Complex Multiplication

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nplex Multiplication 2/18 August 2022

Outline of the talk

- Number-theoretic background
- 2 CM of elliptic curves
- 3 Generalization to abelian varieties
- 4 Acknowledgements

Number-theoretic background

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- Prime ideals \mathfrak{p} in K split in L. For example, in $\mathbb{Q}(i)/\mathbb{Q}$,

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, $(3) = (3)$, $(5) = (2+i)(2-i)$.

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• L/K is unramified if every prime in K is unramified in L

The case over $\mathbb Q$

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Complex Multiplication 4/18

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- Complex Multiplication resolves the case when the base field is an imaginary quadratic field, i.e., $\mathbb{Q}(\sqrt{-D})$ for some D > 0.
- These two are essentially the only known cases. The problem is wide open.
- Slogan: this piece of **arithmetic** information will be extracted from studying **geometric** objects, namely, elliptic curves.

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What is an elliptic curve?

- By an **elliptic curve** E/K, we understand
 - a one-dimensional nonsingular projective variety over K of genus one, together with a special point O ∈ E;
 - naïvely, it is a smooth curve given by a cubic equation

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- An elliptic curve can be endowed with a group structure.



Endomorphism of elliptic curves

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• Question: are there endomorphisms other than maps of form [m]?

CM of elliptic curves

• Write $\operatorname{End}(E)$ for the endomorphism ring of E.

Theorem 3

Let E/\mathbb{C} be an elliptic curve. Then either $\operatorname{End}(E) = \mathbb{Z}$ or $\operatorname{End}(E)$ is isomorphic to a subring R of $\mathbb{Q}(\sqrt{-D})$ for some D > 0.

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• An elliptic curve E/\mathbb{C} has **complex multiplication by** R if $R = \operatorname{End}(E)$ is the subring of an imaginary quadratic field.

Construction of class fields

Now we state the main result of CM of elliptic curves.

Theorem 4

Let K be an imaginary quadratic field. Let E/\mathbb{C} be an elliptic curve with CM by ring of integers \mathcal{O}_K . Then

- K(j(E)) is the maximal unramified extension of K
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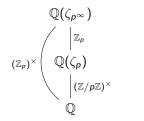
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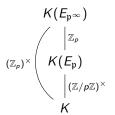
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- K(j(E)) is the maximal unramified extension of K
- $K(j(E), x(E_{tors}))$ is the maximal abelian extension of K.
- Slogan: *j*-invariant and the x-coordinate of torsion points generate abelian extensions of $\mathbb{Q}(\sqrt{-D})$ for some D > 0.

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- Substituting \mathbb{Q} by K an imaginary quadratic field, the role of ζ_{p^n} is played by the \mathfrak{p}^n -torsion points on E.
- We then study the *p*-adic *L*-series attached to an elliptic curve, which helps us understand the BSD conjecture.

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Facts about abelian varieties

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- Elliptic curves are one-dimensional abelian varieties.
- Hence abelian varieties are higher-dimensional analogues of elliptic curves. They have a structure of an abelian group.

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- A CM-algebra is a finite product of CM-fields.
- Roughly speaking, an abelian variety A/\mathbb{C} has CM iff $\operatorname{End}(A) \otimes \mathbb{Q}$ contains a CM-algebra of dimension 2 dim A.
- Sadly, we obtain some (not all!) finite abelian extensions of a CM-field.

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- The most general case, the main theorem over \mathbb{Q} , was proved by Langlands, Tate, and Deligne in the 1980s, called motivic CM theory.
- CM has deep relations with the BSD conjecture and other arithmetic theories.
- It also has fruitful applications to cryptography.

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Acknowledgements

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