac Lane’s study of algebraic topology.

GENERAL THEORY OF NATURAL EQUIVALENCES

BY

SAMUEL EILENBERG AND SAUNDERS MACLANE

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Figure: General Theory of Natural Equivalences

Mathematical objects frequently come with **morphisms** between them.

Objects	Morphisms
sets	functions
groups	group homomorphisms
rings	ring homomorphisms
k -vector spaces	k -linear transformations
topological spaces	continuous map
posets	monotone functions

What do they have in common?

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We denote $f : X \rightarrow Y$ for $f \in \text{Hom}(X, Y)$ and $f \circ g$ for composition.

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⊙ (associativity) if $f \in \text{Hom}(A, B)$, $g \in \text{Hom}(B, C)$, $h \in \text{Hom}(C, D)$, then

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- ⊙ (identity) there exists $\text{id}_X \in \text{Hom}(X, X)$ for each $X \in \mathcal{C}$ such that

$$f \circ \text{id}_A = f = \text{id}_B \circ f$$

for any $f \in \text{Hom}(A, B)$.

Introduction

Right away we have a lot of examples of “big” categories

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However, categories do not have to be big, e.g. \mathbb{N} is a category.

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This allows us to draw diagrams consisting of multiple morphisms.

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ g \downarrow & \nearrow h & \\ Z & & \end{array}$$

Terminology. We say that a diagram such as

$$\begin{array}{ccc} X & \xrightarrow{a} & Y \\ f \downarrow & & \downarrow g \\ R & \xrightarrow{b} & S \end{array}$$

commutes if for each pair of vertices A, B in the diagram, the maps produced following different paths from A to B are the same map (in this case, this means $a \circ g = b \circ f$).

The Yoga of Category Theory

Rule 2. Instead of construction, characterize things by their interactions with other things.

The Yoga of Category Theory

Instead of “injective map”, think “left-cancellative map”, i.e. a map $f : X \rightarrow Y$ s.t.,

$$f \circ g = f \circ h \implies g = h$$

for all $g, h : Z \rightarrow X$. This is called a **monomorphism**.

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Injectivity does not make sense in all categories, but in the ones that do, an injective map is obviously a monomorphism. The converse is not necessarily true!

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Instead of “surjective map”, think “right-cancellative map”, i.e. a map $f : X \rightarrow Y$ s.t.,

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Question. How would you characterize isomorphisms in a category?

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Answer. A morphism $f : X \rightarrow Y$ is an isomorphism if there exists $g : Y \rightarrow X$ s.t.

$$g \circ f = \text{id}_X \quad \text{and} \quad f \circ g = \text{id}_Y$$

in which case g is called the inverse of f .

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Example. Suppose G is a group. Instead of thinking a subgroup H as a subset of G closed under operation and identity, think of it as a pair (H, i) where H is a group and $i : H \rightarrow G$ a monomorphism, up to an equivalence $(H, i) \cong (H', i')$ if exists isomorphism $\phi : H \rightarrow H'$ s.t.

$$\begin{array}{ccc} H & \xrightarrow{i} & G \\ \phi \downarrow & \nearrow i' & \\ H' & & \end{array}$$

commutes, i.e. $i = i' \circ \phi$. In fact, this is how we define subobjects in a any category.

Exercise. Dually, how would you characterize quotient objects of an object in a category?

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Let $A, B \in \mathcal{C}$, where we have a notion of product $A \times B$ e.g. if $\mathcal{C} \in \{\mathbf{Set}, \mathbf{Grp}, \mathbf{Ring}\}$.

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Think of $A \times B$ as (P, π_A, π_B) where $P \in \mathcal{C}$ and

$$\pi_A : P \rightarrow A$$

$$\pi_B : P \rightarrow B$$

are morphisms satisfying the universal property of products.

The Yoga of Category Theory

Universal Property of Products. For all $Q \in \mathcal{C}$ and $\tau_A : Q \rightarrow A$ and $\tau_B : Q \rightarrow B$, there exists a unique morphism $q : Q \rightarrow P$ s.t. $\tau_A = \pi_A \circ q$ and $\tau_B = \pi_B \circ q$.

$$\begin{array}{ccccc} & & Q & & \\ & \swarrow \tau_B & \downarrow q & \searrow \tau_A & \\ B & \xleftarrow{\pi_B} & P & \xrightarrow{\pi_A} & A \end{array}$$

Here (Q, τ_A, τ_B) is a “test” to find the “smallest/universal product” (P, π_A, π_B) .

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Here (Q, τ_A, τ_B) is a “test” to find the “smallest/universal product” (P, π_A, π_B) .

In fact, this is how we define products in an arbitrary category.

Exercise. Try formulating the idea of coproducts, the dual notion to products.

The Yoga of Category Theory

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Here, we will talk about one example: topological groups.

Definition. A **topological group** is a group G with a topology such that the maps

$$\begin{aligned} m : G \times G &\rightarrow G & (g, h) &\mapsto gh \\ \text{inv} : G &\rightarrow G & g &\mapsto g^{-1} \end{aligned}$$

are continuous (where $G \times G$ has the product topology).

The Yoga of Category Theory

Instead of “topological group”, think “group object in **Top**”.

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Definition. A **group object** in a category \mathcal{C} with finite products is an object $G \in \mathcal{C}$ with

$$m : G \times G \rightarrow G$$

$$e : 1 \rightarrow G$$

$$\text{inv} : G \rightarrow G$$

where 1 is the terminal object (the object such that there exists a unique $X \rightarrow 1$ for each X) satisfying the “group axioms”, i.e. the following three diagrams commute.

$$\begin{array}{ccc} G \times G \times G & \xrightarrow{(\text{id}, m)} & G \times G \\ (m, \text{id}) \downarrow & & \downarrow m \\ G \times G & \xrightarrow{m} & G \end{array}$$

$$\begin{array}{ccc} G & \xrightarrow{(\text{id}, e)} & G \times G \\ (e, \text{id}) \downarrow & \searrow \text{id} & \downarrow m \\ G \times G & \xrightarrow{m} & G \end{array}$$

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Universal Properties

Rule 3. Always define things (and think of things) in terms of their universal properties.

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- ⊙ It allows conceptual non-element-wise proofs.
- ⊙ It allows for easier abstractions and analogies.

What do universal properties do?

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Example. (Universal Property of Tensor Products) Let V, W be k -vector spaces, their tensor product is a pair $(V \otimes W, \otimes)$ where $V \otimes W$ is a k -vector space and $\otimes : V \times W \rightarrow V \otimes W$ a bilinear map such that for every pair $(V \otimes' W, \otimes')$ where $V \otimes' W$ is a k -vector space and $\otimes' : V \times W \rightarrow V \otimes' W$ a bilinear map, exists unique $h : V \otimes W \rightarrow V \otimes' W$ s.t. $\otimes' = h \circ \otimes$.

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We use a “test” $(V \otimes' W, \otimes')$ to find “smallest/universal” $(V \otimes W, \otimes)$.

Definition. The initial object of a category \mathcal{C} is an object $I \in \mathcal{C}$ such that for each object $X \in \mathcal{C}$ there exists a unique morphism $I \rightarrow X$. The initial object I is unique up to (a unique) isomorphism. Dually, one could define the terminal object.

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Suppose \mathcal{C} is the category where objects consist of all pairs $(V \otimes W, \otimes)$ (where $V \otimes W$ is a k -vector space and $\otimes : V \times W \rightarrow V \otimes W$ a bilinear map), and a morphism

$$h : (V \otimes W, \otimes) \rightarrow (V \otimes' W, \otimes')$$

is a linear map $h : V \otimes W \rightarrow V \otimes' W$ such that $\otimes' = h \circ \otimes$, then the universal property of tensor products is saying that the tensor product is the initial object in the category \mathcal{C} .

All universal properties are formulated this way!

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In other words, what all universal properties do is finding the initial (or dually, the terminal) object in a particular category, which, in fully generality, is a comma category.

Question. Let \mathcal{C} be the category where objects are (X, ξ, u) where

- ⊙ X is a Banach space
- ⊙ $\xi : X \otimes X \rightarrow X$
- ⊙ $u \in X$

and morphisms are contracting linear maps preserving ξ and u .

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What is the initial object in this category (it does have one)?

Answer. The initial object of \mathcal{C} is $(L^1[0, 1], \gamma, 1)$ where γ is the “concatenation” map, 1 is the constant function with value 1, and $L^1[0, 1]$ the space of integrable functions on $[0, 1]$

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Integrability pops out just by adding two simple pieces of information!

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Exercise. Let \mathbb{Q} be the field of rational numbers, how would you characterize the field extension $\mathbb{Q} \hookrightarrow \mathbb{Q}(\sqrt{2})$ in terms of a universal property?

Many structures defined by universal properties are generalized by limits and colimits.

Functors and Natural Transformations

We would like to go one further level of abstraction.

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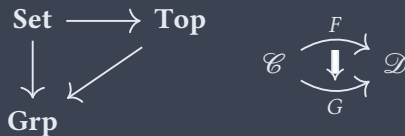


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Functors and Natural Transformations

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- ⊙ functors: “morphisms” between categories,
- ⊙ natural transformations: “morphisms” between “morphisms” between categories.

Functors and Natural Transformations

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A contravariant functor is the same but with arrow reversed. Alternatively, a contravariant functor $\mathcal{C} \rightarrow \mathcal{D}$ is a covariant functor $\mathcal{C}^{\text{op}} \rightarrow \mathcal{D}$, where \mathcal{C}^{op} reverses arrows in \mathcal{C} .

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We will focus on the first three perspectives.

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Example. $\pi_1 : \mathbf{pcTop}_* \rightarrow \mathbf{Grp}$ which sends a path-connected topological space to its fundamental group and a continuous function to its induced map.

Example. Let \mathcal{C} be a category such that each $\mathrm{Hom}(X, Y)$ is a set. Fix $A \in \mathcal{C}$. Define the Hom-functor $\mathrm{Hom}(A, -) : \mathcal{C} \rightarrow \mathbf{Set}$ sending an object $X \mapsto \mathrm{Hom}(A, X)$ and a morphism $f \mapsto [g \mapsto f \circ g]$. We define dually the contravariant functor $\mathrm{Hom}(-, A) : \mathcal{C} \rightarrow \mathbf{Set}$.

Theorem. Given a path-connected topological group G , then $\pi_1(G)$ is abelian.

Proof. The usual proof in textbooks uses Eckmann-Hilton argument, but category theory gives us a more conceptual proof. The fundamental group functor

$$\pi_1 : \mathbf{pcTop} \rightarrow \mathbf{Grp}$$

preserves group objects since it preserves terminal object and products, therefore it sends group objects in \mathbf{pcTop} , the path-connected topological groups, to group objects in \mathbf{Grp} , which the reader may check, are precisely the abelian groups.

Functors and Natural Transformations

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Thus, the diagram F commutes when for each $f, g : X \rightarrow Y$ in \mathcal{C} , we have $Ff = Fg$.

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Exercise. Convince yourself this is equivalent to the usual linear representation of a group G , which is a (ρ, V) where V is a vector space and $\rho : G \rightarrow \text{GL}(V)$ a homomorphism.

Functors and Natural Transformations

Definition. Let $F, G : \mathcal{C} \rightarrow \mathcal{D}$ be functors, then a natural transformation $\Phi : F \rightarrow G$ consists of a morphism $\Phi_X : F(X) \rightarrow G(X)$ for each $X \in \mathcal{C}$ such that each

$$\begin{array}{ccc} F(X) & \xrightarrow{\Phi_X} & G(X) \\ Ff \downarrow & & \downarrow Gf \\ F(Y) & \xrightarrow{\Phi_Y} & G(Y) \end{array}$$

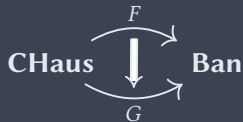
commutes. Let $\text{Fun}(\mathcal{C}, \mathcal{D})$ be the category of functors from \mathcal{C} to \mathcal{D} , where morphisms are natural transformations, the isomorphisms in which are called natural isomorphisms.

Example. Define the natural transformation $\det : \mathrm{GL}_n(-) \rightarrow (-)^\times$ where for each ring R the morphism $\det_R : \mathrm{GL}_n(R) \rightarrow R^\times$ is given by the determinant map. This is a natural transformation because it is defined by the same formula

$$\det_R((a_{i,j})) = \sum_{\sigma \in S_n} \prod_i \mathrm{sgn}(\sigma) a_{i,\sigma(i)}$$

across rings, so it commutes with any ring homomorphism.

Exercise. Express Riesz representation theorem as a natural isomorphism



Question. What does it mean for an equivalence or isomorphism to be natural?

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Functors and Natural Transformations

Given a finite dimensional k -vector space V with dual space $V^* = \text{Hom}(V, k)$.

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Question. Recall that there is an isomorphism

$$\phi : V \longrightarrow V^{**} \quad v \longmapsto [f \mapsto f(v)]$$

Is this isomorphism *natural*?

Answer. This second isomorphism is natural and the first one isn't, because unlike the first one, the second one does not depend on a choice of bases. It is “uniform” across vector spaces (defined by the same formula). In other words, it is **functorial**.

Yoneda Lemma

“The Yoneda lemma is the hardest trivial thing in mathematics.” – Dan Piponi

Theorem. Let \mathcal{C} be a category where each $\text{Hom}(X, Y)$ is a set, and let $A \in \mathcal{C}$. Let $F : \mathcal{C}^{\text{op}} \rightarrow \mathbf{Set}$ be a functor, then there is an isomorphism

$$\text{Hom}(\text{Hom}(-, A), F) \cong F(A)$$

functorial in A and F (natural isomorphism as functors $\mathcal{C} \times \text{Fun}(\mathcal{C}, \mathbf{Set}) \rightarrow \mathbf{Set}$).

Yoneda Lemma

Proof. For $\Phi : \text{Hom}(-, A) \rightarrow F$ and $u = \Phi_A(\text{id}_A)$. If $f : X \rightarrow A$ then

$$\begin{array}{ccc} \text{Hom}(A, A) & \xrightarrow{f_*} & \text{Hom}(X, A) \\ \Phi_A \downarrow & & \downarrow \Phi_X \\ F(A) & \xrightarrow{Ff} & F(X) \end{array}$$

commutes. Thus $\Phi_X(f) = (Ff)(u)$ is determined by u , which gives the isomorphism. This does not depend on any choice based on A or F , thus functorial in A and F .

All information of A is encoded in $\text{Hom}(-, A)$, and vice versa.

Yoneda Lemma

The Yoneda lemma also implies that the functor

$$\begin{aligned}\mathcal{C} &\longrightarrow \mathbf{Fun}(\mathcal{C}^{\mathrm{op}}, \mathbf{Set}) \\ A &\longmapsto \mathrm{Hom}(-, A)\end{aligned}$$

called the Yoneda embedding, is fully faithful, i.e. we have

$$\mathrm{Hom}(\mathrm{Hom}(-, X), \mathrm{Hom}(-, Y)) \cong \mathrm{Hom}(X, Y)$$

for all $X, Y \in \mathcal{C}$.

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Plugging in $\mathrm{Spec}(k)$ in this functor gives k -rational points of X .

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The inverse process of base change is known as descent theory.

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Example. For $0 < k < n$, the Grassmannian $\mathrm{Gr}(k, n) : \mathbf{Sch}^{\mathrm{op}} \rightarrow \mathbf{Set}$

$$\mathrm{Gr}(k, n)(S) = \{\alpha : \mathcal{O}_S^{\otimes n} \rightarrow \mathcal{V}\} / \sim$$

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The Grassmannian is representable by a scheme.

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Example. elliptic curves, more generally algebraic curves of genus g

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Hard Solution. Develop the theory of Artin stacks and Deligne-Mumford stacks.

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and many more.