Category Theory Demystified

A Friendly Introduction to Abstract Nonsense

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Table of contents

Introduction

Rule 1. Don’t be intimidated by categories (or the fancy diagrams or buzzwords).

Introduction

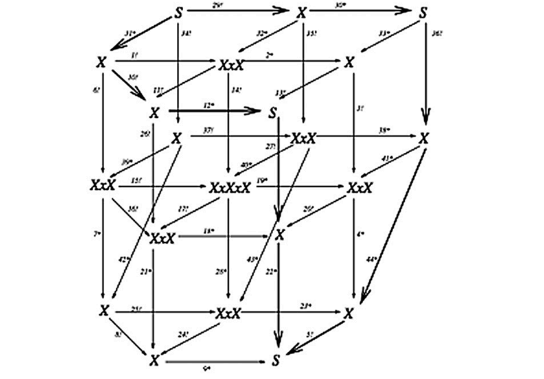

Figure: Voevodsky’s 2-theory

Introduction

Category theory originated from Eilenberg and Mac Lane's study of algebraic topology.

GENERAL THEORY OF NATURAL EQUIVALENCES

BY

SAMUEL EILENBERG AND SAUNDERS MACLANE

CONTENTS

Figure: General Theory of Natural Equivalences

Introduction

Mathematical objects frequently come with **morphisms** between them.

Objects	Morphisms
sets	functions
groups	group homomorphisms
rings	ring homomorphisms
k-vector spaces	k-linear transformations
topological spaces	continuous map
posets	monotone functions

What do they have in common?

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We denote $f : X \to Y$ for $f \in \mathrm{Hom}(X, Y)$ and $f \circ g$ for composition.

Introduction

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- (associativity) if $f \in \mathrm{Hom}(A, B)$, $g \in \mathrm{Hom}(B, C)$, $h \in \mathrm{Hom}(C, D)$, then

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- (identity) there exists $\mathrm{id}_X \in \mathrm{Hom}(X, X)$ for each $X \in \mathscr{C}$ such that

$$f \circ \mathrm{id}_A = f = \mathrm{id}_B \circ f$$

for any $f \in \mathrm{Hom}(A, B)$.

Right away we have a lot of examples of “big” categories

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k-**Vect**	k-vector spaces	k-linear transformations
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However, categories do not have to be big, e.g. $\mathbb{N}$ is a category.

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This allows us to draw diagrams consisting of multiple morphisms.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"'] & Y \\ Z \arrow[ur, "h"'] & \end{tikzcd}$$

Terminology. We say that a diagram such as

$$\begin{tikzcd} X \arrow[r, "a"] \arrow[d, "f"'] & Y \arrow[d, "g"] \\ R \arrow[r, "b"'] & S \end{tikzcd}$$

commutes if for each pair of vertices A, B in the diagram, the maps produced following different paths from A to B are the same map (in this case, this means $a \circ g = b \circ f$).

The Yoga of Category Theory

Rule 2. Instead of construction, characterize things by their interactions with other things.

Instead of “injective map”, think “left-cancellative map”, i.e. a map $f : X \to Y$ s.t.,

$$f \circ g = f \circ h \implies g = h$$

for all $g, h : Z \to X$. This is called a **monomorphism**.

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Injectivity does not make sense in all categories, but in the ones that do, an injective map is obviously a monomorphism. The converse is not necessarily true!

Instead of "surjective map", think "right-cancellative map", i.e. a map $f : X \to Y$ s.t.,

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Answer. A morphism $f : X \to Y$ is an isomorphism if there exists $g : Y \to X$ s.t.

$$g \circ f = \mathrm{id}_X \quad \text{and} \quad f \circ g = \mathrm{id}_Y$$

in which case g is called the inverse of f.

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Example. Suppose G is a group. Instead of thinking a subgroup H as a subset of G closed under operation and identity, think of it as a pair (H, i) where H is a group and $i : H \to G$ a monomorphism, up to an equivalence $(H, i) \cong (H', i')$ if exists isomorphism $\phi : H \to H'$ s.t.

$$\begin{tikzcd} H \arrow[r, "i"] \arrow[d, "\phi"'] & G \\ H' \arrow[ur, "i'"'] & \end{tikzcd}$$

commutes, i.e. $i = i' \circ \phi$. In fact, this is how we define subobjects in a any category.

Exercise. Dually, how would you characterize quotient objects of an object in a category?

Let $A, B \in \mathscr{C}$, where we have a notion of product $A \times B$ e.g. if $\mathscr{C} \in \{\mathbf{Set}, \mathbf{Grp}, \mathbf{Ring}\}$.

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Instead of thinking $A \times B$ as pairs of elements (with possible additional structure),

Think of $A \times B$ as (P, π_A, π_B) where $P \in \mathscr{C}$ and

$$\begin{aligned} \pi_A &: P \to A \\ \pi_B &: P \to B \end{aligned}$$

are morphisms satisfying the universal property of products.

Universal Property of Products. For all $Q \in \mathscr{C}$ and $\tau_A : Q \to A$ and $\tau_B : Q \to B$, there exists a unique morphism $q : Q \to P$ s.t. $\tau_A = \pi_A \circ q$ and $\tau_B = \pi_B \circ q$.

$$\begin{tikzcd}
 & Q \arrow[dl, "\tau_B"'] \arrow[d, dashed, "q"] \arrow[dr, "\tau_A"] & \\
B & P \arrow[l, "\pi_B"'] \arrow[r, "\pi_A"] & A
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Here (Q, τ_A, τ_B) is a "test" to find the "smallest/universal product" (P, π_A, π_B).

In fact, this is how we define products in an arbitrary category.

Exercise. Try formulating the idea of coproducts, the dual notion to products.

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Here, we will talk about one example: topological groups.

Definition. A **topological group** is a group G with a topology such that the maps

$$\begin{aligned} m &: G \times G \to G \quad & (g,h) &\mapsto gh \\ \mathrm{inv} &: G \to G & g &\mapsto g^{-1} \end{aligned}$$

are continuous (where $G \times G$ has the product topology).

Instead of “topological group”, think “group object in **Top**”.

The Yoga of Category Theory

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Definition. A **group object** in a category $\mathscr{C}$ with finite products is an object $G \in \mathscr{C}$ with

$$\begin{aligned} m &: G \times G \to G \\ e &: 1 \to G \\ \mathrm{inv} &: G \to G \end{aligned}$$

where 1 is the terminal object (the object such that there exists a unique $X \to 1$ for each X) satisfying the “group axioms”, i.e. the following three diagrams commute.

$$\begin{tikzcd} G \times G \times G \arrow[r, "(\mathrm{id}, m)"] \arrow[d, "(m, \mathrm{id})"'] & G \times G \arrow[d, "m"] \\ G \times G \arrow[r, "m"'] & G \end{tikzcd} \qquad \begin{tikzcd} G \arrow[r, "(\mathrm{id}, e)"] \arrow[d, "(e, \mathrm{id})"'] \arrow[dr, "\mathrm{id}"] & G \times G \arrow[d, "m"] \\ G \times G \arrow[r, "m"'] & G \end{tikzcd} \qquad \begin{tikzcd} G \arrow[r, "(\mathrm{inv}, \mathrm{id})"] \arrow[d, "(\mathrm{id}, \mathrm{inv})"'] \arrow[dr, "e"] & G \times G \arrow[d, "m"'] \\ G \times G \arrow[r, "m"] & G \end{tikzcd}$$

Universal Properties

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Rule 3. Always define things (and think of things) in terms of their universal properties.

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- It allows conceptual non-element-wise proofs.
- It allows for easier abstractions and analogies.

Universal Properties

What do universal properties do?

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Example. (Universal Property of Tensor Products) Let V, W be k-vector spaces, their tensor product is a pair $(V \otimes W, \otimes)$ where $V \otimes W$ is a k-vector space and $\otimes : V \times W \to V \otimes W$ a bilinear map such that for every pair $(V \otimes' W, \otimes')$ where $V \otimes' W$ is a k-vector space and $\otimes' : V \times W \to V \otimes' W$ a bilinear map, exists unique $h : V \otimes W \to V \otimes' W$ s.t. $\otimes' = h \circ \otimes$.

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We use a "test" $(V \otimes' W, \otimes')$ to find "smallest/universal" $(V \otimes W, \otimes)$.

Definition. The initial object of a category $\mathscr{C}$ is an object $I \in \mathscr{C}$ such that for each object $X \in \mathscr{C}$ there exists a unique morphism $I \to X$. The initial object I is unique up to (a unique) isomorphism. Dually, one could define the terminal object.

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Suppose $\mathscr{C}$ is the category where objects consist of all pairs $(V \otimes W, \otimes)$ (where $V \otimes W$ is a k-vector space and $\otimes : V \times W \to V \otimes W$ a bilinear map), and a morphism

$$h : (V \otimes W, \otimes) \to (V \otimes' W, \otimes')$$

is a linear map $h : V \otimes W \to V \otimes' W$ such that $\otimes' = h \circ \otimes$, then the universal property of tensor products is saying that the tensor product is the initial object in the category $\mathscr{C}$.

All universal properties are formulated this way!

Universal Properties

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In other words, what all universal properties do is finding the initial (or dually, the terminal) object in a particular category, which, in fully generality, is a comma category.

Question. Let $\mathscr{C}$ be the category where objects are (X, ξ, u) where

- X is a Banach space
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What is the initial object in this category (it does have one)?

Universal Properties

Answer. The initial object of $\mathscr{C}$ is $(L^1[0,1], \gamma, 1)$ where γ is the “concatenation” map, 1 is the constant function with value 1, and $L^1[0,1]$ the space of integrable functions on $[0,1]$

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Integrability pops out just by adding two simple pieces of information!

Exercise. Given a set X, how would you characterize the free group $\mathrm{Free}(X)$ generated by elements of X in terms of a universal property?

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Exercise. Let $\mathbb{Q}$ be the field of rational numbers, how would you characterize the field extension $\mathbb{Q} \hookrightarrow \mathbb{Q}(\sqrt{2})$ in terms of a universal property?

Universal Properties

Many structures defined by universal properties are generalized by limits and colimits.

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- functors: "morphisms" between categories,
- natural transformations: "morphisms" between "morphisms" between categories.

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A contravariant functor is the same but with arrow reversed. Alternatively, a contravariant functor $\mathscr{C} \to \mathscr{D}$ is a covariant functor $\mathscr{C}^{\mathrm{op}} \to \mathscr{D}$, where $\mathscr{C}^{\mathrm{op}}$ reverses arrows in $\mathscr{C}$.

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We will focus on the first three perspectives.

Functors and Natural Transformations

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Example. $\pi_1 : \mathbf{pcTop}_* \to \mathbf{Grp}$ which sends a path-connected topological space to its fundamental group and a continuous function to its induced map.

Example. Let $\mathscr{C}$ be a category such that each $\mathrm{Hom}(X, Y)$ is a set. Fix $A \in \mathscr{C}$. Define the Hom-functor $\mathrm{Hom}(A, -) : \mathscr{C} \to \mathbf{Set}$ sending an object $X \mapsto \mathrm{Hom}(A, X)$ and a morphism $f \mapsto [g \mapsto f \circ g]$. We define dually the contravariant functor $\mathrm{Hom}(-, A) : \mathscr{C} \to \mathbf{Set}$.

Theorem. Given a path-connected topological group G, then $\pi_1(G)$ is abelian.

Proof. The usual proof in textbooks uses Eckmann-Hilton argument, but category theory gives us a more conceptual proof. The fundamental group functor

$$\pi_1 \,:\, \mathbf{pcTop} \to \mathbf{Grp}$$

preserves group objects since it preserves terminal object and products, therefore it sends group objects in **pcTop**, the path-connected topological groups, to group objects in **Grp**, which the reader may check, are precisely the abelian groups.

A functor $F : \mathscr{C} \to \mathscr{D}$ can be viewed as a diagram in $\mathscr{D}$ indexed by (or in the shape of) $\mathscr{C}$.

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d] & \bullet \arrow[dl] \\ \bullet & \end{tikzcd} \qquad \longmapsto \qquad \begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[dl, "h"] \\ C & \end{tikzcd}$$

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Thus, the diagram F commutes when for each $f, g : X \to Y$ in $\mathscr{C}$, we have $Ff = Fg$.

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Exercise. Convince yourself this is equivalent to the usual linear representation of a group G, which is a (ρ, V) where V is a vector space and $\rho : G \to \mathrm{GL}(V)$ a homomorphism.

Definition. Let $F, G : \mathscr{C} \to \mathscr{D}$ be functors, then a natural transformation $\Phi : F \to G$ consists of a morphism $\Phi_X : F(X) \to G(X)$ for each $X \in \mathscr{C}$ such that each

$$\begin{tikzcd} F(X) \arrow[r, "\Phi_X"] \arrow[d, "Ff"'] & G(X) \arrow[d, "Gf"] \\ F(Y) \arrow[r, "\Phi_Y"'] & G(Y) \end{tikzcd}$$

commutes. Let $\mathrm{Fun}(\mathscr{C}, \mathscr{D})$ be the category of functors from $\mathscr{C}$ to $\mathscr{D}$, where morphisms are natural transformations, the isomorphisms in which are called natural isomorphisms.

Example. Define the natural transformation $\det : \mathrm{GL}_n(-) \to (-)^\times$ where for each ring R the morphism $\det_R : \mathrm{GL}_n(R) \to R^\times$ is given by the determinant map. This is a natural transformation because it is defined by the same formula

$$\det_R((a_{i,j})) = \sum_{\sigma \in S_n} \prod_i \mathrm{sgn}(\sigma)\, a_{i,\sigma(i)}$$

across rings, so it commutes with any ring homomorphism.

Exercise. Express Riesz representation theorem as a natural isomorphism

$$\begin{tikzcd} \mathbf{CHaus} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow"] & \mathbf{Ban} \end{tikzcd}$$

Question. What does it mean for an equivalence or isomorphism to be natural?

Functors and Natural Transformations

Given a finite dimensional k-vector space V with dual space $V^* = \mathrm{Hom}(V, k)$.

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Question. Recall that there is an isomorphism

$$\phi : V \longrightarrow V^{**} \quad v \longmapsto [f \mapsto f(v)]$$

Is this isomorphism *natural*?

Answer. This second isomorphism is natural and the first one isn't, because unlike the first one, the second one does not depend on a choice of bases. It is "uniform" across vector spaces (defined by the same formula). In other words, it is **functorial**.

Yoneda Lemma

Yoneda Lemma

"The Yoneda lemma is the hardest trivial thing in mathematics." – Dan Piponi

Theorem. Let $\mathscr{C}$ be a category where each $\mathrm{Hom}(X, Y)$ is a set, and let $A \in \mathscr{C}$. Let $F : \mathscr{C}^{\mathrm{op}} \to \mathbf{Set}$ be a functor, then there is an isomorphism

$$\mathrm{Hom}(\mathrm{Hom}(-, A), F) \cong F(A)$$

functorial in A and F (natural isomorphism as functors $\mathscr{C} \times \mathrm{Fun}(\mathscr{C}, \mathbf{Set}) \to \mathbf{Set}$).

Yoneda Lemma

Proof. For $\Phi : \mathrm{Hom}(-, A) \to F$ and $u = \Phi_A(\mathrm{id}_A)$. If $f : X \to A$ then

$$\begin{tikzcd}
\mathrm{Hom}(A, A) \arrow[r, "f_*"] \arrow[d, "\Phi_A"'] & \mathrm{Hom}(X, A) \arrow[d, "\Phi_X"] \\
F(A) \arrow[r, "Ff"'] & F(X)
\end{tikzcd}$$

commutes. Thus $\Phi_X(f) = (Ff)(u)$ is determined by u, which gives the isomorphism. This does not depend on any choice based on A or F, thus functorial in A and F.

Yoneda Lemma

All information of A is encoded in $\mathrm{Hom}(-, A)$, and vice versa.

Yoneda Lemma

The Yoneda lemma also implies that the functor

$$\begin{aligned} \mathscr{C} &\longrightarrow \mathrm{Fun}(\mathscr{C}^{\mathrm{op}}, \mathbf{Set}) \\ A &\longmapsto \mathrm{Hom}(-, A) \end{aligned}$$

called the Yoneda embedding, is fully faithful, i.e. we have

$$\mathrm{Hom}(\mathrm{Hom}(-, X), \mathrm{Hom}(-, Y)) \cong \mathrm{Hom}(X, Y)$$

for all $X, Y \in \mathscr{C}$.

Applications

A scheme X determines and is determined by its functor of points

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Plugging in $\mathrm{Spec}(k)$ in this functor gives k-rational points of X.

Application

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The Grassmannian is representable by a scheme.

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Example. elliptic curves, more generally algebraic curves of genus g

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Hard Solution. Develop the theory of Artin stacks and Deligne-Mumford stacks.

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and many more.