ENSAE : Bayesian statistics

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SUR Model

The linear SUR model involves a set of regression equations with cross-equation param eter restrictions and correlated error terms having differing variances. Algebraically, the SUR model is given by:

$$y_j = X_j\beta_j + u_j, j = 1, \dots m, \text{ with } E[u_i u_j^T] = \begin{cases} -\omega_{ij} I, & (i \neq j) \\ -\omega_i^2 I, & (i = j) \end{cases}$$

we can model the problem with :

$$y = X\beta + u$$

where,

- $u \sim \mathcal{N}(0, \Omega \otimes I)$
- Ω is m × m symmetric matrix with diagonal elements $\{\omega_1^2, \dots \omega_m^2\}$
- X = diag$\{X_1, \dots X_m\}$

we can write the likehood function as :

$$L(y|\beta, \Omega) = \frac{1}{(2\pi)^{\frac{nm}{2}} |\Omega|^{\frac{n}{2}}} \exp[-\frac{1}{2} tr\{R\Omega^{-1}\}]$$

where $R = (r_{ij})$, $r_{ij} = (y_i - X_i\beta_i)^T (y_j - X_j\beta_j)$

Markov chain Monte Carlo Approach

In this approach, we suppose that we don't have a prior knowledge on the parameters. Therefore, we use a Jeffrey's invariant prior :

$$\pi(\beta, \Omega) = \pi(\beta)\pi(\Omega) \propto |\Omega|^{-\frac{m+1}{2}}$$

Where m is the number of different values of β.

Using this prior, we can obtain the joint posterior density function :

$$\pi(\beta, \Omega|D) \propto |\Omega|^{-\frac{n+m+1}{2}} \exp[(-\frac{1}{2}\{\mathrm{tr}(R\Omega^{-1})\}]$$

We can deduce the conditional posteriors $\pi(\beta|\Omega, D)$ and $\pi(\Omega|\beta, D)$:

$$\beta|\Omega, D \sim \mathcal{N}(\hat{\beta}, \hat{\Omega}_\beta) \text{ and } \Omega|\beta, D \sim IW(R, n)$$

where,

$\hat{\beta} = \{X^T(\Omega^{-1} \otimes I)X\}^{-1}X^T(\Omega^{-1} \otimes I)y$

$\hat{\Omega}_\beta = (X^T(\Omega^{-1} \otimes I)X)^{-1}$

We will implement this method using gibbs sampler.

We start by β_0, Ω_0. At each step t :

- We generate a new value β_t from $\mathcal{N}(\hat{\beta}_{t-1}, \hat{\Omega}_{t-1})$.
- We generate a new value Ω_t from $IW(R_t, n)$

In [ ]:
```
from utils import generate_SUR, Sampler, DefaultParams
import numpy as np
from numpy.linalg import inv
from scipy.stats import invwishart, invgamma
import matplotlib.pyplot as plt
import seaborn as sns
from scipy.stats import multivariate_normal as mvnorm
from scipy.stats import expon
import math
```

```python
import pandas as pd
sns.set()

import warnings
warnings.filterwarnings("ignore")
```

In [ ]:
```python
from abc import ABCMeta, abstractclassmethod
import numpy as np

class DefaultParams():
    def __init__(self, beta1 = np.array([3, -2]) , beta2 = np.array([2, 1]), omega = np.array([[0.1, -0.15],
                [-0.15, 0.2]]), n = 100 ) -> None:
        self._beta1 = beta1.reshape(-1,1)
        self._beta2 = beta2.reshape(-1,1)
        self._n = n
        self._omega = omega.reshape(2,2)

    @property
    def beta1(self):
        return self._beta1
    @property
    def beta2(self):
        return self._beta2
    @property
    def n(self):
        return self._n
    @property
    def omega(self):
        return self._omega

    def print(self):
        print("beta1 = ", self._beta1.reshape(-1))
        print("beta2 = ", self._beta2.reshape(-1))
        print("omega = ", self._omega)
        print("number of simulations for each model = ", self._n)



class Sampler(metaclass=ABCMeta):
    def __init__(self, SUR_data:dict) -> None:
        self.SUR_data = SUR_data
    @abstractclassmethod
    def sample(self):
```

```
        pass


def generate_SUR(params:DefaultParams=DefaultParams()):
    n = params.n
    omega = params.omega

    beta = np.concatenate((params.beta1, params.beta2), axis=0)

    x1 = np.random.rand(n,2) * 10 - 5
    x2 = np.random.rand(n,2) * 10 - 5
    X = np.block([
                [x1,              np.zeros((n, 2))],
                [np.ones((n, 2)), x2              ],
    ])
    Omega_combined = np.kron(omega, np.eye(n))
    U = np.random.multivariate_normal(mean=np.zeros(2*n), cov=Omega_combined).reshape(-1,1)
    Y = X @ beta + U
    assert Y.shape == (2*n, 1)

    SUR_data = {}
    SUR_data['X'] = X
    SUR_data['x1'] = x1
    SUR_data['x2'] = x2
    SUR_data['Y'] = Y
    return SUR_data
```

In [ ]:

```
# get the default parameters of SUR
params = DefaultParams()
params.print()
```

```
beta1 =  [ 3 -2]
beta2 =  [2 1]
omega =  [[ 0.1  -0.15]
 [-0.15  0.2 ]]
number of simulations for each model =  100
```

In [ ]:

```
class MCMC_Sampler(Sampler):
    def __init__(self, SUR_data, num_iterations=11000, burn_in_steps=1000):
        super().__init__(SUR_data)
        self.num_iterations = num_iterations
        self.burn_in_steps = burn_in_steps
```

```python
    def _sample_beta(self, omega):
        omega_inv = inv(omega)
        n = params.n
        omega_inv_tensor = np.block([
                        [omega_inv[0,0]*np.eye(n), omega_inv[0,1]*np.eye(n)],
                        [omega_inv[1,0]*np.eye(n), omega_inv[1,1]*np.eye(n)],
        ])
        X = self.SUR_data['X']
        Y = self.SUR_data['Y']
        X_transposed = np.transpose(X)
        beta_hat = inv(X_transposed @ omega_inv_tensor @ X) @ X_transposed @ omega_inv_tensor @ Y
        omega_hat = inv(X_transposed @ omega_inv_tensor @ X)
        beta = np.random.multivariate_normal(mean=beta_hat.reshape(-1,), cov=omega_hat)
        return beta

    def _sample_omega(self, beta):
        y1 = self.SUR_data['Y'][:params.n]
        y2 = self.SUR_data['Y'][params.n:]
        x1 = self.SUR_data['x1']
        x2 = self.SUR_data['x2']
        m1 = y1 - x1 @ beta[:2].reshape(-1,1)
        m2 = y2 - x2 @ beta[2:].reshape(-1,1)
        R = np.array([[(np.transpose(m1) @ m1).item(), (np.transpose(m1) @ m2).item()],
                      [(np.transpose(m2) @ m1).item(), (np.transpose(m2) @ m2).item()]])
        omega = invwishart.rvs(df=params.n, scale=R)
        return omega

    def sample(self):
        beta_simu = []
        omega_simu = []
        omega = np.random.rand(2,2) # initialisation
        for _ in range(self.num_iterations):
            beta = self._sample_beta(omega)
            omega = self._sample_omega(beta)
            beta_simu.append(beta)
            omega_simu.append(omega)
        return np.array(beta_simu[self.burn_in_steps:]), np.array(omega_simu[self.burn_in_steps:])
```

In [ ]:

```python
mcmc_sampler = MCMC_Sampler(SUR_data=generate_SUR())
beta_simu, omega = mcmc_sampler.sample()

beta1 = beta_simu[:,:2]
beta2 = beta_simu[:,2:]
```

In [ ]:
```
result_mcmc = pd.DataFrame()
result_mcmc['beta11'] = beta1[:,0]
result_mcmc['beta12'] = beta1[:,1]
result_mcmc['beta21'] = beta2[:,0]
result_mcmc['beta22'] = beta2[:,1]
result_mcmc['omega11'] = omega[:,0,0]
result_mcmc['omega12'] = omega[:,0,1]
result_mcmc['omega22'] = omega[:,1,1]
result_mcmc['method'] = 'MCMC'
```

In [ ]:
```
def plot_beta(beta, title):
    fig = plt.figure(figsize=(15, 15))
    l = int(np.sqrt(len(beta[:,0])))
    plt.subplot(4, 2, 1)
    plt.plot(beta[:,0], label='beta1_1')
    plt.plot(np.ones(len(beta[:,0]))*params.beta1[0], label='real_beta1_1',linestyle = '-')
    plt.xlabel("#Iterations")
    plt.legend()
    plt.subplot(4,2,2)
    plt.hist(beta[:,0],bins = l , label='beta1_1')
    plt.vlines(x=params.beta1[0],ymin=0,ymax=300,label = 'real_beta1_1',color = 'r')
    plt.legend()

    plt.subplot(4, 2, 3)
    plt.plot(beta[:,1], label='beta1_2')
    plt.plot(np.ones(len(beta[:,0]))*params.beta1[1], label='real_beta1_2',linestyle = '-')
    plt.legend()
    plt.subplot(4,2,4)
    plt.hist(beta[:,1],bins = l, label='beta1_2')
    plt.vlines(x=params.beta1[1],ymin=0,ymax=300,label = 'real_beta1_2',color = 'r')
    plt.legend()

    plt.subplot(4, 2, 5)
    plt.plot(beta[:,2], label='beta2_1')
    plt.plot(np.ones(len(beta[:,2]))*params.beta2[0], label='real_beta2_1',linestyle = '-')
    plt.legend()
    plt.subplot(4,2,6)
    plt.hist(beta[:,2],bins = l, label='beta2_1')
    plt.vlines(x=params.beta2[0],ymin=0,ymax=300,label = 'real_beta2_1',color = 'r')
    plt.legend()

    plt.subplot(4, 2, 7)
```

```
    plt.plot(beta[:,3], label='beta2_2')
    plt.plot(np.ones(len(beta[:,3]))*params.beta2[1], label='real_beta2_2',linestyle = '-')
    plt.subplot(4,2,8)
    plt.hist(beta[:,3],bins = l, label='beta2_2')
    plt.vlines(x=params.beta2[1],ymin=0,ymax=300,label = 'real_beta2_2',color = 'r')
    plt.legend()

    fig.suptitle(title, fontsize=20)
    # plt.ylim(-3, 4)
    fig.tight_layout()

    plt.legend()
    plt.show()
```

In [ ]:

```
def plot_omega(omega, title):
    fig = plt.figure(figsize=(15, 15))
    l = int(np.sqrt(len(omega[:,0,0])))
    plt.subplot(3, 2, 1)
    plt.plot(omega[:,0,0], label='omega1')
    plt.plot(np.ones(len(omega[:,0,0]))*params.omega[0,0], label='real_omega1_1',linestyle = '-')
    plt.xlabel("#Iterations")
    plt.legend()

    plt.subplot(3,2,2)
    plt.hist(omega[:,0,0], bins = l, label='omega1')
    plt.vlines(x=params.omega[0,0],ymin=0,ymax=300,label = 'real_omega1_1',color = 'r')
    plt.legend()


    plt.subplot(3,2,3)
    plt.plot(omega[:,0,1], label='omega1_2')
    plt.plot(np.ones(len(omega[:,0,0]))*params.omega[1,0], label='real_omega1_2',linestyle = '-')
    plt.legend()

    plt.subplot(3,2,4)
    plt.hist(omega[:,0,1], bins = l,label='omega1_2')
    plt.vlines(x=params.omega[0,1],ymin=0,ymax=300,label = 'real_omega1_2',color = 'r')
    plt.legend()

    plt.subplot(3,2,5)
    plt.plot(omega[:,1,1], label='omega2')
    plt.plot(np.ones(len(omega[:,0,0]))*params.omega[1,1], label='real_omega2_2',linestyle = '-')
```

```
    plt.subplot(3,2,6)
    plt.hist(omega[:,1,1], bins = l,label='omega2_2')
    plt.vlines(x=params.omega[1,1],ymin=0,ymax=300,label = 'real_omega2_2',color = 'r')
    plt.legend()

    fig.suptitle(title, fontsize=20)
    fig.tight_layout()
    fig.subplots_adjust(top=0.95)
    # plt.ylim(-0.1, 0.3)
    plt.legend()
    plt.show()
```

In [ ]:
```
print(f'Mean: {np.mean(beta1, axis=0)};')
print(f'real: {params.beta1.reshape(-1)};')
```

```
Mean: [ 3.00026783 -1.98214178];
real: [ 3 -2];
```

In [ ]:
```
print(f'Mean: {np.mean(beta2, axis=0)};')
print(f'real: {params.beta2.reshape(-1)};')
```

```
Mean: [2.00231161 0.95445803];
real: [2 1];
```

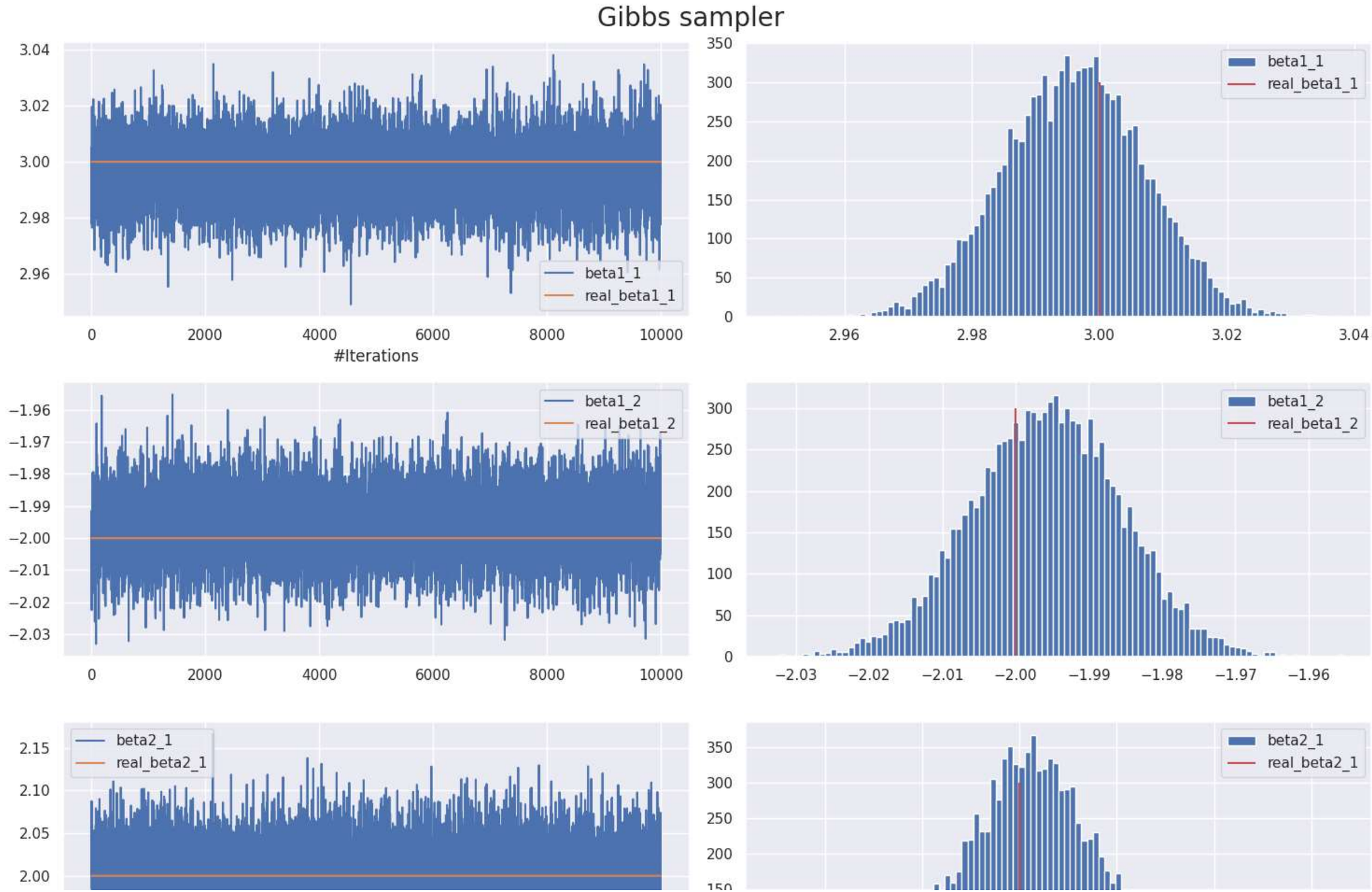
In [ ]:
```
print(f'Mean: {np.mean(beta1, axis=0)};')
print(f'real: {params.beta1.reshape(-1)};')
```

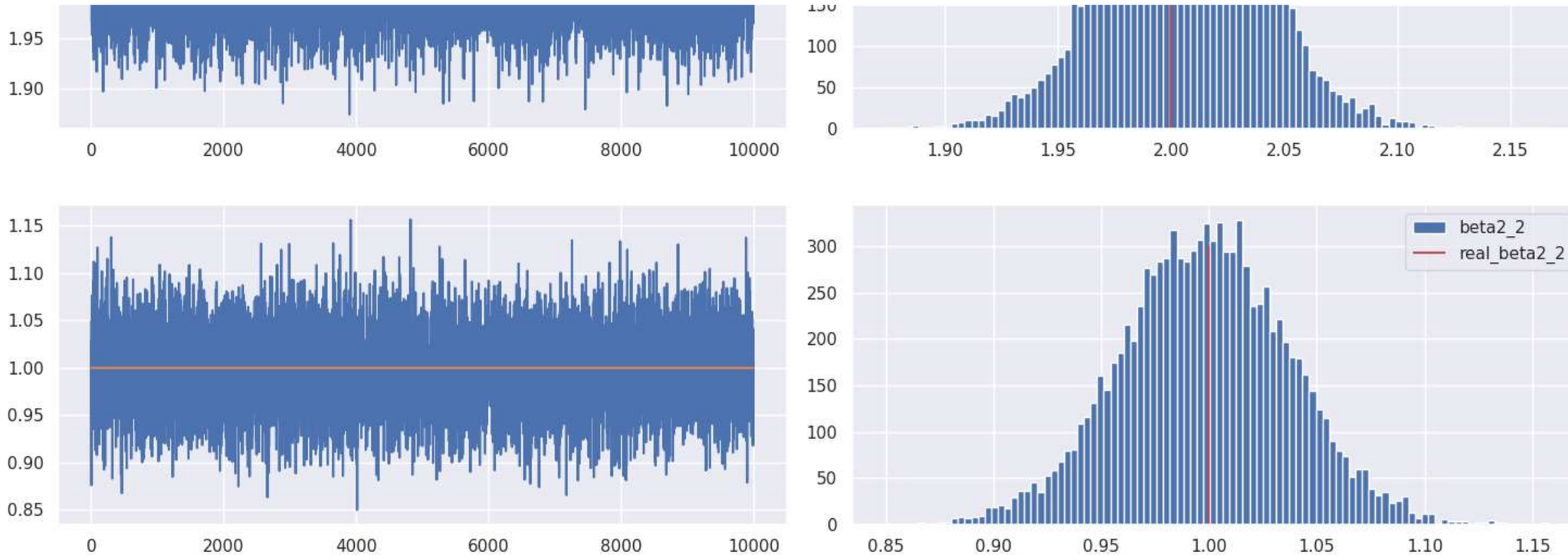
```
Mean: [ 3.00026783 -1.98214178];
real: [ 3 -2];
```

In [ ]:
```
print(f'Mean: {np.mean(omega, axis=0)};')
print(f'real: {params.omega.reshape(-1)};')
```

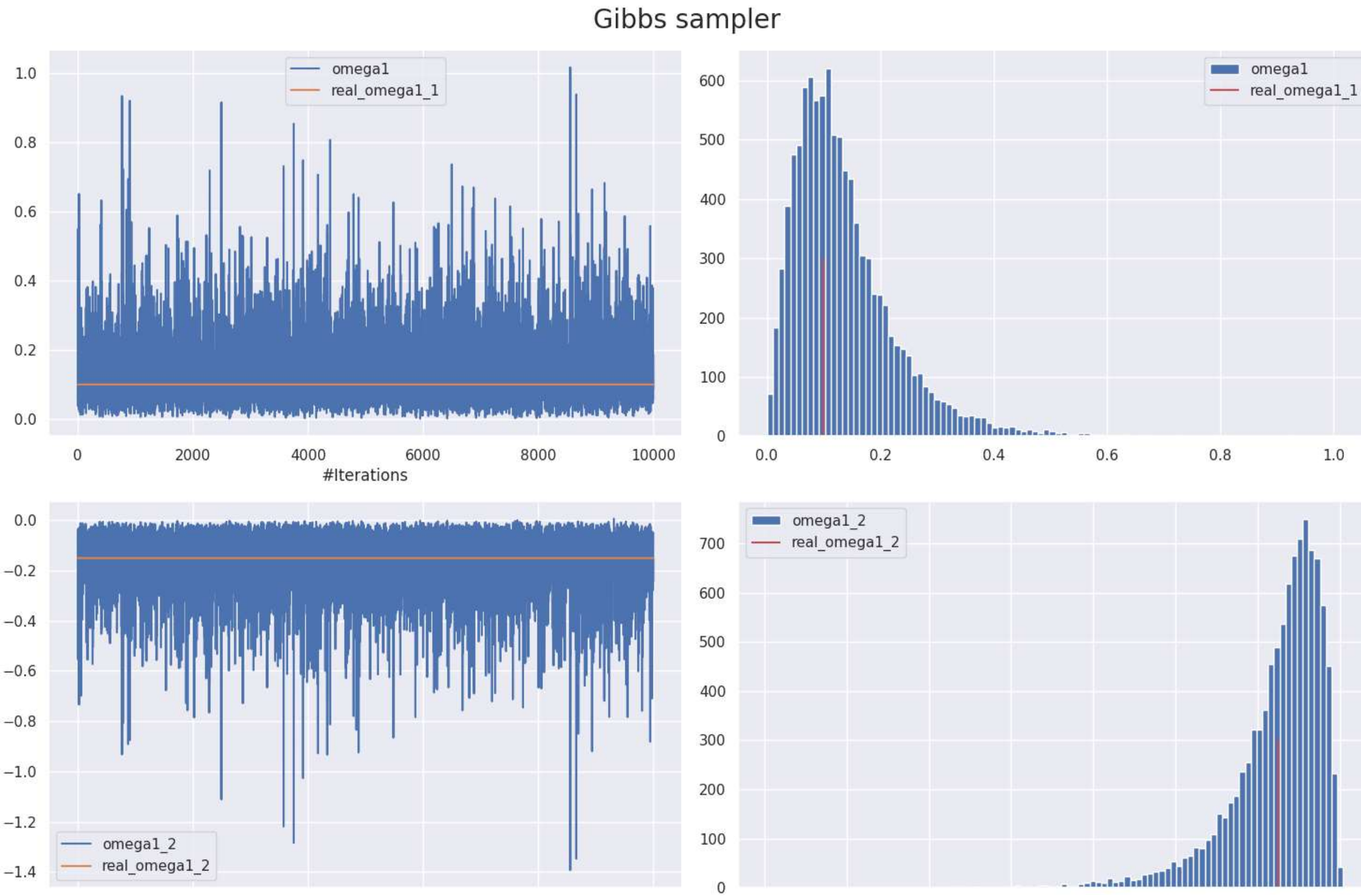
```
Mean: [[ 0.13646275 -0.15300753]
 [-0.15300753  1.14121271]];
real: [ 0.1  -0.15 -0.15  0.2 ];
```

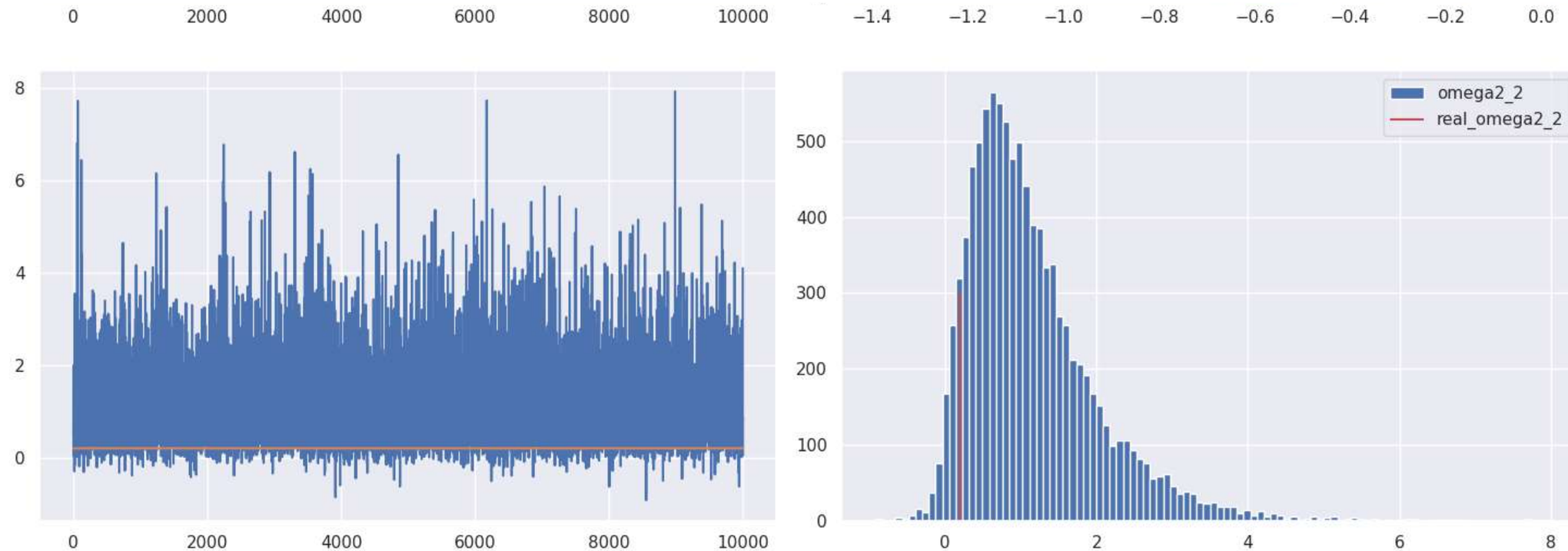
In [ ]:
```
plot_beta(beta=beta_simu, title="Gibbs sampler")
```


Gibbs sampler
beta1_1
real_beta1_1
#Iterations
beta1_2
real_beta1_2
beta2_1
real_beta2_1



In [ ]:
```
plot_omega(omega=omega, title="Gibbs sampler")
```


Gibbs sampler
omega1
real_omega1_1
#Iterations
omega1_2
real_omega1_2



Direct Monte Carlo Approach

In this approach, we apply a DMC on a transformed model.

$$\begin{cases} y_1 = X_1\beta_1 + e_1 \equiv Z_1 b_1 + e_1 \\ y_j = X_j\beta_j + \sum_{l=1}^{j-1} \rho_{jl}(y_l - X_l\beta_l)e_j \equiv Z_j b_j + e_j \end{cases}$$

where the n × (pj + j − 1) matrices Z_j are functions of $\beta_{j-1}, \ldots \beta_1$ and

$$E[e_i e_j^T] = \begin{cases} O, (i \neq j) \\ \sigma_i^2 I, (i = j) \end{cases}$$

and Σ = diag$\{\sigma_1^2, \ldots \sigma_m^2\}$

Note that, there is a one to one relation between the parameters of the SUR and the model above.

This transformation helps us to do direct Monte Carlo simulations.

The prior density :

$$\pi(b,\Sigma) \propto |\Omega(b,\Sigma)|^{-\frac{m+1}{2}}|J| = \prod_{j=1}^{m}(\sigma_j^2)^{-\frac{m+1}{2}}\prod_{j'=1}^{m-1}(\sigma_{j'}^2)^{m-j'} = \prod_{j=1}^{m}(\sigma_j^2)^{\frac{m-2j-1}{2}}$$

The joint posterior density is :

$$\pi(b_j|b_{j-1},\dots b_1,\sigma_j^2,D) = \mathcal{N}(\hat{b}_j,\sigma_j^2(Z_j^TZ_j)^{-1})$$

$$\pi(\sigma_j^2|b_{j-1},\dots b_1,D) = IG(\frac{\hat{\gamma}_j}{2},\frac{\hat{\nu}_j}{2})$$

where,

- $\hat{b}_j = (Z_j^TZ_j)^{-1}Z_j^Ty_j$
- $\hat{\gamma}_j = (y_j - Z_j\hat{b}_j)^T(y_j - Z_j\hat{b}_j)$
- $\hat{\nu}_j = n - m - p_j + j + 1$

To simplify the formula, we will focus on the case m = 2

For m = 2,

- $b_1 = \beta_1$
- $b_2 = (\beta_2, \rho_{21})$

~

- $Z_1 = X_1$
- $Z_2 = (X_2, y_1 - X_1b_1)$

~

- $\omega_1^2 = \sigma_1^2$
- $\omega_2^2 = \rho_{21}\sigma_1^2 + \sigma_2^2$
- $\omega_{12} = \rho_{21}\sigma_1^2$

~

- $\rho_{21} = \frac{\omega_{12}}{\omega_1^2}$

In [ ]:
```
class DMC_Sampler(Sampler):
    def __init__(self, SUR_data, params = DefaultParams(), n_samples=10000) -> None:
        super().__init__(SUR_data)
        self.n_samples = n_samples
        self._params = params

    def sample(self):
        y1 = self.SUR_data['Y'][:self._params.n]
        y2 = self.SUR_data['Y'][self._params.n:]
        x1 = self.SUR_data['x1']
        x2 = self.SUR_data['x2']

        z1 = x1.copy()
        b1_hat = inv(z1.T@z1)@z1.T@y1
        gamma1_hat = ((y1 - z1@b1_hat).T@(y1 - z1@b1_hat)).reshape(-1)
        nu1_hat = self._params.n - 2

        #generate sigma1 and b1
        sigma1 = invgamma.rvs(a = nu1_hat/2,scale=gamma1_hat/2,size = self.n_samples)
        b1 = np.array([np.random.multivariate_normal(mean = b1_hat.reshape(-1), cov = sigma1_k*inv(z1.T@z1)) for sigma1_k in  sigma1])

        #generate sigma2 and b2
        z2 = np.array([np.hstack((x2, y1 - x1@b.reshape(-1,1))) for b in b1])
        b2_hat = np.array([inv(z.T@z)@z.T@y2 for z in z2])
        gamma2_hat = np.array([((y2 - z2[i]@b2_hat[i]).T@(y2 - z2[i]@b2_hat[i])).reshape(-1) for i in range(len(b2_hat))])
        nu2_hat = self._params.n - 1
        sigma2 = np.array([invgamma.rvs(a = nu2_hat/2,scale=g/2) for g in gamma2_hat])
        b2 = np.array([np.random.multivariate_normal(mean = b2_hat[i].reshape(-1), cov = sigma2[i]*inv(z2[i].T@z2[i])) for i in  range(len(sigma2))])

        #get the initial parameters
        beta1 = b1
        beta2 = b2[:,:2]
        rho21 = b2[:,-1]
        omega1 = sigma1
        omega2 = rho21*sigma1 + sigma2
        omega1_2 = rho21*sigma1
        omega = np.array([np.array([[o1,o12],[o21,o22]]) for o1,o12,o21,o22 in zip(omega1,omega1_2,omega1_2,omega2)])
```

```
        return beta1, beta2, omega
```

In [ ]:
```
DMC_sampler = DMC_Sampler(SUR_data=generate_SUR())
beta1, beta2, omega= DMC_sampler.sample()
```

In [ ]:
```
result_dmc = pd.DataFrame()
result_dmc['beta11'] = beta1[:,0]
result_dmc['beta12'] = beta1[:,1]
result_dmc['beta21'] = beta2[:,0]
result_dmc['beta22'] = beta2[:,1]
result_dmc['omega11'] = omega[:,0,0]
result_dmc['omega12'] = omega[:,0,1]
result_dmc['omega22'] = omega[:,1,1]
result_dmc['method'] = 'DMC'
```

In [ ]:
```
print(f'Mean: {np.mean(beta1, axis=0)};')
print(f'real: {params.beta1.reshape(-1)};')
```

```
Mean: [ 3.00023696 -2.00961859];
real: [ 3 -2];
```

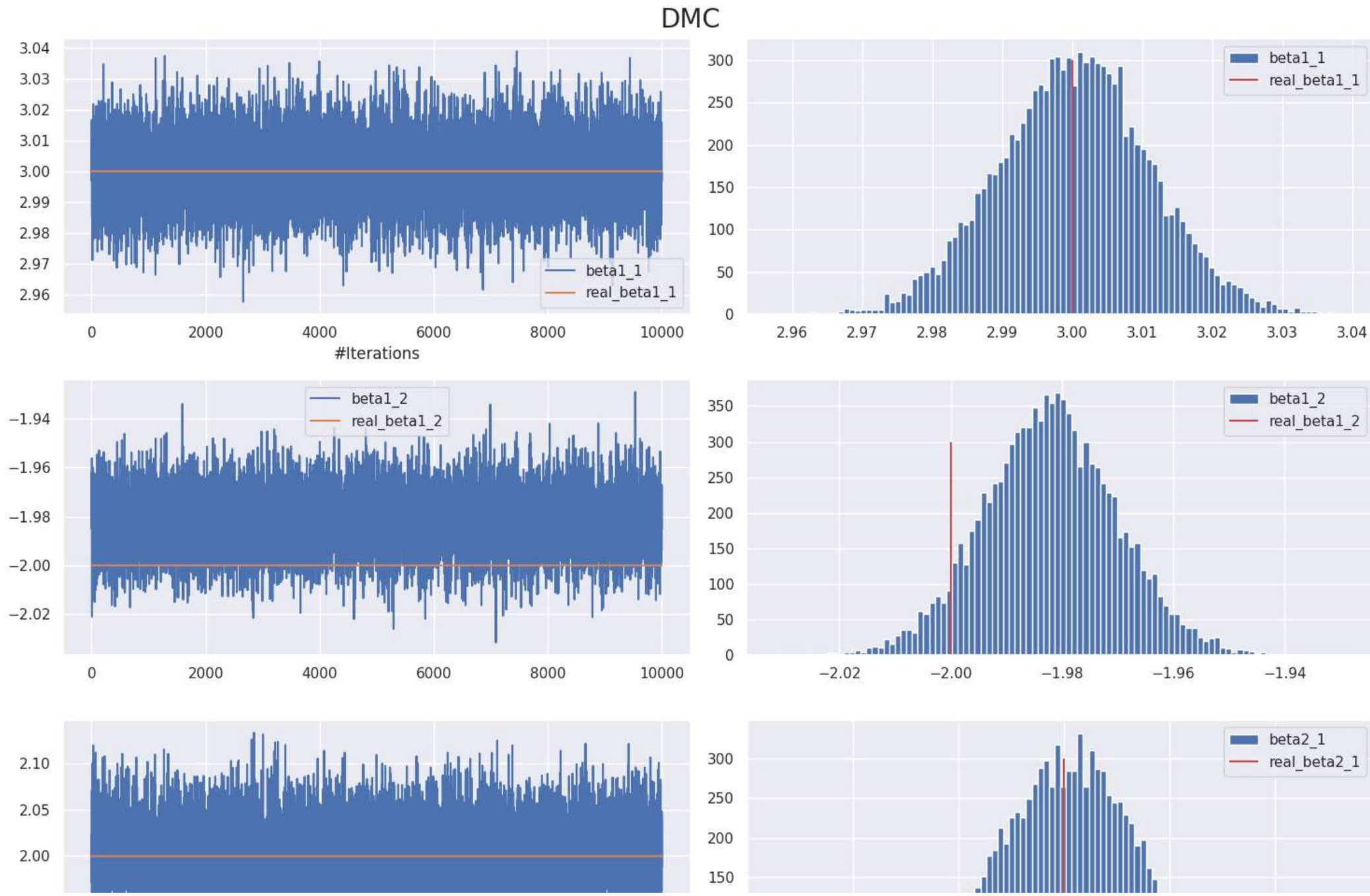
In [ ]:
```
print(f'Mean: {np.mean(beta2, axis=0)};')
print(f'real: {params.beta2.reshape(-1)};')
```

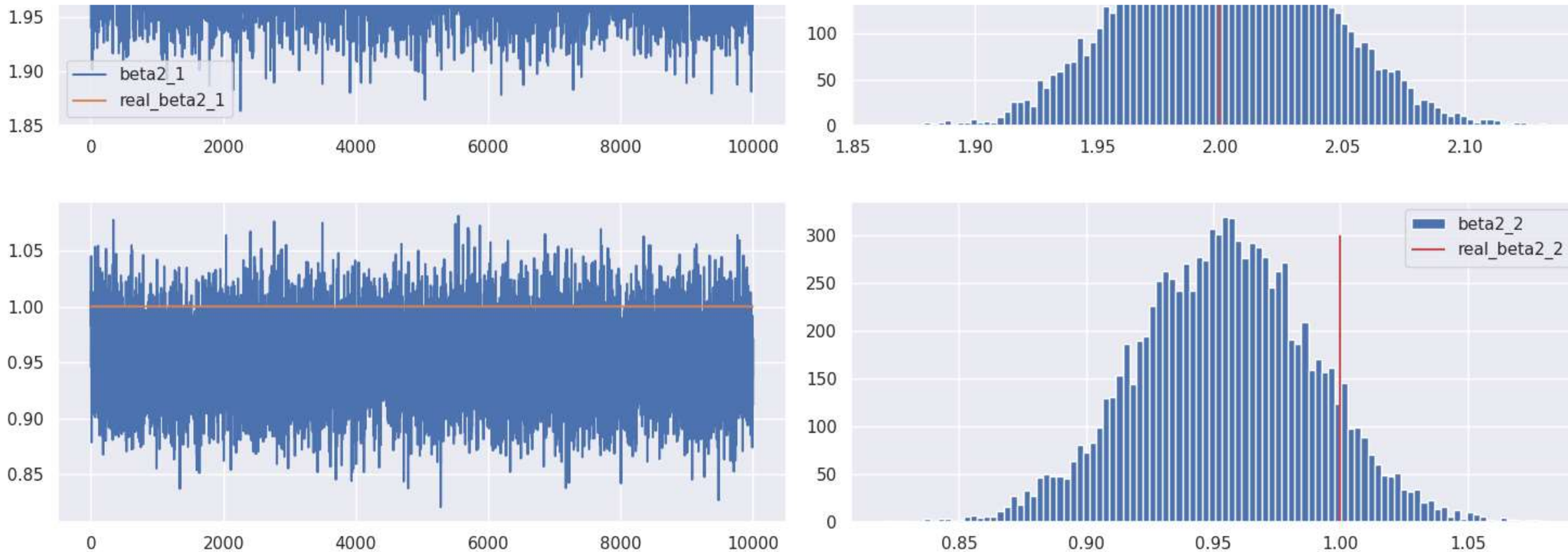
```
Mean: [1.9813913  1.03464623];
real: [2 1];
```

In [ ]:
```
print(f'Mean: {np.mean(omega, axis=0)};')
print(f'real: {params.omega};')
```

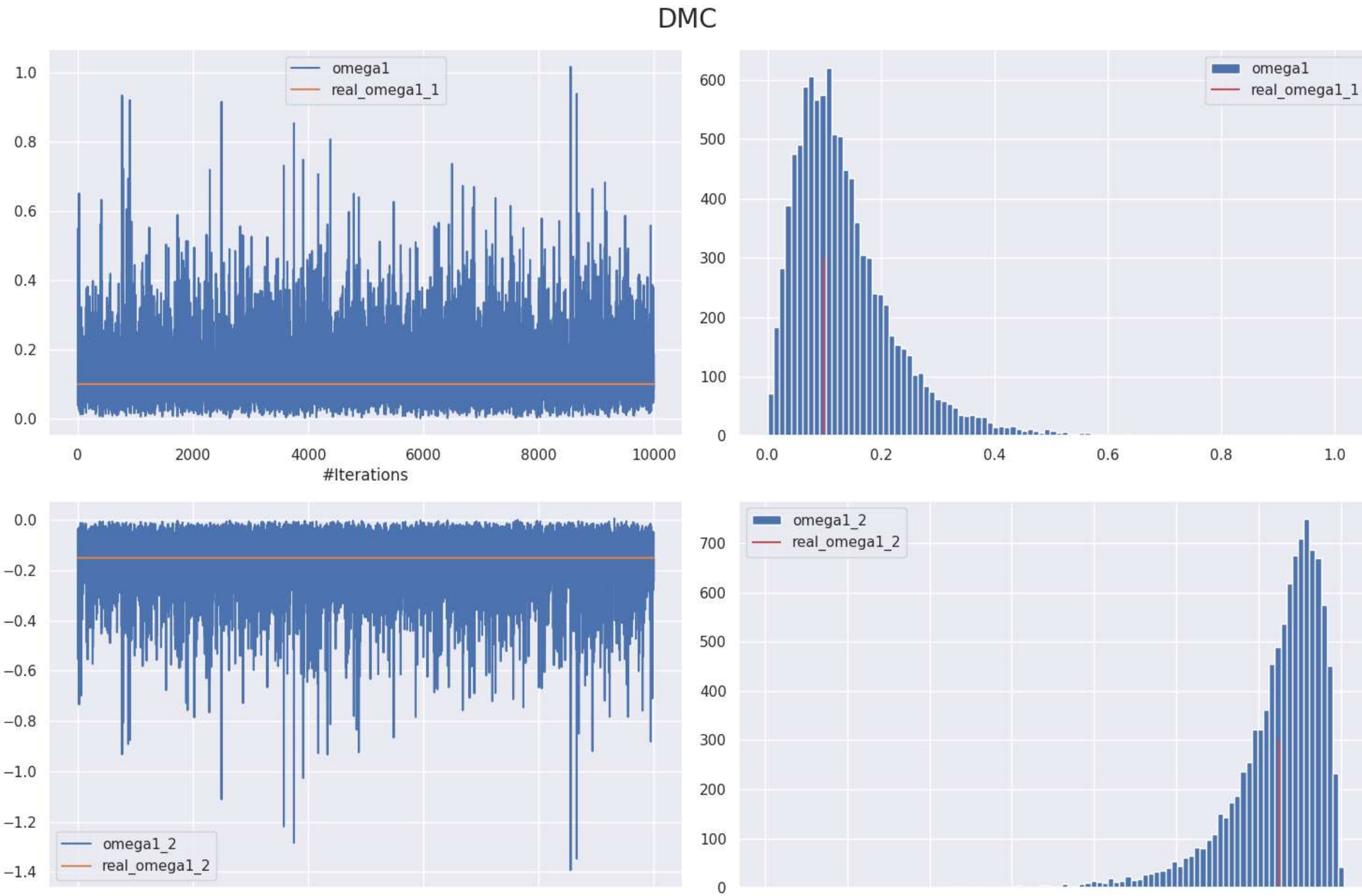
```
Mean: [[ 0.10729699 -0.16143749]
 [-0.16143749  0.89877817]];
real: [[ 0.1  -0.15]
 [-0.15  0.2 ]];
```

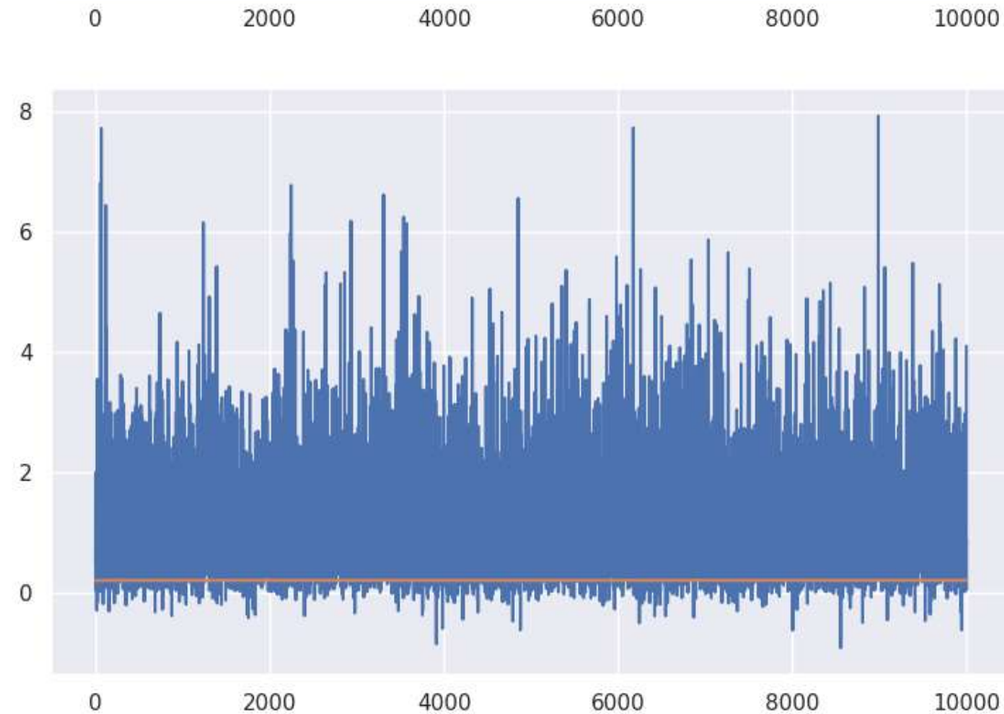
In [ ]:
```
beta = np.hstack((beta1.reshape(-1,2),beta2.reshape(-1,2)))
plot_beta(beta, title="DMC")
```

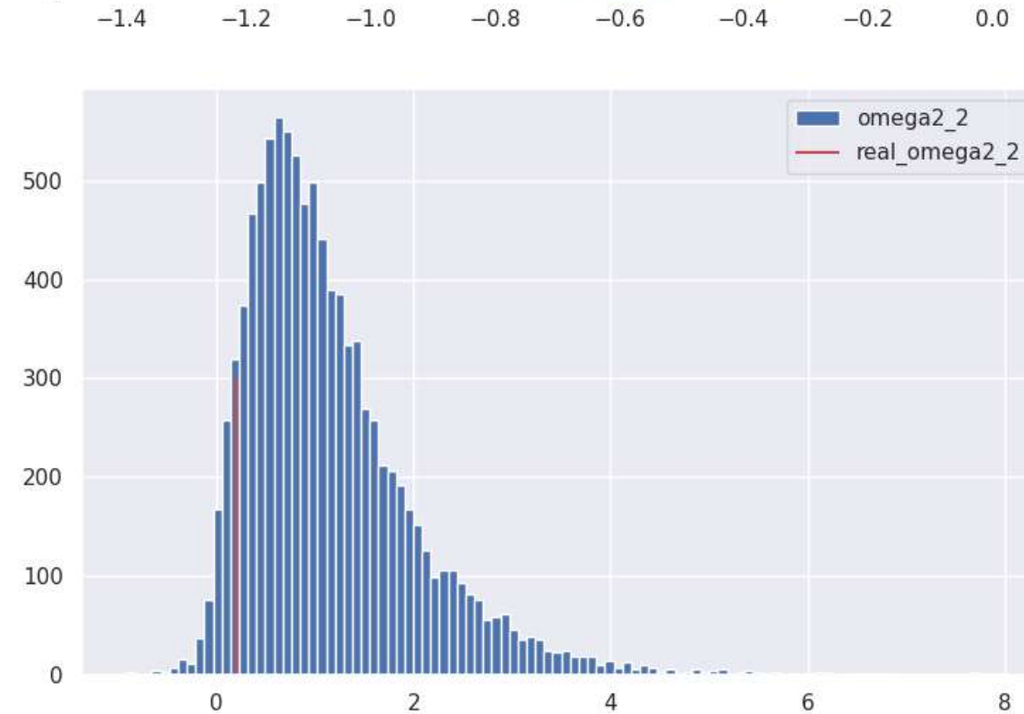

DMC
beta1_1
real_beta1_1
#Iterations
beta1_2
real_beta1_2
beta2_1
real_beta2_1



In [ ]:
```
plot_omega(omega, title="DMC")
```


DMC
omega1
real_omega1_1
#Iterations
omega1_2
real_omega1_2





Direct Monte Carlo Importance Sampling Approach

We use again the transformed model but with hierarchical prior structure.

$$\pi(\beta, \Omega, \lambda) = \pi(\beta|\lambda)\pi(\Omega)\pi(\lambda)$$

- $\pi(\beta|\lambda) \propto \exp\{-\frac{1}{2}(\beta - \beta_0)^T \Gamma(\lambda, A)(\beta - \beta_0)\}$
- $\pi(\Omega) \propto |\Omega|^{-\frac{m+1}{2}}$
- $\pi(\lambda) = \prod_{j=1}^{m} \pi(\lambda_j) \propto \prod_{j=1}^{m} (\lambda_j)^{a_0 - 1} \exp\{-\lambda_j / b_0\}$

The conditional posterior probabilities:

- $\pi(b_j | b_{j-1} \ldots, b_1, \sigma_j^2, \lambda, D) = \mathcal{N}(\hat{b}_j, \sigma_j^2 (Z_j^T Z_j + \gamma_j D_j)^{-1})$
- $\pi(\sigma_j^2 | b_{j-1}, \ldots b_1, \lambda, D) = IG(\hat{a}_j / 2, \hat{h}_j / 2)$

With,

- $\hat{b}_j = (Z_j^T Z_j + \gamma_j D_j)^{-1}[(Z_j^T Z_j)^{-1}\tilde{b} + \lambda_j D_j b_0]$
- $\tilde{b}_j = (Z_j^T Z_j)^{-1} Z_j^T y_j$
- $\hat{a}_j = n - m - 2a_0 + 2j + 1$
- $\hat{h}_j = \gamma_j / b_0 + (n - p + j - 1)\hat{s}_j^2 + (\hat{b}_j - b_{0j})^T (Z_j^T Z_j + \gamma_j D_j)^{-1} (\hat{b}_j - b_{0j})$
- $\hat{s}_j^2 = (y_j - Z_j \tilde{b}_j)^T (y_j - Z_j \tilde{b}_j)/(n - p_j + j - 1)$

To generate λ, we will use importance sampling :

$$w_k = \frac{\pi(b^{(k)}, \Sigma^{(k)}, \lambda^{(k)}|y)}{\pi(b^{(k)}, \Sigma^{(}k)|\lambda^{(k)}, y)\pi(\lambda^{(k)})^s}$$

For simplicity, we will consider the case m = 2

Let us suppose that π_s as inverse gamma of parameter (5,0.005)

In [ ]:
```
class DMC_IS_Sampler(Sampler):
    def __init__(self, SUR_data, params = DefaultParams(), n_samples=1000) -> None:
        super().__init__(SUR_data)
        self.n_samples = n_samples
        self._params = params



    def L(self,y1,y2,b1,b2,z1,z2,sigma1,sigma2):
        likehood = []
        pi = np.arccos(-1)
        for z2_i,b1_i,b2_i,sigma1_i,sigma2_i in zip(z2,b1,b2,sigma1,sigma2):
            f1 = ((1/((2*pi*sigma1_i)**(self._params.n)))*np.exp(-(y1-z1@b1_i).T@(y1-z1@b1_i)/(2*sigma1_i)))
            f2 = ((1/((2*pi*sigma2_i)**(self._params.n)))*np.exp(-(y2-z2_i@b2_i).T@(y2-z2_i@b2_i)/(2*sigma2_i)))
            likehood.append(f2[0]*f1[0])
        return np.array(likehood)

    def pi_s(self,lamda):
        l = invgamma.pdf(x = lamda[:,0], a=1,scale = 0.001)
        r = invgamma.pdf(x = lamda[:,1], a=1,scale = 0.001)
        return l*r
```

```python
def prior(self,b1,b2,sigma1,sigma2,lamda):
    a0 = 5
    b0 = 0.005
    b01 = np.random.rand(2).reshape(-1,1)
    b02 = np.append(np.random.rand(2),0).reshape(-1,1)
    A11 = np.array([[0.1,0],[0,0.2]])
    A12 = np.array([[0.2,0],[0,0.3]])
    A21 = np.array([[0.2,0],[0,0.3]])
    A22 = np.array([[0.5,0],[0,0.2]])
    gamma_1 = lamda[:,0]*sigma1
    gamma_2 = lamda[:,1]*sigma2

    f1 = (sigma1**(1/2 + a0 -2))*(gamma_1**(a0-1))*np.exp(-gamma_1/(sigma1*b0))
    f2 = (sigma2**(1/2 + a0 -1))*(gamma_2**(a0-1))*np.exp(-gamma_2/(sigma2*b0))
    f3 = 1
    return f1*f2*f3


def importance_weights(self,y1,y2,b1,b2,z1,z2,sigma1,sigma2,lamda,posterior_such_lamda):
    return (L(y1,y2,b1,b2,z1,z2,sigma1,sigma2)*prior(b1,b2,sigma1,sigma2,lamda))/(pi_s(lamda)*posterior_such_lamda)


def sample(self):

    SUR_data = generate_SUR()
    y1 = self.SUR_data['Y'][:self._params.n]
    y2 = self.SUR_data['Y'][self._params.n:]
    x1 = self.SUR_data['x1']
    x2 = self.SUR_data['x2']
    a0 = 5
    b0 = 0.005


    b01 = np.random.rand(2).reshape(-1,1)
    b02 = np.append(np.random.rand(2),0).reshape(-1,1)

    A11 = np.array([[0.1,0],[0,0.2]])
    A12 = np.array([[0.2,0],[0,0.3]])
    A21 = np.array([[0.2,0],[0,0.3]])
    A22 = np.array([[0.5,0],[0,0.2]])
```

```
        W1 = A11
        W2 = A22 - A12@(inv(A11))@A21
        #generate lamda
        lamda = invgamma.rvs(a=1,scale = 0.001,size=(self.n_samples,2))

        gamma = lamda.copy() # How can we calculate the gamma if we don't have sigma ?

        #generate sigma1
        z1 = x1.copy()
        a1_hat = self._params.n  - 2*a0 + 1
        b1_tilde = inv(z1.T@z1)@z1.T@y1
        d1 = W1
        s1_hat = (((y1 - z1@b1_tilde).T@(y1 - z1@b1_tilde))/(self._params.n - 2))[0]
        b1_hat = np.array([inv(z1.T@z1+g*d1)@(inv(z1.T@z1)@b1_tilde + l*d1@(b0*np.ones(2).reshape(-1,1))) for g,l in zip(gamma[:,0],lamda[:,0])])
        h1_tilde = np.array([(g/b0 + (self._params.n -2)*s1_hat + (b1-b01).T@inv(z1.T@z1+l*d1)@(b1 - b01))[0] for g,b1,l in zip(gamma[:,0],b1_hat,lamda[:,0])])
        sigma1 = np.array([invgamma.rvs(a=a1_hat/2,scale = h/2) for  h in h1_tilde])
        sigma1_pdf = np.array([invgamma.pdf(x=s,a=a1_hat/2,scale = h/2) for  h,s in zip(h1_tilde,sigma1)])

        #generate b1
        b1 = np.array([np.random.multivariate_normal(mean = b1_hat_i.reshape(-1), cov = sigma1_i*inv(z1.T@z1 + g*d1)) for b1_hat_i,sigma1_i,g in zip(b1_hat,sigm
        b1_pdf = np.array([mvnorm.pdf(x =b1_i, mean = b1_hat_i.reshape(-1), cov = sigma1_i*inv(z1.T@z1 + g*d1)) for b1_i,b1_hat_i,sigma1_i,g in zip(b1,b1_hat,si

        #generate sigma2
        z2 = np.array([np.hstack((x2, y1 - x1@b.reshape(-1,1))) for b in b1])
        a2_hat = self._params.n -2*a0  +1
        b2_tilde = np.array([inv(z.T@z)@z.T@y2 for z in z2])
        d2 = np.pad(W2,[(0,1),(0,1)])
        s2_hat = np.array([(((y2 - z@b).T@(y2 - z@b))/(self._params.n - 1))[0] for z,b in zip(z2,b2_tilde)])
        b2_hat = np.array([inv(z.T@z+g*d2)@(inv(z.T@z)@b + l*d2@(b0*np.ones(3).reshape(-1,1))) for z,g,b,l in zip(z2,gamma[:,1],b2_tilde,lamda[:,1])])
        h2_tilde = np.array([(g/b0 + (self._params.n -1)*s2 + (b2-b02).T@inv(z.T@z+g*d2)@(b2 - b02))[0] for g,b2,s2,z in zip(gamma[:,1],b2_hat,s2_hat,z2)])
        sigma2 = np.array([invgamma.rvs(a=a2_hat/2,scale = h2/2) for  h2 in h2_tilde])
        sigma2_pdf = np.array([invgamma.pdf(x=s,a=a2_hat/2,scale = h/2) for  h,s in zip(h2_tilde,sigma2)])

        #generate b2
        b2 = np.array([np.random.multivariate_normal(mean = b2_hat_i.reshape(-1), cov = sigma2_i*inv(z.T@z + g*d2)) for b2_hat_i,sigma2_i,g,z in zip(b2_hat,sigm
        b2_pdf = np.array([mvnorm.pdf(x =b2_i,mean = b2_hat_i.reshape(-1), cov = sigma2_i*inv(z.T@z + g*d2)) for b2_i,b2_hat_i,sigma2_i,g,z in zip(b2,b2_hat,sig

        #get the initial parameters
        beta1 = b1
        beta2 = b2[:,:2]
        rho21 = b2[:,-1]
        omega1 = sigma1
        omega2 = rho21*sigma1 + sigma2
```

```
        omega1_2 = rho21*sigma1
        omega = np.array([np.array([[o1,o12],[o21,o22]]) for o1,o12,o21,o22 in zip(omega1,omega1_2,omega1_2,omega2)])
        posterior_such_lamda = sigma1_pdf.reshape(-1)*sigma2_pdf.reshape(-1)*b1_pdf*b2_pdf

        return beta1, beta2, omega, lamda
```

In [ ]:
```
DMC_IS_sampler = DMC_IS_Sampler(SUR_data=generate_SUR())
beta1, beta2, omega, lamda = DMC_IS_sampler.sample()
```

In [ ]:
```
print(f'Mean: {np.mean(beta1, axis=0)};')
print(f'real: {params.beta1.reshape(-1)};')
```

```
Mean: [-0.00034093  0.00044798];
real: [ 3 -2];
```

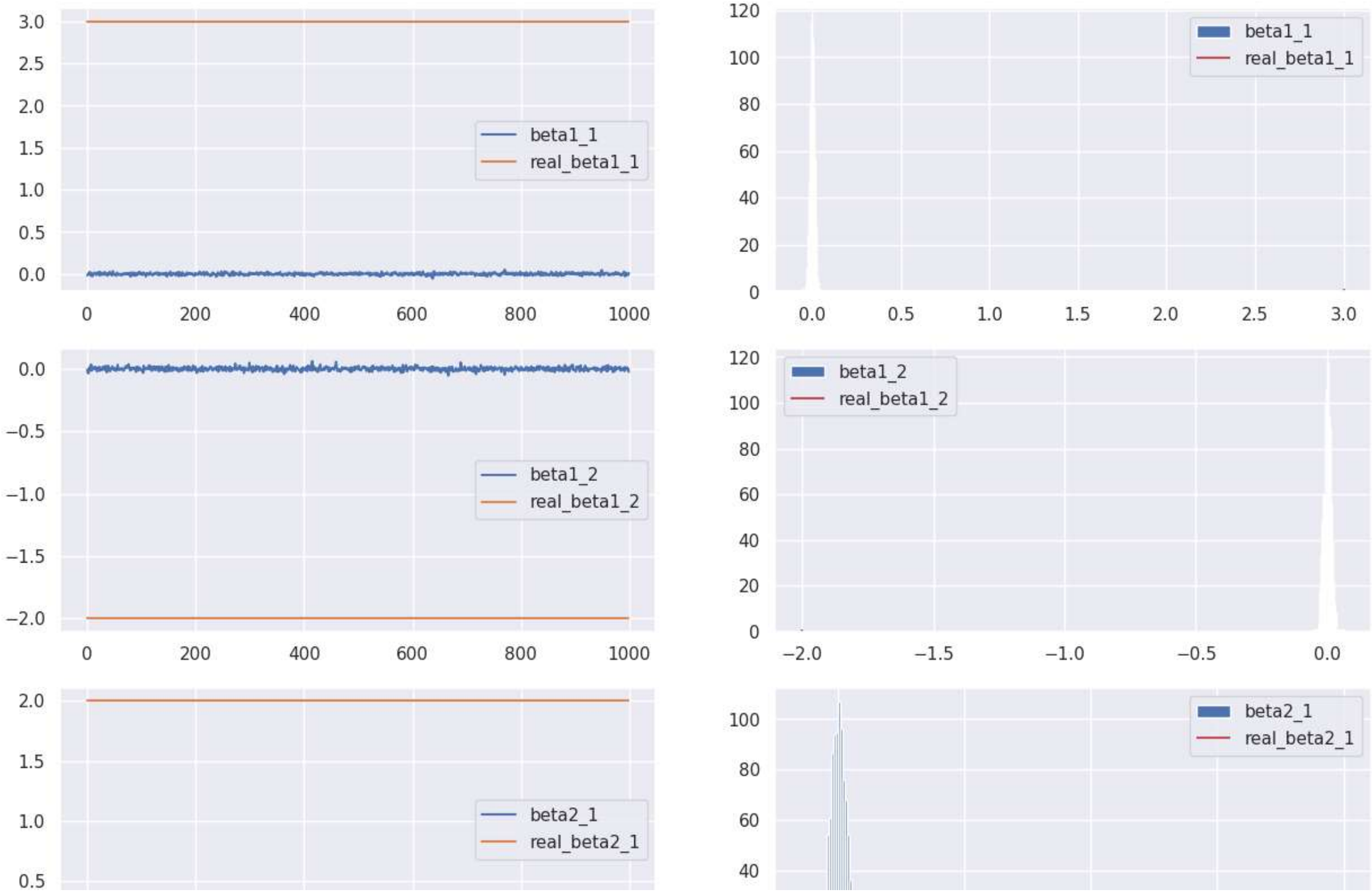
In [ ]:
```
print(f'Mean: {np.mean(beta2, axis=0)};')
print(f'real: {params.beta2.reshape(-1)};')
```

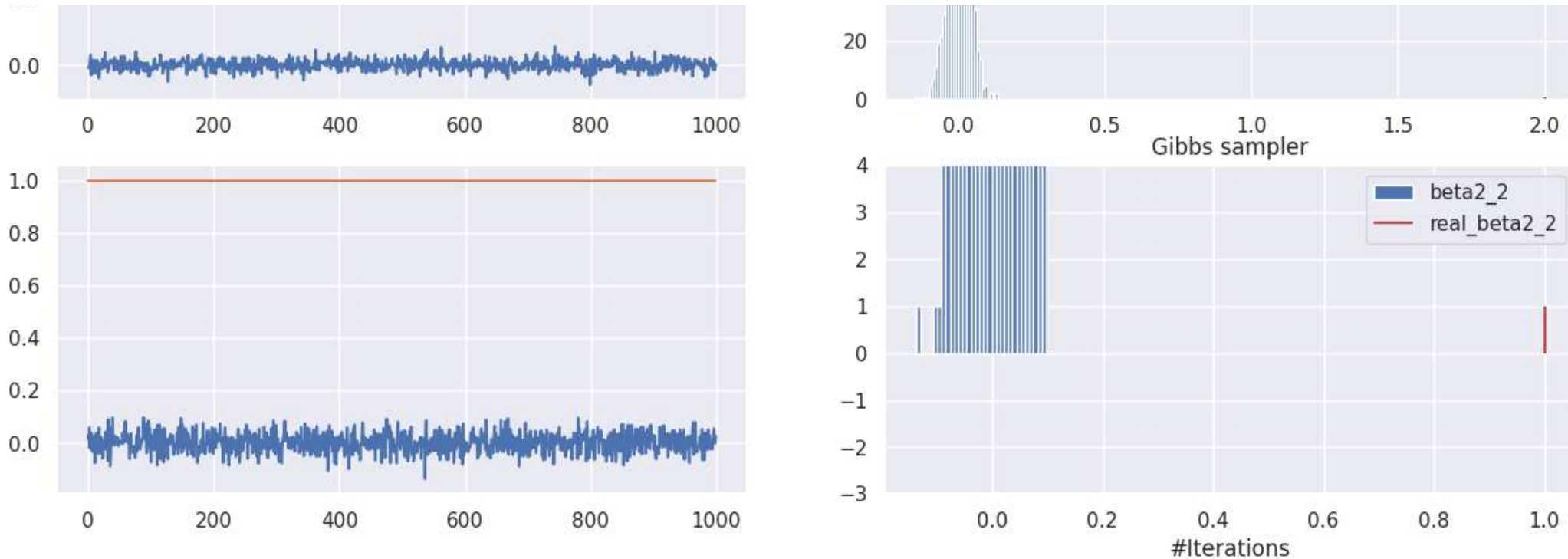
```
Mean: [0.00122583 0.00138709];
real: [2 1];
```

In [ ]:
```
print(f'Mean: {np.mean(omega, axis=0)};')
print(f'real: {params.omega};')
```

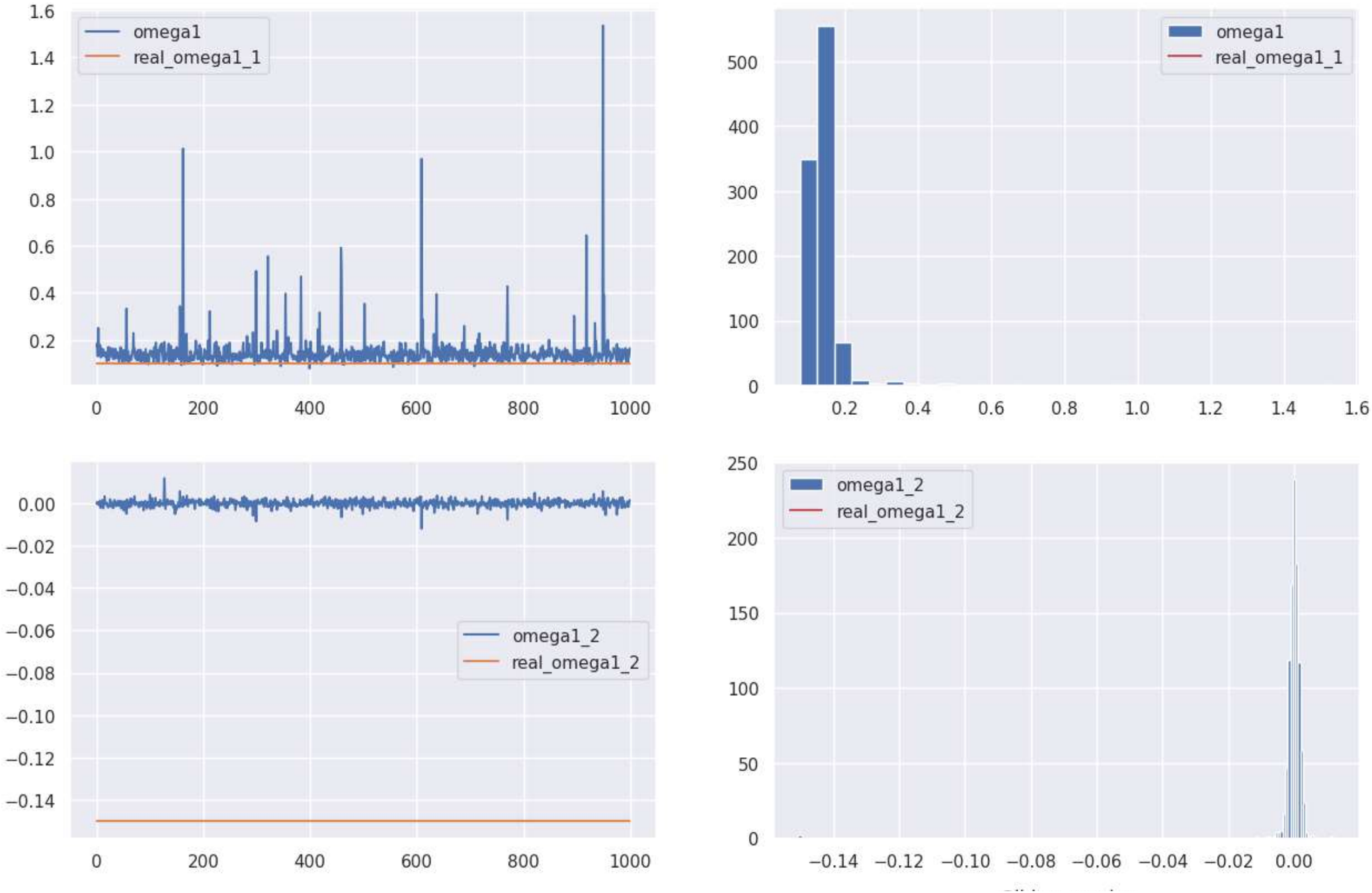
```
Mean: [[ 1.44641854e-01 -2.82382155e-05]
 [-2.82382155e-05  1.17588176e+00]];
real: [[ 0.1  -0.15]
 [-0.15  0.2 ]];
```

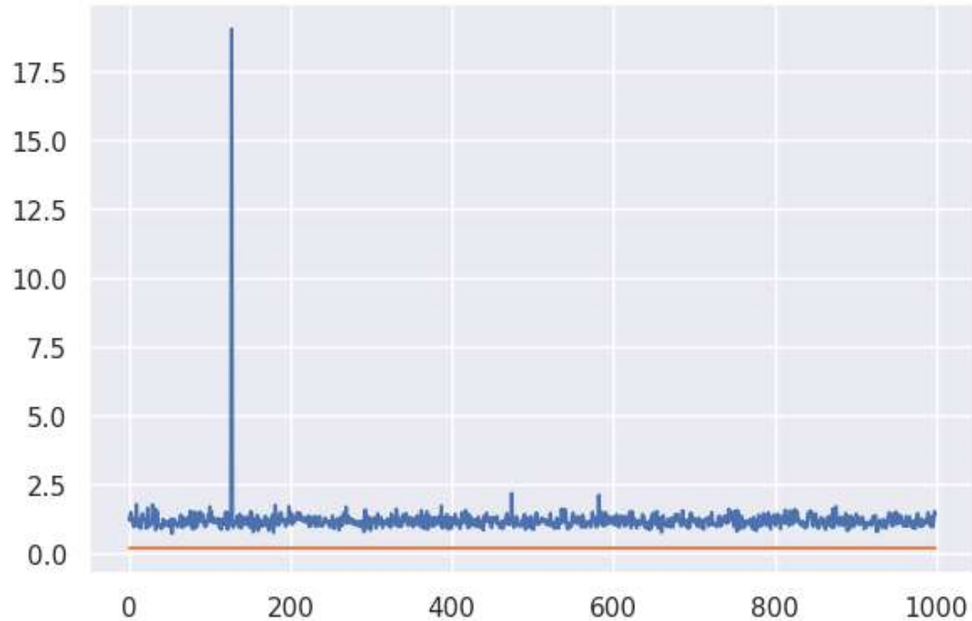
In [ ]:
```
beta = np.hstack((beta1.reshape(-1,2),beta2.reshape(-1,2)))
plot_beta(beta)
```

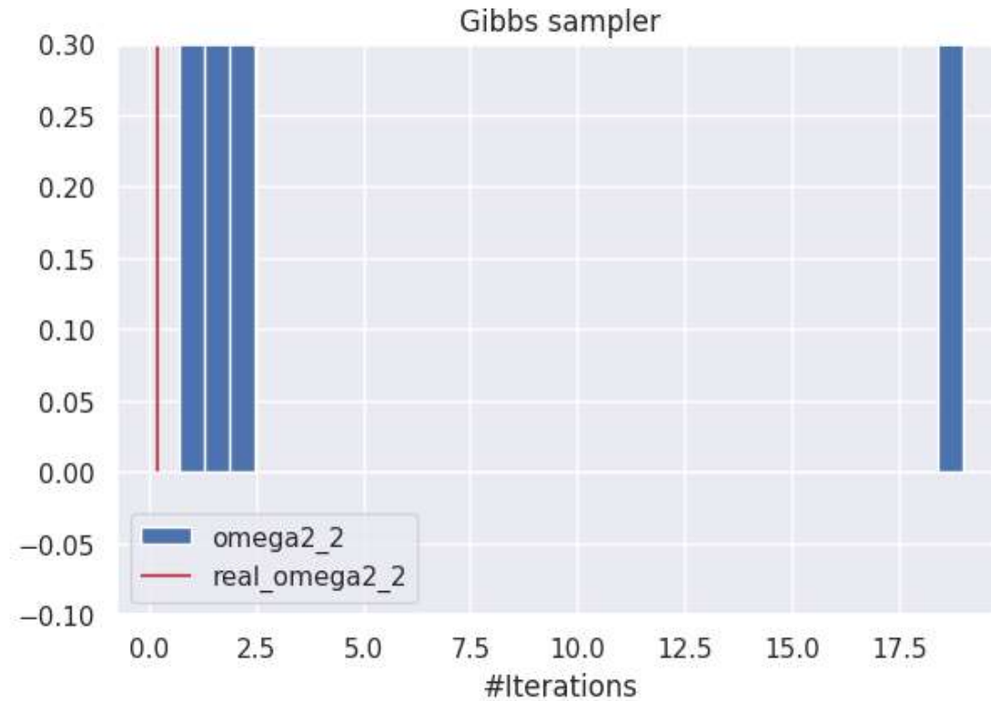

beta1_1
real_beta1_1
beta1_2
real_beta1_2
beta2_1
real_beta2_1



In [ ]:
```
plot_omega(omega)
```


omega1
real_omega1_1
omega1
real_omega1_1
omega1_2
real_omega1_2
omega1_2
real_omega1_2





Direct Monte Carlo Importance Sampling Approach Based On a Posterior Of Degree Of Freedom V

In [ ]:
```
def multivariatet(mu,Sigma,N,M):
    '''
    Output:
    Produce M samples of d-dimensional multivariate t distribution
    Input:
    mu = mean (d dimensional numpy array or scalar)
    Sigma = scale matrix (dxd numpy array)
    N = degrees of freedom
    M = # of samples to produce
    '''
    d = len(Sigma)
    g = np.tile(np.random.gamma(N/2.,2./N,M),(d,1)).T
```

```
    Z = np.random.multivariate_normal(np.zeros(d),Sigma,M)
    return mu + Z/np.sqrt(g)
```

In [ ]:

```
class DMC_IS_Sampler_V(Sampler):
    def __init__(self, SUR_data, params = DefaultParams(), n_samples=10000) -> None:
        super().__init__(SUR_data)
        self.n_samples = n_samples
        self._params = params

    def h(self,v,n):
        pi = np.arccos(-1)
        return (v**(v/2))*(v+n/2)/(pi)*math.gamma(v/2)

    def L(self,y1,y2,b1,b2,z1,z2,v,sigma1,sigma2,n):
        likehood = []
        pi = np.arccos(-1)
        for z2_i,b1_i,b2_i,sigma1_i,sigma2_i,v_i in zip(z2,b1,b2,sigma1,sigma2,v):
            f1 = (self.h(v_i,n)/(sigma1_i**(n/2)))*((v_i + ((y1 - z1@b1_i.reshape(-1,1)).T@(y1-z1@b1_i.reshape(-1,1)))/sigma1_i)**(-(n+v_i)/2))
            f2 = (self.h(v_i,n)/(sigma2_i**(n/2)))*((v_i + ((y2 - z2_i@b2_i.reshape(-1,1)).T@(y2-z2_i@b2_i.reshape(-1,1)))/sigma2_i)**(-(n+v_i)/2))
            likehood.append(f1[0][0]*f2[0][0])

        return np.array(likehood)


    def posterior(self,y1,y2,b1, b2, sigma1, sigma2, n, q1, q2, v, z1, z2):
        f = []
        for b1_i,b2_i,sigma1_i,sigma2_i,v_i,z2_i in zip(b1,b2,sigma1,sigma2,v,z2):
            f1 = (sigma1_i**(-(n+q1)/2))*(v_i + (((y1-z1@b1_i.reshape(-1,1)).T@(y1-z1@b1_i.reshape(-1,1)))/(sigma1_i))**(-(n+v_i)/2))
            f2 = (sigma2_i**(-(n+q2)/2))*(v_i + (((y2-z2_i@b2_i.reshape(-1,1)).T@(y2-z2_i@b2_i.reshape(-1,1)))/(sigma2_i))**(-(n+v_i)/2))

            f.append(f1[0][0]*f2[0][0])
        return np.array(f)


    def prior(self,b1,b2,q1,q2,sigma1,sigma2,v):
        fv = expon(3).pdf(v)
        f1 = sigma1**(-q1/2)
        f2 = sigma2**(-q2/2)
        return f1*f2*fv


    def importance_weights(self,y1,y2,z1,z2,b1,b2,sigma1,sigma2,n,q1,q2,v):
        prior_s = self.prior(b1,b2,q1,q2,sigma1,sigma2,v)
```

```
        post_s = self.posterior(y1,y2,b1, b2, sigma1, sigma2, n, q1, q2, v, z1, z2) + 0.00001
        likehood = self.L(y1,y2,b1,b2,z1,z2,v,sigma1,sigma2,n)

        w = prior_s*likehood/post_s
        w = w/(np.sum(w))
        return w


    def sample(self):

        SUR_data = generate_SUR()
        y1 = self.SUR_data['Y'][:self._params.n]
        y2 = self.SUR_data['Y'][self._params.n:]
        x1 = self.SUR_data['x1']
        x2 = self.SUR_data['x2']

        #generate v from  a prior exponential
        v = []
        for i in range(self.n_samples):
            y = np.random.exponential(3)
            while(y<2):
                y = np.random.exponential(3)
            v.append(y)
        v = np.array(v)

        #generate sigma1
        z1 = x1.copy()
        b1_hat = inv(z1.T@z1)@z1.T@y1
        p1_tilde = 2
        v1_hat  = self._params.n - 2 +  v
        s1 = ((y1 - z1@b1_hat).T@(y1 - z1@b1_hat)/(self._params.n - p1_tilde))[0][0]
        v1_tilde = self._params.n - p1_tilde
        q1 = 1

        temp_sigma1 = np.array([np.random.beta((v_i-q1+2)/2,(self._params.n + q1 - p1_tilde -2)/2) for v_i in v])
        temp_sigma1 = (temp_sigma1)/(1-temp_sigma1)
        sigma1 = (temp_sigma1/v)*(v1_tilde*s1)
        sigma1_tilde = ((v*sigma1 + (v1_hat - v)*s1)/v1_hat)

        #generate b1
        b1 = np.array([multivariatet(b1_hat.reshape(-1),sigma*inv(z1.T@z1),v1_tilde,1) for sigma  in sigma1_tilde]).reshape(-1,2)

        #generate sigma2
```

```
        q2 = 1
        p2_tilde = 3
        v2_hat = self._params.n - p2_tilde + v
        z2 = np.array([np.hstack((x2, y1 - x1@(b.reshape(-1,1)))) for b in b1])
        b2_hat = np.array([inv(z.T@z)@z.T@y2 for z in z2])
        s2 = np.array([(((y2 - z@b).T@(y2 - z@b)/(self._params.n - p2_tilde))[0][0] for b,z in zip(b2_hat,z2)])
        v2_tilde = self._params.n - p2_tilde

        temp_sigma2 = np.array([np.random.beta((v_i-q2+2)/2,(self._params.n + q2 - p2_tilde -2)/2) for v_i in v])
        temp_sigma2 = (temp_sigma2)/(1-temp_sigma2)
        sigma2 = (temp_sigma2/v)*(v2_tilde*s2)
        sigma2_tilde = ((v*sigma2 + (v2_hat - v)*s2)/v2_hat)

        #generate b2
        b2 = np.array([multivariatet(b.reshape(-1),sigma*inv(z.T@z),v2_tilde,1) for sigma,b,z  in zip(sigma2_tilde,b2_hat,z2)]).reshape(-1,3)


        beta1 = b1
        beta2 = b2[:,:2]
        rho21 = b2[:,-1]
        omega1 = sigma1
        omega2 = rho21*sigma1 + sigma2
        omega1_2 = rho21*sigma1
        omega = np.array([np.array([[o1,o12],[o21,o22]]) for o1,o12,o21,o22 in zip(omega1,omega1_2,omega1_2,omega2)])

        w = self.importance_weights(y1,y2,z1,z2,b1,b2,sigma1,sigma2,self._params.n,q1,q2,v)

        return beta1, beta2, omega, w
```

In [ ]:
```
DMC_IS_sampler_V = DMC_IS_Sampler_V(SUR_data=generate_SUR())
beta1, beta2, omega, w = DMC_IS_sampler_V.sample()
```

In [ ]:
```
result_dmc_is = pd.DataFrame()
result_dmc_is['beta11'] = beta1[:,0]
result_dmc_is['beta12'] = beta1[:,1]
result_dmc_is['beta21'] = beta2[:,0]
result_dmc_is['beta22'] = beta2[:,1]
result_dmc_is['omega11'] = omega[:,0,0]
result_dmc_is['omega12'] = omega[:,0,1]
result_dmc_is['omega22'] = omega[:,1,1]
result_dmc_is['method'] = 'DMC-IS'
```

In [ ]:
```
print(f'Mean: {np.mean(beta1, axis=0)};')
print(f'real: {params.beta1.reshape(-1)};')
```

```
Mean: [ 3.00026783 -1.98214178];
real: [ 3 -2];
```

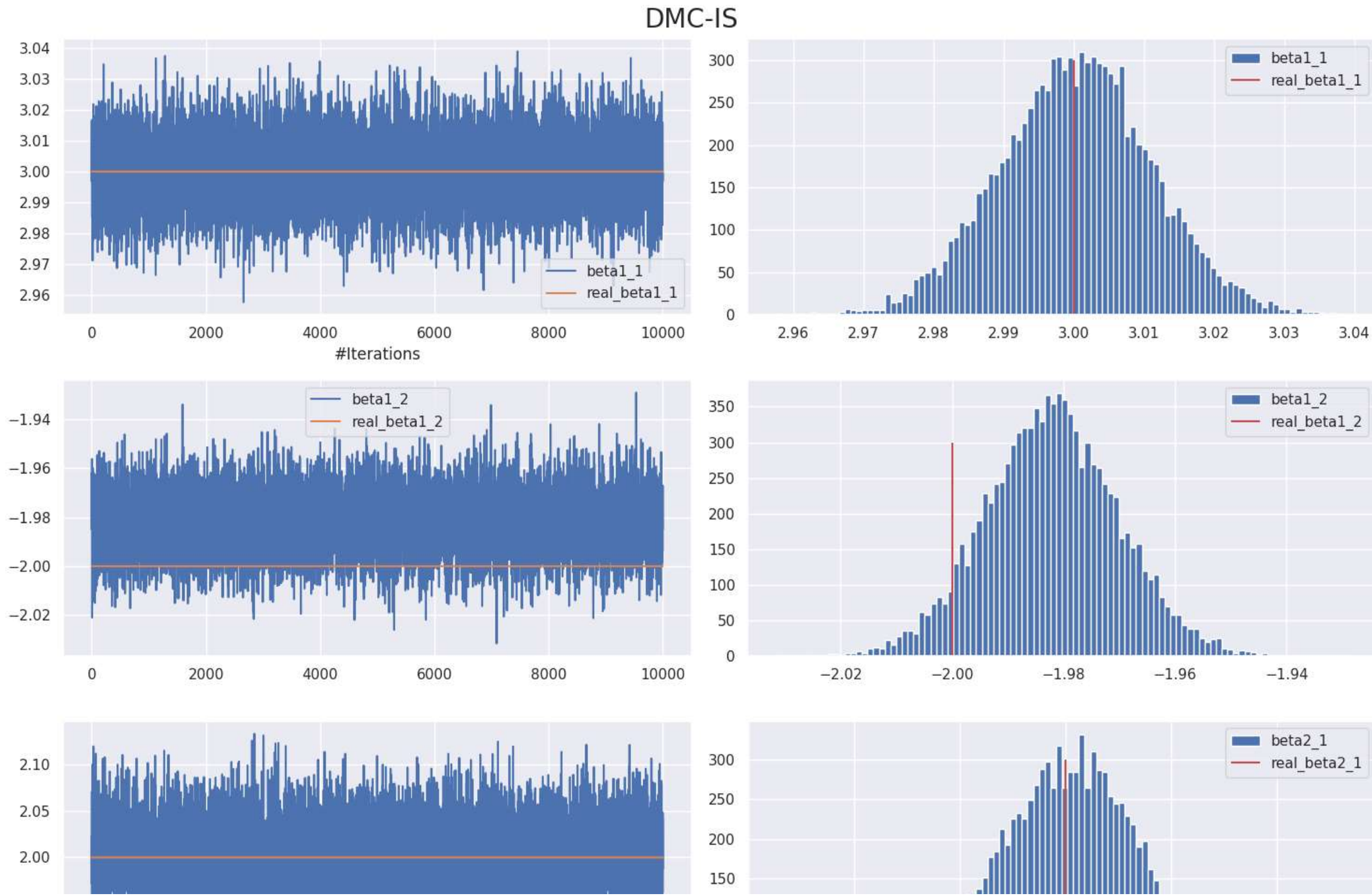
In [ ]:
```
print(f'Mean: {np.mean(beta2, axis=0)};')
print(f'real: {params.beta2.reshape(-1)};')
```

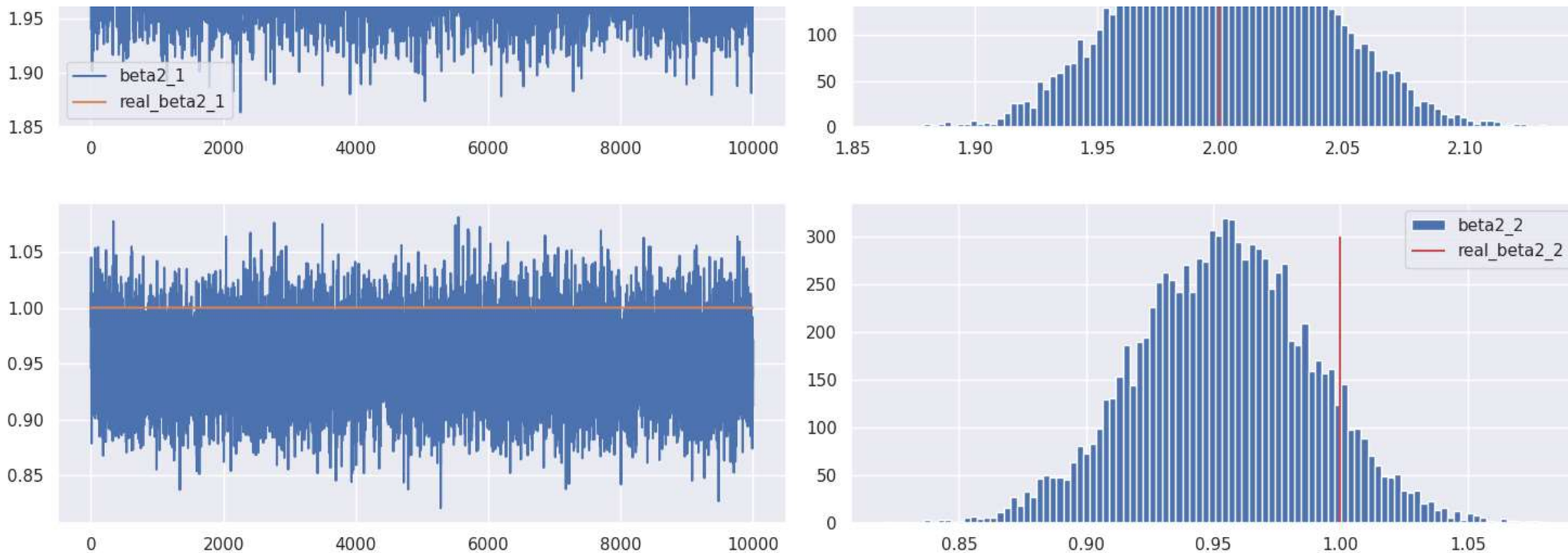
```
Mean: [2.00231161 0.95445803];
real: [2 1];
```

In [ ]:
```
print(f'Mean: {np.mean(omega, axis=0)};')
print(f'real: {params.omega};')
```

```
Mean: [[ 0.13646275 -0.15300753]
 [-0.15300753  1.14121271]];
real: [[ 0.1  -0.15]
 [-0.15  0.2 ]];
```

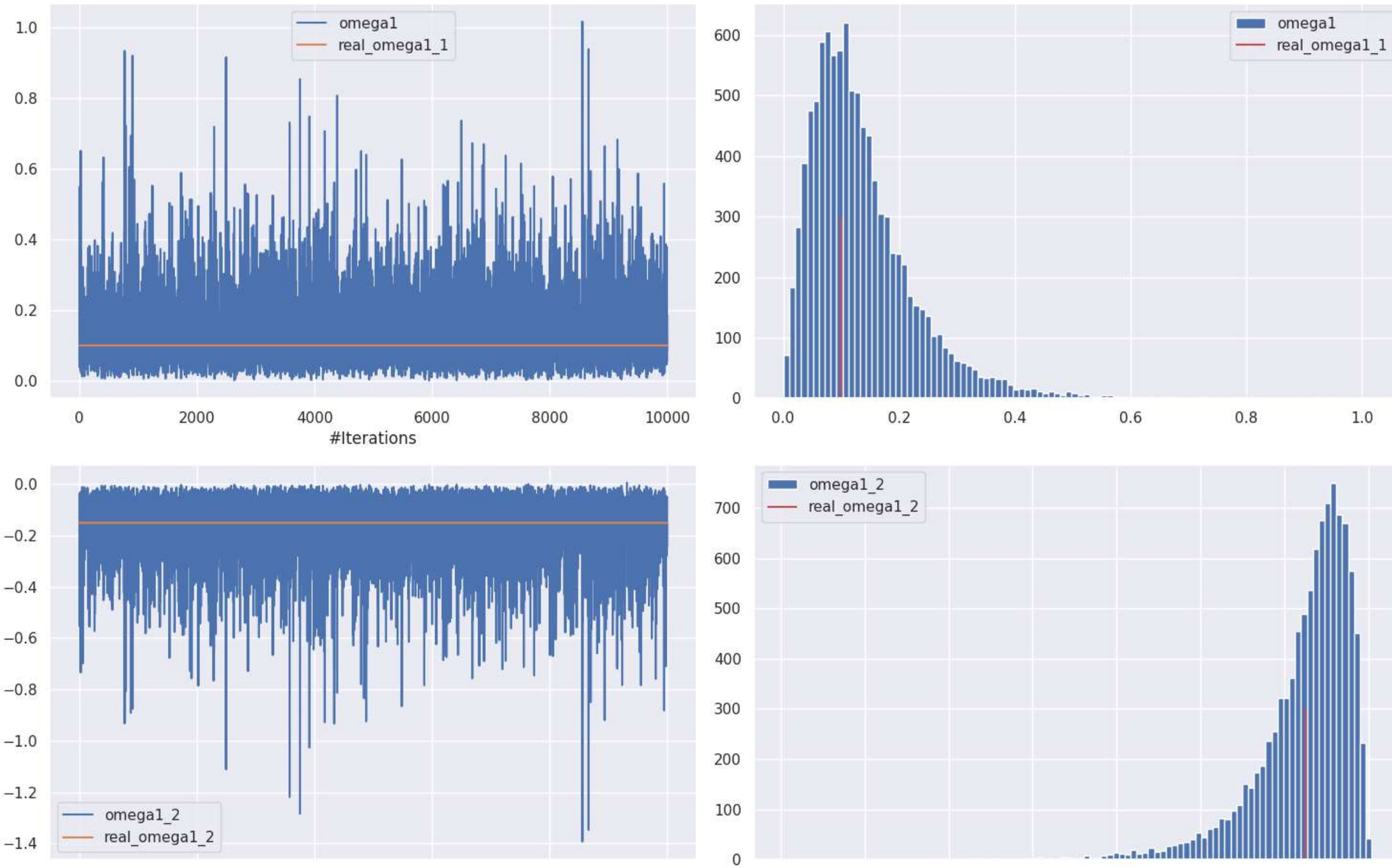
In [ ]:
```
beta = np.hstack((beta1.reshape(-1,2),beta2.reshape(-1,2)))
plot_beta(beta, title="DMC-IS")
```

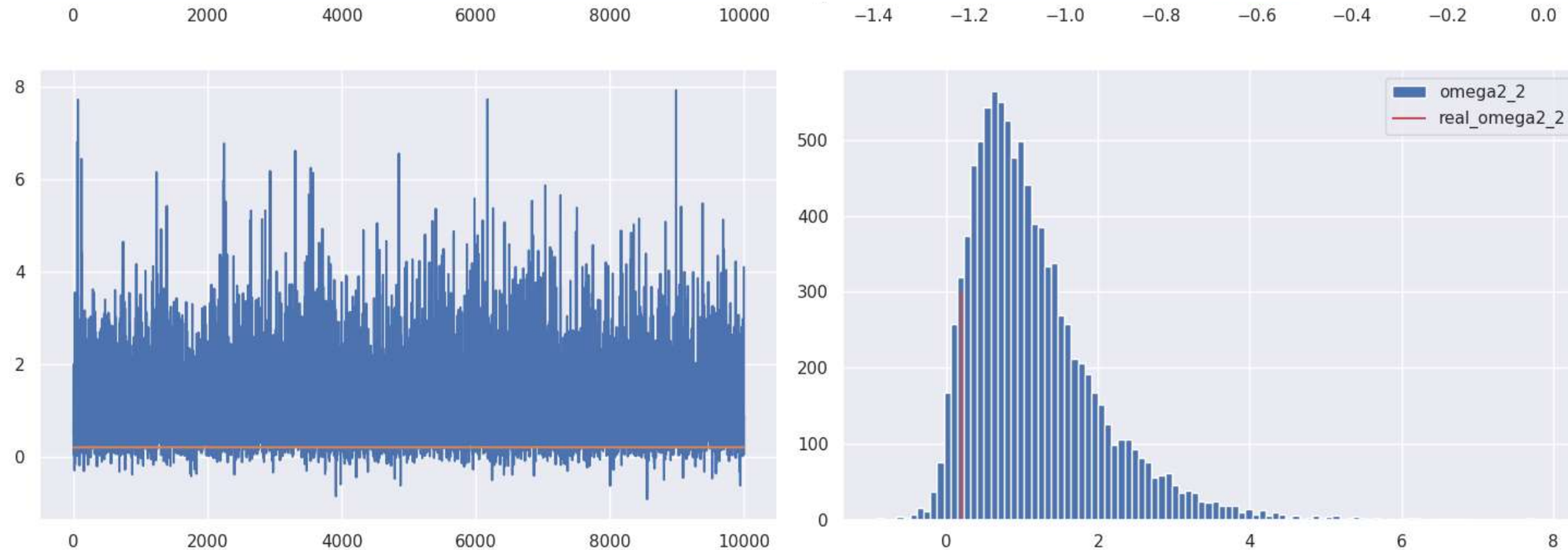

DMC-IS
beta1_1
real_beta1_1
#Iterations
beta1_2
real_beta1_2
beta2_1
real_beta2_1



In [ ]:
```
plot_omega(omega, title="DMC-IS")
```

DMC-IS





In [ ]:
```
result = pd.concat([result_mcmc,result_dmc,result_dmc_is])
```

In [ ]:
```
result.head()
```

Out[ ]:

	beta11	beta12	beta21	beta22	omega11	omega12	omega22	method
0	3.019683	-2.002239	1.985053	1.049338	0.100102	-0.115917	1.527808	MCMC
1	2.994201	-1.991487	2.087773	0.956920	0.112141	-0.140434	1.042604	MCMC
2	2.997580	-2.022459	2.001482	0.971150	0.095671	-0.089073	0.900431	MCMC
3	3.005104	-2.009856	2.039633	0.876046	0.086060	-0.066424	1.280196	MCMC
4	2.988944	-1.995380	2.007683	1.028890	0.105332	-0.181331	1.159984	MCMC

In [ ]:
```
def CI_95_high(x):
    return np.mean(x) + 1.96 * np.std(x)
```

In [ ]:
```
def CI_95_low(x):
    return np.mean(x) - 1.96 * np.std(x)
```

In [ ]:
```
a = result[['method','beta11','beta12']].groupby(['method']).agg(['mean', 'std', CI_95_low, CI_95_high])
```

In [ ]:
```
a.loc['True_value'] = [3, '-', '-', '-', -2, '-', '-', '-']
```

In [ ]:
```
a
```

Out[ ]:

	beta11				beta12			
	mean	std	CI_95_low	CI_95_high	mean	std	CI_95_low	CI_95_high
method								
DMC	3.000237	0.011803	2.977105	3.023369	-2.009619	0.011489	-2.032137	-1.987101
DMC-IS	3.000268	0.010885	2.978934	3.021601	-1.982142	0.011942	-2.005547	-1.958737
MCMC	2.995966	0.011061	2.974288	3.017644	-1.995835	0.010207	-2.015839	-1.975831
True_value	3.000000	-	-	-	-2.000000	-	-	-

In [ ]:
```
params.beta2.reshape(-1)
```

Out[ ]:
```
array([2, 1])
```

In [ ]:
```
a2 = result[['method','beta21','beta22']].groupby(['method']).agg(['mean', 'std', CI_95_low, CI_95_high])
a2.loc['True_value'] = [2, '-', '-', '-', 1, '-', '-', '-']
a2
```

Out[ ]:

	beta21				beta22			
	mean	std	CI_95_low	CI_95_high	mean	std	CI_95_low	CI_95_high
method								
DMC	1.981391	0.039064	1.90483	2.057952	1.034646	0.036735	0.962648	1.106644
DMC-IS	2.002312	0.036883	1.930025	2.074598	0.954458	0.03477	0.886313	1.022603
MCMC	2.005689	0.034997	1.937099	2.074279	0.997878	0.039184	0.921081	1.074674
True_value	2.000000	-	-	-	1.000000	-	-	-

In [ ]:
```
params.omega
```

Out[ ]:
```
array([[ 0.1 , -0.15],
       [-0.15,  0.2 ]])
```

In [ ]:
```
b = result[['method','omega11','omega12','omega22']].groupby(['method']).agg(['mean', 'std', CI_95_low, CI_95_high])
```

In [ ]:
```
b.loc['True_value'] = [0.1, '-', '-', '-', -0.15, '-', '-', '-', 0.2, '-', '-', '-']
b
```

Out[ ]:

	omega11				omega12				omega22			
	mean	std	CI_95_low	CI_95_high	mean	std	CI_95_low	CI_95_high	mean	std	CI_95_low	CI_95_high
method												
DMC	0.107297	0.01554	0.076839	0.137755	-0.161437	0.041592	-0.242954	-0.079921	0.898778	0.161595	0.582067	1.215489
DMC-IS	0.136463	0.094046	-0.047858	0.320783	-0.153008	0.117556	-0.383406	0.077391	1.141213	0.87518	-0.574054	2.85648
MCMC	0.106693	0.015418	0.076476	0.136911	-0.127667	0.039334	-0.204758	-0.050577	1.225412	0.182722	0.867295	1.583529
True_value	0.100000	-	-	-	-0.150000	-	-	-	0.200000	-	-	-

In [ ]: