Category Theory Demystified

A Friendly Introduction to Abstract Nonsense

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Introduction

Rule 1. Don’t be intimidated by categories (or the fancy diagrams or buzzwords).

Introduction

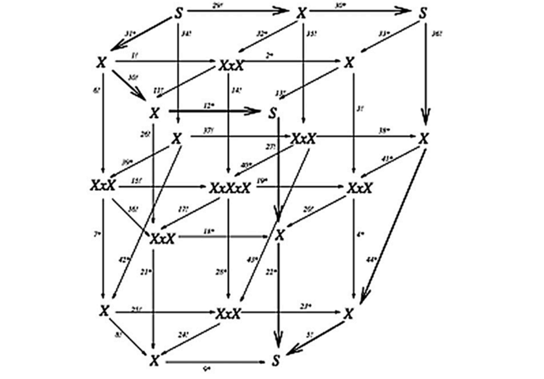

Figure: Voevodsky's 2-theory

Introduction

Category theory originated from Eilenberg and Mac Lane's study of algebraic topology.

GENERAL THEORY OF NATURAL EQUIVALENCES

BY

SAMUEL EILENBERG AND SAUNDERS MACLANE

CONTENTS

Figure: General Theory of Natural Equivalences

Mathematical objects frequently come with **morphisms** between them.

Objects	Morphisms
sets	functions
groups	group homomorphisms
rings	ring homomorphisms
k-vector spaces	k-linear transformations
topological spaces	continuous map
posets	monotone functions

What do they have in common?

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We denote $f : X \to Y$ for $f \in \mathrm{Hom}(X, Y)$ and $f \circ g$ for composition.

Introduction

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- (associativity) if $f \in \mathrm{Hom}(A, B)$, $g \in \mathrm{Hom}(B, C)$, $h \in \mathrm{Hom}(C, D)$, then

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- (identity) there exists $\mathrm{id}_X \in \mathrm{Hom}(X,X)$ for each $X \in \mathscr{C}$ such that

$$f \circ \mathrm{id}_A = f = \mathrm{id}_B \circ f$$

  for any $f \in \mathrm{Hom}(A,B)$.

Right away we have a lot of examples of “big” categories

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k-**Vect**	k-vector spaces	k-linear transformations
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However, categories do not have to be big, e.g. $\mathbb{N}$ is a category.

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This allows us to draw diagrams consisting of multiple morphisms.

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "g"'] & Y \\
Z \arrow[ur, "h"'] &
\end{tikzcd}$$

Terminology. We say that a diagram such as

$$\begin{tikzcd} X \arrow[r, "a"] \arrow[d, "f"'] & Y \arrow[d, "g"] \\ R \arrow[r, "b"'] & S \end{tikzcd}$$

commutes if for each pair of vertices A, B in the diagram, the maps produced following different paths from A to B are the same map (in this case, this means $a \circ g = b \circ f$).

The Yoga of Category Theory

Rule 2. Instead of construction, characterize things by their interactions with other things.

Instead of "injective map", think "left-cancellative map", i.e. a map $f : X \to Y$ s.t.,

$$f \circ g = f \circ h \implies g = h$$

for all $g, h : Z \to X$. This is called a **monomorphism**.

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Injectivity does not make sense in all categories, but in the ones that do, an injective map is obviously a monomorphism. The converse is not necessarily true!

Instead of "surjective map", think "right-cancellative map", i.e. a map $f : X \to Y$ s.t.,

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Answer. A morphism $f : X \to Y$ is an isomorphism if there exists $g : Y \to X$ s.t.

$$g \circ f = \mathrm{id}_X \quad \text{and} \quad f \circ g = \mathrm{id}_Y$$

in which case g is called the inverse of f.

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Example. Suppose G is a group. Instead of thinking a subgroup H as a subset of G closed under operation and identity, think of it as a pair (H, i) where H is a group and $i : H \to G$ a monomorphism, up to an equivalence $(H, i) \cong (H', i')$ if exists isomorphism $\phi : H \to H'$ s.t.

$$\begin{tikzcd} H \arrow[r, "i"] \arrow[d, "\phi"'] & G \\ H' \arrow[ur, "i'"'] & \end{tikzcd}$$

commutes, i.e. $i = i' \circ \phi$. In fact, this is how we define subobjects in a any category.

Exercise. Dually, how would you characterize quotient objects of an object in a category?

Let $A, B \in \mathscr{C}$, where we have a notion of product $A \times B$ e.g. if $\mathscr{C} \in \{\mathbf{Set}, \mathbf{Grp}, \mathbf{Ring}\}$.

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Instead of thinking $A \times B$ as pairs of elements (with possible additional structure),

Think of $A \times B$ as (P, π_A, π_B) where $P \in \mathscr{C}$ and

$$\pi_A : P \to A$$
$$\pi_B : P \to B$$

are morphisms satisfying the universal property of products.

Universal Property of Products. For all $Q \in \mathscr{C}$ and $\tau_A : Q \to A$ and $\tau_B : Q \to B$, there exists a unique morphism $q : Q \to P$ s.t. $\tau_A = \pi_A \circ q$ and $\tau_B = \pi_B \circ q$.

$$
\begin{tikzcd}
 & Q \arrow[dl, "\tau_B"'] \arrow[d, dashed, "q"] \arrow[dr, "\tau_A"] & \\
B & P \arrow[l, "\pi_B"] \arrow[r, "\pi_A"'] & A
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Here (Q, τ_A, τ_B) is a “test” to find the “smallest/universal product” (P, π_A, π_B).

In fact, this is how we define products in an arbitrary category.

Exercise. Try formulating the idea of coproducts, the dual notion to products.

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Here, we will talk about one example: topological groups.

Definition. A **topological group** is a group G with a topology such that the maps

$$\begin{aligned} m &: G \times G \to G \quad & (g,h) &\mapsto gh \\ \mathrm{inv} &: G \to G & g &\mapsto g^{-1} \end{aligned}$$

are continuous (where $G \times G$ has the product topology).

The Yoga of Category Theory

Instead of “topological group”, think “group object in **Top**”.

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Definition. A **group object** in a category $\mathscr{C}$ with finite products is an object $G \in \mathscr{C}$ with

$$\begin{aligned} m &: G \times G \to G \\ e &: 1 \to G \\ \mathrm{inv} &: G \to G \end{aligned}$$

where 1 is the terminal object (the object such that there exists a unique $X \to 1$ for each X) satisfying the “group axioms”, i.e. the following three diagrams commute.

$$\begin{tikzcd} G \times G \times G \arrow[r, "(\mathrm{id}, m)"] \arrow[d, "(m, \mathrm{id})"'] & G \times G \arrow[d, "m"] \\ G \times G \arrow[r, "m"'] & G \end{tikzcd} \quad \begin{tikzcd} G \arrow[r, "(\mathrm{id}, e)"] \arrow[d, "(e, \mathrm{id})"'] \arrow[dr, "\mathrm{id}"] & G \times G \arrow[d, "m"] \\ G \times G \arrow[r, "m"'] & G \end{tikzcd} \quad \begin{tikzcd} G \arrow[r, "(\mathrm{inv}, \mathrm{id})"] \arrow[d, "(\mathrm{id}, \mathrm{inv})"'] \arrow[dr, "e"] & G \times G \arrow[d, "m"] \\ G \times G \arrow[r, "m"] & G \end{tikzcd}$$

Universal Properties

Rule 3. Always define things (and think of things) in terms of their universal properties.

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- It is better to define an object by what it does instead of what it is concretely (and to giving a concrete construction, it suffice to check it satisfies the universal property), and this is often more elegant and conceptual than a concrete construction.
- It allows conceptual non-element-wise proofs.
- It allows for easier abstractions and analogies.

Universal Properties

What do universal properties do?

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Example. (Universal Property of Tensor Products) Let V, W be k-vector spaces, their tensor product is a pair $(V \otimes W, \otimes)$ where $V \otimes W$ is a k-vector space and $\otimes : V \times W \to V \otimes W$ a bilinear map such that for every pair $(V \otimes' W, \otimes')$ where $V \otimes' W$ is a k-vector space and $\otimes' : V \times W \to V \otimes' W$ a bilinear map, exists unique $h : V \otimes W \to V \otimes' W$ s.t. $\otimes' = h \circ \otimes$.

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We use a "test" $(V \otimes' W, \otimes')$ to find "smallest/universal" $(V \otimes W, \otimes)$.

Universal Properties

Definition. The initial object of a category $\mathscr{C}$ is an object $I \in \mathscr{C}$ such that for each object $X \in \mathscr{C}$ there exists a unique morphism $I \to X$. The initial object I is unique up to (a unique) isomorphism. Dually, one could define the terminal object.

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Suppose $\mathscr{C}$ is the category where objects consist of all pairs $(V \otimes W, \otimes)$ (where $V \otimes W$ is a k-vector space and $\otimes : V \times W \to V \otimes W$ a bilinear map), and a morphism

$$h : (V \otimes W, \otimes) \to (V \otimes' W, \otimes')$$

is a linear map $h : V \otimes W \to V \otimes' W$ such that $\otimes' = h \circ \otimes$, then the universal property of tensor products is saying that the tensor product is the initial object in the category $\mathscr{C}$.

Universal Properties

All universal properties can be formulated this way, in other words, what all universal properties do is finding the initial (or dually, the terminal) object in a particular category. This category, in full generality, is the comma category $(X \downarrow F)$ (or dually $(F \downarrow X)$).

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Exercise. Let $\mathbb{Q}$ be the field of rational numbers, how would you characterize the field extension $\mathbb{Q} \hookrightarrow \mathbb{Q}(\sqrt{2})$ in terms of a universal property?

Universal Properties

Many structures defined by universal properties are generalized by limits and colimits.

Functors and Natural Transformations

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A contravariant functor is the same but with arrow reversed. Alternatively, a contravariant functor $\mathscr{C} \to \mathscr{D}$ is a covariant functor $\mathscr{C}^{\mathrm{op}} \to \mathscr{D}$, where $\mathscr{C}^{\mathrm{op}}$ reverses arrows in $\mathscr{C}$.

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We will focus on the first three perspectives.

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Example. $\pi_1 : \mathbf{pcTop}_* \to \mathbf{Grp}$ which sends a path-connected topological space to its fundamental group and a continuous function to its induced map.

Example. Let $\mathscr{C}$ be a category such that each $\mathrm{Hom}(X, Y)$ is a set. Fix $A \in \mathscr{C}$. Define the Hom-functor $\mathrm{Hom}(A, -) : \mathscr{C} \to \mathbf{Set}$ sending an object $X \mapsto \mathrm{Hom}(A, X)$ and a morphism $f \mapsto [g \mapsto f \circ g]$. We define dually the contravariant functor $\mathrm{Hom}(-, A) : \mathscr{C} \to \mathbf{Set}$.

Theorem. Given a path-connected topological group G, then $\pi_1(G)$ is abelian.

Proof. The usual proof in textbooks uses Eckmann-Hilton argument, but category theory gives us a more conceptual proof. The fundamental group functor

$$\pi_1 : \mathbf{pcTop} \to \mathbf{Grp}$$

preserves group objects since it preserves terminal object and products, therefore it sends group objects in **pcTop**, the path-connected topological groups, to group objects in **Grp**, which the reader may check, are precisely the abelian groups.

Functors and Natural Transformations

A functor $F : \mathscr{C} \to \mathscr{D}$ can be viewed as a diagram in $\mathscr{D}$ indexed by (or in the shape of) $\mathscr{C}$.

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d] & \bullet \arrow[dl] \\ \bullet & \end{tikzcd} \quad \longmapsto \quad \begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[dl, "h"] \\ C & \end{tikzcd}$$

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Thus, the diagram F commutes when for each $f, g : X \to Y$ in $\mathscr{C}$, we have $Ff = Fg$.

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Definition. A linear representation of G in k-Vect is a functor $F : \mathbf{B}G \to k\text{-Vect}$.

Let G be a group, then we can view G as a category $\mathbf{B}G$ called the delooping groupoid.

The category $\mathbf{B}G$ has one object $*$ and $\mathrm{Hom}(*, *) = G$ with group operation as composition.

Definition. A linear representation of G in k-Vect is a functor $F : \mathbf{B}G \to k$-Vect.

Exercise. Convince yourself this is equivalent to the usual notion of a representation of a group G, which is a (ρ, V) where V is a vector space and $\rho : G \to \mathrm{GL}(V)$ a homomorphism.

Definition. Let $F, G : \mathscr{C} \to \mathscr{D}$ be functors, then a natural transformation $\Phi : F \to G$ consists of a morphism $\Phi_X : F(X) \to G(X)$ for each $X \in \mathscr{C}$ such that each

$$\begin{tikzcd} F(X) \arrow[r, "\Phi_X"] \arrow[d, "Ff"'] & G(X) \arrow[d, "Gf"] \\ F(Y) \arrow[r, "\Phi_Y"'] & G(Y) \end{tikzcd}$$

commutes. Let $\mathrm{Fun}(\mathscr{C}, \mathscr{D})$ be the category of functors from $\mathscr{C}$ to $\mathscr{D}$, where morphisms are natural transformations, the isomorphisms in which are called natural isomorphisms.

Natural Transformations and Isomorphisms

Example. Define the natural transformation $\det : \mathrm{GL}_n(-) \to (-)^\times$ where for each ring R the morphism $\det_R : \mathrm{GL}_n(R) \to R^\times$ is given by the determinant map. This is a natural transformation because it is defined by the same formula

$$\det_R((a_{i,j})) = \sum_{\sigma \in S_n} \prod_i \operatorname{sgn}(\sigma)\, a_{i,\sigma(i)}$$

across rings, so it commutes with any ring homomorphism.

Question. What does it mean for an equivalence or isomorphism to be natural?

Given a finite dimensional k-vector space V with dual space $V^* = \mathrm{Hom}(V, k)$.

Functors and Natural Transformations

Given a finite dimensional k-vector space V with dual space $V^* = \mathrm{Hom}(V, k)$.

Question. Recall that, by choosing bases, there is an isomorphism

$$\phi : V \longrightarrow V^*$$

Is this isomorphism *natural*?

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Question. Recall that, by choosing bases, there is an isomorphism

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Question. Recall that there is an isomorphism

$$\phi : V \longrightarrow V^{**} \quad v \longmapsto [f \mapsto f(v)]$$

Is this isomorphism *natural*?

Answer. This second isomorphism is natural and the first one isn't, because unlike the first one, the second one does not depend on a choice of bases. It is "uniform" across vector spaces (defined by the same formula). In other words, it is **functorial**.

Yoneda Lemma

Yoneda Lemma

"The Yoneda lemma is the hardest trivial thing in mathematics." – Dan Piponi

Yoneda Lemma

Theorem. Let $\mathscr{C}$ be a category where each $\mathrm{Hom}(X, Y)$ is a set, and let $A \in \mathscr{C}$. Let $F : \mathscr{C} \to \mathbf{Set}$ be a functor, then there is an isomorphism

$$\mathrm{Hom}(\mathrm{Hom}(-, A), F) \cong F(A)$$

functorial in A and F (natural isomorphism as functors $\mathscr{C} \times \mathrm{Fun}(\mathscr{C}, \mathbf{Set}) \to \mathbf{Set}$).

Proof. For $\Phi : \mathrm{Hom}(-, A) \to F$ and $u = \Phi_A(\mathrm{id}_A)$. If $f : X \to A$ then

$$\begin{tikzcd}
\mathrm{Hom}(A, A) \arrow[r, "f_*"] \arrow[d, "\Phi_A"'] & \mathrm{Hom}(X, A) \arrow[d, "\Phi_X"] \\
F(A) \arrow[r, "Ff"'] & F(X)
\end{tikzcd}$$

commutes. Thus $\Phi_X(f) = (Ff)(u)$ is determined by u, which gives the isomorphism. This does not depend on any choice based on A or F, thus functorial in A and F.

Yoneda Lemma

All information of A is encoded in $\mathrm{Hom}(-, A)$, and vice versa.

Yoneda Lemma

The Yoneda lemma also implies that the functor

$$\begin{aligned} \mathscr{C} &\longrightarrow \mathrm{Fun}(\mathscr{C}^{\mathrm{op}}, \mathbf{Set}) \\ A &\longmapsto \mathrm{Hom}(-, A) \end{aligned}$$

called the Yoneda embedding, is fully faithful, i.e. we have

$$\mathrm{Hom}(\mathrm{Hom}(-, X), \mathrm{Hom}(-, Y)) \cong \mathrm{Hom}(X, Y)$$

for all $X, Y \in \mathscr{C}$.

Yoneda Lemma

A scheme determines and is determined by its functor of points

$$\mathrm{Hom}(-, A) \,:\, \mathbf{Sch}^{\mathrm{op}} \to \mathbf{Set}$$

Yoneda Lemma

A scheme determines and is determined by its functor of points

$$\mathrm{Hom}(-, A) \ : \ \mathbf{Sch}^{\mathrm{op}} \to \mathbf{Set}$$

This perspective gives rise Grothendieck's relative point of view. We can use the functor of points to create (fine) moduli spaces, and this led to the theory of stacks.

Other Applications

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- Representation theory
- Mathematical logic, type theory, functional programming
- Mathematical physics

Final words

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This is a new rap on the oldest of stories -

Functors on abelian categories.

If the functor is left exact

You can derive it and that's a fact

But first you must have enough injective

Objects in the category to stay active.

If that's the case - no time to lose;

Resolve injectively any way you choose.

Final words

Apply the functor and don't be sore -

The sequence ain't exact no more.

Here comes the part that is the most fun, Sir,

Take homology to get the answer.

On resolution it don't depend

All are chain homotory equivalent.

Hey, Mama, when your algebra shows a gap

Go over this Derived Functor Rap.

Paul Bressler, 1988

References

References